Complex Analysis

Homework 13: 3.3) 24, 40

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24 a) Show the function $f(z) = x^2 - x + y + i(y^2 - 5y - x)$ is not analytic at any point but is differentiable along the curve $y = x + 2$

We see $u(x, y) = x^2 - x + y$ and $v(x, y) = y^2 - 5y - x$. So

$$\begin{aligned} u_x &= 2x - 1 \\ u_y &= 1 \\ v_x &= -1 \\ v_y &= 2y - 5 \end{aligned}$$

In order for the function to be differentiable at a point z, the C-R equations must hold. It is true for all points in the complex plane that $u_y = -v_x$, but let us examine the second C-R equation, $u_x = v_y$.

$$\begin{aligned} & 2x - 1 = 2y - 5 \\ \iff & 2x + 4 = 2y \\ \iff & x + 2 = y \end{aligned}$$

Therefore, the C-R equations hold at a point z if and only if z is on the line $y = x + 2$. Thus, f is not differentiable (and thus not analytic) at any point not on the line $y = x+2$. Also, since every neighborhood of every point on the line $y = x + 2$ contains a point not on the line $y = x + 2$, every neighborhood of every point on the line contains a point which is not differentiable, thus f is not analytic at any point on the line $y = x + 2$. Therefore, f is nowhere analytic.

Note that u, v, u_x, u_y, v_x, and v_y are all polynomial (or constant) functions. Then all u and v and their first order partial derivatives are all continuous. Since this is true, and the C-R equations hold for points on the line $y = x + 2$, the function f is differentiable on the line $y = x + 2$.

24 b) Find the derivative of $f(z) = x^2 - x + y + i(y^2 - 5y - x)$ on the curve $y = x + 2$. Using the fact that $f'(z) = \frac{\partial u}{\partial z} + i\frac{\partial v}{\partial x}$ we find the derivative:

$$\begin{aligned} f'(z) &= \tfrac{\partial u}{\partial z} + i\tfrac{\partial v}{\partial x} \\ &= 2x - 1 + i(-1) \\ &= 2x - 1 - i \end{aligned}$$

40) Consider the function

$$f(z) = \begin{cases} 0 & z = 0 \\ \frac{z^5}{|z^4|} & z \neq 0 \end{cases}$$

a) Express f in the form $f(z) = u(x, y) + iv(x, y)$

We first express $\frac{z^5}{|z^4|}$ in this form:

$$\begin{aligned} \frac{z^5}{|z^4|} &= \frac{(x+iy)^5}{|(x+iy)^4|} \\ &= \frac{x^5 + 5x^4iy + 10x^3i^2y^2 + 10x^2i^3y^3 + 5xi^4y^4 + i^5y^5}{|x^4 + 4x^3iy + 6x^2i^2y^2 + 4xi^3y^3 + i^4y^4|} \\ &= \frac{(x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)}{|(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)|} \\ &= \frac{(x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)}{\sqrt{(x^4 - 6x^2y^2 + y^4)^2 + (4x^3y - 4xy^3)^2}} \\ &= \frac{(x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)}{\sqrt{x^8 + 4x^6y^2 + 6x^4y^4 + 4x^2y^6 + y^8}} && \text{After a bit of algebra} \\ &= \frac{(x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)}{\sqrt{(x^2+y^2)^4}} \\ &= \frac{(x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)}{(x^2+y^2)^2} \\ &= \frac{(x^5 - 10x^3y^2 + 5xy^4)}{(x^2+y^2)^2} + i\frac{(5x^4y - 10x^2y^3 + y^5)}{(x^2+y^2)^2} \end{aligned}$$

Now we have $f(z)$ given by

$$f(z) = \begin{cases} 0 & z = 0 \\ \left(\frac{x^5 - 10x^3y^2 + 5xy^4}{(x^2+y^2)^2}\right) + i\left(\frac{5x^4y - 10x^2y^3 + y^5}{(x^2+y^2)^2}\right) & z \neq 0 \end{cases}$$

b) Show that f is not differentiable at the origin.

Consider the limit as f approaches 0 along the imaginary axis ($y = 0$):

$$\begin{aligned}
\lim_{x \to 0} \frac{(x + i(0))^5}{|(x + i(0))^4|} &= \lim_{x \to 0} \frac{x^5}{|x^4|} \\
&= \lim_{x \to 0} \frac{x^5}{x^4} \\
&= \lim_{x \to 0} x \\
&= 0
\end{aligned}$$

Now consider the limit as f approaches 0 along the line $y = x + 1$:

$$\begin{aligned}
\lim_{x \to 0} \frac{(x + i(x + 1))^5}{|(x + i(x + 1))^4|} &= \lim_{x \to 0} \frac{(0 + i(0 + 1))^5}{|(0 + i(0 + 1))^4|} \\
&= \lim_{x \to 0} \frac{i^5}{|i^4|} \\
&= \lim_{x \to 0} \frac{i^5}{1} \\
&= i^5
\end{aligned}$$

So we get different values for the limit depending on which path of approach we choose, thus the limit does not exist. This means that the f is discontinuous at 0, thus f is not differentiable at 0.

c) Show that the Cauchy-Riemann equations are satisfied at the origin.

We begin by finding the partial derivatives u_x, u_y, v_x and v_y at the point 0. At 0, $u = v = 0$. Using the limit definition for partial derivatives, we see

$$\begin{aligned} u_x &= \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} \\ &= \lim_{\Delta x \to 0} 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} u_y &= \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} \\ &= \lim_{\Delta y \to 0} 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} v_x &= \lim_{\Delta x \to 0} \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{0 - 0}{\Delta x} \\ &= \lim_{\Delta x \to 0} 0 \\ &= 0 \end{aligned}$$

$$\begin{aligned} v_y &= \lim_{\Delta y \to 0} \frac{v(x, y + \Delta y) - v(x, y)}{\Delta y} \\ &= \lim_{\Delta y \to 0} \frac{0 - 0}{\Delta y} \\ &= \lim_{\Delta y \to 0} 0 \\ &= 0 \end{aligned}$$

Since $0 = 0 =$ -0, the Cauchy-Riemann equations are satisfied for f at $z = 0$.