

## I. ENERGY

$$\langle K \rangle = -\frac{1}{2} \int \frac{\nabla_{r_1}^2 \psi}{\psi} |\psi|^2 d^3 r_1 d^3 r_2 - \frac{1}{2} \int \frac{\nabla_{r_2}^2 \psi}{\psi} |\psi|^2 d^3 r_1 d^3 r_2 \quad (1)$$

$$= -\frac{1}{2} \int_0^\infty \left( -\frac{2\alpha}{r_1} + \alpha^2 \right) \exp(-2\alpha r_1) 4r_1^2 dr_1 \int_0^\infty \exp(-2\alpha r_2) 4r_2^2 dr_2 + \dots \quad (2)$$

$$= -\frac{1}{2} \left( -\frac{1}{\alpha} \right) \frac{1}{\alpha^3} - \frac{1}{2} \left( -\frac{1}{\alpha} \right) \frac{1}{\alpha^3} \quad (3)$$

$$= \boxed{\frac{1}{\alpha^4}}. \quad (4)$$

$$\langle V_{\text{en}} \rangle = -Z \int \frac{1}{r_1} \exp(-2\alpha r_1) \exp(-2\alpha r_2) d^3 r_1 d^3 r_2 - \dots \quad (5)$$

$$= -Z \int_0^\infty \frac{1}{r_1} \exp(-2\alpha r_1) 4r_1^2 dr_1 \int_0^\infty \exp(-2\alpha r_2) 4r_2^2 dr_2 - \dots \quad (6)$$

$$= -Z \left( \frac{1}{\alpha^2} \right) \left( \frac{1}{\alpha^3} \right) - Z \left( \frac{1}{\alpha^2} \right) \left( \frac{1}{\alpha^3} \right) \quad (7)$$

$$= \boxed{-\frac{2Z}{\alpha^5}}. \quad (8)$$

## II. GRADIENT

The wavefunction is

$$\psi(r_1, r_2) = \exp(-\alpha r_1) \exp(-\alpha r_2). \quad (9)$$

$$\psi(x_1, y_1, z_1, x_2, y_2, z_2) = \exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \exp\left(\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right). \quad (10)$$

The gradient is

$$\nabla\psi = \left[ \frac{\partial\psi}{\partial x_1}, \frac{\partial\psi}{\partial y_1}, \frac{\partial\psi}{\partial z_1}, \frac{\partial\psi}{\partial x_2}, \frac{\partial\psi}{\partial y_2}, \frac{\partial\psi}{\partial z_2} \right]^T. \quad (11)$$

Evaluate each element.

$$\frac{\partial\psi}{\partial x_1} = -\alpha \exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \frac{1}{2}(x_1^2 + y_1^2 + z_1^2)^{-1/2} 2x_1 \exp\left(\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right) = -\alpha \frac{x_1}{r_1} \psi \quad (12)$$

Thus, the gradient is

$$\boxed{\frac{\nabla\psi}{\psi} = -\alpha \left[ \frac{x_1}{r_1}, \frac{y_1}{r_1}, \frac{z_1}{r_1}, \frac{x_2}{r_2}, \frac{y_2}{r_2}, \frac{z_2}{r_2} \right]^T}. \quad (13)$$

## III. LAPLACIAN

$$\frac{\partial^2\psi}{\partial x_1^2} = -\alpha \exp\left(-\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right) \frac{\partial}{\partial x_1} \left[ x_1 \frac{1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} \exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \right] \quad (14)$$

$$= -\alpha \exp\left(-\alpha\sqrt{x_2^2 + y_2^2 + z_2^2}\right) \frac{\partial}{\partial x_1} A. \quad (15)$$

$$(16)$$

$$\frac{\partial A}{\partial x_1} = \frac{1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} \exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \quad (17)$$

$$+ x_1 \left(-\frac{1}{2}\right) (x_1^2 + y_1^2 + z_1^2)^{-\frac{3}{2}} 2x_1 \exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \quad (18)$$

$$+ x_1 \frac{1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} (-\alpha) \frac{1}{2} \frac{2x_1}{\sqrt{x_1^2 + y_1^2 + z_1^2}} \exp\left(-\alpha\sqrt{x_1^2 + y_1^2 + z_1^2}\right) \quad (19)$$

$$= \left[ \frac{1}{r_1} - \frac{x_1^2}{r_1^3} - \alpha \frac{x_1^2}{r_1^2} \right] \exp(-\alpha r_1). \quad (20)$$

$$(21)$$

$$\frac{\partial^2 \psi}{\partial x_1^2} = -\alpha \exp(-\alpha r_2) \left[ \frac{1}{r_1} - \frac{x_1^2}{r_1^3} - \alpha \frac{x_1^2}{r_1^2} \right] \exp(-\alpha r_1) \quad (22)$$

$$= -\frac{\alpha}{r_1} \psi \left[ 1 - \frac{x_1^2}{r_1^2} - \alpha \frac{x_1^2}{r_1} \right]. \quad (23)$$

Thus, the laplacian is

$$\frac{\nabla^2 \psi}{\psi} = \left[ \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial y_1^2} + \frac{\partial^2 \psi}{\partial z_1^2}, \frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^2 \psi}{\partial y_2^2} + \frac{\partial^2 \psi}{\partial z_2^2} \right] \quad (24)$$

$$= \left[ -\frac{\alpha}{r_1} \left( 3 - \frac{x_1^2 + y_1^2 + z_1^2}{r_1^2} - \alpha \frac{x_1^2 + y_1^2 + z_1^2}{r_1} \right), -\frac{\alpha}{r_2} \left( 3 - \frac{x_2^2 + y_2^2 + z_2^2}{r_2^2} - \alpha \frac{x_2^2 + y_2^2 + z_2^2}{r_2} \right) \right] \quad (25)$$

$$= \left[ -\frac{\alpha}{r_1} (2 - \alpha r_1), -\frac{\alpha}{r_2} (2 - \alpha r_2) \right] \quad (26)$$

$$= \boxed{\left[ -\frac{2\alpha}{r_1} + \alpha^2, -\frac{2\alpha}{r_2} + \alpha^2 \right]}. \quad (27)$$