BOUNDED-RANK TENSORS ARE DEFINED IN BOUNDED DEGREE

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ABSTRACT. Matrices of rank at most k are defined by the vanishing of polynomials of degree $k+1$ in their entries (namely, their $(k+1)\times(k+1)$-subdeterminants), regardless of the size of the matrix. We prove a qualitative analogue of this statement for tensors of arbitrary dimension, where matrices correspond to two-dimensional tensors. More specifically, we prove that for each k there exists an upper bound $d = d(k)$ such that tensors of *border rank* at most k are defined by the vanishing of polynomials of degree at most d, regardless of the dimension of the tensor and regardless of its sizes in each dimension. Our proof involves passing to an infinite-dimensional limit of tensor powers of a vector space, whose elements we dub *infinite-dimensional tensors*, and exploiting the symmetries of this limit in crucial way.

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1. INTRODUCTION

Tensor rank is an important notion in such diverse areas as multilinear algebra and algebraic geometry [10, 11, 25]; algebraic statistics [1, 17], in particular with applications to phylogenetics [2, 9, 8, 30], and complexity theory [6, 7, 14, 21, 22, 23, 28, 29]. We will prove a fundamental, qualitative property of tensor rank that we believe could have substantial impact on theoretical questions in these areas. On the other hand, a future computational counterpart to our theory may also lead to practical use of the aforementioned property.

To set the stage, let $p, n_1, \ldots, n_p$ be natural numbers and let K be an infinite field. An $n_1 \times \ldots \times n_p$-tensor ω (or p-tensor, for short) over K is a p-dimensional array of scalars in K. It is said to be *pure* if there exist vectors $x_1 \in K^{n_1}, \ldots, x_p \in$

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K^{n_p} such that $\omega(i_1, \ldots, i_p) = x_1(i_1) \cdots x_p(i_p)$ for all relevant indices. The *rank* of a general tensor ω is the minimal number of terms in any expression of ω as a sum of pure tensors. When p equals 2, this notion of rank coincides with that of matrix rank. As a consequence, linear algebra provides an alternative characterisation of rank for 2-tensors: the 2-tensors of rank at most k are precisely those for which all $(k+1) \times (k+1)$-subdeterminants vanish. When p is larger than 2, no such clean alternative characterisation seems to exist. One immediate reason for this is that having rank at most k is typically not a Zariski-closed condition, that is, not equivalent to the vanishing of some polynomials; the following classical example illustrates this.

Example 1.1. Take $K = \mathbb{C}$, $p = 3$, and $n_1 = n_2 = n_3 = 2$. A tensor ω defines a linear map ϕ from $\mathbb{C}^2$ to the space $M_2(\mathbb{C})$ of 2×2-matrices over $\mathbb{C}$ by

$$\phi : (x_1, x_2) \mapsto \begin{bmatrix} x_1\omega(1,1,1) + x_2\omega(1,1,2) & x_1\omega(1,2,1) + x_2\omega(1,2,2) \\ x_1\omega(2,1,1) + x_2\omega(2,1,2) & x_1\omega(2,2,1) + x_2\omega(2,2,2) \end{bmatrix}.$$

If $\operatorname{im}\phi$ is zero, then ω has rank 0. If $\operatorname{im}\phi$ is one-dimensional, then the rank of ω equals the rank of any non-zero matrix in $\operatorname{im}\phi$. If $\operatorname{im}\phi$ is spanned by two linearly independent matrices of rank 1, then the rank of ω is two. This is the "generic" situation. In the remaining case, the rank is 3. The 3-tensors of the latter type are contained and dense in the zero set of Cayley's *hyperdeterminant*. This quartic polynomial in the entries of ω equals the discriminant of determinant of $\phi(x)$, seen as a homogeneous quadratic polynomial in x_1, x_2 with coefficients that are homogeneous quadratic polynomials in the entries of $\omega(i, j, k)$. Hence the set of tensors of rank at most 2 is dense in all $2 \times 2 \times 2$-tensors, and its complement is non-empty and locally closed. Moreover, working over $\mathbb{R}$ instead of $\mathbb{C}$, the tensors whose hyperdeterminant is negative also have rank 3. This serves to show that tensor rank is a delicate notion.

This situation leads to the notion of *border rank* of a tensor ω, which is the minimal k such that all polynomials that vanish identically on the set of tensors of rank at most k also vanish on ω. In other words, ω has border rank at most k if it lies in the Zariski-closure of the set of tensors of rank at most k. Thus all $2 \times 2 \times 2$-tensors over $\mathbb{C}$ have border rank at most 2, and the same is true over $\mathbb{R}$. Furthermore, over $\mathbb{C}$ the tensors of border rank at most k is also the ordinary (Euclidean) closure of the set of tensors of rank at most k. This explains the term border rank: tensors of border rank at most k are those that can be approximated arbitrarily well by tensors of rank at most k.

There is an important situation where rank and border rank do agree, and that is the case of rank at most one, i.e., pure tensors. These are always defined by the vanishing of all 2×2-subdeterminants of *flattenings* of the p-tensor ω into matrices. Such a flattening is obtained by partitioning the p dimensions $n_1, \ldots, n_p$ into two sets, for example $n_1, \ldots, n_q$ and $n_{q+1}, \ldots, n_p$, and viewing ω as a $(n_1 \cdots n_q) \times (n_{q+1} \cdots n_p)$-matrix. In fact, it is well known that the 2×2-subdeterminants of all flattenings even generate the ideal of all polynomials vanishing on pure tensors.

Among the most advanced results on tensor rank is the fact that a p-tensor has border rank at most 2 if and only if all 3×3-subdeterminants of all its flattenings have rank at most 2 [24]; the *GSS-conjecture* in [17] that these subdeterminants generate the actual ideal has recently been proved by Raicu [26]. In any case, having border rank at most 2 is a property defined by polynomials of degree 3, regardless

k	0	1	2	3	4	$(3l-1)/2$ ($l \geq 3$ odd)	k
$d(k)$	1	2	3	≥ 4	≥ 9	$\geq 3l$	$\geq k+1$

TABLE 1. Degree bounds: $d(0) = 1$ is trivial; $d(1) = 2$ is classical—2×2-subdeterminants of flattenings generate the ideal of pure tensors; $d(2) = 3$ is due to [24] and scheme-theoretically to [26]; for $k = 3$ there are already quartic polynomials in the ideal for $3 \times 3 \times 3$-tensors (while there are no 4×4-subdeterminants of flattenings) [24, Proposition 6.1]; these are due to Strassen [29, Section 4] and together they generate the ideal if $p = 3$ [25, Theorem 1.3] but it is unknown whether they define border-rank ≤ 3 tensors for larger p; $d((3l-1)/2) \geq 3l$ and its special case $d(4) \geq 9$ follows from Strassen's degree-$3l$ equation for the hypersurface of $l \times l \times 3$-tensors of border rank at most $(3l-1)/2$ [29, Theorem 4.6] (see also [22]); and $d(k) \geq k+1$ follows from a general fact about ideals of secant varieties [24, Corollary 3.2] applied to the variety of pure tensors.

of the dimension of the tensor. Replacing 2 by k and 3 by "some degree bound", we arrive the statement of our first main theorem, to which the title of this paper refers.

Theorem 1.2 (Main Theorem I)**.** *For fixed $k \in \mathbb{N}$, there exists a $d \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ and $n_1, \ldots, n_p \in \mathbb{N}$ the set of $n_1 \times \ldots \times n_p$ tensors over K of border rank at most k is defined by the vanishing of a number of polynomials of degree at most d.*

Table 1 shows what is known about the smallest $d(k)$ satisfying the conclusion of the theorem for K of characteristic 0.

Remark 1.3. *A priori*, the bound d in Main Theorem I depends on the infinite field K, as well as on k. But take L to be any algebraically closed field of the same characteristic as K. Then, by Hilbert's Basis theorem, d from Main Theorem I applied to L has the property that for all $p, n_1, \ldots, n_p$ the ideal of all polynomials vanishing on tensors over L of rank at most k is the radical of its sub-ideal generated by elements of degree at most d. This latter statement (certain ideals given as kernels of L-algebra homomorphisms defined over $\mathbb{Z}$ are the radicals of their degree-at-most-d parts) can be phrased as a linear algebra statement over the common prime field of L and K. Hence any d that works for L also works for K. This shows the existence of a degree bound that depends only on k and on the *characteristic* of K. It seems likely that a uniform bound exists that does not depend on the characteristic either, but that would require more subtle arguments.

An interesting consequence of Main Theorem I is the following, for whose terminology we refer to [2].

Corollary 1.4. *For every fixed k, there exists a uniform degree bound on equations needed to cut out the k-taxa general Markov model on any phylogenetic tree (in which vertices of any degree are allowed).*

Indeed, in [2] it is proved that the k-taxa general Markov model on any tree is cut out by certain $(k+1)\times(k+1)$-determinants together with the equations coming from star models (and this also holds scheme-theoretically [15]). These star models, in turn, are varieties of border-rank-k tensors. Note that set-theoretic equations for the 4-taxa model were recently found in [16] and, using numerical algebraic geometry, in [3].

This paper is organised as follows. Section 2 recalls a coordinate-free version of tensors, together with the fundamental operations of contraction and flattening, which do not increase (border) rank. That section also recalls a known reduction to the case where all sizes n_i are equal to any fixed value n greater than k (Lemma 2.3), as well as an essentially equivalent reformulation of Main Theorem I (Main Theorem II). In Section 3 we introduce a natural projective limit A_∞ of the spaces of $n \times n \times \ldots \times n$-tensors (with the number of factors tending to infinity), which contains as an (infinite-dimensional) subvariety the set $X_\infty^{\leq k}$ of ∞-*dimensional tensors of border rank at most* k. This variety is contained in the *flattening variety* $Y_\infty^{\leq k}$ defined by all $(k+1)\times(k+1)$-subdeterminants of flattenings. In Section 4 we prove that $Y_\infty^{\leq k}$ is defined by finitely many orbits of equations under a group G_∞ of natural symmetries of $A_\infty, X_\infty^{\leq k}, Y_\infty^{\leq k}$ (Proposition 4.2). In Section 5 we recall some basic properties of equivariantly Noetherian topological spaces, and prove that $Y_\infty^{\leq k}$ is a G_∞-Noetherian topological space with the Zariski topology (Theorem 5.6). This means that any G_∞-stable closed subset is defined by finitely many G_∞-orbits, and hence in particular this holds for $X_\infty^{\leq k}$ (Main Theorem III below). In Section 6 we show how this implies Main Theorems I and II, and in Section 7 we speculate on finiteness results for the entire ideal of $X_\infty^{\leq k}$, rather than just set-theoretic finiteness.

2. Tensors, flattening, and contraction

For most of the arguments in this paper—with the notable exception of the proof of Lemma 5.7—a coordinate-free notion of tensors will be more convenient than the array-of-numbers interpretation from the Introduction. Hence let $V_1, \ldots, V_p$ be finite-dimensional vector spaces over a fixed infinite field K. Setting $[p] := \{1, \ldots, p\}$, we write $V_I := \bigotimes_{i\in I} V_i$ for the tensor product of the V_i with $i \in I \subseteq [p]$. The rank of a tensor ω in $V_{[p]}$ is the minimal number of terms in any expression of ω as a sum of pure tensors $\bigotimes_{i\in[p]} v_i$ with $v_i \in V_i$.

Given a p-tuple of linear maps $\phi_i : V_i \to U_i$, where $U_1, \ldots, U_p$ are also vector spaces over K, we write $\phi_{[p]} := \bigotimes_{i\in[p]} \phi_i$ for the linear map $V_{[p]} \to U_{[p]}$ determined by $\bigotimes_{i\in[p]} v_i \mapsto \bigotimes_{i\in[p]} \phi_i(v_i)$. Clearly $\operatorname{rk} \pi_{[p]}\omega \leq \operatorname{rk}\omega$ for any $\omega \in V_{[p]}$, and this inequality carries over to the border rank. In the particular case where a single U_j equals K, U_i equals V_i for all $i \neq j$, and $\phi_i = \mathrm{id}_{V_i}$ for all $i \neq j$, we identify $U_{[p]}$ with $V_{[p]-\{j\}}$ in the natural way, and we call the map $\phi_{[p]} : V_{[p]} \to V_{[p]-\{j\}}$ the *contraction* along the linear function ϕ_j. The composition of contractions along linear functions ϕ_j on V_j for all j in some subset J of $[p]$ is a linear map $V_{[p]} \to V_{[p]-J}$, called a contraction along the pure $|J|$-tensor $\bigotimes_{j\in J} \phi_j$, that does not increase (border) rank. Now we can phrase a variant of our Main Theorem I.

Theorem 2.1 (Main Theorem II.)**.** *For all $k \in \mathbb{N}$ there exists a p_0 such that for all $p \geq p_0$ and for all vector spaces $V_1, \ldots, V_p$ a tensor $\omega \in V_1 \otimes \cdots \otimes V_p$ has border rank at most k if and only if all its contractions along pure $(p - p_0)$-tensors have border rank at most k.*

Remark 2.2. If p_0 is known explicitly, this theorem gives rise to the following theoretical randomised algorithm for testing whether a given p-tensor has border rank at most k: for each $(p-p_0)$-subset J of $[p]$ contract along a random pure tensor in the dual of V_J, and on the resulting p_0-tensor evaluate the polynomials cutting out tensors of border rank at most k. If each of these tests is positive, then one can be fairly certain that the given tensor is, indeed, of border rank at most k. While this might not be a practical algorithm, its number of arithmetic operations is polynomial in the number of entries in the tensor, as long as we take k constant.

In our proofs of Main Theorems I and II we will use a second operation on tensors, namely, *flattening*. Suppose that I, J form a partition of $[p]$ into two parts. Then there is a natural isomorphism $\flat = \flat_{I,J} : V_{[p]} \to V_I \otimes V_J$. The image $\flat\omega$ is a 2-tensor called a flattening of ω. Its rank (as a 2-tensor) is a lower bound on the border rank of ω. The first step in our proof below is a reduction to the case where all V_i have the same dimension.

Lemma 2.3. *Let p, k, n be natural numbers with $n \geq k+1$, and let $V_1, \ldots, V_p$ be vector spaces over K. Then a tensor $\omega \in \bigotimes_{i\in[p]} V_i$ has rank (respectively, border rank) at most k if and only if for all p-tuples of linear maps $\phi_i : V_i \to K^n$ the tensor $(\bigotimes_{i\in[p]} \phi_i)\omega$ has rank (respectively, border rank) at most k.*

This lemma is well known, and even holds at the scheme-theoretic level [2, Theorem 11]. We include a proof for completeness.

Proof. The "only if" part follows from the fact that $\phi_{[p]}$ does not increase rank or border rank. For the "if" part assume that ω has rank strictly larger than k, and we argue that there exist $\phi_1, \ldots, \phi_p$ such that $\phi_{[p]}\omega$ still has rank larger than k. It suffices to show how to find ϕ_1; the remaining ϕ_i are found in the same manner. Let U_1 be the image of ω regarded as a linear map from the dual space $V^*_{[p]-\{1\}}$ to V_1. If U_1 has dimension at most n, then by elementary linear algebra there exist linear maps $\phi_1 : V_1 \to K^n$ and $\psi_1 : K^n \to V_1$ such that $\psi_1 \circ \phi_1$ is the identity map on U_1. Set $\omega' := \phi_1 \otimes (\bigotimes_{i>1} \mathrm{id}_{V_i})\omega$, so that by construction ω itself equals $\psi_1 \otimes (\bigotimes_{i>1} \mathrm{id}_{V_i})\omega'$. By the discussion above, we have the inequalities $\mathrm{rk}\,\omega \geq \mathrm{rk}\,\omega' \geq \mathrm{rk}\,\omega$, so that both ranks are equal and larger than k, and we are done. If, on the other hand, U_1 has dimension larger than n, then let $\phi_1 : V_1 \to K^n$ be any linear map that maps U_1 surjectively onto K^n. Defining ω' as before, we find that the image of ω' regarded as a linear map $V^*_{[p]-\{1\}} \to K^n$ is all of K^n, so that the flattening $\flat_{1,[p]-\{1\}}\omega'$ already has rank $n > k$. This implies that ω' itself has rank larger than k. $\square$

3. Infinite-dimensional tensors and their symmetries

The space of $\mathbb{N} \times \mathbb{N}$-matrices can be realised as the projective limit of the spaces of $p \times p$-matrices for $p \to \infty$, where the projection from $(p+1) \times (p+1)$-matrices to $p \times p$-matrices sends a matrix to the $p \times p$-matrix in its upper left corner. In much the same way, we will construct a space of *infinite-dimensional tensors* as a projective limit of all tensor powers $V^{\otimes p}$, $p \in \mathbb{N}$ of the vector space $V = K^n$ for some fixed natural number n. To this end, fix a linear function $x_0 \in V^*$, and denote by π_p the contraction $V^{\otimes p} \to V^{\otimes p}$ determined by

$$\pi_p : v_1 \otimes \cdots \otimes v_p \otimes v_{p+1} \mapsto x_0(v_{p+1}) \cdot v_1 \otimes \cdots \otimes v_p.$$

Dually, this surjective map gives rise to the injective linear map

$$(V^*)^{\otimes p} \to (V^*)^{\otimes p+1},\ \xi \mapsto \xi \otimes x_0.$$

Let T_p be the coordinate ring of $V^{\otimes p}$. We identify T_p with the symmetric algebra $S((V^*)^{\otimes p})$ generated by the space $(V^*)^{\otimes p}$, and embed T_p into T_{p+1} by means of the linear inclusion $(V^*)^{\otimes p} \to (V^*)^{\otimes p+1}$ above. Now the projective limit $A_\infty := \lim_{\leftarrow p} V^{\otimes p}$ along the projections π_p is a vector space whose coordinate ring T_∞ is the union $\bigcup_{p\in\mathbb{N}} T_p$. Here by coordinate ring we mean that A_∞ corresponds naturally (via evaluation) to the set of K-valued points of T_∞ (ie. K-algebra homomorphisms $T \to K$). Not every vector space has a coordinate ring in that sense, as it is equivalent to being the dual space of a vector space: A_∞ is naturally the dual space of the space ${T_\infty}^1$ of homogeneous polynomials of degree 1 in T_∞. For example, as ${T_\infty}^1$ has a countably infinite basis, it is not a dual space and has itself no coordinate ring in that sense.

At a crucial step in our arguments we will use the following more concrete description of T_∞. Extend x_0 to a basis $x_0, x_1, \ldots, x_{n-1}$ of V^*, so that $(V^*)^{\otimes p}$ has a basis in bijection with words $w = (i_1, \ldots, i_p)$ over the alphabet $\{0, \ldots, n-1\}$, namely, $x_w := x_{i_1} \otimes \cdots \otimes x_{i_p}$. The algebra T_p is the polynomial algebra in the variables x_w with w running over all words of length p. In T_∞, the coordinate x_w is identified with the variable $x_{w'}$ where w' is obtained from w by appending a 0 at the end. We conclude that T_∞ is a polynomial ring in countably many variables that are in bijective correspondence with infinite words $(i_1, i_2, \ldots)$ in which all but finitely many characters are 0. The finite set of positions where such a word is non-zero is called the *support* of the word.

Now for symmetry: the symmetric group S_p acts on $V^{\otimes p}$ by permuting the tensor factors:

$$\pi(v_1 \otimes \cdots \otimes v_p) = v_{\pi^{-1}(1)} \otimes \cdots \otimes v_{\pi^{-1}(p)}.$$

This leads to the contragredient action of S_p on the dual space $(V^*)^{\otimes p}$ by

$$\pi(x_1 \otimes \cdots \otimes x_p) = x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(p)},$$

which extends to an action of S_p on all of T_p by means of algebra automorphisms. In terms of words, πx_w equals the coordinate $x_{w'}$ where the q-th character of w' is the $\pi^{-1}(q)$-th character of w. Let S_∞ denote the union $\bigcup_{p\in\mathbb{N}} S_p$, where S_p is embedded in S_{p+1} as the subgroup fixing $p+1$. Hence S_∞ is the group of all bijections $\pi : \mathbb{N} \to \mathbb{N}$ whose set of fixed points has a finite complement. This group acts on A_∞ and on T_∞ by passing to the limit.

The action of S_∞ on T_∞ has the following fundamental property: for each $f \in T_\infty$ there exists a $p \in \mathbb{N}$ such that whenever $\pi, \sigma \in S_\infty$ agree on the initial segment $[p]$ we have $\pi f = \sigma f$. Indeed, we may take p equal to the maximum of the union of the supports of words w for which x_w appears in f. In this situation, there is a natural left action of the *increasing monoid* $\operatorname{Inc}(\mathbb{N}) = \{\pi : \mathbb{N} \to \mathbb{N} \mid \pi(1) < \pi(2) < \ldots\}$ by means of injective algebra endomorphisms on T_∞; see [19, Section 5]. The action is defined as follows: for $f \in T_\infty$, let p be as above. Then to define πf for $\pi \in \operatorname{Inc}(\mathbb{N})$ take any $\sigma \in S_\infty$ that agrees with π on the interval $[p]$ (such a σ exists) and set $\pi f := \sigma f$. In terms of words: πx_w equals the coordinate $x_{w'}$, where the q-th character of w' is the $\pi^{-1}(q)$-th character of w if $q \in \operatorname{im} \pi$, and 0 otherwise. By construction, the $\operatorname{Inc}(\mathbb{N})$-orbit of any $f \in T_\infty$ is contained in the S_∞-orbit of f. Note that the left action of $\operatorname{Inc}(\mathbb{N})$ on T_∞ gives rise to a *right* action of $\operatorname{Inc}(\mathbb{N})$ by means of surjective linear maps $A_\infty \to A_\infty$. Since we cannot take inverses, there is no corresponding left action on A_∞, while of course the left action of S_∞ can be made into a right action by taking inverses. A crucial argument in Section 5 uses a map that is not equivariant with respect to S_∞ but is equivariant relative to $\mathrm{Inc}(\mathbb{N})$.

Apart from S_p, the group $\mathrm{GL}(V)^p$ acts linearly on $V^{\otimes p}$ by

$$(g_1, \dots, g_p)(v_1 \otimes \cdots v_p) = (g_1 v_1 \otimes \cdots \otimes g_p v_p),$$

and this action gives a right action on $(V^*)^{\otimes p}$ by

$$(z_1 \otimes \cdots \otimes z_p)(g_1, \dots, g_p)((z_1 \circ g_1) \otimes \cdots \otimes (z_p \circ g_p)).$$

Regarding $\mathrm{GL}(V)^p$ as the subgroup of $\mathrm{GL}(V)^{p+1}$ consisting of all tuples $(g_1, \dots, g_p, 1)$, we obtain a (left) action of the union of all $\mathrm{GL}(V)^p$ on A_∞, as well as an action on T_∞ by means of algebra automorphisms.

The group generated by S_p and $\mathrm{GL}(V)^p$ in their representations on $V^{\otimes p}$ is the semidirect (wreath) product $S_p \ltimes \mathrm{GL}(V)^p$, which we denote by G_p. The embeddings $S_p \to S_{p+1}$ and $\mathrm{GL}(V)^p \to \mathrm{GL}(V)^{p+1}$ together determine an embedding $G_p \to G_{p+1}$, and the union G_∞ of all G_p, $p \in \mathbb{N}$ has an action on A_∞ and on T_∞ by means of automorphisms.

Now we come to tensors of bounded border rank. Write $X_p^{\leq k}$ for the subvariety of $V^{\otimes p}$ consisting of tensors of border rank at most k. Since π_p is a contraction along a pure tensor, it maps $X_{p+1}^{\leq k}$ into $X_p^{\leq k}$; let $X_\infty^{\leq k} \subseteq A_\infty$ be the projective limit of the $X_p^{\leq k}$. The elements of G_p stabilise $X_p^{\leq k}$, and hence the elements of G_∞ stabilise $X_\infty^{\leq k}$. We can now state our third Main Theorem, which will imply Main Theorems I and II.

Theorem 3.1 (Main Theorem III)**.** *For any fixed natural number k, the set $X_\infty^{\leq k}$ is the common zero set of finitely many G_∞-orbits of polynomials in T_∞.*

We emphasise here that Main Theorem III only concerns the variety $X_\infty^{\leq k}$ as a set of K-valued points. We do not claim that the *ideal* of $X_\infty^{\leq k}$ is generated by finitely many G_∞-orbits of polynomials; see Section 7.

4. The flattening variety

In this section we discuss the *flattening variety* $Y_\infty^{\leq k} \subseteq A_\infty$, which contains $X_\infty^{\leq k}$ and is defined by explicit equations. These equations are determinants obtained as follows. Given any partition of $[p]$ into I, J we have the flattening $V^{\otimes p} \to V^{\otimes I} \otimes V^{\otimes J}$. Composing this flattening with a $(k+1) \times (k+1)$-subdeterminant of the resulting two-tensor gives a degree-$(k+1)$ polynomial in T_p. The linear span of all these equations for all possible partitions I, J is a G_p-submodule U_p of T_p, which by π_p^* is embedded into U_{p+1}. Let $Y_p^{\leq k}$ denote the subvariety of $V^{\otimes p}$ defined by the U_p. This is a G_p-stable variety, and π_p maps $Y_{p+1}^{\leq k}$ into $Y_p^{\leq k}$. The projective limit $Y_\infty^{\leq k} \subseteq A_\infty$ is the zero set of the space $U := \bigcup_p U_p$. The variety $Y_\infty^{\leq k}$ consists of all infinite-dimensional tensors all of whose flattenings to 2-tensors have rank at most k. In particular, since flattening does not increase rank, $Y_\infty^{\leq k}$ contains $X_\infty^{\leq k}$.

For later use we describe these determinants in more concrete terms in the coordinates x_w, where w runs through the infinite words over $\{0, 1, \dots, n-1\}$ of finite support. Let $\mathbf{w} = (w_1, \dots, w_l)$ be an l-tuple of pairwise distinct infinite words with finite support. Let $\mathbf{w}' := (w'_1, \dots, w'_m)$ be another such tuple, and assume that the support of each w_i is disjoint from that of each w'_j. Then we let $x[\mathbf{w}; \mathbf{w}']$ be the

$l \times m$ matrix with (i,j)-entry equal to $x_{w_i+w'_j}$. The space U is spanned by the determinants of all matrices $x[\mathbf{w};\mathbf{w}']$ with $l=m=k+1$.

Example 4.1. Take $w_1=1, w_2=2, w'_1=01, w'_2=02$, where we follow the convention that the (infinitely many) trailing zeroes of words may be left out, and where we have left out brackets and separating commas for brevity. Then

$$x[\mathbf{w};\mathbf{w}'] = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}.$$

Apart from the inclusion $X_\infty^{\leq k} \subseteq Y_\infty^{\leq k}$, we will need the following crucial property of $Y_\infty^{\leq k}$, whose proof will take up the remainder of this section.

Proposition 4.2. *For every fixed natural number k the flattening variety $Y_\infty^{\leq k}$ is the common zero set of finitely many G_∞-orbits of $(k+1)\times(k+1)$-determinants* $\det x[\mathbf{w};\mathbf{w}']$ *with* $\mathbf{w},\mathbf{w}'$ *as above.*

The proposition is an immediate consequence of the following related result:

Lemma 4.3. *Let $k \geq 0$ be fixed. There is an integer p_0 such that whenever $p > p_0$ and whenever $v \in V^{\otimes p}$ is an element such that for all pure contractions $\phi\colon V^{\otimes p} \to V^{\otimes [p]-I} \cong V^{\otimes p_0}$ with $|I| = p - p_0$, $\phi(v) \in Y_{p_0}^{\leq k}$, then $v \in Y_p^{\leq k}$.*

Proof of Proposition 4.2. Let p_0 be the integer whose existence is asserted by Lemma 4.3. Let $f_1, f_2, \ldots, f_N \in T_{p_0}$ be the finitely many $(k+1)\times(k+1)$-determinants that generate the ideal of $Y_{p_0}^{\leq k}$. Of course, in the inclusion $T_{p_0} \subset T_\infty$, each f_i is one of the $\det x[\mathbf{w};\mathbf{w}']$ for $\mathbf{w},\mathbf{w}'$ each lists of $k+1$ words supported in $[p_0]$.

We will now show that $v \in A_\infty$ is an element of $Y_\infty^{\leq k}$ if and only if $f_i(gv) = 0$ for all i and all $g \in G_\infty$. Note that $f_i(gv)$ is equal to $f_i((gv)_{p_0})$ where $(gv)_{p_0}$ is the image of v in $V^{\otimes p_0}$ under the canonical projection $A_\infty \to V^{\otimes p_0}$. Now if $v \in Y_\infty^{\leq k}$, then obviously so is gv for each $g \in G_\infty$, and hence $(gv)_{p_0}$ is an element of $Y_{p_0}^{\leq k}$. This shows the only if part.

For the converse, suppose that $f_i(gv) = 0$ for all i and all $g \in G_\infty$. We need to show that $v \in Y_\infty^{\leq k}$. Equivalently, we need to show that for all p, the image $v_p \in V^{\otimes p}$ of v lies in $Y_p^{\leq k}$. Recall that $f_i \in T_{p_0}$ is identified in T_p with f_i precomposed with the pure contraction ϕ of the last $p - p_0$ factors along $x_0^{\otimes(p-p_0)}$. Now if $\phi'\colon V^{\otimes p} \to V^{\otimes p_0}$ is any other (nonzero) pure contraction, then there is $g \in G_p$ such that $\phi'(v) = \phi(gv)$ for all $v \in V^{\otimes p}$; indeed, the symmetric group S_p can be used to ensure that the same factors are being contracted, and $\mathrm{GL}(V)^p$ can be used to ensure that the contraction takes place along the same pure tensor. If $g \in G_p$, then $(gv)_p = gv_p$, and it follows that our assumption implies that $f_i(\phi'(v_p)) = 0$ for all i and all pure contractions $\phi'\colon V^{\otimes p} \to V^{\otimes p_0}$. By the lemma this means that $v_p \in Y_p^{\leq k}$, and we are done. □

The proof of the lemma itself—while not difficult—requires some algebraic geometry. As it will turn out, the crucial point is the following observation.

Lemma 4.4. *Let $k \geq 0$. There is an integer p_1 such that whenever $p > p_1$ and $W \subseteq V^{\otimes p}$ is a subspace the following holds: if for all i and all pure contractions $\phi\colon V^{\otimes p} \to V^{\otimes [p]-\{i\}}$, $\dim \phi(W) \leq k$, then* $\dim W \leq k$.

Proof. We will show that $p_1 = (k+1)\lfloor \log_2(k+1) \rfloor$ works. Let $G(d, V^{\otimes p})$ denote the Grassmannian of d-planes in $V^{\otimes p}$, which is a projective algebraic variety over

K. Set

$$Z(d,k) := \{W \in G(d, V^{\otimes p}) \mid \dim \phi(W) \leq k \text{ for all contractions } \phi\},$$

a closed subvariety of $G(d, V^{\otimes p})$. The assertion of the lemma is equivalent to the statement that $Z(d,k) = \emptyset$ if $d > k$ and $p > p_1$. To prove this, we proceed by induction on d. If $W \in Z(d,k)$, then all hyperplanes of W are elements of $Z(d-1,k)$, hence if the latter variety is empty, then so is $Z(d,k)$. Thus, in the following we may assume that $d = k+1$ and force a contradiction.

So suppose that $Z(k+1,k)$ is nonempty. We will use that it is stable under $\mathrm{GL}(V)^p \subseteq G_p$. Denote by B the group of upper triangular matrices in $\mathrm{GL}(V)$ with respect to the basis $e_0, \ldots, e_{n-1}$ of V dual to $x_0, \ldots, x_{n-1}$. Then B^p is a connected solvable algebraic group and by Borel's Fixed Point Theorem ([4], Theorem 15.2), B^p must have a fixed point W on the projective algebraic variety $Z(d,k)$. This means that W is a B^p-stable subspace of $V^{\otimes p}$.

Let $D \subseteq \mathrm{GL}(V)$ denote the subset of diagonal matrices. Then $D \subseteq B$ and hence W is also D^p-stable. Any D^p-stable subspace is spanned by common eigenvectors for D^p (any algebraic representation of D^p is diagonalisable). Now $v \in V^{\otimes p}$ is a D^p-eigenvector if and only if $v = e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_p}$ (up to a nonzero scalar) for some $i_1, \ldots, i_p$ with $0 \leq i_k \leq n-1$. If such a v is in W, then also the linear span of $B^p v$ is. We claim that

$$\langle B^p v \rangle = E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_p} =: E$$

where $E_i = \langle e_0, e_1, \ldots, e_i \rangle \subseteq V$ is the B-stable subspace generated by e_i. Notice that E is clearly B^p-stable. The only non-trivial assertion is therefore that $\langle B^p v \rangle$ contains E. This is seen most easily by considering the action of the Lie algebra $\mathfrak{b}^p$ of B^p where $\mathfrak{b}$ is the Lie algebra of B, ie. the space of all upper triangular matrices. Since B^p maps W to itself, so does $\mathfrak{b}^p$. An element $X = (X_1, X_2, \ldots, X_p)$ of $\mathfrak{b}^p$ acts on $V^{\otimes p}$ by

$$X(v_1 \otimes v_2 \otimes \cdots \otimes v_p) = \sum_{i=1}^{p} v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \otimes X_i(v_i) \otimes v_{i+1} \otimes \cdots \otimes v_p.$$

Thus, if for $i \leq j$ we denote the element of $\mathfrak{b}$ that maps e_j to e_i and everything else to zero by E_{ij}, then if $j_1 \leq i_1$, $j_2 \leq i_2$, $\ldots$, $j_p \leq i_p$

$$e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_p} = (E_{j_1 i_1}, 0, \ldots, 0)(0, E_{j_2 i_2}, 0, \ldots, 0) \cdots (0, \ldots, 0, E_{j_p i_p})(v).$$

It follows that indeed, $E \subseteq W$. Since W is by assumption spanned by $k+1$ such eigenvectors v_i, and hence $W = \sum_i \langle B^p v_i \rangle$, we find that

$$W = \sum_{a=1}^{k+1} F_a$$

where each F_a is a tensor product of some of the E_j. Now for a fixed a, let $F_a = E_{i_1} \otimes \cdots \otimes E_{i_p}$. Then most i_k must be equal to 1: indeed, the dimension of F_a is $(i_1+1)(i_2+2)\cdots(i_p+1) \leq k+1$, hence at most $\lfloor \log_2(k+1) \rfloor$ of the i_j can be 1 or larger. We conclude that there are at most $p_1 = (k+1)\lfloor \log_2(k+1) \rfloor$ indices where at least one F_a has a factor of dimension at least 2. Since $p > p_1$, there is at least one index i where all F_a have a one-dimensional factor E_0. Thus, in the flattening corresponding to $\{i\}$, $[p] - \{i\}$, W corresponds to a space of the form

$$W = E_0 \otimes H$$

where $\dim H = \dim W$. Now any linear function on V that is not zero on E_0 yields a contraction of W of dimension equal to $\dim H = \dim W = k+1$ – a contradiction. □

Proof of Lemma 4.3. Let $p_0 = 2p_1 = 2(k+1)\lfloor \log_2(k+1) \rfloor$. We will proceed by induction on $p > p_0$. So let $v \in V^{\otimes p}$ be an element such that for all subsets $I \subset [p]$ with $p - p_0$ elements, the image of v under any contraction $V^{\otimes p} \to V^{\otimes [p]-I} \cong V^{\otimes p_0}$ is an element of $Y_{p_0}^{\leq k}$. Note that if $p > p_0 + 1$, then for the image v' of v under any contraction $V^{\otimes p} \to V^{\otimes [p]-\{i\}}$, any pure contraction $V^{\otimes [p]-\{i\}} \to V^{\otimes [p]-\{i\}-I}$ takes v' to an element of $Y_{p_0}^{\leq k}$, where now I ranges over all subsets with $p-1-p_0$ elements. As a consequence, if $p > p_0 + 1$, then we may assume by induction that v' is an element of $Y_{p-1}^{\leq k}$. Thus, we need to show the following: Given $v \in V^{\otimes p}$ such that all pure contractions of v in $V^{\otimes [p]-\{i\}}$ are elements of $Y_{p-1}^{\leq k}$, then $v \in Y_p^{\leq k}$, provided $p > p_0$.

In order to see this, let $[p] = I \cup J$ be any partition and consider the corresponding flattening

$$\flat \colon V^{\otimes p} \to V^{\otimes I} \otimes V^{\otimes J}.$$

After exchanging the roles of I and J if necessary, we may assume that $|J| > p_1$. The statement that all $(k+1)\times(k+1)$-subdeterminants on $\flat v$ are zero is equivalent to the statement that same as saying that $\flat v$ has rank at most k when regarded as a linear map from $(V^{\otimes I})^*$ to $V^{\otimes J}$, or, in other words, that the image $W \subseteq V^{\otimes J}$ of this map has dimension at most k.

Since $|J| > p_1$ we may apply Lemma 4.4 to W. Indeed, for each $j \in J$, all contractions in factor j of $V^{\otimes J}$ map W to subspaces of $V^{\otimes J - \{j\}}$ of dimension at most k. This follows form the fact that this subspace is equal to the image W' of the map $(V^{\otimes I})^* \to V^{\otimes J-j}$ obtained by first applying $\flat v$ and then contracting $V^{\otimes J} \to V^{\otimes J-i}$. This on the other hand, is nothing but the map $\flat'(v')$ where v' is the image of v under the same contraction but applied to $V^{\otimes p}$, and $\flat'$ is the flattening of $[p] - \{j\}$ along $I, J-j$. Since v' gives rise to a map of rank at most k by assumption, $\dim W' \leq k$ as claimed. Now this holds for all contractions and all factors and we may conclude that, indeed, $\dim W \leq k$, and $\flat v$ has rank at most k. □

5. Equivariantly Noetherian rings and spaces

We briefly recall the notions of equivariantly Noetherian rings and topological spaces, and proceed to prove the main result of this section, namely, that $Y_\infty^{\leq k}$ is G_∞-Noetherian (Theorem 5.6).

If a monoid Π acts by means of endomorphisms on a commutative ring R (with 1), then we call R *equivariantly Noetherian*, or Π-Noetherian, if every chain $I_1 \subseteq I_2 \subseteq \ldots$ of Π-stable ideals stabilises. This is equivalent to the statement that every Π-stable ideal in R is generated by finitely many Π-orbits. Similarly, if Π acts on a topological space X by means of continuous maps $X \to X$, then we call X equivariantly Noetherian, or Π-Noetherian, if every chain $X_1 \supseteq X_2 \supseteq \ldots$ of Π-stable closed subsets stabilises. If R is an K-algebra then we can endow the set X of K-valued points of R, i.e., K-algebra homomorphisms $R \to K$ (sending 1 to 1), with the Zariski topology. An endomorphism $\Psi : R \to R$ gives a continuous map $\psi : X \to X$ by pull-back, and if R has a left Π-action making it equivariantly Noetherian, then this induces a right Π-action on X making X equivariantly

Noetherian. This means, more concretely, that any Π-stable closed subset of X is defined by the vanishing of finitely many Π-orbits of elements of R. If Π happens to be a group, then we can make the right action into a left action by taking inverses. Here are some further easy lemmas; for their proofs we refer to [13].

Lemma 5.1. *If X is a Π-Noetherian topological space, then any Π-stable subset of X is Π-Noetherian with respect to the induced topology.*

We will only need this statement for closed subsets, for which the proof is contained in [13]. For completeness, we prove the general case here.

Proof. Let Y be any Π-stable subset of X, and let $Y_1 \supseteq Y_2 \supseteq \ldots$ be a sequence of Π-stable closed subsets of Y. Define X_i as the closure of Y_i in X, so that $X_i \cap Y = Y_i$ by properties of the induced topology. A straightforward verification shows that each X_i is Π-stable, hence as X is Π-Noetherian we find that the sequence $X_1 \supseteq X_2 \supseteq \ldots$ stabilises. But then the sequence $Y_1 = X_1 \cap Y \supseteq Y_2 = X_2 \cap Y \supseteq \ldots$ stabilises, as well. $\square$

Lemma 5.2. *If X and Y are Π-Noetherian topological spaces, then the disjoint union $X \cup Y$ is also Π-Noetherian with respect to the disjoint union topology and the natural action of Π.*

Lemma 5.3. *If X is a Π-Noetherian topological space, Y is a topological space with Π-action (by means of continuous maps), and $\psi : X \to Y$ is a Π-equivariant continuous map, then $\operatorname{im}\psi$ is Π-Noetherian with respect to the topology induced from Y.*

Lemma 5.4. *If Π is a group and $\Pi' \subseteq \Pi$ a subgroup acting from the left on a topological space X', and if X' is a Π'-Noetherian, then the orbit space $X := (\Pi \times X')/\Pi'$ is a left-Π-Noetherian topological space.*

In this lemma, $\Pi \times X'$ carries the direct-product topology of the discrete group Π and the topological space X', the right action of Π' on it is by $(\pi, x')\sigma = (\pi\sigma, \sigma^{-1}x)$, and the topology on the quotient is the coarsest topology that makes the projection continuous. The left action of Π on the quotient comes from left-action of Π on itself. As a consequence, closed Π-stable sets in X are in one-to-one correspondence with closed Π'-stable sets in X', whence the lemma. Next we recall a fundamental example of an equivariantly Noetherian ring, which will be crucial in what follows.

Theorem 5.5 ([12, 19])**.** *For any Noetherian ring Q and any $l \in \mathbb{N}$, the ring $Q[x_{ij} \mid i = 1, \ldots, l;\ j = 1, 2, 3, 4, \ldots]$ is equivariantly Noetherian with respect to the action of $\operatorname{Inc}(\mathbb{N})$ by $\pi x_{ij} = x_{i\pi(j)}$.*

Main Theorems I, II, and III will be derived from the following theorem, whose proof needs the rest of this section.

Theorem 5.6. *For every natural number k the variety $Y_\infty^{\leq k}$ is a G_∞-Noetherian topological space.*

We will proceed by induction on k. For $k = 0$ the variety $Y_\infty^{\leq k}$ consists of a single point, the zero tensor, and the theorem trivially holds. Now assume that the theorem holds for $k-1$. By Proposition 4.2 there exist k-tuples $\mathbf{w}_1, \ldots, \mathbf{w}_N, \mathbf{w}'_1, \ldots, \mathbf{w}'_N$ of words with finite support, such that each word in $\mathbf{w}_a$ and each word in $\mathbf{w}'_a$ have disjoint supports and such that $Y_\infty^{\leq k-1}$ is the common zero set of the polynomials in $\bigcup_{a=1}^N G_\infty \det(x[\mathbf{w}_a;\mathbf{w}'_a])$. For each $a=1,\ldots,N$ let Z_a denote the open subset of $Y_\infty^{\leq k}$ where not all elements of $G_\infty \det(x[\mathbf{w}_a;\mathbf{w}'_a])$ vanish; hence we have

$$Y_\infty^{\leq k} = Y_\infty^{\leq k-1} \cup Z_1 \cup \ldots \cup Z_N.$$

We will show that each $Z_a,\ a=1,\ldots,N$ is a G_∞-Noetherian topological space, with the topology induced from the Zariski topology on A_∞. Together with the induction hypothesis and Lemmas 5.2 and 5.3, this then proves that $Y_\infty^{\leq k}$ is G_∞-Noetherian, as claimed.

To prove that $Z := Z_a$ is G_∞-Noetherian, consider $\mathbf{w} := \mathbf{w}_a = (w_1,\ldots,w_k)$ and $\mathbf{w}' := \mathbf{w}'_a = (w'_1,\ldots,w'_k)$. Let p be the maximum of the union of all supports of the w_i and the w'_j, and let Z' denote the open subset of $Y_\infty^{\leq k}$ where $\det(x[\mathbf{w};\mathbf{w}'])$ is non-zero. This subset is stable under the group S'_∞ of all permutations in S_∞ that restrict to the identity on $[p]$.

Lemma 5.7. *The open subset $Z' \subseteq Y_\infty^{\leq k}$ is an S'_∞-Noetherian topological space.*

Proof. We will prove that it is $\mathrm{Inc}(\mathbb{N})'$-Noetherian, where $\mathrm{Inc}(\mathbb{N})'$ is the set of all increasing maps $\pi \in \mathrm{Inc}(\mathbb{N})$ that restrict to the identity on $[p]$; consult Section 3 for the action of $\mathrm{Inc}(\mathbb{N})$. Since the $\mathrm{Inc}(\mathbb{N})'$-orbit of an equation is contained in the corresponding S'_∞-orbit, this implies that Z' is S'_∞-Noetherian.

We start with the polynomial ring R in the variables x_w, where w runs over all infinite words over $\{0,1,\ldots,n-1\}$ with the property that the support of w contains at most 1 position that is larger than p. Among these variables there are n^p for which the support is contained in $[p]$, and the remaining variables are labelled by $[n]^p \times \{1,\ldots,n-1\} \times (\mathbb{N}-[p])$. On these variables acts $\mathrm{Inc}(\mathbb{N})'$, fixing the first n^p variables and acting only on the last (position) index of the last set of variables. By Theorem 5.5 with Q the ring in the first n^p variables and $l = n^p \times (n-1)$, the ring R is $\mathrm{Inc}(\mathbb{N})'$-Noetherian. Let $S = R[\det(x[\mathbf{w};\mathbf{w}'])^{-1}$ be the localisation of R at the determinant $\det x[\mathbf{w},\mathbf{w}']$; again, S is $\mathrm{Inc}(\mathbb{N})'$-Noetherian. We will construct an $\mathrm{Inc}(\mathbb{N})'$-equivariant map ψ from the set of K-valued points of S to A_∞ whose image contains Z'. We do this, dually, by means of an $\mathrm{Inc}(\mathbb{N})'$-equivariant homomorphism Ψ from T_∞ to S.

To define Ψ recursively, we first fix a partition I, J of $[p]$ such that each w_i has support contained in I and each w'_j has support contained in J. Now if $x_w \in T_\infty$ is one of the variables in R, then we set $\Psi x_w := x_w$. Suppose that we have already defined Ψ on variables x_w such that $\mathrm{supp}(w) - [p]$ has cardinality at most b, and let w be a word for which $\mathrm{supp}(w) - [p]$ has cardinality $b+1$. Let q be the maximum of the support of w, and write $w = w_{k+1} + w'_{k+1}$, where the support of w'_{k+1} is contained in $J \cup \{q\}$ and the support of w_{k+1} is contained in $I \cup \{p+1,\ldots,q-1\}$. Consider the determinant of the matrix

$$x[w_1,\ldots,w_k,w_{k+1};w'_1,\ldots,w'_{k+1}].$$

This determinant equals

$$\det(x[w_1,\ldots,w_k;w'_1,\ldots,w'_k]) \cdot x_w - P,$$

where $P \in T_\infty$ is a polynomial in variables that are of the form $x_{w_i+w'_j}$ with $i,j \leq k+1$ but not both equal to $k+1$. All of these $w_i + w'_j$ have supports containing at most b elements outside the interval $[p]$, so $\Psi(P)$ has already been defined. Then we set

$$\Psi x_w := \det(x[\mathbf{w},\mathbf{w}'])^{-1}\psi(P).$$

The map Ψ is $\mathrm{Inc}(\mathbb{N})'$-equivariant by construction.

The set $Z' \subseteq Y_\infty^{\leq k}$ is contained in the image of the map ψ. Indeed, this follows directly from the fact that the determinant of the matrix

$$x[w_1, \ldots, w_k, w_{k+1}; w_1', \ldots, w_{k+1}']$$

vanishes on Z' while $\det(x[\mathbf{w}, \mathbf{w}'])$ does not. More precisely, Z' equals the intersection of $Y_\infty^{\leq k}$ with $\operatorname{im}\psi$, and hence by Lemmas 5.3 and 5.1 it is $\mathrm{Inc}(\mathbb{N})'$-Noetherian. We already pointed out that this implies that Z' is S_∞'-Noetherian. □

Remark 5.8. While it is possible to describe the map Ψ in the proof of the theorem also by its action on points, it is rather involved and apparently yields no additional insight. A point worth mentioning, though, is the the following fact: if, for $i > p$, we denote the contraction $V^{\otimes [p] \cup \{i\}} \to V^{\otimes p}$ (along x_0 in the last factor) by ∂_i, the ring R mentioned in the proof is the coordinate ring of the affine scheme whose K-points are the (infinite) sequences $(v_1, v_2, \ldots)$ such that $v_i \in V^{\otimes [p] \cup \{i\}}$ and $\partial_i(v_i) = \partial_j(v_j)$ for all i, j, and S is the coordinate ring of the open subset of sequences where a certain $k \times k$-subdeterminant is nonzero. So Ψ may be thought of as a "glueing together" of such a sequence to form an element in A_∞.

Example 5.9. The following shows that the map Ψ in the proof above is not S_∞'-equivariant, which justifies the detour via $\mathrm{Inc}(\mathbb{N})'$. Take $\mathbf{w}, \mathbf{w}'$ as in Example 4.1, and take $w = 0012$. The matrix in the proof above equals

$$\begin{bmatrix} x_{11} & x_{12} & x_{1002} \\ x_{21} & x_{22} & x_{2002} \\ x_{011} & x_{021} & x_{0012} \end{bmatrix}$$

and gives rise to

$$\Psi x_{0012} = (x_{11}x_{22} - x_{12}x_{21})^{-1} \cdot (x_{2002}(x_{11}x_{021} - x_{12}x_{011}) - x_{1002}(x_{21}x_{021} - x_{22}x_{011})).$$

On the other hand, take $w = 0021$, obtained from w by permuting positions 3 and 4. The matrix in the proof above equals

$$\begin{bmatrix} x_{11} & x_{12} & x_{1001} \\ x_{21} & x_{22} & x_{2001} \\ x_{012} & x_{022} & x_{0021} \end{bmatrix}$$

and gives rise to

$$\Psi x_{0021} = (x_{11}x_{22} - x_{12}x_{21})^{-1} \cdot (x_{2001}(x_{11}x_{022} - x_{12}x_{012}) - x_{100}(x_{21}x_{022} - x_{22}x_{012})),$$

which is *not* obtained from the expression above by permuting positions 3 and 4.

Now that Z' is S_∞'-Noetherian, Lemma 5.4 implies that the G_∞-space $(G_\infty \times Z')/S_\infty$ is G_∞-Noetherian. The map from this space to A_∞ sending (g, z') to gz' is G_∞-equivariant and continuous, and its image is the open set $Z \subseteq Y_\infty^{\leq k}$. Lemma 5.3 now implies that Z is S_∞-Noetherian. We conclude that, in addition to the closed subset $Y_\infty^{\leq k-1} \subseteq Y_\infty^{\leq k}$, also the open subsets $Z_1, \ldots, Z_N$ are S_∞-Noetherian. As mentioned before, this implies that $Y_\infty^{\leq k} = Y_\infty^{\leq k-1} \cup Z_1 \cup \ldots \cup Z_N$ is S_∞-Noetherian, as claimed in Theorem 5.6.

6. Proofs of main theorems

We are now in a position to prove our Main Theorems I, II, and III. We start with the latter.

Proof of Main Theorem III.. As $X_\infty^{\leq k}$ is a closed G_∞-stable subset of $Y_\infty^{\leq k}$, and as $Y_\infty^{\leq k}$ is a G_∞-Noetherian topological space (Theorem 5.6), $X_\infty^{\leq k}$ is cut out from $Y_\infty^{\leq k}$ by finitely many G_∞-orbits of equations. Moreover, $Y_\infty^{\leq k}$ itself is cut out from A_∞ by finitely many G_∞-orbits of Equations (Proposition 4.2), and hence the same is true for $X_\infty^{\leq k}$. □

In the proofs of Main Theorems I and II we will use inclusion maps

$$\tau_p : V^{\otimes p} \to V^{\otimes p+1},\ \omega \mapsto \omega \otimes e_0,$$

where e_0 is an element of V such that $x_0(e_0) = 1$. This map sends $X_p^{\leq k}$ into $X_{p+1}^{\leq k}$ and satisfies $\pi_p \circ \tau_p = \mathrm{id}_{V^{\otimes p}}$.

Proof of Main Theorem I.. By Lemma 2.3 it suffices to prove that for fixed $k \in \mathbb{N}$ there exists a $d \in \mathbb{N}$ such that for all $p \in \mathbb{N}$ the variety $X_p^{\leq k}$ of tensors of border rank at most k is defined in $V^{\otimes p}$ by polynomials of degree at most d. By Main Theorem III there exists a d such that $X_\infty^{\leq k}$ is defined in A_∞ by polynomials of degree at most d; we prove that the same d suffices in Main Theorem I. Indeed, suppose that all polynomials of degree at most d in the ideal of $X_p^{\leq k}$ vanish on a tensor $\omega \in V^{\otimes p}$. Let ω_∞ be the element of A_∞ obtained from ω by successively applying $\tau_p, \tau_{p+1}, \ldots$. More precisely, for any coordinate $x_w \in T_q,\ q \geq p$ we have $x_w(\omega_\infty) = x_w(\tau_{q-1}\tau_{q-2}\cdots\tau_p\omega)$, and this determines ω_∞. We claim that ω_∞ lies in $X_\infty^{\leq k}$. Indeed, otherwise some T_q contains a polynomial f of degree at most d that vanishes on $X_q^{\leq k}$ but not on ω_∞. Now q cannot be smaller than p, because then f vanishes on $X_p^{\leq k}$ but not on ω. But if $q \geq p$, then $f \circ \tau_{q-1} \circ \cdots \circ \tau_p$ is a polynomial in T_p of degree at most d that vanishes on $X_p^{\leq k}$ but not on ω. This contradicts the assumption on ω. □

The proof of Main Theorem II is slightly more involved.

Proof of Main Theorem II. By Lemma 2.3 it suffices to show that for fixed $k \in \mathbb{N}$ there exists a p_0 such that a tensor in $V^{\otimes p},\ p \geq p_0$ is of border rank at most k as soon as all its contractions along pure $(p - p_0)$-tensors to $V^{\otimes p_0}$ have border rank at most k. By Main Theorem III there exists a p_0 such that the G_∞-orbits of the equations of $X_{p_0}^{\leq k}$ define $X_\infty^{\leq k}$. We claim that this value of p_0 suffices for Main Theorem II, as well. Indeed, suppose that $\omega \in V^{\otimes p}$ has the property that all its contractions along pure tensors to $V^{\otimes p_0}$ lie in $X_{p_0}^{\leq k}$, and construct $\omega_\infty \in A_\infty$ as in the proof of Main Theorem I. We claim that ω_∞ lies in $X_\infty^{\leq k}$. For this we have to show that for each f in the ideal of $X_{p_0}^{\leq k}$ and each $g \in G_\infty$ the polynomial gf vanishes on ω_∞. Let $q \in \mathbb{N}$ be such that $g \in G_q$. By construction, $f \in T_{p_0}$ is identified with the function in T_q obtained by precomposing f with the contraction $V^{\otimes q} \to V^{\otimes p_0}$ along the pure tensor $x_0^{\otimes q-p_0}$ on the last $q - p_0$ factors. Hence gf is the same as contraction $V^{\otimes q} \to V^{\otimes p_0}$ along *some* pure tensor (in some of the factors), followed by $g'f$ for some $g' \in G_{p_0}$. Evaluating gf at the tensor ω_∞ is the same as evaluating it at

$$\omega \otimes (e_0)^{\otimes q-p},$$

and boils down to contracting some, say l, of the factors e_0 with a pure tensor in $(V^*)^{\otimes l}$, and $q-p_0-l$ of the remaining factors V with a pure tensor μ in $(V^*)^{\otimes q-p_0-l}$, and evaluating $g'f$ at the result. But this is the same thing as contracting ω with μ in $(V^*)^{\otimes q-p_0-l}$ to obtain a $\omega' \in V^{\otimes p-q+p_0+l}$ and evaluating $g'f$ at $\omega' \otimes e_0^{q-p-l}$. Now by assumption ω' lies in $X_{p-q+p_0+l}^{\leq k}$ (since $p-q+p_0+l \leq p_0$), and hence $\omega' \otimes e_0^{q-p-l}$ lies in $X_{p_0}^{\leq k}$. This proves that $g'f$ vanishes on it, so that gf vanishes on ω_∞, as claimed. Hence ω_∞ lies in $X_\infty^{\leq k}$. But the projection $A_\infty \to V^{\otimes p}$ sends ω_∞ to ω and $X_\infty^{\leq k}$ to $X_p^{\leq k}$. Hence $\omega \in X_p^{\leq k}$, as required. □

7. All equations?

In contrast with Main Theorem III, we do not believe that the full ideal of $X_\infty^{\leq k}$ is generated by finitely many G_∞-orbits; and in fact this fails already if $k=1$ where $X_\infty^{\leq 1} = Y_\infty^{\leq 1}$. Since it takes some space, the proof of this negative result is deferred to the appendix. To obtain finiteness results for the full ideal of $X_\infty^{\leq k}$ we propose to use further non-invertible symmetries, provided by the following monoid.

Definition 7.1. The *substitution monoid* $\mathrm{Subs}(\mathbb{N})$ consists of infinite sequences $\sigma = (\sigma(1), \sigma(2), \ldots)$ of finite, pairwise disjoint subsets of $\mathbb{N}$. The multiplication in this monoid is given by

$$(\sigma\pi)(p) = \bigcup_{q \in \pi_p} \sigma(q).$$

A straightforward computation shows that $\mathrm{Subs}(\mathbb{N})$ is indeed an associative monoid with unit element $1 = (\{1\}, \{2\}, \{3\}, \ldots)$. It has a natural left action on infinite words over $\{0, 1, \ldots, n-1\}$ with finite support: $\sigma w := w'$ where $w'(p) = 0$ if p does not lie in any of the $\sigma(q)$, and $w'(p) = w(q)$ if p lies in $\sigma(q)$. We let $\mathrm{Subs}(\mathbb{N})$ act on T_∞ by algebra homomorphisms determined by $\sigma x_w := x_{\sigma w}$. This gives rise to a *right* action of $\mathrm{Subs}(\mathbb{N})$ on A_∞. The following lemma explains our interest in this monoid of non-invertible symmetries.

Lemma 7.2. *The ideal of $X_\infty^{\leq k}$ is stable under the* $\mathrm{Subs}(\mathbb{N})$*-action.*

Proof. First, for pure tensors, a straightforward calculation shows that for $\mathbf{w} = (w_1, w_2)$ and $\mathbf{w}' = (w_1', w_2')$ pairs of words as before, we have $\sigma \det x[\mathbf{w}, \mathbf{w}'] = \det x[\sigma\mathbf{w}, \sigma\mathbf{w}']$ where σ acts component-wise on $\mathbf{w}$ and $\mathbf{w}'$. Hence $\mathrm{Subs}(\mathbb{N})$ stabilises $X_\infty^{\leq 1}$. Since $\mathrm{Subs}(\mathbb{N})$ acts linearly on A_∞, this means that $\mathrm{Subs}(\mathbb{N})$ also stabilises the set of sums of k elements of $X_\infty^{\leq 1}$. But then it also stabilises the Zariski-closure $X_\infty^{\leq k}$ of this set. □

We have the following conjecture.

Conjecture 7.3. *For every fixed k, the ideal of $X_\infty^{\leq k}$ is generated by finitely many* $\mathrm{Subs}(\mathbb{N})$*-orbits of equations.*

This conjecture certainly holds for $k=1$ and $k=2$: then the ideal is generated by subdeterminants of flattenings, which by the following lemma form a single $\mathrm{Subs}(\mathbb{N})$-orbit.

Lemma 7.4. *For fixed k, the $k \times k$-sub-determinants* $\det x[\mathbf{w}, \mathbf{w}']$ *in T_∞ form a single* $\mathrm{Subs}(\mathbb{N})$*-orbit.*

Proof. Let us write the elements of $\mathbf{w} = (w_1, w_2, \ldots, w_k)$ and $\mathbf{w}' = (w'_1, w'_2, \ldots, w'_k)$ in a $2k \times \mathbb{N}$-table $T_{\mathbf{w},\mathbf{w}'}$ with the entry at (i, j) equal to $w_i(j)$ if $i \leq k$ and equal to $w'_{i-k}(j)$ if $k < i \leq 2k$. Note that we require that each w_i has support disjoint from that of each w'_j, so that each column either starts with k zeroes or ends with k zeroes. Fix $\mathbf{u} = (u_1, \ldots, u_k)$ and $\mathbf{u}' = (u'_1, \ldots, u'_k)$ such that in the corresponding table $T_{\mathbf{u},\mathbf{u}'}$ every non-zero column starting or ending with k zeroes occurs exactly once (hence the number of non-zero columns is $2(n^k - 1)$). For general $\mathbf{w}, \mathbf{w}'$ as above, define $\sigma \in \mathrm{Subs}(\mathbb{N})$ as follows: $\sigma(j) := \emptyset$ if the j-th column of $T_{\mathbf{u},\mathbf{u}'}$ is zero, and

$$\sigma(j) := \{l \in \mathbb{N} \mid \text{ the } l\text{-th column of } T_{\mathbf{w},\mathbf{w}'} \text{ equals the } j\text{-th column of } T_{\mathbf{u},\mathbf{u}'}\}$$

otherwise. By construction we find $\sigma u_i = w_i$ and $\sigma u'_i = w'_i$ for all i, and therefore $\sigma \det x[\mathbf{u}, \mathbf{u}'] = \det x[\mathbf{w}, \mathbf{w}']$. □

We point out two interesting submonoids of $\mathrm{Subs}(\mathbb{N})$. First, the submonoid of $\mathrm{Subs}(\mathbb{N})$ consisting of σ for which each $\sigma(i)$ is non-empty. This submonoid acts by means of injective endomorphisms on T_∞, and satisfies a variant of the previous lemma with "a single orbit" replaced by "finitely many orbits". The same is true for an even smaller submonoid, denoted $\mathrm{Subs}_<(\mathbb{N})$ where we require $\sigma(i) \neq \emptyset$ for all i *and* $\max \sigma(1) < \max \sigma(2) < \ldots$—again at the cost of increasing the number of orbits. Using this submonoid we prove the following finiteness result.

Proposition 7.5. *For every natural number d, the polynomials in T_∞ of degree at most d form a Noetherian* $\mathrm{Subs}(\mathbb{N})$*-module, that is, every* $\mathrm{Subs}(\mathbb{N})$*-submodule is generated by a finite number of* $\mathrm{Subs}(\mathbb{N})$*-orbits.*

We prove the stronger statement, where $\mathrm{Subs}(\mathbb{N})$ is replaced by the smaller monoid $\mathrm{Subs}_<(\mathbb{N})$.

Proof. For any natural number m, define a partial order on infinite words over $\{0, 1, \ldots, m\}$ of finite support by $w \leq w'$ if and only if there exists a $\sigma \in \mathrm{Subs}_<(\mathbb{N})$ such that $\sigma w = w'$. By induction on m, we prove that this is a well-quasi-order in the sense of [20]. For $m = 0$ there is only one word, and $\leq$ is trivially a well-quasi-order. Suppose next that it is a well-quasi-order for $m - 1$, and consider a sequence $w_1, w_2, \ldots$ of words over $\{0, 1, \ldots, m\}$ of finite support. After passing to a subsequence we may assume that all w_a use exactly the same alphabet. If this alphabet has less than $m + 1$ elements, then by the induction hypothesis there exist $a < b$ such that $w_a \leq w_b$, and we are done. Hence we may assume that the common alphabet of all words is $\{0, 1, \ldots, m\}$. For each w_a, let s_a be the symbol in w_a whose *last* occurrence precedes the last occurrences of all other symbols, that is, the tail of w_a after the last occurrence of s_a uses an alphabet of cardinality $m - 1$. By passing to a subsequence we may assume that all s_a are equal, and without loss of generality that they are all equal to m.

Now there is a weaker partial order on words of finite support over the alphabet $\{0, \ldots, m\}$, defined by $w \leq' w'$ if and only if there exists a $\pi \in \mathrm{Inc}(\mathbb{N})$ such that the symbol at position j of w equals the symbol at position $\pi(j)$ of w'. This partial order *is* a well-quasi-order by Higman's lemma [18]. Hence after passing to a subsequence we may assume that $w_1 \leq' w_2 \leq' \ldots$.

Returning to our main argument, for each w_a let w'_a denote the word obtained from it by deleting the head up to and including the last occurrence of $s_a = m$, so w'_a is a word that uses exactly the alphabet $\{0, \ldots, m - 1\}$ (w'_a is indexed by

$1, 2, 3, \ldots$). By the induction hypothesis, there exist $a < b$ such that $w'_a \leq w'_b$. We claim that then also $w_a \leq w_b$. We construct $\sigma \in \mathrm{Subs}_<(\mathbb{N})$ witnessing this fact as follows. Let $\pi \in \mathrm{Inc}(\mathbb{N})$ be a witness of $w_a \leq' w_b$, i.e., for all $j \in \mathbb{N}$ we have $(w_a)(j) = (w_b)(\pi(j))$; let j_a, j_b be the positions of the last occurrences of m in w_a, w_b, respectively; and let $\gamma \in \mathrm{Subs}_<(\mathbb{N})$ be a witness of $w'_a \leq w'_b$. Then define σ by

$$\sigma(j) = \begin{cases} \{\pi(j)\} \text{ if } j < j_a, \\ \{j_b\} \cup \{i \in [j_b - 1] \mid w_b(i) = m \text{ and } i \notin \pi[j_a - 1]\} \text{ if } j = j_a, \\ \{j_b + i \mid i \in \gamma(j - j_a)\} \cup \{i \in [j_b - 1] \mid w_b(i) = w_a(j) \text{ and } i \notin \pi[j_a - 1]\} \\ \quad \text{if } j > j_a \text{ and } j - j_a \text{ is the first position of } w_a(j) \text{ in } w'_a, \text{ and} \\ \{j_b + i \mid i \in \gamma(j - j_a)\} \\ \quad \text{if } j > j_a \text{ and } j - j_a \text{ is not the first position of } w_a(j) \text{ in } w'_a. \end{cases}$$

A straightforward computation—see below for an example—shows that $\sigma w_a = w_b$, so that $w_a \leq w_b$, as required. This shows that $\leq$ is indeed a well-quasi-order.

We use this to prove that for every infinite sequence $u_1, u_2, \ldots$ of monomials in the x_w of degree at most d there exists $a < b$ and $\sigma \in \mathrm{Subs}_<(\mathbb{N})$ such that $\sigma u_a = u_b$. Since there exists a degree $e \leq d$ such that infinitely many of the u_a have degree equal to e, we may assume that all u_a have the same degree e. To each u_a we then associate an $e \times \mathbb{N}$ table T_a, whose rows record the words w corresponding to variables x_w occurring in u_a, including their multiplicities (in any order). The columns of T_a form a word of finite support over the alphabet $\{0, 1, \ldots, n-1\}^e$. Hence by the above, there exist $a < b$ and $\pi \in \mathrm{Subs}_<(\mathbb{N})$ such that $\pi T_a = T_b$. But then also $\pi u_a = u_b$, as claimed.

Now finally let W be any $\mathrm{Subs}_<(\mathbb{N})$-submodule of the space of polynomials in the x_w of degree at most d. Choose any well-order on monomials in the x_w that is preserved by the $\mathrm{Subs}_<(\mathbb{N})$—for instance, lexicographic with respect to the order on the variables defined by $x_w \geq x_{w'}$ if $\sum_{j=1}^{\infty} w(j) n^j \geq \sum_{j=1}^{\infty} w'(j) n^j$ (an inequality between the numbers represented by w and w' in the n-ary number system). By the above, there exist $w_1, \ldots, w_l \in W$ such that for all $w \in W \setminus \{0\}$ there exist $i \in [l]$ and $\pi \in \mathrm{Subs}_<(\mathbb{N})$ for which the leading monomial of πw_i (with respect to the well-order) equals the leading monomial of w. This implies that the $\mathrm{Subs}_<(\mathbb{N})$-orbits of $w_1, \ldots, w_l$ span W. $\square$

Example 7.6. Here is an example of the construction of σ in the proof above. Let $m = 2$ and

$$\begin{aligned} w_a &= 0010212011 & j_a &= 7 & w'_a &= 011 \\ w_b &= 0020102012001011 & j_b &= 10 & w'_b &= 001011 \\ \pi &= (1, 2, 5, 6, 7, 9, 10, 11, 13, 15, 17, 18, 19, \ldots) \\ \gamma &= (\{1, 2, 4\}, \{5\}, \{3, 6\}, \{7\}, \{8\}, \ldots) \end{aligned}$$

so that $\gamma w'_a = w'_b$ and $w_a(i) = w_b(\pi(i))$ for all i (of course γ and π with these properties are in general not unique). Then σ above equals

$$\sigma = (\{1\}, \{2\}, \{5\}, \{6\}, \{7\}, \{9\}, \{3, 10\}, \{4, 8, 11, 12, 14\}, \{15\}, \{13, 16\}, \{17\}, \{18\}, \ldots).$$

Note how all the gaps in $\mathrm{im}\, \pi$ are filled by the second half of σ.

We now have the following immediate corollary, which makes our conjecture more plausible. In fact, Andrew Snowden pointed out to us that his beautiful theory of Δ-modules [27] yields a similar equivalence.

Corollary 7.7. *Conjecture 7.3 is equivalent to the statement that the ideal of $X_\infty^{\leq k}$ is generated by polynomials of bounded degree.* □

Finally, we point out that the action of $\mathrm{Subs}_<(\mathbb{N})$ on the coordinate ring T_∞ allows for *equivariant Gröbner Basis* methods as in [5]. We conjecture that for a fixed value of k (and of $n > k$) Conjecture can in principle be proved by a finite equivariant Gröbner basis computation. It would be interesting to try and do so for the GSS-conjecture, even if the computer proof would not be nearly as elegant as Raicu's proof.

Appendix A. The ideal of $X_\infty^{\leq 1}$ is not finitely generated

As announced in Section 7, we will now show that the ideal of $X_\infty^{\leq 1}$ is not G_∞-finitely generated, at least in characteristic zero. We write T_∞^i for the space of elements of degree i in T_∞, and throughout K is algebraically closed of characteristic zero. Let $H_p \subset G_p$ be the connected component of 1 (ie. $H_p = \mathrm{GL}(V)^p$), and let $H_\infty \subseteq G_\infty$ be their union. Similarly T_∞ denotes the union of all D^p where D is the group of diagonal matrices in H_p, and finally, B_∞ is the union of all B^p with B, contrary to before, the group of *lower* triangular matrices in H. In order to use representation theory, we need to use lower triangular matrices to make sure that the embedding $T_p \mapsto T_{p+1}$ (given by tensoring with x_0) maps eigenvectors of B_p to eigenvectors for B_{p+1}. (In fact the union of upper triangular matrices in H_∞ has no eigenvectors at all in T_∞.) Next, let $X = X(D_\infty)$ be the character group of D_∞ which we may identify with $X(D)^{\mathbb{N}}$ (and recall that $X(D) = \mathbb{Z}^n$, generated by the weights of the e_i, which we will denote by ε_i). We call a H_∞-representation M a *lowest weight module* of weight $\lambda \in X$, if M is generated by an eigenvector of B_∞ that has D_∞-weight λ. For example, T_∞^1 is itself a lowest weight module with lowest weight $(\varepsilon_0, \varepsilon_0, \dots)$ and eigenvector x_0 for B_∞.

If M is any representation for H_∞, we write $M[k]$ for the representation obtained by shifting k-places. That is, $h = (h_1, h_2, \dots) \in H_\infty$ acts on m as $(h_{k+1}, h_{k+2}, \dots)m$. Similarly, for a weight $\lambda \in X$, we define $\lambda[k]$ to be the weight obtained from X by inserting k zeros in the beginning. If M is a lowest weight module with lowest weight λ, then so is $M[k]$ and its weight is $\lambda[k]$.

For the purposes of this appendix, an *ind-finite representation* W of H_∞ is a directed union $W = \bigcup_p W_p$, where $W_1 \subseteq W_2 \subseteq \dots$ are nested finite-dimensional vector spaces and W_p is a H_p-representation such that the inclusions $W_p \to W_{p+1}$ are H_p-equivariant, and moreover, any B^p-eigenvector is mapped to a B^{p+1}-eigenvector.

Lemma A.1. *Let W be an ind-finite H_∞-representation. Then W is semi-simple, i.e. generated by irreducible subrepresentations, and each simple submodule is a lowest weight module with a uniquely determined lowest weight.*

Proof. This is an immediate consequence of the same fact that W is ind-finite and the lemma is true for finite dimensional representations of H_p. We skip the details. □

Let $\lambda_0 = (2\varepsilon_0, 2\varepsilon_0, \dots) \in X$. It is the lowest weight of the span of squares of pure tensors in T^2_∞ (in the finite dimensional setting this would be called the Cartan-component). We denote this sub-representation by C.

For any lowest weight λ we define its *complexity* as the number of positions where it differs from λ_0. For all lowest weights in T^2_∞, this is finite. The group S_∞ acts on H_∞ by permuting factors and on X by permuting entries. Consequently, if $M \subseteq T^2_\infty$ is a lowest-weight module of weight λ, then πM is a lowest weight module of weight $\pi\lambda$; in particular, this S_∞-action preserves the complexity of these weights.

Now let $x_v x_w$ be any monomial of degree 2 and let M be the H_∞-module generated by $x_v x_w$.

Lemma A.2. *The complexity of every lowest weight appearing in M is at most the number of positions where the words v and w differ. In particular, it is bounded.*

Proof. Any lowest weight vector in M is in the image of a lowest weight vector under the multiplication map $m\colon T^1_\infty \otimes T^1_\infty \to T^2_\infty$ and in fact this vector may be assumed to be an element of the module generated by $x_v \otimes x_w$. Next, the claim is invariant under the S_∞-action, so we may assume that v, w differ precisely at the first k positions and then coincide.

The space T^1_∞ is a direct limit of the tensor products $(V^*)^{\otimes p}$; after reordering we get that $T^1_\infty \otimes T^1_\infty$ is isomorphic (as an H_∞-representation) to

$$(V^{*\otimes k} \otimes V^{*\otimes k}) \otimes (T^1_\infty[k] \otimes T^1_\infty[k])$$

where H_∞ acts by its first k entries (diagonally) on $V^{*\otimes k} \otimes V^{*\otimes k}$. With this notation in place, $x_v \otimes x_w$ has the form $(u_1 \otimes u_2) \otimes (r \otimes r)$ where $u_1, u_2 \in V^{*\otimes k}$ and r is a pure tensor in $T^1_\infty[k]$; hence this is also true for any H_∞-translate of $x_v \otimes x_w$. Here we call "pure tensor" an element that is the image of a pure tensor in one of the $V^{*\otimes p}$.

Note that $r \otimes r$ in $T^1_\infty[k] \otimes T^1_\infty[k]$ generates an irreducible H_∞-representation of weight $\lambda_0[k]$, since $r \otimes r$ is in the orbit of $x_0 \otimes x_0$, and thus, a representation isomorphic to $C[k]$ (under the map m). It follows that M is isomorphic to a sub-representation of $(V^{*\otimes k} \otimes V^{*\otimes k}) \otimes C[k]$. Any lowest weight vector here is the tensor product of one in $(V^{*\otimes k}) \otimes V^{*\otimes k})$ and the one in $C[k]$. From this it follows easily that any lowest weight in the module generated by $x_v \otimes x_w$ (and hence M) has the form $\omega + \lambda_0[k]$ where ω is a lowest weight of $V^{*\otimes k} \otimes V^{*\otimes k}$ (and hence corresponds to just k weights of D and all zeros afterwards). It follows that the complexity is at most k as claimed. □

The crucial observation is now the following lemma.

Lemma A.3. *The weights of the lowest weight modules in T^2_∞ have arbitrarily high complexity. In other words, for each integer N, there exits a lowest weight of complexity at least N.*

Proof. Note that T^2_∞ is equivariantly isomorphic to the second symmetric power $S^2(T^1_\infty)$. For any $k > 0$, the canonical map $S^2(V^{*\otimes k}) \otimes C[k] \to T^2_\infty$ is an equivariant embedding where H_∞ acts on $S^2(V^{*\otimes k})$ by means of the projection $H_\infty \to H_k$.

Standard representation theory for H_k shows that $S^2(V^{*\otimes k})$ contains an H_k-sub-representation isomorphic to $(\bigwedge^2 V^*)^{\otimes k}$ whenever k is even. To see this, we observe that on on $W = V^{*\otimes k} \otimes V^{*\otimes k}$ the group S_{2k} acts by permuting the tensor factors. This allows us to identify $S^2(V^{*\otimes k})$ with the symmetric tensors in W

(those fixed by the product of k-transpositions $(1\,k+1)(2\,k+2)\cdots(k-1\,2k)$). Then $(\bigwedge^2 V^*)^{\otimes k}$ is isomorphic to the subrepresentation given by those tensors w in W that satisfy $(i\,k+i)w = -w$ for all i. If k is even, any such tensor is symmetric. The lowest weight for B^k of this representation is $(\varepsilon_0+\varepsilon_1,\ldots,\varepsilon_0+\varepsilon_1)$ (k entries). So $(\bigwedge^2 V^*)^{\otimes k} \otimes C[k]$ is an irreducible representation of T^2_∞ and its lowest weight is

$$(\underbrace{\varepsilon_0+\varepsilon_1,\varepsilon_0+\varepsilon_1,\ldots,\varepsilon_0+\varepsilon_1}_{k},2\varepsilon_0,2\varepsilon_0,\ldots).$$

The complexity of this weight is k. So for any N, there is a lowest weight of complexity N or $N+1$. □

As a corollary we obtain the following result:

Proposition A.4. *T^2_∞, as a G_∞-module, is not finitely generated. Moreover, the ideal of $X^{\leq 1}_\infty = Y^{\leq 1}_\infty$ is not G_∞-finitely generated.*

Proof. The G_∞-module generated by finitely many elements of T^2_∞ will be contained in the module generated by all monomials $x_v x_w$ appearing in any of the generators. Since S_∞ preserves the complexity of lowest weights, any appearing irreducible H_∞-module will have a lowest weight of complexity at most equal to the highest number of places where the two words v, w differ in any of the generating monomials by Lemma A.2. Since T^2_∞ contains lowest weight module of arbitrary complexity, this proves the first claim.

As for the second, if the ideal I of $X^{\leq 1}_\infty$ were G_∞-finitely generated it could be generated by finitely many homogeneous elements of degree 2. It therefore suffices to show that $I \cap T^2_\infty$ contains lowest weight modules of arbitrarily high complexity. In fact, as in the finite dimensional case, every lowest weight module other than C will be contained in the ideal: if f is a lowest weight vector of weight λ, say. Then there is p such that $f \in T_p$ and hence $f \in I$ if and only if f vanishes on $X^{\leq 1}_p$. Set $v := e_0^{\otimes p}$, the highest weight vector in $V^{\otimes p}$. Since $\lambda \neq 2\varepsilon_0$, $f \neq x_0^2$ and so $f(v) = 0$. But then also $f(bv) = \lambda(b)^{-1} f(v) = 0$ for all $b \in B^p$. Since $B^p v$ is open and dense in $X^{\leq 1}_p$, it follows that $f \in I$ and we are done.

In fact, this shows that as in the case of p finite, the ideal spans a complement to C. □

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