Limits

$\lim_{x \to a} f(x)$ → when x approaches to a then what is the value of $f(x)$

$f(a)$ may or may not be defined

For limit to exist LHL = RHL हो होना चाहिए।

LHL

$$\lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

RHL

$$\lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

Factorisation Method

1. $$\lim_{x \to 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1} = \lim_{x \to 1} \frac{\sqrt{x}\left((\sqrt{x})^3 - 1\right)}{\sqrt{x} - 1}$$

$\frac{0}{0}$ form

$$= \lim_{x \to 1} \frac{\sqrt{x}\,\cancel{(\sqrt{x}-1)}\,(x + 1 + \sqrt{x})}{\cancel{(\sqrt{x}-1)}}$$

$$= 3$$

Rationalization Method

$$\lim_{x \to 0} \frac{x^3}{\sqrt{1+x^3} - \sqrt{1-x^3}} \quad \left(\frac{0}{0}\right) = \lim_{x \to 0} \frac{x^3\left(\sqrt{1+x^3} + \sqrt{1-x^3}\right)}{\cancel{1} + x^3 - \cancel{1} + x^3}$$

$$= \frac{1}{2} \times 2$$

$$= 1$$

* Let $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$

then

1.) $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = l \pm m$

2.) $\lim_{x \to a} f(x) \times g(x) = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x) = l \times m$

3) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = l/m$ ⇒ ये 3rd वाला तभी होगा जब $m \neq 0$

4) $\lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} f(x)^{\lim_{x \to a} g(x)} = l^m$ if $\lim_{x \to a} f(x) = l$, $l > 0$

5.) $\lim_{x\to a} f\circ g(x) = f\left(\lim_{x\to a} g(x)\right) = f(m)$ provided $f(x)$ is continous at $g(x)=m$

Remember these limit (L'Hospital से आसानी से निकाल लेंगे)

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{e^x-1}{x} = 1$$

$$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1 \qquad \lim_{x\to 0} \frac{a^x-1}{x} = \log_e a$$

* $\frac{d(a^x)}{dx} = a^x \ln a$

Q $\lim_{x\to 0} \frac{15^x - 5^x - 3^x + 1}{\sin^2 x} \left(\frac{0}{0}\right)$

$$\lim_{x\to 0} \frac{(15^x-1)-(5^x-1)-(3^x-1)}{\left(\frac{\sin x}{x}\right)^2 x^2} \qquad \left(\frac{\sin^2 x}{x^2}\right)\times x \quad \to 1$$

$$\lim_{x\to 0} \frac{5^x 3^x - 5^x - 3^x + 1}{x^2} = \lim_{x\to 0} \frac{5^x(3^x-1)-1(3^x-1)}{x^2}$$

$$\lim_{x\to 0} \frac{(5^x-1)(3^x-1)}{x^2} = \ln 5 \cdot \ln 3$$

Q $\lim_{x\to 0} \frac{P(e^x - q)}{2x} = 1$ Find $P+q$

$\lim_{x\to 0} \frac{e^x-1}{x} = 1$ $\qquad P=2$, $q=1$ $\qquad P+q=3$

→ solving $\frac{\infty}{\infty}$ form

$$f(x) = \lim_{x\to\infty} \frac{x^m + x^{m-1} + x^{m-2} + \cdots}{x^n + x^{n-1} + x^{n-2} + \cdots}$$

if $m>n$ $\quad f(x) = \infty$

$m=n$ $\quad f(x) = K$

$m<n$ $\quad f(x) = 0$

coefficient का ratio

denominator, numerator की power देख लो नहीं तो highest power common ले लो individually

$\lim_{n\to\infty} \frac{n^2}{1+2+\cdots+n}$ $\quad \lim_{n\to\infty} \frac{n^2}{n(n+1)} = 1$

$\lim_{x\to-\infty} \frac{\sqrt{3x^2+2}}{x-2}$

$-\infty$ substitute कर देते $y = -1/x$ $\quad x \to -\infty,\ y \to 0$

$\lim_{y\to 0} \frac{\sqrt{\frac{3}{y^2}+2}}{-1/y-2} = \lim_{y\to 0} \frac{\sqrt{3+2y^2}}{-1-2y} = -\sqrt{3}$

$\frac{\infty}{\infty}$ highest power common करते से हो जाता है.

$\Rightarrow$ solving $\infty-\infty$ form $\Rightarrow$ इसको $\frac{0}{0}$ या $\frac{\infty}{\infty}$ form में convert करते हैं

$\lim_{x\to\infty} \left(\frac{x^2+x+1}{x+1} - ax - b \right) = 4$

sol

$\lim_{x\to\infty} \frac{x^2+x+1-ax^2-ax-bx-b}{x+1} = 4$

limit 4 ह आया है मतलब $a=1$

$\lim_{x\to\infty} \frac{\cancel{x}+1-\cancel{x}-bx-b}{x+1} = 4$

$\lim_{x\to\infty} \frac{x(1/x-b-b/x)}{x(1+1/x)} = 4 \Rightarrow b=-4$

$\lim_{x\to\infty} \left(\sqrt{x^2+2x} - x \right)$

$\lim_{x\to\infty} \frac{\sqrt{x^2+2x}+x}{\cancel{x^2}+2x-\cancel{x^2}} \Rightarrow \lim_{x\to\infty} \frac{x\left(\sqrt{1+2/x}+1\right)}{\cancel{x}\ 2}$

$=1$

→ solving 1^∞ form

$\lim\limits_{x\to a} \left(1+f(x)\right)^{g(x)}$ is of form 1^∞

$f(x) \to 0$ then $e^{\lim\limits_{x\to a} f(x)\cdot g(x)}$

$x\to a$ $g(x) \to \infty$

log लेके करते हैं

$$l = \lim_{x\to a} \left(1+f(x)\right)^{g(x)}$$

$$\log l = \lim_{x\to a} g(x)\log\left(1+f(x)\right)$$

$$\log l = \lim_{x\to a} g(x)\left\{\frac{\log(1+f(x))}{f(x)}\right\} f(x) \qquad \left\{\lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right\}$$

$\Downarrow$

$x \to a$

$f(x) \to 0$

$\therefore \lim\limits_{x\to a} \dfrac{\log(1+f(x))}{f(x)} = 1$

$$\log l = \lim_{x\to a} f(x)\, g(x)$$

$$l = e^{\lim\limits_{x\to a} f(x)\, g(x)}$$

* $\lim\limits_{x\to a} f(x)^{g(x)} = e^{\lim\limits_{x\to a} (f(x)-1)\, g(x)}$

$f(x) \to 1$

$g(x) \to \infty$

Q $\lim\limits_{x\to 0} (1+x)^{1/x}$ 1^∞

$e^{\lim\limits_{x\to 0} x \cdot 1/x} = e$

Q $\lim\limits_{x\to\infty} \left(\dfrac{x^2+5x+3}{x^2+x+3}\right)^x$ (1^∞) form

$\lim\limits_{x\to\infty} \dfrac{x^2+5x+3}{x^2+x+3} = 1$

$$e^{\lim_{x\to\infty}\left(\frac{x^2+5x+3}{x^2+x+3}-1\right)x}$$

$$e^{\lim_{x\to\infty}\frac{4x}{x^2+x+3}\cdot x}$$

$$e^4$$

L'Hopitals Rule

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$ → Differentiate numerator and denominator till $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form is removed

Q $\lim_{x\to 0}\frac{e^x-1-x}{x^2} \Rightarrow \lim_{x\to 0}\frac{e^x-1}{2x} = \frac{1}{2}$

Q $f(x)=|x|$

Discuss the limit of $f(x)$ at $x=0$

Sol. $f(x)=\begin{cases} x & x\geq 0 \\ -x & x<0 \end{cases}$

LHL

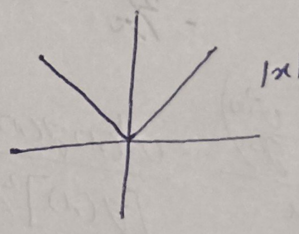
$\lim_{h\to 0}|x-h| = \lim_{h\to 0}|-h|$

$= 0$

RHL

$\lim_{x\to 0^+}|x| = \lim_{h\to 0}|0+h| = \lim_{h\to 0} h = 0$

$\because$ LHS = RHL $\therefore$ limit exist.



* Modulous function $|x-a|$ is continuous in its domain

Standard limits:-

1) $\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}$
2) $\lim_{x\to 0}\frac{e^{mx}-1}{x} = m$
3) $\lim_{x\to 0}\frac{a^x-1}{x} = \ln a$
4) $\lim_{x\to 0}[1+ax]^{1/x} = e^a$
5) $\lim_{x\to 0}[1+a/x]^x = e^a$
6) $\lim_{x\to 0}\frac{\sin x}{x} = 1$
7) $\lim_{x\to 0}\frac{\tan x}{x} = 1$
8) $\lim_{x\to\infty}\frac{\sin x}{x} = 0$
9) $\lim_{x\to\infty}\frac{\cos x}{x} = 0$
10) $\lim_{x\to 0}\frac{\sin mx}{x} = m$

11. $\lim_{x\to 0}\left[\frac{a^x+b^x}{2}\right]^{1/x} = \sqrt{ab}$

12) $\lim_{x\to 0}\left[\frac{a^x+b^x+c^x}{3}\right]^{1/x} = \sqrt[3]{abc}$

13) $\lim_{x\to 0}\left[\cos x + a\sin bx\right]^{1/x} = e^{ab}$

14) $\lim_{x\to 0}\left[\frac{1-\cos ax}{x^2}\right] = \frac{a^2}{2}$

→ solving $0\times\infty$ form इसमे $0/0$ या ∞/∞ form मे लाते है

Q $\lim_{x\to 1}(x-1)\tan\left(\frac{\pi x}{2}\right)$

sol $\lim_{x\to 1}\frac{(x-1)}{\cot(\pi x/2)}$

$\lim_{x\to 1}\frac{1}{-\operatorname{cosec}^2 \pi x/2 \cdot \pi/2}$

$-2/\pi$

$\frac{d(\sin x)}{dx} = \cos x$

$\frac{d(\tan x)}{dx} = \sec^2 x$

$\frac{d(\operatorname{cosec} x)}{dx} = -\cot x \operatorname{cosec} x$

$\frac{d(\cos x)}{dx} = -\sin x$

$\frac{d(\cot x)}{dx} = -\operatorname{cosec}^2 x$

$\frac{d(\sec x)}{dx} = \sec x\tan x$

$\frac{d\left(\frac{f(x)}{g(x)}\right)}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$

→ solving 0^0 form ⇒ इसको भी $0/0$ form मे लाओ।

$\frac{d(a^x)}{dx} = a^x \ln(a)$

$\frac{d(\log_a(x))}{dx} = \frac{1}{x\ln a}, x>0$

Q $\lim_{x\to 0} x^x$

Let $y = x^x$

$\log y = x\log x$

$\log y = \lim_{x\to 0} x\log x = \lim_{x\to 0}\frac{\log x}{1/x}$

$= \lim_{x\to 0}\frac{1/x}{-1/x^2}$

$= \lim_{x \to 0} -\frac{x^2}{x} = 0$

$\therefore \log y = 0$

$y = e^0 = 1$

* $f(g(x))$ is continuous at $x = a$ if $g(x)$ is continuous at $x = a$ & $f(x)$ is continuous at $g(a)$

Q $\lim_{x \to 0} x^{\sin x}$

Let $y = x^{\sin x}$

$\log y = \sin x \log x$

$\underbrace{\lim_{x \to 0} \log y}_{\text{y variable है}} = \lim_{x \to 0} \sin x \log x$

x का limit है कोई दिक्कत नहीं हटा दो

$\Rightarrow \log y = \lim_{x \to 0} \frac{\log x}{\operatorname{cosec} x}$

$= \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec} x \cot x}$

$= \lim_{x \to 0} \frac{-\sin x \tan x}{x}$

$= 1 \cdot 0 = 0$

$\therefore \log y = 0$

$y = e^0 = 1$

Continuity of a function :-

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x \to a} f(x) = f(a)$

LHL = RHL = value of a function at $x = a$

$\Downarrow$

$\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)$

If a function $f(x)$ is not continuous at $x = a$, we say $f(x)$ is discontinuous at $x = a$

$f(x)$ will be discontinuous at $x = a$ if

$\rightarrow \lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$

$\rightarrow \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \neq f(a)$

→ $f(a)$ is not defined

→ Atleast one the limit doesn't exists

इनमें से कोई 1 condition satisfy हो गई discontinuous बोल देंगे।

Q If $f(x) = \frac{|x|}{x}$, is it continuous at $x=0$? बोल देंगे discontinuous है।

$\frac{0}{0}$ या $\frac{\infty}{\infty}$ form हो तो सीधा बोल देंगे discontinuous है।

x की जो value है उसे put करके देख लो कौन सा form है

LHL at $x \to 0$

let $x = 0-h$

$$\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} \frac{|0-h|}{(0-h)}$$

$$= -\frac{\cancel{h}}{\cancel{h}} = -1$$

RHL at $x \to 0$

Let $x = 0+h$

$$\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} \frac{|0+h|}{(0+h)}$$

$$= \frac{\cancel{h}}{\cancel{h}} = 1$$

LHL ≠ RHL

∴ $f(x)$ is discontinuous

Q If $f(x) = \begin{cases} 2x+3 & \text{when } x<0 \\ 0 & \text{when } x=0 \\ x^2+3 & \text{when } x>0 \end{cases}$ is it continuous at $x=0$?

LHL.

$$\lim_{h\to 0} f(0-h) = \lim_{h\to 0} 2(0-h)+3$$

$$= \lim_{h\to 0} -2h+3 = 3$$

RHL

$$\lim_{h\to 0} f(0+h) = \lim_{h\to 0} (0+h)^2+3$$

$$= 3$$

∴ ~~Discontinuous~~

$L.H.L = R.H.S \neq f(0)$

$\therefore$ $f(x)$ is discontinuous

→ Continuity at end Points

In an interval $[a, b]$, $f(x)$ is continuous if it satisfies 3 conditions

i) $f(x)$ is continuous $\forall x \in (a, b)$

ii) $f(x)$ should be right continous at initial point

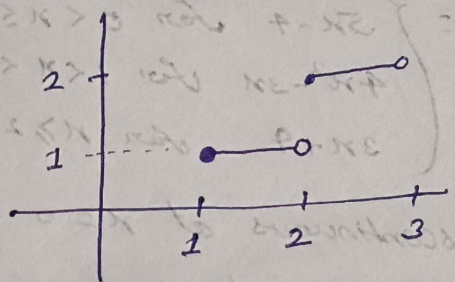
$$\lim_{x \to a^+} f(x) = f(a)$$

iii) $f(x)$ should be left continous at final point

$$\lim_{x \to b^-} f(x) = f(b)$$

Q If $f(x) = [x]$, where $[\cdot]$ denotes greatest integer functions. Then check continuity on $[1, 2]$

sol graph बना ले



interval $[1, 2]$ है

मतलब $x = 1$ & $x = 2$

1 पर तो continuous है

2 पर नहीं है

$\therefore$ $f(x)$ is discontinous over interval $[1, 2]$

$f(x)$ is continous over interval $[1, 2)$

मतलब $1 \leq x < 2$

Q $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3} & x \neq 3 \\ 3 & x = 3 \end{cases}$ Check continuity at $x = 3$

sol. $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3} = 6 \neq 3$ $\therefore$ discontinuous

Q If $f(x) = \begin{cases} (x+1)^{\cot x} & \text{for } x \neq 0 \\ K & \text{for } x = 0 \end{cases}$ is continuous at $x=0$ then find K?

Sol. Given,

$$\lim_{x \to a} f(x) = f(a)$$

$$\lim_{x \to 0} f(x) = f(0)$$

$$\lim_{x \to 0} (x+1)^{\cot x} = K$$

$$e^{\lim_{x \to 0} \cot x \cdot x} = K$$

$$e^{\lim_{x \to 0} \frac{x}{\tan x}} = K$$

$$\boxed{K = e}$$

Q If $f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 5x-4 & \text{for } 0 < x \leq 1 \\ 4x^2-3x & \text{for } 1 < x < 2 \\ 3x+4 & \text{for } x \geq 2 \end{cases}$ then which of the following is true.

a) $f(x)$ is discontinuous at $x=0$.

b) $f(x)$ is continuous at $x=1$

c) $f(x)$ is left continuous at $x=2$

d) All of the above

Sol. $x=0$

$f(x)=0$ LHL $=0$ RHL $=-4$ discontinuous

$x=1$

$f(x)=1$ L.H.L $=1$ R.H.L $=1$ continuous

$x=2$

left continuous मतलब $f(x) = \lim_{x \to 2^-} f(x)$

$4 \times 4 - 3 \times 2$

$16 - 6 = 10$

$f(2) = 10$

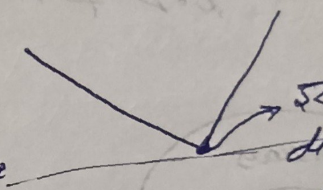
Differentiability

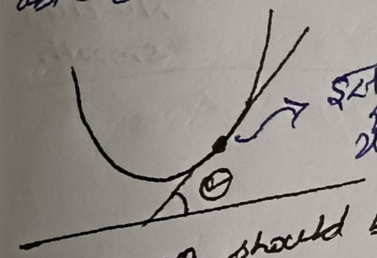
Let $f(x)$ be real valued function on an open interval (a,b) where $c \in (a,b)$. Then $f(x)$ is said to be differentiable (or) derivable at $x=c$

iff $\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ Exists and finite

This limit is derivative of $f(x)$ and it is denoted by $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

→ Smooth Curve find करते है
किसी point पर differentiable है या नहीं ये पता करने के लिए उसका graph draw कर लो और उस point पर । ही tangent possible है तब बोलेंगे Differentiable है





→ Necessary and sufficient conditions for a function to be differentiable at a point is existence of finite left hand derivative and finite right hand derivative and equality of both of them

$$L(f'(c)) = R(f'(c))$$

Left Hand derivative (L.D) | Right hand derivative (RD)

$\lim_{x \to c^-} \frac{f(x)-f(c)}{x-c} = \lim_{h \to 0} \frac{f(c-h)-f(c)}{-h}$ | $\lim_{x \to c^+} \frac{f(x)-f(c)}{x-c} = \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$

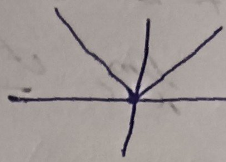
$F(x)$ is differentiable at $x=c$

$$L.D(f(x)) = RD(f(x))$$

* Every differentiable function is continuous but a continuous function need not be differentiable.

Ex 1 $f(x) = |x|$, Check differentiability at $c=0$.

$f(x) = |x|$ Continuous है @ $x=0$
but differentiable नहीं है

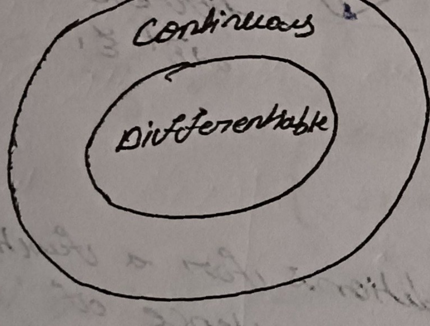


$$LD = \lim_{h \to 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \to 0} \frac{|-h| - 0}{-h} = \frac{h}{-h} = -1$$

$$RD = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h} = 1$$

$$L.D \neq R.D$$

Continuity -	No jumps, No cuts
Differentiability -	No jumps, No cuts, Smooth



Q Let $f(x) = x|x|$, $x \in R$ then $f(x)$ at $x=0$ is ___

a) continuous, differentiable ✓
b) continuous, not differentiable
c) differentiable but not continuous
d) Neither differentiable nor continuous

Sol.

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x = 0 \\ -x^2 & x < 0 \end{cases}$$

$f(0^-) = f(0^+) = f(0) \Rightarrow$ continuous

Q $f(x) = \dfrac{1}{1+|x|}$ Check differentiability at $x=0$

Sol. $f(x) = \begin{cases} \dfrac{1}{1+x} & x>0 \\ 1 & x=0 \\ \dfrac{1}{1-x} & x<0 \end{cases}$

$\cancel{LD = \lim_{h\to 0} \frac{h}{1+(0-h)}}$

$LD = \lim_{h\to 0} \dfrac{\dfrac{1}{1-(0-h)} - 1}{0-h} = \lim_{h\to 0} \dfrac{\dfrac{1}{1+h} - 1}{-h}$

$= \dfrac{-h}{-h(1+h)} = 1$

$RD = \lim_{h\to 0} \dfrac{\dfrac{1}{1+(0+h)} - 1}{0+h} = \lim_{h\to 0} \dfrac{\dfrac{1}{1+h} - 1}{h}$

$= \lim_{h\to 0} \dfrac{-h}{h(1+h)}$

$= -1$

$LD \neq RD$

$f(x)$ is not differentiable at $x=0$

Q Let $f(x) = |x-1| + |x+1|$ Discuss the continuity & differentiability of the function

Sol $f(x) = \begin{cases} -(x-1) - (x+1) & x<-1 \\ -(x-1) + x+1 & -1 \le x \le 1 \\ x-1 + x+1 & x>1 \end{cases}$

$f(x) = \begin{cases} -2x & x<-1 \\ 2 & -1 \le x \le 1 \\ 2x & x>1 \end{cases}$

continuous $\bar{c}$ $f(x)$ $x=-1$ & $x=1$

$$f'(x) = \begin{cases} -2 & x < -1 \\ 0 & -1 \le x \le 1 \\ 2 & x > 1 \end{cases}$$

⇒ Clearly दिख रहा है differentiable नहीं है. at $x=-1$ & $x=1$

* $|x-a|$ is not differentiable at $x=a$

$|ax+b|$ is not differentiable at $x=-b/a$

Q $f(x) = [\sin x]$ → G.I.F when $x \in [0, 2\pi]$ which of the following is true

a) $f(x)$ is discontinuous at $x=\pi/2$

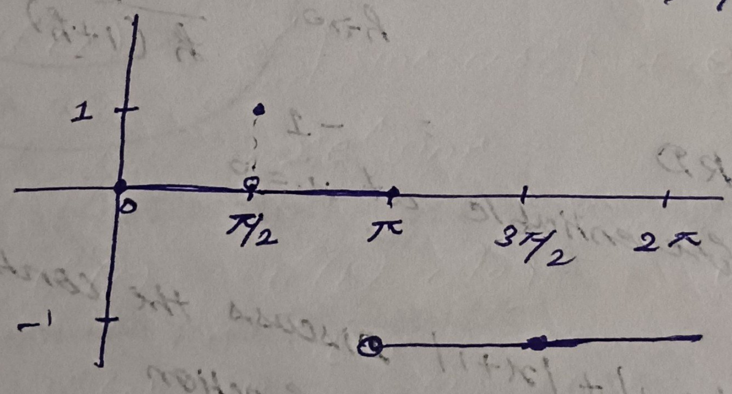
b) $f(x)$ is discontinuous at $x=\pi$

c) $f(x)$ is continuous at $x=3\pi/2$

d) ✓ All of the above

∴ All of the above

sol.



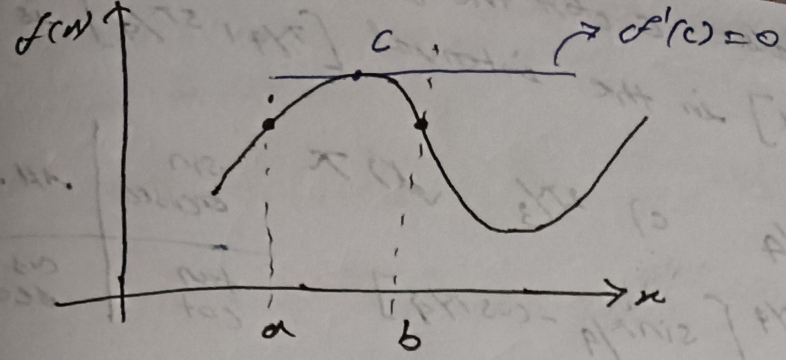
Mean Value Theorems

1) Rolle's Theorem :- Let $f(x)$ be defined in $[a,b]$ such that it satisfies 3 conditions

i) $f(x)$ is continuous $[a,b]$

ii) $f(x)$ is differentiable (a,b)

iii) $f(a) = f(b)$ then $\exists$ atleast one point $c \in (a,b)$ such that $f'(c) = 0$



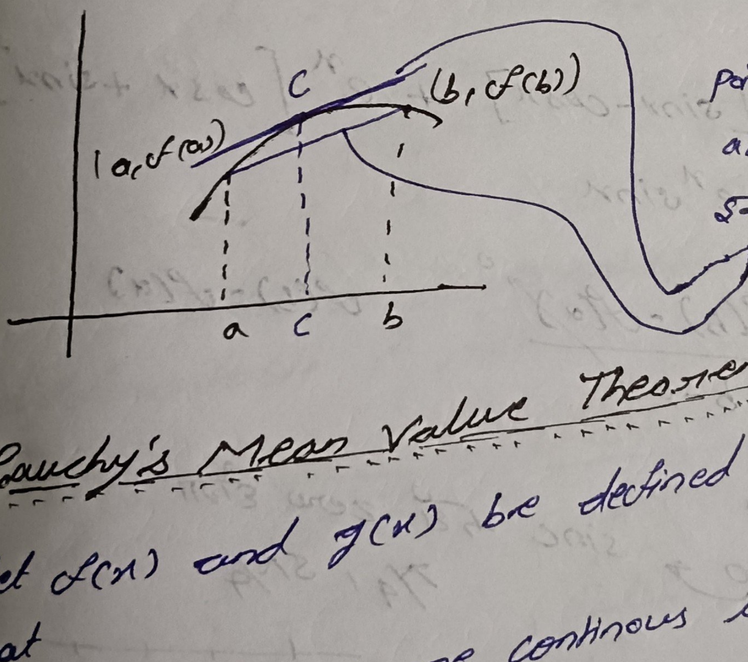
Lagranges Mean Value Theorem

Let $f(x)$ be defined in $[a,b]$ such that it satisfies 2 conditions

i) $f(x)$ is continuous in $[a,b]$

ii) $f(x)$ is differentiable in (a,b) then $\exists$ atleast one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



point c का tangent a& b के को connect करेंगे इन दोनो का slope equal होगा।

Cauchy's Mean Value Theorem

Let $f(x)$ and $g(x)$ be defined in $[a,b]$ such that

i) $f(x)$ and $g(x)$ are continous in $[a,b]$

ii) $f(x)$ and $g(x)$ are differentiable in (a,b)

iii) $g'(x) \neq 0 \ \forall \ x \in (a,b)$ then $\exists$ atleast one Point $c \in (a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Q. The mean value c for the function $e^x[\sin x - \cos x]$ in the interval $[\pi/4, 5\pi/4]$ is -

a) 0 b) $3\pi/4$ c) $2\pi/3$ d) π ✓

sin cosec	All +ve
tan cot	cos sec

$f(\pi/4) = e^{\pi/4}[\sin \pi/4 - \cos \pi/4]$

$= e^{\pi/4}[1/\sqrt{2} - 1/\sqrt{2}] = 0$

$f(5\pi/4) = e^{5\pi/4}[\sin(5\pi/4) - \cos 5\pi/4]$

$-1/\sqrt{2} - (-1/\sqrt{2})$

$= 0$

$f(a) = f(b)$ so Rolles theorem लगा सकते है

continous है $[\pi/4, 5\pi/4]$ पर & differentiable

है $(\pi/4, 5\pi/4)$

$f'(x) = e^x[\sin x - \cos x] + e^x[\cos x + \sin x]$

$= 2e^x \sin x$

$f'(c) = \dfrac{f(b) - f(a)}{b-a} \to 0$ $f(b) = f(a)$

$f'(c) = 0$

$2e^c \sin c = 0$ sinc कहाँ zero होगा

$\sin c = 0$ $\pi/4, 5\pi/4$

$\therefore c = \pi$

$\pi/4 \quad \pi/2 \quad \pi \quad 5\pi/4$

Q. Find mean value c for the function

$f(x) = x(x+3)e^{-x/2}$ on $[-3, 0]$

$f(-3) = 0$ $f(0) = 0$

continuity & differentiability की tension मत लो
differentiate कर पा रहे हो तो कर दो मतलब
differentiable है

$$f'(x) = (x+3)e^{-x/2} + xe^{-x/2} - \frac{1}{2}x(x+3)e^{-x/2}$$

$$= e^{-x/2}\left(x+3+x-\frac{x^2}{2}-\frac{3x}{2}\right)$$

$$= e^{-x/2}\left(3+\frac{x}{2}-\frac{x^2}{2}\right)$$

$$f'(c) = 0$$

$$e^{-c/2}\left(3+\frac{c}{2}-\frac{c^2}{2}\right) = 0 \Rightarrow 6+c-c^2=0$$

$$c^2-c-6=0$$

$$c^2-3c+2c-6=0$$

$$(c-3)(c+2)=0$$

$$c = 3, -2$$

$\therefore$ $c = -2$

Q The mean value c for the function $f(x) = \sqrt{x^2-4}$ is $[2,4]$

a) $\sqrt{6}$ b) $2\sqrt{2}$ c) $8/3$ d) None

sol. $f'(x) = \dfrac{2x}{2\sqrt{x^2-4}} = \dfrac{x}{\sqrt{x^2-4}}$

finite $\forall\ x > 2$

$\therefore$ $f(x)$ is differentiable in $(2,4)$ थी हो जाएगा
differentiable हो गया तो continous

$$f'(c) = \frac{c}{\sqrt{c^2-4}} = \frac{f(b)-f(a)}{b-a}$$

$$\frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{12}}{\sqrt{4}} \quad \sqrt{3}$$

$$\frac{c^2}{c^2-4} = 3$$

$$c^2 = 3c^2 - 12$$

$$\Rightarrow 2c^2 = 12$$

$$c = \sqrt{6}$$

continuity & differentiability की tension मत लो differentiate कर पा रहे हो तो कर दो मतलब differentiable है

$$f'(x) = (x+3)e^{-x/2} + xe^{-x/2} - \frac{1}{2}x(x+3)e^{-x/2}$$

$$= e^{-x/2}\left(x+3+x-\frac{x^2}{2}-\frac{3x}{2}\right)$$

$$= e^{-x/2}\left(3+\frac{x}{2}-\frac{x^2}{2}\right)$$

$f'(c) = 0$

$e^{-c/2}\left(3+\frac{c}{2}-\frac{c^2}{2}\right) = 0 \Rightarrow 6+c-c^2=0$

$c^2-c-6=0$

$c^2-3c+2c-6=0$

$(c-3)(c+2)=0$

$c=3,-2$

$\therefore c=-2$

Q The mean value c for the function $f(x)=\sqrt{x^2-4}$ is $[2,4]$

a) $\sqrt{6}$ b) $2\sqrt{2}$ c) $8/3$ d) None

sol. $f'(x) = \frac{2x}{2\sqrt{x^2-4}} = \frac{x}{\sqrt{x^2-4}}$

finite $\forall\ x>2$

$\therefore f(x)$ is differentiable is $(2,4)$ जब हो जाएगा differentiable हो गया तो continous

$f'(c) = \frac{c}{\sqrt{c^2-4}} = \frac{f(b)-f(a)}{b-a}$

$\therefore \frac{c}{\sqrt{c^2-4}} = \frac{\sqrt{12}}{\sqrt{4}}\ \sqrt{3}$

$\therefore \frac{c^2}{c^2-4} = 3$

$\Rightarrow c^2 = 3c^2-12$

$\Rightarrow 2c^2 = 12$

$c = \sqrt{6}$

Q Find mean value c for the function
$f(x) = lx^2+mx+n$, $(l \neq 0)$ in $[a,b]$

Sol differentiate कर पा रहे हो तो differentiable है

$f'(x) = 2lx + m$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$2lc+m = \frac{lb^2+mb+\cancel{n}-la^2-ma-\cancel{n}}{(b-a)}$$

$$= \frac{l\cancel{(b-a)}(b+a)+m\cancel{(b-a)}}{\cancel{(b-a)}}$$

$$2lc+\cancel{m} = l(b+a)+\cancel{m}$$

$$2lc = lb+la$$

$$2c = b+a$$

$$c = \frac{a+b}{2}$$

Q The mean value c for the function $f(x) = \sin x$ and $g(x) = \cos x$ in $[-\pi/2, 0]$ is ____

question को पढ़ के लग रहा है cauchy लगाना है because $f(x)$, $g(x)$ involve है

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\frac{\cos c}{-\sin c} = \frac{0+1}{1-0}$$

$$\cos c = -\sin c$$

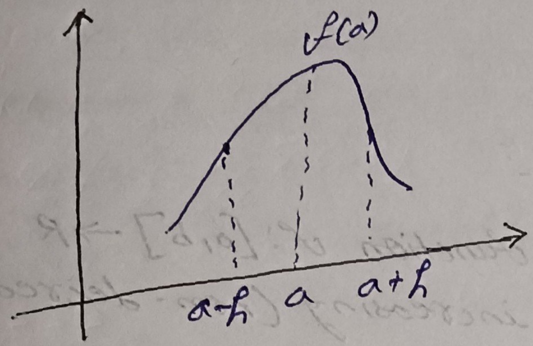
$$\tan c = -1 \qquad -\pi/4 \in [-\pi/2, 0]$$

$$c = -\pi/4$$

Maxima & Minima

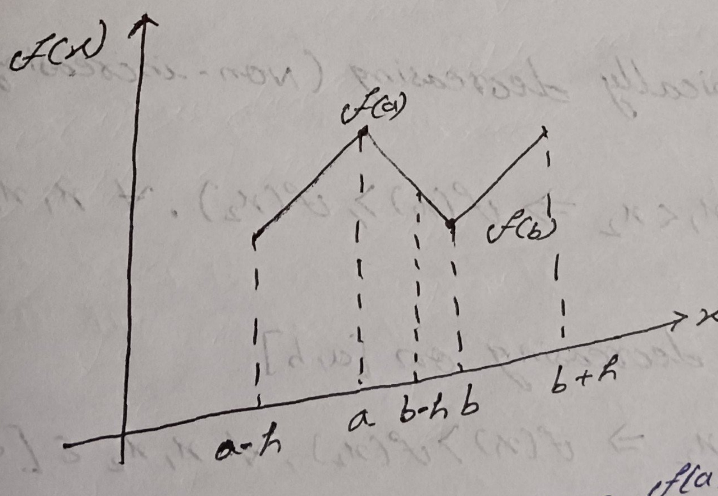
A function $f(x)$ is said to have maximum at $x=a$ If $f(a)$ greatest of all values in neighbourhood of a where a is an interior Point in the domain of $f(x)$

$$f(a) \geq f(a+h), \quad f(a) \geq f(a-h)$$



A function $y=f(x)$ is said to be minimum at $x=b$ if $f(b)$ smallest of all values in suitably small neighbourhood of b where b is an interior point in domain of $f(x)$.

$$f(b) \leq f(b+h), \quad f(b) \leq f(b-h)$$



Stationary Point :- If $f'(a)=0$ then $f(a)$ is stationary Value and the stationary point is $(a, f(a))$. $f(a)$ exist

ऐसी value जहाँ $f'(x)=0$ है $x=a$ then a को stationary point कहलाते हैं।

Critical point:- We say that $x=a$ is a critical point of the function $f(x)$, if $f(a)$ exists and either of the following are true

1.) $f'(a)=0$ → यह stationary point का condition है।

2.) $f'(a)$ does not exist

सारे stationary point को critical point बोल सकते हैं।

Monotonic function:-

A function $f:[a,b]\to R$ is said to be i) Monotonically increasing (non-decreasing) on $[a,b]$

if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)\ \forall\ x_1, x_2 \in [a,b]$

ii) strictly increasing on $[a,b]$

if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2),\ \forall\ x_1, x_2 \in [a,b]$

iii) Monotonically decreasing (Non-increasing) on $[a,b]$

if $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2),\ \forall\ x_1, x_2 \in [a,b]$

iv) strictly decreasing on $[a,b]$

If $x_1 < x_2 \Rightarrow f(x_1) > f(x_2),\ \forall\ x_1, x_2 \in [a,b]$

→ Greatest & Least values:-

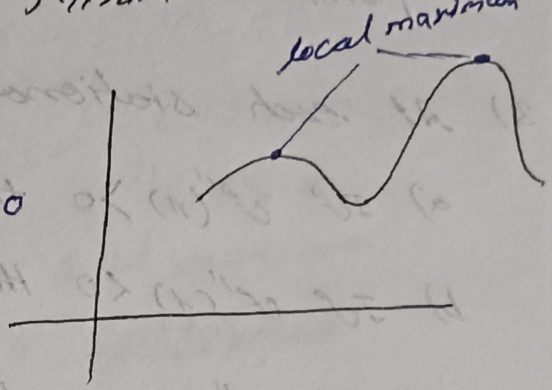
Let f be a function defined on set A and $l \in f(A)$, then l is said to be

i) The maximum value or greatest value or absolute max or global max of f in A if

$f(x) \leq l \;,\; \forall x \in A$

ii) The minimum value or least value or absolute minimum or global minimum of f in A if $f(x) \geq l \;\; \forall x \in A$.

→ Local Maximum & Local Minimum



Let f be a function.

Local Maximum: If $\exists a, \delta > 0$ such that $f(x) < f(a)$, $\forall x \in (a-\delta, a) \cup (a, a+\delta)$

Local Minimum:- If $\exists a, \delta > 0$ such that $f(x) > f(a)$ $\forall x \in (a-\delta, a) \cup (a, a+\delta)$

→ Global Maxima & Minima

Inorder to find the global maximum of $f(x)$ in $[a,b]$ find out all critical points of $f(x)$ in $[a,b]$ (i.e all points at which $f'(x) = 0$)

Let $c_1, c_2, \ldots c_n$ be the critical points at which $f'(x) = 0$ and

Let $f(c_1), f(c_2) \ldots f(c_n)$ be the values of the function at critical points then

Global maximum $= \max\{f(a), f(c_1), \ldots f(c_n), f(b)\}$

Global minimum $= \min\{f(a), f(c_1), \ldots f(c_n), f(b)\}$

Method to finding Maxima and Minima

1.) Find $f'(x)$

2.) Equal $f'(x)$ to zero for obtaining the stationary points

3.) At each stationary points find $f''(x)$

a) If $f''(x) > 0$ then function has min value

b) If $f''(x) < 0$ then function has max value

c) If $f''(x) = 0$ then we need to find

$f'''(x)$ at $x = x_0$

If $f'''(x_0) \neq 0$ then we get neither maximum nor minimum

But If $f'''(x_0) = 0$ then find $f^{IV}(x_0)$

If $f^{IV}(x_0) = +ve$, $f(x)$ is minimum at $x = x_0$

If $f^{IV}(x_0) = -ve$, $f(x)$ is maximum at $x = x_0$

and so on, process is repeated till point is decided.

observation है odd derivative उस point पर zero होगा तभी even derivative वाला पाएगा maximum या minimum.

Note:- It must be remembered that this method is not applicable to those critical points where $f'(x)$ remains undefined.

* Inflextion point :- neither maximum nor minimum मिले।

Q The function $f(x) = 3x^4 - 4x^3 + 10$ has minimum value at x?

Sol $f'(x) = 12x^3 - 12x^2$

$x^3 - x^2 = 0$

$x^2(x-1) = 0 \Rightarrow x = 0, x = 1,$

$f''(x) = 12 \cdot 3 \cdot x^2 - 12 \cdot 2x$

$= 12(3x^2 - 2x)$

$f''(0) = 0$

$f''(1) > 0$

1 is minima

$f'''(x) = 12(6x - 2)$

$f'''(0) \neq 0$

मतलब inflextion point

on

Q $f(x) = x(x-1)^2$, find the points at which $f(x)$ assumes maximum & minimum.

Sol.

$f'(x) = (x-1)^2 + 2x(x-1)$

$f'(x) = 0$

$(x-1)(\underbrace{x-1+2x}_{3x-1}) = 0 \Rightarrow x = 1, 1/3$

$f''(x) = 2(x-1) + 2(x-1) + 2x$

$= 6x - 4$

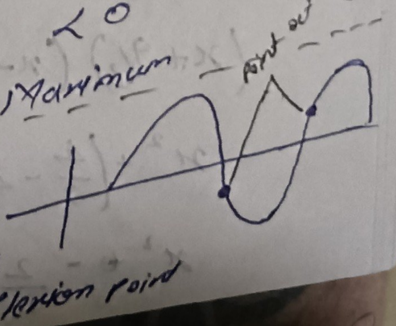
$f''(1/3) = 6 \times 1/3 - 4 = 2 - 4$

< 0

$f''(1) = 2 > 0$

Minimum

Maximum



Inflexion or inflection point

If function is changing it's curvature at that point then it is called inflexion point

* For the function $f(x) = x^2 e^{-x}$ the maximum value appears at $x = $?

Sol. $f(x) = x^2 e^{-x}$

$f'(x) = 2xe^{-x} - x^2 e^{-x}$

$= e^{-x}(2x - x^2)$

$f'(x) = 0$

$e^{-x}(2x - x^2) = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \,/\, x = 2$

$f''(x) = e^{-x}(2-2x) - e^{-x}(2x - x^2)$

$= e^{-x}\left[2 - 2x - 2x + x^2\right]$

$= e^{-x}\left[2 - 4x + x^2\right]$

$f''(0) = 1 \times [2 - 0 + 0] > 0$ Minimum

$f''(2) = e^{-2}[2 - 8 + 4] < 0$ Maximum.

* $\boxed{\dfrac{d(|x|)}{x} = \dfrac{|x|}{x}}$

* If $y = a\log|x| + bx^2 - x$ has extreme values at $x = -4/3,\ x = 2$ then, find a, b

$y' = \dfrac{a}{|x|}\dfrac{|x|}{x} + 2bx - 1$

$f'(x) = 0$

$a + 2bx^2 - x = 0$

$(x + 4/3)(x - 2) = 0$

$x^2 + \left(\frac{4}{3} - 2\right)x - 8/3 = 0$

$x^2 - \frac{2}{3}x - 8/3 = 0 \Rightarrow \frac{3}{2}x^2 - x - \frac{8}{2} = 0$

$a = -4$

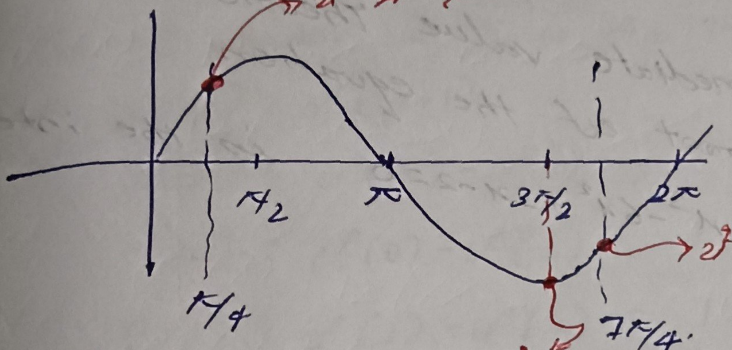
$2b = 3/2$

$b = 3/4$

Gate 2012

Q. Consider the function $f(x) = \sin x$ in the interval $[\pi/4, 7\pi/4]$. The number of location of the local minima of this function are.

Sol. Graph बना के कर लो



Q. Find local maxima & local minima of given function.

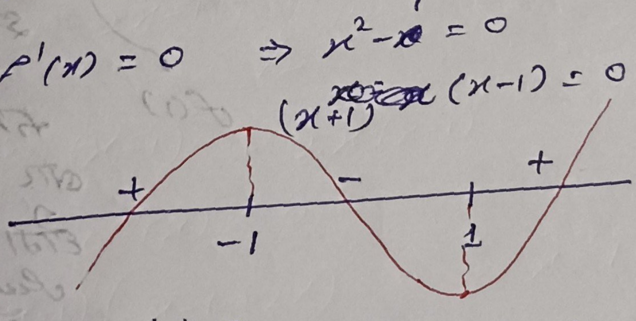
$$f(x) = x^3 - 3x + 3$$

$$f'(x) = 3x^2 - 3$$

$$f'(x) = 0 \Rightarrow x^2 - 1 = 0$$

$$(x+1)(x-1) = 0$$

Graph, $f'(x)=0$ का root a, b, c मान्या और a पर Graph का sign नहीं change हो रहा तो उसे point of inflection बोलते



−1 is local minima

1 is local maxima

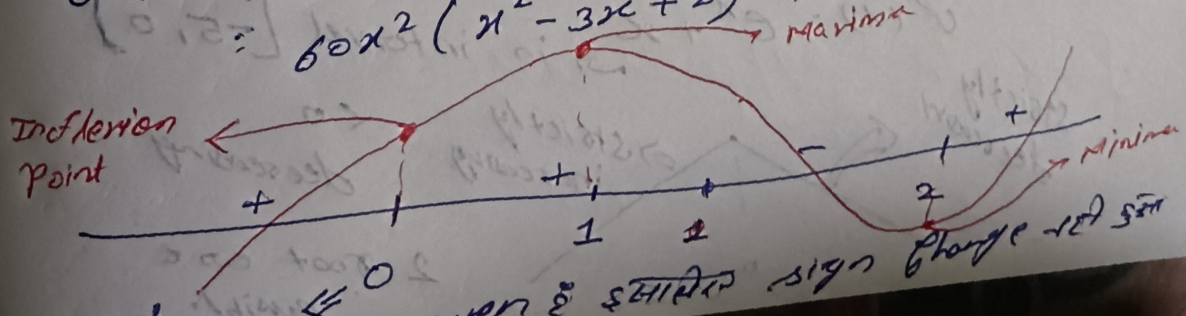
$f(-1) = -1 + 3 + 3$
$= 5$

$f(1) = 1 - 3 + 3$
$= 1$

Q. $f(x) = 12x^5 - 45x^4 + 40x^3 + 40$

$$f'(x) = 60x^4 - 180x^3 + 120x^2$$

$$= 60x^2(x^2 - 3x + 2) \Rightarrow 60x^2(x-1)(x-2) = 0.$$



x^2 की Power even है इसलिए sign change नहीं हुआ

→ Intermediate Value Theorem

If f is a continuous function on the closed interval $[a, b]$ and if d is between $f(a)$ and $f(b)$, then there is a number $c \in [a, b]$ with

$$f(c) = d$$

⊙ use the Intermediate value Theorem to show that there is root of the equation

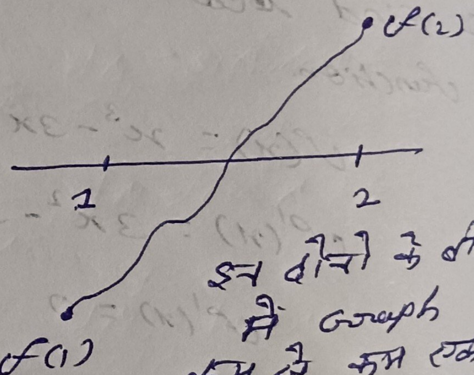
$$4x^3 - 6x^2 + 3x - 2 = 0$$ in the interval $[1, 2]$

sol.

$$f(1) = 4 - 6 + 3 - 2 = -2$$

$$f(2) = 4 \times 8 - 6 \times 4 + 3 \times 2 - 2$$
$$= 32 - 24 + 6 - 2$$
$$= 12$$

$f(1) < 0 < f(2)$



इन दोनों के बीच में Graph कम से कम एक बार तो cut किया होगा x-axis से function continuous है इसलिए।

⊙ The number of roots of $e^x + 0.5x^2 - 2 = 0$ in the range $[-5, 5]$ is.

sol

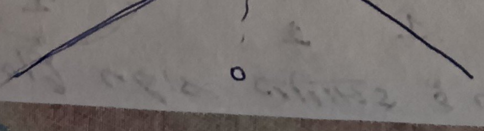
$$f(x) = e^x + 0.5x^2 - 2$$

$$f'(x) = e^x + x \rightarrow \text{in interval } [-5, 0]$$

strictly decreasing

→ strictly increasing

< 0 decreasing



2 root are possible.

~~Integration~~

Q. A function $f(x)$ is continuous in the interval $[0, 2]$. It is known that $f(0) = f(2) = -1$ and $f(1) = 1$ which of the following statement must be true?

a) There exist a y in the interval $(0,1)$ such that $f(y) = f(y+1)$

$$g(y) = f(y) - f(y+1)$$

$$g(0) = f(0) - f(1) = -1 - 1 = -2$$

$$g(1) = f(1) - f(2) = 1 - (-1) = 2$$

$\because$ $g(0)\, g(1) < 0 \Rightarrow$ This implies

$\exists\, y \in (0,1)$ such that $g(y) = 0$

$\therefore$ $f(y) - f(y+1) = 0$

$f(y) = f(y+1)$

ये important point था इस question में बाकी पूरा question हैंडल तो Gate का PYQ है।

Integration

The process of finding a function, given its derivative is called anti-differentiation (or) integration

⇒ If $F'(x) = f(x)$, we say $F(x)$ is an anti-derivative of $f(x)$

we write $\int f(x) = F(x) + c$

Formula

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$\int \cos x \, dx = \sin x + c$ $\qquad$ $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

$\int \sin x \, dx = -\cos x + c$ $\qquad$ $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

$\int \sec^2 x \, dx = \tan x + c$

$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$ $\qquad$ $\int \frac{-1}{\sqrt{1-x^2}} dx = \cos^{-1} x + c$

$\int \sec x \tan x = \sec x + c$ $\qquad$ $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$

$\int \frac{1}{x} dx = \ln x + c$ $\qquad$ $\int -\frac{1}{1+x^2} dx = \cot^{-1} x + c$

$\int e^x dx = e^x + c$ $\qquad$ $\int a^x dx = \frac{a^x}{\log_e a} + c$

$\int \cot x \, dx = \log|\sin x| + c$

$\int \tan x \, dx = -\log|\cos x| + c$
$= \log|\sec x| + c$

$\int \sec x \, dx = \log|(\sec x + \tan x)| + c$

$\int \operatorname{cosec} x \, dx = \log|\operatorname{cosec} x - \cot x| + c$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}(x/a) + c$$

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \cos^{-1}(x/a) + c$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$$

Example 1:- $\int \frac{x^3-1}{x^2}\,dx$

$$\int\left(\frac{x^3}{x^2} - \frac{1}{x^2}\right)dx = \int x\,dx - \int \frac{1}{x^2}\,dx$$

$$= \frac{x^2}{2} + c_1 - \int x^{-2}\,dx$$

$$= \frac{x^2}{2} + c_1 - \left[\frac{x^{-2+1}}{-2+1} + c_2\right]$$

$$= \frac{x^2}{2} + c_1 - \left[-\frac{1}{x} + c_2\right]$$

$$= \frac{x^2}{2} + \frac{1}{x} + \underbrace{c_1 - c_2}_{c}$$

$$= \frac{x^2}{2} + \frac{1}{x} + c$$

Q $\int \cos 2x\,dx$

$t = 2x$

$\frac{dt}{dx} = 2 \Rightarrow dt = 2dx$

$$\int \frac{\cos t\,dt}{2}$$

$$\frac{\sin t}{2} + c \Rightarrow \frac{\sin 2x}{2} + c$$

Q $\int \operatorname{cosec} x(\operatorname{cosec} x + \cot x)\,dx$

Sol $\int(\operatorname{cosec}^2 x + \operatorname{cosec} x \cot x)\,dx$

$$-\cot x - \operatorname{cosec} x + c$$

Methods of Integration

1) Integration by substitution

$\int f(x)$ can be transformed into another form by changing independent variable 'x' to 't' by substituting $x = g(t)$

Consider $\quad I = \int f(x)\,dx$

put $x = g(t)$

$$\frac{dx}{dt} = g'(t) \Rightarrow dx = g'(t)\,dt$$

$$\therefore I = \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

Q $\int \tan x\,dx$.

$$\int \frac{\sin x}{\cos x}\,dx$$

cos x का differentiation ऊपर है ∴ substitution method लगा दो

$t = \cos x$

$\frac{dt}{dx} = -\sin x$

$dt = -\sin x\,dx$

$$\int \frac{-dt}{t}$$

$\Rightarrow -\log|t| + c$

$\Rightarrow -\log|\cos x| + c$

$\Rightarrow \log|\sec x| + c$

$$\int \tan x\,dx = \log|\sec x| + c$$

$t = \cos x$

$\frac{dt}{dx} = -\sin x$

$dt = -\sin x\,dx$

Q $\int \sin^3 x \cos^2 x\,dx$

$\cancel{t = \sin x}$

$\cancel{\frac{dt}{dx} = \cos x}$

$\cancel{dt = \cos x}$

Sol. $\int (1-\cos^2 x)\cos^2 x \sin x\,dx$

$$\int -(1-t^2)\,t^2\,dt$$

$$\int (t^4 - t^2)\,dt = \frac{t^5}{5} - \frac{t^3}{3} + c$$

$$\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + c$$

2.) Integration by Parts:-

This method is used to integrate the product of two functions

if $f(x)$ and $g(x)$ be two integrable functions then

$$\int f(x) g(x)\,dx = f(x) \int g(x)\,dx - \int \left(f'(x) \int g(x)\,dx \right) dx$$

$f(x) \Rightarrow$ ILATE को choose करेंगे जैसे लिखा हुआ है वैसे ही order होगा।

Inverse Trignometric Function

Logarithmic Function

Algebraic Function

Trignometric Function

Exponential Function

Q $\int e^x x\,dx$

$$\int e^x x\,dx = x e^x - \int e^x\,dx$$

$$= x e^x - e^x \Rightarrow e^x (x-1) + C$$

Q $\int \frac{\log x}{x^2}$

$$-\frac{\log x}{x} + \int \frac{1}{x} \cdot \frac{1}{x}\,dx$$

$$-\frac{\log x}{x} + \int \frac{dx}{x^2} \Rightarrow -\frac{\log x}{x} - \frac{1}{x} + c$$

Q $\int x \cos x\,dx$

$$\Rightarrow x \sin x - \int \sin x\,dx$$

$$\Rightarrow x \sin x + \cos x + c$$

3) Integration by Partial fractions:-

when we have the ration of two polynomials in the form of $\frac{P(x)}{Q(x)}$, where $P(x)$, $Q(x)$ are polynomials in x and $Q(x) \neq 0$

Ex:- $\int \frac{1}{x(x+1)} dx$.

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$$

$$\frac{1}{x(x+1)} = 1/x - 1/x+1$$

$$\int \frac{1}{x} dx - \int \frac{1}{x+1} dx$$

$$\log|x| - \log|x+1| + C$$

o Find $\int \frac{x^2}{(x^2+1)(x^2+4)} dx$

put $x^2 = y$

$$\int \frac{y}{(y+1)(y+4)} \qquad \frac{y}{(y+1)(y+4)} = \frac{A}{(y+1)} + \frac{B}{(y+4)}$$

$$A = -1/3$$

$$B = \frac{-4}{-3} = 4/3$$

$$= \int -\frac{dx}{3(x^2+1)} + \int \frac{4\,dx}{3(x^2+4)}$$

$$= -\frac{1}{3}\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$$

$$= -1/3 \tan^{-1}x + \frac{1}{2}\cdot 4/3 \tan^{-1} x/2 + C$$

Definite Integrals

Let $f(x)$ be continuous function in $[a, b]$ and $F(x)$ be the anti-derivative of $f(x)$ then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Ex:- $\int_0^{\pi/2} \cos x\,dx = \sin x \Big]_0^{\pi/2} = \sin \pi/2 - \sin 0$

$= 1 - 0 = 1$

Properties of Definite Integrals

1.) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

2.) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad \forall c \in (a,b)$

3.) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

4.) $\int_a^b \frac{f(x)\,dx}{f(x) + f(a+b-x)} = \frac{b-a}{2}$

5.) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

6.) $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$

7.) $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$ if $f(a+x) = f(x)$

8.) $\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$ if $f(a-x) = f(x)$

9.) $\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx$

$$= \left[\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots\frac{2}{3} \text{ or } \frac{1}{2}\right]k$$

$$\text{where } k = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \pi/2 & \text{if } n \text{ is even} \end{cases}$$

10.) $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{[(m-1)(m-3)\cdots 2 \text{ (or) } 1][(n-1)(n-3)\cdots 2 \text{ or } 1]}{(m+n)(m+n-2)\cdots 2 \text{ or } 1}k$$

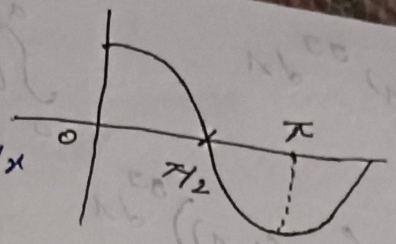
where $m, n \in Z^+$ and $k = \begin{cases} \pi/2 & \text{when } m,n \text{ are even} \\ 1 & \text{others} \end{cases}$

Q $\int_0^{\pi/2} \frac{\tan x}{\tan x + \cot x}\,dx$ → इसपे हम लोग $\int_a^b \frac{f(x)\,dx}{f(x)+f(a+b-x)} = \frac{b-a}{2}$ x^2 property लगा सकते हैं

$$\int_0^{\pi/2} \frac{\tan x}{\tan x + \tan(0+\pi/2-x)}$$

$$= \frac{\pi/2 - 0}{2} = \pi/4$$

$$\int_0^{\pi} |\cos x|\, dx$$

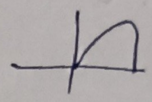


$$\int_0^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{\pi} |\cos x|\, dx$$

$$\int_0^{\pi/2} \cos x\, dx + \int_{\pi/2}^{\pi} -\cos x\, dx$$

$\cos(\pi - x) = -\cos x$

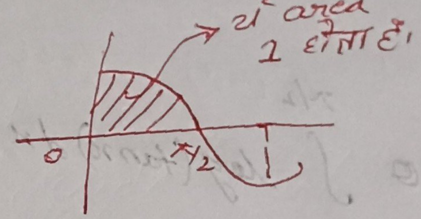
$$\sin x \Big]_0^{\pi/2} - \sin x \Big]_{\pi/2}^{\pi}$$



$$\left[\sin \tfrac{\pi}{2} - \sin 0\right] - \left[\sin \pi - \sin \tfrac{\pi}{2}\right]$$

$$-1\ \cancel{-2[1+0] = 0}$$

$$1 - [-1] = 2$$



$$\int_0^{n} [x]\, dx$$

$$\int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \cdots \int_{n-1}^{n} (n-1)\, dx$$

$$0 + 1[2-1] + 2[3-2] + \cdots (n-1)[n-(n-1)]$$

$$0 + 1 + 2 + \cdots (n-1)$$

$$1 + 2 + \cdots (n-1)$$

$$\frac{(n-1)(n-1+1)}{2} = \frac{(n-1)n}{2}$$

$$\int_0^1 x(1-x)^{99}\,dx \qquad \int_0^a f(x)\,d(x) = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 (1-x)\left(1-(1-x)\right)^{99} dx$$

$$\int_0^1 (1-x)\,x^{99}\,dx = \int_0^1 x^{99}\,dx - \int_0^1 x^{100}\,dx$$

$$= \left.\frac{x^{100}}{100}\right]_0^1 - \left.\frac{x^{101}}{101}\right]_0^1$$

$$= \frac{1}{100} - \frac{1}{101} = \frac{1}{10100}$$

$$\int_0^{\pi/2} \log(\tan x)\,dx$$

$$I = \int_0^{\pi/2} \log(\tan x)\,dx$$

$$I = \int_0^{\pi/2} \log\left(\tan(\pi/2 - x)\right)dx$$

$$I = \int_0^{\pi/2} \log(\cot x)\,dx$$

$$I + I = 2I = \int_0^{\pi/2} \left[\log(\tan x) + \log(\cot x)\right]dx$$

$$\log(\tan x \cdot \cot x)$$

$$\log(1) = 0$$

$$= 0$$

$$I = 0$$

Q $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$

$x = \tan t$

$\dfrac{dx}{dt} = \sec^2 t$

$x \to 0 \quad x \to 1$

$t \to 0 \quad t \to \pi/4$

$$\int_0^{\pi/4} \frac{\log(1+\tan t)}{(\cancel{1+\tan^2 t})}\,\cancel{\sec^2 t}\,dt$$

$$I = \int_0^{\pi/4} \log(1+\tan t)\,dt \quad \text{---}\ (1)$$

$$I = \int_0^{\pi/4} \log(1+\tan(\pi/4 - t))$$

$$I = \int_0^{\pi/4} \log\left(1 + \frac{1-\tan t}{1+\tan t}\right) dt$$

$$I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan t}\right) dt \quad \text{----}\ (2)$$

Equ^n 1 + equ^n 2

$$2I = \int_0^{\pi/4} \log(1+\tan t)\,dt + \log\left(\frac{2}{1+\tan t}\right) dt$$

$$2I = \int_0^{\pi/4} \log 2\,dt \Rightarrow 2I = \log 2\ \pi/4$$

$$I = \frac{\pi}{8}\log 2$$

Q If $I_1 = \displaystyle\int_0^{\pi} f(\cos^2 x)\,dx$

$$I_2 = \int_0^{3\pi} f(\cos^2 x)\,dx$$

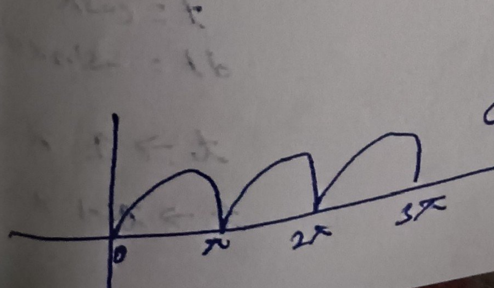
$$\int_0^{na} f(x)\,dx = n\int_0^{a} f(x)\,dx$$

if $f(a+x) = f(x)$

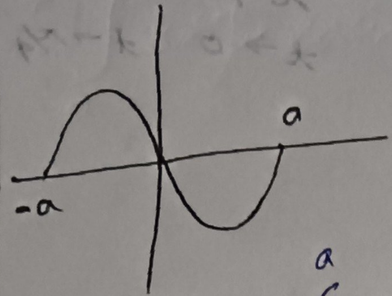
$$f(\cos^2 x) = f(\cos^2(\pi + x))$$

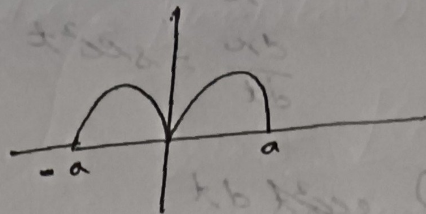
$$\therefore I_2 = 3\int_0^{\pi} f(\cos^2 x)\,dx$$

$$\boxed{I_2 = 3 I_1}$$



$\cos^2 x$ का graph repeat हो रहा





$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

Q $\displaystyle\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx$

Sol. $f(-x) = \cos(-x) \log\left(\frac{1-x}{1+x}\right)$

$= \cos x \log\left(\frac{1+x}{1-x}\right)^{-1}$

$= -\cos x \log\left(\frac{1+x}{1-x}\right)$

$= -f(x)$

$\therefore$ odd function

$\therefore \displaystyle\int_{-1/2}^{1/2} \cos x \log\left(\frac{1+x}{1-x}\right) dx = 0$

Q $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\,dx$

$\displaystyle\int_0^a x f(x)\,dx = \frac{a}{2}\int_0^a f(x)\,dx$

if $f(a-x) = f(x)$

$f(x) = \dfrac{\sin x}{1+\cos^2 x}$

~~f(-x)~~ $=$

$f(\pi - x) = \dfrac{\sin(\pi - x)}{1+\cos^2(\pi - x)} = \dfrac{\sin x}{1+\cos^2 x} = f(x)$

$I = \pi/2 \displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$

$t = \cos x$

$dt = -\sin x\,dx$

$t \to 1 \quad x \to 0$

$t \to -1 \quad x \to \pi$

$$I = \frac{\pi}{2}\int_{1}^{-1} -\frac{1}{1+t^2}\,dt$$

$$I = -\pi/2 \; \cdot \tan^{-1}t\Big]_{1}^{-1}$$

$$I = -\pi/2\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$= -\pi/2 \cdot \left[-\pi/4 - \pi/4\right]$$

$$= \pi/2 \cdot \pi/2 = \frac{\pi^2}{4}$$

$$* \int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \left[\frac{n-1}{n}\;\frac{n-3}{n-2}\;\frac{n-5}{n-4}\cdots\frac{2}{3}\,(\text{or})\,\frac{1}{2}\right] K$$

where $K = \begin{cases} \frac{\pi}{2} & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

either you reach to 2/3 or 1/2 skip

Q $$\int_0^{\pi/2} \sin^8 x\,dx = \left[\frac{7}{8}\cdot\frac{8-3}{8-2}\cdot\frac{8-5}{8-4}\cdot\frac{8-7}{8-6}\right]\frac{\pi}{2}$$

$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2$$

$$= \frac{35\pi}{256}$$

Q $$\int_0^{\pi/2} \sin^5 x\,dx = \left[\frac{4}{5}\cdot\frac{5-3}{5-2}\right]\cdot 1 = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$$

Q $$\int_0^{\pi/2} \cos^3 x\,dx = \frac{3-1}{3}\cdot 1 = \frac{2}{3}$$

Q $$\int_0^{\pi/2} \cos^6 x\,dx = \left[\frac{6-1}{6}\cdot\frac{6-3}{6-2}\cdot\frac{6-5}{6-4}\right]\pi/2$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\pi/2 = \frac{15\pi}{96}$$

* $$\int_0^{\pi/2} \sin^m x \cos^n x \, dx = \frac{[(m-1)(m-3)\cdots 2 \text{ or } 1]\cdot[(n-1)(n-3)\cdots 2 \text{ or } 1]}{(m+n)(m+n-2)\cdots 2 \text{ or } 1}\cdot K$$

where $K = \begin{cases} \pi/2 & \text{when } m, n \text{ are even} \\ 1 & \text{otherwise.} \end{cases}$

Q1 $$\int_0^{\pi/2} \sin^8 x \cos^3 x \, dx = \frac{[(8-1)(8-3)(8-5)(8-7)]\cdot[(3-1)]}{11\cdot(11-2)(11-4)(11-6)(11-8)(11-10)}$$

$$= \frac{(\cancel{7\cdot 5\cdot 3\cdot 1})\, 2}{11\cdot 9\cdot \cancel{7\cdot 5\cdot 3\cdot 1}} = \frac{2}{99}.$$

Q2 $$\int_0^{\pi} \sin^4 x \cdot \cos^3 x \, dx = \quad 0 \text{ for है}$$

1) $\pi/2$ में convert करना होगा

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

$$f(\pi - x) = \sin^4(\pi - x)\cos^3(\pi - x)$$
$$= -\sin^4 x \cos^3 x$$
$$= -f(x)$$

$\therefore \quad I = 0$

Q3 $\int_{-2\pi}^{2\pi} \sin^4 x \cos^8 x \, dx$

$f(-x) = \sin^4(-x)\cos^8(-x) = \sin^4 x \cos^8 x = f(x)$

even. $\therefore \int_0^a f(x)\,dx$

$I = \int_{-2\pi}^{2\pi} \sin^4 x \cos^8 x \, dx = 2\int_0^{2\pi} \sin^4 x \cos^8 x \, dx$

$f(\cancel{2}\pi + x) = \sin^4(\pi+x)\cos^8(\pi+x) = \sin^4 x \cos^8 x$

$\therefore$

$I = 2 \cdot 2 \int_0^{\pi} \sin^4 x \cos^8 x \, dx$

Here also

$f(\pi - x) = f(x)$.

$I = \cancel{2} \int_0^{\pi/2} \sin^4 x \cos^8 x \, dx$

$\cancel{8}\left[\frac{(\cancel{3}\cdot 1)(7\cdot 5\cdot \cancel{3}\cdot 1)}{\underset{4}{\cancel{12}}\cdot 10\cdot \cancel{8}\cdot \underset{2}{\cancel{6}}\cdot 4\cdot 2} \right] \times \pi/2$

$= \frac{735\pi}{\underset{256}{\cancel{1280}}} = \frac{7\pi}{256}$

<u>GATE 2009</u> $\int_0^{\pi/4} \frac{1-\tan x}{1+\tan x} = \int_0^{\pi/4} \frac{\tan \pi/4 - \tan x}{\tan \pi/4 + \tan x}$

$= \int_0^{\pi/4} \tan(\pi/4 - x)\,dx$

$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$= \int_0^{\pi/4} \tan(\pi/4 - (\pi/4 - x))\,dx$

$= \int_0^{\pi/4} \tan(x)\,dx$

$$\int_0^{\pi/4} \tan x \, dx = \log|\sec x| \Big]_0^{\pi/4}$$

$$= \log\sqrt{2} - \log 1$$

$$= \log\sqrt{2} = \frac{1}{2}\log 2$$

Q $\displaystyle\int_0^{\pi/2} \frac{\cos x + i\sin x}{\cos x - i\sin x} dx$, given $i = \sqrt{-1}$, what will be the evaluation of the integral.

Sol. $\displaystyle\int_0^{\pi/2} \frac{e^{ix}}{e^{-ix}} dx$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

$$\int_0^{\pi/2} e^{2ix} dx = \frac{1}{2i} e^{2ix}\Big]_0^{\pi/2}$$

$$= \frac{1}{2i}\left[e^{2i\pi/2} - e^{2i\cdot 0}\right]$$

$$= \frac{1}{2i}\left(e^{i\pi} - e^{0}\right)$$

$$= \frac{1}{2i}\left[\cos\pi + i\sin\pi - \cos 0\right]$$

$$= \frac{1}{2i}\left[-1-1\right] = -\frac{1}{i} = i$$

GATE 2015

Q $\displaystyle\int_{1/\pi}^{2/\pi} \frac{\cos(1/x)}{x^2} dx$ $\qquad$ Let $1/x = t$

$\qquad dt/dx = -1/x^2$

$\qquad x \to 1/\pi \quad x \to 2/\pi$

$\qquad t \to \pi \quad t \to \pi/2$

$$\int_{\pi}^{\pi/2} -\cos t \, dt$$

we know $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$$\int_{\pi/2}^{\pi} \cos t\,dt = \sin t\Big]_{\pi/2}^{\pi} = \sin\pi - \sin\pi/2$$

$$= 0 - 1$$

$$= -1$$