Complex Analysis

Homework 11: 3.2) 22, 32, 38

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22) Show that the function $f(z) = |z|$ is nowhere differentiable.

Let $z = x + iy$ be any point in the complex plain, and let $\Delta z = \Delta x + i\Delta y$. Then

$$\begin{aligned}
\frac{f(z+\Delta z) - f(z)}{\Delta z} &= \frac{|z+\Delta z| - |z|}{\Delta z} \\
&= \frac{|z+\Delta z| - |z|}{\Delta z} \cdot \frac{|z+\Delta z| + |z|}{|z+\Delta z| + |z|} \\
&= \frac{|z+\Delta z|^2 - |z|^2}{\Delta z(|z+\Delta z|^2 + |z|^2)} \\
&= \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\overline{z}}{\Delta z(|z+\Delta z| + |z|)} \\
&= \frac{z\overline{z} + z\overline{\Delta z} + \overline{z}\Delta z + \Delta z\overline{\Delta z} - z\overline{z}}{\Delta z(|z+\Delta z| + |z|)} \\
&= \frac{z\overline{\Delta z} + \overline{z}\Delta z + \Delta z\overline{\Delta z}}{\Delta z(|z+\Delta z| + |z|)} \\
&= \frac{z\overline{\Delta z} + \overline{z\overline{\Delta z}} + \Delta z\overline{\Delta z}}{\Delta z(|z+\Delta z| + |z|)} \\
&= \frac{2\mathrm{Re}(z\overline{\Delta z}) + \Delta z\overline{\Delta z}}{\Delta z(|z+\Delta z| + |z|)} \\
&= \frac{2(x\Delta x + y\Delta y) + \Delta x^2 + \Delta y^2}{(x+iy)(\sqrt{(x+\Delta x)^2 + (y+\Delta y)^2} + \sqrt{x^2+y^2})}
\end{aligned}$$

That is, $\frac{f(z+\Delta z)-f(z)}{\Delta z} = \frac{2(x\Delta x+y\Delta y)+\Delta x^2+\Delta y^2}{(x+iy)(\sqrt{(x+\Delta x)^2+(y+\Delta y)^2}+\sqrt{x^2+y^2})}$. Now consider the limit as Δz goes to 0 of this fraction. First, let us approach along a line parallel to the real axis. Then $\Delta y = 0$, and we have

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2}{\Delta x(\sqrt{(x+\Delta x)^2+y^2}+\sqrt{x^2+y^2})} &= \lim_{\Delta x\to 0}\frac{2x+\Delta x}{(\sqrt{(x+\Delta x)^2+y^2}+\sqrt{x^2+y^2})}\\ &= \frac{2x}{\sqrt{x^2+y^2}+\sqrt{x^2+y^2}}\\ &= \frac{x}{\sqrt{x^2+y^2}}\end{aligned}$$

That is, $\lim_{\Delta x\to 0}\frac{2x\Delta x+\Delta x^2}{\Delta x(\sqrt{(x+\Delta x)^2+y^2}+\sqrt{x^2+y^2})} = \frac{x}{\sqrt{x^2+y^2}}$. Now let us approach along a line parallel to the imaginary axis. Then $\Delta x = 0$, and we have

$$\begin{aligned}\lim_{\Delta y\to 0}\frac{2y\Delta y+\Delta y^2}{i\Delta y(\sqrt{x^2+(y+\Delta y)^2}+\sqrt{x^2+y^2})} &= \lim_{\Delta y\to 0}\frac{2y+\Delta y}{i(\sqrt{x^2+(y+\Delta y)^2}+\sqrt{x^2+y^2})}\\ &= \frac{2y}{i(\sqrt{x^2+y^2}+\sqrt{x^2+y^2})}\\ &= \frac{y}{i\sqrt{x^2+y^2}}\end{aligned}$$

That is, $\lim_{\Delta y\to 0}\frac{2y\Delta y+\Delta y^2}{i\Delta y(\sqrt{x^2+(y+\Delta y)^2}+\sqrt{x^2+y^2})} = \frac{y}{i\sqrt{x^2+y^2}}$. So approaching along a line parallel to the real axis we get a real number as the limit, but approaching along a line parallel to the imaginary axis we get an imaginary number. No real number is equal to an imaginary number, so this means the limit does not exist, and since z was an arbitrary complex number, this implies the limit does not exist for any complex number, so the function $f(z) = |z|$ is nowhere differentiable.

32) Prove that $\frac{\mathrm{d}}{\mathrm{d}z}[f(z)+g(z)] = f'(z)+g'(z)$.

Let $z = x+iy$ be a complex number and let f and g be functions differentiable at z. Then $lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z} = f'(z)$ and $lim_{\Delta z\to 0}\frac{g(z+\Delta z)-g(z)}{\Delta z} = g'(z)$ exist. We see

$$\begin{aligned}lim_{\Delta z\to 0}\frac{f(z+\Delta z)+g(z+\Delta z)-f(z)-g(z)}{\Delta z} &= lim_{\Delta z\to 0}\left(\frac{f(z+\Delta z)-f(z)}{\Delta z}+\frac{g(z+\Delta z)-g(z)}{\Delta z}\right)\\ &= lim_{\Delta z\to 0}\left(\frac{f(z+\Delta z)-f(z)}{\Delta z}\right)\\ &+ lim_{\Delta z\to 0}\left(\frac{g(z+\Delta z)-g(z)}{\Delta z}\right)\\ &= f'(z)+g'(z)\end{aligned}$$

Therefore, since f'(z) and g'(z) exist, the limit exists, and thus the derivative of the sum of functions is the sum of their derivatives (at a point z).

38 a) Let $f(z) = z^2$. Write down the real and imaginary parts of f and f'. What do you observe?

$\text{Re}(f) = x^2 - y^2$ and $\text{Im}(f) = 2ixy$. Also, $\text{Re}(f') = 2x$ and $\text{Im}(f') = 2iy$. I notice that the real and imaginary parts of f' are the derivative with respect to x of the real and imaginary parts of f.

38 b) Repeat part **a** for $f(z) = 3iz + 2$.

$$3iz + 2 = 3i(x + iy) + 2 = (-3y + 2) + i3x$$

$\text{Re}(f) = -3y + 2$ and $\text{Im}(f) = i3x$. Also, $\text{Re}(f') = 0$ and $\text{Im}(f') = 3i$. I notice the same thing as in part **a**.

38 c) Make a conjecture about the relationship between real and imaginary parts of f versus f'.

The real and imaginary parts of f' are the derivatives with respect to the x of the real and imaginary parts of f respectively.