Complex Analysis

Homework 6: 2.2) 4, 18, 27

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Due September 21, 2015

4) Find the image S' of the set $S = \{z | 2 \leq \mathrm{Re}(z) \leq 3\}$ under the complex mapping $f(z) = 3iz$.

Let $z = x + iy$ be an element of S. We see

$$\begin{aligned} f(z) &= 3iz \\ &= 3i(x + iy) \\ &= -3y + i3x \end{aligned}$$

Since $2 \leq x \leq 3$ we have that $6 \leq \mathrm{Im}(f(z)) \leq 9$, and there is no restriction on y, so the image of S under $f(z) = 3iz$ is

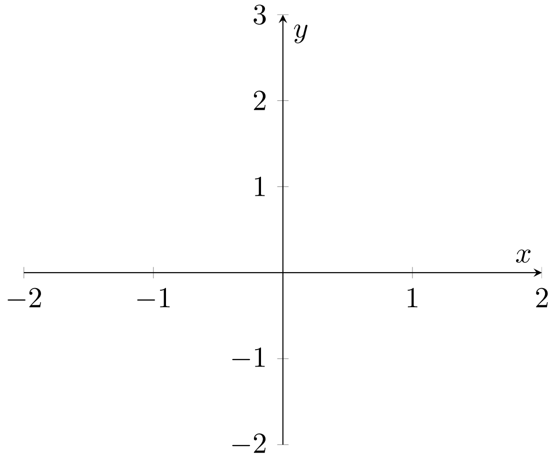
$$S' = \{z | 6 \leq \mathrm{Im}(z) \leq 9\}$$

18a) Plot the parametric curve C given by $z(t) = i + e^{it}, 0 \leq t \leq \pi$ and describe the curve in words.

Note that $z(t)$ can be rewritten as

$$\begin{aligned} z(t) &= i + \cos(t) + i\sin(t) \\ &= \cos(t) + i(1 + \sin(t)) \end{aligned}$$

This is familiar as it is the equation of a circle with a 1 added to the imaginary part. Thus, $z(t)$ for $0 \leq t \leq \pi$ forms the top half of a circle of radius 1 centered at $(0, 1)$.



18b) Find the parametrization of the image C' of C under the mapping $f(z) = (z-i)^2$

$$\begin{aligned} f(z(t)) &= ((i + e^{it}) - 1)^2 \\ &= e^{2it} \end{aligned}$$

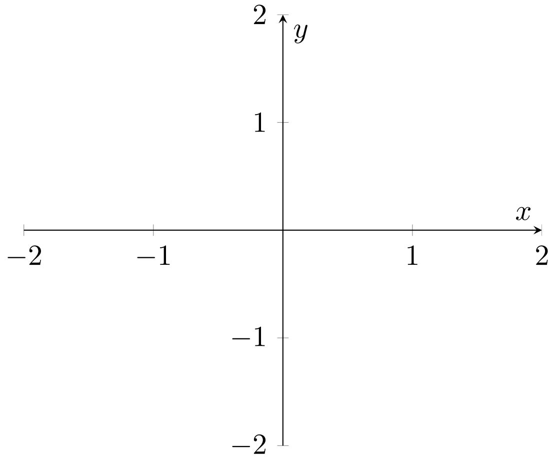
Thus the curve C' is given by $f(z(t)) = e^{2it}$

18c) Plot the parametric curve C' found in part b and describe the curve in words.

Note that C' can be represented by

$$e^{2it} = \cos(2t) + i\sin(2t)$$

since $0 \leq t \leq \pi$, this forms a full circle centered at the origin of radius 1.



27) In this problem we find the image of the line $x = 1$ under the complex mapping $w = 1/z$

a. The line $x = 1$ consists of all points $z = 1 + iy$ where $-\infty < y < \infty$. Find the real and imaginary parts u and v of $f(z) = 1/z$ at a point $z = 1 + iy$ on this line.

$$\begin{aligned} f(1+iy) &= \frac{1}{1+iy} \\ &= \frac{1-iy}{(1+iy)(1-iy)} \\ &= \frac{1-iy}{1+y^2} \\ &= \frac{1}{1+y^2} + i\frac{-y}{1+y^2} \end{aligned}$$

Thus we have $\mathrm{Re}(f(1+iy)) = \frac{1}{1+y^2}$ and $\mathrm{Im}(f(1+iy)) = -\frac{y}{1+y^2}$.

b. Show that $\left(u - \frac{1}{2}\right)^2 + v^2 = \frac{1}{4}$ for the functions u and v from part **a**.

$u = \frac{1}{1+y^2}$ and $v = -\frac{y}{1+y^2}$. Plugging in, we have

$$\begin{aligned}\left(u - \frac{1}{2}\right)^2 + v^2 &= \left(\frac{1}{1+y^2} - \frac{1}{2}\right)^2 + \left(-\frac{y}{1+y^2}\right)^2 \\ &= \frac{1}{\left(y^2+1\right)^2} - \frac{1}{y^2+1} + \frac{1}{4} + \frac{y^2}{\left(y^2+1\right)^2} \\ &= \frac{y^2+1}{\left(y^2+1\right)^2} - \frac{1}{y^2+1} + \frac{1}{4} \\ &= \frac{1}{4}\end{aligned}$$

c. Based on part **b**, describe the image of the line $x = 1$ under the complex mapping $w = 1/z$

The equation from part **b** is that of a circle centered at $(\frac{1}{2}, 0)$ with radius $\frac{1}{2}$

d. Is there a point on the line $x = 1$ that maps onto 0? Do you want to alter your description of the image in part **c**?

If there were a point which mapped to 0, that would imply that 0 times some number would equal something non-zero, which is a contradiction. Therefore I would like to alter myanswer from part **c** to be the same, but without the point $(0, 0)$.