Complex Analysis

Homework 2: 1.2) 20, 32, 48

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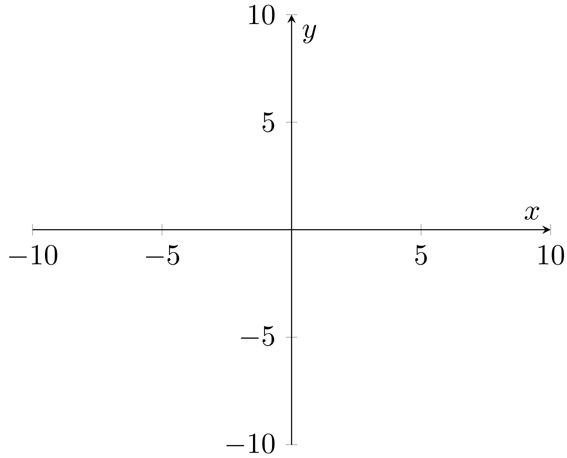
20) Describe the set of points z in the complex plane that satisfy the equation $\bar{z} = z^{-1}$. Include a plot of the set of points in the complex plane.

Let $z = x + iy$. We see

$$\begin{aligned} \bar{z} = z^{-1} &\iff x - yi = \frac{1}{x+yi} \\ &\iff (x-yi)(x+yi) = 1 \\ &\iff x^2 + y^2 = 1 \end{aligned}$$



32) Find a number $z = x + iy$ that satisfies the equation $|z|^2 + 1 + 12i = 6z$.

$$\begin{aligned} & |z|^2 + 1 + 12i = 6z \\ \implies & x^2 + y^2 + 1 + 12i = 6x + 6yi \\ \implies & (x^2 + y^2 - 6x + 1) + (12 - 6y)i = 0 + 0i \end{aligned}$$

By the definition of equality of complex numbers, we have

$$x^2 + y^2 - 6x + 1 = 0 \qquad \text{and} \qquad 12 - 6y = 0$$

Using the second equation, we find the value of y:

$$
\begin{aligned}
&12-6y=0\\
\implies &12=6y\\
\implies &2=y
\end{aligned}
$$

And plugging into the first equation we find x:

$$
\begin{aligned}
&x^2+y^2-6x+1=0\\
\implies &x^2+4-6x+1=0\\
\implies &(x-5)(x-1)=0\\
\implies &x\in\{1,5\}
\end{aligned}
$$

Therefore, we can choose either of two values for x to find a z, and thus the number $z=1+2i$ satisfies the equation $|z|^2+1+12i=6z$.

48) For any two comlex numbers z_1 and z_2, show that

$$|z_1+z_2|^2+|z_1-z_2|^2=2(|z_1|^2+|z_2|^2)$$

Let $z_1=x_1+y_1i$ and $z_2=x_2+y_2i$ be complex numbers. We see

$$
\begin{aligned}
|z_1+z_2|^2+|z_1-z_2|^2&=|(x_1+x_2)-(y_1+y_2)i|^2+|(x_1-x_2)+(y_1-y_2)i|^2\\
&=(x_1+x_2)^2+(y_1+y_2)^2+(x_1-x_2)^2+(y_1-y_2)^2\\
&=x_1^2+2x_1x_2+x_2^2+y_1^2+2y_1y_2+y_2^2+x_1^2-2x_1x_2+x_2^2+y_1^2-2y_1y_2+y_2^2\\
&=2x_1^2+2y_1^2+2x_2^2+2y_2^2\\
&=2\left(\sqrt{x_1^2+y_1^2}^2+\sqrt{x_2^2+y_2^2}^2\right)\\
&=2\left(|z_1|^2+|z_2|^2\right)
\end{aligned}
$$

That is, $|z_1+z_2|^2+|z_1-z_2|^2=2\left(|z_1|^2+|z_2|^2\right)$.