Number Theory

Homework 5

Kenny Roffo

Due February 29, 2016

1) Let p and q be distinct odd primes. Prove that $n = pq$ is not perfect.

Since p and q are odd, there exist integers x and y such that $p = 2x+1$ and $q = 2y+1$. Thus we can find n in terms of x and y:

$$\begin{aligned} n &= pq \\ &= (2x+1)(2y+1) \\ &= 4xy + 2x + 2y + 1 \end{aligned}$$

Now, for the sake of contradiction, let us assume that n is perfect. That would mean that n is equal to the sum of its positive divisors except itself. Since p and q are prime, they along with 1 and n are the only positive divisors of n (by the uniqueness of prime factorizations). Thus $n = 1 + p + q$ and we examine this equality:

$$\begin{aligned} & n = 1 + p + q \\ \implies & 4xy + 2x + 2y + 1 = 1 + (2x+1) + (2y+1) \\ \implies & 2(2xy + x + y) + 1 = 2(x + y + 1) + 1 \\ \implies & 2xy + x + y = x + y + 1 \\ \implies & 2xy = 1 \\ \implies & x = \frac{1}{2y} \end{aligned}$$

So x is not an integer, since 1 divided by any positive integer other than itself is not an integer. This contradicts that x is actually an integer, thus our assumption that n is perfect must be false, so in fact n is not perfect.

2) Find the digits X and Y if

$$495 | 273X49Y5$$

We begin by examining 495. Its prime factorization is $495 = 3^2 \cdot 5 \cdot 11$. Thus 273X49Y5 must be divisible by 3, 5 and 11. It is obviously divisible by 5, but what about 3 and 11? We discussed several decimal representation divisibility rules in class, including when a number is divisible by 11 or 3.

273X49Y5 is divisible by 11 exactly the alternating sum of its digits is divisible by 11. We see:

$$5 - Y + 9 - 4 + X - 3 + 7 - 2 = 12 + X - Y$$

Thus we know $11|12+X-Y$, so $11n = 12+X-Y$, and thus $11m = 1+X-Y$, where $m = n-1$. Now, since X and Y are digits, the maximum value of $1+X-Y$ is 10 and the minimum is -8. This implies $m = 0$, which further implies $Y = X+1$.

Now, what about 3? Well, since there are actually 2 threes in the prime factorization of 495, it will prove more helpful to use $3^2 = 9$. 273X49Y5 is divisible by 9 exactly when the sum of its digits is divisible by 9. We see:

$$5 + Y + 9 + 4 + X + 3 + 7 + 2 = 30 + X + Y$$

Thus $9s = 30 + X + Y$ so $9t = 3 + X + Y$ where $t = s - 3$. Plugging in $X + 1$ for Y we have $9t = 3 + 2X + 1$. Examining, it becomes clear that there is only one digit that fits this description: 7. Thus $X = 7$ and $Y = 8$.

3) Prove that there is no integer n for which $\phi(n) = 14$.

Let n be an integer. Then n has prime factorization $p_1^{e_1} p_2^{e_2} ... p_k^{e_k}$ where all p_i are prime and e_i are positive integers. Thus

$$\begin{aligned} \phi(n) &= \phi\left(p_1^{e_1}\right)\phi\left(p_2^{e_2}\right)...\phi\left(p_k^{e_k}\right) \\ &= p_1^{e_1-1}\left(p_1-1\right)p_2^{e_2-1}\left(p_2-1\right)...p_k^{e_k-1}\left(p_k-1\right) \end{aligned}$$

Now for the sake of contradiction let us assume $\phi(n) = 14$. 14 has prime factorization $2^1 \cdot 7^1$, thus some p_s must be 7:

$$\phi(n) = p_1^{e_1-1}\left(p_1-1\right)p_2^{e_2-1}\left(p_2-1\right)...7^{e_s-1}\left(6\right)...p_k^{e_k-1}\left(p_k-1\right)$$

But now we have $14 = 2 \cdot 7 = 2 \cdot 3 \cdot m$ where m is equal to all of the above product except the 6, and is a positive integer, since a prime either minus 1 or raised to a power of 0 or more will both always be a positive integer. But this contradicts the uniqueness of prime factorizations, so our assumption that $\phi(n) = 14$ must be impossible. Therefore there is no integer n such that $\phi(n) = 14$.

(Note that non-positive integers are not in the domain of ϕ)

4) Suppose that n is an even perfect number. Prove that $\phi(n)$ is not perfect.

Since n is an even perfect number, it can be written as

$$n = 2^{m-1}\left(2^m - 1\right)$$

where m is an integer, and $2^m - 1$ is prime. Thus 2^{m-1} and $2^m - 1$ are relatively prime, so by a theorem shown in class

$$\phi(n) = \phi\left(2^{m-1}\left(2^m-1\right)\right) = \phi\left(2^{m-1}\right)\phi\left(2^m-1\right)$$

We know that $\phi(p^k) = p^{k-1}(p-1)$ where p is prime, $k \in \mathbb{Z}$, thus

$$\phi\left(2^{m-1}\right) = 2^{m-2}\left(2-1\right) \qquad \text{and} \qquad \phi\left(2^m-1\right) = \left(2^m-1\right)^0\left(2^m-2\right)$$

Thus

$$\begin{aligned} \phi(n) &= 2^{m-2}\left(2^m-2\right) \\ &= 2^{m-1}\left(2^{m-1}-1\right) \\ &= 2\left(2^{m-2}\left(2^{m-1}-1\right)\right) \end{aligned}$$

Clearly $\phi(n)$ is even, so it is certainly not an odd perfect number. Now assume it is an even perfect number. Then there exists an integer s such that $\phi(n) = 2^{s-1}(2^s - 1)$ where $2^s - 1$ is prime. Before proceeding, we must show that s is smaller than m. Clearly s cannot be equal to m. Now assume $s > m$. Then $2^{s-1} > 2^{m-1}$ and $2^s - 1 > 2^{m-1} - 1$ so it is impossible for $2^{m-1}\left(2^{m-1} - 1\right) = \phi(n) = 2^{s-1}(2^s - 1)$ Thus s must be smaller than m

$$\begin{aligned} 2^{s-1}(2^s - 1) &= 2^{m-1}\left(2^{m-1} - 1\right) \\ \Longrightarrow 2^s - 1 &= 2^{m-s}\left(2^{m-1} - 1\right) \end{aligned}$$

So $2^s - 1$ can be written as a product of integers other than 1 and itself (since $s < m$, 2^{m-s} is an integer and is greater than 1), which contradicts that it is prime. Thus $\phi(n)$ cannot be an even perfect number. So $\phi(n)$ is not a perfect number.