Complex Analysis

Homework 20: 5.1) 14, 28, 32

Kenny Roffo

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14) Evaluate the line integrals $\int_C G(x,y)dx, \int_C G(x,y)dy$ and $\int_C G(x,y)ds$ on the curve

$$G(x,y)\frac{x^2}{y^3}; \qquad 2y = 3x^{3/2} \qquad 1 \le x \le 8$$

$$\begin{aligned}
\int_C G(x,y)dx &= \int_1^8 \left(\frac{x^2}{y^3}\right) dx \\
&= \int_1^8 \left(\frac{x^2}{\left(\frac{3}{2}x^{3/2}\right)^3}\right) dx \\
&= \frac{3^3}{2^3}\int_1^8 \left(\frac{x^2}{x^{9/2}}\right) dx \\
&= \frac{8}{27}\int_1^8 x^{2-9/2} dx \\
&= \frac{8}{27}\int_1^8 x^{-5/2} dx \\
&= \frac{8}{27}\left[-\frac{2}{3}x^{-3/2}\right]_1^8 \\
&= \frac{16}{81}\left[-8^{-3/2} + 1^{-3/2}\right] \\
&= \frac{16}{81}\left[1 - \frac{1}{8\sqrt{8}}\right] \\
&\approx 0.1888
\end{aligned}$$

$$
\begin{aligned}
\int_C G(x,y)dy &= \int_1^8 \frac{x^2}{y^3}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{2}x^{3/2}\right)dx \\
&= \int_1^8 \frac{x^2}{\left(\frac{3}{2}x^{3/2}\right)^3}x^{1/2}dx \\
&= \frac{8}{27}\int_1^8 x^{2+1/2-9/2}dx \\
&= \frac{8}{27}\int_1^8 x^{-2}dx \\
&= \frac{8}{27}\left[-x^-1\right]_1^8 \\
&= \frac{8}{27}\left[1-\frac{1}{8}\right] \\
&\approx 0.259
\end{aligned}
$$

$$
\begin{aligned}
\int_C G(x,y)ds &= \int_1^8 \frac{x^2}{\left(\frac{3}{2}x^{3/2}\right)^3}\sqrt{1+\left[\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3}{2}x^{3/2}\right)\right]^2}dx \\
&= \int_1^8 \frac{x^2}{\left(\frac{3}{2}x^{3/2}\right)^3}\sqrt{1+\left[x^{1/2}\right]^2}dx \\
&= \frac{8}{27}\int_1^8 \frac{x^2}{x^{9/2}}\sqrt{1+x}dx \\
&= \frac{8}{27}\left[-\frac{2}{3}\frac{(x+1)^{3/2}}{x^{3/2}}\right]_1^8 && \text{via WolframAlpha.com} \\
&= 0.323
\end{aligned}
$$

28) Evaluate $\oint_C (x^2+y^2)dx - 2xydy$ on the closed curve $C = C_1 \cup C_2$ where

$C_1 = \left\{(x,y)|y = x^2, 0 \le x \le 1\right\}$ and $C_2 = \{(x,y)|y = \sqrt{x}, 0 \le x \le 1\}$

$$\begin{aligned}
\oint_C \left(x^2+y^2\right) dx - 2xydy &= \oint_{C_1} \left(x^2+y^2\right) dx - 2xydy + \oint_{C_2} \left(x^2+y^2\right) dx - 2xydy \\
&= \oint_0^1 \left(x^2 + \left(x^2\right)^2\right) dx - 2x(x^2)(2xdx) + \oint_0^1 (x^2 + (\sqrt{x})^2)dx - 2x\sqrt{x}\left(\frac{1}{2}x^{-1/2}dx\right) \\
&= \oint_0^1 \left(x^2+x^4\right) - 4x^4 + \left(x^2+x\right) - x\ dx \\
&= \oint_0^1 -3x^4 + 2x^2\ dx \\
&= \left[-\frac{3}{5}x^5 + \frac{2}{3}x^2\right]_0^1 \\
&= \frac{2}{3} - \frac{3}{5} \\
&= \frac{19}{15} \\
&\approx 3.267
\end{aligned}$$

32) Evaluate $\int_{-C} y dx - x dy$ where C is given by $x = 2\cos t, y = 3\sin t, 0 \leq t \leq \pi$

$$\begin{aligned}
\int_{-C} y dx - x dy &= \int_{\pi}^{0} (3\sin t)\,(-2\sin t)dt - (2\cos t)\,(3\cos t)dt \\
&= \int_{\pi}^{0} -6\sin^2 t - 6\cos^2 t dt \\
&= -6\int_{\pi}^{0} 1 dt \\
&= 6\pi
\end{aligned}$$