An Analysis of Pairwise Preference

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... seek not the depths of your knowledge with staff or sounding line. For self is a sea boundless and measureless. Say not, "I have found the truth," but rather, "I have found a truth."
- Kahlil Gibran, *"Self Knowledge"*

0.1 Introduction

This thesis will explore the notion of *pairwise preference*, a concept with applications in political science, economics, social science, and machine learning (Wauthier, Jordan, and Jojic 2013, Arrow 1970). A pairwise preference is simply *a preference for one item over another, given two items.* We will show that this representation is highly general, and that problems posed in this framework are amenable to many kinds of analysis. More importantly, we believe that such a representation may be capable of representing human subjectivity accurately and precisely.

The choice to explore this particular representation was not arbitrary. In the **historical context** section, we will review aspects of the history of science, philosophy, and politics, and attempt to discern some key themes. This section will motivate the investigation of pairwise preference and predict some desirable theoretical properties.

In the **definitions** section, we will present the basic elements of socrata graphs. We will show how a large class of preference resolution problems can be set up within this general framework.

In **applications**, we will demonstrate various algorithms for analyzing these types of graphs, and discuss their strengths and limitations.

In **future directions**, we will identify additional avenues of exploration. There are particularly interesting possibilities involving blockchain-based virtual machine (BBVM) technologies like Ethereum.

0.2 Theoretical Context

This section will attempt to introduce some broad and fundamental ideas, ultimately rediscovering some key aspects of what is known as "process philosophy".

To begin, we will develop the problem's context, drawing on work from a variety of fields, including cognitive science, computer science, mathematics, philosophy, and political economy. In addition, we will look at past and current events to attempt to bring the current historical moment into focus.

Our argument will proceed as follows. First, we will attempt to view the problem of human social organization through the lens of resources and communication. Then, we will describe a fundamental link between representation and computability. We will then turn to the role of analysis in society, and discuss ways in which analysis can fail. Next, we will connect ideas from philosophy and cognitive science, casting light on an important property of subjective mental concepts. Finally, we will argue that pairwise preferences are well-suited to the task of representing subjective preference.

This is an ambitious undertaking; there is a high probability that some aspect or another of this history is being (seriously) misread. The author is not an authority on these subjects. The goal is not to convince the reader of the superiority or correctness of this interpretation, only the validity. From a perspective of falsifiability, we will assess the value of the ideas developed in context of the performance of the theory in application. If the theory developed brings no empirically verifiable advantages, then we will revise or abandon it.

0.2.1 A Schematic View

> *Man is not an ant, conveniently equipped with an inborn pattern of social instincts. On the contrary, he seems to be strongly endowed with a self-centered nature. If his relatively weak physique forces him to seek cooperation, his inner drives constantly threaten to disrupt his social working partnerships.*
>
> - Robert Heilbroner, *The Worldly Philosophers*, p18

Let us consider the problem of nonviolent coordination at scale. Specifically, let us view this is a problem of preference resolution. Let us view

preference resolution as a problem of information flow.

Small communities, such as groups of friends, information flows easily across the human medium of language, and these communities are generally seen as capable of peaceful, mutually-beneficial coordination. They resolve preferences easily, with few resources, and the results are generally satisfactory Deacon 1998.

As communities grow larger, the amount of total information increases and preferences become more complex and difficult to resolve. Further, language loses efficiency as meaning fluctuates across the group. This increase in problem scale and decrease in language efficiency lead to a need for some form of new structure to manage this process and coordinate the members Hobbes 1982. Structures require additional resources, or a more efficient utilization of existing resources. If the community cannot acquire new resources or technology, we can expect the nature of coordination to become more oppressive, at least for some members of the community Eisenberg, Muckenhirn, and Rudrane 1972.

The use of the term "non-violent" might (reasonably) seem to suggest that its absence implies violence; we choose to interpret it less dramatically as a loss of personal freedom. Consider a bleak workplace, in which workers have relatively little control over their work B. Y.-J. Lin et al. 2013. Consider a bronze-age empire, in which large public works projects were built by coercing large segments of the population into service Heilbroner 1999.

This is the schematic relationship: scale and nonviolence are opposed, given a fixed level of resources and technology. Additional resources or more efficient structure can allow nonviolent coordination at a larger scale. "Structure" can refer to both objective social forms (such as democratic institutions), as well as the organization of mental concepts (such as the notion of "democracy").

This last century has seen great advances both in terms of resources and technology. The majority of our existing preference-resolution structures predate these developments. In light of this, it would seem reasonable that there exist some number of viable preference-resolution frameworks waiting to be developed. Experiments in developing these frameworks, falling under the banner of "liquid democracy", are ongoing. Recent success includes the use of pol.is in Taiwan Barry 2016. In this example, pol.is was used as part of the "vTaiwan" initiative to gather fine-grained public opinion to inform the legislative process.

It is just such a framework that we will attempt to develop. We will study

it using standard tools and evaluate its merits objectively.

0.2.2 Representation

Our modern base-10 numbering system, commonly known as the "Arabic" numbering system, has roots in in both the Middle East and India. Originally developed in India, this numbering system was brought to the Middle East by (among others) the 9th-century Persian mathematician Al-Khwarizmi, via his *On the Calculation with Hindu Numerals*. Fibonacci, a 13th-century Italian mathematician, became aware of this text and became an advocate for this numbering system, arguing for its adoption in place of Roman numerals in his *Liber Abaci* Ore 1988 Ferguson 2009.

Prior to the adoption of Arabic numerals, mathematics was done using Roman numerals Heilbroner 1999 Gowers, Barrow-Green, and Leader 2008. Roman numerals, while adequate for counting and basic addition and subtraction, were unwieldy for more complex operations like multiplication and division; hence the widespread use of the abacus as a computational tool. To practitioners of this era, such operations would likely have been seen as "advanced"; problems involving these operations would have been "difficult". The adoption of the Arabic system allowed for previously challenging analysis to be performed quickly, easily, and accurately; leading to an overall acceleration in the pace of mathematical development. In the parlance of machine learning, we could argue that the abacus represents an optimization within a local optimum; the adoption of the new numbering system represents an escape from that optimum. In this historical anecdote, we see a demonstration of a fundamental relationship: that between **representation** and **analysis**. Analytical methods are defined in relation to one or another representation; changes in representation imply a change in the set of available analytic operations. In addition, observe that changes in representation have no impact on the underlying world; nothing about the world changed to make multiplication easier. Change occurred only in the mind.

The study of this type of history of thought came into popularity during the 20th century, best exemplified by the work of the post-modern philosophers. Michel Foucault was the first to attempt an "excavation" of our culture, an effort to detect the hidden history of our understanding Foucault 1994. Thomas Kuhn achieved a parallel, if not more impressive, excavation of science, showing how the idealized perfection of the objective scientific method fails when implemented by human beings Kuhn 1996.

The notion of transforming from one representation to another appears often in the computer science and industrial engineering literature; in the former, pertaining to problems and the algorithms which solve them, and in the latter, to optimization. Often, abstract notions of representation have concrete implications in terms of computability; this will become a key theme moving forward. In computer science, a problem is said to *reduce* to another if it can be shown that an instance of the first can be transformed into an instance of the second, while preserving truth conditions. An problem that may be difficult to analyze in one form may become easy to analyze if converted into a different, but provably equivalent, form. In optimization, there exists tremendous knowledge on how to solve certain kinds of optimization problems, if represented in specific constrained (convex) forms. Problems defined as optimizations of convex objective functions over convex sets can be solved by computer relatively quickly and precisely. Much of the skill in this field is being able to identify how a problem represented in some (arbitrary) form can be transformed into an equivalent convex optimization problem (such as a linear, quadratic, or semidefinite program). The ability to discern these relationships among problems is a fundamental skill for researchers in these fields.

As observed by British computer scientist Philip Wadler Wadler 2015:

> *Powerful insights arise from linking two fields of study previously thought separate. Examples include Descartes's coordinates, which links geometry to algebra, Planck's Quantum Theory, which links particles to waves, and Shannon's Information Theory, which links thermodynamics to communication.*

For a more current example, we can look at recent development in machine learning. Graphical models, a formalism for representing and analyzing complex joint probability distributions as graphs, allowed for the mixing of analytic techniques from both statistics and computer science. Problems that are difficult to solve for a probability distributions are may be easy to solve for an equivalent graph, and vice versa Wainwright and Jordan 2008.

An important clarification is that for problems which can *in principle* be solved in several representations, one representation may allow for *faster* solutions. This is important because problems requiring hours or months of computation to solve are essentially unsolvable for applications needing results in minutes or days.

An interesting (but speculative) example comes to us via studies of the Wason selection task. In this task, participants are asked to solve logical reasoning problems by flipping cards. Experiments have shown that such problems are difficult when presented abstractly, but become significantly easier when presented in terms of common social reasoning tasks Cosmides and Tooby 1992.

We can extend this notion of representation and analysis to more general domains. Natural language is a representation; the written word can be read, but not analyzed with the precision of mathematics. Images, music, and so on are also representations; each representation permits some modes of analysis and eliminates others.

In machine learning and optimization, it is very common to represent data as points in high-dimensional space. A car, possessing *weight, acceleration, horsepower, mpg*, and *year*, can be thought of as a point in five-dimensional space, denoted $\mathbb{R}^5$ (more specifically, in the positive orthant of this space, $\mathbb{R}^5_+$). Such representations allow for analysis using all of the tools of geometry and linear algebra. These tools are powerful; much research has been conducted on methods for *embedding* non-numeric data types into these types of space. Examples of such techniques include collaborative filters and word vectors Mikolov et al. 2013 Koren, Bell, and Volinsky 2009.

The above discussion has been qualitative; let's place the concepts of representation and analysis on more rigorous footing.

0.2.3 The Data Processing Inequality

One of the first results in the field of Information Theory goes as follows:

If you have three variables, $\{X, Y, Z\}$, existing in a Markov relationship such that X affects Y, and Y affects Z:

$$X \to Y \to Z$$

Then the *mutual information* (intuitively, the amount one variable tells you about another) between Z and X can never be more than the mutual information between Y and X. This is known as the *data processing inequality*Cover and Thomas 2006, because we can think of X as some sort of "true" world, Y as some data (measurements) taken of the world, and $Z = f(X)$ is the result of some analysis process f we perform on those measurements. The data processing inequality states that no amount of analysis can produce

information about the world not already present in the data itself. This can be stated formally as:

$$I(X;Y) \geq I(X;Z)$$

At first glance, this may seem incorrect. After all, what is the point of data analysis if we can't learn anything new? To understand why this makes sense, we need to think of an analysis f not as providing new *information*, but rather as taking existing information and *converting* it into a more applicable form. Consider an average over n measurements:

$$f(X) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

The average contains less *information* than the original data (for example, by discarding all information concerning variance), but is nonetheless a concise and useful summary of an important aspect of the data.

This result has several implications. First, it allows us to frame the general problem of optimization and machine learning as the exploration of the space of possible data analyses. To illustrate this, let us present the data processing inequality in the language of machine learning:

$$Y \to X \to \hat{Y} \Rightarrow I(X;Y) \geq I(\hat{Y};Y)$$

Here, we have the standard notation of Y representing the true, unobservable world; X represents some data set of measurements taken from that world, and $\hat{Y} = f(X)$ representing the result of some analysis f, which may be, among other things, prediction, classification, or structural description of the data X.

Problems in optimization and machine learning are generally represented in a common form: we have some goal, represented via a real-valued *objective function*. This objective function (also known as as *loss function*) calculates some key metric, like the likelihood of a prediction or the magnitude of an error. With this function, we can then test various candidate data analyses and see how they perform with regard to this metric. The analysis which does the best (by minimizing or maximizing the metric) is our answer. Formally, if we think of L as our objective function, and f being the analysis, then we are looking for an optimal analysis f^* such that:

$$L(f^*(X)) \geq L(f(X))$$

$\forall f \in \mathcal{F}$, with $\mathcal{F}$ representing the set of all possible analyses. When L is some type of error metric, we are likely optimizing some (potentially convex) function. When L is a type of likelihood, we are likely performing some form of statistical parameter estimation. The role of L is important, in that it is often the derivatives of L that allows us to explore $\mathcal{F}$. Much research in these fields can be seen as concerning itself with 1) developing new methods for exploring $\mathcal{F}$, and 2) developing new functions L to facilitate progress with (1).

0.2.4 Measurement

Ultimately, our goal is knowledge: information about the world. When speaking of measurement, we treat the term very generally. Taking the temperature with a thermometer, tagging animals in wildlife preserves, and writing essays can all be seen as efforts to represent symbolically some aspect of the world. A first key idea is that measurement and representation are fundamentally linked; a measurement is only interpretable as a form of symbolic representation. A second key idea is that advances in measurement fundamentally expands the space of possible analyses; conversely, the power of analyses is upper-bounded by the power of measurement.

For an historical example, consider the history of oncology. In attempting to study cancer, progress was made most rapidly for Leukemia, largely due to the ease with which cancer could be measured in the blood Mukherjee 2011.

For a timely real-world example, consider the ubiquity of mobile phone cameras and the influence such cameras have had on police accountability. Ostensibly, police have been abusing minority populations for decades, if not for centuries or even millennia. Progress on this issue was slow, in large part due to difficulty in measuring the problem. Soon after mobile phone cameras became commonplace, reporting (measurement) of police violence increased dramatically, giving rise to national protest and the well-organized and influential Black Lives Matter movement. In this example, it is easy to see how the critical factor was change in the underlying world, nor exclusively the development of better organizing techniques, but rather innovations in methods of measurement.

Formally, let us think of a representation $r \in \mathcal{R}$ as a process applied to the world Y, resulting in some objective measurement (data) $X \triangleq r(Y)$. The world is always at least as complex as the measurement (consider the example of a map vs a territory: a perfect map would necessarily be the size of the entire territory). This means than all measurements are information-discarding. This means that we can compare two measurements $r^*, r \in \mathcal{R}$ by comparing their mutual information. If r^* captures more information, then:

$$I(Y; r^*(Y)) > I(Y; r(Y)).$$

If would prefer to think of r as a random function (to incorporate the notion of measurement error), then the relation becomes:

$$I(Y; \mathbb{E}[r^*(Y)]) > I(Y; \mathbb{E}[r(Y)]).$$

We cannot actually evaluate $I(Y; r(Y))$, as Y is not available for analysis except through $r(Y)$; it is necessarily a unknowable object. This does not, however, mean that r is beyond study. Rather, we will evaluate r indirectly, by seeing how it impacts downstream analysis. The key point is that it may always be possible to find a better r^*; *the existence of such an r^* cannot be disproven.*

Recalling the data processing inequality and our historical examples, we see how transforming between representations does not increase information; rather, it allows for new kinds of analysis of existing information. We can think of the process of transformation as applying a function

$$g : X \to X'$$

$g \in \mathcal{G}$, which maps representation X to X'. If this function is *injective* (or "one-to-one"), then both representations contain the same amount of information. The conversion between graphs and matrices is an example of this type of transformation. If the function is not injective, then information will be lost during the transformation. This transformation may still be desirable (and often is) if the target representation allows for more valuable task-specific analysis. The conversion of natural language into a bag of words is an example of this type of transformation.

Formally, we can say that

$$I(Y; X) \geq I(Y; g(X))$$

$\forall g \in \mathcal{G}$. With this result in hand, it may seems like transforming between symbolic representations is pointless. However, it may be the case that a transformed representation permits analysis in away the original does not. Analysis of text often falls in this category. Many techniques for textual analysis rely on conversions of large blocks of text into many small entities: either single words (bags of words) or local word groups (n-grams). These entities are then counted, and the statistical properties of these counts are analyzed. These types of analyses have yielded valuable results: Markov models of language and topic models are just two examples Blei, Ng, and Jordan 2003. We can also think about recommendation systems, in which individuals and items are typically embedded into high-dimensional Cartesian space, as a type of representation transformation Koren, Bell, and Volinsky 2009.

The utility of these models is hard to capture in the language of mutual information, as it is always true that:

$$I(Y; r(Y)) \geq I(Y; g(r(Y))) \geq I(Y; f(g(r(Y))))$$

To represent the utility of task-specific analysis, we will posit a task-specific *utility function* U_t, whose domain is any arbitrary analysis, and range is $\mathbb{R}$. This utility function is left intentionally general and can incorporate many factors, such as computability. The subscript t reflects the notion that utility is interpreted as a function of time (or more concretely, computer instructions).

For a concrete example, let us consider the classic problem of sorting. Consider an unsorted array X of n integers, which we would like to sort. We will compare two algorithms: insertion sort, denoted f_i, and quicksort, denoted f_q. As insertion sort runs in time $O(n^2)$ and quicksort runs in time $O(nlogn)$, we can say that:

$$U_{O(nlogn)}(f_q(X)) = U_{O(n^2)}(f_i(X))$$

Further, we can say that:

$$U_{O(nlogn)}(f_q(X)) \geq U_{O(nlogn)}(f_i(X))$$

This example illustrates another important point: although we introduce f as a very general function, utility is evaluated only on concrete implementations. Observe that an analysis $f \in \mathcal{F}$ assumes a particular representation as input; we make our notation more precise by subscripting: $f_g \in \mathcal{F}_g$.

For a second concrete example, let us consider the basic problem of indexing. We have a linked list X of n elements, and will need to repeatedly access arbitrary elements by position. We have three functions: g_{la}, which converts a linked list to a contiguous array, f_l, which indexes over a linked list, and f_a, which indexes over an array. g_{la} has complexity $O(n)$, f_l has complexity $O(n)$, and f_a has complexity $O(1)$. We can see that, for a single access:

$$U_{O(n+1)}(f_a(g_{la}(X))) = U_{O(n+1)}(f_l(X))$$

If we need to access elements k times, however, we will find that (assuming that $g_{la}(X)$ is evaluated only once):

$$U_{O(n+k)}(f_a(g_{la}(X))) = U_{O(nk)}(f_l(X))$$

We see how the transformation of representation can yield benefits in terms of utility over time. Complexity analysis is nothing new; the value of this notation is the easy extension to approximate algorithms, which converge arbitrarily close to the "true" answer over time. This notation also allows us to compare different classes of algorithms. In image recognition, for example, deep neural networks have been shown to outperform most other algorithms, in terms of classification accuracy These algorithms, however, are relatively slow to train. U_t allows us to compare these algorithms in a new way: a neural network might have more utility over a long time horizon, but a simpler algorithm could have higher utility if time were limited.

The problem is therefore one of finding a transformation and analysis *pipeline* f_g^* such that:

$$U_t(f_g^*(X)) \geq U_t(f_g(X))$$

$\forall f_g \in \mathcal{F}_g$.

In optimization, for example, problem constraints are often "relaxed" to allow for fast solutions which assume convexity. We can think of these relaxations as information-discarding transformations which increase *overall* utility by allowing for fast analysis. Incorporating notation from earlier, we are ultimately interested in functions r^* and f_g^* such that:

$$U(f_g^*(r^*(Y))) \geq U(f_g(r(Y)))$$

$\forall r \in \mathcal{R}$, $\forall f_g \in \mathcal{F}_g$. Ranging across all g and f, we see that for each task-specific utility U, it may always be possible to develop some new r^* such that:

$$U(f_g(r^*(Y))) \geq U(f_g(r(Y))).$$

In other words, in addition research in methods of analysis (f, L), we can conceive of research in methods of representation and measurement (r, g). The development of new measurements has the potential to unlock more powerful types of downstream analysis, by capturing more information about the underlying world. Note that we have *not* shown that:

$$I(Y; r'(Y)) \geq I(Y; r(Y)) \rightarrow U_t(f'_g(r'(Y))) \geq U_t(f_g(r(Y)))$$

for some $f'_g, f_g \in \mathcal{F}_g$. This is due to the varied interaction between representation and computability; there is no guarantee that an information-capturing representation will be easier to analyze; the opposite may be true. However, a high information-capturing representation can always be converted into a simpler form; better measurements can only improve overall analytic ability.

Much of the theory we will develop in subsequent sections will be an attempt to discover exactly such a representation.

In 1984, statisticians Persi Diaconis and David Freedman published a paper on the topic of "projection pursuit". In this paper, they show that an arbitrary projection of a high-dimensional random variable into a lower-dimensional space will with high probability exhibit a Gaussian distribution, regardless of the distribution of the original random variable Diaconis and Freedman 1984. The Gaussian, or "normal", distribution, is the maximum-entropy distribution for a random variable, given the variance of that variable. Put another way, a variable that is normally distributed contains less information than one with any other continuous distribution. Put yet another way, projection into lower dimensions very likely discards information. Put a final way, *the process of measurement itself can be seen as a process of projecting information from some complex, unknown space (the true world) onto a finite-dimensional representational space*; necessarily, information is lost.

This is unfortunate. The benefits of projection (which can be thought of as data compression) are great: data becomes smaller and easier to store,

transmit, and analyze. If the original data has special structure which can be exploited, compression can occur (using techniques such as Principal Components Analysis or gZip) with as little as zero information loss. In other cases, as first shown by Johnson and Lindenstrauss, random data compression can occur with acceptable loss of information Dasgupta and Gupta 2002.

The takeaway is that compression is more effective when it can exploit any *special structure* of the object in the original space. In the context of this argument, we will say that measurement is more effective when the representation captures accurately any special structure of the aspect of the world being measured.

These are general statements. To continue to develop these ideas, we will turn to briefly to economic history.

0.2.5 Economic History

Middle-age economies were simple. These economies, based largely on barter and regional currencies, required participants to possess special domain knowledge of the region and the items involved in the exchange; participants without this knowledge struggled to participate Heilbroner 1999. The innovation of money price transformed economies by allowing for a standard, numeric representation of value.

This standard, consistent representation created the foundation for further economic innovations. In the domain of bookkeeping, numerical prices allowed for the development of double-entry bookkeeping, enable the development and maintenance of business ventures on a larger scale Ferguson 2009. In the domain of finance, numerical prices allowed for the development of mathematical models of risk, which themselves allowed for the development of systems of loans and credit. The development of credit can be seen as enabling the coordination of economic activity across not only space, but time Ferguson 2009. These innovations – made possible also by improved communications technology – greatly increased the sophistication of human economic affairs.

The interpretation of price as a low-dimensional representation of information has long been appreciated (although not phrased in those terms). Via the mechanism of price, disparate independent entities are able to coordinate activities; a revolution in economics came about when this type price-based coordination was argued to contribute to communal progress. In his classic

Wealth of Nations Smith 2000, Adam Smith described the "invisible hand" which emerges from decentralized self-interested economic activity to guide overall economic development.

Recognizing the challenge of a centralized measuring and analyzing of economic information, Hayek and his conservative contemporaries defended price-based free markets against Communist alternatives on the grounds that price systems allowed information to flow rapidly throughout large and decentralized economies Hayek 1945. Central planners Hayek argued, would fail to aggregate enough information to make optimal planning decisions.

In idealized settings, market-based coordination can be shown to be optimal (with regards to some accepted optimality criteria, of course). In his classic paper "The Problem of Social Cost", Nobel prize-winning economist Ronald Coase demonstrated that, in the presence of perfect information and zero transaction costs, problem involving externalities (like pollution) can be solved optimally by market forces Coase 1960. In his paper, Coase argues that, with perfect information about the externalities and no transaction costs, entities will bargain amongst themselves and ultimately arrive at a Pareto-efficient equilibrium, independent of the initial allocation of property rights. He goes on to argue that, given real-world conditions of limited information about externalities and non-zero transaction costs, such equilibriums cannot be expected to emerge solely from bargaining between entities. Additional structures (such as governments and legal systems) will be necessary, and such structures are biased towards initial holders of property rights, meaning that initial allocations of property rights will influence final outcomes (as historical experience has shown to be the case).

In this example, we can also see an example of one of our schematic relationships: that between cost and technology. A problem for Coase's theory is that externalities can be difficult, if not impossible to measure. To accurately price and account for negative externalities like pollution, these externalities must be accurately measured. Sufficiently precise measurement may be impossible; else prohibitively expensive. Improvements in technology (for example, in air or water sampling) should allow for more accurate measurement of externalities as lower cost, which by our proposed theory will allow more effective peaceful large-scale coordination (via a market system).

The strengths of markets are often appreciated; as are their drawbacks. Observers of England's industrial revolution were horrified by the living con-

ditions of urban workers. Although admittedly overlooking the bleakness of rural life, critics of industrialization pointed out how little of the human element seemed to be present in the economic calculus of industry Heilbroner 1999. Observing the aftermath of the United State's Great Depression, economist John Maynard Keynes realized that unaided free markets would not necessarily bring about long-term economic growth Heilbroner 1999. The instability of markets and their tendency towards crashes was observed by other economists Minsky 2008. Overall, we seem justified in concluding that the measure of price was a radicaldevelopment, allowing for vastly improved communication and coordination. However, the efficient metric fails to capture very important information, and is therefore by itself insufficient for coordinating human activity.

0.2.6 The Proxy Gap

Imagine we have a library, and we would like to know how many books the library contains. We count the number of books (discrete objects, implicitly defined as units of paper, binding, etc), and return the sum. Overall, a straightforward process.

Now, imagine we have visitors to this library. These visitors read books, and we would like to know how much they liked the books they read. We might ask them to rate each book on a scale of 1-5. For each book, we average these scores to return an aggregate score.

In the first example, there is no question about the legitimacy of the measurement; the result can be seen as authoritative. In the second case, however, there is room to question: how well do these measurements reflect the true subjectivity of the readers? In the first case, we were concerned with objective quantities; in the latter, subjective experiences, which are generally accepted as being difficult to measure.

Given the difficulty of representing and analyzing subjective experience and personal qualities, a common practice is to rely on easily-measurable objective *proxies*. American mathematician Cathy O'Neil discusses this at length, reviewing the use of measurement proxies in domains as diverse as education, hiring, credit, and criminal justice O'Neil 2016. One of her main conclusions is that poorly-designed proxies facilitate the perpetuation of existing race and class inequalities. Many have begun reaching similar conclusions in the last few years; research into "algorithmic fairness" is new and

ongoing Tramèr et al. 2015 Friedler, Scheidegger, and Venkatasubramanian 2016.

Moving forward, we will refer to this gap between a measurement proxy and the stated object of measurement as the *proxy gap*. Returning to the library example, the book count has a proxy gap of zero, as the measurement exactly captures the aspect of the world it purports to measure. The book ratings, however, have a non-zero proxy gap: there are aspects of book preference which are not captured on a 1-5 scale.

Statistician David Freedman discusses an analogous concept in his 1995 paper, "Some Issues in the Foundation of Statistics" Freedman 1995:

> *Regression models are widely used by social scientists to make causal inferences; such models are now almost a routine way of demonstrating counter-factuals. However, the "demonstrations" generally turn out to depend on a series of untested, even unarticulated, technical assumptions. Under the circumstances, reliance on model outputs may be quite unjustified. Making the ideas of validation somewhat more precise is a serious problem in the philosophy of science. That models should correspond to reality is, after all, a useful but not totally straightforward idea — with some history to it. Developing models, and testing their connection to the phenomena, is a serious problem in statistics.*

By definition, the proxy gap is a qualitative, not quantitative, property. As such, we will use it as a tool for developing the arguments to follow, but will abstain from attempting to incorporate it formally into analysis. Still, the notion will be useful, as it provides a common frame with which to reason about potential failures of mathematical models of the world.

Education

As an example, let us consider some aspects of the American higher education system.

Prospective undergraduates are faced with a choice of several hundred possible universities – too many to reasonably research and evaluate. Clearly, there was a need for some sort of summary assessment of school quality. In 1983, in response to a perceived need, the *US News & World Report* released the first edition of their now-infamous college rankings (oneil). Taking into

account a number of factors, such as student test scores, acceptance and retention rates, and alumni giving, *US News* generated a score, which it used to rank schools. This ranking has, for better or worse, become a standard measure by which colleges are judged.

These rankings became so influential that schools began going to extreme (even illegal) lengths to improve their scores: lying about student test scores, building multi-million dollar athletic facilities (increasing tuition costs accordingly), and rejecting candidates deemed "unlikely to attend" O'Neil 2016. Observe that these changes have unclear, if not outright negative, impact on "school quality".

In this example, the proxy gap is the gap between the actual measurements (student statistics) and the aspect of the world we are interested in ("school quality"). Because of this gap, incentives are distorted and it becomes possible to game the system. Schools are able to rise in the rankings without improving their quality, while other schools might fall in the rankings even as quality of education improves — if the improvements do not manifest in specific ways.

Standardized testing plays a huge role in controlling access to higher education. These tests exist at all levels of education (SAT for undergraduates, GRE, LSAT, MCAT, GRE, and GMAT for graduate students). These tests purport to measure academic aptitude for their respective courses of study; they are each a proxy with an associated proxy gap. This proxy gap has lead to large test preparation industries. In all cases, prospective students can be found eagerly signing up for test preparation courses, drilling practice questions and studying the standard structures of the test in question. Students with access to resources (such as preparation courses) generally perform better; thus, critics say that these tests are at least in part measuring socioeconomic status Zumbrin 2014.

In these examples, we see how the issue of *legitimacy* of measurements becomes a major issue. Measurements exhibiting large proxy gaps are easier to challenge as illegitimate. This illegitimacy becomes a source of tension, as the illegitimacy of measurements calls into doubt the legitimacy of the systems built on those measurements. As data and data processing becomes increasingly central to our activities, we should be attuned to the issue of legitimacy in modeling.

What insights can we draw from the examples given? A general theme is

a desire to measure subjective qualities. In light of the earlier discussion on representation and measurement, it would seem we would benefit from some representation able to capture this subjective quality. Representations of subjectivity exist already: language and art being two important examples. In fact, language was the first symbolic representation, far predating any mathematical symbolism.

What these representations lack, however, is formality. The formalisms of mathematics have allowed for the precise communication of fantastically complex ideas. It should seem that we would like a representation combining the formality of math with the subjectivity of language.

0.2.7 The Dialectic

To develop our argument, we must first introduce some ideas of Hegel. A german "idealist", Hegel believed that concepts, and human conceptual structures, played a key role in determining the course taken by society. Contrast these ideas with those of materialists, such as Marx, who believed that resource constraints ultimately shaped social forms. Historical experience has, fortunately, shown that both views contains much truth; the experience of the Anabaptists in Munster following the Protestant ReformationCarlin 2013 shows clearly the power of ideas in shaping events, while the experience of the Israeli Kibbutzim points to the power of material conditions.

In developing his theory of ideas, Hegel articulated the concept of a dialectic. In a dialectical process, we begin with an idea, known as the *thesis*. In contraposition to the thesis, there emerges an opposing idea, known as the *antithesis*. Thesis and antithesis engage in tension, the space between them one of paradox. This tension is ultimately resolved into a new idea, the *synthesis*. This synthesis then plays the role of thesis to begin a new dialectical process. Hegel contended that this process was helical, and thus contributed a forward momentum to human affairs.

We note that Hegel himself did not use the language of thesis/antithesis/synthesis, but rather spoke of abstract/negative/concrete.

An important aspect of a dialectic is that the thesis and antithesis are in opposition to each other, but neither is "correct" or "incorrect". We experience dialectics regularly in our lived experience:

1. Freedom vs Security
2. Individual vs Community

3. Change vs Stability

4. Process vs Outcome

The recurrence of these themes in theater and literature attests to the fundamental role the dialectical structure plays in our shared and individual experiences.

Fundamentally, a dialectic is a paradoxical space between two contradictory extremes. Resolving these tensions is an important part of our decision-making process. As an example, we can view any political process (or smaller scale group decision-making processes) as being exercises in resolving these tensions into concrete actions.

Contrast a dialectic dimension with something like $\mathbb{R}$, the real line. $\mathbb{R}$ is also a space in between two extremes: $-\infty$ and ∞. *The key difference is that numerical space is well-defined, while the dialectical space is by definition contradictory.*

We argue that concepts are ultimately relational in nature, and that attempts to represent them in reference to some absolute scale inevitably obfuscates important aspects of their character.

The Hegelian dialectic is not without critics. Karl Popper, an eminent philosopher of science, has attacked dialectical thinkers for their willingness to accept, if not outright invite, contradictions Popper 2002. Popper, famous for his notion of "falsifiability" as a prerequisite property of valid scientific theories, felt that such conceptions of history were impossible to invalidate.

We resonate with Popper's emphasis on falsifiability. Further, we agree with Popper's argument that contradictions are valuable primarily for their value in helping organizing our collective effort towards their resolution. We believe a dialectical understanding is not in conflict with the scientific method, each being appropriate for making sense of one or another aspect of our experience. Going further, we speculate that by continuously attempting to understand and resolve these paradoxes, dialectical distinctions can be made successively more nuanced. We can imagine this as a *recursive process*, by which distinctions are made nuanced, giving rise to further distinctions and further opportunities for nuance; this can be seen as the ideation process itself.

Returning to our earlier political example, it is easy to see dialectical tensions in the two-party system of the United States. In the legislature,

major dialectical themes such as "progress vs. stability" shape debate. For the development of a specific policy, however, we should abhor contradiction; the budgets should balance.

The popular Myers-Briggs Type Indicator provides a useful case study of the power and pitfalls of the dialectical understanding. This test asks a subject a series of questions, and then places them into one of two binary categories, along four axes. Subjects are encouraged to see these categorizations as providing insight into their personality and their interactions with others. This test appears often in popular culture and in business, and this popularity self-evidences the test's appeal. Clearly, many find the test's structuring to be a valuable aid in their own thinking.

The test is not without critics. Primary criticisms include low test-retest reliability, and of oversimplification and misunderstanding of complex personality traits. These criticisms point to two important aspects of dialectical thinking:

1. Any particular dialectic relationship represents a particular level of abstraction; any dialectical relationship can be made more nuanced. This relates to the dynamic, process-like nature of the dialectical relationship.

2. Individuals experience reality differently. Descriptions of dialectic relationships are at best *modal*, in that they describe common aspects of the experience of many, but not all.

In seeking dialectical structure, the goal cannot be to learn a fixed and absolute structure. Rather, the goal must be to learn the particular task-specific structure (in terms of level of nuance) appropriate for the entities involved.

0.2.8 Concept and Distinction

Having introduced the sweeping notion of the dialectical process, how might we continue forward in developing a concrete theory? It is too much to hope to be able to represent and formally analyze grand and abstract dialectical relationships. What do we take with us?

First, we take the idea of *distinction*, and take the notion of making distinctions between concepts as a fundamental operation. Recalling momentarily our history of economics, we observe that it was not until the

economic realm was *distinguished* from the social and political realm could economic thinking take full flight Heilbroner 1999. Next, we accept the idea of a fundamentally process-like and relational nature of concepts; we should be cautious of approaches which attempt to embed these concepts in an absolute space. Then, we ask whether these notions of distinction and relation can be applied to concepts more concrete than the grand abstractions often seen in introductions to dialectics.

Greek philosophers were preoccupied with understanding the true nature of things. Heraclitus, a prominent pre-Socratic, believed that the true nature of things was *change*: "you cannot step in the same river twice." Socrates and his students Plato and Aristotle felt differently: they felt that things were fundamentally constant and unchanging nature: "if Socrates gets sick, he is still Socrates."

In his "Allegory of the Cave", Plato defends this notion, arguing that the wide variation of things seen in the world is due to of physical objects being randomly-perturbed instantiations of constant, idealized types. This Aristotelian orientation towards understanding phenomena in terms of fixed and constant properties has had significant influence on the development of science Pirsig 2006.

Leibniz had a dream. He dreamed of a language, *characteristica universalis*, which could represent perfectly a wide variety of concepts in the world. This language, Couturat writes Couturat 1901, "would express the composition of concepts by the combination of signs representing their simple elements, such that the correspondence between composite ideas and their symbols would be natural and no longer conventional." Leibniz envisioned a future where disputes were settled by representing the problem as sentences in this universal language, and then simply *calculating* the answer. "Calculemus!", he would say: let us calculate.

Many logicians attempted to develop rigorous logical systems for reasoning about discrete concepts and categories. Ludwig Wittgenstein, however, recognized the impossibility of necessary and sufficient logical conditions for categories, and instead developed the idea of softer "family resemblances" among groups of things. As an aside, we can see unsupervised approaches in machine learning (K-nearest neighbors, gaussian mixture models, and kernel-based approaches to classification being notable examples) as attempts to emphasize this relational quality.

An especially illustrative example comes to us in the form of Alfred North Whitehead. A British mathematician and professor, Whitehead wrote the seminal *Principia Mathematica* with his student, Bertrand Russell. In *Principia*, Whitehead and Russell sought to describe axioms and inference rules via which all mathematical truths could be proven Doxiadis and Papadimitriou 2009. A monumental endeavor, it was nonetheless a failure. Kurt Gödel, the German mathematician, showed that such systems were impossible: that any formal system contains unprovable truths Hofstadter 1999. Later in his career, Whitehead came to believe in the superiority of a *process-based* ontology, in which objects in the world are understood not in terms of fixed, absolute characteristics, but rather as continuously undergoing change processes. He would go on to write was has become the seminal text of process philosophy, *Process and Reality* Whitehead 1979.

These lines of thinking eventually transitioned from philosophy to cognitive linguistics, being pursued in the 1970s by the American linguist Eleanor Rosch, eventually culminating in the development of *Prototype Theory* Rosch 1973 Rosch 1975.

What Rosch found was that concepts tended to organize into hierarchies, with general *prototypes* being more readily available for cognition than specific instances. These prototypes in general describe the salient aspects of the object: the prototype *tree* captures a great deal of information about both subordinate types *pine* and *elm*. Superordinate types fail to contain enough information: *plant* fails to capture enough salient properties.

A frequent characteristic of prototypes is a mono-syllabic name. Consider *car* and *tree*, both succinct aural representations. Flower, while being bi-syllabic, is descended from the monosyllabic Old French *flor*.

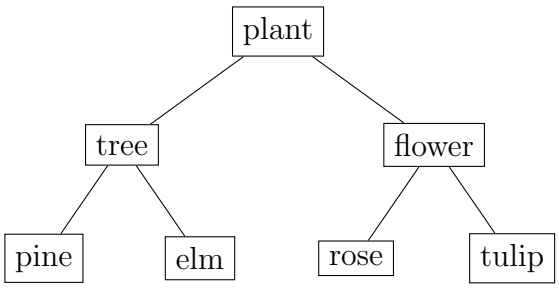


Figure 1: Example conceptual tree

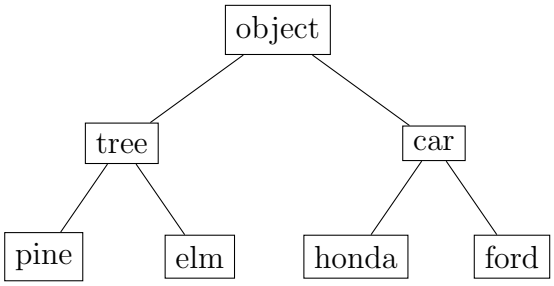


Figure 2: Alternative set of relationships

Rosch and others went on to show that culture and environment can effect what individuals come to understand as "prototypical"; in other words, how they come to distinguish experience. For an individual raised in a forest, for example, the categories of *pine* and *elm* may in fact be prototypical categories. For this individual, the differences between the two species are more salient than their similarities. We can see this also in many professions: professionals communicate in jargon, making distinctions unfamiliar to those outside their field.

Rosch's work sheds a clarifying light on the relational nature of concepts. What distinguishes *tree* from *elm* is not any particular property of *elm*, but rather the presence of *pine*. If *pine* did not exist, then *elm* would add no information above and beyond that provided by *tree*; *elm* would not even exist.

American cognitive linguist George Lakoff develops Rosch's ideas in new directions, making central the idea of *metaphor* in building understanding George Lakoff 2003. In Lakoff's view, complex ideas (distinctions) are understood by analogy from more basic ideas (distinctions). A child, for instance, first comes to understand ideas of *up* and *down*, and *warm* and *cold*. These first ideas are rooted in physical experience (the child has a body and experiences physical phenomena). The child later comes to associate warmth with affection and cold with rejection (by observing the relationship between physical warmth and closeness to a parent, for example).

Once established, a child can come to understand more complex ideas by seeing them in terms of the more basic metaphors. For example, imagine a child coming to understand a tumultuous friendship through the ideas of "hot" and "cold". For example, imagine a couple coming to understand their

relationship as a journey they are on together. The metaphoric nature of our understanding shapes politics, Lakoff argues: candidates shape messages and choose words with care in their attempt to create advantageous associations in the minds of the electorate Lakoff 2014.

With these examples, we wish to show that the principles of distinction and opposition developed with regards to the dialectic can be applied more generally to mundane and everyday concepts.

A Thought Experiment

As a thought experiment, let's imagine a new intelligent agent, such as a human infant, or some hypothetical AI, has just come into existence. Having been instantiated without assumptions, this agent possesses just one single, unified concept, extending infinitely in all directions.

The agent begins to experience a constant stream of stimulus, and must somehow learn how to navigate and act in the environment. What might this agent do? What operations are possible?

If the agent's entire understanding consists of a single concept, then the first thing the agent might do it **separate** that concept into two concepts. The separation might be along some **dimension**, such that the two resulting concepts exist in some relation to each other. Having successfully performed the separate operation, the agent could separate the resulting concepts again and again, recursively ad infinitum, each time along some new dimension, achieving an arbitrarily refined conceptual structure.

In understanding the **separate** operation, we have the notion of dimension. Recalling Lakoff's basic metaphor, an early separation could be between left and right, or up and down – basic distinctions needed to navigate physical environments. As each separation is performed in sequence, we can also imagine a *hierarchy* of separations, with earlier separations representing more fundamental distinctions, with later separations representing more fine-grained distinctions (within the framework established by earlier separations).

As a brief digression, it is thought-provoking to imagine this process of recursive conceptual definition as a sort of inverse of a traditional Buddhist or Hindu meditation practice. In a meditation practice, one attempts to cease making distinctions in their experience. Here, we intentionally develop a system of distinctions recursively out of undifferentiated experience.

0.2.9 Ordering and Pairwise Preference

If we exhibit prejudice towards numerical representation, what alternatives might there be? Any candidate representation should exhibit the following desiderata:

1. The space between objects is left undefined.

Fortunately, item *ordering* makes no statement about the nature of the relationship between the items, apart from their relative relations to each other.

Nobel prize-winning economist Kenneth Arrow has done extensive work analyzing voting systems. Throughout much of his career, he advocated for ordinal representation of preference Bianchi 2014, pointing out that relative preference is all that is naturally observed:

> *The only evidence of an individual's utility function is supplied by his observable behavior, specifically the choices he makes in the course of maximizing the function. But such choices are defined by the preference order and must therefore be the same for all utility functions compatible with that ordering.* ***Hence there is no quantitative meaning of utility for an individual.***
>
> - Kenneth Arrow 1973; p104, emphasis added

However, Arrow would also show the limits of such a representation. In his *impossibility theorem*, Arrow proved that no ranked-choice system could satisfy all of a set of voting system criteria Arrow 1970. In light of this, Arrow would later amend his views and come to tolerate cardinal representations of preference, which contemplate real-valued distance between items, on the grounds that such representations provide additional information Hamlin 2012. Indeed, certain limitations of ordinal preference representation (such as the possibility of intransitive preference) are absent given cardinal preference representation. However, Arrow cautioned against such systems, observing that wide ranges made it more likely that voters would misrepresent their preferences Hamlin 2012: "The trouble with methods where you have three or four classes, I think if people vote sincerely they may well be very satisfactory. The problem is the incentive to misrepresent your vote may be high."

Interpreting Arrow's comments through our framework of representation and measurement, it seems as though we can interpret cardinal preferences

as capable of representing more information. We can see this is the case, given that we can always convert cardinal preferences to ordinal, but not vice versa.

This would seem to be a challenge for this thesis, which has been prejudiced against numerical representation of subjectivity. If the question was only general expressiveness of representation, then the challenge would be severe. Arrow's comments on error, however, point to a deeper tension: measurement error is almost certain to be higher for cardinal representations.

Returning to our formalisms, if we denote ordinal measurements with r_o, cardinal measurements with r_c, and the reduction of cardinal measurements to ordinal with g_{co}, we see that:

$$I(Y; r_c(Y)) \geq I(Y; g_{co}(r_c(Y)))$$

but

$$I(Y; r_c(Y)) \stackrel{?}{=} I(Y; r_o(Y))$$

The latter uncertainty is due to the unknown trade off between expressiveness and measurement error.

We argue that pairwise preference has the advantage of being able to *directly represent* subjectivity: a preference (or distinction) between two concepts. This property emerges from the structured and limited nature of the pairwise preference: two items, and a preference for one over the other. This representation does not by itself contain a huge amount of information, but what information is contained is an accurate reflection of the aspect of the world being represented: subjective preference. We argue that, due to this direct representation, pairwise preferences are robust against measurement error; using terminology developed earlier, we can think of a pairwise preference as having a proxy gap of near-zero. Regarding potential intransitivity of preference, seen as a major weakness for ordinal preference representation, we contend that such intransitivity is a feature of subjective experience, and we and seek to develop methods for recognizing and responding to intransitivity. In the parlance of software engineering, "it's not a bug, its a feature".

It is exactly for this ability of pairwise preference to directly represent subjectivity (and the contradictions that sometimes appear) that we choose to explore it.

0.3 Mechanics

0.3.1 Elements

Having concluded our survey of material and philosophical history, we are ready to develop the theory. Our goal will be to develop a representation of subjective experience that is amenable to formal analysis.

Central to this theory will be the notion of pairwise preferences, which we have attempted to justify as a valid, yet formal, representation of subjective experience.

Before we can assess preference, we must establish some criterion to evaluate preferences against. Among the same candidate set of answers, different questions can easily lead to different preferences. For example, ranging over the items in your fridge, the questions "what to eat for breakfast" and "what to eat for dinner" will (likely) lead to different preferences among the same set of options.

We define questions with maximum generality, considering a general "question object" Q. The representation of Q is intentionally unspecified; the specific from of Q will affect only the interpretation of the results, not the underlying theory.

We anticipate that Q will often be represented via a string of characters (as in the fridge example). However, Q could be an image, a sound, an equation, or some new object yet unimagined.

In order to ask questions, we need candidate answers. These candidate answers are members of an *answer set* A. As with the question object Q, the specific representation of the elements of A are intentionally unspecified.

In order to represent subjective experience, we need an entity capable of subjective experience. Discussions of subjectivity inevitably involve discussions of the notoriously elusive topic consciousness. While the exact nature of consciousness is not known, researchers generally agree that conscious experience is attributed to discrete entities: as an entity I have an experience of consciousness that is separate from the experiences of other entities.

To resolve group preferences, we need a set of such discrete entities. This set of entities is denoted E. Unsurprisingly and in the spirit of this exposition, we leave the specific nature of these entities intentionally unspecified. We anticipate that these entities will often be people. Later in this work we will

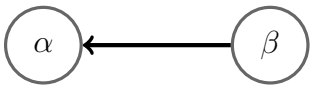
introduce potential new directions for this theory, for which we may want to define these entities differently. Specifically, we will see how the problem of defining entities can be understood as selecting an "access policy", the choice of which will shape the interpretation of the results.

Having introduced questions, answers, and entities, we are ready to introduce the "preference", conceived of as a basic unit of subjective experience. As the name suggests, a preference represents an *answer to a question*. Preferences have five components:

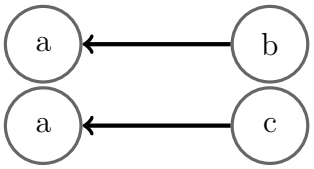
1. A question object Q.
2. An entity $e \in E$.
3. A candidate answer $\alpha \in A$.
4. A second candidate answer $\beta \in A$.
5. The preference $p \in \{-1, 1\}$, where 1 corresponds to preferring α.

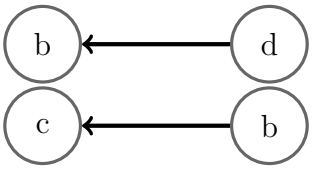
It is not immediately obvious how we might operate on this representation. Recalling that preferences are only defined relative to a question Q, we can make Q implicit. Further, for the moment let us assume that we are interested only in the preferences of a single entity e.

Now, we see that the salient attributes of a preference are the two options α, β, as well as the preference p. If we imagine α and β as nodes, then p can be represented as a directed edge between the two. Specifically, we create an edge (β, α) if $p = -1$ or (α, β) if $p = 1$ (the edge flows from loser to winner):

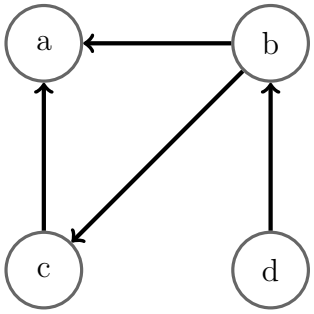


This graphical representation is desirable, as it suggests a natural way of aggregating preference. Imagine we have $A = \{a, b, c, d\}$. The entity generates the following preferences:

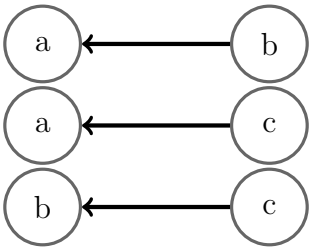




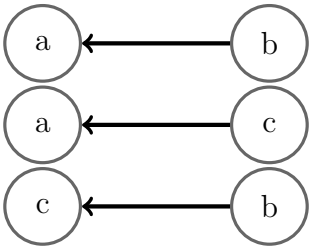
We can aggregate these preferences into the following *preference graph* S:



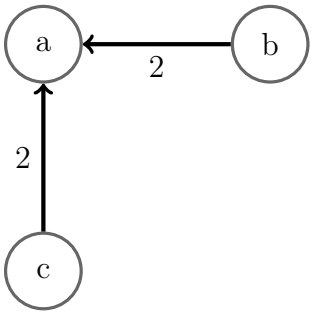
Aggregating preferences across multiple entities presents little difficulty. If we have entity α generating the following (an ordered preference of a, b, c):



And entity β generating the following (an ordered preference of a, c, b):



We can assign each preference a weight of 1 and combine the edges. Parallel edges sum, while antiparallel edges cancel:



We denote this weight $w(u, v) \in \mathbb{N}^+$.

It is instructive to see what occurs in the instance of two entities with opposing preferences. Say we α with preferences (a, b, c), and β with preferences (c, b, a). We might say we could resolve this by selecting b, as this seems the mutually-agreeable option.

Is this principled? Selecting b would cause both entities to have their second preference, a forfeiting of one item each. Selecting a or c, on the other hand, would cause one entity to have its first preference, and the other third — a forfeiture of two items. In a sense, this pair of opposing preferences render all answers equal.

Observe what happens in the corresponding preference graph (Figure 3).

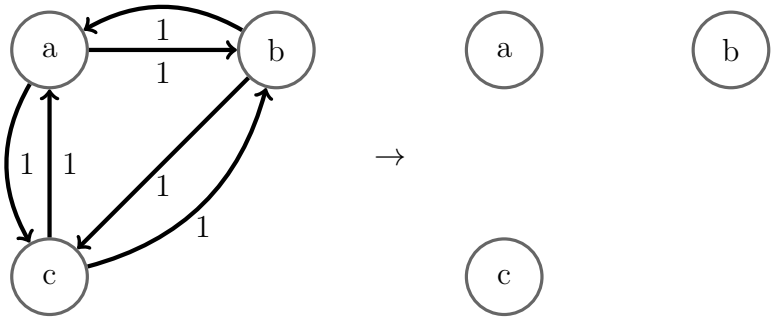


Figure 3: Cancellation of opposing preferences.

The answers are disconnected; we interpret as an absence of preference.

A note on method: we have been very intentional in our avoidance of assigning numerical value to subjective preference. For an *individual* entity (with an emphasis on the literal meaning of "individual" as non-divisible), preference is non-numeric. Populations have magnitudes, however, and so we

can comfortably reason in terms of sums and ratios when considering groups of entities.

0.3.2 A Probabilistic View

Let us now consider some structural properties of preference graphs, and interpret them through the lens of preference resolution. In this discussion, we will consider graphs in which we have observed each ... TODO

In the language of computer science, we denote graphs as $G = (V, E)$, with graph G consisting of some set V of *vertices* (or *nodes*) and some set E of *edges* between the vertices. Edges can be directed or undirected, and are denoted $(u, v) \in E$, with $u, v \in V$.

When analyzing complexity, we use $n = |V|$, the number of vertices, and $m = |E|$, the number of edges. Since an edge may exist between any pair of edges (even a self-pair), we see that $m \leq n^2$. Note that $(u, v), (v, u), (u, u)$ may count as three separate edges. In our case, there are $\frac{n^2-n}{2}$ possible preferences (per entity).

A vertex can be called a *sink* if all of its edges point inward towards it. In the context of preference, a sink is an option that is preferred over all others. As a first pass, we can think of the problem of resolving preference as a problem of finding a sink. Sinks can be found in O(n) time.

It might seem as the problem of preference resolution is simple: observe preferences, and find the sink. Unfortunately, there is no guarantee that a sink will exist. As n increases, the likelihood of a sink existing in a random graph decreases exponentially. Every vertex has $n-1$ edges. For vertex u, to be a sink, all of these edges must point towards it. If we consider a random graph with a uniform distribution on all edges:

$$p((u, v) \in E) = \frac{1}{2}$$

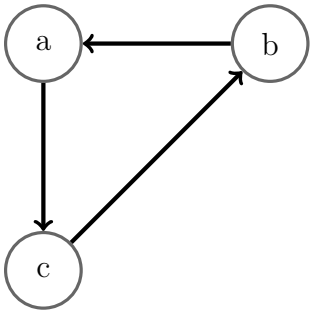
then

$$p(u_{sink}) = \prod_{v \in V} p((u, v) \in E) = \frac{1}{2^{n-1}}$$

Since the relationships between any pair of vertices is independent of all other pairs, the probability of *any* sink is the sum of the probabilities of the individual vertices being sinks:

$$p(G_{sink}) = \sum_{u \in V} p(u_{sink}) = \frac{n}{2^{n-1}}$$

The likelihood of a sink existing in a random graph diminishes quickly as n increases. If a graph has no sink, then there must exist at least one *cycle* in the graph. A cycle is a subset of vertices and edges such that there exists a "path" of edges such that any vertex in the cycle is reachable from any other. Here is the simplest cycle between three vertices:



It is easy to see that this graph has no sink. *From the perspective of preference, we interpret a cycle as a set of options which are preferred equally*; alternatively, we can say that they are *indistinguishable*.

If there are n vertices, then there are $(n^2 - n)/2$ pairs. Since there are two possible states for each pair, there are a total of

$$2^{(n^2-n)/2}$$

possible graphs. As an illustration, let's consider a triple, where $n = 3$.For every triple of vertices, there are $2^3 = 8$ possible permutations of edges:

In the case of three vertices, we see cycles in two out of eight possible graphs. A cycle exists between any set of k vertices if all k corresponding edges have the same orientation, something which occurs with likelihood $\frac{1}{2^n}$. The likelihood of a cycle among any k vertices shrinks exponentially in k, but note that number of subsets of k increases combinatorically in the number of nodes: there are $\binom{n}{k}$ such subsets. For example, given $n = 5$ and $k = 3$, there are

$$\binom{5}{3} = \frac{5!}{3!2!} = \frac{20}{2} = 10$$

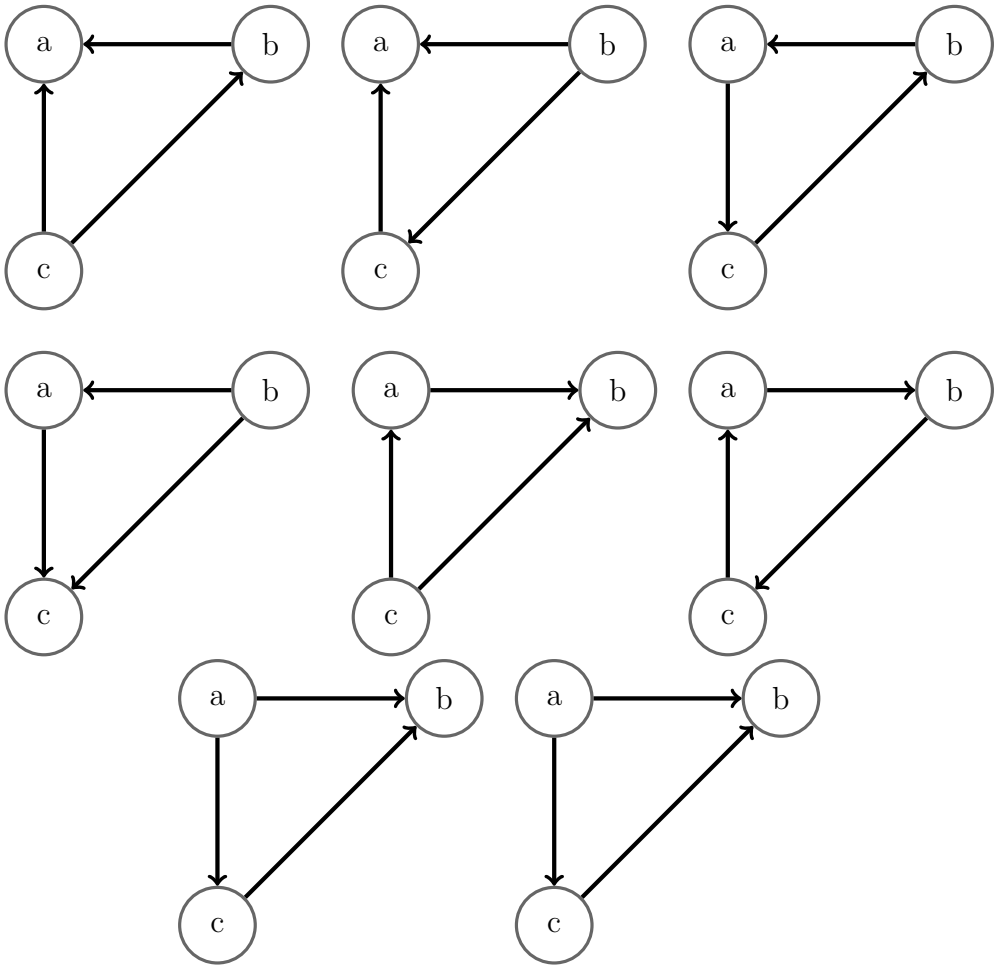
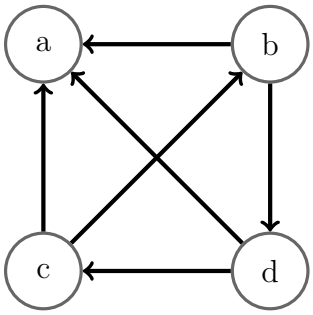


Figure 4: All possible 3-vertex directed graphs

such subsets.

It is worth noting that while the absence of cycles necessarily implies the presence of a sink, the presence of a cycle does not necessarily imply the absence of a sink. To see this, consider the following four-vertex directed graph:



In this graph, vertices $\{c, b, d\}$ form a cycle, but a is still a sink. As discussed earlier, however, the likelihood of a sink existing in a random graph, even allowing for cycles, decreases exponentially in n. We raise this point to underscore that the presence of cycles does not imply the absence of meaningful structure; simply that meaningful structure will be more challenging to discern.

What is "structure"? To understand what constitutes structure, it will be helpful to consider an example of the absence of structure: a completely random graph. To describe this graph, we will introduce some concepts of probability. Since preference graphs consist of directed edges, we will say that a "completely random" graph is one where, for any two vertices, there is an equal probability of the edge between them pointing in one direction or the other. Formally, we say that edges are Bernoulli distributed:

$$edge(u, v) \sim Bern(0.5)$$

with

$$edge(u, v) = 1 \rightarrow (u, v) \in E$$

$$edge(u, v) = -1 \rightarrow (v, u) \in E.$$

If we consider incoming edges as having a value of 1, and outgoing edges as having a value of -1, the the expected value of any edge is:

$$\mathbb{E}[edge(u, v)] = 0$$

and the sum of ingoing and outgoing edges for any node u in the completely random graph, assuming edge independence, is:

$$\mathbb{E}\left[\sum_{v \neq u} edge(u, v)\right] = \sum_{v \neq u} \mathbb{E}[edge(u, v)] = 0$$

This graph has no meaningful structure; no preferences can be learned. Considering this example, it seems as though *structure* can be thought of in terms of deviations from this random case.

0.3.3 A Linear Algebra View

A graph can be represented as a matrix, with the values of the matrix corresponding to the values of the edges between nodes. Let us denote the raw connection matrix as C. By setting the values of the diagonal equal to the sum of their corresponding columns ($C_{ii} = \sum_j C_{ji}$), and normalizing the rows of this matrix ($\sum_i C_{ji} = 1$), we can construct a Markovian "transition matrix", denoted M (Figure 5).

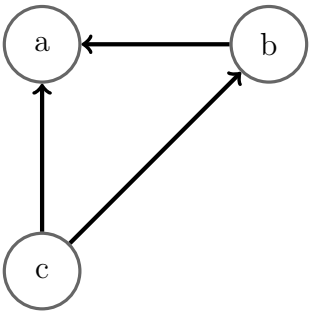
This matrix M encodes the probability of "transitioning" from one item to another, with the values on the diagonal representing the likelihood of "sticking with" an item. If we imagine a vector $x_t \in \triangle^{V-1}$ representing the state at time t as a distribution over all possible items, then

$$x_{t+1} = x_t^T M$$

gives us the distribution over items at time $t + 1$. If M has certain properties (*irreducible*, meaning that any item can be eventually reached from any other item, and *aperiodic*, meaning that there are no stable loops among sets of states), then it can be shown that eventually x will converge to a "steady state" distribution, in which $x = x^T M$ (H. W. Lin and Tegmark 2016). Further, it can be shown that this steady state distribution, denoted x_∞, is equivalent to the principal eigenvector of the matrix M, here denoted v_1. The normalization of M ensures that the principal eigenvector, denoted λ_1, is always equal to 1.

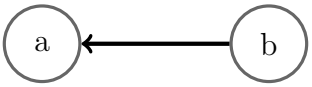
Interpreting x_∞ as a distribution over items, then the components of x_∞ with the largest values (probability) can be seen as the "most preferred" items (Paisley 2015). The Perron-Frobenius theorem forms the foundation of these results, and use of this method has a long history (Keener 1993) and many applications, including the ranking of sports teams (Landau 1915) and websites (Brin and Page 1998).

With these concepts in hand, we can now ask what these steady states might look like for a series of simple preference graphs (Figures 6, 7, 8, 9, 10, 11).



$$C = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 0 & 0 \\ .5 & .5 & 0 \\ .5 & .5 & 0 \end{bmatrix}$$

Figure 5: Preference graph, connection matrix C, and corresponding transition matrix M for a transitive preference among three items.



$$M = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

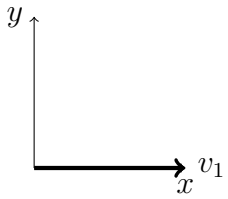
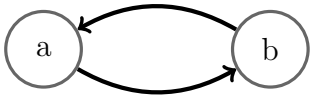


Figure 6: Preference graph, transition matrix M, and corresponding principal eigenvector v_1 for a two-node transitive preference. Steady state achieved after one iteration.

This series of basic examples illustrate how the language of graphs and matrices allows us to make meaningful statements about preference in the presence of challenging structures like cycles, which manifest as contradictions when represented in terms of linear ordering.



$$M = \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$$

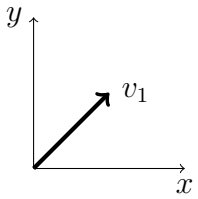
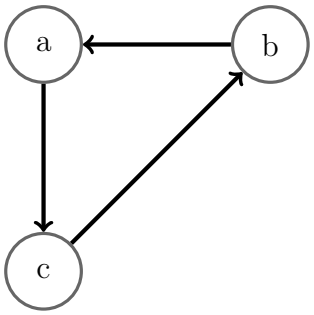


Figure 7: Preference graph, transition matrix M, and corresponding principal eigenvector v_1 for a two-node cycle. Intuitively, we set the probability of each item equal to the average of the prior preferences. The normalizing restriction on x ensures that these averages continue to represent a distribution over the items. Note also that the steady state (a uniform distribution) is achieved after only one iteration.



$$M = \begin{bmatrix} 1 & 0 & 0 \\ .5 & .5 & 0 \\ .5 & .5 & 0 \end{bmatrix}$$

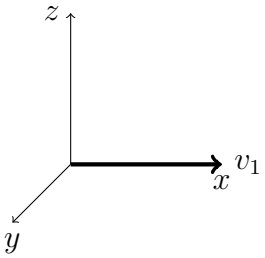
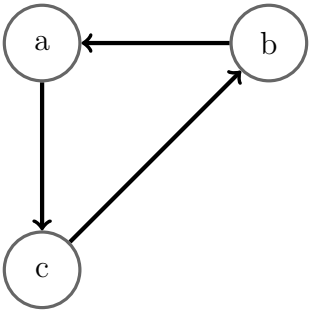


Figure 8: Preference graph, transition matrix M, and corresponding principal eigenvector v_1 for a three-node transitive preference. In this case, the steady state is achieved in the limit. At every iteration, C sends its probability mass to A and B evenly, and has no mass after the first iteration. B splits its mass between itself and A, while A directs all of its mass towards itself. The limiting convergence is due to the logarithmic reallocation of mass from B to A (a bit of a Zeno-style paradox).



$$M = \begin{bmatrix} .5 & 0 & .5 \\ .5 & .5 & 0 \\ 0 & .5 & .5 \end{bmatrix}$$

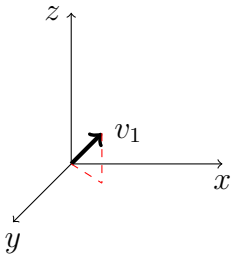
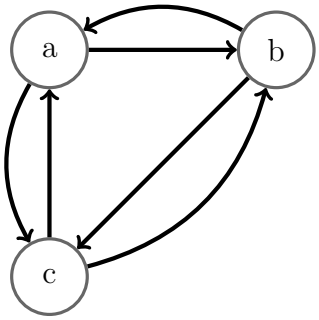


Figure 9: Preference graph, transition matrix M, and corresponding principal eigenvector v_1 for a three-node cycle. Here the reallocation of probability mass across iterations exhibits more complex dynamics, and converges to v_1 in the limit.



$$M = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$$

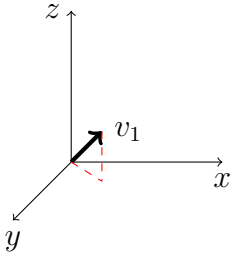
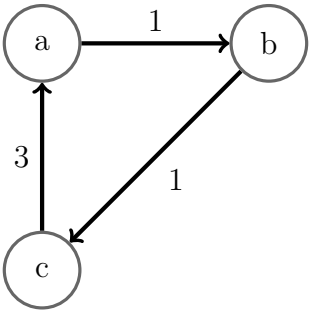


Figure 10: Preference graph, transition matrix M, and corresponding principal eigenvector v_1 for a three-node double cycle.



$$M = \begin{bmatrix} .75 & .25 & 0 \\ 0 & .5 & .5 \\ .75 & 0 & .25 \end{bmatrix}$$

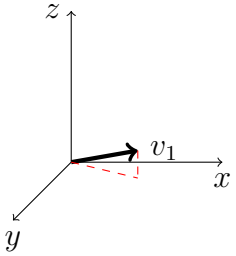


Figure 11: Preference graph, transition matrix M, and corresponding principal eigenvector v_1 for a three-node cycle with edge weights. Note how the introduction of variable weights repositions v_1 within the simplex, allowing us to recover a transitive ordering $a > b > c$.

0.3.4 Connections to Social Choice Theory

Those readers familiar with the economics literature will see many parallels to the Social Choice Theory presented in Arrow 1970. In this work, Arrow proposed three criteria for good voting systems, and went on to famously prove that no voting system could exist which satisfied all three. These criteria are:

- **Unanimity**: if all voters prefer a to b, the group prefers a to b.
- **Non-dictatorship**: there is no individual voter whose preferences always prevail.
- **Independent of Irrelevant Alternatives (IIA)**: the group preference between a and b should be determined only by individual preferences between a and b (and not, for example, c).

Arrows impossibility theorem shows that these three criteria taken together can lead to contradiction. We will give a sketch of the proof here (inspired by Geanakoplos 2005), as well as reinterpret the contradiction through the framework of preference graphs.

First, imagine we have a set V of N voters, asked to each submit a linear ordering of three items: a, b, and c. The voters are partitioned into three sets as follows: $S_1 = \{V_1, ..., V_{k-1}\}$, $K = \{V_k\}$, and $S_2 = \{v_{k+1}, ..., V_N\}$, and all voters in each set always vote the same way. Set K consists of one voter, V_k, who we will refer to as the "pivotal voter" in that if V_k votes the same way as either S_1 or S_2, then that preference prevails for the group. In Figure 12, we see a sequence of three states of preference. A contradiction emerges in the third state, when the preference reversals of S_1 and S_2 have no impact on the group preference.

In Figure 13, we see the same sequence of preferences represented graphically. In this view, we see that the "paradox" is simply the inability to represent cyclical preference as a linear ordering. We also see how the IIA assumption becomes problematic, as the ordering of $b > a$ and $a > c$ inhibits the group ordering of $c > b$. If we relax the IIA assumption and allow cycles, we see that the final preference graph of this proof has the same structure as our example graph in Figure 11, and therefore has a steady state distribution of $c > a > b$. Note that the eigenvector-based methods described earlier imply IIA relaxation.

S_1	K	S_2	Group
b	a	a	a
c	b	b	b
a	c	c	c

$\rightarrow$

S_1	K	S_2	Group
b	**b**	a	**b**
c	**a**	b	**a**
a	c	c	c

$\rightarrow$

S_1	K	S_2	Group
c	b	a	b
b	a	**c**	a
a	c	**b**	c

Figure 12: Sequence of preference orderings for voter subsets S_1, K, and S_2, and final group preference. V_k is a dictator in that the group prefers $b > c$ even though both S_1 and S_2 prefer $c > b$.

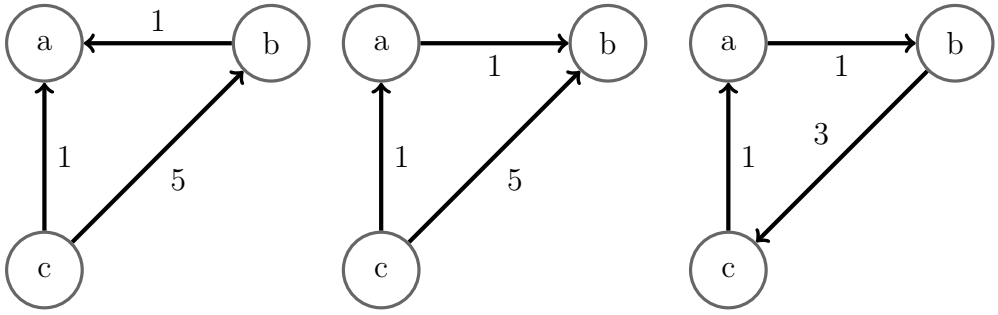


Figure 13: Same sequence of preference orderings, shown as preference graphs. We set $|S_1| = |S_2| = 2$. S_1 and S_2's reversal of preference between b and c in the third graph creates a cycle.

0.4 Applications

Ultimately, our goal is to make statements about the *global* relationships which exist in the graph, using information which comes to us via *local* relationships between nodes. Arrow's IIA criteria requires the consideration of local relationships only, but as we have seen, this criteria is problematic in the presence of cyclic preference structure.

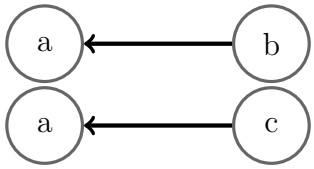
The graphical approach taken here relaxes this criterion and considers all items and preferences collectively. However, accounting for global relationships between nodes (relationships involving three or more nodes) is computationally challenging due to the rapid increase in the number of subsets to consider. For example, a graph of ten items has 45 pairs, 120 triples, and 210 sets of four. Fully accounting for all possible interactions among items will likely be computationally prohibitive for large graphs; as a result, techniques for approximately learning preference structure, or for pruning or otherwise constraining the number of items, will become valuable.

0.4.1 Linear-Time Methods

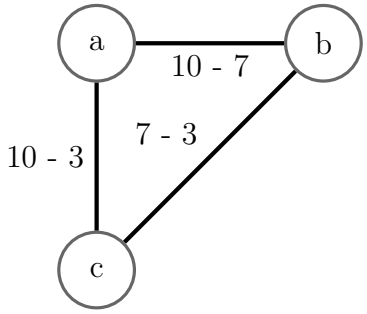
If a graph is not fully connected (some vertices lack edges), then it is possible for there to exist multiple sinks. If a graph is fully connected, however, then there can exist at most one sink. **Proof**: if there were two sinks, one must point towards the other: a contradiction $\rightarrow\leftarrow$.

The first example we will consider is the most fundamental of preference resolution systems: the **plurality voting system**. In this system, entities (which we will call voters) are allowed to vote for one of n candidates, and the candidate with the most votes win.

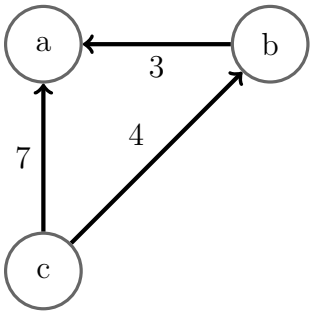
To set up this system in the language of preference graphs, we will define the following: Our *access policy* is that every voter is allowed to cast one vote in the election. Each vote will be translated into $n-1$ preferences, with an edge pointing to the chosen candidate from every other candidate. For example, given three candidates $\{a, b, c\}$, a vote for a would translate into the following:



We aggregate the preferences using the additive rule described above. The winner of the election will be the sink of the resulting graph. The presence of at least one sink is guaranteed by the structure of the access policy. Consider the following result: a receives 10 votes, b receives 7, and c receives 3. We combine these into the following graph:

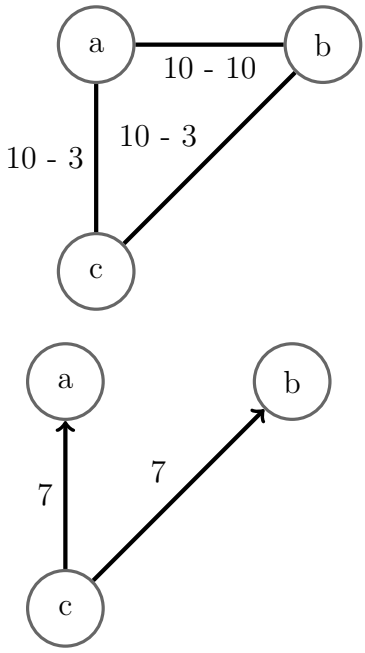


Then taking the differences and creating the directed edges:



We run the sink-finding algorithm and return a, the winner. It is worth nothing how the complexity of this preference-resolution scheme is $O(n)$, the same as the complexity of simply taking the maximum of vote counts for n candidates.

What happens in the case of a tie? Let's say b receives 10 votes:



In this case, we will observe two sinks in the resulting graph, representing two possible winners. This corresponds nicely with our intuition of what should happen in this case.

Earlier, we showed how the preference resolution problem can be as simple as finding a sink in a directed graph, an $O(n)$ operation. Without a guarantee

of acyclicity, however, we must turn to alternative methods. Intuitively, we would like to say that a node which has a lot of incoming edges is preferred, even if that node possesses some number of outgoing edges.

The simplest approach in this vein would be rank items by the value of their incoming edges, with the winner being the item which satisfies $argmax_{v \in V}[in(v)]$. If the preferences are input as ordered rankings, this method is equivalent to the **Borda count** election system, a method used in practice by governments today (Reilly 2002).

A slight variation on this approach would be to take the difference of incoming and outgoing edges, and return the node with the largest difference. If preferences are input as ordered rankings of all items, this method is equivalent to the method of ranking only the incoming edges (for any pair, the sum of incoming and outgoing edges is always equal to the number of votes, therefore $argmax_{v \in V}[in(v) - out(v)] = argmax_{v \in V}[in(v)]$). If there is a different access policy, this method may return different results.

These algorithms, which both run in $O(m+n)$ (linear) time, utilize local (pairwise) information between nodes, but incorporate no information about global relationships among nodes.

0.4.2 Eigenvector Methods

Google's first **PageRank** algorithm, designed by founders Sergey Brin and Larry Page, was designed to solve a similar problem: given a directed graph of websites, can one determine which sites are most relevant for a given query? Brin and Page's solution was to model the web as a random directed graph, and to imagine a "random surfer" who would randomly click on links (represented as directed edges from one site to the next). As this surfer traversed the web, she would be more likely to arrive at pages which had more inbound links; these pages were *preferred*. Links from preferred pages are worth more than links from peripheral pages, as the popularity of the preferred page meant it was more likely that a surfer would be travel elsewhere via that page. Brin and Page 1998 describes PageRank as follows:

> *We assume page A has pages T1...Tn which point to it (i.e., are citations). The parameter d is a damping factor which can be set between 0 and 1. We usually set d to 0.85. There are more details about d in the next section. Also C(A) is defined as the*

number of links going out of page A. The PageRank of a page A is given as follows:

PR(A) = (1-d) + d (PR(T1)/C(T1) + ... + PR(Tn)/C(Tn))

Note that the PageRanks form a probability distribution over web pages, so the sum of all web pages PageRanks will be one. PageRank or PR(A) can be calculated using a simple iterative algorithm, and corresponds to the principal eigenvector of the normalized link matrix of the web.

The principle difference between the web graph of Brin and Page and the preference graphs we consider here is the variable value associated with the edges. On the web, a link is a link; there is no notion of a link being worth "more" or "less", apart from the page the link originated from.

In our case, edges have weight independent of the popularity of their corresponding nodes. We can naturally interpret this weight as the strength of the relative preference, and so should seek to allocate preference mass according to the strength of these preferences. Using the notation of Brin and Page, we introduce a new term $W(A, T_i) \in \mathbb{R}^+$, corresponding to the (normalized) weight of the edge from A to T_i:

$$W(A, T_i) \triangleq \frac{w(A, T_i)}{\sum_{j=1}^{n} w(A, T_j)}$$

Now, we present a modified PageRank, which we call **PrefRank**:

$$PR(A) \leftarrow (1 - d) + d \sum_{i=1}^{n} \frac{PR(T_i) \times W(A, T_i)}{C(T_i)}$$

We use the right arrow instead of the equals sign to emphasize that this expression is an *assignment*, updating the values associated with each node at each iteration. The normalization is important to ensure that the sum of the PR scores remains constant over time.

This approach differs from the eigenvector methods described earlier in that self-edges are not added.

Empirical Results

We evaluate this algorithm on simulated data as follows.

For V items and preference strength B:

- for $n \in \{1, ..., N\}$:
  - Draw p_n, q_n randomly from V.
  - Draw $\mathbb{1}[p_n < q_n] \sim Bernoulli(B)$

We assess quality of ordering with Spearman's footrule (as discussed in Wauthier, Jordan, and Jojic 2013), a sum of the per-item position displacements between true ranking and recovered ranking (Figure 14).

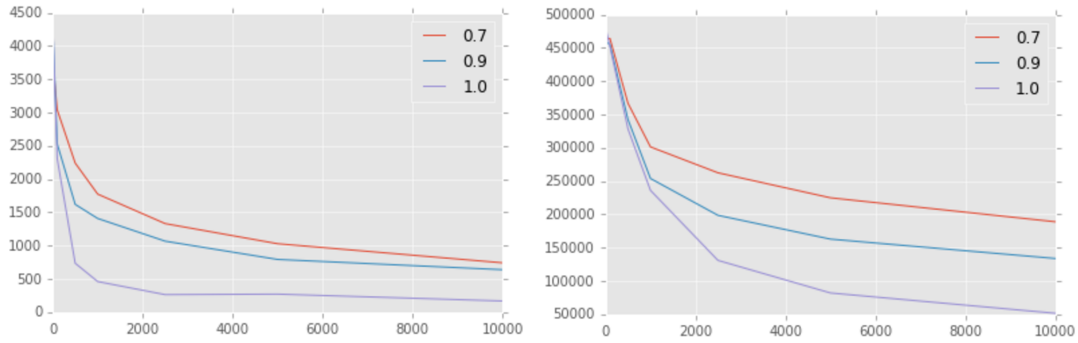


Figure 14: Spearman footrule as a function of number of observations, varying preference strengths B. Left plot is V=100, right is V=1000.

0.4.3 Bayesian Methods

The complexity of the algorithms discussed above are all, in some form, $O(f(n))$. Finding some way of reducing n would allow all of these techniques to be applied more quickly.

One way to reduce n would be to identify elements of V which are closely related, and to collapse them into more general "categories" or "prototypes," which can be treated as though they were single nodes in a graph. Rosch 1973 provides theoretical justification for this approach, arguing that human cognition utilizes abstract "prototypes" in order to reason heuristically about the world. Identifying these prototypes is conceptually similar to identifying other types of graphical structures, such as communities in social networks.

Community-finding is a major problem in computer science, and much work has been done on this problem. Here we present a model, based on the

Mixed-Membership Stochastic Blockmodel (MMSB) of Airoldi et al. 2008. This is a "Bayesian model" in that we first assert a model for our data, in which latent factors (hidden variables) interact and ultimately bring about the data we observe. Inference in this model amounts to learning the optimal ("posterior") values of these hidden variables, based on the data.

Model Specification

In this model, we assume each item is perceived as a mixture of one or more abstract "prototypes". We then assume there is a fixed "interaction matrix" B, governing preferences between prototypes, where B_{gh} indicates that probability that an item of prototype g is preferred over an item of prototype h.

The generative process is as follows:

- For items $p \in V$:
  - Draw a K-dimensional membership vector $\pi_p \sim Dirichlet(\alpha)$.
- For each observation $x_n = (p_n, q_n, y_n) \in X$:
  - Draw item type $z_{p_n \to q_n} \sim Multinomial(\pi_{p_n})$
  - Draw item type $z_{q_n \to p_n} \sim Multinomial(\pi_{q_n})$
  - Draw $y_n \sim Bernoulli(z^T_{p_n \to q_n} B z_{q_n \to p_n})$

The MMSB we propose extends the work of Airoldi et al. 2008 by imposing symmetric structure on the matrix B. Specifically, we enforce that $B_{gh} = 1 - B_{hg}\ \forall g, h$ (note that this implies $B_{gg} = 0.5\ \forall g$). Unlike other mixed-membership stochastic blockmodels, which emphasize intra-community connective patterns, our model exclusively considers inter-community connective patterns.

This model does not attempt to learn distinct preferences per entity. This was intentional, as this model is attempting to capture and represent preference in aggregate. That said, this work could be extended by learning a different interaction matrix B per user. We leave this for future work.

Inference

Our goal is to learn posterior values for $\pi_p, z_{p_n \to q_n}, z_{q_n \to p_n}$, and B. $\pi_p, z_{p_n \to q_n}$, and $z_{q_n \to p_n}$ are random variables, and we will learn posterior values via mean-field variational inference (Wainwright and Jordan 2008, Blei, Kucukelbir, and McAuliffe 2016). B is a matrix of parameters, and so we learn posterior values via variational Expectation-Maximization.

We first assume the following posterior "q" distributions $q(\pi_p), q(z_{p_n \to q_n})$, and $q(z_{q_n \to p_n})$:

- $\pi_p \sim Dirichlet(\gamma_p)$
- $z_{p_n \to q_n} \sim Multinomial(\phi_{p_n \to q_n})$
- $z_{q_n \to p_n} \sim Multinomial(\phi_{q_n \to p_n})$

The essence of variational inference (specifically coordinate-ascent VI) is that we can learn the optimal distribution of each variable *given the other variables*. We iterate over the variables, updating their distributions in turn, with each iteration bringing the q distributions closer to the true posterior.

The update equations are as follows:

$$\hat{\gamma_{p,k}} = \alpha + \sum_{n \in N} \mathbb{1}(p = p_n)\phi_{p_n \to q_n, k} + \sum_{n \in N} \mathbb{1}(p = q_n)\phi_{q_n \to p_n, k}$$

$$\hat{\phi_{p_n \to q_n, g}} \propto exp\Big\{ \mathbb{E}_q\big[log(\pi_{p,g})\big] + \sum_h \phi_{q_n \to p_n, h} \mathbb{E}_q\big[logp(y_n | B_{gh})\big] \Big\}$$

$$\hat{\phi_{q_n \to p_n, h}} \propto exp\Big\{ \mathbb{E}_q\big[log(\pi_{q,h})\big] + \sum_g \phi_{p_n \to q_n, g} \mathbb{E}_q\big[logp(y_n | B_{gh})\big] \Big\}$$

$$\hat{B_{gh}} = \frac{\sum_{n \in N} \phi_{p_n \to q_n, g}\phi_{q_n \to p_n, h} y_n + \phi_{p_n \to q_n, h}\phi_{q_n \to p_n, g}(1 - y_n)}{\sum_{n \in N} \phi_{p_n \to q_n, g}\phi_{q_n \to p_n, h} + \phi_{p_n \to q_n, h}\phi_{q_n \to p_n, g}}$$

Additional details of these derivations are given in Appendix 0.6.1.

Empirical Results

We evaluated the MMSB model in three ways: on simulated data, on the MovieLens dataset, and on survey data.

Simulations

In order to validate our implementation and the validity of the model, we fit our MMSB to simulated data. For small-to-medium sized graphs, our implementation recovers (with some variation) the true prototype distributions π and interaction matrix B, given enough observations. See Figures 15 and 16.

MovieLens Data

Given that the MovieLens dataset we work with is constructed based on the user ratings, the preferences we observe for a single user must be transitive. As such, for a single user, we expect to be able to learn five "prototypes", each corresponding to a different rating, with the interaction matrix B to encoding a transitive ordering among these ratings. We find that this occurs: when setting $K = 5$, the model learns a strict ordering among the prototypes, and is able to correctly predict this user preferences with 98% accuracy on the heldout dataset (Figure 17).

With multiple users, we are no longer guaranteed a single shared transitive ordering. We see in Figure 18 that the heldout accuracy of the MMSB plateaus at around 76% when trained on the data from 30 users. Adding more prototypes, beyond $K = 4$, does not improve the performance of the model. This suggests that the model is simply assigning each movie to the prototype corresponding to its average rating; thus, having more than 5 prototypes is not useful.

Survey Data

We fit the MMSB to a survey of beer preferences, first considering only the answers from the single opinionated user. We fit the model to 900 training observations, and measured predictive accuracy on the remaining 344. We varied K, the number of prototypes, from 1 to 15, but found that predictive accuracy was very stable for K ¿ 1, hovering around 78%. With $K = 3$, our model learned a transitive ordering of preferences among prototypes. See Figures 20 and 19.

We next considered the answers coming from all other participants. We fit the model to 150 training observations and measured accuracy on the remaining 74. We varied K in the same way as before, and observed both more variable and overall weaker predictive accuracy, rarely surpassing 60%.

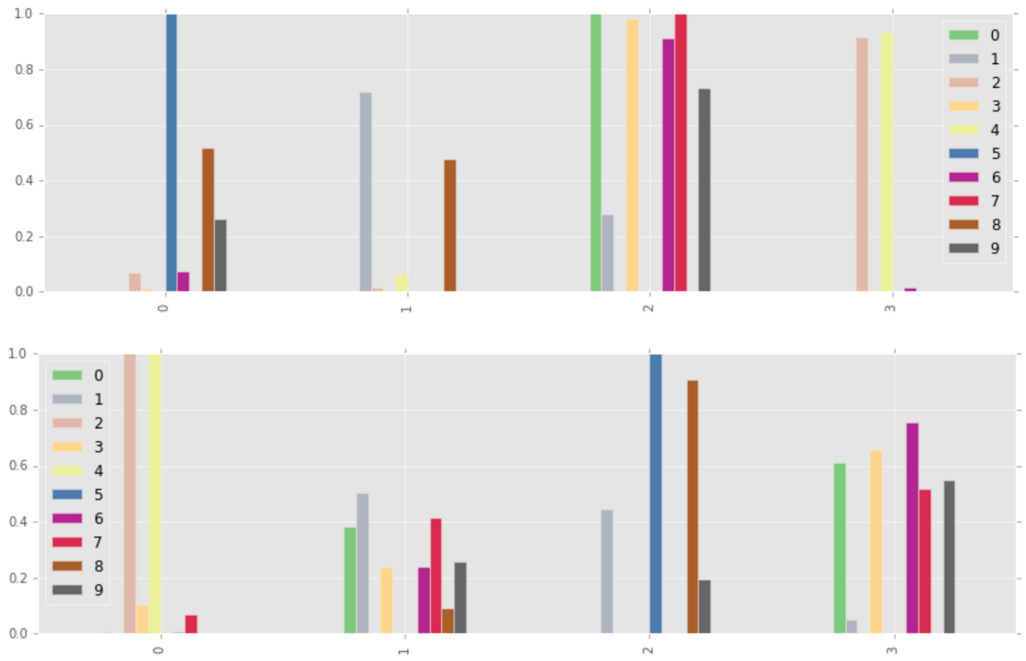


Figure 15: True (top) vs. recovered (bottom) prototype assignments, K=4, V=10, N=10000. We see how the model has correctly grouped the items into their prototypes. Note the label-switching — this illustrates the multi-modal nature of the joint probability distribution.

We conclude that this model is capable of capturing prototype interaction structure, but it will not perform well given small samples, weakly structured, or very noisy data.

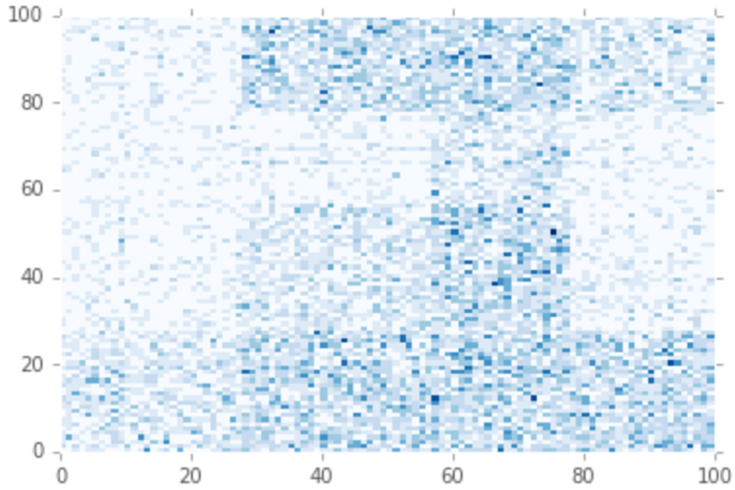


Figure 16: Simulated interaction matrix, items sorted by most likely prototype, K=4, V=100, N=10000. The visible blocks show that items coming from similar prototypes interact in similar ways to items coming from other prototypes. The diagonal is gray, indicating that intra-prototype comparisons are 50/50 chance.

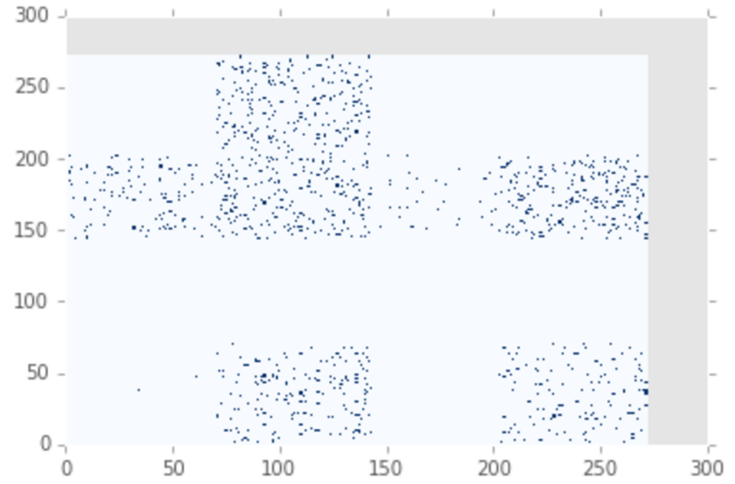


Figure 17: Movie interaction matrix, one user, items sorted by most likely prototype, K=5, V=272, N=1499

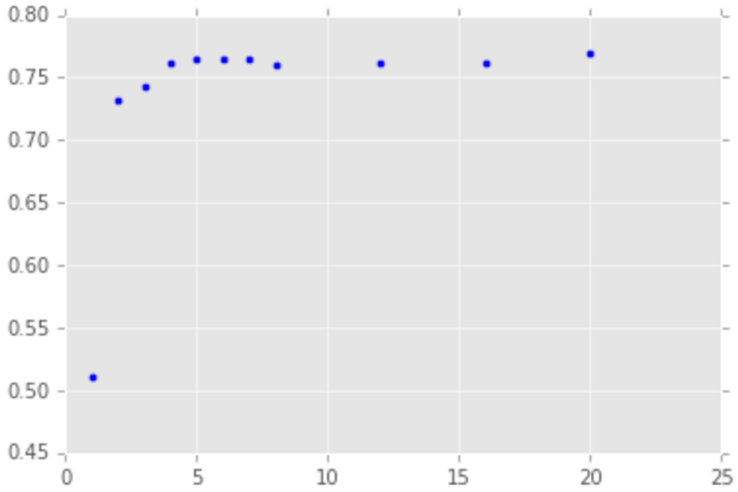


Figure 18: Predictive accuracy against held-out data, 200 films and 20 users, function of number of prototypes. Accuracy plateaus at K = 4, indicating that the model is assigning each movie to a prototype corresponding to an average rating.

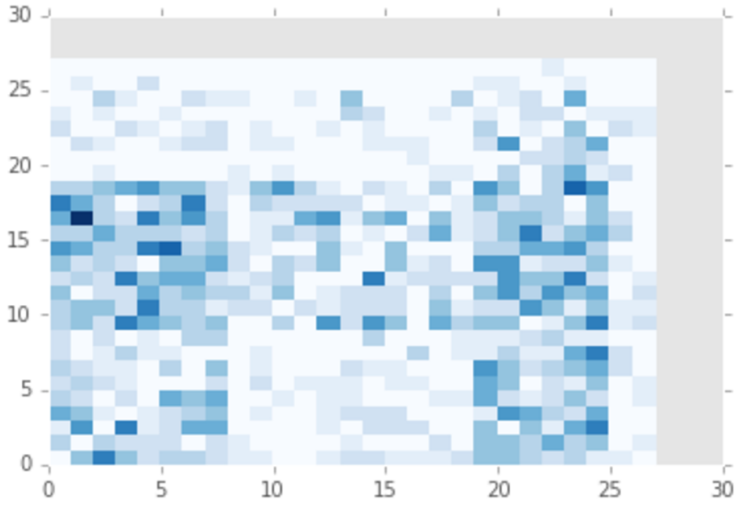


Figure 19: Beers interaction matrix, one user, items sorted by most likely prototype, K=3, V=27, N=1244

	0	1	2
0	0.50	0.10	0.11
1	0.90	0.50	0.89
2	0.89	0.11	0.50

Figure 20: B matrix for beer survey, one user. Ordering is transitive (1 ¡ 2 ¡ 0).

0.5 Future Directions

0.5.1 Deployment to a BBVM Environment

0.5.2 Active Learning and Item Pruning

0.5.3 Optimal Committee Discovery

0.5.4 Comparison to alternative representations

The problem of representing subjectivity is not new; rather, it represents one of the fundamental challenges of the social sciences. It will be illustrative to compare this proposed system to common standards.

A *Likert scale* is a type of survey, consisting of a number of *Likert items*, each asking the participant to answer a question by checking boxes such as "Strongly Agree", "Strongly Disagree" and so on. This method attempts to project subjectivity onto a range, for later interpretation via a number line.

In well-designed Likert scales, the responses distribute uniformly across the range, which can consist of any number of degrees of feeling (five to ten being common). This distribution of answers allows a researcher to justify interpreting the responses as though they fell along some sort of number line, and perform analysis accordingly.

The use of socrata avoids some of the problems observed with Likert scales.

GENERALIZE TO LIKERT SCALE?

1 -¿ 2

1 -¿ 3

2 -¿ 3

4 -¿ 3

5 -¿ 3

5 -¿ 4

An important consideration is the treatment of contradiction (specifically, intransitivity) among preferences. In the political science literature, intransitivity is seen as problematic Arrow 1970. We, however, view this type of contradiction as a property of subjective experience and do not attempt to eliminate it. Rather, we allow for this possibility and seek to interpret it.

0.5.5 Relations to other branches of social science

- Study of social influences on voting (music preference study) The preferences measured via this approach will likely exhibit the same phenomenon as studied by.

There are many possible extensions of this theory.

Item clustering for algorithmic efficiency

Recursive decomposition of participants

Studying how question phrasing affects answers. By using the same answer set A for multiple questions, and studying how the responses differ, it should be possible to gain a deeper understanding about the relationship between the questions.

Beyond economics, political theorist Hannah Arendt has written about the need for a "public sphere", in which there exist methods and structures to allow the achievement of collective freedom via the construction of a common world. CITE.

0.6 Appendices

0.6.1 Details of MMSB Derivation

Joint probability and ELBO

Here is the joint probability for this model:

$$p(Y, \pi_{1:P}, Z_{p\to q}, Z_{q\to p}|B, \alpha) =$$
$$\prod_{n=1}^{N} p(y_n|z_{p_n\to q_n}, z_{q_n\to p_n}, B)p(z_{p_n\to q_n}|\pi_{p_n})p(z_{q_n\to p_n}|\pi_{q_n}) \prod_{p=1}^{V} p(\pi_p|\alpha)$$

Inference will involve learning posterior values for $\pi_{1:P}, Z_{p\to q}, Z_{q\to p}, B$. We will learn $\pi_{1:P}, Z_{p\to q}, Z_{q\to p}$ through variational inference, and B through variational expectation-maximization (as it is not a random variable).

We introduce the following q distributions for the latent variables:

$$\begin{aligned} q(\pi_p) &\sim Dirichlet(\gamma_p) \\ q(z_{p_n \to q_n}) &\sim Multinomial(\phi_{p_n \to q_n}) \\ q(z_{q_n \to p_n}) &\sim Multinomial(\phi_{q_n \to p_n}) \end{aligned}$$

Note that the matrix Γ will be $V \times K$, while matrices $\Phi_{p \to q}$ and $\Phi_{q \to p}$ are $N \times K$. We will learn all parameters by maximizing the ELBO:

$$\begin{aligned} ELBO \quad & (\Gamma, \Phi_{p \to q}, \Phi_{q \to p}; Y, \pi_{1:P}, Z_{p \to q}, Z_{q \to p}, B, \alpha) = \\ & \mathbb{E}_q \quad \Bigg[\sum_{n=1}^{N} \Bigg(\log p(y_n | z_{p_n \to q_n}, z_{q_n \to p_n}, B) + \log p(z_{p_n \to q_n} | \pi_{p_n}) + \log p(z_{q_n \to p_n} | \pi_{q_n}) \Bigg) \\ & \quad + \sum_{p=1}^{V} \log p(\pi_p | \alpha) \Bigg] \\ - \quad & \mathbb{E}_q \quad \Bigg[\sum_{n=1}^{N} \Bigg(\log q(z_{p_n \to q_n} | \phi_{p_n \to q_n}) + \log q(z_{q_n \to p_n} | \phi_{q_n \to p_n}) \Bigg) + \sum_{p=1}^{V} \log q(\pi_p | \gamma_p) \Bigg] \end{aligned}$$

The updates for $\gamma, \phi_{p \to q}, \phi_{q \to p}$ are exactly as given in Airoldi et al. 2008, with the modification that we iterate over N observations, rather than $V \times V$ pairs.

Learning B Matrix

Our model differs from the MMSB specified by Airoldi et al. 2008 , in that we introduce restrictions on the matrix B. The first thing to note is that the symmetric restriction on B implies that we must learn and store only the upper-triangle of the matrix; the lower-triangle can be generated from the upper. We formalize this symmetry with the following likelihood distribution on y_n (with g referring to the index corresponding to the one-hot vector $z_{p_n \to q_n}$, and h corresponding to the same for $z_{q_n \to p_n}$):

$$\begin{aligned} & p(y_n | z_{p_n \to q_n} = g, z_{q_n \to p_n} = h, B) \\ = \; & p(y_n | B_{gh})^{\mathbb{1}[g<h]} p(y_n | 1 - B_{hg})^{\mathbb{1}[g \geq h]} \\ = \; & \left(B_{gh}^{y_n} (1 - B_{gh})^{(1-y_n)} \right)^{\mathbb{1}[g<h]} \left((1 - B_{hg})^{y_n} B_{hg}^{(1-y_n)} \right)^{\mathbb{1}[g \geq h]} \end{aligned}$$

Values of B are learned through variational EM, in which we set the values using maximum-likelihood, using the values of the variational parameters learned during CAVI. To derive the update for B, we take the gradient of the ELBO with respect to B, set it to 0, and solve. Note that we only need to consider the terms in the ELBO with depend on B; we hide all other terms in the constant C. Additionally, we let $g_n = z_{p_n \to q_n}$, and $h_n = z_{q_n \to p_n}$:

$$\begin{aligned}
ELBO(B) &= C + \mathbb{E}_q\Big[\sum_{n=1}^{N} \log p(y_n | z_{p_n \to q_n}, z_{q_n \to p_n}, B)\Big] \\
&= C + \mathbb{E}_q\Big[\sum_{n=1}^{N} \mathbb{1}[g_n < h_n]\Big(y_n \log(B_{g_n h_n}) + (1-y_n)\log(1 - B_{g_n h_n})\Big) \\
&\quad + \mathbb{1}[g_n \geq h_n]\Big(y_n \log(1 - B_{h_n g_n}) + (1-y_n)\log(B_{h_n g_n})\Big)\Big] \\
&= C + \sum_{n=1}^{N}\Big[\sum_{g_n < h_n}\Big(p(g_n)p(h_n)\big(y_n \log(B_{g_n h_n}) + (1-y_n)\log(1 - B_{g_n h_n})\big)\Big) \\
&\quad + \sum_{g_n \geq h_n}\Big(p(g_n)p(h_n)\big(y_n \log(1 - B_{h_n g_n}) + (1-y_n)\log(B_{h_n g_n})\big)\Big)\Big] \\
&= C + \sum_{n=1}^{N}\Big[\sum_{g_n < h_n}\Big(\phi_{p_n \to q_n, g_n}\phi_{q_n \to p_n, h_n}\big(y_n \log(B_{g_n h_n}) + (1-y_n)\log(1 - B_{g_n h_n})\big)\Big) \\
&\quad + \sum_{g_n \geq h_n}\Big(\phi_{p_n \to q_n, g_n}\phi_{q_n \to p_n, h_n}\big(y_n \log(1 - B_{h_n g_n}) + (1-y_n)\log(B_{h_n g_n})\big)\Big)\Big]
\end{aligned}$$

We now take the derivative with respect to B_{gh}, assuming that $g < h$:

$$\begin{aligned}
\frac{\partial ELBO}{\partial B_{gh}} &= \sum_{n=1}^{N}\Big[\phi_{p_n \to q_n, g}\phi_{q_n \to p_n, h}\Big(\frac{y_n}{B_{gh}} - \frac{1-y_n}{1-B_{gh}}\Big) \\
&\quad + \phi_{p_n \to q_n, h}\phi_{q_n \to p_n, g}\Big(\frac{-y_n}{1-B_{gh}} + \frac{1-y_n}{B_{gh}}\Big)\Big]
\end{aligned}$$

Setting this expression to 0, and solving for B_{gh}, gives the following closed-form solution:

$$\hat{B}_{gh} = \frac{\sum_{n=1}^{N} \phi_{p_n \to q_n, g}\phi_{q_n \to p_n, h} y_n + \phi_{p_n \to q_n, h}\phi_{q_n \to p_n, g}(1-y_n)}{\sum_{n=1}^{N} \phi_{p_n \to q_n, g}\phi_{q_n \to p_n, h} + \phi_{p_n \to q_n, h}\phi_{q_n \to p_n, g}}$$

0.6.2 Datasets

MovieLens-100k

We now describe the way we generate pairwise preference data from the MovieLens-100k dataset. At a high level, if a user u rated M_u movies, we randomly sample $10 * M_u$ pairs of movies rated by that user. For each randomly sampled pair, if the user gave different ratings to the 2 movies, we add the user's pairwise preference to our dataset. The exact algorithm is shown in below. Importantly, we run the above process for both the training data and the test data. We use the file 'u5.base' as our training set, and 'u5.test' as our test set, from the MovieLens-100k dataset, which we downloaded at `http://grouplens.org/datasets/movielens/100k/`. There are 80,000 ratings in this training set, and 20,000 ratings in this test set. After running the above procedure on the training set, we separate 20% of the data as heldout, and use the rest for training.

Data-generating Algorithm for MovieLens data

For a given user u, let M_u denote the set of movies rated by this user, and let r_{um} denote the rating user u gave to movie m.

- Let $D = \emptyset$, $n = 1$.
- For each user $u \in U$
  - For $n \in \{1, ..., 10 \cdot |M_u|\}$
    - Randomly select $p_n, q_n \in M_u$, where $p_n < q_n$.
    - Let $x_n = (p_n, q_n, u_n)$, and $y_n = \mathbb{1}\left[r_{up_n} < r_{uq_n}\right]$
    - If $(x_n, y_n) \notin D$ and $r_{up_n} \neq r_{uq_n}$
      - $D = D \cup (x_n, y_n)$.
      - $n = n + 1$.

All Our Ideas

All Our Ideas is an online platform enabling for the creation and distribution of "wiki surveys" (Salganik and Levy 2015) — in which users are prompted to make pairwise preferences with items drawn from a dynamic answer pool.

As an academic project, All Our Ideas makes raw survey data available to download. These data include information about the candidate answers and every comparison made, giving hashed IP addresses to preserve user anonymity.

This survey we consider, which ran over the summer of 2012, was used to answer a critical question: what beer to serve at the author's undergraduate going-away party. The survey considered 27 beers and had 1468 responses from 17 IP addresses, with a majority of responses (1244) coming from a single IP address (not the author's).

Bibliography

Airoldi, Edoardo M. et al. (2008). "Mixed Membership Stochastic Blockmodels". In: *Journal of Machine Learning Research*. URL: http://dblp.uni-trier.de/rec/bib/journals/jmlr/AiroldiBFX08.

Arrow, Kenneth J. (1970). *Social Choice and Individual Values*. 2nd ed. Yale University Press. ISBN: 9780300013641.

Barry, Liz (2016). "vTaiwain: Public Participation Methods on the Cyberpunk Frontier of Democracy". In: *Civicist*. URL: http://civichall.org/civicist/vtaiwan-democracy-frontier/.

Bianchi, Marina (2014). *The Active Consumer: Novelty and Surprise in Consumer Choice*. Routledge. ISBN: 9781138007147.

Blei, David M., Alp Kucukelbir, and Jon D. McAuliffe (2016). "Variational Inference: A Review for Statisticians". In: *arXiv*. URL: https://arxiv.org/pdf/1601.00670.pdf.

Blei, David M., Andrew Y. Ng, and Michael I. Jordan (2003). "Latent Dirichlet Allocation". In: *Journal of Machine Learning Research*. URL: https://www.cs.princeton.edu/~blei/papers/BleiNgJordan2003.pdf.

Brin, Sergey and Lawrence Page (1998). "The Anatomy of a Large-Scale Hypertextual Web Search Engine". In: *Computer Networks and ISDN Systems*. URL: http://infolab.stanford.edu/pub/papers/google.pdf.

Carlin, Dan (2013). *Prophets of Doom*. URL: http://www.dancarlin.com/product/hardcore-history-48-prophets-of-doom/.

Coase, Ronald H. (1960). "The Problem of Social Cost". In: *The Journal of Law and Economics*. URL: http://econ.ucsb.edu/~tedb/Courses/UCSBpf/readings/coase.pdf.

Cosmides, Leda and John Tooby (1992). "Cognitive Adaptations for Social Exchange". In: *The Adapted Mind: Evolutionary Psychology and the Gen-*

eration of Culture. Ed. by L. Cosmides, J. Tooby, and J. Barkow. Oxford University Press. Chap. 3.

Couturat, Louis (1901). *La Logique de Leibniz*. Felix Alcan.

Cover, Thomas M. and Joy A. Thomas (2006). *Elements of Information Theory*. 2nd ed. Wiley-Interscience. ISBN: 9780471241959.

Dasgupta, Sanjoy and Anupam Gupta (2002). "An Elementary Proof of a Theorem of Johnson and Lindenstrauss". In: *Wiley Periodicals*. URL: http://cseweb.ucsd.edu/~dasgupta/papers/jl.pdf.

Deacon, Terrence W. (1998). *The Symbolic Species: The Co-evolution of Language and the Brain*. W. W. Norton & Company. ISBN: 9780393317541.

Diaconis, Persi and David Freedman (1984). "Asymptotics of Graphical Projection Pursuit". In: *The Annals of Statistics*. URL: http://statweb.stanford.edu/~cgates/PERSI/papers/freedman84.pdf.

Doxiadis, Apostolos and Christos Papadimitriou (2009). *Logicomix: An Epic Search for Truth*. 1st ed. Bloomsbury USA. ISBN: 9781596914520.

Eisenberg, J. F., N. A. Muckenhirn, and R. Rudrane (1972). "The Relation between Ecology and Social Structure in Primates". In: *Science*. URL: http://www.jstor.org/stable/1733777.

Ferguson, Niall (2009). *The Ascent of Money: A Financial History of the World*. Penguin Books. ISBN: 9780143116172.

Foucault, Michel (1994). *The Order of Things: An Archaeology of the Human Sciences*. Reissue. Vintage. ISBN: 9780679753353.

Freedman, David (1995). "Some Issues in the Foundation of Statistics". In: *Foundations of Science*. URL: https://mangellabs.soe.ucsc.edu/sites/default/files/16/freedman_antibayes.pdf.

Friedler, Sorelle A., Carlos Scheidegger, and Suresh Venkatasubramanian (2016). "On the (im)possibility of fairness". In: *arXiv*. URL: https://arxiv.org/abs/1609.07236.

Geanakoplos, John (2005). "Three Brief Proofs of Arrow's Impossibility Theorem". In: *Economic Theory*. URL: http://www.jstor.org/stable/25055941.

George Lakoff, Mark Johnson (2003). *Metaphors We Live By*. 1st ed. University Of Chicago Press. ISBN: 9780226468013.

Gowers, Timothy, June Barrow-Green, and Imre Leader (2008). *The Princeton Companion to Mathematics*. Princeton University Press. ISBN: 9780691118802.

Hamlin, Aaron (2012). *Podcast 2012-10-06: Interview with Nobel Laureate Dr. Kenneth Arrow*. URL: https://electology.org/podcasts/2012-10-06_kenneth_arrow.

Hayek, Friedrich A. (1945). "The Use of Knowledge in Society". In: *American Economic Review*. URL: http://www.econlib.org/library/Essays/hykKnw1.html.

Heilbroner, Robert L. (1999). *The Worldly Philosophers: the Lives, Times, and Ideas of the Great Economic Thinkers*. Touchstone. ISBN: 9780684862149.

Hobbes, Thomas (1982). *Leviathan*. Penguin Classics. ISBN: 9780140431957.

Hofstadter, Douglas R. (1999). *Gödel, Escher, Bach: An Eternal Golden Braid*. 20th Anniversary. Basic Books. ISBN: 9780465026562.

Keener, James P. (1993). "The Perron-Frobenius Theorem and the Ranking of Football Teams". In: *SIAM Review*. URL: http://www.jstor.org/stable/2132526.

Koren, Yehuda, Robert Bell, and Chris Volinsky (2009). "Matrix Factorization Techniques for Recommender Systems". In: *Computer*. URL: http://dl.acm.org/citation.cfm?id=1608614.

Kuhn, Thomas S. (1996). *The Structure of Scientific Revolutions*. 3rd ed. University of Chicago Press. ISBN: 9780226458083.

Lakoff, George (2014). *The ALL NEW Don't Think of an Elephant!: Know Your Values and Frame the Debate*. 2nd ed. Chelsea Green Publishing. ISBN: 9781603585941.

Landau, Edmund (1915). "Über Preisverteilung bei Spielturnieren". In: *Zeitschrift für Mathematik und Physik*. URL: http://iris.univ-lille1.fr/handle/1908/2031.

Lin, Blossom Yen-Ju et al. (2013). "Job autonomy, its predispositions and its relation to work outcomes in community health centers in Taiwan". In: *Health Promotion International*. URL: http://heapro.oxfordjournals.org/content/28/2/166.

Lin, Henry W. and Max Tegmark (2016). "Critical Behavior from Deep Dynamics: A Hidden Dimension in Natural Language". In: *arXiv*. URL: https://arxiv.org/abs/1606.06737.

Mikolov, Tomas et al. (2013). "Efficient Estimation of Word Representations in Vector Space". In: *arXiv*. URL: https://arxiv.org/abs/1301.3781.

Minsky, Hyman P. (2008). *Stabilizing an Unstable Economy*. 1st ed. McGraw-Hill Education. ISBN: 9780071592994.

Mukherjee, Siddhartha (2011). *The Emperor of All Maladies: A Biography of Cancer*. Scribner. ISBN: 9781439170915.

O'Neil, Cathy (2016). *Weapons of Math Destruction*. Crown. ISBN: 9780553418811.

Ore, Oystein (1988). *Number Theory and its History*. Dover Publications. ISBN: 9780486656205.

Paisley, John (2015). *Lecture 20: Sequential Data*. COMS w4721 Lecture. URL: http://www.columbia.edu/~jwp2128/Teaching/W4721/Spring2015/slides/lecture_4-14-15.pdf.

Pirsig, Robert M. (2006). *Zen and the Art of Motorcycle Maintenance: An Inquiry Into Values*. HarperTorch. ISBN: 9780060589462.

Popper, Karl R. (2002). *Conjectures and Refutations: The Growth of Scientific Knowledge*. 2nd ed. Routledge. ISBN: 9780415285940.

Reilly, Benjamin (2002). "Social Choice in the South Seas: Electoral Innovation and the Borda Count in the Pacific Island Countries". In: *International Political Science Review*. URL: http://journals.sagepub.com/doi/abs/10.1177/0192512102023004002.

Rosch, Eleanor (1973). "Natural Categories". In: *Cognitive Psychology*. URL: http://philpapers.org/rec/ROSNC.

— (1975). "Cognitive Representations of Semantic Categories". In: *Journal of Experimental Psychology*. URL: http://psycnet.apa.org.ezproxy.cul.columbia.edu/journals/xge/104/3/192.pdf.

Salganik, Matthew J. and Karen E. C. Levy (2015). "Wiki Surveys: Open and Quantifiable Social Data Collection". In: *PLoS ONE*. URL: http://dx.doi.org/10.1371/journal.pone.0123483.

Smith, Adam (2000). *An Inquiry into the Nature and Causes of the Wealth of Nations*. Modern Library. ISBN: 9780679783367.

Tramèr, Florian et al. (2015). "FairTest: Discovering Unwarranted Associations in Data-Driven Applications". In: *arXiv*. URL: https://arxiv.org/abs/1510.02377.

Wadler, Philip (2015). "Propositions as Types". In: *Communications of the ACM*. URL: http://homepages.inf.ed.ac.uk/wadler/papers/propositions-as-types/propositions-as-types.pdf.

Wainwright, Martin J. and Michael I. Jordan (2008). "Graphical Models, Exponential Families, and Variational Inference". In: *Foundations and Trends in Machine Learning*. URL: https://people.eecs.berkeley.edu/~wainwrig/Papers/WaiJor08_FTML.pdf.

Wauthier, Fabian L., Michael I. Jordan, and Nebojsa Jojic (2013). "Efficient Ranking from Pairwise Comparisons". In: URL: https://people.eecs.berkeley.edu/~jordan/papers/wauthier-jordan-jojic-icml13.pdf.

Whitehead, Alfred N. (1979). *Process and Reality (Gifford Lectures Delivered in the University of Edinburgh During the Session 1927-28)*. 2nd ed. Free Press. ISBN: 9780029345702.

Zumbrin, Josh (2014). "SAT Scores and Income Inequality: How Wealthier Kids Rank Higher". In: *The Wall Street Journal*. URL: http://blogs.wsj.com/economics/2014/10/07/sat-scores-and-income-inequality-how-wealthier-kids-rank-higher/.