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On the Laws of a Metabelian Variety

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1. INTRODUCTION

Many varieties of groups are known to have a finite basis for their laws (for the definition and general properties of varieties, see [*6*]–[*8*]). Among such varieties are any variety consisting of nilpotent groups of bounded class [*4*] and any variety generated by a finite group [*9*]. It is not known whether all varieties have a finite basis for their laws. In this paper we prove the following theorem.

THEOREM A. *Any variety consisting of metabelian groups has a finite basis for its laws.*

Because of the relationship between varieties of groups and verbal subgroups of free groups, Theorem A is equivalent to

THEOREM B. *The free metabelian group of countable rank has maximum condition on verbal subgroups.*

Our main result is stronger than this.

THEOREM C. *A free metabelian group of arbitrary rank has maximum condition on characteristic subgroups.*

This may be compared with a result of Hall [*2*], which states that a finitely generated metabelian group has maximum condition on normal subgroups.

The proof uses a matrix representation, due to Magnus [*5*], to translate the problem to one of ring theory. The ring-theoretic result may be regarded as a generalization to an infinite number of variables of the result that the polynomial ring in a finite number of variables over a Noetherian ring is itself Noetherian. We begin with an interesting proposition on abstract closure relations, communicated to me by G. Higman, which greatly simplifies my original proofs of the ring-theoretic results.

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2. Ring Theory

An *algebraic closure relation* on a set C is a rule assigning to each subset X of C a subset $\mathrm{cl}\, X$ of C subject to the conditions (i) $X \subseteq \mathrm{cl}\, X$, (ii) if $X \subseteq Y$, then $\mathrm{cl}\, X \subseteq \mathrm{cl}\, Y$, (iii) $\mathrm{cl}\,\mathrm{cl}\, X = \mathrm{cl}\, X$, (iv) if $x \in \mathrm{cl}\, X$, then $x \in \mathrm{cl}\, X_0$, for some finite $X_0 \subseteq X$. Such a relation may be defined by giving the closed subsets of C, and the most important example for us occurs when C is a commutative ring and the closed sets are the ideals of C. An algebraic closure relation has *finite basis property* (f.b.p.) if every closed set is the closure of a finite subset. This is equivalent, by the well-known argument, to the maximum condition on closed sets.

A partially ordered set is said to be *partially well-ordered* if every infinite sequence of elements contains a (nonstrictly)-increasing subsequence (see [3] for equivalent definitions).

The following example will be needed. Let V be the set of all finite sequences of nonnegative integers, the empty set being regarded as a sequence of zero length. We define two orderings on V,

$(i_1, \dots, i_m) < (j_1, \dots, j_n)$ if $m < n$, or $m = n$ and, for some r, $i_r < j_r$ but $i_s = j_s$ for all $s > r$;

$(i_1, \dots, i_m) \preccurlyeq (j_1, \dots, j_n)$ if $m \leqslant n$, and, for some one-one order preserving map φ of $(1, \dots, m)$ into $(1, \dots, n)$, $i_r \leqslant j_{\varphi r}$ for all r. It is easy to check that $(V, \leqslant)$ is well-ordered, and that the identity map is an order-homomorphism from $(V, \preccurlyeq)$ to $(V, \leqslant)$. By theorem 4.3 of [3], $(V, \preccurlyeq)$ is partially well-ordered.

Now let C be any set with an algebraic closure relation and P any partly ordered set. We define on $C \times P$ an algebraic closure relation, which we say is *induced* by the closure relation on C and the partial ordering on P, by the requirement: $(c, p) \in \mathrm{cl}\, X$ iff $\exists (c_1, p_1), \dots, (c_n, p_n) \in X$ such that

$$c \in \mathrm{cl}(c_1, \dots, c_n)$$

and $p_i \leqslant p$, $i = 1, \dots, n$. (It is easy to check that this is an algebraic closure relation on $C \times P$.)

Proposition 1. *If the closure operation on C has* f.b.p., *and P is partially well-ordered, then the induced operation on $C \times P$ has* f.b.p.

Proof. Let X be any closed subset of $C \times P$. For any $p \in P$, define $X(p) \subseteq C$ by $X(p) \times p = X \cap (C \times p)$. By the definition of the operation in $C \times P$, $X(p)$ is a closed subset of C, and, if $p < q$ then $X(p) \subseteq X(q)$. We order the collection of indexed sets $X(p)$ for all p by $X(p) < X(q)$ iff $p < q$ and $X(p) = X(q)$ as sets.

Then, for any p, $\exists q$ with $X(p) > X(q)$, and $X(q)$ minimal. For otherwise there would be an infinite sequence $X(p_1) > X(p_2) > \cdots$, which is impossible since P, being partially well-ordered, contains no strictly-decreasing infinite sequence. Also there can be only a finite number of minimal elements. For if there were an infinite number, we could find an infinite sequence $p_1 < p_2 < \cdots$ such that each $X(p_i)$ is minimal. But then, if $i < j$ we have $X(p_i) \subseteq X(p_j)$ because $p_i < p_j$, and also $X(p_i) \neq X(p_j)$ because $X(p_j)$ is minimal. This is impossible since C has f.b.p.

Let, then, $X(p_1),\ldots, X(p_r)$ be the minimal elements. As C has f.b.p. we can find elements c_{ij}, $i = 1,\ldots, n_j$, such that $X(p_j) = \mathrm{cl}\{\bigcup_i c_{ij}\}$. Now take any $(c, p) \in X$. Then $c \in X(p)$, and, for some j, $X(p_j) < X(p)$, so that $c \in X(p_j)$. From the definition of the closure operation in $C \times P$, it follows that $X = \mathrm{cl}\{\bigcup_{i,j} c_{ij}\}$, as required.

We can now proceed to the ring theory. Let J denote the positive integers, and let R be a commutative ring and $S = R[x_1, x_2, \ldots]$. Let Φ denote the set of all one–one order preserving maps of J into itself.

Definition 1. *An ideal I of S is called a Φ-ideal if whenever the polynomial $p(x_1, x_2, \ldots) \in I$, then $p(x_{\phi 1}, x_{\phi 2}, \ldots) \in I$ for any $\varphi \in \Phi$.*

Definition 2. *Let M_0 be the free R-module with basis $t_1, t_2, \ldots$. A submodule N_0 of M_0 is called a Φ-submodule if whenever*

$$\sum r_i t_i \in N_0 \qquad \textit{then} \qquad \sum r_i t_{\varphi i} \in N_0 \qquad \textit{for any} \qquad \varphi \in \Phi.$$

Definition 3. *Let M be the free S-module with basis $t_1, t_2, \ldots$. A submodule N of M is called a Φ-submodule if whenever*

$$\sum p_i(x_1, x_2, \ldots)\, t_i \in N \quad \textit{then} \quad \sum p_i(x_{\varphi 1}, x_{\varphi 2}, \ldots)\, t_{\varphi i} \in N \quad \textit{for any} \quad \phi \in \Phi.$$

Proposition 2. *If R is a Noetherian domain then S has maximum condition on Φ-ideals.*

Proof. We define the *weight* of a monomial $ax_1^{i_1} \cdots x_n^{i_n}$ to be the sequence $(i_1, \ldots, i_n) \in V$. The *leading term* of a polynomial is the term of maximal weight in $(V, \leqslant)$ and the weight of a polynomial is the weight of its leading term. We can define a map $\theta : S \to R \times V$ by $\theta p = (b, wtp)$, where b is the leading coefficient of p. The closure operation on R in which the closed sets are ideals has f.b.p. because R is Noetherian. By Proposition 1, the closure operation on $R \times V$ induced from this operation on R and the ordering $\leqslant$ on V has f.b.p. It is easy to see that if I is a Φ-ideal of S, then θI is a closed subset of $R \times V$.

Let $\theta I = \text{cl}\{\theta p_1, ..., \theta p_r\}$. It is easy to check that for any $p \in I$, $\exists q$ in the Φ-ideal spanned by $p_1, ..., p_r$ such that $\theta q = \theta p$, and that this implies $wt(p - q) < wtp$. Since $(V, \leqslant)$ is well-ordered it follows by induction that I is the Φ-ideal spanned by $p_1, ..., p_r$. So we have shown any Φ-ideal is finitely spanned as Φ-ideal, which is equivalent to the proposition.

PROPOSITION 3. *If R is a Noetherian domain then M has maximum condition on Φ-submodules.*

Proof. We define a map $\pi : M \to R \times V \times J$ by $\pi 0 = 0 \times V \times J$, and, if $u = \sum_{i=1}^{n} p_i t_i$ where $p_n \neq 0$ and p_n has leading coefficient a, $\pi u = (a, wtp_n, n)$. We define two orderings on $V \times J$: 1. $((i_1, ..., i_m), r) \leqslant ((j_1, ..., j_n), s)$ iff $r < s$ or $r = s$ and $(i_1, ..., i_m) \leqslant (j_1, ..., j_n)$; 2. $((i_1, ..., i_m), r) \preccurlyeq ((j_1, ..., j_n), s)$ iff there is a one–one order-preserving map $\varphi : J \to J$ with $\varphi r = s$, $\varphi m \leqslant n$, and $i_k \leqslant j_{\varphi k}$, $k = 1, ..., m$. It is easy to see that $(V \times J, \leqslant)$ is well-ordered and that the identity is an order-homomorphism from $(V \times J, \preccurlyeq)$ to $(V \times J, \leqslant)$. If we can show that $(V \times J, \preccurlyeq)$ is partially well-ordered, we can complete the proof exactly as the proof of Proposition 2, with $V \times J$ replacing V.

So let us take an infinite sequence (α_i, r_i) in $V \times J$. Taking a subsequence if necessary, we may assume that $r_1 \leqslant r_2 \leqslant \cdots$ and that either α_k contains fewer than r_k elements for all k or that α_k contains at least r_k elements for all k. In the former case, a subsequence (α_{k_n}, r_{k_n}) such that α_{k_n} is an increasing subsequence of $(V, \preccurlyeq)$ [which can be found, since $(V, \preccurlyeq)$ is partially well-ordered] will be an increasing subsequence of $V \times J$. For the latter case, when we have an element (α, r) where $\alpha = (i_1, ..., i_m)$ with $m \geqslant r$, we shall call $(i_1, ..., i_{r-1})$ the *initial part* of (α, r), $(i_{r+1}, ..., i_m)$ the *final part* (the initial and final parts may be empty), and i_r the *pivotal element*. Then, given a sequence (α_k, r_k) of the second kind, we can take a subsequence such that the initial parts form an increasing sequence in $(V, \preccurlyeq)$; then a subsequence of this such that the pivotal elements increase; then a subsequence of this latter such that the final parts form an increasing sequence in $(V, \preccurlyeq)$. The resulting subsequence of our original sequence is an increasing sequence in $(V \times J, \preccurlyeq)$, as required.

COROLLARY. *If R is a Noetherian domain, M_0 has maximum condition on Φ-submodules.*

Let N_0 be a Φ-submodule of M_0. Then SN_0 is a Φ-submodule of M, and $M_0 \cap SN_0 = N_0$ (retracting S onto R and M onto M_0 by mapping x_i to 1, all i), whence the result. The corollary can also be proved directly, using a map $M_0 \to R \times J$.

Proposition 1 can also be used to give a variant of the standard proof that $R[x]$ is Noetherian if R is, or to prove various extensions of Proposition 2.

3. Group Theory

Let F be a free group, $A = F/F'$, and $B = F/F''$ the corresponding free abelian and free metabelian groups. We will use X to denote a basis of F, A, or B.

We proceed to the proof of Theorem C. When B has finite rank, the theorem is an immediate consequence of Hall's theorem so we need only consider the case when X is infinite. Suppose X is not countable and let X_0 be a countable subset of X. Let K be a characteristic subgroup of B, and let B_0 be the subgroup of B generated by X_0. Any $k \in K$ is of form $k = \theta k_0$ with $k_0 \in K \cap B_0$ and θ an automorphism of B induced by a permutation of X, i.e., K is determined by $K \cap B_0$. Now $K \cap B_0$ is a characteristic subgroup of B_0, since any automorphism of B_0 can be extended to an automorphism of B mapping X–X_0 identically. Hence the theorem is true for B if it is true for B_0, and we shall assume for the remainder of the proof that X is countable.

Magnus [5] shows that if R is a normal subgroup of F, and $G = F/R$, $\Gamma = ZG$ the integral group ring of G, and T the free Γ-module with basis $t_1, t_2, \ldots$, then F/R' is isomorphic to a group of matrices of the form $\left[\begin{smallmatrix} g & t \\ 0 & 1 \end{smallmatrix}\right]$ with $g \in G$, $t \in T$, namely to that subgroup of the group of all such matrices which is generated by the matrices

$$\begin{bmatrix} \bar{x}_i & t_i \\ 0 & 1 \end{bmatrix},$$

where $\bar{x}_i$ is the image in G of x_i in F. Under this isomorphism an element is in R/R' if and only if its image is of form $\left[\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right]$ for some $t \in T$. The matrix representation is just a convenient way of expressing F/R' as an extension of R/R' by F/R. Magnus states the result in full generality, but only proves a special case. However, in the language of the free differential calculus [1], a homomorphism of F into the matrix groups is defined by

$$f \to \begin{bmatrix} \bar{f} & \sum \overline{\dfrac{\partial f}{\partial x_i}}\, t_i \\ 0 & 1 \end{bmatrix},$$

where $\bar{f} \in G$ is the image of $f \in F$, and R' is the kernel of this homomorphism ([1], Proposition 4.9).

We apply this with $R = F'$. Let $S^* = Z[x_1, x_1^{-1}, \ldots]$ be the group ring of A, M^* the free S^*-module with basis $t_1, t_2, \ldots$, $S = Z[x_1, x_2, \ldots]$, and M the S-submodule of M^* spanned by $t_1, t_2, \ldots$. Then we will regard B as a group of matrices of form $\left[\begin{smallmatrix} a & u \\ 0 & 1 \end{smallmatrix}\right]$ with $a \in A = B/B'$, $u \in M^*$, and the matrices $\left[\begin{smallmatrix} x_i & t_i \\ 0 & 1 \end{smallmatrix}\right]$ form a basis of B.

Let K be a characteristic subgroup of B. Then $K \cap B'$ consists of matrices $\begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix}$. Let $N^* = \{u \in M^*; \begin{bmatrix} 1 & u \\ 0 & 1 \end{bmatrix} \in K \cap B'\}$. Then N^* is an S^*-submodule of M^* because $K \cap B'$ is normal in B. Let N consist of those elements of N^* containing only positive powers of the x_i. Plainly N is an S-submodule of M with $N^* = S^*N$. Now any permutation of the basis elements of B defines an automorphism of B. Since $K \cap B'$ is characteristic in B, and any $\varphi \in \Phi$ has the same effect on any finite set of natural numbers as some permutation of the natural numbers, we see easily that N is a Φ-submodule of M.

Suppose $K_1 \subseteq K_2 \subseteq \cdots$ is an increasing sequence of characteristic subgroups of B. Then $N_1 \subseteq N_2 \subseteq \cdots$ is an increasing sequence of Φ-submodules of M. By Proposition 3, this sequence is ultimately constant, i.e., the sequence $K_1 \cap B' \subseteq K_2 \cap B' \subseteq \cdots$ is ultimately constant. Also B/B' is a free Z-module with basis $x_1, x_2, \ldots$, and $K_1B'/B' \subseteq K_2B'/B' \subseteq \cdots$ is an increasing sequence of Φ-submodules of B/B', so is ultimately constant by the corollary to Proposition 3. Hence the sequence $K_1 \subseteq K_2 \subseteq \cdots$ is ultimately constant, and Theorem C is proved.

Remark 1. When B has finite rank, a simplification of the above argument shows that B (and hence any finitely generated metabelian group) has maximal condition for normal subgroups. This proof is not significantly different from that of Hall.

Remark 2. It is plainly possible to define Φ-subgroups of B with respect to a basis X, and show that B has maximum condition on normal Φ-subgroups. However, characteristic subgroups are Φ-subgroups with respect to any basis, whereas, in general, a Φ-subgroup for one basis will not be a Φ-subgroup for a different basis.

Remark 3. I do not know if Theorems B and/or C apply to free-Abelian-by-nilpotent groups. The methods of this paper do not extend directly, since the free-Abelian-by-class-two group does not have maximum condition on Φ-subgroups.

Remark 4. The laws of a nilpotent variety may be chosen from a particularly simple infinite set. I do not know if such a set can be found for metabelian varieties. There is a metabelian variety which cannot be defined by a finite number of the laws n:

$$x_1^n = 1, [[x_1, x_2], [x_3, x_4]] = 1, \qquad \text{and} \qquad [x_1^{n_1}, x_2^{n_2}, x_{i_1}^{n_3}, x_{i_2}^{n_4}, \ldots]^r = 1$$

for arbitrary $i_1, i_2, \ldots, n_1, n_2, \ldots, r$.

References

1. Fox, R. H. Free differential calculus I. *Ann. Math.* **57** (1953), 547-560.
2. Hall, P. Finiteness conditions for soluble groups. *Proc. London Math. Soc.* **4** (1954), 419-436.
3. Higman, G. Ordering by divisibility in abstract algebras. *Proc. London Math. Soc.* **2** (1952), 326-336.
4. Lyndon, R. C. Two notes on nilpotent groups. *Proc. Am. Math. Soc.* **3** (1952), 579-583.
5. Magnus, W. On a theorem of Marshall Hall. *Ann. Math.* **40** (1939), 764-876.
6. Neumann, B. H. Identical relations in groups. *Math. Ann.* **114** (1937), 506-525.
7. Neumann, H. On varieties of groups and their associated near-rings. *Math. Zeit.* **65** (1956), 36-69.
8. Neumann, H. Varieties of groups. (Duplicated lecture notes). Manchester College of Science and Technology, Manchester, England, 1963.
9. Oates, S. and Powell, M. B. Identical relations in finite groups. *J. Algebra* **1** (1964), 11-59.