Number Theory

Homework 4

Kenny Roffo

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1) Show that 3, 5, 7 is the only prime triplet.

We begin by seeing that 9 plus a multiple of 6 is never prime.

$$\begin{aligned} 9 + 6n &= 3(3) + 3 \cdot 2(n) \\ &= 3\,(3 + 2n) \end{aligned}$$

Now we examine the odd triplets. Let $P(n)$ be the statement that one of $n, n+2, n+4$ is 9 plus a multiple of 6. $P(5)$ is true since 5, 7, 9 contains 9 and $9 = 9 + 6(0)$. Now let k be an odd integer greater than or equal to 5 and assume $P(k)$ to be true. Then $k, k+2$ or $k+4$ is 9 plus a multiple of 6. If it is k, then $k = 9 + 6(m)$ for some integer m and the next prime triplet contains $k + 6 = (k+2) + 4$. We see

$$k + 6 = 9 + 6(m) + 6 = 9 + 6(m + 1)$$

so $(k+2)+4$ is 9 plus a multiple of 6, so $P(k+2)$ is true. Now if it is $k+2$ or $k+4$ is 9 plus a multiple of 6, then the next prime triplet (which contains both $k+2$ and $k+4$) contains a number which is 9 plus a multiple of 6 thus $P(k+2)$ is true. Thus we have shown that if $P(k)$ is true, then $P(k+2)$ is true. Therefore by the principle of mathematical induction $P(n)$ is true for all odd integers greater than or equal to 5. Thus there exist no prime triplets that begin with a number greater than or equal to 5. Since no number below 2 is prime, and 2, 4, 6 is not a prime triplet, we have shown that 3, 5, 7 is the only prime triplet.

2) Let $p, p+2$ be twin primes with $p > 3$. Prove the sum is divisible by 12.

All numbers can be written in one of the following forms:

$$\begin{aligned} &6n+0 = 2(3n+0) \\ &6n+1 \\ &6n+2 = 2(3n+1) \\ &6n+3 = 3(2n+1) \\ &6n+4 = 2(3n+2) \\ &6n+5 \end{aligned}$$

As shown above, no number of the forms $6n+0$, $6n+2$, $6n+3$ and $6n+4$ can be prime, because they have divisors other than 1 and themselves (and thus more than 2 divisors). Note also, that a number of the form $6n+5$ may also be written (for a different n) as $6n-1$. Thus all primes are of the form either $6n-1$ or $6n+1$. It is obvious that a pair of twin primes cannot both be of the same form, thus any pair of twin primes has the form $6n-1$ and $6n+1$. Summing twin primes, we see

$$(6n-1) + (6n+1) = 12n$$

thus for any pair of twin primes, 12 is a divisor of their sum.

3) Find all perfect squares of the form $17p+1$ where p is prime.

Let p be a prime such that $17p+1 = x^2$ where $x \in \mathbb{Z}$. Manipulating this equality, we see

$$\begin{aligned} &17p + 1 = x^2 \\ \implies &17p = x^2 - 1 \\ \implies &17p = (x-1)(x+1) \end{aligned}$$

We now have a product of two primes equal to a product of two integers. If the right side contained a non-prime integer, then it could be rewritten as a product of two or more primes. In this case, we would have two distinct prime factorizations equal to each other, a contradiction. Thus $x-1$ and $x+1$ must be prime, and furthermore one of the two must be 17. If $x-1 = 17$, then $x+1 = 19$, which is prime, so $p = 19$ is a solution. If $17 = x+1$, then $p = x-1 = 15$ is not prime, thus is not a solution. Therefore $p = 19$ is the only possible prime solution to the equation $17p+1 = x^2$.

4) Show that no number of the form $8^n + 1$ is prime where n is a positive integer.

This becomes simple when we realize that this is a sum of perfect cubes:

$$\begin{aligned} 8^n + 1 &= 2^{3n} + 1^3 \\ &= (2^n + 1)\left(2^{2n} - 2^n + 1\right) \end{aligned}$$

By the closure laws, both polynomials are integers, and of course $1 < 2^n + 1 < 8^n + 1$, so $8^n + 1$ has more than two positive divisors, thus it is not prime.