Complex Analysis

Homework 3: 1.3) 34, 44
1.4) 18

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34) Use de Moivre's formula with $n = 3$ to find trigonometric identities for $\cos 3\theta$ and $\sin 3\theta$.

De Movire's formula says $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$. Applying $n = 3$ to the formula, we see

$$\begin{aligned}
\cos 3\theta + i\sin 3\theta &= (\cos\theta + i\sin\theta)^3 \\
&= \left(\cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta\right)(\cos\theta + i\sin\theta) \\
&= \cos^3\theta - 3\sin^2\theta\cos\theta + i\left(-\sin^3\theta + 3\cos^2\theta\sin\theta\right)
\end{aligned}$$

By the definition of equality of complex numbers, this implies

$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$ and $\sin 3\theta = -\sin^3\theta + 3\cos^2\theta\sin\theta$

44) Describe the set of points z in the complex plane that satisfy $\arg(z) = \frac{\pi}{4}$.

All $z \in \mathbb{C}$ can be written in polar form as $r(\cos\theta + i\sin\theta)$. In our particular case, we know $\theta = \frac{\pi}{4}$, so we must describe the set of all z of the form

$$r\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = r\frac{\sqrt{2}}{2} + ir\frac{\sqrt{2}}{2}$$

. This tells us that the set of interest is the set of all $z \in \mathbb{C}$ such that the real and imaginary parts are equal.

18) Use the fact that $8i = (2+2i)^2$ to find all solutions of the equation $z^2 - 8z + 16 = 8i$.

To solve this problem, we will simply find all the 2^{nd} roots of $(2+2i)^2$. First we must express the complex number $(2+2i)$ in polar form. We see the radius is

$$\begin{aligned} r &= \sqrt{2^2 + 2^2} \\ &= 2\sqrt{2} \end{aligned}$$

and the angle is found using basic trigonometry:

$$\begin{aligned} \theta &= \tan^{-1}\frac{2}{2} \\ &= \frac{\pi}{4} \end{aligned}$$

Therefore we can write $(2+2i) = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Now we know from how powers of complex numbers work

$$(2+2i)^2 = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

Finally, we can use the formula

$$\phi = \frac{\theta + 2k\pi}{n}$$

with values of k from 0 up to $n-1$ (in this case 1) to find all distinct 2^{nd} roots of $(2+2i)^2$ by the formula $w_k = \sqrt{r}\,(\cos\phi + i\sin\phi)$.

$$\begin{aligned} k = 0 : w_0 &= \sqrt{8}\left(\cos\left(\frac{\frac{\pi}{2} + 2(0)\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2(0)\pi}{2}\right)\right) \\ &= \sqrt{8}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\ k = 1 : w_1 &= \sqrt{8}\left(\cos\left(\frac{\frac{\pi}{2} + 2(1)\pi}{2}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2(1)\pi}{2}\right)\right) \\ &= \sqrt{8}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right) \end{aligned}$$

Looking back at the original problem, we realize that $z^2 - 8z + 16$ is really just $(z-4)^2$, so if we add 4 to each of our roots of $8i$ we will have our z's, thus the solutions to the equation are

$$z = \sqrt{8}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) + 4 \qquad \text{and} \qquad z = \sqrt{8}\left(\cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)\right) + 4$$