# Finiteness theorems and algorithms for permutation invariant chains of Laurent lattice ideals

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## 1. Introduction

In commutative algebra, finiteness plays a significant role both theoretically and computationally. An important example is Hilbert's basis theorem, which states that any ideal $I \subseteq R$ in a polynomial ring $R = \mathbb{C}[x_1, \ldots, x_n]$ over the complex numbers $\mathbb{C}$ (or more generally, over any field $\mathbb{K}$) has a finite set of generators $G = \{g_1, \ldots, g_m\}$:

$$I = \langle G \rangle_R := g_1 R + \cdots + g_m R.$$

In other words, $\mathbb{C}[x_1, \ldots, x_n]$ is a *Noetherian* ring. Equivalently, any ascending chain of ideals $I_1 \subseteq I_2 \subseteq \cdots$ in $\mathbb{C}[x_1, \ldots, x_n]$ stabilizes (i.e., there exists an $N$ such that $I_N = I_{N+1} = \cdots$). This result has many applications in the algebraic theory of polynomial rings (e.g. the existence of finite resolutions (Eisenbud, 1995, p. 340)), but it is also a fundamental fact underlying computational algebraic geometry (e.g. termination of Buchberger's algorithm in the theory of Gröbner bases (Cox et al., 2007, p. 90)).

In many contexts, however, finiteness is observed even though Hilbert's basis theorem does not directly apply. A motivating example is the (non-Noetherian) ring $R = \mathbb{C}[x_1, x_2, \ldots]$ of polynomials in an *infinite*

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number of indeterminates $X = \{x_1, x_2, \ldots\}$, equipped with a permutation action on indices. More precisely, the *symmetric group* $\mathfrak{S}_{\mathbb{P}}$ of all permutations of $\mathbb{P} := \{1, 2, \ldots\}$ acts naturally on $R$ via:

$$\sigma f(x_{s_1}, \ldots, x_{s_\ell}) := f(x_{\sigma(s_1)}, \ldots, x_{\sigma(s_\ell)}), \quad \sigma \in \mathfrak{S}_{\mathbb{P}},\ f \in R. \tag{1}$$

Although many ideals in the ring $R$ are not finitely generated, an important subclass still admit finite presentations. Call an ideal $I$ *permutation-invariant* if it is fixed under the action of $\mathfrak{S}_{\mathbb{P}}$:

$$\mathfrak{S}_{\mathbb{P}} I := \{\sigma f : \sigma \in \mathfrak{S}_{\mathbb{P}}, f \in I\} = I.$$

It is known that for every such permutation-invariant $I \subseteq R$, there is a finite set of generators $G = \{g_1, \ldots, g_m\} \subset I$ giving it a presentation of the form:

$$I = \langle \mathfrak{S}_{\mathbb{P}} G \rangle_R.$$

As a simple example, the ideal $M \subset \mathbb{C}[x_1, x_2, \ldots]$ of polynomials without constant term has the finite presentation $M = \langle \mathfrak{S}_{\mathbb{P}} x_1 \rangle_{\mathbb{C}[x_1, x_2, \ldots]}$ even though it is not finitely generated.

The above finiteness property for the ring $\mathbb{C}[x_1, x_2, \ldots]$ was first discovered by Cohen in the context of group theory Cohen (1967), but seems to have gone unnoticed until its independent rediscovery recently by the authors of Aschenbrenner and Hillar (2007). Generalizations and extensions of this result have since been applied to unify several finiteness results in algebraic statistics Hillar and Sullivant (2011) as well as help prove open conjectures in that field (notably, the independent set conjecture Hoşten and Sullivant (2007); Hillar and Sullivant (2011), finiteness for the $k$-factor model Draisma (2010), and, more recently, that bounded-rank tensors are defined in bounded degree Draisma and Kuttler (2011)).

In this paper, we derive new finiteness properties for certain classes of polynomial ideals that are invariant under a symmetric group action. Motivated by an algebraic question of Dress and Sturmfels in chemistry (Aschenbrenner and Hillar, 2007, Section 5), we prove that invariant chains of lattice ideals stabilize up to monomial localization (see Theorem 3 below). This general result gives evidence for Conjecture 5.10 in Aschenbrenner and Hillar (2007) (stated as Conjecture 25 below). Moreover, for the specific chains studied there (in (Aschenbrenner and Hillar, 2007, Section 5.1)), we present an algorithm for explicitly constructing these generators (see Theorem 7 and Algorithm 1 below). Our results also have potential implications for algebraic statistics. To prepare for the precise statements, however, we need to introduce some notation.

Given a set $S$, let $\mathfrak{S}_S$ denote the group of permutations of $S$. We shall focus our attention primarily on the sets $S = [n] := \{1, 2, \ldots, n\}$ and $S = \mathbb{P} := \{1, 2, \ldots\}$, the set of positive integers. In these cases, we write $\mathfrak{S}_n$ and $\mathfrak{S}_{\mathbb{P}}$, respectively, for the symmetric groups.[2] Given a positive integer $k \geq 1$, let $[S]^k$ be the set of all ordered $k$-tuples $u = (u_1, \ldots, u_k)$, and let $\langle S \rangle^k$ be the subset of those with pairwise distinct $u_1, \ldots, u_k$. When $S = [n]$, we write $[n]^k$ and $\langle n \rangle^k$ for $[S]^k$ and $\langle S \rangle^k$, respectively.

The symmetric group $\mathfrak{S}_S$ acts on $[S]^k$ naturally via

$$\sigma(u_1, \ldots, u_k) := (\sigma(u_1), \ldots, \sigma(u_k)), \quad \text{for } \sigma \in \mathfrak{S}_S; \tag{2}$$

and this action restricts to an action on $\langle S \rangle^k$.

Write $X_S := \{x_s : s \in S\}$ for the set of indeterminates indexed by a set $S$, and let $\mathbb{K}[X_S]$ denote the polynomial ring with coefficients in a field $\mathbb{K}$ (e.g., $\mathbb{C}$ or $\mathbb{R}$) and indeterminates $X_S$. The action of any group $\mathfrak{S}$ on $S$ induces an action on $X_S$, which we extend to an action on $\mathbb{K}[X_S]$ as in (1).

We are interested here in the highly structured $\mathfrak{S}$-*invariant* ideals of $\mathbb{K}[X_S]$ (simply called *invariant* ideals below if the group $\mathfrak{S}$ is understood); these are ideals $I \subseteq \mathbb{K}[X_S]$ for which $\mathfrak{S} I = I$.[3] Guised in various

[2] We embed $\mathfrak{S}_n$ into $\mathfrak{S}_m$ for $n \leq m$ in the natural way.

[3] In the language of Aschenbrenner and Hillar (2009), invariant ideals are also the $\mathbb{K}[X_S] * \mathfrak{S}$-submodules of $\mathbb{K}[X_S]$, where $\mathbb{K}[X_S] * \mathfrak{S}$ is the skew group ring associated to $\mathbb{K}[X_S]$ and $\mathfrak{S}$.

forms, invariant ideals of polynomial rings arise naturally in many contexts. For instance, they appear in applications of polynomial algebra to chemistry Ruch et al. (1967); Aschenbrenner and Hillar (2007); Draisma (2010), finiteness of statistical models in algebraic statistics and toric algebra Santos and Sturmfels (2003); Sturmfels and Sullivant (2005); Kuo (2006); Drton et al. (2007); Hoşten and Sullivant (2007); Aschenbrenner and Hillar (2007); Scala and Levandovskyy (2009); Brouwer and Draisma (2011); Aoki et al. (2010); Hara and Takemura (2010a,b); Draisma (2010); Snowden (2010); Draisma and Kuttler (2011); Haws et al. (2011); Hillar and Sullivant (2011), and the algebra of tensor rank Draisma and Kuttler (2011).

Given an ideal $I \subseteq R$ of a polynomial ring $R = \mathbb{K}[X_S]$, let $I^{\pm}$ denote the localization $I \hookrightarrow I^{\pm}$ of $I$ with respect to the multiplicative set of monomials of $R$ (including the monomial 1). In particular, $R^{\pm}$ is the ring of *Laurent polynomials* in the indeterminates of $R$, and any ideal $I \subseteq R$ lifts to an ideal $I^{\pm} \subseteq R^{\pm}$, which we call a *Laurent ideal*. In simple terms, the ideal $I^{\pm}$ consists of elements of the form $gh^{-1}$ where $g \in I$ and $h$ is a monomial of $R$ (see e.g. Eisenbud (1995)). An action of any group $\mathfrak{S}$ on $R$ extends naturally to an action on $R^{\pm}$: for $\sigma \in \mathfrak{S}$ and $gh^{-1} \in R^{\pm}$, we can define $\sigma(gh^{-1}) := \sigma(g)\sigma(h)^{-1} \in R^{\pm}$. In this way, any $\mathfrak{S}$-invariant ideal $I$ lifts to an $\mathfrak{S}$-invariant ideal $I^{\pm} \subseteq R^{\pm}$. As above, for a subset $G \subseteq R$, we let $\langle G \rangle_R$ denote the ideal generated by $G$ over $R$.

In this paper, we work with localized (Laurent) ideals because they allow us to prove very general finiteness theorems in cases where no other known techniques are able to produce such results.

In what follows, we are primarily concerned with the polynomial rings (and their localizations):

$$\mathcal{R}_n := \mathbb{K}[X_{[n]^k}], \quad \mathcal{R}_{\mathbb{P}} := \mathbb{K}[X_{[\mathbb{P}]^k}] = \bigcup_{n \in \mathbb{P}} \mathcal{R}_n; \quad R_n := \mathbb{K}[X_{\langle n \rangle^k}], \quad R_{\mathbb{P}} = \bigcup_{n \in \mathbb{P}} R_n; \quad T_n := \mathbb{K}[t_1, \ldots, t_n], \tag{3}$$

in which $k$ is a fixed positive integer. Since the set $[n]^k$ sits naturally inside $[m]^k$ for $n \leq m$, we have an embedding of rings $\mathcal{R}_n \subseteq \mathcal{R}_m$; similarly, $R_n \subseteq R_m$. Our main objects of interest will be *ascending chains* $I_\circ$ of ideals $I_n \subseteq \mathcal{R}_n$ (simply called *chains* below):

$$I_\circ := I_1 \subseteq I_2 \subseteq \cdots . \tag{4}$$

In general, a chain of ideals (4) will not stabilize in the sense of Hilbert's basis theorem because the number of indeterminates in $\mathcal{R}_n$ increases with $n$. However, if the ideals comprising a chain are $\mathfrak{S}$-invariant, we may still be able to find an $N$ such that all the ideals $I_N, I_{N+1}, \ldots$ are the same "up to symmetry". We now make these notions precise (with corresponding definitions for Laurent ideals and the rings $R_n$).

**Definition 1.** A chain $I_\circ := I_1 \subseteq I_2 \subseteq \cdots$ of ideals $I_n \subseteq \mathcal{R}_n$ is an *invariant chain* if

$$\mathfrak{S}_m I_n \subseteq I_m, \quad \text{for all } m \geq n.$$

**Definition 2.** An invariant chain $I_\circ$ *stabilizes* if there is an integer $N$ such that

$$\langle \mathfrak{S}_m I_N \rangle_{\mathcal{R}_m} = I_m, \quad \text{for all } m \geq N.$$

Such an $N$ is a *stabilization bound* for the chain, and generators for $I_N$ are called *generators* for $I_\circ$.

In words, an invariant chain stabilizes when its fundamental structure is contained in a finite number of ideals comprising the chain. When $k = 1$, every invariant chain of ideals in $\{\mathcal{R}_n\}_{n \in \mathbb{P}}$ stabilizes Aschenbrenner and Hillar (2007); Hillar and Sullivant (2011). However, the corresponding fact fails to hold for $k \geq 2$ (e.g., see (Aschenbrenner and Hillar, 2007, Proposition 5.2) or (Hillar and Sullivant, 2011, Example 3.8)), and more refined methods are required to detect chain stabilization.

In many applications, the invariant chains consist of toric ideals, so we shall focus our attention here on the slightly more general class of lattice ideals (see Section 3 for definitions). For instance, the independent set conjecture in algebraic statistics (Hoşten and Sullivant, 2007, Conj. 4.6) concerns stabilization for a large family of toric chains.

Our first main result asserts that invariant chains of lattice ideals stabilize locally, and it is similar to a chain stabilization result used in a recent proof Hillar and Sullivant (2011) of the independent set conjecture. We prove this result in Section 3 using ideas from order theory as described in Section 2.

**Theorem 3.** *Every invariant chain $I^{\pm}_{\circ} := I^{\pm}_1 \subseteq I^{\pm}_2 \subseteq \cdots$ of Laurent lattice ideals $I^{\pm}_n \subseteq \mathcal{R}^{\pm}_n$ (resp. $I^{\pm}_n \subseteq R^{\pm}_n$) stabilizes.*

Although this result is quite general, our proof is nonconstructive. In applications, however, one usually desires bounds on chain stabilization. Our second main result restricts to the rings $R_n$ and provides a stabilization bound for the special case of Laurent toric chains induced by a monomial (Aschenbrenner and Hillar, 2007, Section 5.2), which we study in Section 4. These toric ideals appear in applications to algebraic statistics García-García et al. (2010); Hillar and Sullivant (2011) and voting theory Daugherty et al. (2009).

**Theorem 4.** *Let $f \in \mathbb{K}[y_1, \dots, y_k]$ be a monomial of degree $d$. For each $n \geq k$, consider the (toric) map:*

$$\phi_n : R_n \to T_n, \quad x_{(u_1,\dots,u_k)} \mapsto f(t_{u_1}, \dots, t_{u_k}).$$

*Let $I_n = \ker \phi_n$, and let $I^{\pm}_n$ be the corresponding Laurent ideal. Then $N = 2d$ is a stabilization bound for the invariant chain $I^{\pm}_{\circ} = I^{\pm}_k \subseteq I^{\pm}_{k+1} \subseteq \cdots$ of Laurent ideals.*

**Example 5.** Let $k = 2$ and suppose that $f = y_1^2 y_2 \in \mathbb{K}[y_1, y_2]$. For every $n \geq 2$, the map $\phi_n$ is defined by $\phi_n(x_{(i,j)}) = t_i^2 t_j$ for $(i,j) \in \langle n \rangle^2$. Theorem 4 asserts that if $N = 2 \cdot \deg(f) = 6$, then the generators of $I^{\pm}_6$ form a generating set for the whole chain $I^{\pm}_{\circ}$ up to the action of the symmetric group $\mathfrak{S}_m$; that is, for all $m \geq 6$, we have $\langle \mathfrak{S}_m I^{\pm}_6 \rangle_{R_m} = I^{\pm}_m$. For instance, when $m \geq 9$, we observe that $x_{(3,9)}x_{(7,9)} - x_{(3,7)}x_{(9,7)} \in I_m$ (thus, in $I^{\pm}_m$) since

$$\phi_n(x_{(3,9)}x_{(7,9)}) = t_3^2 t_7^2 t_9^2 = \phi_n(x_{(3,7)}x_{(9,7)}).$$

Thus, by Theorem 4, there exist permutations $\sigma_1, \dots, \sigma_r \in \mathfrak{S}_m$, elements $g_1, \dots, g_r \in I^{\pm}_6$, and polynomials $h_1, \dots, h_r \in R^{\pm}_m$, such that $x_{(3,9)}x_{(7,9)} - x_{(3,7)}x_{(9,7)} = h_1\sigma_1 g_1 + \cdots + h_r \sigma_r g_r$. Theorem 7 below, provides a method for finding such polynomial combinations in general; in this case, one possibility is $r = 1$, $h_1 = 1$, $\sigma_1 = (1\,3\,9\,2\,7) \in \mathfrak{S}_m$, and $g_1 = x_{(1,3)}x_{(2,3)} - x_{(1,2)}x_{(3,2)} \in I^{\pm}_6$. For more details on this example (including an explicit set of generators for $I^{\pm}_6$), see Section 4. □

**Remark 6.** Rather surprisingly, it is still an open question whether the (non-Laurent) toric chain $I_{\circ}$ stabilizes in Example 5, and more generally, for any monomial $f$ that is not square-free. Section 6 discusses more open problems of this nature.

In the development of the proof of Theorem 4, we also found an algorithm for computing these generators.

**Theorem 7** (Algorithm 1)**.** *There is an effective algorithm to compute a finite set of generators for the Laurent chains $I^{\pm}_{\circ}$ in Theorem 4.*

The first step of the algorithm in Theorem 7 is to embed a toric ideal into a Veronese ideal in a larger polynomial ring and use the fact that the latter is generated by quadratic binomials. A second procedure replaces the extra indeterminates of the larger ring by special quotients of monomials involving only indeterminates of the original polynomial ring. In turn, this reduces to an integer programming problem, which we solve explicitly. The following example illustrates some of the main ideas involved.

**Example 8.** (Continuing Example 5). Consider the polynomial rings $R'_n := R_n[x_{(1,2,3)}]$ in an extra indeterminate $x_{(1,2,3)}$, and extend $\phi_n$ to a map $\phi'_n : R'_n \to T_n$ by setting $\phi'_n(x_{(1,2,3)}) = t_1 t_2 t_3$. Notice that if $h \in I_n$, then $h \in \ker \phi'_n$, and also that

$$\phi'_n(x^2_{(1,2,3)}) = \phi'_n(x_{(1,3)}x_{(2,3)}) = \phi'_n(x_{(1,2)}x_{(3,2)}) = t_1^2 t_2^2 t_3^2.$$

Thus, $p_1 := x_{(1,3)}x_{(2,3)} - x_{(1,2,3)}^2$ and $p_2 := x_{(1,2)}x_{(3,2)} - x_{(1,2,3)}^2$ lie in $\ker \phi'_n$ (for $n \geq 3$). Consider any generating set for $\ker \phi'_n$ which contains $p_1, p_2$; then, each $g \in I_n$ can be expressed in terms of these generators. For instance,

$$g = x_{(1,3)}x_{(2,3)} - x_{(1,2)}x_{(3,2)} = (x_{(1,3)}x_{(2,3)} - x_{(1,2,3)}^2) - (x_{(1,2)}x_{(3,2)} - x_{(1,2,3)}^2) \in \ker \phi'_n.$$

Next, notice that

$$\phi'_n(x_{(1,2,3)}) = t_1t_2t_3 = \frac{\phi_n(x_{(1,2)})\phi_n(x_{(3,1)})}{\phi_n(x_{(1,3)})} = \phi_n\left(\frac{x_{(1,2)}x_{(3,1)}}{x_{(1,3)}}\right). \tag{5}$$

Therefore, if we replace $x_{(1,2,3)}$ by $\frac{x_{(1,2)}x_{(3,1)}}{x_{(1,3)}}$ in the two generators $p_1$ and $p_2$ above, we obtain two elements $\hat{p}_1$, $\hat{p}_2 \in I_n^{\pm}$ which also generate $g$. More generally, if we can find a finite set of generators for the chain of ideals $\ker \phi'_n$, then we would have generators for the chain of ideals $I_n$ up to monomial inversion.

Identity (5) was discovered by solving the following integer programming problem (described more fully in Example 28). The exponent vector of $t_1t_2t_3$ is $u = (1, 1, 1, 0, \ldots, 0) \in \mathbb{Z}^n$ and for any $(i, j) \in \langle n \rangle^2$, the exponent vector of $\phi_n(x_{(i,j)}) = t_i^2 t_j$ is

$$w_{i,j} := (0, \ldots, 0, 2, 0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^n,$$

in which the nonzero components of $w_{i,j}$ are the $i$th and $j$th with respective values 2 and 1. To find an expression such as (5), we needed to write $u$ as an integer linear combination of the vectors $w_{i,j}$ (this is done in general in Lemma 27). □

The most recent finiteness result along the lines of Theorems 3 and 4 can be found in the work of Draisma and Kuttler Draisma and Kuttler (2011). There, they prove set-theoretically that for any fixed positive integer $r$, there exists $d \in \mathbb{N}$ such that for all $p \in \mathbb{N}$, the set of $p$-tensors (elements of $V_1 \otimes \cdots \otimes V_p$, where each $V_i$ is a finite dimensional $\mathbb{K}$-vector space) of border rank at most $r$ are defined by the vanishing of finitely many polynomials of degree at most $d$ (when $r = 1$ these polynomials define the toric ideals). The authors of Draisma and Kuttler (2011) also realized the ideals defined by these polynomial equations as invariant chains under the action of $\mathfrak{S}_{\mathbb{P}}$, and they conjectured (Draisma and Kuttler, 2011, Conjecture 7.3) stabilization. They also provide a proof for the cases $r = 1$ and $r = 2$ (Draisma and Kuttler, 2011, Lemma 7.4), although the case $r = 1$ was first proved by Snowden in Snowden (2010). The results of Draisma and Kuttler (2011) extend those of Landsberg and Manivel in Landsberg and Manivel (2004), where they show set-theoretically that $p$-tensors of rank at most 2 are defined by polynomials of degree 3 (the $(3 \times 3)$-subdeterminants of all the *flattenings*) regardless of the dimension of the tensor. We note that an ideal-theoretic proof of this last fact was recently discovered by Raicu Raicu (2010).

While the general problem of deciding which chains of ideals stabilize seems difficult, it is possible that every invariant chain of (non-Laurent) lattice or toric ideals stabilizes, and Theorem 3 provides evidence. However, even for the special case studied here of a toric chain induced by a monomial, this is not known (Aschenbrenner and Hillar, 2007, Conjecture 5.10) and appears to be a difficult problem (although it is true for square-free monomials (Aschenbrenner and Hillar, 2007, Theorem 5.7)). We pose the following open question.

**Problem 9.** Does every invariant chain of lattice ideals (resp. toric ideals) stabilize?

The outline of this paper is as follows. In Section 2, we introduce the order theory required for proving Laurent lattice stabilization (Theorem 3) in Section 3. Next, Section 4 contains a proof of Theorem 4 using some ideas from toric algebra and integer programming. Also found there is an another approach to finding Laurent chain generators in Theorem 4 (e.g., the generators alluded to in Example 5) which can produce smaller generating sets than those produced by Algorithm 1. Section 5 contains a discussion of Theorem 7 and Algorithm 1. Finally, in Section 6 we present some open problems and conjectures arising from our computational investigations.

## 2. Nice Orderings

In this section, we explain the ideas from the theory of partial orderings that are needed to prove Theorem 3. A *well-partial-ordering* $\leq$ on a set $S$ is a partial order such that (1) there are no infinite antichains and (2) there are no infinite strictly decreasing sequences. One can check that this naturally generalizes the notion of "well-ordering" to orders $\leq$ which are not total.

Let $\mathfrak{S}$ be a group acting on a set $S$ (a $\mathfrak{S}$-*set*), and suppose that $\leq$ is a well-ordering of $S$. For $s \in S$ and $\sigma \in \mathfrak{S}$, let $s_< := \{t \in S : t < s\}$ and $\sigma s_< := \{\sigma t : t < s\}$. We define a partial ordering $\preceq$ on $S$ as follows:

$$s \preceq t \quad :\Longleftrightarrow \quad s \leq t \text{ and there exists } \sigma \in \mathfrak{S} \text{ such that } \sigma s = t \text{ and } \sigma \cdot s_< \subseteq t_<. \tag{6}$$

A group element $\sigma \in \mathfrak{S}$ verifying (6) is called a *witness* of the relation $s \preceq t$. An example of this construction can be found in Example 10.

Call the well-ordering $\leq$ of $S$ a *nice* ordering if $\preceq$ is a well-partial-ordering. Many naturally occurring $\mathfrak{S}$-sets have nice orderings. For instance, the set of $k$-element subsets of $\mathbb{P}$ with the natural action of $\mathfrak{S} = \mathfrak{S}_{\mathbb{P}}$ has a nice ordering Ahlbrandt and Ziegler (1984). Camina and Evans studied the ring-theoretic consequences of nice orderings in Camina and Evans (1991), inspired by the ideas in Ahlbrandt and Ziegler (1984). They showed that if $S$ has a nice ordering, then the $\mathbb{K}[\mathfrak{S}]$-module $\mathbb{K}S$ is Noetherian over the group ring $\mathbb{K}[\mathfrak{S}]$ for any field $\mathbb{K}$ (Camina and Evans, 1991, Theorem 2.4). We shall prove that $[\mathbb{P}]^k$ also has a nice ordering; however, our application (Theorem 3) requires a more refined version of this statement. This refinement is given by Theorem 19 below. Before proving this theorem, we first define a nice ordering of $[\mathbb{P}]^k$ with special properties.

Consider $\mathfrak{S}_{\mathbb{P}}$ acting on $[\mathbb{P}]^k$ as described in (2). We first give a total well-ordering $\leq_{dlex}$ on $[\mathbb{P}]^k$ as follows. Given $w = (w_1, \ldots, w_k) \in [\mathbb{P}]^k$, set $|w|_\infty := \max\{w_1, \ldots, w_k\}$. Define the *degree lexicographic* total ordering on $[\mathbb{P}]^k$ by

$$v \leq_{dlex} w \quad :\Longleftrightarrow \quad |v|_\infty < |w|_\infty \text{ or } |v|_\infty = |w|_\infty \text{ and } v <_{lex} w. \tag{7}$$

Here, $<_{lex}$ is the natural *lexicographic* ordering of elements of $[\mathbb{P}]^k$ given by $(u_1, \ldots, u_k) <_{lex} (w_1, \ldots, w_k) :\Longleftrightarrow u_1 = w_1, \ldots, u_{r-1} = w_{r-1}$ and $u_r < w_r$ for some $r \in [k]$.

Notice that for every $w \in [\mathbb{P}]^k$ there are only finitely many $v \in [\mathbb{P}]^k$ such that $v <_{dlex} w$; hence, $\leq_{dlex}$ is a well-ordering of $[\mathbb{P}]^k$. The well-ordering $\leq_{dlex}$ induces the partial order $\preceq_{dlex}$ as in (6).

**Example 10.** With the above definition of $\leq_{dlex}$ for $[\mathbb{P}]^2$, we have $(2,3) \leq_{dlex} (2,4)$ and $(2,3) \leq_{dlex} (3,1)$. Moreover, when $[\mathbb{P}]^2$ is equipped with the action of $\mathfrak{S}_{\mathbb{P}}$, we claim that $(2,3) \preceq_{dlex} (2,4)$. Represent the elements of $\mathfrak{S}_{\mathbb{P}}$ in cyclic notation so that $(3\,4) \cdot (2,3) = (2,4)$, and observe that

$$\begin{aligned}
(3\,4) \cdot (2,3)_{<_{dlex}} &= (3\,4) \cdot \{(1,3),(2,1),(2,2),(1,2)\} \\
&= \{(1,4),(2,1),(2,2),(1,2)\} \\
&\subseteq \{(1,1),(1,4),(3,1),(1,3),(2,3),(3,2),(3,3),(2,1),(2,2),(1,2)\} \\
&= (2,4)_{<_{dlex}}.
\end{aligned}$$

On the other hand, we have $(2,3) \not\preceq_{dlex} (3,1)$. To see this, let $\sigma \in \mathfrak{S}_{\mathbb{P}}$ be such that $\sigma \cdot (2,3) = (3,1)$; thus $\sigma(1) \geq 2$. Notice that for $(2,1) \in (2,3)_{<_{dlex}}$, we have $\sigma \cdot (2,1) = (\sigma(2), \sigma(1)) = (3, \sigma(1))$. It follows that $(3,2) \leq_{dlex} \sigma \cdot (2,1)$. Since

$$(3,1)_{<_{dlex}} = \{(2,3),(1,3),(2,1),(1,2),(1,1),(2,2)\},$$

we see that $(3,2) \notin (3,1)_{<_{dlex}}$; therefore, $\sigma \cdot (2,3)_{<_{dlex}} \not\subseteq (3,1)_{<_{dlex}}$. $\square$

Although not needed for our main result, a solution to the following problem would likely be useful in converting the methods of this section into computational tools.

**Problem 11.** Give a computationally efficient criteria for determining if $u \preceq_{dlex} v$ for $u, v \in [\mathbb{P}]^k$.

One may also ask the following open-ended problem.

**Problem 12.** Let $S$ be an $\mathfrak{S}$-set. Characterize those total well-orderings $\leq$ which are nice.

We are now in position to show that the ordering $\leq_{dlex}$ is nice.

**Proposition 13.** *The ordering $\preceq_{dlex}$ of $[\mathbb{P}]^k$ is a well-partial-ordering.*

The proof of this proposition uses a special case of a result of Higman Higman (1952); Nash-Williams (1963), which we state in the following lemma. Recall that a *strictly increasing* map $\varphi : [m] \to [n]$ satisfies $\varphi(i) < \varphi(i+1)$ for all $i$.

**Lemma 14** (Higman (1952))**.** *Let $\Sigma$ be a finite set. The following ordering $\leq_H$ on the set $\Sigma^*$ of all finite sequences of elements of $\Sigma$ is a well-partial-ordering:*

$$(x_1, \ldots, x_m) \leq_H (y_1, \ldots, y_n) \quad :\Longleftrightarrow \quad \left\{ \begin{array}{l} \exists\ \varphi\colon [m] \to [n] \textit{ such that } \varphi \textit{ is strictly} \\ \textit{increasing and } x_i = y_{\varphi(i)} \textit{ for all } i \in \\ [m] \end{array} \right.$$

*Proof of Proposition 13.* Let $\Sigma := \{0, 1, \ldots, k\}$. First order $[\mathbb{P}]^k \times \Sigma^*$ by the product of the orderings $\leq_{dlex}$ and $\leq_H$ on $[\mathbb{P}]^k$ and $\Sigma^*$, respectively. Then $[\mathbb{P}]^k \times \Sigma^*$ is well-partial-ordered, by Higman's Lemma (the product ordering of two well-partial-orderings is a well-partial ordering). For $w = (w_1, \ldots, w_k) \in [\mathbb{P}]^k$, set $n := |w|_\infty$; also, let $w^* := (w_1^*, \ldots, w_n^*) \in \Sigma^*$ be given by

$$w_i^* := \sum_{w_j = i} j \quad \text{ for } i = 1, \ldots, n.$$

To prove that $\preceq_{dlex}$ is a well-partial ordering on $[\mathbb{P}]^k$, it suffices to show that the map $w \mapsto (w, w^*)\colon [\mathbb{P}]^k \to [\mathbb{P}]^k \times \Sigma^*$ is an *order-embedding*; that is, if $v \leq_{dlex} w$ and $v^* \leq_H w^*$, then $v \preceq_{dlex} w$ for all $v, w \in [\mathbb{P}]^k$.

Suppose that $v \leq_{dlex} w$ and $v^* \leq_H w^*$, and let $m = |v|_\infty$, $n = |w|_\infty$; then there exists a function $\varphi\colon [m] \to [n]$ strictly increasing such that $v_i^* = w_{\varphi(i)}^*$ for all $1 \leq i \leq m$. Since $\varphi$ is injective, it can be extended to a permutation $\sigma \in \mathfrak{S}_{\mathbb{P}}$. We claim that $v \preceq_{dlex} w$ via witness $\sigma$ so that $\sigma v = w$ and $\sigma v_{<_{dlex}} \subseteq w_{<_{dlex}}$.

We first verify that $\sigma v = w$. For $i \in \{1, \ldots, k\}$, let $l = v_i \leq m$. Notice that $w_{\sigma(l)}^* = v_l^*$, and so together with the definition of $v^*$, we have

$$v_l^* = i + \sum_{\substack{v_j = l \\ i \neq j}} j \Longrightarrow w_{\sigma(l)}^* = i + \sum_{\substack{w_j = \sigma(l) \\ i \neq j}} j.$$

In particular, $w_i = \sigma(l)$; thus, $\sigma(v_i) = \sigma(l) = w_i$ and so $\sigma v = w$.

Now, suppose $u \leq_{dlex} v$. Since $\sigma$ and $\varphi$ agree on $\{v_1, \ldots, v_k\}$, it follows that $|\sigma u|_\infty \leq |\sigma v|_\infty = |w|_\infty = n$. To show $\sigma u \leq_{dlex} w$, it suffices to verify this when $|u|_\infty = |v|_\infty$, as the other case follows from $\varphi$ being strictly increasing. If $|u|_\infty = m = |v|_\infty$ and $u \leq_{dlex} v$, there is an $r \in [k]$ such that $u_1 = v_1, \ldots, u_{r-1} = v_{r-1}$ and $u_r < v_r$. Therefore, $\sigma(u_1) = w_1, \ldots, \sigma(u_{r-1}) = w_{r-1}$ and $\sigma(u_r) < \sigma(v_r) = w_r$ as $\sigma$ is strictly increasing. Thus, $\sigma u \leq_{dlex} w$ and so $\sigma v_{<_{dlex}} \subseteq w_{<_{dlex}}$ as required. $\square$

**Remark 15.** Higman's lemma is also a key element in all known proofs of the finiteness result for $\mathfrak{S}_{\mathbb{P}}$-invariant ideals of $\mathbb{C}[x_1, x_2, \ldots]$ that was mentioned in the introduction.

The following result also follows from the proof of Proposition 13.

**Corollary 16.** *The ordering $\preceq_{dlex}$ of $\langle\mathbb{P}\rangle^k$ is a well-partial-ordering.*

*Proof.* The same proof as Proposition 13 works just by noticing that, in this case, $w_i^* = j$ if $w_j = i$ or 0 otherwise. $\square$

Not all natural orders are nice as the following example demonstrates.

**Example 17.** Define the *reverse lexicographic ordering* $\leq_{revlex}$ on $[\mathbb{P}]^k$ as follows:

$$(u_1, \ldots, u_k) \leq_{revlex} (w_1, \ldots, w_k) \;:\Longleftrightarrow\; u_k = w_k, \ldots, u_{k-r} = w_{k-r} \;\text{ and }\; w_{k-r-1} < u_{k-r-1}, \tag{8}$$

for some $r \in [k]$. In contrast to Proposition 13, the partial order $\leq_{revlex}$ is not nice. For instance, we have in $[\mathbb{P}]^2$ the following infinite strictly decreasing sequence:

$$\cdots \preceq_{revlex} (6,3) \preceq_{revlex} (5,3) \preceq_{revlex} (4,3). \qquad \square$$

The nice ordering $\preceq_{dlex}$ is useful theoretically because of the following property.

**Lemma 18.** *Let $\preceq_{dlex}$ be the well-partial-ordering (6) induced by the nice ordering $\leq_{dlex}$ of $[\mathbb{P}]^k$. Also, let $s, t \in [\mathbb{P}]^k$ satisfy $s \preceq_{dlex} t$ and $|t|_\infty \leq M$ for some $M \in \{0, 1, \ldots\}$. Then there is a $\sigma \in \mathfrak{S}_M$ witnessing $s \preceq_{dlex} t$.*

*Proof.* Since $s \preceq_{dlex} t$, there exists $\tau \in \mathfrak{S}_{\mathbb{P}}$ such that $\tau s = t$ and $\tau s_{<_{dlex}} \subseteq t_{<_{dlex}}$. Let $M = |t|_\infty$. Construct $\sigma \in \mathfrak{S}_M$ by setting $\sigma(i) := \tau(i)$ if $\tau(i) \leq M$ and then extending $\sigma$ to a permutation of $[M]$. We claim that $\sigma s = t$ and $\sigma s_{<_{dlex}} \subseteq t_{<_{dlex}}$. Since $s \leq_{dlex} t$, we have $|s|_\infty \leq |t|_\infty = M$. Therefore writing $s = (s_1, \ldots, s_k) \in [\mathbb{P}]^k$, it follows that $\tau(s_i) \leq M$ for each $i$; thus, $\sigma(s) = \tau(s) = t$. Notice also that $\tau(w) \leq M$ for all $w \in s_{<_{dlex}}$ because for all $w \in s_{<_{dlex}}$, we have $w <_{dlex} s$ which implies $|w|_\infty \leq |s|_\infty \leq M$, and the same holds for all $u \in t_{<_{dlex}}$. Therefore, $\tau(w_i) \leq |t|_\infty = M$ for all $w \in s_{<_{dlex}}$ and each $i = 1, \ldots, k$. Thus, $\sigma(w) = \tau(w)$; therefore, $\sigma s_{<_{dlex}} \subseteq t_{<_{dlex}}$. $\square$

If $A$ is a commutative ring and $S$ an $\mathfrak{S}$-set, we let $AS$ denote the free $A$-module with basis $S$. Also, let $A[\mathfrak{S}]$ be the (left) group ring (whose elements are formal linear combinations of elements in $\mathfrak{S}$ with coefficients in $A$ Lam (2001)). The natural linear action of $A[\mathfrak{S}]$ on $AS$ makes it into an $A[\mathfrak{S}]$-module. The following is the refinement of the Noetherianity result from Camina and Evans (1991) that we will use to prove Theorem 3.

**Theorem 19.** *Let $A$ be a Noetherian commutative ring. For every $A[\mathfrak{S}_{\mathbb{P}}]$-submodule $B \subseteq A[\mathbb{P}]^k$, there exists a finite set $G \subseteq B$ such that*

$$f \in B \cap A[m]^k \iff \exists\, \sigma_1, \ldots, \sigma_\ell \in \mathfrak{S}_m;\, g_1, \ldots, g_\ell \in G;\, a_1, \ldots, a_\ell \in A \;\; \textit{with} \;\; f = \sum_{i=1}^{\ell} a_i \sigma_i g_i.$$

*Proof.* Let $\preceq_{dlex}$ be the well-partial-ordering of $[\mathbb{P}]^k$ (by Proposition 13) induced by the total well-order $\leq_{dlex}$ from (7). A *final segment* of the partial order $\preceq_{dlex}$ is a set $F \subseteq [\mathbb{P}]^k$ such that $u \in F$ and $u \preceq_{dlex} v$ implies that $v \in F$. A well-known characterization of well-partial-orderings (see e.g. Kruskal (1972)) is that final segments are finitely generated. That is, for every final segment $F$, there is a finite set $T \subseteq F$ such that $F = \{v : \exists\, u \in T \text{ with } u \preceq_{dlex} v\}$.

If $f \in A[\mathbb{P}]^k$, we define the *head* of $f$, $\mathrm{Head}(f)$, to be the largest nonzero element in $[\mathbb{P}]^k$ (with respect to $\leq_{dlex}$) in the *support* of $f$ (those elements of $[\mathbb{P}]^k$ occurring in $f$ with nonzero coefficient).

For the $A[\mathfrak{S}_{\mathbb{P}}]$-submodule $B$, let $J \subseteq A$ be the ideal generated by the (leading) coefficients of $\mathrm{Head}(f)$ as $f$ ranges over elements of $B$. By Noetherianity of $A$, we have $J = \langle c_1, \ldots, c_r \rangle_A$ for some $c_i \in A$. Also, since $\preceq_{dlex}$ is a well-partial-order, the final segment $F = \{\mathrm{Head}(f) : f \in B\}$ is finitely generated by $T = \{\mathrm{Head}(b_1), \ldots, \mathrm{Head}(b_{|T|})\}$ for some $b_j \in B$. Consider now the finite set,

$$G := \{c_i b_j : 1 \leq i \leq r, 1 \leq j \leq |T|\} \subseteq B.$$

We claim that $G$ is a subset of $B$ fulfilling the requirements of the theorem statement.

Let $f \in B \cap A[m]^k$. Then, $\mathrm{Head}(h_1) \preceq_{dlex} \mathrm{Head}(f)$ for some $h_1 \in \{b_1, \ldots, b_{|T|}\}$ with witness $\sigma_1 \in \mathfrak{S}_m$ (by Lemma 18). There are $a_1, \ldots, a_r \in A$ such that

$$f_1 := f - \sum_{i=1}^{r} a_i c_i \sigma_1 h_1 \in B$$

has a strictly smaller (with respect to $\leq_{dlex}$) head than $f$. Continuing in this manner we can produce a sequence $f_1, f_2, \ldots$ of elements in $B$ such that

$$\cdots \leq_{dlex} \mathrm{Head}(f_2) \leq_{dlex} \mathrm{Head}(f_1) \leq_{dlex} \mathrm{Head}(f).$$

Since $\leq_{dlex}$ is a well-ordering, it follows that $f_p = 0$ for some $p \in \mathbb{P}$ which gives an expansion for $f$ as in the statement of the theorem. $\square$

**Corollary 20.** *$A[\mathbb{P}]^k$ and $A\langle\mathbb{P}\rangle^k$ are Noetherian $A[\mathfrak{S}_{\mathbb{P}}]$-modules.*

**Remark 21.** It turns out that Corollary 20 holds when $A\langle\mathbb{P}\rangle^k$ is replaced by $AS$ and $A[\mathfrak{S}_{\mathbb{P}}]$ by $A[\mathfrak{S}]$ for any $\mathfrak{S}$-set $S$ with a nice ordering (this follows from the argument above). However, to prove Theorem 3 in the next section, we need the more refined statement found in Theorem 19, which asks for witnesses $\sigma$ to (6) having special properties.

## 3. Laurent chain stabilization

In this short section, we prove that invariant chains of Laurent lattice ideals stabilize (this is Theorem 3 from the introduction). The proof uses the order theory from the previous section and a few properties of lattice ideals. Some basic material on lattice and toric ideals can be found in (Miller and Sturmfels, 2005, Chapter 7) and Sturmfels (1996), respectively, and a more general reference for binomial ideals is Eisenbud and Sturmfels (1996).

Let $\mathcal{G}$ be a finitely generated abelian group and let $a_1, \ldots, a_d$ be distinguished generators of $\mathcal{G}$. Let $L$ denote the kernel of the surjective homomorphism $\mathbb{Z}^d$ onto $\mathcal{G}$. The *lattice ideal* associated with $L$ is the following ideal in $\mathbb{K}[z_1, \ldots, z_d]$:

$$I_L = \langle \mathbf{z}^u - \mathbf{z}^v : u, v \in \mathbb{N}^d \text{ with } u - v \in L \rangle.$$

Here, we use the shorthand $\mathbf{z}^u = z_1^{u_1} \cdots z_d^{u_d}$ for $u = (u_1, \ldots, u_d) \in \mathbb{Z}^d$. A *toric ideal* is the special case of a lattice ideal in which the group $\mathcal{G}$ is torsion-free; in this case, the ideal $I_L$ is also prime.

Notice that if $S = \{s_1, \ldots, s_d\}$ is a set with $d$ elements, there is a natural isomorphism between $\mathbb{Z}^d$ and the free $\mathbb{Z}$-module $\mathbb{Z}S$ with basis $S$ given by:

$$(a_1, \ldots, a_d) \in \mathbb{Z}^d \mapsto \sum_{i=1}^{d} a_i s_i \in \mathbb{Z}S.$$

Although simple, this identification will be useful for us below.

**Example 22.** In the case $S = \langle 3 \rangle^2 = \{(1,2),(1,3),(2,1),(2,3),(3,1),(3,2)\}$, the integer vector $(-2,2,1,0,-1,0) \in \mathbb{Z}^6$ is also represented by $-2 \cdot (1,2) + 2 \cdot (1,3) + (2,1) - (3,1) \in \mathbb{Z}\langle 3 \rangle^2$. $\square$

For simplicity of exposition, we focus our attention on lattice ideals in the polynomial rings $\mathcal{R}_n$ (equipped with the action of $\mathfrak{S}_n$) from (3), each of which has $d_n = n^k$ indeterminates. Let $L_n \subseteq L_{n+1}$ be an increasing sequence of subgroups of $\mathbb{Z}^{d_n} \subseteq \mathbb{Z}^{d_{n+1}}$ and let $I_n := I_{L_n} \subseteq \mathcal{R}_n$ (resp. $I_n^{\pm} \subseteq \mathcal{R}_n^{\pm}$) be the corresponding lattice (resp. Laurent lattice) ideals.

The basic idea in our proof of Theorem 3 is to view $L = \bigcup_{n \in \mathbb{P}} L_n$ as an $\mathfrak{S}_{\mathbb{P}}$-invariant subgroup of the free abelian group $\mathbb{Z}[\mathbb{P}]^k = \bigcup_{n \in \mathbb{P}} \mathbb{Z}[n]^k$, which has free basis $[\mathbb{P}]^k$ over $\mathbb{Z}$. The set $L$ has a finite generating set up to $\mathfrak{S}_{\mathbb{P}}$-symmetry (using Theorem 19 and the fact that $\mathbb{Z}$ is Noetherian), and these vectors are all contained in $L_N$ for some integer $N$. The remainder of the proof converts this fact back to the level of ideals. The complete details are as follows.

Given an integer vector $h \in \mathbb{Z}^d$, we set $h_+ \in \mathbb{N}^d$ and $h_- \in \mathbb{N}^d$ to be the nonnegative and nonpositive part of $h$, respectively (so that $h = h_+ - h_-$). The following is elementary.

**Lemma 23.** *Suppose that $v, h_1, \ldots, h_m \in \mathbb{Z}^d$ and set $u = v + \sum_{i=1}^m h_i$. There exists a monomial $\mathbf{z}^c \in \mathbb{K}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]$ such that $\mathbf{z}^c(\mathbf{z}^u - \mathbf{z}^v) \in \langle \mathbf{z}^{h_{i+}} - \mathbf{z}^{h_{i-}} : i = 1, \ldots, m \rangle_{\mathbb{K}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]}$.*

*Proof.* We shall induct on $m$, the base case being vacuously true. Consider the identity:

$$(\mathbf{z}^u - \mathbf{z}^v) = \mathbf{z}^{h_m}(\mathbf{z}^{u-h_m} - \mathbf{z}^v) + \mathbf{z}^{v-h_{m-}}(\mathbf{z}^{h_{m+}} - \mathbf{z}^{h_{m-}}). \tag{9}$$

As $u' = u - h_m$ has fewer terms, the proof follows by induction. $\square$

Collecting these facts together, we can now prove the main result of this section.

*Proof of Theorem 3.* The submodule $L \subseteq \mathbb{Z}[\mathbb{P}]^k$ is finitely generated over $\mathbb{Z}[\mathfrak{S}_{\mathbb{P}}]$ by Theorem 19. Set $H = G \cup -G$ for a finite set of generators $G \subseteq L$ satisfying the property in Theorem 19, and let $N$ be such that $H \subseteq \mathbb{Z}^{d_N}$. Consider two vectors $u, v \in \mathbb{N}^{d_m}$ such that $u - v \in L_m = L \cap [m]^k$, with $m \geq N$. By assumption, the vector $u - v$ is a $\mathbb{Z}$-linear combination of $\mathfrak{S}_m$-permutations of elements in $H$. By Lemma 23, it follows that $\mathbf{z}^u - \mathbf{z}^v$ is a monomial multiple of an element in the ideal (of $\mathcal{R}_m^{\pm}$) generated by permutations (in $\mathfrak{S}_m$) of $\{\mathbf{z}^{h_+} - \mathbf{z}^{h_-} : h \in H\}$. Thus, $I_m^{\pm} \subseteq \langle \mathfrak{S}_m I_N \rangle_{\mathcal{R}_m^{\pm}}$ and the chain stabilizes. $\square$

## 4. Stabilization of chains induced by monomials

We now focus on the polynomial rings $R_n$ from (3) and the corresponding chains of toric ideals encountered in the statement of Theorem 4.

**Definition 24.** Let $k \in \mathbb{P}$ and $f \in \mathbb{K}[y_1, \ldots, y_k]$. For for each $n \geq k$, consider

$$\phi_n : R_n \to T_n, \qquad x_{(u_1, \ldots, u_k)} \mapsto f(t_{u_1}, \ldots, t_{u_k}).$$

Let $I_n = \ker \phi_n$. The invariant chain $I_k \subseteq I_{k+1} \subseteq \cdots$ is called the invariant chain of ideals *induced by the polynomial $f$*.

The ideals in Definition 24 appear in voting theory Daugherty et al. (2009), algebraic statistics Sturmfels and Sullivant (2005); Hillar and Sullivant (2011); Draisma (2010); García-García et al. (2010), and toric algebra Aschenbrenner and Hillar (2007); Hillar and Sullivant (2011). When $f$ is a monomial, each $I_n = \ker \phi_n$ is a homogeneous toric ideal. The following was conjectured in Aschenbrenner and Hillar (2007).

**Conjecture 25** (Aschenbrenner and Hillar (2007))**.** *The chain of ideals induced by any monomial stabilizes.*

The authors of Aschenbrenner and Hillar (2007) verified the special case of Conjecture 25 when $f$ is a square-free monomial. Underlying their proof is the fact that for every $n \geq k$, the ideals $I_n$ are generated by quadratic binomials (Sturmfels, 1996, Theorem 14.2). Unfortunately, the corresponding statement is false when $f$ is not square-free. Although a proof for the general conjecture is not known, Theorem 3 shows (albeit nonconstructively) that the Laurent versions of these chains stabilize.

The main goal of this section is to provide an effective version of Theorem 3 for this situation that allows for explicit computation of generators (this is Theorem 4 from the introduction). In the next section, we describe this algorithm and give a reference to an implementation of it in software. We also explain another approach to finding these generators at the end of this section.

Our running example throughout will be the case $f = y_1^2 y_2$, and all computations were performed using `Macaulay2` Grayson and Stillman (????) and `4ti2` 4ti2 team (????). If $I_\circ$ is the chain of ideals induced by $y_1^2 y_2$, then Theorem 3 guarantees stabilization of $I_\circ^\pm$. Moreover, Theorem 4 provides a stabilization bound $N = 2 \cdot \deg(f) = 6$. Using Algorithm 1 from Section 5, the following is a generating set for $I_\circ^\pm$ (below, we use a shorthand notation for indices; e.g., $x_{16} = x_{(1,6)}$):

$$\begin{aligned} G^\pm \quad = \quad \big\{ & x_{16}x_{21}^2x_{54}x_{65} - x_{14}x_{15}x_{26}^2x_{56}, && x_{16}^2x_{21}^4x_{43}x_{65} - x_{13}x_{14}^2x_{15}x_{26}^4, && x_{13}x_{43} - x_{14}x_{34}, \\ & x_{16}^2x_{21}^4x_{45}x_{65} - x_{14}^2x_{15}^2x_{26}^4, && x_{16}x_{21}^2x_{34}x_{65} - x_{14}x_{15}x_{26}^2x_{36}, && x_{13}x_{24} - x_{14}x_{23}, \\ & x_{16}x_{21}^2x_{36} - x_{13}^2x_{26}^2, && x_{16}^2x_{21}^2x_{32} - x_{12}x_{13}^2x_{26}^2 \big\}. \end{aligned}$$

Therefore, the chain of Laurent ideals $I_\circ^\pm$ induced by $y_1^2 y_2$ is generated by these 8 elements of $G^\pm$ up to the action of the symmetric group. It is important to remark that these binomials are not generators of the original ideal $I_6$, nor of the chain $I_\circ$. Moreover, this generating set is not smallest possible, as shown in Section 4.2, where we study the combinatorial structure of this special case and find a generating set with only 4 elements for the Laurent chain $I_\circ^\pm$.

### *4.1. Proof of Theorem 4*

First observe that the inclusion $R_n \hookrightarrow R_n^\pm$ gives us for every $n \geq k$ an extension of $\phi_n$ given by the homomorphism $\psi_n : R_n^\pm \to T_n^\pm$ satisfying $\psi_n(x_u) = \phi_n(x_u)$ and $\psi_n(x_u^{-1}) = \phi_n(x_u)^{-1}$ for all $u \in \langle n \rangle^k$. Notice that $I_n^\pm = \ker \psi_n$ and that we have the following commutative diagram:

$$\begin{tikzcd} R_n \arrow[r, hook] \arrow[dr, "\phi_n"'] & R_n^\pm \arrow[d, "\psi_n"] \\ & T_n^\pm \end{tikzcd} \tag{10}$$

Let $\alpha \in \mathbb{N}^k$ be the exponent vector of a (non-constant) monomial $f = \mathbf{y}^\alpha = y_1^{\alpha_1} \cdots y_k^{\alpha_k}$, and consider

$$\mathcal{A}_n := \{\sigma(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)^\top \in \mathbb{Z}^n : \sigma \in \mathfrak{S}_n\}.$$

The set of column vectors $\mathcal{A}_n$ can be represented as an $n \times \binom{n}{k} k!$ matrix with rows indexed by the indeterminates $t_i$ (for $i = 1, \ldots, n$) and columns indexed by the indeterminates $x_w$ (for $w \in \langle n \rangle^k$). The matrix $\mathcal{A}_n$ defines a semigroup homomorphism $\mathbb{N}^{\binom{n}{k}k!} \to \mathbb{N}^n$ which lifts to the homomorphism $\phi_n$. The kernel $I_n$ is generated by the set:

$$\left\{ \mathbf{x}^a - \mathbf{x}^b : \mathcal{A}_n(a) = \mathcal{A}_n(b),\, a, b \in \mathbb{N}^{\binom{n}{k}k!} \right\}.$$

For more details about toric ideals and their generating sets, see Sturmfels (1996).

**Example 26.** Let $k = 2$, $n = 3$, and $\alpha = (2, 1)$. The following represents the matrix $\mathcal{A}_3$ associated to the homomorphism $\phi_3$ defined by $f = y_1^2 y_2$.

| | $x_{12}$ | $x_{13}$ | $x_{21}$ | $x_{23}$ | $x_{31}$ | $x_{32}$ |
|---|---|---|---|---|---|---|
| $t_1$ | 2 | 2 | 1 | 0 | 1 | 0 |
| $t_2$ | 1 | 0 | 2 | 2 | 0 | 1 |
| $t_3$ | 0 | 1 | 0 | 1 | 2 | 2 |

The ideal $I_3$ is generated by binomials: $\{x_{13}x_{21}^2 - x_{12}^2x_{23},\ x_{13}^2x_{21} - x_{12}^2x_{31},\ x_{21}x_{31} - x_{12}x_{32},\ x_{21}^2x_{32} - x_{12}x_{23}^2,\ x_{13}x_{23} - x_{12}x_{32},\ x_{13}x_{21}x_{32} - x_{12}x_{23}x_{31},\ x_{13}^2x_{32} - x_{12}x_{31}^2,\ x_{23}x_{31}^2 - x_{13}x_{32}^2,\ x_{23}^2x_{31} - x_{21}x_{32}^2\}$. $\square$

Next, we argue that it suffices to study those maps $\phi_n : R_n \to T_n$ defined by an exponent vector $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ with $\gcd(\alpha) := \gcd(\alpha_1, \ldots, \alpha_k) = 1$. To see this, suppose that $\gcd(\alpha) = \ell > 1$, and consider $\alpha' = \ell^{-1} \cdot \alpha$. Let $\phi_n$ and $\phi'_n$ be the homomorphisms given by $\phi_n(x_w) = t_{w_1}^{\alpha_1} \cdots t_{w_k}^{\alpha_k}$ and $\phi'_n(x_w) = t_{w_1}^{\alpha'_1} \cdots t_{w_k}^{\alpha'_k}$, respectively. Note that $\phi_n(x_w) = (\phi'_n(x_w))^\ell$ for all $w \in \langle n\rangle^k$, so if $a, b \in \mathbb{N}^{\binom{n}{k}k!}$, then $\phi_n(\mathbf{x}^a) = \phi_n(\mathbf{x}^b) \iff \phi'_n(\mathbf{x}^a)^\ell = \phi'_n(\mathbf{x}^b)^\ell \iff \phi'_n(\mathbf{x}^a) = \phi'_n(\mathbf{x}^b)$ (as $\phi'_n(\mathbf{x}^a)$ and $\phi'_n(\mathbf{x}^b)$ are monomials in $T_n$); thus, $\mathbf{x}^a - \mathbf{x}^b \in \ker \phi_n$ if and only if $\mathbf{x}^a - \mathbf{x}^b \in \ker \phi'_n$.

Our first basic tool is a combinatorial lemma describing the $\mathbb{Z}$-linear column span of $\mathcal{A}_n$ inside $\mathbb{Z}^n$. For $\alpha \in \mathbb{Z}^k$, we set $|\alpha| := \sum_{i=1}^k \alpha_i$.

**Lemma 27.** *Let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ with $\gcd(\alpha_1, \ldots, \alpha_k) = 1$. The integral span of the columns $\mathcal{A}_n$ $(n > k)$ is:*

$$\mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n) = \{\beta \in \mathbb{Z}^n : |\beta| \equiv 0 \mod |\alpha|\}.$$

*Proof.* Let $\mathfrak{A} = \{\beta \in \mathbb{Z}^n : |\beta| \equiv 0 \mod |\alpha|\}$; clearly, $\mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n) \subseteq \mathfrak{A}$. By assumption, $\gcd(\alpha) = 1$; thus, there are integers $b_1, \ldots, b_k \in \mathbb{Z}$ with $b_1\alpha_1 + \cdots + b_k\alpha_k = 1$.

For every $j = 1, \ldots, n-1$ and $i = 1, \ldots, k$, let $\sigma_{ij}$ be the transposition $(i\, j) \in \mathfrak{S}_n$, and consider the vector

$$h_j = b_1(\sigma_{1j}\alpha) + \cdots + b_k(\sigma_{kj}\alpha).$$

Notice that $h_j$ is a vector whose $j$th entry is 1 and $h_j \in \mathrm{Span}\{\mathcal{A}_n\}$. Consider also the transposition $\tau_j = (j\, n)$, and the vector

$$h'_j = \tau_j h_j = b_1(\tau_j\sigma_{1j}\alpha) + \cdots + b_k(\tau_j\sigma_{kj}\alpha).$$

For every $i$ and $j$, the composition $\tau_j\sigma_{ij}$ is the transposition $(i\, n) \in \mathfrak{S}_n$; thus, $h'_j$ is obtained from $h_j$ by changing the 1 from position $j$ to position $n$. Naturally, $h'_j \in \mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n)$. Let $\widehat{h_j} = h_j - h'_j \in \mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n)$. Notice that $\widehat{h_j}$ is the vector with 1 in the $j$th position, $-1$ in the $n$th, and zeroes elsewhere.

Now, let $\beta = (\beta_1, \ldots, \beta_n) \in \mathfrak{A}$. By assumption, there exists $q \in \mathbb{Z}$ such that $|\beta| = q|\alpha|$. For every $j = 1, \ldots, n-1$, there is $r_j \in \mathbb{Z}$ with $\beta_j = q\alpha_j + r_j$. Set

$$\gamma = q\alpha + \sum_{j=1}^{n-1} r_j\widehat{h_j} \in \mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n).$$

It is easy to check that $\beta = \gamma$, and so $\beta \in \mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n)$ as desired. $\square$

**Example 28.** Consider $\alpha = (2, 1)$ and $n = 3$. Since $\gcd(2, 1) = 1$, we can write $1 = (1)2 + (-1)1$. The vectors $\widehat{h}_1, \widehat{h}_2 \in \mathrm{Span}_{\mathbb{Z}}\{\mathcal{A}_3\}$ from the proof of Lemma 27 are precisely $(1, 0, -1)^\top$, $(0, 1, -1)^\top$. Therefore, the vector $u = (1, 1, 1)^\top \in \mathfrak{A}$ can be written as

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in \mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_3). \qquad \square$$

One immediate consequence of Lemma 27 is that the toric ideals in this section are not normal. This likely contributes to the difficulty of proving stabilization for chains induced by a non-square-free monomial.

**Corollary 29.** *Let $\mathbf{y}^\alpha$ be a non-square-free monomial in $\mathbb{K}[y_1, \ldots, y_k]$. For every $n \geq |\alpha|$, the toric ideal $I_n$ induced by the monomial $\mathbf{y}^\alpha$ is not normal.*

Recall from (Sturmfels, 1996, Proposition 13.5) that a toric ideal $I_\mathcal{A}$ is normal if and only if $pos(\mathcal{A}) \cap \mathrm{Span}_{\mathbb{Z}}(\mathcal{A}) = \mathrm{Span}_{\mathbb{N}}(\mathcal{A})$, where $pos(\mathcal{A})$ is the polyhedral cone defined by the columns of $\mathcal{A}$.

*Proof.* Let $\alpha \in \mathbb{N}^k$ with $\gcd(\alpha) = 1$, and let $\tau \in \mathfrak{S}_n$ be the cyclic permutation $\tau = (1\,2 \cdots |\alpha|)$. Realize $\alpha \in \mathbb{Z}^n$ by $\alpha = (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)^\top \in \mathbb{Z}^n$. Consider the following identity:

$$(1, \ldots, 1, 0, \ldots, 0)^\top = \frac{1}{|\alpha|}(\alpha + \tau\alpha + \cdots + \tau^{|\alpha|-2}\alpha + \tau^{|\alpha|-1}\alpha).$$

By construction $z = (1, \ldots, 1, 0, \ldots, 0)^\top \in pos(\mathcal{A}_n)$ and $|z| = |\alpha|$; thus, by Lemma 27 we see that $z \in pos(\mathcal{A}_n) \cap \mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n)$. However, since $\mathbf{y}^\alpha$ is not square-free, we must have $z \notin \mathrm{Span}_{\mathbb{N}}(\mathcal{A}_n)$. $\square$

Although not required for the proof of Theorem 4, the Smith normal form (SNF) of the matrices $\mathcal{A}_n$ can be easily computed from Lemma 27. For basic properties and algorithms involving the SNF over a principal ideal domain, we refer the reader to Hazewinkel et al. (2004); Yap (2000).

**Corollary 30.** *Let $\alpha \in \mathbb{N}^k$ such that $\gcd(\alpha) = 1$. For $n > k$ consider the matrix*

$$\mathcal{A}_n = \left(\sigma(\alpha_1, \ldots, \alpha_k, 0, \ldots, 0)^\top \in \mathbb{Z}^n : \sigma \in \mathfrak{S}_n\right).$$

*The Smith normal form for $\mathcal{A}_n$ is* $\mathrm{diag}(1, \ldots, 1, |\alpha|)$.

*Proof.* Use vectors $\widehat{h_j}$ from the proof of Lemma 27 to reduce the matrix $\mathcal{A}_n$ to SNF $\mathrm{diag}(1, \ldots, 1, d)$, for some $d \in \mathbb{N}$. From Lemma 27, we know that $\mathrm{Span}_{\mathbb{Z}}(\mathcal{A}_n)$ is $\mathfrak{A} = \{\beta \in \mathbb{Z}^n : |\beta| \equiv 0 \mod |\alpha|\}$. Since $\mathbb{Z}^n/\mathfrak{A}$ is a finitely generated $\mathbb{Z}$-module, the fundamental decomposition theorem for modules (Hazewinkel et al., 2004, Theorem 7.8.2) implies that

$$\mathbb{Z}^n/\mathfrak{A} \cong \mathbb{Z}/d\mathbb{Z}.$$

On the other hand, $\mathfrak{A}$ is the kernel of the map $\mathbb{Z}^n \to \mathbb{Z}/|\alpha|\mathbb{Z}$ given by $\beta \mapsto |\beta| \mod |\alpha|$; therefore,

$$\mathbb{Z}^n/\mathfrak{A} \cong \mathbb{Z}/|\alpha|\mathbb{Z},$$

as $\mathbb{Z}$-modules. Hence, $\mathbb{Z}/|\alpha|\mathbb{Z} \cong \mathbb{Z}/d\mathbb{Z}$, which implies $d = |\alpha|$. $\square$

Let $d = \deg f = |\alpha|$ and set $r = \max\{\alpha_1, \ldots, \alpha_k\}$. Consider now

$$\mathcal{B}_n := \{(a_1, \ldots, a_n)^\top \in \mathbb{Z}^n : a_1 + \cdots + a_n = d,\ 0 \leq a_1, \ldots, a_n \leq r\}. \tag{11}$$

There is a natural bijection between elements of $\mathcal{B}_n$ and multisubsets of $[n]$ of cardinality $d$ with at most $r$ repetitions. Let $\Gamma_n$ be the set of such multisubsets. Every $a = (a_1, \ldots, a_n) \in \mathcal{B}_n$ is in bijection with $\widetilde{a} \in \Gamma_n$ via:

$$a = (a_1, \ldots, a_n) \longleftrightarrow \widetilde{a} = \{1^{a_1}, 2^{a_2}, \ldots, n^{a_n}\}. \tag{12}$$

Let $\widetilde{R}_n := \mathbb{K}\,[X_{\Gamma_n}]$. When $\mathcal{B}_n$ is viewed as a matrix with rows indexed by $t_i$ (for $i \in [n]$) and columns indexed by $x_{\widetilde{a}}$ (for $\widetilde{a} \in \Gamma_n$), it defines a semigroup homomorphism that lifts to a homomorphism of $\mathbb{K}$-algebras:

$$\widetilde{\phi}_n : \widetilde{R}_n \longrightarrow T_n.$$

By definition, $\mathcal{A}_n \subseteq \mathcal{B}_n$, and this inclusion gives an embedding $\eta : R_n \hookrightarrow \widetilde{R}_n$. Also, $\widetilde{\phi}_n$ extends the map $\phi_n$ in the sense that $\phi_n = \widetilde{\phi}_n \circ \eta$. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
R_n \arrow[r, hook, "\eta"] \arrow[d, "\phi_n"'] & \widetilde{R}_n \arrow[dl, "\widetilde{\phi}_n"] \\
T_n &
\end{tikzcd}$$

**Example 31.** Let $n = 3$ and $\alpha = (2, 1)$. Then $\widetilde{R}_3 = \mathbb{K}[x_{123}, x_{112}, x_{113}, x_{122}, x_{223}, x_{133}, x_{233}]$, and the following table represents the matrix $\mathcal{B}_3$ associated to the homomorphism $\widetilde{\phi}_3$:

| | $x_{123}$ | $x_{112}$ | $x_{113}$ | $x_{122}$ | $x_{223}$ | $x_{133}$ | $x_{233}$ |
|---|---|---|---|---|---|---|---|
| $t_1$ | 1 | 2 | 2 | 1 | 0 | 1 | 0 |
| $t_2$ | 1 | 1 | 0 | 2 | 2 | 0 | 1 |
| $t_3$ | 1 | 0 | 1 | 0 | 1 | 2 | 2 |

□

We next note the following fact, easily derived using (Sturmfels, 1996, Theorem 14.2).

**Lemma 32.** *The ideal $\widetilde{I}_n \subseteq \widetilde{R}_n$ is generated by the quadratic binomials of any Gröbner basis.*

In particular, since finite Gröbner bases always exist, $\widetilde{I}_n$ has a finite set of quadratic binomials generating it.

We now explain the key idea in our proof of Theorem 4. Since the map $\widetilde{\phi}_n$ extends $\phi_n$, we have $I_n \hookrightarrow \widetilde{I}_n$. Suppose that $\widetilde{I}_n = \langle \widetilde{G}_n \rangle$ for some set $\widetilde{G}_n \subseteq \widetilde{R}_n$, and that we can find a $\mathbb{K}$-algebra homomorphism $\mu$ making the following diagram commutative:

$$\begin{tikzcd}
R_n \arrow[rr, hook] \arrow[dr, hook, "\eta"] \arrow[ddr, "\phi_n"'] & & R_n^{\pm} \arrow[ddl, "\psi_n"] \\
& \widetilde{R}_n \arrow[ur, dashed, "\mu"] \arrow[d, "\widetilde{\phi}_n"] & \\
& T_n^{\pm} &
\end{tikzcd} \tag{13}$$

Then, as is easily checked, $\mu(\widetilde{G}_n)$ will be a generating set for $I_n^{\pm}$. If, in addition, the $\widetilde{G}_n$ can themselves be finitely generated up to symmetry and $\mu$ is equivariant,[4] then we have generated the whole Laurent chain $I_{\circ}^{\pm}$ up to the symmetric group. As the proof of the following proposition explains, the existence of such a $\mu$ is guaranteed by Lemma 27.

[4] The term *equivariant* for the map $\mu$ signifies that $\mu(\sigma h) = \sigma\mu(h)$ for any $\sigma \in \mathfrak{S}_n$ and $h \in \widetilde{R}_n$.

**Proposition 33.** *Fix $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^n$ and let $f = \mathbf{y}^\alpha \neq 1$. For each $n > k$, there exists an equivariant $\mathbb{K}$-algebra homomorphism $\mu : \widetilde{R}_n \to R_n^\pm$ that makes the diagram (13) commute.*

*Proof.* Consider a multisubset $\widetilde{a} \in \Gamma_n$. If $x_{\widetilde{a}} \in \eta(R_n) \subseteq \widetilde{R}_n$, then define $\mu(x_{\widetilde{a}}) := \eta^{-1}(x_{\widetilde{a}})$. Assume $x_{\widetilde{a}} \notin \eta(R_n)$. Since $\widetilde{a}$ is in bijection with $a \in \mathcal{B}_n$ as in (12), we have $|a| = |\alpha|$. By Lemma 27, we can find integers $B = \{b_1, \ldots, b_M\} \subset \mathbb{Z}$ such that

$$a = \sum_{i=1}^{M} b_i u_i,$$

with $M = \binom{n}{k} k!$ and $u_i \in \mathcal{A}_n$. Let $B^+ = \{b_i \in B : b_i \in \mathbb{Z}_{>0}\}$ and $B_- = \{b_i \in B : b_i \in \mathbb{Z}_{<0}\}$. Consider the fraction

$$\mathfrak{q} := \frac{\prod_{b_i \in B^+} x_{u_i}^{b_i}}{\prod_{b_i \in B_-} x_{u_i}^{-b_i}}. \tag{14}$$

Clearly $\mathfrak{q} \in R_n^\pm$, so we can define $\mu(x_{\widetilde{a}}) := \mathfrak{q} \in R_n^\pm$.

Extend $\mu$ to $\widetilde{R}_n$ by linearity. By construction, $\mu$ makes the diagram (13) commute since for $\widetilde{a} \in \Gamma_n$, one can verify that $\psi_n(\mu(x_{\widetilde{a}})) = \widetilde{\phi}_n(x_{\widetilde{a}}) = \prod_{i=1}^n t_i^{a_i}$. □

**Remark 34.** The above construction of $\mu$ is not necessarily unique as it depends on the representation of $\mathfrak{q}$.

**Example 35.** Continuing from Example 28, we want to map $x_{123} \in \widetilde{R}_3$ to a fraction in $R_3^\pm$ that only involves the indeterminates of $\widetilde{R}_3$ corresponding to those of $R_3$:

$$\mu(x_{123}) = \frac{x_{112}x_{331}}{x_{113}}.$$

We also want this fraction to have the same image under $\psi_3$ as $x_{123}$ has under $\widetilde{\phi}_3$. Indeed, as one can check, we have $\widetilde{\phi}_3(x_{123}) = t_1t_2t_3$ and

$$\psi_3(\mu(x_{123})) = \frac{\psi_3(x_{112}x_{331})}{\psi_3(x_{113})} = \frac{\phi_3(x_{12})\phi_3(x_{31})}{\phi_3(x_{13})} = \frac{(t_1^2t_2)(t_3^2t_1)}{(t_1^2t_3)} = t_1t_2t_3.$$

□

We are finally in position to prove Theorem 4.

*Proof of Theorem 4.* Let $\widetilde{I}_n = \ker \widetilde{\phi}_n$; this ideal is generated by binomials of the form

$$x_{\widetilde{a}}x_{\widetilde{b}} \cdots x_{\widetilde{c}} - x_{\widetilde{a}'}x_{\widetilde{b}'} \cdots x_{\widetilde{c}'},$$

in which $\widetilde{a} \cup \widetilde{b} \cdots \cup \widetilde{c} = \widetilde{a}' \cup \widetilde{b}' \cdots \cup \widetilde{c}'$ as a union of multisets (Sturmfels, 1996, Remark 14.1). From Lemma 32, there is a finite generating set $\mathcal{G}_n$ of $\widetilde{I}_n$ consisting of quadratic binomials. Let $G_n$ be a finite set of generators for $I_n$. Note that $\eta(I_n) \subseteq \widetilde{I}_n$ and so $\eta(G_n) \subseteq \widetilde{I}_n$. For $g \in G_n$, we can write

$$\eta(g) = \sum_{\widetilde{p} \in \mathcal{G}_n} h_{\widetilde{p}}\widetilde{p}, \quad \text{with } h_{\widetilde{p}} \in \widetilde{R}_n. \tag{15}$$

We know $G_n$ is a generating set for $I_n^\pm$, but we give another generating set for $I_n^\pm$ in terms of $\mathcal{G}_n$.

Applying the map $\mu$ from Proposition 33 to both sides of expression (15), we have

$$g = \mu(\eta(g)) = \sum_{\widetilde{p} \in \mathcal{G}_n} \mu(h_{\widetilde{p}})\mu(\widetilde{p}).$$

Moreover, $\mu(\widetilde{p}) \in I_n^{\pm}$. It follows that $I_n^{\pm} = \langle \mu(\widetilde{p}) : \widetilde{p} \in \mathcal{G}_n \rangle_{R_n^{\pm}}$. Since $\widetilde{p} \in \mathcal{G}_n$ is a quadratic binomial,

$$\widetilde{p} = x_{\widetilde{a}} x_{\widetilde{b}} - x_{\widetilde{a}'} x_{\widetilde{b}'}, \quad \text{with } \widetilde{a} \cup \widetilde{b} = \widetilde{a}' \cup \widetilde{b}' \text{ as multisets.}$$

The cardinality of each of $\widetilde{a}$, $\widetilde{b}$ is $d = |\alpha|$, and so the number of distinct numbers in $\widetilde{a} \cup \widetilde{b}$ is at most $2d$. In particular, $\mu(\widetilde{p}) \in \langle \mathfrak{S}_n I_{2d}^{\pm} \rangle_{R_n^{\pm}}$ for $n \geq 2d$. Thus, $I_\circ^{\pm}$ stabilizes with bound $N = 2d$. □

**Example 36.** Continuing with Example 5, let $g = x_{39}x_{79} - x_{37}x_{97} \in I_9$. Under the inclusion $\eta : R_6 \hookrightarrow \widetilde{R}_6$, we have $\eta(g) = x_{339}x_{779} - x_{337}x_{799}$. From Lemma 32, the ideal $\widetilde{I}_9$ is generated by quadratic binomials. We can write $\eta(g)$ in terms of those generators; in this case,

$$\eta(g) = (x_{339}x_{779} - x_{379}^2) - (x_{337}x_{799} - x_{379}^2) \in \widetilde{I}_9.$$

Let $\widetilde{p}_1 := x_{337}x_{779} - x_{379}^2$ and $\widetilde{p}_2 := x_{337}x_{799} - x_{379}^2$. We have $\widetilde{p}_1 = \sigma\widetilde{q}_1$ and $\widetilde{p}_2 = \sigma\widetilde{q}_2$ for the following $\widetilde{q}_1, \widetilde{q}_2 \in \widetilde{I}_6$ (actually $\widetilde{I}_3$ in this case) and $\sigma = (1\,3\,9)(2\,7) \in \mathfrak{S}_9$:

$$\widetilde{q}_1 = x_{113}x_{223} - x_{123}^2, \quad \widetilde{q}_2 = x_{112}x_{233} - x_{123}^2.$$

Thus, $\mu(\widetilde{p}_1) = \sigma\mu(q_1)$ and $\mu(\widetilde{p}_2) = \sigma\mu(q_2)$ since $\mu$ is equivariant and $\mu(q_1)$ and $\mu(q_2)$ generate $g$ up to symmetry:

$$g = \sigma\left(x_{12}x_{23} - \frac{x_{12}^2 x_{31}^2}{x_{13}^2}\right) - \sigma\left(x_{12}x_{32} - \frac{x_{12}^2 x_{31}^2}{x_{13}^2}\right).$$

□

### 4.2. *Toric ideals induced by $y_1^2 y_2$*

Theorem 4 provides evidence that chains of ideals induced by monomials stabilize. The simplest (unknown) case is when $f = y_1^2 y_2$. Here, we present an explicit computation of the generators for the corresponding Laurent chain that is different from Algorithm 1. We hope to illustrate some of the complexity of the general problem and also to elaborate on other approaches for tackling Conjecture 25.

For $n \geq 2$, let $I_n$ be the toric ideal induced by the monomial $y_1^2 y_2$. Let $\mathcal{A}_n \in \mathbb{Z}^{n \times \binom{n}{k} k!}$ be the matrix that defines the semigroup homomorphism $\phi_n$ such that $I_n = \ker \phi_n$ (recall Definition 24). For example, when $n = 5$ we have

$$\mathcal{A}_5 = \begin{pmatrix} 2&2&2&2&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&0&0&2&0&0&0&2&2&2&1&1&1&0&0&0&0&0&0 \\ 0&1&0&0&0&2&0&0&1&0&0&2&0&0&2&2&1&1&0&0 \\ 0&0&1&0&0&0&2&0&0&1&0&0&2&0&1&0&2&0&2&1 \\ 0&0&0&1&0&0&0&2&0&0&1&0&0&2&0&1&0&2&1&2 \end{pmatrix}.$$

When the columns of $\mathcal{A}_n$ are ordered lexicographically, a basis for $\ker_{\mathbb{Z}}(\mathcal{A}_n)$ as a $\mathbb{Z}$-module can be described as follows:

$$\ker_{\mathbb{Z}}(\mathcal{A}_n) = \begin{pmatrix} \mathcal{K}_n \\ \mathcal{I}_{d-n} \end{pmatrix}, \tag{16}$$

where $\mathcal{I}_{d-n}$ is the $(d-n) \times (d-n)$ identity matrix and $\mathcal{K}_n$ is a matrix whose structure we now describe. Let $c_r \in \mathbb{Z}^{n-2}$ be the row vector whose entries are all equal to $r$. Then,

$$\mathcal{K}_n = \begin{pmatrix} \mathcal{L}_1 & \mathcal{L}_2 & \mathcal{L}_3 & \mathcal{L}_4 \end{pmatrix},$$

in which

$$-\mathcal{L}_1 = \begin{pmatrix} c_{-2} \\ 2 \cdot \mathcal{I}_{n-2} \\ c_1 \end{pmatrix}, \quad -\mathcal{L}_2 = \begin{pmatrix} c_{-2} \\ \mathcal{I}_{n-2} \\ c_2 \end{pmatrix}, \quad -\mathcal{L}_3 = \begin{pmatrix} c_{-3} \\ 2 \cdot \mathcal{I}_{n-2} \\ c_2 \end{pmatrix}, \quad -\mathcal{L}_4 = \begin{pmatrix} c_{-4} \\ \mathcal{A}_{n-2} \\ c_2 \end{pmatrix}.$$

For instance, when $n = 5$, the integer kernel of $\mathcal{A}_5$ has the following $\mathbb{Z}$-basis

$$\ker(\mathcal{A}_5) = \begin{pmatrix}
2 & 2 & 2 & 2 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 \\
-2 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & 0 & -2 & -2 & -1 & -1 & 0 & 0 \\
0 & -2 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & -1 & 0 & -2 & 0 & -2 & -1 \\
0 & 0 & -2 & 0 & 0 & -1 & 0 & 0 & -2 & 0 & -1 & 0 & -2 & -1 & -2 \\
-1 & -1 & -1 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 & -2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$

For each $n$, the elements of $\ker_{\mathbb{Z}}(\mathcal{A}_n)$ are $\mathbb{Z}$-linear combinations of the columns of the matrix (16). For each $i = 1, \ldots, 4$, we can realize the columns of $\mathcal{L}_i$ as the first column of $\mathcal{L}_i$ after applying a permutation $\sigma \in \mathfrak{S}_n$ to it. For instance, when $n = 5$, the first column of $\mathcal{L}_1$ is the vector $(2, -2, 0, 0, -1, 1, 0, \ldots, 0)^\top \in \mathbb{Z}^{20}$, which corresponds to the binomial $x_{12}^2 x_{31} - x_{13}^2 x_{21}$. If we apply the transposition $(3\,5) \in \mathfrak{S}_5$ to this element, we get $x_{12}^2 x_{51} - x_{15}^2 x_{21}$, whose corresponding integer vector is precisely the third column of $\mathcal{L}_1$; namely, $(2, 0, 0, -2, -1, 0, 0, 1, 0, \ldots, 0)^\top \in \mathbb{Z}^{20}$.

In general, for every $n$ and for $i = 1, 2, 3$, the transposition $(3\,j)$ with $4 \leq j \leq n$ applied to the binomial corresponding to the first column of $\mathcal{L}_i$ will be equal to the binomial whose support corresponds to the $(j-2)$-th column of $\mathcal{L}_i$. For $\mathcal{L}_4$, instead of transpositions, we use those permutations that send the pair $(3, 4)$ to $(i, j)$ for $3 \leq i \neq j \leq n$ to write those binomials corresponding to the columns of $\mathcal{L}_4$ in terms of the first column of $\mathcal{L}_4$. For instance, the binomial $x_{12}^4 x_{34} - x_{14} x_{13}^2 x_{21}^2$ has support the first column of $\mathcal{L}_4$. When we apply the permutation $(3\ 4\ 5) \in \mathfrak{S}_5$ to this binomial, we get $x_{12}^4 x_{45} - x_{15} x_{14}^2 x_{21}^2$, which has support the 5th column of $\mathcal{L}_4$. Consider the set

$$H^{\pm} = \{x_{12}^2 x_{31} - x_{13}^2 x_{21}, x_{12}^2 x_{23} - x_{13} x_{21}^2, x_{12}^3 x_{32} - x_{13}^2 x_{21}^2, x_{12}^4 x_{34} - x_{13}^2 x_{14} x_{21}^2\}$$

of binomials corresponding to the first column of each $\mathcal{L}_i$. The action of $\mathfrak{S}_5$ on $H^{\pm}$ produces generators for the Laurent ideal $I_5^{\pm}$ corresponding to the toric ideal $I_5$, by Lemma 23. In general for $n \geq 5$, the action of $\mathfrak{S}_n$ on $H^{\pm}$ produces generators for the Laurent ideal $I_n^{\pm}$. We thus obtain a generating set for the chain $I_\circ^{\pm}$

that depends only on the description of $\ker_{\mathbb{Z}}(\mathcal{A}_n)$ and is independent from the methods used in the proof of Theorem 4.

Unfortunately, we could not generalize this technique to other cases as the combinatorics that describe $\ker_{\mathbb{Z}}(\mathcal{A}_n)$ in general becomes more complicated. We also remark that the set $H^{\pm}$ fails to be a generating set for the (non-Laurent) chain of ideals induced by $y_1^2 y_2$.

## 5. Algorithms

The proof of Theorem 4 suggests an algorithm to find the generators of a chain of Laurent toric ideals induced by a monomial $\mathbf{y}^\alpha$. We stated the existence of such an algorithm in Theorem 7 from the introduction. In this section we describe this algorithm and argue its correctness. A full implementation in `Macaulay2` Grayson and Stillman (????) can be found in Hillar and Martín del Campo (2010).

**Algorithm 1** [Theorem 7]

**Input:** Exponent vector $\alpha \in \mathbb{N}^k$
**Output:** Generators for the chain of Laurent ideals defined by $\mathbf{y}^\alpha$ up to symmetry
1: $d := 2 \cdot |\alpha|$
2: Compute the matrix $\mathcal{B}_d$ (11)
3: Compute the Gröbner basis $\mathcal{G}$ of the toric ideal $I_{\mathcal{B}_d}$
4: **for all** $g \in \mathcal{G}$ **do**
5:     **for all** indeterminates $x_w$ in $g$ **do**
6:         **if** $x_w$ is not indexed by a permutation of $\alpha$ **then**
7:             $g :=$ replace $x_w$ in $g$ by the monomial quotient $\mu(x_w)$
8:         **end if**
9:     **end for**
10: **end for**
11: Remove redundant generators from $\mathcal{G}$
12: **return** $\mathcal{G}$

Given an exponent vector $\alpha \in \mathbb{N}^k$, the algorithm computes a set of generators for the chain of Laurent ideals defined by $\mathbf{y}^\alpha$ up to the action of the symmetric group. In the first steps, it considers all the integer partitions of $d = 2 \cdot |\alpha|$ with parts at most $\max \alpha := \max\{\alpha_1, \ldots, \alpha_n\}$, and then constructs the matrix $\mathcal{B}_d$ by taking all the permutations of such partitions.

In step 3, the algorithm constructs the toric ideal $I_d$ that corresponds to the matrix $\mathcal{B}_d$ and computes its Gröbner basis $\mathcal{G}$ (with respect to any term order). This Gröbner basis computation is the most expensive step for large ideals. We decided to use the `Macaulay2` package `FourTiTwo`, which invokes one of the fastest routines, `4ti2`, specializing in computing Gröbner bases for toric ideals 4ti2 team (????).

Step 11 removes the redundant generators from $\mathcal{G}$. Using Lemma 32, we start by removing all the non-quadratic generators from $\mathcal{G}$. We then remove the symmetric orbit of each of the remaining generators. To illustrate how drastically the number of generators is decreased after this step, consider once more the running example of Section 4. When $\mathbf{y}^\alpha = y_1^2 y_2$, the Laurent toric chain has a stabilization bound at $n = 6$; for this value of $n$, the toric ideal $I_6 \subset R_6$ has 270 minimal generators. When we lift to the ideal $\widetilde{I}_6 \subset \widetilde{R}_6$, we obtain 849 minimal generators, but only 13 modulo the action of the symmetric group. From those, we find that 11 generate the corresponding Laurent ideal modulo the symmetric group. But after clearing denominators and common monomial factors, we found that only 8 from those 11 (exactly those 8 that are presented in the introduction of Section 4) form a generating set of the Laurent ideal $I_6^{\pm}$ modulo the action of the symmetric group. Since the number of generators increase when passing to the ring $\widetilde{R}_n$, one way to improve speed on this orbit removal step is to remove the orbit after Step 3 and again in Step 11.

The core of our algorithm is Step 7, where we turn Lemma 27 and Proposition 33 into a computational tool. We unfold this step in Algorithm 2 below. This algorithm expresses every element of the column span of $\mathcal{B}_d$ as a linear combination of the column span of the matrix $\mathcal{A}_d$ using the construction found in the proof of Lemma 27, and detailed in Algorithm 3 below. This integer decomposition is then used to create the map $\mu$ in (10).

---

**Algorithm 2** Construction of the map $\mu$

---

**Input:** indeterminate $x_w$, exponent vector $\alpha \in \mathbb{N}^k$ with $\gcd(\alpha) = 1$
**Output:** monomial quotient $\mu(x_w)$

Set $V := \{\sigma\alpha \mid \sigma \in \mathfrak{S}_n\}$
**if** $u \notin V$ **then**
    Write $w = b_1 v_1 + \cdots + b_r v_r$ with $b_i \in \mathbb{Z}$ and $v_i \in V$ for all $i \in [r]$ (Lemma 27)
    $indexB_+ := \{i \in [r] \mid b_i > 0\}$ and $indexB_- := \{i \in [r] \mid b_i < 0\}$
    $numerator := 1$ and $denominator := 1$
    **for all** $i \in indexB_+$ **do**
        $numerator = numerator \cdot x_{w_i}^{b_i}$
    **end for**
    **for all** $i \in indexB_-$ **do**
        $denominator = denominator \cdot x_{w_i}^{-b_i}$
    **end for**
    **return** $numerator/denominator$
**else**
    **return** $x_w$
**end if**

---

**Algorithm 3** Integer decomposition of $\beta$ in terms of $\alpha$

---

**Input:** Integer vector $\beta \in \mathbb{Z}^n$, exponent vector $\alpha \in \mathbb{N}^k$ with $\gcd(\alpha) = 1$ and $|\alpha|$ dividing $|\beta|$
**Output:** List $\{\{a_\sigma, v_\sigma\} \mid \sigma \in \mathfrak{S}_n\}$ such that $\beta = \sum_{\sigma \in \mathfrak{S}_n} a_\sigma v_\sigma$, where $v_\sigma = \sigma \cdot \alpha$, and $a_\sigma \in \mathbb{Z}$

Write $1 = b_1\alpha_1 + \cdots + b_k\alpha_k$, where $\alpha = (\alpha_1, \ldots, \alpha_k)$ and $b_i \in \mathbb{Z}$
$q := |\beta|/|\alpha|$
$L := \{\{q, v_\alpha\}\}$, where $v_\alpha = (\alpha_1, \ldots, \alpha_k, 0, \ldots, 0) \in \mathbb{Z}^n$
**for** $j$ from 1 to $n-1$ **do**
    **if** $\beta_j - \alpha_j \cdot q \neq 0$ **then**
        **for** $i$ from 1 to $k$ **do**
            $L = L \cup \{\{(\beta_j - \alpha_j \cdot q)b_i, v_{\sigma_{ij}}\}, \{-b_i(\beta_j - \alpha_j \cdot q), v_{\tau_j \sigma_{ij}}\}\}$, where $\sigma_{ij}$ and $\tau_j$ are as in the proof of Lemma 27
        **end for**
    **end if**
**end for**
**return** $L$

---

## 6. Open problems and conjectures

Stabilization of chains of ideals is unexpected and important for applications. However, the problem of deciding whether a chain is stable under the action of a group seems difficult, even for the special case of the symmetric group. In this section, we present some conjectures based on computational evidence. We focus first when the ideals comprising the chain are toric ideals as they tend to have rich combinatorial structure; later, we turn to a more general setting and close with some problems that develop this topic further.

Motivated by the study of bounds on the Castelnuovo-Mumford regularity in algebraic geometry, Bayer and Mumford introduced in Bayer and Mumford (1993) the *degree-complexity* of a homogeneous ideal $I$ with respect to a term order $\preceq$ as the maximal degree in a reduced Gröbner basis of $I$, and this is the largest degree of a minimal generator of $\mathrm{in}_{\preceq}(I)$. In our context, degree-complexity is important because it is closely related to stabilization of chains of ideals. For instance, in the proof of Theorem 4, we exploited the fact that the ideal $\widetilde{I}_n$ has degree-complexity 2 for every $n$. On the other hand, if the ideals comprising a chain induced by a monomial do not have a degree-complexity bound, then stabilization for such chain is impossible (this is elementary; for instance, see (Aschenbrenner and Hillar, 2007, Lemma 5.4)).

We pose the following problems based on our observations in Table 6 (computed using our software Hillar and Martín del Campo (2010)).

**Conjecture 37.** *Let $\alpha = (\alpha_1, \alpha_2)$ with $\gcd(\alpha_1, \alpha_2) = 1$ (suppose $\alpha_1 \geq \alpha_2$). The degree-complexity of $I_n$ is of the form $2\alpha_1 - \alpha_2$ for all (but possible finitely many) ideals $I_n$ in the chain of (non-Laurent) toric ideals induced by the monomial $\mathbf{y}^\alpha$.*

**Problem 38.** Let $\alpha \in \mathbb{N}^k$ with $\gcd(\alpha) = 1$; is the degree-complexity of $I_n$ constant as $n \to \infty$?

Recall the set $\mathcal{B}_n$ from (11). Using the fact that $I_{\mathcal{B}_n}$ is generated by quadratics we show in this paper that for $\mathcal{A}_n \subseteq \mathcal{B}_n$, the chain of ideals $I_{\mathcal{A}_n}$ has a corresponding Laurent chain $I^{\pm}_{\mathcal{A}_n}$ that is stable under the action of $\mathfrak{S}_{\mathbb{P}}$. On the other hand, Conjecture 25 makes the stronger claim that the chain $I_{\mathcal{A}_n}$ stabilizes. While it is difficult to find subsets $\mathcal{C}_n \subseteq \mathcal{B}_n$ for which the chain of ideals $I_{\mathcal{C}_n}$ is stable under the action of $\mathfrak{S}_{\mathbb{P}}$, one might get some indications by solving the following problem.

**Problem 39.** Determine subsets $\mathcal{C}_n \subseteq \mathcal{B}_n$ such that the toric ideal $I_{\mathcal{C}_n}$ has constant degree-complexity as $n$ grows.

This problem is of particular interest in algebraic statistics. For instance, in (Haws et al., 2011, Conjecture 7.3), it is conjectured that for any $T \geq 3$ and a fixed $S \geq 3$, the toric ideals of the homogeneous Markov chain model on $S$ states are generated by polynomials of degree at most $S-1$. This is an instance of Problem 39, as for each $T \geq 3$, the design matrix of such a model is precisely a subset of the matrix $\mathcal{B}_n$ for $n = T$.

The early stabilization of structured chains appears to be common. It would be interesting to construct examples of chains with nontrivial lower bounds on stabilization.

**Problem 40.** Let $f(d)$ be an increasing function $f : \mathbb{P} \to \mathbb{P}$. Find a family of invariant chains $\left\{ I_\circ^{(d)} \right\}_{d=1}^{\infty}$ (over $\mathcal{R}_{\mathbb{P}}$ or $R_{\mathbb{P}}$) which have stabilization bound at least $f(d)$.

More specifically, we ask whether a linear lower bound holds for the chains in Theorem 4 and their Laurent counterparts.

**Problem 41.** Is there a constant $C > 0$ such that the chains $\left\{ I_\circ^{(d)} \right\}_{d=1}^{\infty}$ from Theorem 4 must have stabilization bounds at least $f(d) = Cd$.

| $\alpha \setminus n$ | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|
| $(1,1)$ | 1 | 2 | 2 | 2 | 2 | 2 |
| $(2,1)$ | 3 | 3 | 3 | 3 | 3 | 3 |
| $(3,1)$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $(4,1)$ | 7 | 7 | 7 | 7 | 7 | 7 |
| $(5,1)$ | 9 | 9 | 9 | 9 | 9 | 9 |
| $(6,1)$ | 11 | 11 | 11 | 11 | 11 | - |
| $(7,1)$ | 13 | 13 | 13 | 13 | 13 | - |
| $(8,1)$ | 15 | 15 | 15 | 15 | - | - |
| $(3,2)$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $(4,2)$ | 3 | 3 | 3 | 3 | 3 | 3 |
| $(5,2)$ | 8 | 8 | 8 | 8 | 8 | 8 |
| $(6,2)$ | 5 | 5 | 5 | 5 | 5 | 5 |
| $(7,2)$ | 12 | 12 | 12 | 12 | 12 | - |
| $(1,3,2)$ | 3 | 3 | 3 | 3 | - | - |
| $(4,3,2)$ | 3 | 5 | 5 | 5 | - | - |

**Table 1.** Degree-complexity of the toric ideal $I_n$ defined by $\mathbf{y}^\alpha$

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