Number Theory

Homework 9

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Due April 11, 2016

1) Prove that in a primitive Pythagorean triple x, y, z the product xy is divisible by 12, hence $60|xyz$.

Let x, y, z be a primitive Pythagorean triple defined by integers s and t. Note that if xy is divisible by 3 and by 4, then xy must be divisible by 12. We see

$$xy = 2st(s^2 - t^2)$$

Since one of s and t is even, xy is either $4at(s^2 - t^2)$ or $4bs(s^2 - t^2)$ where a and b are integers. Thus 4 divides xy. We must now prove that 3 divides xy. Note that if $3|s$ or $3|t$ then $3|x$ hence $3|xy$. However, if 3 divides neither of s and t, then by Fermat's Theorem we have

$$s^2 \equiv 1 \pmod 3 \qquad \text{and} \qquad t^2 \equiv 1 \pmod 3$$

and thus

$$y = s^2 - t^2 \equiv 0 \pmod 3$$

and so $3|y$, which means $3|xy$. Therefore, both 3 and 4 divide xy, so 12 divides xy.

2) Verify that 3, 4, 5 is the only primitive Pythagorean triple involving consecutive positive integers.

Let $n, n+1, n+2$ three consecutive positive integers. For these to form a Pythagorean triple, it would have to be the case that

$$n^2 + (n+1)^2 = (n+2)^2$$

Solving for n, we see:

$$\begin{aligned}
& n^2 + (n+1)^2 = (n+2)^2 \\
\Longrightarrow \quad & n^2 + n^2 + 2n + 1 = n^2 + 4n + 4 \\
\Longrightarrow \quad & n^2 - 2n - 3 = 0 \\
\Longrightarrow \quad & (n-3)(n+1) = 0 \\
\Longrightarrow \quad & n \in \{-1, 3\}
\end{aligned}$$

Using basic algebra along with the fact that n must be positive, we find that n must be 3. Therefore, since we know 3,4,5 is a primitive Pythagorean triple we know that 3, 4, 5 is the only primitive Pythagorean triple invovling consecutive positive integers.

3) Find all primitive Pythagorean triples containing 60.

Since 60 is even, we must have $x = 2st = 60 \implies st = 30$. Now we examine possibilities for s and t. Exactly one of the two must be even, and they must be relatively prime. Note that we will also make s the larger of the two since otherwise we will repeat a PPT with y negative. $s = 30, t = 1$:

$$\begin{aligned} x &= 2st = 2(30)(1) = 60 \\ y &= s^2 - t^2 = 30^2 - 1^2 = 899 \\ z &= s^2 + t^2 = 30^2 + 1^2 = 901 \end{aligned}$$

$s = 15, t = 2$:

$$\begin{aligned} x &= 2st = 2(15)(2) = 60 \\ y &= s^2 - t^2 = 15^2 - 2^2 = 221 \\ z &= s^2 + t^2 = 15^2 + 2^2 = 229 \end{aligned}$$

$s = 10, t = 3$:

$$\begin{aligned} x &= 2st = 2(10)(3) = 60 \\ y &= s^2 - t^2 = 10^2 - 3^2 = 91 \\ z &= s^2 + t^2 = 10^2 + 3^2 = 109 \end{aligned}$$

$s = 6, t = 5$:

$$\begin{aligned} x &= 2st = 2(6)(5) = 60 \\ y &= s^2 - t^2 = 6^2 - 5^2 = 11 \\ z &= s^2 + t^2 = 6^2 + 5^2 = 61 \end{aligned}$$

4) Find all primitive Pythagorean triples containing 61.

Since 61 is odd, either $y = 61$ or $z = 61$. First we will consider when $y = 61$. Since 61 is prime,

$$y = (s - t)(s + t) = 61$$

implies that $s - t = 1$ and $s + t = 61$. Thus $2s = 62 \implies s = 31$, thus $t = 30$. In this case we have

$$\begin{aligned} x &= 2st = 2(31)(30) = 1860 \\ y &= s^2 - t^2 = 31^2 - 30^2 = 61 \\ z &= s^2 + t^2 = 31^2 + 30^2 = 1861 \end{aligned}$$

Now we consider the case where $z = 61$. We must find s and t such that exactly one of them is even, and they are relatively prime, and the sum of their squares is 61 (since $z = s^2 + t^2$). To find these we simply start from $s = 1$ and find what t must be for $s^2 + t^2$ to be equal to 61, and if all the above conditions are satisfied we have found a PPT. Then we check $s = 2, 3, ...$ up to $s = 7$ since $s > 7$ gives $s^2 > 61$. Following this process, we have the following PPT: $s = 6, t = 5$:

$$\begin{aligned} x &= 2st = 2(6)(5) = 60 \\ y &= s^2 - t^2 = 6^2 - 5^2 = 11 \\ z &= s^2 + t^2 = 6^2 + 5^2 = 61 \end{aligned}$$