Quantum Spin

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- Orbital momentum is momentum due to revolution, spin momentum is due to rotation
- Spin momentum is really just the sum of the orbital momentum of all the particles which make up the object

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- Instead we consider particles to have *intrinsic* angular momentum, **S**
- Elementary Particles each have a specific spin which never changes
- A few examples:

Particle	Spin
pi meson	0
e^-, p, n	$\frac{1}{2}$
photon	1
delta	$\frac{3}{2}$
graviton	2

Defining the Spin Operators

- We begin with the commutation relations:

$$[S_x, S_y] = i\hbar S_z \qquad [S_y, S_z] = i\hbar S_x \qquad [S_z, S_x] = i\hbar S_y$$

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and thus

$$S_\pm |sm\rangle = \hbar\sqrt{s(s+1) - m(m+1)} |s(m+1)\rangle \tag{2}$$

Here $S_\pm = S_x \pm iS_y$

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$$\chi = \begin{bmatrix} a \\ b \end{bmatrix} = a\chi_+ + b\chi_-$$

Here we have

$$\chi_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

represents spin up and

$$\chi_- = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

represents spin down

Spin 1/2

- From equation 1, we have:

$$\mathbf{S}^2\chi_+ = \frac{3}{4}\hbar^2\chi_+ \qquad \mathbf{S}^2\chi_- = \frac{3}{4}\hbar^2\chi_-$$

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- Solving this system we see that

$$\mathbf{S}^2 = \frac{3}{4}\hbar^2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Spin 1/2

- From $\mathbf{S}_z \chi_+ = \frac{\hbar}{2}\chi_+$ and $\mathbf{S}_z \chi_- = -\frac{\hbar}{2}\chi_-$ it follows:

$$\mathbf{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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- Referring back to equation 2, we have:

$$\mathbf{S}_+\chi_- = \hbar\chi_+ \qquad \mathbf{S}_-\chi_+ = \hbar\chi_- \qquad \mathbf{S}_+\chi_+ = \mathbf{S}_-\chi_- = 0$$

so

$$\mathbf{S}_+ = \hbar\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \mathbf{S}_- = \hbar\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

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- Recall $S_\pm = S_x + iS_y$. It follows that

$$\mathbf{S}_x = \frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad \mathbf{S}_y = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$$

Spin 1/2

- Since $\mathbf{S}_x$, $\mathbf{S}_y$ and $\mathbf{S}_z$ all have a factor of $\frac{\hbar}{2}$ we will let σ_x be the matrix in $\mathbf{S}_x$, and σ_y, σ_z defined similarly. Then $\mathbf{S} = \frac{\hbar}{2}\sigma$ and

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

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- These are known as the **Pauli Spin Matrices**

Addition of Angular Momenta

- Let's consider two spin-1/2 particles: Electron and Proton
- There are four possible configurations

$$\uparrow\downarrow, \uparrow\uparrow, \downarrow\downarrow, \downarrow\uparrow$$

- How would we calculated the total angular momentum of the atom? We let

$$S = S^{(1)} + S^{(2)}$$

- Each of the four states represent an eigenstate of S_z
- Therefore, the z components add

$$S_z \chi_1 \chi_2 = (S_z^1 + S_z^2) \chi_1 \chi_2 = (S_z^1 \chi_1) \chi_2 + \chi_1 (S_z^2 \chi_2)$$

Addition of Angular Momenta

- Continued...

$$S_z\chi_1\chi_2 = (S_z^1 + S_z^2)\chi_1\chi_2 = (S_z^1\chi_1)\chi_2 + \chi_1(S_z^2\chi_2)$$
$$= (\hbar m_1\chi_1)\chi_2 + \chi_1(\hbar m_2\chi_2) = \hbar(m_1 + m_2)\chi_1\chi_2$$

- The quantum number m is just $m_1 + m_2$

$$\uparrow\uparrow : m = 1$$
$$\uparrow\downarrow : m = 0$$
$$\downarrow\uparrow : m = 0$$
$$\downarrow\downarrow : m = -1$$

Spin States

- We can apply the lowering operator to the state $\uparrow\uparrow$ to find the states of S:

$$S_-(\uparrow\uparrow) = (S_-^{(1)}\uparrow)\uparrow + \uparrow(S_-^{(2)}\uparrow)$$
$$= (\hbar\downarrow)\uparrow + \uparrow(\hbar\downarrow) = \hbar(\downarrow\uparrow + \uparrow\downarrow)$$

- We can do this to obtain three states with $s = 1$
- In $|sm\rangle)$ notation they are:

$$\left\{ \begin{array}{ll} |1\,1\rangle & = \uparrow\uparrow \\ |1\,0\rangle & = \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ |1\,-1\rangle & = \downarrow\downarrow \end{array} \right\} \quad s = 1 \text{ (triplet).}$$

- This makes $s = 1$ a triplet state

Singlet State

- On the contrary, if we look at $s = 0$ we have only

$$\left\{ |0\,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow) \right\} \quad s = 0 \text{ (singlet).}$$

- This makes $s = 0$ a singlet state
- If we apply the raising or lowering operator to this we obtain zero
- Then, the total spin that two spin-1/2 particles can carry is either 1 or 0
- This is true if the triplet states are eigenvectors of S^2 with eigen value $2\hbar^2$
- The singlet state must also be an eigenvector of S^2 with eigenvalue of 0

Spin State Eigenvectors

- Let's check:

$$spinsquared S^2 = (S^{(1)}+S^{(2)})\cdot(S^{(1)}+S^{(2)}) = (S^{(1)})^2+(S^{(2)})^2+2S^{(1)}\cdot \tag{3}$$

- From previous equations we have

$$\begin{aligned} S^{(1)} \cdot S^{(2)}(\uparrow\downarrow) &= (S_x^{(1)}\uparrow)(S_x^{(2)}\downarrow) + (S_y^{(1)}\uparrow)(S_y^{(2)}\downarrow) + (S_z^{(1)}\uparrow)(S_z^{(2)}\downarrow \\ &= (\frac{\hbar}{2}\downarrow)(\frac{\hbar}{2}\uparrow) + (\frac{\imath\hbar}{2}\downarrow)(\frac{-\imath\hbar}{2}\uparrow) + (\frac{\hbar}{2}\uparrow)(\frac{-\hbar}{2}\downarrow) \\ &= \frac{\hbar^2}{4}(2\downarrow\uparrow - \uparrow\downarrow) \end{aligned}$$

- As we might expect, the opposite orientation gives:

$$S^{(1)} \cdot S^{(2)}(\downarrow\uparrow) = \frac{\hbar^2}{4}(2\uparrow\downarrow - \downarrow\uparrow)$$

Proof Continued

- If we check:

$$S^{(1)} \cdot S^{(2)}|10\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2\downarrow\uparrow - \uparrow\downarrow + 2\uparrow\downarrow - \downarrow\uparrow) = \frac{\hbar^2}{4}|10\rangle$$

and

$$S^{(1)} \cdot S^{(2)}|00\rangle = \frac{\hbar^2}{4}\frac{1}{\sqrt{2}}(2\downarrow\uparrow - \uparrow\downarrow - 2\uparrow\downarrow + \downarrow\uparrow) = -\frac{3\hbar^2}{4}|00\rangle$$

- Then from **??** we conclude that

$$S^2|10\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{\hbar^2}{4}\right)|10\rangle = 2\hbar^2|10\rangle$$

and

$$S^2|00\rangle = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + 2\frac{3\hbar^2}{4}\right)|00\rangle = 0$$

Spin State Eigenvectors

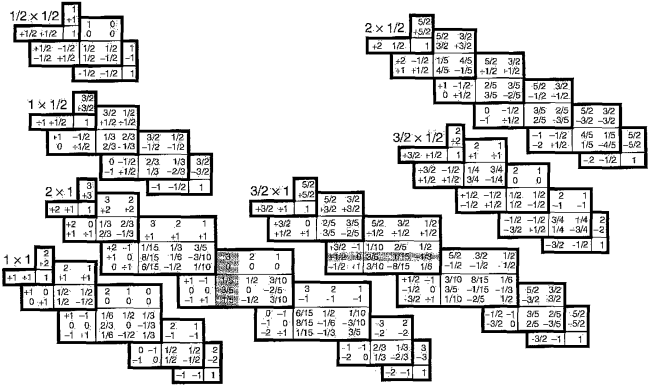
- We have just shown that if you combine spin $1/2$ with spin $1/2$ you will obtain spin 1 and 0 states.
- Spin 1 state is an eigenvector of S^2 with eigenvalue $2\hbar^2$ and spin 0 state is an eigenvector of S^2 with eigenvalue 0.
- If we are to expand this to the general case we see that by combing spin s_1 with spin s_2, we get a total spin of

$$s = (s_1 + s_2), (s_1 + s_2 - 1), (s_1 + s_2 - 2), ..., |s_1 - s_2|$$

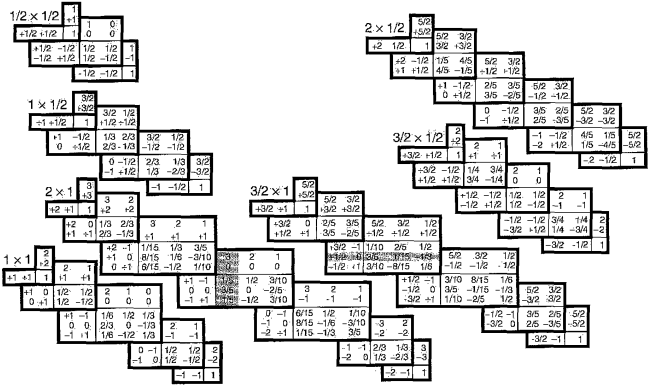
- This gives rise to the combined state of total spin s and z-component m as a linear combination of composite states:

$$|sm\rangle = \sum_{m_1+m_2=m} C^{s_1 s_2 s}_{m_1 m_2 m} |s_1 m_1\rangle |s_2 m_2\rangle$$

- The Clebsch-Gordan coefficients (a square root sign is present for every entry):



$$|30\rangle = \frac{1}{\sqrt{5}}|21\rangle|1-1\rangle + \sqrt{\frac{3}{5}}|20\rangle|10\rangle + \frac{1}{\sqrt{5}}|2-1\rangle|11\rangle$$



The End

Thanks for listening!