# Number Theory

**Homework 1:**
**2.2) 8**
**2.3) 2c, 2d, 4a**

Kenny Roffo

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**8)** Prove that no integer in the following sequence is a perfect square: 11, 111, 1111, 11111, ...

This sequence can be defined by $a_i$ where $a_1 = 11$ and $a_{i+1} = a_i \times 10 + 1$. Let $P(n)$ be the statement $a_n$ is of the form $4k+3$. $P(1)$ is true, since 11, is of the form $4k+3$ ($k = 2$). Now let $j \geq 1$ and assume $P(j)$ to be true. Then $a_j = 4k+3$ for some integer $k$. Thus we see

$$\begin{aligned} a_{j+1} &= (4k+1) \cdot 10 + 1 \\ &= 40k + 8 + 3 \\ &= 4(10k+2) + 3 \end{aligned}$$

so $a_{j+1}$ is of the form $4k+3$, thus by mathematical induction every term of the sequence is of the form $4k+3$. It was shown in class that all perfect squares are of the form either $4k$ or $4k+1$, thus no number in the sequence is a perfect square. (This was shown by squaring the general forms $4k, 4k+1, 4k+2$ and $4k+3$, one of which any given number can be written as, and showing that the squares of these can only be of the form either $4k$ or $4k+1$).

**2c)** Given integers $a, b, c$ show $a|b$ if and only if $ac|bc$, where $c \neq 0$.

Assume $a|b$. Then $a \neq 0$ and $ax = b$ for some $x \in \mathbb{Z}$. Clearly $(ax)c = (b)c$, thus $(ac)x = bc$. Since neither $a$ nor $c$ are zero, $ac$ is not zero, therefore $ac|bc$. Now assume $ac|bc$. Then $(ac)y = bc$ for some $y \in \mathbb{Z}$. Dividing by $c$, we see $ay = b$. Since $ac|bc$, $a$ cannot be zero, thus $a|b$. Since each condition implies the other, we have shown that given integers $a, b, c$ show $a|b$ if and only if $ac|bc$, where $c \neq 0$.

**2d)** Given integers $a, b, c, d$, show if $a|b$ and $c|d$ then $ac|bd$.

Assume $a|b$ and $c|d$. Then $ax = b$ and $cy = d$ for integers $x$ and $y$, and $a$ and $c$ are non-zero. We see

$$\begin{aligned} bd &= (ax)(cy) \\ &= (ac)(xy) \end{aligned}$$

Thus $ac|bd$.

**4a)** For $n \geq 1$, use mathematical induction to establish that $8|5^{2n} + 7$.

Let $P(n)$ be the statement $8|5^{2n} + 7$. We see $5^{2(1)} + 7 = 32$, which is divisible by 8 since $8(4) = 32$, so $P(1)$ is true. Now let $k \geq 1$ and assume $P(k)$ to be true. Then $8|5^{2k} + 7$, so $8t = 5^{2k} + 7$ for some integer $t$. We see

$$\begin{aligned} (5^2)8t &= 5^2\left(5^{2k} + 7\right) \\ \implies 200t - 168 &= 5^{2(k+1)} + 7 \\ \implies 8\,(25t - 21) &= 5^{2(k+1)} + 7 \end{aligned}$$

Thus $8|5^{2(k+1)} + 7$, so $P(k+1)$ is true! Therefore, by the principle of mathematical induction $P(n)$ is true for all positive $n$, so for $n \geq 1$, $8|5^{2n} + 1$.