Complex Analysis

Homework 12: 3.2) 33, 34

Kenny Roffo

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33) Complete the proof to show that if functions f and g are analytic at a point z_0 and $f(z_0) = 0$, and $g(z_0) = 0$, but $g'(z_0) \neq 0$, then $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$.

We begin with the hypothesis that f and g are analytic at a point z_0. Analyticity at z_0 implies f and g are differentiable at z_0. Hence both limits

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \qquad \text{and} \qquad g'(z_0) = \lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0}$$

exist. But since $f(z_0) = 0, g(z_0) = 0$, the foregoing limits are the same as

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z))}{z - z_0} \qquad \text{and} \qquad g'(z_0) = \lim_{z\to z_0} \frac{g(z)}{z - z_0}$$

Now examine $\lim_{z\to z_0} \frac{f(z)}{g(z)}$ and finish the proof.

$$\begin{aligned}
\lim_{z\to z_0} \frac{f(z)}{g(z)} &= \lim_{z\to z_0} \frac{f(z)}{g(z)} \cdot \frac{\frac{1}{z-z_0}}{\frac{1}{z-z_0}} \\
&= \lim_{z\to z_0} \frac{\frac{f(z)}{z-z_0}}{\frac{g(z)}{z-z_0}} \\
&= \frac{\lim_{z\to z_0} \frac{f(z)}{z-z_0}}{\lim_{z\to z_0} \frac{g(z)}{z-z_0}} \\
&= \frac{f'(z_0)}{g'(z_0)} && \text{(Since } g'(z_0) \neq 0\text{)}
\end{aligned}$$

34 a) Justify the equality

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}z}[f(z)g(z)] &= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)}{\Delta z}\\ &= \lim_{\Delta z\to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}g(z+\Delta z)+f(z)\frac{g(z+\Delta z)-g(z)}{\Delta z}\right]\end{aligned}$$

I will start with the second form, and multiply through and simplify:

$$\begin{aligned}&\lim_{\Delta z\to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}g(z+\Delta z)+f(z)\frac{g(z+\Delta z)-g(z)}{\Delta z}\right]\\ &= \lim_{\Delta z\to 0}\left[\frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z+\Delta z)}{\Delta z}+\frac{f(z)g(z+\Delta z)-f(z)g(z)}{\Delta z}\right]\\ &= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z+\Delta z)+f(z)g(z+\Delta z)-f(z)g(z)}{\Delta z}\\ &= \lim_{\Delta z\to 0}\frac{f(z+\Delta z)g(z+\Delta z)-f(z)g(z)}{\Delta z}\end{aligned}$$

Thus the equality is justified.

34 b) Use the definition of a complex function being continuous at a point to justify $\lim_{\Delta z\to 0} g(z+\Delta z)=g(z)$.

Since g is differentiable at z, it is continuous at z, so $\lim_{z'\to z} g(z')=g(z)$. As $\Delta z\to 0$, we see $(z+\Delta z)\to z$, so we have

$$\begin{aligned}\lim_{\Delta z\to 0} g(z+\Delta z) &= \lim_{z'\to z} g(z')\\ &= g(z)\end{aligned}$$

34 c) Use that the limit of a sum is the sum of the limits, and the limit of a produt is the product of the limits to finish the proof.

$$\begin{aligned}&\lim_{\Delta z\to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}g(z+\Delta z)+f(z)\frac{g(z+\Delta z)-g(z)}{\Delta z}\right]\\ &= \lim_{\Delta z\to 0}\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right]\lim_{\Delta z\to 0}\left[g(z+\Delta z)\right]+\lim_{\Delta z\to 0}\left[f(z)\right]\lim_{\Delta z\to 0}\left[\frac{g(z+\Delta z)-g(z)}{\Delta z}\right] && \text{By limit laws}\\ &= f'(z)g(z)+f(z)g'(z) && \text{By part b}\end{aligned}$$