Complex Analysis

Homework 9: 3.1) 47, 52

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20) Consider the limit $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2}i\right)$
a. What value does the limit approach as z approaches along the line $y = x$?

Along the line $y = x$ the function becomes

$$\begin{aligned}\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2}i &= \frac{2x^2}{x^2} - \frac{x^2-x^2}{x^2}i \\ &= 2 - 0i\end{aligned}$$

Thus the limit $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2}i\right)$ as z approaches along the line $y = x$ is $2 - 0i$.

b. What value does the limit approach as z approaches 0 along the line $y = -x$?

Along the line $y = -x$, we know that $y^2 = x^2$, and the same thing happens as in part a, and we find that the limit $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2}i\right)$ as z approaches along the line $y = -x$ is $2 - 0i$.

c. Do the answers from parts a and b imply that the limit exists?

Of course not! The limit only exists if it is the same along every possible path.

d. What value does the limit approach as z approaches 0 along the line $y = 2x$?

Along the line $y = 2x$, $y^2 = 4x^2$ and the function becomes $\frac{8x^2}{x^2} - \frac{x^2-4x^2}{4x^2}i$. To solve for this limit, we will have to make use of L'Hospital's Rule on the real an imaginary parts.

First we look at the real part.

$$\begin{aligned}
\lim_{z\to 0} \frac{8x^2}{x^2} &= \lim_{z\to 0} \frac{16x}{2x} && \text{By L'Hospital's Rule} \\
&= \lim_{z\to 0} \frac{16}{2} && \text{By L'Hopital's Rule} \\
&= 8
\end{aligned}$$

Now we apply L'Hopital's Rule to find the limit of the imaginary part:

$$\begin{aligned}
\lim_{z\to 0} \frac{x^2 - 4x^2}{4x^2} i &= \lim_{z\to 0} \frac{-6x}{8x} i && \text{By L'Hospital's Rule} \\
&= \lim_{z\to 0} \frac{-6}{8} i && \text{By L'Hospital's Rule} \\
&= -\frac{3}{4}
\end{aligned}$$

Synthesizing these results, we find that $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2} i\right)$ as z approaches along the line $y = 2x$ is $8 - \frac{3}{4}i$.

e. What can you say about $\lim_{z\to 0}\left(\frac{2y^2}{x^2} - \frac{x^2-y^2}{y^2} i\right)$.

Since the limit is not the same across every path of approach, the limit does not exist.

30) Show that $f(z) = \frac{z-3i}{z^2+2z-1}$ is continuous at the point $z_0 = 1 + i$.

Since $f(z)$ is a rational function, we know it is continuous at all points. Thus, $f(z)$ is continuous at z_0.

40) Show that the $f(z) = \begin{cases} \frac{z}{|z|} & z \neq 0 \\ 1 & z = 0 \end{cases}$ is discontinuous at $z = 0$.

Consider the limit as z approaches along the line $y = x$. We have

$$\begin{aligned}
\lim_{z\to 0} f(z) &= \lim_{z\to 0} \frac{x+iy}{|x+iy|} \\
&= \lim_{z\to 0} \frac{x+ix}{|x+ix|} \\
&= \lim_{z\to 0} \frac{x+ix}{\sqrt{x^2+x^2}} \\
&= \lim_{z\to 0} \frac{x+ix}{\sqrt{2}x} \\
&= \lim_{z\to 0} \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}
\end{aligned}$$

Therefore, if the limit exists it must be $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$, and if it does not exist, then $f(z)$ is discontinuous at $z_0 = 0$. Assuming, however, that it does exist, we see $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \neq 0 = f(z_0)$, and thus $f(z)$ is discontinuous at z_0. Therefore, no matter what the case, $f(z)$ is discontinuous at z_0.