I. GRADIENT

The (unnormalized) wavefunction is

$$\psi(r_1, r_2) = \exp(-\alpha r_1)\exp(-\alpha r_2). \tag{1}$$

$$\psi(x_1, y_1, z_1, x_2, y_2, z_2) = \exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right)\exp\left(\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right). \tag{2}$$

The gradient is

$$\nabla\psi = \left[\frac{\partial\psi}{\partial x_1}, \frac{\partial\psi}{\partial y_1}, \frac{\partial\psi}{\partial z_1}, \frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial y_2}, \frac{\partial\psi}{\partial z_2}\right]^T. \tag{3}$$

Evaluate each element.

$$\frac{\partial\psi}{\partial x_1} = -\alpha\exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right)\frac{1}{2}(x_1^2 + y_1^2 + z_1^2)^{-1/2}2x_1\exp\left(\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right) \quad = -\alpha\frac{x_1}{r_1}\psi \tag{4}$$

Thus, the gradient is

$$\boxed{\frac{\nabla\psi}{\psi} = -\alpha\left[\frac{x_1}{r_1}, \frac{y_1}{r_1}, \frac{z_1}{r_1}, \frac{x_2}{r_2}, \frac{y_2}{r_2}, \frac{z_2}{r_2}\right]^T.} \tag{5}$$

II. LAPLACIAN

$$\frac{\partial^2\psi}{\partial x_1^2} = -\alpha\exp\left(-\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right)\frac{\partial}{\partial x_1}\left[x_1\frac{1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}\exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right)\right] \tag{6}$$

$$= -\alpha\exp\left(-\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right)\frac{\partial}{\partial x_1}A. \tag{7}$$

$$\tag{8}$$

$$\frac{\partial A}{\partial x_1} = \frac{1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}\exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \tag{9}$$

$$+ x_1\left(-\frac{1}{2}\right)(x_1^2 + y_1^2 + z_1^2)^{-\frac{3}{2}}2x_1\exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \tag{10}$$

$$+ x_1\frac{1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}(-\alpha)\frac{1}{2}\frac{2x_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}}\exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \tag{11}$$

$$= \left[\frac{1}{r_1} - \frac{x_1^2}{r_1^3} - \alpha\frac{x_1^2}{r_1^2}\right]\exp(-\alpha r_1). \tag{12}$$

$$\tag{13}$$

$$\frac{\partial^2\psi}{\partial x_1^2} = -\alpha\exp\left(-\alpha r_2\right)\left[\frac{1}{r_1} - \frac{x_1^2}{r_1^3} - \alpha\frac{x_1^2}{r_1^2}\right]\exp(-\alpha r_1) \tag{14}$$

$$= -\frac{\alpha}{r_1}\psi\left[1 - \frac{x_1^2}{r_1^2} - \alpha\frac{x_1^2}{r_1}\right]. \tag{15}$$

Thus, the laplacian is

$$\frac{\nabla^2\psi}{\psi} = \left[\frac{\partial^2\psi}{\partial x_1^2} + \frac{\partial^2\psi}{\partial y_1^2} + \frac{\partial^2\psi}{\partial z_1^2}, \frac{\partial^2\psi}{\partial x_2^2} + \frac{\partial^2\psi}{\partial y_2^2} + \frac{\partial^2\psi}{\partial z_2^2}\right] \tag{16}$$

$$= \left[-\frac{\alpha}{r_1}\left(3 - \frac{x_1^2 + y_1^2 + z_1^2}{r_1^2} - \alpha\frac{x_1^2 + y_1^2 + z_1^2}{r_1}\right), -\frac{\alpha}{r_2}\left(3 - \frac{x_2^2 + y_2^2 + z_2^2}{r_2^2} - \alpha\frac{x_2^2 + y_2^2 + z_2^2}{r_2}\right)\right] \tag{17}$$

$$= \left[-\frac{\alpha}{r_1}(2 - \alpha r_1), -\frac{\alpha}{r_2}(2 - \alpha r_2)\right] \tag{18}$$

$$= \boxed{\left[-\frac{2\alpha}{r_1} + \alpha^2, -\frac{2\alpha}{r_2} + \alpha^2\right].} \tag{19}$$

III. ENERGY

The normalized wavefunction is

$$\psi(r_1, r_2) = \alpha^3 \exp(-\alpha r_1) \exp(-\alpha r_2). \tag{20}$$

Using the expression of laplacian from the previous section, the kinetic energy is

$$\begin{aligned}
\langle K \rangle &= -\frac{1}{2} \int \frac{\nabla_{r_1}^2 \psi}{\psi} |\psi|^2 d^3 r_1 d^3 r_2 - \frac{1}{2} \int \frac{\nabla_{r_2}^2 \psi}{\psi} |\psi|^2 d^3 r_1 d^3 r_2 && (21) \\
&= -\frac{1}{2} \int_0^\infty \left(-\frac{2\alpha}{r_1} + \alpha^2 \right) \alpha^6 \exp(-2\alpha r_1) 4 r_1^2 dr_1 \int_0^\infty \exp(-2\alpha r_2) 4 r_2^2 dr_2 + \cdots && (22) \\
&= -\frac{1}{2} \left(-\frac{1}{\alpha} \right) \frac{1}{\alpha^3} \alpha^6 - \frac{1}{2} \left(-\frac{1}{\alpha} \right) \frac{1}{\alpha^3} \alpha^6 && (23) \\
&= \boxed{\alpha^2}. && (24)
\end{aligned}$$

The electron-nuclear potential is

$$\begin{aligned}
\langle V_{\text{en}} \rangle &= -Z \int \frac{\alpha^6}{r_1} \exp(-2\alpha r_1) \exp(-2\alpha r_2) d^3 r_1 d^3 r_2 - \cdots && (25) \\
&= -Z \int_0^\infty \frac{1}{r_1} \exp(-2\alpha r_1) 4 r_1^2 dr_1 \int_0^\infty \exp(-2\alpha r_2) 4 r_2^2 dr_2 - \cdots && (26) \\
&= -Z \left(\frac{1}{\alpha^2} \right) \left(\frac{1}{\alpha^3} \right) \alpha^6 - Z \left(\frac{1}{\alpha^2} \right) \left(\frac{1}{\alpha^3} \right) && (27) \\
&= \boxed{-2Z\alpha}. && (28)
\end{aligned}$$