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ANALOG OF HILBERT BASIS THEOREM FOR INFINITELY GENERATED COMMUTATIVE ALGEBRAS

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Let F[X]be the free commutative and associative algebra (with unity) over a field F generated by the infinite set X. We give a new proof of the theorem of Aschenbrenner - Hillar: The set of all ideals of F[X]closed under the permutations of the generators from X satisfies the ascending chains condition.

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1. Introduction

Let F[X]be the free commutative and associative algebra (with unity element) over a field F generated by the set X.

Denote by $S(X)$ the group of finite permutations of the infinite set X. If $f = f(x_1, \dots, x_m)$ is a polynomial ($x_i \in X, f \in F[X]$), $\sigma \in S(X)$, then we put

$$\sigma f = f(\sigma(x_1), \dots, \sigma(x_m)).$$

An ideal I of the algebra F[X]is said to be an S-ideal iff $\sigma f \in I$ for every polynomial $f \in I$ and every permutation $\sigma \in S(X)$.

Very recently M. Aschenbrenner and C. J Hillar [2] proved the following

Theorem 1.1. *The set of all S-ideals satisfies the ascending chains condition.*

In the present paper we give a new proof of this theorem.

2. Reductions

Let $u \in F[X]$ be a monomial,

$$u = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_m}^{n_m},$$

where $n_1 \geq n_2 \geq \cdots \geq n_m > 0, x_{i_k} \in X, x_{i_j} \neq x_{i_k}$ for $j \neq k$. The sequence

$$t(u) = (n_1, n_2, \ldots, n_m)$$

is called the type of the monomial u. If $u = 1$ then we put $t(u) = (0)$. We order the types lexicographically. We denote by $[X]$ the set of all monomials. For a given nonzero polynomial $f \in F[X]$,

$$f = \sum_{u \in [X]} \alpha_u u, \qquad \alpha_u \in F,$$

we define the $t(f)$ as the maximal $t(u)$ such that $\deg u = \deg f$ and $\alpha_u \neq 0$. As usual we call the polynomial

$$\overline{f} = \sum_{t(u)=t(f)} \alpha_u u$$

the senior part of the polynomial f.

In the paper we often use the following evident remark

Remark 1.

1. Let $f \in F[X]$,

$$f = \sum_{u \in [X]} \alpha_u u,$$

$t(f) = (n_1, n_2, \ldots, n_m), v = x_1^{a_1} \cdots x_k^{a_k}, k \leq m, a_1 \geq a_2 \cdots \geq a_k \geq 0$. Assume the monomial $x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}$ appears in f with nonzero coefficient then

$$\overline{fv} = \sum_{u \in G} \alpha_u uv,$$

where G is a set of all monomials of type $t(f)$ which are divisible by $x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}$.

2. If the polynomial f does not depend on the variables $x_1, \ldots x_k$ then $\overline{fv} = \overline{f}v$.

Lemma 1. *For every nonzero S-ideal I there exist numbers r, m, c such that for every $n_1, \ldots, n_m$, satisfying the properties $n_i - n_{i+1} \geq r, i = 1, \ldots, m-1$, $n_1 \geq n_2 \geq \cdots \geq n_m \geq c$ the ideal I contains a polynomial whose senior part equals*

$$u = x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m},$$

where $x_k \in X, x_j \neq x_k$ for $j \neq k$.

Proof. We take a polynomial $f \in I$, $f \neq 0$. Let $t(f) = (d_1, \cdots, d_m)$. Since I is an S-ideal, we may assume that $\overline{f}$ contains the monomial $x_1^{d_1} \cdots x_m^{d_m}$ with nonzero coefficient α. We put

$$r = \max_{i \leq m-1} d_i - d_{i+1}, \qquad c = d_m$$

and prove that r, m, c are the desired numbers. Take arbitrary $n_1, \cdots, n_m$, $n_i - n_{i+1} \geq r$, $n_1 \geq n_2 \geq \cdots \geq n_m \geq c$. Put $a_i = n_i - d_i$. Since $n_i - n_{i+1} \geq r \geq d_i - d_{i+1}$ then $a_i = n_i - d_i \geq n_{i+1} - d_{i+1} = a_{i+1}$. Besides $a_i \geq 0$ for every

i because $n_m - d_m \geq n_m - c = 0$. Put $v = x_1^{a_1} \cdots x_m^{a_m}$. By Remark 1 $\overline{f}v = \alpha x_1^{d_1+a_1} \ldots x_m^{d_m+a_m} = \alpha x_1^{n_1} \ldots x_m^{n_m}$. The lemma is proved. $\square$

Let $t = (n_1, n_2, \ldots, n_m)$ be an arbitrary nonzero type. We can write the type t in the form

$$t = (d_1, d_1, \ldots, d_1, d_2, d_2, \ldots, d_2, \ldots, d_h, d_h, \ldots, d_h),$$

where $d_1 > \cdots > d_h > 0$. We put $h(t) = h$ and call this number the height of type t. We call also the height of the polynomial f the number $h(t(f))$.

Lemma 2. *For any nonzero S-ideal I there exists number h such that every polynomial is equal modulo I to some polynomial whose height is less than or equal to h.*

Proof. Obviously, it is sufficient to prove the conclusion of lemma only for an arbitrary monomial u. We put $h = (m-1)r + c$ where m, r, c are taken from the conclusion of Lemma 1. We prove that h is the desired number. Assume that the set M of the monomials for which the conclusion of lemma fails is not empty. We choose $v \in M$ so that $t(v) = (l_1, \ldots, l_k)$ is minimal. Since $v \in M$ then $h(t(v)) > h = (m-1)r + c$. It follows from this that the type $t(v)$ contains a subsequence $(n_1, n_2, \ldots, n_m)$ with property $n_i - n_{i+1} \geq r, i = 1, \ldots, m-1, n_1 \geq n_2 \geq \cdots \geq n_m \geq c$. By lemma 1 v can be written modulo I as a linear combination of monomials u_j with $t(u_j) < t(v)$. Since $t(u_j) < t(v)$, then $u_j \notin M$, and each u_j can be written modulo I as a linear combination of monomials whose heights are less than or equal to h. This contradicts the assumption that $v \in M$. So $M = \emptyset$ and the lemma is proved. $\square$

We also use in our further reductions the following evident lemma which does not concern polynomials

Lemma 3. *Let $b_n^{(i)}, i = 1, \ldots, m, n = 1, 2, \ldots$ be m infinite sequences of natural numbers. Then there exists an increasing sequence of numbers $n_1 < n_2 < \ldots$ such that for every $i \leq m$ the subsequence $b_{n_k}^{(i)}$ is either constant or monotone increasing.*

Proof. We consider the first sequence $b_n^{(1)}$. Since the members of this sequence are the natural numbers we can choose an infinite subsequence which is either constant (if the sequence is bounded) or monotone increasing (if the sequence is unbounded). We denote the chosen subsequense by $b_{l_r}^{(1)}$ and consider the sequence $b_{l_r}^{(2)}$. As above we choose an infinite constant or monotone increasing subsequence $b_{s_q}^{(2)}$. Continue this process, finally we obtain the desired sequence $n_1 < n_2 < \ldots$. The lemma is proved. $\square$

Assume that the set of all S-ideals does not satisfy the ascending chains condition. It means that there exists an infinite independent sequence of polynomials $f_1, f_2, \ldots$, i.e. for every n S-ideal I_n generated by $f_1, \ldots, f_{n-1}$ does not contain f_n.

If I is an S-ideal then we define a relative type $t_I(f)$ as a minimal type $t(g)$, where $g \in f + I$. Evidently $t_I(f) \leq t(f)$.

We may assume the sequence of polynomials $f_1, f_2, \ldots$ satisfies the following

Property 1. For every n the polynomial f_n has a minimal type among the polynomials in the coset $f_n + I_n$ i.e. $t_{I_n}(f_n) = t(f_n)$.

Remark 2. Any subsequence $f_{n_1}, f_{n_2}, \ldots$ of the sequence $f_1, f_2, \ldots$ also satisfies Property 1.

Indeed, denote by I'_k the S-ideal generated by $f_{n_1}, f_{n_2}, \ldots, f_{n_{k-1}}$. Evidently $I'_k \subseteq I_{n_k}$. Then we have

$$t(f_{n_k}) \geq t_{I'_k}(f_{n_k}) \geq t_{I_{n_k}}(f_{n_k}) = t(f_{n_k}).$$

By Lemma 2 applied to S-ideal $I = I_2$ the sequence $h(t(f_n))$ is bounded. Then we can choose an infinite subsequence of polynomials f_{n_k} with constant $h(t(f_{n_k}))$. So, by Property 1 and Remark 2 we may assume that the sequence of polynomials $f_1, f_2, \ldots$ satisfies the following

Property 2. There exists the natural constant h such that $h(t(f_n)) = h$ for every n.

Let

$$t(f_n) = (d_n^{(1)}, \ldots d_n^{(1)}, \ldots, d_n^{(h)}, \ldots, d_n^{(h)}).$$

Let $\delta_n^{(i)} = d_n^{(i)} - d_n^{(i+1)}$, $i = 1, \ldots, h-1$. Applying Lemma 3 to the defined sequences and using the Property 1 and Remark 2 as above we may assume that the sequence of polynomials $f_1, f_2, \ldots$ satisfies the following

Property 3. The sequences $\delta_n^{(i)}$ are monotone increasing for

$$i = m_1, \ldots, m_{k-1}, m_1 < m_2 < \ldots m_{k-1} \leq h,$$

and constant for the other i (if all these sequences are constant then by definition $k = 1$).

Let $m_1, \ldots, m_{k-1}$ be the numbers defined right now, $m_0 = 0, m_k = h$. We put $n_i = m_i - m_{i-1}, 1 \geq i \geq k$. It is evident $\sum\limits_{i=1}^{k} n_i = h$. Denote by λ the partition $(n_1, n_2, \ldots, n_k)$ of the number h. Let

$$t = (d_1, \ldots, d_1, d_2, \ldots, d_2, \ldots, d_h, \ldots, d_h), d_1 > d_2 > \ldots > d_h > 0$$

be an arbitrary type of height h. Put

$$\lambda_i(t) = (d_{m_{i-1}+1}, \ldots, d_{m_{i-1}+1}, d_{m_{i-1}+2}, \ldots, d_{m_{i-1}+2}, \ldots, d_{m_i}, \ldots, d_{m_i}),$$

where $i = 1, \ldots, k$. If u is a monomial of type t and x is a variable from X, $\deg_x u > 0$, then we define λ-degree of monomial u with respect to x as follows: $\lambda \deg_x u = \lambda_i(t)$ iff the number $\deg_x u$ occurs in $\lambda_i(t)$.

Definition Let $f = f(x_1, \ldots, x_n)$ be a polynomial. We call f λ-homogeneous iff there exists some type t of height h such that f can be represented as a linear combination of monomials of type t having equal λ-degrees with respect to each variable x_i. These λ-degrees are called respectively the λ-degrees of f in x_i.

Divide the set of variables X into k infinite mutually exclusive subsets $X_1, \ldots, X_k$. We denote by $S(X_1, \ldots, X_k)$ the subgroup of the group $S(X)$ consisting of all the finite permutations σ with the property: $\sigma(X_i) = X_i, i = 1, \ldots, k$.

Let $d_1 > d_2 > \cdots > d_h > 0$. Denote by $H(d_1, \ldots, d_h)$ the set of all λ-homogeneous polynomials f satisfying the properties:

1. $t(f) = (d_1, \ldots d_1, \ldots, d_h, \ldots, d_h)$;

2. If $\lambda \deg_x f = \lambda_i(t(f))$ then $x \in X_i$.

We call a subset $\Delta \subseteq H(d_1, \ldots, d_h)\Delta$ an SI-set if it satisfies the following properties:

1. If the polynomials $f, g \in \Delta$ have equal λ-degrees in each variable $x \in X, \alpha, \beta \in F$ then $\alpha f + \beta g \in \Delta$;

2. If a monomial u belongs to $H(d_1, \ldots, d_h), f \in \Delta$ and the polynomial f does not depend on the variables appearing in u then $uf \in \Delta$;

3. If $f \in \Delta, \tau \in S(X_1, \ldots X_k)$ then $\tau f \in \Delta$.

Remark 3. Let $d'_1 > \ldots d'_h > 0$. We establish one-to-one correspondence between the sets $H(d_1, \ldots, d_h)$ and $H(d'_1, \ldots, d'_h)$ by the rule: we change the degrees d_i by d'_i in every polynomials from the set $H(d_1, \ldots, d_h)$. It is evident that SI-sets of $H(d_1, \ldots, d_h)$ will correspond to the SI-sets of $H(d'_1, \ldots, d'_h)$. In particular, if the set of all SI-sets of $H(d_1, \ldots, d_h)$ satisfies the ascending chains condition then the same will be true for the SI-sets of $H(d'_1, \ldots, d'_h)$.

Proposition 2.1. *If the set of all SI-sets satisfies the ascending chains condition then the set of all S-ideals also satisfies the ascending chains condition.*

Proof. Assume the contrary. Then there exists an infinite sequence of polynomials f_n satisfying Properties 1, 2, 3 formulated above.

Let

$$t(f_n) = (d_n^{(1)}, \ldots d_n^{(1)}, \ldots, d_n^{(h)}, \ldots, d_n^{(h)}).$$

For $n > m$ we put $a_{n,m}^{(i)} = d_n^{(i)} - d_m^{(i)}$ and prove the inequality

$$a_{n,m}^{(i)} \geq a_{n,m}^{(i+1)}, \qquad \text{for } i < h. \tag{1}$$

Indeed

$$a_{n,m}^{(i)} - a_{n,m}^{(i+1)} = (d_n^{(i)} - d_n^{(i+1)}) - (d_m^{(i)} - d_m^{(i+1)}) = \delta_n^{(i)} - \delta_m^{(i)}.$$

By Property 3 the sequence $\delta_n^{(i)}$ is monotone nondecreasing. Hence $\delta_n^{(i)} - \delta_m^{(i)} \geq 0$ and the inequality (1) is proved.

We put also $d_i = d_1^{(i)}, a_n^{(i)} = a_{n,1}^{(i)}$. We have an evident equality

$$a_{n,m}^{(i)} = a_n^{(i)} - a_m^{(i)}. \tag{2}$$

Let H be the set of all λ-homogeneous polynomials whose type equals $t(f_n)$ for some n. We define the mapping $\phi : H \to H(d_1, \ldots, d_h)$ as follows. Let $g \in H, t(g) = t(f_n)$ then evidently there exists a permutation $\sigma \in S(x)$ such that

$$\sigma g = h(x_1^{(1)}, \ldots, x_{s_1}^{(1)}, \ldots, x_1^{(h)}, \ldots, x_{s_h}^{(h)}),\ \ x_j^{(i)} \in X_i,\ \ \lambda \deg_{x_j^{(i)}} \sigma g = \lambda_i(t(g)).$$

The polynomial σg is divisible by the monomial

$$w = (x_1^{(1)} \cdots x_{s_1}^{(1)})^{a_n^{(1)}} \cdots (x_1^{(h)} \cdots x_{s_h}^{(h)})^{a_n^{(h)}}. \tag{3}$$

It is easy to see

$$\widetilde{g} = \frac{\sigma g}{w} \in H(d_1, \ldots, d_h).$$

We put $\phi(g) = \widetilde{g}$.

The senior part $\overline{f}_n$ of the polynomial f_n can be written as a sum of λ-homogeneous polynomials $g_n^{(i)}$, where $i = 1, \ldots, r_n$. Denote by Δ_j an SI-set generated by all the polynomials $\widetilde{g}_l^{(i)}$ where $l < j$. By the assumption $\Delta_{n+1} = \Delta_n$ for some n. We prove that for every $i \le r_n$ there exists a polynomial f in the S-ideal I_n generated by $f_1, \ldots, f_{n-1}$ such that $\overline{f} = g_n^{(i)}$.

Since $\widetilde{g}_n^{(i)} \in \Delta_n$ then $\widetilde{g}_n^{(i)}$ can be written as a finite sum of the form

$$\widetilde{g}_n^{(i)} = \sum_{j<n, i\le r_j} \sum_{\tau \in S(X_1, \ldots, X_k)} \sum_{u \in M_{j,i}} \alpha_{j,i,\tau,u} \tau u \widetilde{g}_j^{(i)},$$

where $\alpha_{j,i,\tau,u} \in F, M_{j,i}$ is the set of all monomials u satisfying the property: polynomial $\widetilde{g}_j^{(i)}$ does not depend on the variables occurring in u. Multiplying both sides of this equality by some monomial w of form (3) we obtain that it is sufficient to prove that for every $j < n$ and $i \le r_j$ the S-ideal I_n contains a polynomial whose senior part equals $\widetilde{g}_j^{(i)} w'$, where w' is a monomial of form (3) and the polynomial $\widetilde{g}_j^{(i)}$ depends on each variable $x_s^{(r)}, r \le h, s \le s_r$.

Indeed, take the polynomial f_j. We have the equality

$$\overline{f}_j = \sum_{i=1}^{r_j} g_j^{(i)}.$$

Since $f_j \in I_n$ and I_n is an S-ideal we may assume that

$$g_j^{(i)} = g_j^{(i)}(x_1^{(1)}, \ldots, x_{s_1}^{(1)}, \ldots, x_1^{(h)}, \ldots, x_{s_h}^{(h)}).$$

Put

$$w_1 = (x_1^{(1)} \cdots x_{s_1}^{(1)})^{a_{n,j}^{(1)}} \cdots (x_1^{(h)} \cdots x_{s_h}^{(h)})^{a_{n,j}^{(h)}}.$$

By (1), Remark 1 and (2)

$$\overline{f}_j w_1 = g_j^{(i)} w_1 = \widetilde{g}_j^{(i)} w'.$$

So we have proved that there exists a polynomial in the S-ideal I_n whose senior part equals $g_n^{(i)}$.

It folows from the proof that $\overline{f}_n = \overline{f}$ for some $f \in I_n$. This contradicts Property 1 of the sequence of polynomials f_n because $f_n - f \in f_n + I_n$ and $t(f_n - f) < t(f_n)$. Proposition is proved. $\square$

3. Higman's Lemma

In this very short section we recall the famous Higman's Lemma and some of its applications.

Let A be a partially ordered set. The set A is called *well-ordered* iff for every infinite sequence $\{a_i\}_{i\geq 1}$, $a_i \in A$ there exist numbers i and j such that $i < j$ and $a_i \leq a_j$.

Consider the set $S(A)$ of all finite sequences of elements from the set A. We introduce the following partial ordering in $S(A)$: $(a_1, a_2, \ldots, a_k) \leq (b_1, b_2, \ldots, b_l)$ iff there exist indices $i_1 < i_2 < \ldots < i_k$ satisfying the property $a_r \leq b_{i_r}$, $r = 1, \ldots, k$.

Lemma 4. ***(Higman*** [1]*) If A is well-ordered, then $S(A)$ is also well-ordered.*

Let B be a finite set and $\langle B\rangle$ be the free associative semigroup generated by B. $\langle B\rangle$ is a partially ordered set with respect to the following ordering: $u \leq v$, iff $u = b_1 b_2 \ldots b_k$ and $v = w_1 b_1 w_2 b_2 \ldots b_k w_{k+1}$ for some $b_1, \ldots, b_k \in B$ and $w_1, \ldots, w_{k+1} \in \langle B\rangle$.

Corollary 1. *For every infinite sequence $\{u_i\}_{i\geq 1}$, $u_i \in \langle B\rangle$ there exist $i < j$ such that $u_i \leq u_j$.*

Proof. The statement of the lemma is a direct consequence of Higman's lemma applied to an unordered set B. $\square$

Corollary 2. *For every infinite sequence $\{w_i\}_{i\geq 1}$, $w_i \in \underbrace{\langle B\rangle \times \ldots \times \langle B\rangle}_{k}$ there exist $i < j$ such that $w_i \leq w_j$, i.e. $w_i = (u_i^{(1)}, \ldots, u_i^{(k)})$, $w_j = (u_j^{(1)}, \ldots, u_j^{(k)})$ and $u_i^{(l)} \leq u_j^{(l)}$, $l = 1, \ldots, k$.*

Proof. From the previous corollary it follows that $\langle B\rangle$ is a well-ordered set with respect to the introduced ordering. Applying Higman's lemma to the well-ordered set $\langle B\rangle$ we obtain the statement of the corollary. $\square$

4. Proof of the Theorem

Now we can prove the theorem.

Theorem 1.1. The set of all S-ideals satisfies the ascending chains condition.

Proof. By Proposition 1 it is sufficient to prove that the set of all SI-sets satisfies the ascending chains condition.

Let $X_i = \{x_1^{(i)}, x_2^{(i)}, \dots\}$.

Denote by H' the subset of $H(d_1, \dots, d_h)$ consisting of all polynomials f satisfying the property: if f depends on the variable $x_s^{(i)}$ then f depends on each variable $x_j^{(i)}$ for $j < s$. Let Δ be an SI-set. Evidently $\Delta_1 \cap H' = \Delta_2 \cap H'$ if and only if $\Delta_1 = \Delta_2$.

Let $Y_1, \dots, Y_k$ be sets of variables, $Y_j = \{y_1^{(j)}, \dots, y_{n_j}^{(j)}\}, F\langle Y_i\rangle$ be the free associative algebra with unity element generated by the set Y_i. Put

$$R = F\langle Y_1\rangle \otimes \ldots \otimes F\langle Y_k\rangle.$$

Denote by T the set of all tensors t of the form

$$t = u^{(1)} \otimes \ldots \otimes u^{(k)},$$

where $u^{(j)} \in \langle Y_j\rangle$ (we recall $\langle B\rangle$ is a free semigroup generated by B). Every element $g \in R$ can be written uniquely as a linear combination of the tensors from T

$$g = \sum_{t \in T} \alpha_t t, \quad \alpha_t \in F. \tag{4}$$

Let $t \in T, t = u^{(1)} \otimes \ldots \otimes u^{(k)}$. We define the degree of this tensor as follows: deg $t = (q_1, \dots q_k)$, where $q_j = \deg u_j$. If $t \in T, t = u^{(1)} \otimes \ldots \otimes u^{(k)}$, then we put

$$w(t) = u^{(1)} \cdots u^{(k)} \in \langle Y\rangle, \quad Y = Y_1 \cup \cdots \cup Y_k.$$

We call the element (4) homogeneous in each variable if the associative polynomial

$$\sum_{t \in T} \alpha_t w(t) \in F\langle Y\rangle$$

is homogeneous in each variable. Denote by H the set of all elements of R which are homogeneous in each variable. If the element $g \in H$ is written in the form (4) and deg $t = (q_1, \dots, q_k)$ for every tensors t occurring in (4) with nonzero coefficient then we put $\deg g = (q_1, \dots, q_k)$.

Let $S(q)$ be the symmetric group of permutations acting on the set $\{1, \dots, q\}, \tau \in S(q)$ and $w = y_{i_1} \cdots y_{i_q} \in \langle Y\rangle$. We put

$$w\tau = y_{i_{\tau(1)}} \cdots y_{i_{\tau(q)}}.$$

If $t \in T, \deg t = (q_1, \dots, q_k), \sigma \in S(q_1) \oplus \cdots \oplus S(q_k), \sigma = (\tau_1, \dots, \tau_k), \quad \tau_j \in S(q_j)$ then we put

$$t\sigma = u_i^{(1)}\tau_1 \otimes \ldots \otimes u_i^{(k)}\tau_k.$$

If an element $g \in H$ is written in the form (4) and $\deg g = (q_1, \dots, q_k)$ then we put

$$g\sigma = \sum_{t \in T} \alpha_t t\sigma.$$

We establish one-to-one correspondence $\psi : H \to H'$ as follows. Let $u_j \in \langle Y_j\rangle$, $u_j = y_{s_1}^{(j)} \ldots y_{s_q}^{(j)}$. We put

$$\psi_j(u_j) = (x_1^{(j)})^{d_{m_j+s_1}} \ldots (x_q^{(j)})^{d_{m_j+s_q}}$$

(numbers m_j are defined in Property 3). If $t = u_1 \otimes \ldots \otimes u_k \in T$, then we put

$$\psi(t) = \psi_1(u_1) \ldots \psi_k(u_k),$$

if an element $g \in H$ is written in the form (4) then we put

$$\psi(g) = \sum_{t \in T} \alpha_t \psi(t).$$

We call a two-sided ideal I of the algebra R an H-ideal if I is generated by the set $I \cap H$. We call the ideal I an SH-ideal if I is an H-ideal and satisfies the property: if $g \in I \cap H, \deg g = (q_1, \ldots, q_k), \sigma \in S_{q_1} \oplus \ldots \oplus S_{q_k}, \sigma = (\tau_1, \ldots, \tau_k)$, where $\tau_j \in S(q_j)$, then $g\sigma \in I \cap H$.

Prove that if Δ is an SI-set then for some SH-ideal I

$$\psi^{-1}(\Delta \cap H') = I \cap H.$$

Let I be the ideal generated by the set $\psi^{-1}(\Delta \cap H'), g \in I, \deg g = (q_1, \ldots, q_k)$. If $\tau = (\tau_1, \ldots, \tau_k) \in S_{q_1} \oplus \ldots \oplus S_{q_k}$, then $\psi(g\tau) = \tau' \psi(g)$, where τ' is a permutation from $S(X_1, \ldots, X_k)$ which maps $x_i^{(j)}$ into $x_{\tau^{-1}(i)}^{(j)}$, so by Property 3 in the definition of SI-sets, $\psi(g\tau) \in \Delta \cap H'$. Hence, it is sufficient to prove that for any tensor t of the form $t = 1 \otimes \ldots \otimes y_i^{(j)} \otimes \ldots \otimes 1$ $\psi(tg) \in \Delta \cap H'$ and $\psi(gt) \in \Delta \cap H'$. Indeed, $\psi(gt) = \psi(g)(x_{q_j+1}^{(j)})^{d_{m_j+i}} \in \Delta \cap H'$ by Property 2 in the definition of SI-sets. We also have $\psi(tg) = \tau(\psi(g)(x_{q_j+1}^{(j)})^{m_j+i})$, where τ is a cycle permutation from $S(X_1, \ldots, X_k)$ which maps $x_1^{(j)}, x_2^{(j)}, \ldots, x_{q_j+1}^{(j)}$ into $x_2^{(j)}, \ldots, x_{q_j+1}^{(j)}, x_1^{(j)}$ respectively. So, by properties 2 and 3 in the definition of SI-sets $\psi(tg)$ also belongs to $\Delta \cap H'$.

So it remains to prove that the set of all SH-ideals satisfies the ascending chains condition.

We introduce the following ordering on the sets Y_j: $y_{i_1}^{(j)} \geq y_{i_2}^{(j)}$ iff $i_1 \geq i_2$, and order the monomials from $\langle Y_j \rangle$ lexicographically. We also introduce the following ordering of the elements from T: let $t_1, t_2 \in T$, $t_i = u_i^{(1)} \otimes \ldots \otimes u_i^{(k)}$, we put $t_1 \geq t_2$ iff $(u_1^{(1)}, \ldots, u_1^{(k)}) \geq (u_2^{(1)}, \ldots, u_2^{(k)})$, sequences compared lexicographically.

Let g be an arbitrary monomial from H represented in the form (4). We call the greatest t occurring in the right side of (4) with nonzero coefficient the senior tensor of g and denote it by $\overline{g}$.

Assume that the set of all SH-ideals does not satisfy the ascending chains condition. It means that there exists an infinite independent sequence of elements $g_1, g_2, \ldots \in H$, i.e. for every n SH-ideal I_n generated by $g_1, \ldots, g_{n-1}$ does not contain g_n.

We may also assume that g_n has the minimal senior tensor among the polynomials in the coset $g_n + I_n$.

Let $\overline{g_n} = u_n^{(1)} \otimes \ldots \otimes u_n^{(k)}$. We consider the sequence $\{(u_n^{(1)}, \ldots, u_n^{(k)}\}_{n \geq 1}$. By Corollary 2 of Higman's lemma, there exist numbers n, m such that $u_n^{(j)} = b_1^{(j)} b_2^{(j)} \ldots b_{l_j}^{(j)}$ and $u_m^{(j)} = w_0^{(j)} b_1^{(j)} w_1^{(j)} b_2^{(j)} \ldots b_{l_j}^{(j)} w_{l_j}^{(j)}$ for some $b_i^{(j)} \in Y_j, w_i^{(j)} \in \langle Y_j \rangle$. Denote by w the tensor $w_0^{(1)} w_1^{(1)} \ldots w_{l_1}^{(1)} \otimes \ldots \otimes w_0^{(k)} w_1^{(k)} \ldots w_{l_k}^{(k)}$, and let $\deg g_m =$

$(q_1, \dots q_k)$. It is easy to see that $(\overline{g_n}w)\sigma = \overline{g_m}$ for an appropriate $\sigma \in S_{q_1} \oplus \dots \oplus S_{q_k}$. Hence, there exists $\alpha \in F$ such that

$$\overline{g_m - (\alpha g_n)w\sigma} < \overline{g_m}.$$

This contradicts the minimality of $\overline{g_m}$. So the set of all SH-ideals satisfies the ascending chains condition. The theorem is proved. □

References

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