Coherence of twisted commutative algebras
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1. Definitions

(1.1) Let $\mathcal{C}$ be a directed category. Let $R(\mathcal{C})$ be a set of representatives for the isomorphism classes of $\mathcal{C}$. For $x \in R(\mathcal{C})$, let $\mathcal{P}(x)$ be the poset of morphisms with domain x and target some $y \in R(\mathcal{C})$.

Given $g, g' \in \mathcal{P}(x)$, let $\mathrm{syz}(g, g')$ be the set of pairs (φ, φ') so that $\varphi g = \varphi' g'$. We put a poset structure on $\mathrm{syz}(g, g')$ by saying that $(\varphi, \varphi') \le (\psi\varphi, \psi\varphi')$. Let $\mathrm{syz}_0(g, g')$ be the set of minimal elements. We say that $\mathcal{C}$ is **syzygy finite** if $\mathrm{syz}_0(g, g')$ is finite for every pair g, g'.

(1.2) *Example.* Suppose $\mathcal{C}$ is a category of ordered finite sets such that the morphisms are increasing injections of finite sets together with some kind of decoration on the complement of the image (the different types of decorations for fixed sets is required to be finite). Then we claim that $\mathcal{C}$ is syzygy finite. To see this, fix a finite set S and morphisms $f, f' \colon S \to T$. Let $\psi, \psi' \colon T \to U$ be such that $\psi f = \psi' f'$. Then ψ and ψ' have to put the same decoration on $U \setminus (\psi(T) \cup \psi'(T))$. From this, we see that $\mathrm{syz}_0(f, f')$ is contained in $\coprod_U \mathrm{Hom}(T, U)^{\times 2}$ where $|U| \le 2|T|$, which in particular, is finite.

This applies to all of the ordered versions of the categories attached to finite-type twisted commutative algebras.

Fix a field $\mathbf{k}$. We consider functors $\mathcal{C} \to \mathrm{Mod}_{\mathbf{k}}$. For $x \in R(\mathcal{C})$, let P_x be the projective module $P_x(y) = \mathbf{k}[\mathrm{Hom}_{\mathcal{C}}(x, y)]$.

2. Syzygies and Gröbner bases

This material is adapted from [E, Chapter 15].

(2.1) Let $P = \bigoplus_x P_x$ and fix an admissible term ordering $<$ on P. We think of the generator of each P_x as a basis vector of P. Let $M \subseteq P$ be a submodule with a set of generators $\{g_i\}$ where $g_i \in P(y_i)$. Furthermore, we assume that $\{g_i\}$ is reduced, in the sense that $\{\mathrm{init}(g_i)\}$ is irredundant.

Set $P' = \bigoplus_i P_{y_i}$ and let ε_i be the generator of P_{y_i}. Define a term ordering $<$ on P' as follows: $\varphi\varepsilon_i < \psi\varepsilon_j$ if

- $\mathrm{init}(\varphi g_i) < \mathrm{init}(\psi g_j)$, or
- $\mathrm{init}(\varphi g_i) = \mathrm{init}(\psi g_j)$ and $i > j$.

It is clear that this an admissible term ordering. Define $\pi \colon P' \to M$ by $\varepsilon_i \mapsto g_i$.

Pick i, j so that $\mathrm{init}(g_i)$ and $\mathrm{init}(g_j)$ involve the same basis vector x of P. Given $(\varphi_i, \varphi_i) \in \mathrm{syz}_0(\mathrm{init}(g_i), \mathrm{init}(g_j))$, suppose that we have an expression

$$\varphi_i g_i - \varphi_j g_j = \sum_k c_k \psi_k g_k$$

where $\mathrm{init}(\varphi_i g_i - \varphi_j g_j) \ge \mathrm{init}(\psi_k g_k)$ for all k ($c_k \in \mathbf{k}$, the ψ_k are morphisms in $\mathcal{C}$, and we allow the g_k to repeat). We call this expression a **GB-witness** for (φ_i, φ_j). This gives us an element

$$\sigma_{i,j}(\varphi_i, \varphi_j) = \varphi_i\varepsilon_i - \varphi_j\varepsilon_j - \sum_k c_k \psi_k \varepsilon_k$$

in $\ker \pi$. Let $\mathcal{S}$ be the set of such elements we can construct as we vary over $i, j, \varphi_i, \varphi_j$.

Define $\mathcal{S}'$ as the set of $\sigma'_{i,j}(\varphi_i,\varphi_j) = \varphi_i\varepsilon_i - \varphi_j\varepsilon_j$. These are kernel elements for the map $\pi' \colon P' \to M$ defined by $\varepsilon_i \mapsto \mathrm{init}(g_i)$.

Lemma. *$\mathcal{S}'$ generates $\ker\pi'$.*

Proof. The map π' is defined in terms of the monomial basis, so it is clear that $\ker\pi'$ has a basis given by $\psi_i\varepsilon_i - \psi_j\varepsilon_j$ where $(\psi_i,\psi_j) \in \mathrm{syz}(\mathrm{init}(g_i), \mathrm{init}(g_j))$. In particular, there exists $(\varphi_i,\varphi_j) \in \mathrm{syz}_0(\mathrm{init}(g_i),\mathrm{init}(g_j))$ and ψ such that $\psi_i = \psi\varphi_i$ and $\psi_j = \psi\varphi_j$. So $\psi_i\varepsilon_i - \psi_j\varepsilon_j$ is generated by $\sigma'_{i,j}(\varphi_i,\varphi_j) \in \mathcal{S}'$. ∎

(2.2) Theorem (Buchberger's criterion). *A set $\{g_i\}$ of elements in P is a Gröbner basis if and only if for every i,j and $(\psi,\psi') \in \mathrm{syz}_0(\mathrm{init}(g_i),\mathrm{init}(g_j))$, we have an expression*

$$\psi g_i - \psi' g_j = \sum_k c_k\varphi_k g_k$$

where φ_k is a morphism in $\mathcal{C}$ and $c_k \in \mathbf{k}$ such that $\mathrm{init}(\varphi_k g_k) \le \mathrm{init}(\psi g_i - \psi' g_j)$ (we allow the g_k to repeat).

Proof. If the $\{g_i\}$ form a Gröbner basis, it is clear that such expressions exist: find g_k such that $\mathrm{init}(g_k)$ generates $\mathrm{init}(\psi g_i - \psi' g_j)$ and subtract an appropriate amount to make the initial term smaller and repeat. Since the term ordering does not have infinite descending chains, this algorithm will terminate.

Conversely, suppose that these expressions exist and suppose that the $\{g_i\}$ is not a Gröbner basis. Then there exists a finite sum $f = \sum_k \alpha_k\varphi_k g_k$ where $\mathrm{init}(f) \notin \mathrm{init}(M)$ and the φ_k are morphisms in $\mathcal{C}$ (we allow the g_k to repeat). Let $m = \max \mathrm{init}(\varphi_k g_k)$. We can choose f so that m is minimal. Let $f' = \sum'_k \alpha_k\varphi_k g_k$ where the sum is over all terms such that $\alpha_k\varphi_k\mathrm{init}(g_k) = \mathrm{init}(\alpha_k\varphi_k g_k) = m$ (up to scalar).

We have $\sum'_k \alpha_k\varphi_k\mathrm{init}(g_k) = 0$, or else it would be equal to $\mathrm{init}(f)$. By Lemma 2.1, we can write $\sum'_k \alpha_k\varphi_k\varepsilon_k = \sum_j \beta_j\psi_j\sigma'_{a,b}(\varphi_a,\varphi_b)$ where $\mathrm{init}(\pi'(\psi_j\sigma'_{a,b}(\varphi_a,\varphi_b))) < m$ for all terms. Now apply π to get $f' = \sum_j \gamma_j\rho_j g_j$. Then $f = f - f' + \sum_j \gamma_j\rho_j g_j$ is an alternative expression for f where we have made m smaller, which is a contradiction. ∎

Corollary. *If $\{g_i\}$ is a Gröbner basis for M, then $\mathcal{S}$ is a Gröbner basis for $\ker\pi$.*

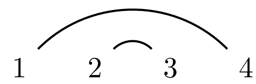
Proof. Pick $\tau \in \ker\pi$ and write $\tau = \sum_i c_i\varphi_i\varepsilon_i$ where the φ_i are morphisms in $\mathcal{C}$ and we allow the ε_i to repeat. Then $\mathrm{init}(\tau) = \mathrm{init}(c_i\varphi_i\varepsilon_i)$ for some i. Let τ' be the sum of the $c_j\varphi_j\varepsilon_j$ such that $\mathrm{init}(c_j\varphi_j g_j) = \mathrm{init}(c_i\varphi_i g_i)$ (up to scalar). By definition of our term ordering, we have $j \ge i$ for each j. Then $\tau' \in \ker\pi'$ since $\pi'(\varphi_j g_j) < \varphi_i\mathrm{init}(g_i)$ if j is not in the set selected for the sum τ'. So by Lemma 2.1, τ' is generated by the elements $\sigma'_{u,v}(\varphi_u,\varphi_v)$ in $\mathcal{S}'$ where $u,v \ge i$. The ones that involve ε_i are the $\sigma'_{i,j}(\varphi_i,\varphi_j)$ where $j > i$. Its initial term is $\varphi_i\varepsilon_i$. In particular, $\mathrm{init}(\tau)$ is generated by such terms. ∎

3. Examples

(3.1) Let $\mathcal{C}$ be the category whose objects are ordered finite sets and whose morphisms are order-preserving injections together with a perfect matching on the complement on the image. This is an ordered version of the model for the tca $\mathrm{Sym}(\mathrm{Sym}^2)$. We focus on P_0. Fix $n \ge 1$ and let f be the sum of all perfect matchings on $[2n]$. We want to compute the Gröbner basis of the submodule f generates.

When $n = 1$, there is nothing to do because f is already a monomial.

When $n = 2$, we first determine the set $\mathrm{syz}_0(f, f)$. The initial term of f is

$f_0 =$  .

We will describe a larger set and only describe it up to $\mathbf{Z}/2$-symmetry. First there is the element $(\mathrm{id}, \mathrm{id})$. In the next degree, we have the five pairs (a morphism $[4] \to [6]$ is specified by just giving a pair of elements $i, j \in [6]$, which we write as $\psi_{i,j}$):

$$(\psi_{2,3}, \psi_{4,5}), (\psi_{2,4}, \psi_{3,5}), (\psi_{2,6}, \psi_{1,5}), (\psi_{1,6}, \psi_{2,5}), (\psi_{1,6}, \psi_{3,4}).$$

If we fix a partition $6 = S \cup S'$ into sets of size 3, we have the following relation

$$\sum_{i,j \in S} \psi_{i,j} f = \sum_{i,j \in S'} \psi_{i,j} f.$$

We can use this to construct GB-witnesses by rearranging terms:

- $S = \{1, 2, 4\}$, $S' = \{3, 5, 6\}$ gives a GB-witness for $(\psi_{2,3}, \psi_{4,5})$ and $(\psi_{2,4}, \psi_{3,5})$.
- $S = \{1, 4, 6\}$, $S' = \{2, 3, 5\}$ gives a GB-witness for $(\psi_{1,6}, \psi_{2,5})$ and $(\psi_{1,6}, \psi_{3,4})$.

Also, we have the following relation

$$\sum_{2 \le i \le 5} \psi_{1,i} f = \sum_{2 \le i \le 5} \psi_{i,6} f$$

which gives a GB-witness for $(\psi_{2,6}, \psi_{1,5})$.

Now we consider GB-witnesses for the next degree syzygies. Let $\psi_{ij|k\ell}$ be the map $[4] \to [8]$ which sends 4 to the complement of $\{i, j, k, \ell\}$ and puts the matching $\{\{i, j\}, \{k, \ell\}\}$. A map $[4] \to [8]$ is the same as a subset $S = \{s_1 < s_2 < s_3 < s_4\}$. Let $T = \{t_1 < t_2 < t_3 < t_4\}$ be the complement of S. We get a syzygy pair for every map $S \colon [4] \to [8]$ as $(\psi_{s_1 s_4 | s_2 s_3}, \psi_{t_1 t_4 | t_2 t_3})$. The relation

$$\psi_{s_1 s_2 | s_3 s_4} f + \psi_{s_1 s_3 | s_2 s_4} f + \psi_{s_1 s_4 | s_2 s_3} f = \psi_{t_1 t_2 | t_3 t_4} f + \psi_{t_1 t_3 | t_2 t_4} f + \psi_{t_1 t_4 | t_2 t_3} f$$

gives a GB-witness for this syzygy pair.

In conclusion, f is a Gröbner basis for the submodule that it generates.

[$\star$ Steven: I would guess that f is still a Gröbner basis for general n, but I haven't figured out an easy way to organize the calculations. $\star$]

4. Coherence properties

(4.1) Let $P = \bigoplus_x P_x$ (finite sum). A submodule $M \subseteq P$ is **GB-finite type** if there is an admissible term ordering on P so that M has a finite Gröbner basis. We say that P is **GB-coherent** if for every GB-finite type submodule, there exists $P' = \bigoplus_y P_y$ and a surjection $P' \to M$ so that the kernel is GB-finite type. Finally, the category of $\mathcal{C}$-modules is **GB-coherent** if every $\bigoplus_x P_x$ is GB-coherent.

(4.2) Theorem. *Suppose $\mathcal{C}$ has at least one admissible term ordering. If $\mathcal{C}$ is syzygy finite, then the category of $\mathcal{C}$-modules is GB-coherent.*

Proof. This follows directly from Corollary 2.2: the assumption that $\mathcal{C}$ is syzygy finite implies that the set $\mathcal{S}$ is finite. $\square$

Corollary. *Let* $P = \bigoplus_x P_x$ *and* $M \subseteq P$ *be a submodule which has a finite Gröbner basis with respect to some admissible term ordering. Then* M *has a projective resolution by finitely generated projectives, i.e.,* M *is a module of type* FP_∞.

References

[E] David Eisenbud, *Commutative Algebra*