Graphs and Eigenvalue Distributions

Kelly Romer

Tulane University

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Outline

Finite Models for Arithmetical Quantum Chaos

Motivation

In quantum mechanics, some energy systems are so complex that it becomes impossible to calculate all their energy levels. Physicists have discovered that the spectra of eigenvalues of various matrices, when normalized between 0 and 1, may model the behaviour of such energy systems (atomic spectra, for example).

- We will consider eigenvalues derived from adjacency matrices of special graphs, from random matrices, etc.

Research Overview

While Terras' paper goes into great depth on the analysis and conjectures of his research, the main focus of this presentation today is to highlight my research inspired by this paper.

Methodology:

1. Generate eigenvalues
    - Adjacency matrices of special graphs
    - Random matrices from probability distributions
    - And more!
2. Analyze eigenvalues
3. Make observations

Tools:

- Code written in Python and SageMath, displayed in Jupyter Notebook

Example: Eigenvalues from n-gons

- Generate n-gon

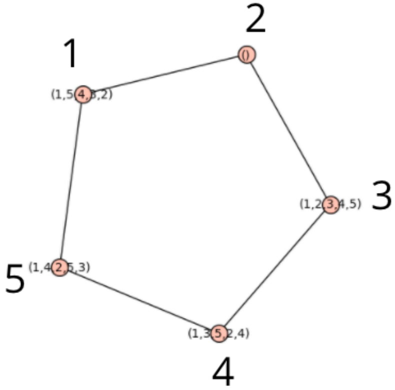
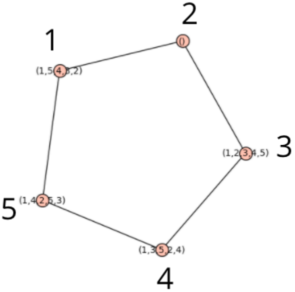


Figure: 5-gon

Example: Eigenvalues from n-gons

- Create adjacency matrix
  - From the graph of size n, create an nxn matrix
  - Each row is a vertex from 1 to n
  - Each column is a vertex from 1 to n
  - Insert a 1 for every edge between 2 vertices and a zero otherwise



$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Example: Eigenvalues from n-gons

- Obtain eigenvalues from the adjacency matrix
- Sort eigenvalues from least to greatest, $\lambda_1 \dots \lambda_n$
- Scale the eigenvalues to be between 0 and 1. This is important for comparing the eigenvalues between matrices; it makes it easier to find patterns

Normalization Equation

$$\forall i = 1, 2, \dots, n, \qquad \lambda_i = \frac{\lambda_i - \lambda_1}{\lambda_n - \lambda_1}$$

Example: Eigenvalues from n-gons

- Analyze
  - Create histograms of normalized eigenvalues
  - Look at statistics
  - Look for patterns and significance

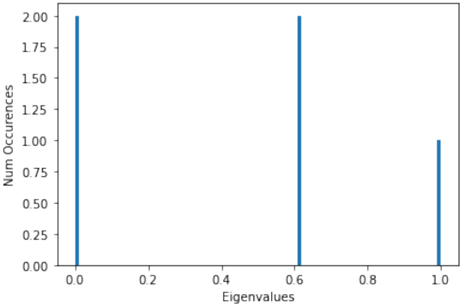


Figure: Histogram of normalized eigenvalues for adjacency matrix of 5-gon

Statistics

- **Average:** Indicates left/right skew of eigenvalues
- **Standard Deviation:** Indicates distance of eigenvalues from the average

$$\frac{\sqrt{\sum_{i=1}^{n-1}\left(\lambda_i - \bar{\lambda}\right)^2}}{n}$$

- For **centralization**, replace $\bar{\lambda}$ with $\frac{1}{2}$. This detects distance from the center, rather than distance from the average.

Statistics

- Detect large spacings between successive lines

$$\frac{\sqrt{\sum_{i=1}^{n-1}\left(\lambda_i-\lambda_{i+1}\right)^2}}{n}$$

- Detect large spacings between all lines

$$\frac{\sqrt{\sum_{1<i<j<n}\left(\lambda_i-\lambda_j\right)^2}}{\binom{n}{2}}$$

Observations

Symmetry in n-gon eigenvalues

- The average is always centered around 0.5
- Visually, a symmetry should definitely be noted

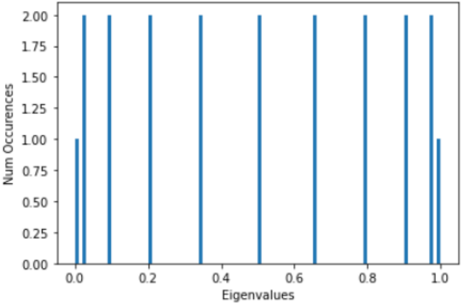


Figure: Histogram of normalized eigenvalues from adjacency matrix of 20-gon

Observations

- Eigenvalues grow closer together at extrema, gaps decrease as n increases
  - Centralization = 0.08 for n = 20 versus .05 for n = 50

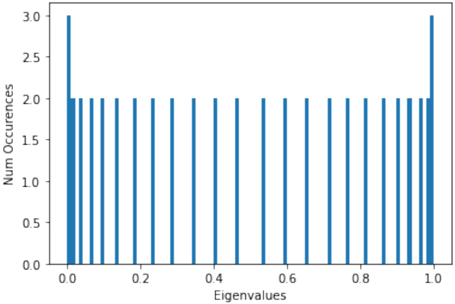


Figure: Histogram of normalized eigenvalues from adjacency matrix of 50-gon

Background Knowledge: Binomial Distribution

To produce the random adjacency matrices in the following slides, I used **binomial distribution**.

- This is because adjacency matrices are composed of *zeros and ones*. Binomial distribution chooses from either 0 or 1.
- The **success rate** p dictates the likelihood of the number 1 being chosen, i.e. *success*

Matrices that are randomly generated, such as these random adjacency matrices, are made symmetric to only yield real eigenvalues

Observations

A random adjacency matrix, chosen with binomial distribution

- As the success rate increases, eigenvalue spread decreases
- There is always 1 eigenvalue that is remarkably larger than the rest

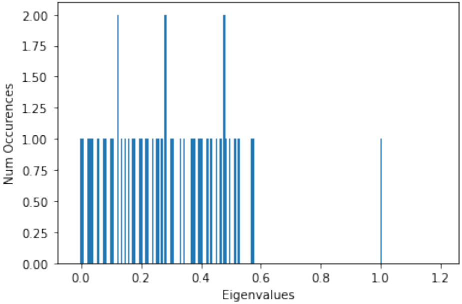


Figure: Histogram of normalized eigenvalues for random adjacency matrix, $p = 0.25$

Observations

A random adjacency matrix, chosen with binomial distribution

- Notice how the second-largest scaled eigenvalue is now around 0.28, rather than 0.59
- Furthermore, when $p = 0.75$, the histogram looks much like that of a 50x50 real symmetric matrix chosen with uniform distribution

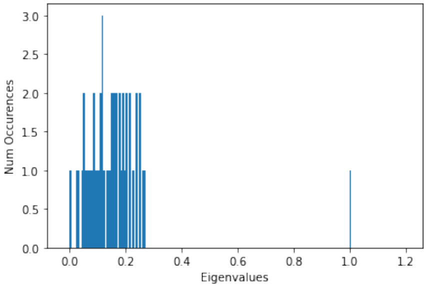


Figure: Histogram of normalized eigenvalues for random adjacency matrix, $p = 0.75$

Observations

Uniform Distribution chooses a number *in the continuous range* [0,1]. The probability of any given number being chosen is equal. These matrices are similar to adjacency matrices, but their values are between 0 and 1 rather than either 0 or 1

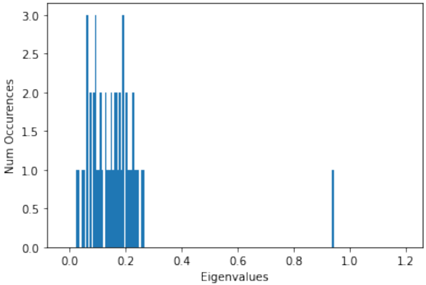


Figure: Histogram of normalized eigenvalues for 50x50 symmetric matrix chosen with uniform distribution

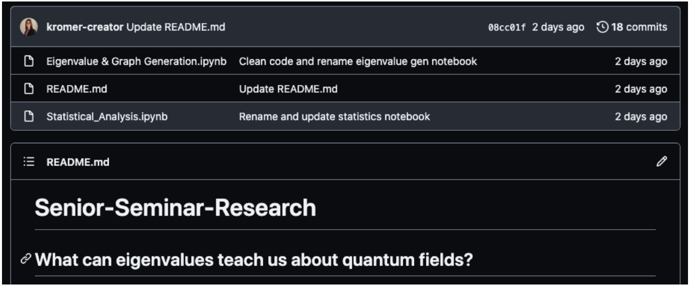
What's Next

The next step in this project is to dig into analysis, and refer back to the Terras' paper *Finite Models for Arithmetical Quantum Chaos*. I will look at the observations Terras has made, and compare it with my data to see if:

- The data I produced reinforces Terras' findings
- I can add onto any of Terras' findings
- I am inspired to do further research

Next presentation, you will see more observations and conclusions tied back to Terras' paper

Thank you! –Resources



- All of my code, as well as a link to my Google drive with data can be viewed on my Senior Seminar Github repository: `https://github.com/kromer-creator/Senior-Seminar-Research`
- The paper *Finite Models for Arithmetical Quantum Chaos* by Audrey Terras can be accessed here: `https://mathweb.ucsd.edu/~aterras/newchaos.pdf`