Complex Analysis

Homework 15: 4.1) 32, 48, 49

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32) Write the principal value of $\text{Ln}\left[(1+i)^4\right]$ in the form $a+ib$:

We know for a complex number z that $\text{Ln}(z) = \log_e|z| + i\text{Arg}(z)$. First let us find $\log_e|z|$.

$$\begin{aligned}\log_e|z| &= \log_e\left|(1+i)^4\right| \\ &= \log_e\left[|1+i|^4\right] \\ &= \log_e\sqrt{2}^4 \\ &= \log_e 4\end{aligned}$$

We will keep this in exact form to preserve meaning. Now we find $\text{Arg}(z)$. First we find $\text{Re}(z)$.

$$\begin{aligned}(1+i)^4 &= 1+4i+6i^2+4i^3+1 \\ &= 2-6+4i-4i \\ &= -4\end{aligned}$$

So $\text{Re}(z) = -4$. Now we find $\text{Arg}(z)$ using $\text{Re}(z) = |z|\cos(\theta)$ (Note that we already found $|z| = 4$):

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\text{Re}(z)}{|z|}\right) \\ &= \cos^{-1}\left(\frac{-4}{4}\right) \\ &= \cos^{-1}(-1) \\ &= \pi\end{aligned}$$

Thus $Arg(z) = \pi$, and so the principal value of $\text{Ln}\left[(1+i)^4\right]$ is

$$\text{Ln}(z) = \log_e 4 + \pi i$$

48) Using de Moivre's formula, $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$, prove that $(e^z)^n = e^{nz}$ where $n \in \mathbb{Z}$.

Let $z = x + iy$ be an element of $\mathbb{C}$ and $n \in \mathbb{Z}$. We see

$$\begin{aligned}
(e^z)^n &= \Big(e^x\left(\cos(y) + i\sin(y)\right)\Big)^n \\
&= e^{nx}\left(\cos(y) + i\sin(y)\right)^n \\
&= e^{nx}\left(\cos(ny) + i\sin(ny)\right) \\
&= e^{nx+iny} \\
&= e^{n(x+iy)} \\
&= e^{nz}
\end{aligned}$$

That is, $(e^z)^n = e^{nz}$.

49) Determine where the complex function $e^{\overline{z}}$ is analytic.

We see

$$\begin{aligned}
e^{\overline{z}} &= e^{x-iy} \\
&= e^x\cos(y) + ie^x\sin(y)
\end{aligned}$$

Thus for $f(z) = e^{\overline{z}}$, $u(x,y) = e^x\cos(y)$ and $v(x,y) = e^x\sin(y)$. We see the first order partial derivatives:

$$\begin{aligned}
u_x &= e^x\cos(y) & v_x &= e^x\sin(y) \\
u_y &= -e^x\sin(y) & v_y &= e^x\cos(y)
\end{aligned}$$

We see the Cauchy-Riemann equations state that

$$u_x = e^x\cos(y) = e^x\cos(y) = v_y$$

and

$$u_y = -e^x\sin(y) = -e^x\sin(y) = -v_x$$

thus the Cauchy-Riemann equations are satisfied everywhere in the complex plane.

Note that the first order partial derivatives of u and v are also continuous. Let us examine the second order partial derivatives:

$$\begin{aligned}
u_{xx} &= e^x\cos(y) & v_{xx} &= e^x\sin(y) \\
u_{yy} &= -e^x\cos(y) & v_{yy} &= -e^x\sin(y) \\
u_{xy} &= -e^x\sin(y) & v_{xy} &= e^x\cos(y)
\end{aligned}$$

Since all of the first and second order partials of u and v are continuous, and u and v satisfy the C-R equations, u and v are harmonic conjugates. Therefore, $f(z) = u(x,y) + iv(x,y) = e^{\overline{z}}$ is analytic in $\mathbb{C}$.