Abstract Algebra

Homework 8: 1, 12.1, 12.4, 12.13

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Due April 27

1) Prove $Q_8 \ncong D_8$.

Assume for the sake of contradiction $Q_8 \cong D_8$. Then there exists a function, call it f, such that $f : Q_8 \to D_8$ is one-to-one and onto.

Q_8 contains one element of order 2 (-1), while D_8 contains 3 elements of order 2 (r^2, s and sr^2). Since an isomorphism must be one-to-one, $f(-1)$ can only be one of these three, and thus some element of Q_8 of order not equal to 2 must map to r^2. But since these two elements have different order, this implies by our theorems that f is not an isomorphism, thus Q_8 and D_8 are not isomorphic to each other.

12.1) Which of the following mappings are homomorphisms? Monomorphisms? Epimorphisms? Isomorphisms?

a) $G = (\mathbb{R} - \{0\}, \cdot), H = (\mathbb{R}^+, \cdot); \varphi : G \to H$ is given by $\varphi(x) = |x|$

Let $x, y \in G$. We see $\varphi(xy) = |xy| = |x||y| = \varphi(x)\varphi(y)$, so φ is a homomorphism. Now let $a = 2$, $b = -2$. We see $a, b \in G$ and $\varphi(a) = |2| = 2 = |-2|\varphi(b)$, so φ is not one-to-one. Now let $z \in H$. Then z is positive, thus $z = |z|$ by definition of absolute value. Also, since $z \in \mathbb{R}^+$, then $z \in \mathbb{R}$, and $\varphi(z) = |z| = z$, so φ is onto. Therefore φ is an epimorphism.

b) $G = (\mathbb{R}^+, \cdot); \varphi : G \to G$ is given by $\varphi(x) = \sqrt{x}$.

Let $x, y \in G$. Then $\varphi(xy) = \sqrt{xy} = \sqrt{x}\sqrt{y} = \varphi(x)\varphi(y)$, so φ is a homomorphism. Now let $a, b \in G$ such that $\varphi(a) = \varphi(b)$. Then $\sqrt{a} = \sqrt{b}$, and squaring both sides we see $a = b$, thus φ is one-to-one. Now let $z \in G$. Then $z^2 \in G$, since $\mathbb{R}^+$ is closed under multiplication, and we see $\sqrt{z^2} = z$. Thus φ is onto, so φ is an isomorphism. (Actually, since φ is an isomorphism from G onto itself, φ is an automorphism)

c) G = group of polynomials $p(x)$ with real coefficients, under addition of polynomials; $\varphi : G \to G$ is given by $\varphi[p(x)] = p(1)$.

Let $f, g \in G$. Then $\varphi(f+g) = (f+g)(1) = f(1) + g(1) = \varphi(f)\varphi(g)$, thus φ is a homomorphism. Now let $A = x^3$ and $B = x^7$. Then $\varphi(A) = 1 = \varphi(B)$, so φ is not one-to-one. Now let $z \in \mathbb{R}^+$, and consider the function $Y(x) = z$, which is in G. Then $\varphi(Y) = z$, thus φ is onto, so φ is an epimorphism.

d) G is as in **c**; $\varphi : G \to G$ is given by $\varphi[p(x)] = p'(x)$.

Let $f, g \in G$. Then $\varphi(f+g) = \frac{\mathrm{d}}{\mathrm{d}x}(f+g) = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{\mathrm{d}g}{\mathrm{d}x} = \varphi(f) + \varphi(g)$, so φ is a homomorphism. Now let $A = x^2$ and $B = x^2 + 2$. Then $\varphi(A) = 2x = \varphi(B)$, but $A \neq B$, so φ is not one-to-one. Now let $Y \in G$. Then Y can be written in the form:

$$Y = a_1 x^n + a_2 x^{n-1} + ... + a_{n-1} x + a_n$$

Here, n is the degree of Y. Consider the function Z as defined by:

$$Z = \frac{a_1}{n} x^n + \frac{a_2}{n-1} x^{n-1} + ... + \frac{a_{n-1}}{2} x^2 + a_n x$$

Note that $Z \in G$, and $\varphi(Z) = Y$. This implies that φ is onto. Therefore φ is an epimorphism.

e) $G =$ the group of subsets of $\{1,2,3,4,5\}$ under symmetric difference; $A = \{1,3,4\}$ and $\varphi : G \to G$ is given by $\varphi(B) = A \triangle B$, for all $B \subset \{1,2,3,4,5\}$.

(I still have to work this one out)

12.4) In each case, determine whether or not the two given groups are isomorphic:
a) $(\mathbb{Z}_{12}, \bigoplus)$ and $(\mathbb{Q}^+, \cdot)$

Since $|\mathbb{Z}_{12}| = 12 \neq \infty = |\mathbb{Q}^+|$, there cannot exist a bijection between these two sets, thus there cannot be an isomorphism between these two groups.

b) $(2\mathbb{Z}, +)$ and $(3\mathbb{Z}, +)$

These two groups are isomorphic by the function $\varphi : 2\mathbb{Z} \to 3\mathbb{Z}$ as defined by $\varphi(x) = \frac{3}{2}x$. Let $x, y \in 2\mathbb{Z}$. We see

$$\begin{aligned}\varphi(x+y) &= \frac{3}{2}(x+y) \\ &= \frac{3}{2}x + \frac{3}{2}y \\ &= \varphi(x) + \varphi(y)\end{aligned}$$

Thus φ is a homomorphism. Now assume $\varphi(x) = \varphi(y)$. Then $\frac{3}{2}x = \frac{3}{2}y$, and this implies $x = y$, so φ is a monomorphism. Now let $z \in 3\mathbb{Z}$. Then $z = 3a$ for some $a \in \mathbb{Z}$. Note that

$2a \in 2\mathbb{Z}$, and $\varphi(2a) = 3a = z$. So φ is an epimorphism. Therefore φ is an isomorphism.

c) $(\mathbb{R} - \{0\}, \cdot)$ and $(\mathbb{R}, +)$
No isomorphism exists between these two groups since $-1 \in \mathbb{R} - \{0\}$ and $o(-1) = 2$, but no element of $(\mathbb{R}, +)$ has order 2.

d) V and $\mathbb{Z}_2 \times \mathbb{Z}_2$
These two groups are isomorphic by the function $\varphi : V \to \mathbb{Z}_2$ as defined by

$$\begin{aligned}
\varphi(e) &= e \\
\varphi(a) &= (1, 0) \\
\varphi(b) &= (0, 1) \\
\varphi(c) &= (1, 1)
\end{aligned}$$

One can check that this is a homomorphism by going through each case, and I have done so. Due to the extensiveness of writing this out, I will not be doing so here. Once you are convinced that it is a homomorphism, note that this definition is obviously one-to-one and onto, so it is an isomorphism.

e) $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_9$
No isomorphism exists between these two groups. Recall $o(x) = o(\varphi(x))$ for an isomorphism φ. $5 \in \mathbb{Z}_9$ has $o(5) = 9$, but no element of $\mathbb{Z}_3 \times \mathbb{Z}_3$ has order 9 (in fact all of them except the identity have order 3). Therefore no isomorphism can exist between these two groups.

f) $(\mathbb{R} - \{0\}, \cdot)$ and $(\mathbb{R}^+, \cdot) \times (\mathbb{Z}_2, \bigoplus)$
These groups are isomorphic under the function $\varphi : \mathbb{R} - \{0\} \to (\mathbb{R}^+, \cdot) \times (\mathbb{Z}_2, \bigoplus)$ as defined by $\varphi(x) = \begin{cases} (|x|, 1) & x < 0 \\ (|x|, 0) & x > 0 \end{cases}$ Let $x, y \in \mathbb{R} - \{0\}$. Either x and y are both negative or positive together, or they have different signs. Consider when both x and y are positive:

$$\begin{aligned}
\varphi(xy) &= (|xy|, 0) \\
&= (|x||y|, 0) \\
&= (|x|, 0)(|y|, 0) \\
&= \varphi(x)\varphi(y)
\end{aligned}$$

so this case satisfies the definition of a homomorphism. Now consider when x and y are negative (remember, the product of two negatives is positive):

$$\begin{aligned}\varphi(xy) &= (|xy|, 0)\\ &= (|x||y|, 0)\\ &= (|x|, 1)(|y|, 1)\\ &= \varphi(x)\varphi(y)\end{aligned}$$

so this case also satisfies the definition of a homomorphism. Now consider the case where one is positive, and one is negative. Without loss of generality we will consider x to be the positive one:

$$\begin{aligned}\varphi(xy) &= (|xy|, 1)\\ &= (|x||y|, 1)\\ &= (|x|, 0)(|y|, 1)\\ &= \varphi(x)\varphi(y)\end{aligned}$$

so every case satisfies the definition of a homomorphism, so φ is a homomorphism. Now assume $\varphi(x) = \varphi(y)$. Then (with $a, b \in \{0, 1\}$):

$$\begin{aligned}&(|x|, a) = (|y|, b)\\ \implies &|x| = |y| a = b\end{aligned}$$

This is true if either $x = y$ or $x = -y$. But since $a = b$, we know x and y have the same sign, so $x = y$. Therefore φ is one-to-one. Now consider $(z, c) \in (\mathbb{R}^+, \cdot) \times (\mathbb{Z}_2, \bigoplus)$. If $c = 0$, then $z \in \mathbb{R} - \{0\}$ and $\varphi(z) = (z, 0)$. If, however, $z = 1$, then $-z \in \mathbb{R} - \{0\}$ and $\varphi(-z) = (z, 1)$. Therefore φ is onto. Therefore we have shown that φ is an isomorphism.

g) $(\mathbb{Z}, +)$ and $(\mathbb{Z}, *)$ where $a * b = a + b - 1$
These two groups are isomorphic by the function $\varphi : (\mathbb{Z}, +) \to (\mathbb{Z}, *)$ as defined by $\varphi(x) = x + 1$. Let $x, y \in \mathbb{Z}$. We see:

$$\begin{aligned}\varphi(x + y) &= x + y + 1\\ &= (x + 1) + (y + 1) - 1\\ &= \varphi(x)\varphi(y)\end{aligned}$$

so φ is a homomorphism. Now assume $\varphi(x) = \varphi(y)$. Then $x + 1 = y + 1 \implies x = y$, so φ is one-to-one. Now let $z \in \mathbb{Z}$. Then $z - 1 \in \mathbb{Z}$ and $\varphi(z - 1) = z - 1 + 1 = z$. Therefore φ is onto, so φ is an isomorphism.

h) G and $G \times G$ where $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ...$
Note that all elements of G have the form $(a_1, a_2, a_3, ...)$. These two groups are isomorphic by the function $\varphi : G \to G \times G$ as defined by $\varphi[(a_1, a_2, a_3, ...)] = \big((a_1, a_3, a_5, ...), (a_2, a_4, a_6, ...)\big)$.

Let $(a_1, a_2, a_3, ...), (b_1, b_2, b_3, ...) \in G$. We see:

$$\begin{aligned}
\varphi[(a_1, a_2, a_3, ...)(b_1, b_2, b_3, ...)] &= \varphi(a_1 + b_1, a_2 + b_2, a_3 + b_3, ...) \\
&= \big((a_1 + b_1, a_3 + b_3, ...), (a_2 + b_2, a_4 + b_4, ...)\big) \\
&= \big((a_1, a_3, ...), (a_2, a_4, ...)\big) \big((b_1, b_3, ...), (b_2, b_4, ...)\big) \\
&= \varphi[(a_1, a_2, a_3, ...)]\varphi[(b_1, b_2, b_3, ...)]
\end{aligned}$$

so φ is a homomorphism. Now assume $\varphi[(a_1, a_2, a_3, ...)] = \varphi[(b_1, b_2, b_3, ...)]$. Then $\big((a_1, a_3, ...), (a_2, a_4, ...)\big) = \big((b_1, b_3, ...), (b_2, b_4, ...)\big)$, which means that $a_1 = b_1$, $a_2 = b_2$, $a_3 = b_3$, ... so $(a_1, a_2, a_3, ...) = (b_1, b_2, b_3, ...)$ and φ is one-to-one. Now let $\big((z_1, z_3, z_5, ...), (z_2, z_4, z_5, ...)\big) \in G \times G$. Then $(z_1, z_2, z_3, ...) \in G$ and $\varphi[(z_1, z_2, z_3, ...)] = \big((z_1, z_3, z_5, ...), (z_2, z_4, z_5, ...)\big)$ Thus φ is onto, therefore φ is an isomorphism.

i) $(\mathbb{R} - \{0\}, \cdot)$ and $(\mathbb{R} - \{1\}, *)$ where $a * b = a + b - ab$
These two groups are isomorphic by the function $\varphi : \mathbb{R} - \{0\} \to (\mathbb{R} - \{1\}, *)$ as defined by $\varphi(x) = 1 - x$. Let $x, y \in (\mathbb{R} - \{0\}, \cdot)$. We see:

$$\begin{aligned}
\varphi(xy) &= 1 - xy \\
&= (1 - x) + x + (1 - y) + y - 1 - xy \\
&= (1 - x) + (1 - y) - (1 - x - y + xy) \\
&= (1 - x) + (1 - y) - (1 - x)(1 - y) \\
&= \varphi(x)\varphi(y)
\end{aligned}$$

so φ is a homomorphism. Now assume $\varphi(x) = \varphi(y)$. Then $1-x = 1-y \implies x = y$, so φ is one-to-one. Now let $z \in \mathbb{R} - \{1\}$. Then $1 - z \in \mathbb{R} - \{0\}$ and $\varphi(1 - z) = 1 - (1 - z) = z$, so φ is onto. Therefore, φ is an isomorphism.

12.13) Let $\varphi : G \to H$ be a homomorphism.

a) Show that if H is abelian and φ is one-to-one, then G is abelian.

Assume H is abelian and φ is one-to-one. Let $g_1, g_2 \in G$. Then $\varphi(g_1g_2) = \varphi(g_1)\varphi(g_2) = \varphi(g_2)\varphi(g_1) = \varphi(g_2g_1)$. That is, $\varphi(g_1g_2) = \varphi(g_2g_1)$. But since φ is one-to-one, this means that $g_1g_2 = g_2g_1$, so G is abelian.

b) Show that if G is abelian and φ is onto, then H is abelian.

Assume G is abelian and φ is onto. Since φ is onto, every element of H can be written as $\varphi(g)$ for some $g \in G$. Thus for all $h_1 = \varphi(g_1), h_2 = \varphi(g_2) \in H$ we have

$$\begin{aligned}
h_1h_2 &= \varphi(g_1)\varphi(g_2) \\
&= \varphi(g_1g_2) \\
&= \varphi(g_2g_1) \\
&= \varphi(g_2)\varphi(g_1) \\
&= h_2h_1
\end{aligned}$$

That is, $h_1 h_2 = h_2 h_1$, so H is abelian.

c) Show that if φ is an isomorphism, then G is abelian if and only if H is abelian.

Assume φ is an isomorphism. Then φ is one-to-one and by **a** H abelian $\implies$ G abelian. Also, φ is onto, and by **b** G abelian $\implies$ H abelian. Thus when φ is an isomorphism H and G must be abelian together, thus G is abelian if and only if H is abelian.