Complex Analysis

Homework 16: 3.4) 38, 39

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38) Suppose $x = r\cos\theta, y = r\sin\theta$ and $f(z) = u(x, y) + iv(x, y)$. Show that

$$u_r = u_x \cos\theta + u_y \sin\theta \qquad u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

and

$$v_r = v_x \cos\theta + v_y \sin\theta \qquad v_\theta = -v_x r\sin\theta + v_y r\cos\theta$$

then deduce the Cauchy-Riemann equations in polar coordinates.

From multivariable calculus we know that for a function $g(x, y)$,

$$\frac{\partial g}{\partial z} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial z}$$

.

Thus we have

$$\begin{aligned} u_r &= \tfrac{\partial u}{\partial x}\tfrac{\partial x}{\partial r} + \tfrac{\partial u}{\partial y}\tfrac{\partial y}{\partial r} \\ &= u_x \cos\theta + u_y \sin\theta \end{aligned}$$

By the exact same calculation, $v_r = v_x \cos\theta + v_y \sin\theta$. We also see

$$\begin{aligned} u_\theta &= \tfrac{\partial u}{\partial x}\tfrac{\partial x}{\partial \theta} + \tfrac{\partial u}{\partial y}\tfrac{\partial y}{\partial \theta} \\ &= -u_x r\sin\theta + u_y r\cos\theta \end{aligned}$$

and by the same calculation $v_\theta = -v_x \sin\theta + v_y \cos\theta$.

Now we must find the Cauchy-Riemann equations' polar form. The C-R equations state $u_x = v_y$ and $u_y = -v_x$. We see

$$
\begin{aligned}
u_r &= u_x \cos\theta + u_y \sin\theta \\
&= v_y \cos\theta - v_x \sin\theta \\
&= \frac{1}{r}\left(-v_x r \sin\theta + v_y r \cos\theta\right) \\
&= \frac{1}{r} v_\theta
\end{aligned}
$$

and also

$$
\begin{aligned}
v_r &= v_x \cos\theta + v_y \sin\theta \\
&= -u_y \cos\theta + u_x \sin\theta \\
&= -\frac{1}{r}\left(-u_x r \sin\theta + u_y r \cos\theta\right) \\
&= -\frac{1}{r} u_\theta
\end{aligned}
$$

so the C-R equations in polar form are

$$
u_r = \frac{1}{r} v_\theta \qquad \text{and} \qquad v_r = -\frac{1}{r} u_\theta
$$

39) Suppose the function $f(z) = u(r,\theta) + iv(r,\theta)$ is differentiable at a point z whose polar coordinates are (r,θ). Solve the the equations derived in **38** for u_x and v_x respectively. Then show that the derivative of f at (r,θ) is

$$
f'(z) = (\cos\theta - i\sin\theta)(u_r + iv_r) = e^{-i\theta}(u_r + iv_r)
$$

Let us start by examining u_x. Solving u_r for u_x we see

$$
u_x = \frac{u_r - u_y \sin\theta}{\cos\theta}
$$

and since $-u_y = v_x$ we can plug in v_x

$$
u_x = \frac{u_r + v_x \sin\theta}{\cos\theta}
$$

Now let's find v_x from v_r:

$$
v_x = \frac{v_r - v_y \sin\theta}{\cos\theta} = \frac{v_r - u_x \sin\theta}{\cos\theta}
$$

Now we substitute for v_x:

$$u_x = \frac{u_r + \left[\frac{v_r - u_x \sin\theta}{\cos\theta}\right]\sin\theta}{\cos\theta} = \frac{u_r}{\cos\theta} + \frac{v_r \sin\theta}{\cos^2\theta} - \frac{u_x \sin^2\theta}{\cos^2\theta}$$

Now we do a bit of algebra to get u_x:

$$\begin{aligned} u_x \cos^2\theta &= u_r \cos\theta + v_r \sin\theta + u_x \sin^2\theta \\ \implies u_x \left(\cos^2\theta + \sin^2\theta\right) &= u_r \cos\theta + v_r \sin\theta \\ \implies u_x &= u_r \cos\theta + v_r \sin\theta \end{aligned}$$

Now we must find v_x. We already found an expression for v_x from v_r. Since $v_y = u_x$, we can simply plug our expression for u_x from u_r into v_x:

$$v_x = \frac{v_r - \left[\frac{u_r + v_x \sin\theta}{\cos\theta}\right]\sin\theta}{\cos\theta} = \frac{v_r - \frac{u_r \sin\theta}{\cos\theta} + \frac{v_x \sin^2\theta}{\cos\theta}}{\cos\theta}$$

Now we again just do some algebra to find v_x:

$$\begin{aligned} v_x \cos^2\theta &= v_r \cos\theta - u_r \sin\theta + v_x \sin^2\theta \\ \implies v_x \left(\cos^2\theta + \sin^2\theta\right) &= v_r \cos\theta - u_r \sin\theta \\ \implies v_x &= v_r \cos\theta - u_r \sin\theta \end{aligned}$$

Now that we have u_x and v_x, we can use the fact that for a function $f(x, y) = u(x, y) + iv(x, y)$ which is differentiable at a point z, the derivative at z is given by

$$f'(z) = u_x + iv_x$$

to find the derivative of a function which is differentiable at a point z in polar coordinates:

$$\begin{aligned} f'(z) &= u_x + iv_x \\ &= (u_r \cos\theta + v_r \sin\theta) + i\,(v_r \cos\theta - u_r \sin\theta) \\ &= (\cos\theta - i\sin\theta)(u_r + iv_r) \\ &= \big(\cos\theta + i\sin(-\theta)\big)\,(u_r + iv_r) \\ &= e^{-i\theta}(u_r + iv_r) \end{aligned}$$

Therefore for a function $f(r, \theta) = u(r, \theta) + iv(r, \theta)$ the derivative at a point $z = re^{i\theta}$ (assuming f is differentiable at z) is given by

$$f'(z) = e^{-i\theta}(u_r + iv_r)$$