Combinations

CS231
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Permutations vs. Combinations

- Both are ways to count the possibilities
- The difference between them is whether order matters or not
- Consider a poker hand:
  - A♦, 5♥, 7♣, 10♠, K♠
- Is that the same hand as:
  - K♠, 10♠, 7♣, 5♥, A♦
- Does the order the cards are handed out matter?
  - If yes, then we are dealing with permutations
  - If no, then we are dealing with combinations

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Permutations

- An *r*-permutation is an ordered arrangement of *r* elements of the set
  - A♦, 5♥, 7♣, 10♠, K♠ is a 5-permutation of the set of cards
- The notation for the number of *r*-permutations: *P*(*n*,*r*)

$$P(n,r) = \frac{n!}{(n-r)!}$$

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Combinations

- What if order *doesn't* matter?
- In poker, the following two hands are equivalent:
  - A♦, 5♥, 7♣, 10♠, K♠
  - K♠, 10♠, 7♣, 5♥, A♦
- The number of *r*-combinations of a set with *n* elements, where *n* is non-negative and 0 ≤ *r* ≤ *n* is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

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Combinations example

- How many different poker hands are there (5 cards)?

$$C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52\times 51\times 50\times 49\times 48}{5\times 4\times 3\times 2\times 1} = 2{,}598{,}960$$

- How many different (initial) blackjack hands are there?

$$C(52,2) = \frac{52!}{2!(52-2)!} = \frac{52!}{2!50!} = \frac{52\times 51}{2\times 1} = 1{,}326$$

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Combination formula proof

- Let *C*(52,5) be the number of ways to generate unordered poker hands
- The number of ordered poker hands is *P*(52,5) = 311,875,200
- The number of ways to order a single poker hand is *P*(5,5) = 5! = 120
- The total number of unordered poker hands is the total number of ordered hands divided by the number of ways to order each hand
- Thus, *C*(52,5) = *P*(52,5)/*P*(5,5)

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Combination formula proof

- Let $C(n,r)$ be the number of ways to generate unordered combinations
- The number of ordered combinations (i.e. r-permutations) is $P(n,r)$
- The number of ways to order a single one of those r-permutations $P(r,r)$
- The total number of unordered combinations is the total number of ordered combinations (i.e. r-permutations) divided by the number of ways to order each combination
- Thus, $C(n,r) = P(n,r)/P(r,r)$

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Combination Formula

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

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Bit Strings

- How many bit strings of length 10 contain:
- Exactly four 1's?
  - Find the positions of the four 1's
  - Does the order of these positions matter?
  - Thus, the answer is $C(10,4) = 210$
- At most four 1's?
  - There can be 0, 1, 2, 3, or 4 occurrences of 1
  - $C(10,0)+C(10,1)+C(10,2)+C(10,3)+C(10,4)$
    = 1+10+45+120+210
    = 386

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Bit Strings

- How many bit strings of length 10 contain:
- At least four 1's?
  - There can be 4, 5, 6, 7, 8, 9, or 10 occurrences of 1
  - $C(10,4)+C(10,5)+C(10,6)+C(10,7)+C(10,8)+C(10,9)+C(10,10)$
    = 210+252+210+120+45+10+1
    = 848
  - Alternative answer: subtract from 2^{10} the number of strings with 0, 1, 2, or 3 occurrences of 1
- An equal number of 1's and 0's?
  - Thus, there must be five 0's and five 1's
  - Find the positions of the five 1's
  - Thus, the answer is $C(10,5) = 252$

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Corollary 1

- Let n and r be non-negative integers with $r \le n$. Then $C(n,r) = C(n,n-r)$
- Proof:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!} = \frac{n!}{r!(n-r)!}$$

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Corollary example

- There are C(52,5) ways to pick a 5-card poker hand
- There are C(52,47) ways to pick a 47-card hand
- P(52,5) = 2,598,960 = P(52,47)
- When dealing 47 cards, you are picking 5 cards to not deal
  - As opposed to picking 5 card to deal
  - Again, the order the cards are dealt in does matter

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Note

- An alternative (and more common) way to denote an *r*-combination:

$$C(n,r)=\binom{n}{r}$$

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Choosing Teams

- Choosing team of 5 among 12
- Two members must work as a pair
  - # of teams that contain both: C(10, 3) = 120
  - # of teams that don't: C(10, 5) = 252
  - addition rule
- Two members must be kept apart
  - # of teams that have either: 2xC(10, 4) = 420
  - # of teams that don't: C(10, 5) = 252

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Choosing Teams

- We have 5 men and 7 women
- How many 5-person groups can be chosen that
  - consist of 3 men and 2 women?
    - C(5, 3) x C(7, 2) = 210
  - have at least one man?
    - C(12, 5) – C(7, 5) = 771
  - at most one man?
    - C(7, 5) + C(5, 1) x C(7, 4) = 196

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r-Combinations with Repetitions

- How many *2*-combinations can be selected from {1, 2, 3}, if repetitions are allowed?
  - {1, 1}, {1, 2}, {1, 3}, {2, 2}, {2, 3}, {3, 3}

1	2	3	selection	string
XX			{1,1}	xx||
X	X		{1,2}	x|x|
X		X	{1,3}	x||x
	XX		{2,2}	|xx|
	X	X	{2,3}	|x|x
		XX	{3,3}	||xx

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r-Combinations with Repetitions

- xx||, x|x|, x||x, |xx|, |x|x, ||xx
- Strings of 4 symbols with 2 x's and 2 |'s
- Notice that once the positions of the x's are fixed, the |'s just go between
- C(4, 2) = 4x3/2 = 6

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r-Combinations with Repetitions

- The number of *r*-Combinations with repetition allowed that can be selected from a set of *n* elements is: C(*r*+*n*-1, *r*)

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Soda Distribution

- Select 15 cans of soft drinks from 5 types
  - How many different selections?
    - C(5+15-1, 15) = C(19, 15) = 3,876
  - If Diet Coke is one of the types, how many selections include at least 6 cans Diet Coke?
    - choose the DCs first, then the rest
    - C(5+9-1, 9) = C(13, 9) = 715
  - If the store only has 5 cans of DC, but at least 15 cans of all others, how many selections?

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Ways to Count

- Choosing *k* elements from *n*

	order matters	order doesn't matter
Repetition allowed	n^k	C(k+n-1, k)
No repetition	P(n, k)	C(n, k)

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Circular seatings

- How many ways are there to seat 6 people around a circular table, where seatings are considered to be the same if they can be obtained from each other by rotating the table?

- First, place the first person in the north-most chair
  - Only one possibility
- Then place the other 5 people
  - There are P(5,5) = 5! = 120 ways to do that
- By the product rule, we get 1*120 =120

- Alternative means to answer this:
- There are P(6,6)=720 ways to seat the 6 people around the table
- For each seating, there are 6 "rotations" of the seating
- Thus, the final answer is 720/6 = 120

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Horse races

- How many ways are there for 4 horses to finish if ties are allowed?
  - Note that order does matter!
- Solution by cases
  - No ties
    - The number of permutations is P(4,4) = 4! = 24
  - Two horses tie
    - There are *C*(4,2) = 6 ways to choose the two horses that tie
    - There are *P*(3,3) = 6 ways for the "groups" to finish
      - A "group" is either a single horse or the two tying horses
    - By the product rule, there are 6*6 = 36 possibilities for this case
  - Two groups of two horses tie
    - There are *C*(4,2) = 6 ways to choose the two winning horses
    - The other two horses tie for second place
  - Three horses tie with each other
    - There are *C*(4,3) = 4 ways to choose the three horses that tie
    - There are *P*(2,2) = 2 ways for the "groups" to finish
    - By the product rule, there are 4*2 = 8 possibilities for this case
  - All four horses tie
    - There is only one combination for this
  - By the sum rule, the total is 24+36+6+8+1 = 75

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Counting Triples

- How many (i, j, k) such that $1 \le i \le j \le k \le n$?
- If n=5, represent (3, 3, 4) as ||xx|x|
- If n=7, represent (2, 4, 5) as |x||x|x||
- How many |'s?
- How many x's?
- C(3+n-1, 3) = (n+2)!/3!x(n-1)!
  = (n+2)(n+1)n/6

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Nested `for` loop

```
for (k := 1 to n)
  for (j := 1 to k)
    for (i := 1 to j)
      //body
    next i
  next j
next k
```

- How many times will the innermost loop body be executed?
- For each iteration, there is a different combination of the indices (i, j, k), $1 \le i \le j \le k \le n$

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