Complex Analysis

Homework 19: 5.1) 12, 31

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12) Evaluate the line integrals $\int_C G(x,y)dx, \int_C G(x,y)dy$ and $\int_C G(x,y)ds$ on the curve

$$G(x,y)x^3 + 2xy^2 + 2x; \qquad x = 2t \qquad y = t^2 \qquad 0 \le t \le 1$$

$$\begin{aligned}
\int_C G(x,y)dx &= \int_C \Big(x^2 + 2xy^2 + 2x\Big)\, dx \\
&= \int_0^1 \Big((2t)^3 + 2(2t)(t^2)^2 + 2(2t)\Big)\,(2dt) \\
&= 8\int_0^1 2t^3 + t^5 + tdt \\
&= 8\left[\frac{1}{2}t^4 + \frac{1}{6}t^6 + \frac{1}{2}t^2\right]_0^1 \\
&= 8\left[\frac{1}{2} + \frac{1}{6} + \frac{1}{2} - 0 - 0 - 0\right] \\
&= \frac{7}{6} \\
&\approx 1.167
\end{aligned}$$

$$
\begin{aligned}
\int_C G(x,y)dy &= \int_C \left(x^3 + 2xy^2 + 2x\right) dy \\
&= \int_0^1 \left((2t)^3 + 2(2t)(t^2)^2 + 2(2t)\right)(2tdt) \\
&= 8\int_0^1 2t^4 + t^6 + t^2 dt \\
&= 8\left[\frac{2}{5}t^5 + \frac{1}{7}t^7 + \frac{1}{3}t^3\right]_0^1 \\
&= 8\left[\frac{2}{5} + \frac{1}{7} + \frac{1}{3} - 0 - 0 - 0\right] \\
&= (\frac{92}{105}) \\
&\approx 0.876
\end{aligned}
$$

$$
\begin{aligned}
\int_C G(x,y)ds &= \int_C \left(x^3 + 2xy^2 + 2x\right) ds \\
&= \int_0^1 \left((2t)^3 + 2(2t)(t^2)^2 + 2(2t)\right)\left(\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} dt\right) \\
&= \int_0^1 \left((2t)^3 + 2(2t)(t^2)^2 + 2(2t)\right)\left(\sqrt{(2)^2 + (2t)^2} dt\right) \\
&= \int_0^1 \left((2t)^3 + 2(2t)(t^2)^2 + 2(2t)\right)\left(\sqrt{4 + 4t^2} dt\right) \\
&= 8\int_0^1 \left(2t^3 + t^5 + t\right)\left(\sqrt{1 + t^2} dt\right) \\
&= 8\int_0^1 \left(2t^2 + t^5 + t\right)\sqrt{1 + t^2} dt \\
&\approx 1.4734 \qquad\qquad \text{Using WolframAlpha}
\end{aligned}
$$

31) Evaluate $\oint_C \left(x^2 - y^2\right) ds$, where C is given by $x = 5\cos t, y = 5\sin t, 0 \leq t \leq 2\pi$
We see

$$\begin{aligned}
ds &= \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} dt \\
&= \sqrt{25\sin^2 t + 25\sin^2 t} dt \\
&= 5dt
\end{aligned}$$

So we have

$$\begin{aligned}
\oint_C \left(x^2 - y^2\right) ds &= 5\oint_0^{2\pi} \left((5\cos t)^2 - (5\sin t)^2\right) dt \\
&= 5\oint_0^{2\pi} \left(25\cos^2 t - 25\sin^2 t\right) dt \\
&= 125\oint_0^{2\pi} \left(\cos^2 t - \sin^2 t\right) dt \\
&= 125\left[\frac{1}{2}\left(t + \sin t\cos t\right) + frac12\left(-t + \sin t\cos t\right)\right]_0^{2\pi} \\
&= 125\left[\sin t\cos t\right]_0^{2\pi} \\
&= 125\left[0 - 0\right] \\
&= 0
\end{aligned}$$