Abstract Algebra

Homework 1 Redo: 0.15, 0.18, 0.21, 1.3, 1.6

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1 0.15 (Unchanged)

Problem: By trying a few cases, guess at a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+...+\frac{1}{(n-1)\cdot n}, \qquad n\geq 2.$$

$n=2: f(2)=\frac{1}{1\cdot 2}=1/2$

$n=3: f(3)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}=1/2+1/6=4/6=2/3$

$n=3: f(3)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}=1/2+1/6+1/12=3/4$

Based on the above inputs and outputs it seems that the formula is $f(x)=\frac{x}{x-1}$.

2 0.18 (Improved)

Problem: The *Fibonacci sequence* $f_1, f_2, f_3, ...$ is defined as follows:

$$f_1=f_2=1, f_3=2, f_4=3, f_5=5, f_6=8, ...,$$

and in general

$$f_n=f_{n-1}+f_{n-2}$$

for all $n\geq 3$. Prove that f_{5k} is divisible by 5 for every $k\geq 1$, that is, 5 divides every 5th member of the sequence.

Proof: Let $P(n)$ be the statement "$5 \mid F_{5n}$". $P(1)$ is true since $F_5=5$ and $5 \mid 5$. Let $k\geq 1$ and assume $P(k)$ to be true. That is, assume $5 \mid F_{5k}$. By definition of the

Fibonacci sequence, we see:

$F_{5k+1} = F_{5k} + F_{5k-1}$
$F_{5k+2} = F_{5k+1} + F_{5k} = (F_{5k} + F_{5k-1}) + F_{5k} = 2F_{5k} + F_{5k-1}$
$F_{5k+3} = F_{5k+2} + F_{5k+1} = (2F_{5k} + F_{5k-1}) + (F_{5k} + F_{5k-1}) = 3F_{5k} + 2F_{5k-1}$
$F_{5k+4} = F_{5k+3} + F_{5k+2} = (3F_{5k} + 2F_{5k-1}) + (2F_{5k} + F_{5k-1}) = 5F_{5k} + 3F_{5k-1}$
$F_{5k+5} = F_{5(k+1)} = F_{5k+4} + F_{5k+3} = (5F_{5k} + 3F_{5k-1}) + (3F_{5k} + 2F_{5k-1}) = 8F_{5k} + 5F_{5k-1}$

That is, $F_{5(k+1)} = 8F_{5k} + 5F_{5k-1}$. Then $P(k+1)$ is true if and only if $5 \mid (8F_{5k} + 5F_{5k-1})$ which is true if $5 \mid 8F_{5k}$ and $5 \mid 5F_{5k-1}$. Thus $5 \mid 8F_{5k}$ by the inductive hypothesis and clearly $5 \mid 5F_{5k-1}$, so $P(k+1)$ is true. Thus $P(k)$ true implies $P(k+1)$ is true, so by mathematical induction $P(n)$ is true $\forall n \in \mathbb{Z}$. Therefore F_{5k} is divisible by 5 for all $n \geq 1$.

3 0.21 (Improved)

Problem: The nth Fermat number is $F_n = 2^{(2^n)} + 1$. Prove that for every $n \geq 1$

$$F_0 F_1 F_2 ... F_{n-1} = F_n - 2.$$

Proof: Let $P(n)$ be the statement $F_0 F_1 F_2 ... F_{n-1} = F_n - 2$. $P(1)$ is true since $F_1 = 5$ implies $F_1 - 2 = 3$ and $F_0 = 3$. Let $k \geq 1$ and assume $P(k)$ is true. Then $F_0 F_1 F_2 ... F_{k-1} = F_k - 2$. Multiplying both sides by F_k, we have $F_0 F_1 ... F_k = (2^{2^k} - 1)(2^{2^k} + 1) = (2^{2^{k+1}} - 1)$. Therefore, $P(k+1)$ is true. Thus, by mathematical induction, $P(n)$ is true for all n. Therefore $F_0 F_1 F_2 ... F_{n-1} = F_n - 2$ for all $n \geq 1$.

4 1.3 (Improved)

Problem: In each case, determine whether or not the given * is a binary operation on the given set S.

$a)S = \mathbb{Z}, a * b = a + b^2$ is a binary operation.

$b)S = \mathbb{Z}, a * b = a^2b^3$ is a binary operation.

$c)S = \mathbb{R}, a * b = \dfrac{a}{a^2 + b^2}$ is not a binary operation.

$d)S = \mathbb{Z}, a * b = \dfrac{a^2 + 2ab + b^2}{a + b}$ is not a binary operation.

$e)S = \mathbb{Z}, a * b = a + b - ab$ is a binary operation.

$f)S = \mathbb{R}, a * b = b$ is a binary operation.

$g)S = \{1, -2, 3, 2, -4\}, a * b = |b|$ is not a binary operation.

Consider $a = 1, b = -4$. Then $a * b = 4$ which is not in the given set.

$h)S = \{1, 6, 3, 2, 18\}, a * b = ab$ is not a binary operation.

Consider $a = 2, b = 18$. Then $a * b = 36$ which is not in the given set.

$i)S =$ the set of all 2×2 matrices with real entries, and if a and b are 2×2 matrices, then $a * b$ is the 2×2 matrix containing the sum their corresponding elements. Then $a * b$ is a binary operation.

$j)S =$ the set of all subsets of a set $X.$, $A * B = (A\Delta B)\Delta B$ is a binary operator.

5 1.6 (Improved)

Problem: For each case in 1.3 in which * is a binary operation on S, determine whether * is commutative and whether it is associative.

a)$a * b = a + b^2$ is not commutative. Consider $a = 1, b = 2$. Then $a * b = 1 + 2^2 = 5$ and $b * a = 2 + 1^2 = 3$, and $5 \neq 3$. Also, $a * b$ is not associative. Consider $a = 1, b = 2, c = 3$. $(a * b) * c = (1 + 2^2) + 3^2 = 14$ and $a * (b * c) = 1 + (2 + 3^2)^2 = 122$ and clearly $14 \neq 122$.

b)$a * b = a^2b^3$ is not commutative. Consider $a = 1, b = 2$. Then $a * b = 1^2 \cdot 2^3 = 8$ and $b * a = 2^2 \cdot 1^3 = 4$ and $4 \neq 8$. Also, $a * b$ is not associative. Consider $a = 1, b = 2, c = 3$. $(a*b)*c = (1^2 \cdot 2^3)^2 \cdot 3^3 = 1728$ and $a*(b*c) = 1^2 \cdot (2^2 \cdot 3^3)^3 = 1259712$ and $1728 \neq 1259712$.

— c and d are not binary operators —

e)$a * b = a + b - ab$ is commutative.

Proof: We want to show $a * b = a + b - ab$ is commutative. That is, we want to show $a * b = b * a$. Plugging in to the equation, we have $a + b - ab = b + a - ba$, so $a * b$ is commutative.

Also, $a * b$ is associative.

Proof: We want to show $a*b = a+b-ab$ is associative. Let a, b and c be integers. We must show $(a+b-ab)+c-(a+b-ab)c = a+(b+c-bc)-a(b+c-bc)$. Distributing and removing parentheses we have $a+b+c-ab-ac-bc+abc = a+b+c-bc-ab-ac+abc$, which is clearly a true statement, thus $a * b$ is associative.

f)$a * b = b$ is not commutative. Consider $a = 1, b = 2$. $a * b = 2$ and $b * a = 1$, but clearly $2 \neq 1$. However, $a * b$ is associative.

Proof: We want to show $a * b = b$ is associative. Let a, b, and c be given. $a * b = b$ is associative if and only if $(a * b) * c = a * (b * c)$. Simplyfing both sides, we see $(a * b) * c = a * (b * c)$ is true if and only if $b * c = a * c$ which is true if and only if $c = c$. Clearly $c = c$, so $a * b = b$ is associative.

— g and h are not binary operators —

i)$a*$b as defined by the question is commutative.

Proof: Let $a = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}$ and $b = \begin{pmatrix} r_5 & r_6 \\ r_7 & r_8 \end{pmatrix}$ where $r_i \in \mathbb{R}$. Then $a*b = \begin{pmatrix} r_1 + r_5 & r_2 + r_6 \\ r_3 + r_7 & r_4 + r_8 \end{pmatrix}$ and $b * a = \begin{pmatrix} r_5 + r_1 & r_6 + r_2 \\ r_7 + r_3 & r_8 + r_4 \end{pmatrix}$. Since addition is commutative on real numbers, it follows that $a*b = b*a$, thus $a*b$ is commutative on the set of 2×2 matrices with real entries.

Also, $a * b$ is associative.

Proof: Let $a = \begin{pmatrix} r_1 & r_2 \\ r_3 & r_4 \end{pmatrix}$, $b = \begin{pmatrix} r_5 & r_6 \\ r_7 & r_8 \end{pmatrix}$, and $c = \begin{pmatrix} r_9 & r_{10} \\ r_{11} & r_{12} \end{pmatrix}$ where $r_n \in \mathbb{R}\forall n$. Then $(a*b)*c = \begin{pmatrix} (r_1 + r_5) + r_9 & (r_2 + r_6) + r_{10} \\ (r_3 + r_7) + r_{11} & (r_4 + r_8) + r_{12} \end{pmatrix}$ and $a*(b*c) = \begin{pmatrix} r_1 + (r_5 + r_9) & r_2 + (r_6 + r_{10}) \\ r_3 + (r_7 + r_{11}) & r_4 + (r_8 + r_{12}) \end{pmatrix}$. The exact same matrix results in both cases since addition is associative, thus $(a*b)*c = a * (b * c)$ so $a * b$ is associative.

j)$A * B$ as defined by the question can be simplified using that symmetric difference of sets is commutative associative. $(A\Delta B)\Delta B = A\Delta(B\Delta B) = A\Delta\emptyset = A$. Thus $A * B$ is not commutative. Consider the sets $A = 1, 2, B = 2, 3$. We see $A*B = (1, 2\Delta 2, 3)\Delta 2, 3 = 1, 3\Delta 2, 3 = 1, 2$, but $B * A = (2, 3\Delta 1, 2)\Delta 1, 2 = 1, 3\Delta 1, 2 = 2, 3$. Clearly $A * B \neq B * A$, so $A * B$ is not commutative. However, $A * B$ is associative:

Proof: Let A and B be sets. We want to show $(A * B) * C = A * (B * C)$. We see $(A * B) * C = ((A\Delta B)\Delta B) * C = A * C = (A\Delta C)\Delta C = A$. Likewise, we see

$A*(B*C) = A*((B\Delta C)\Delta C) = A*B = (A\Delta B)\Delta B = A$. Thus $(A*B)*C = A*(B*C)$, so $A*B$ is associative.