2012 TAMS Tournament

Calculus (math with $\frac{1}{\infty}$)

11.6.2012

This is a 10 question, 50 minute test. Questions 1-3 are worth 3 points; 4-6 are worth 5 points; 7-10 are worth 10 points. There is no guessing penalty. Give all answers as a closed form expression.

1. Differentiate $\sqrt{1 + \tan^2 2x}$.

2. Find $\lim\limits_{t \to 0} \dfrac{10(\sin t - t)}{t^3}$

3. Differentiate $\cos(\cos(\cos(\sin(x))))$.

4. Compute $\lim\limits_{x \to +} x^{\sin x}$.

5. Find the derivative of $\sqrt{x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}}}$ at $x = \frac{\pi}{4}$.

6. Mickey learns Chopin's Revolutionary Etude, Op. 10 No. 12. Suppose the probability of making a mistake during a performance at any given time $t \geq 0$ (in hours) is equal to $P(t)$, where $P'(t) = \left(\sqrt{x^2 + 1} + x\right)^{-4}$. Luckiliy for him, he practices a lot and $P(t)$ approaches 0 as t approaches infinity. What is the probabilty that he makes a mistake after 0 hours of practice?

7. Find the area bounded between the graph of $f(x) = x^{2012}e^{-x}$ and the x-axis for $x > 0$.

8. Suppose $f(x)$ is any everywhere positive and continuous function such that $\int_{-\infty}^{+\infty} f(x)\,dx = 1$. Let $\mu = \int_{-\infty}^{+\infty} x f(x)\,dx$ and $\sigma^2 = \int_{-\infty}^{+\infty} (x - \mu^2)\,dx$. Find, in terms of k, the maximum lower bound for $\int_{\mu - k\sigma}^{\mu + k\sigma} f(x)\,dx$.

9. Evaluate the definite integral $\int_{-1}^{1} \dfrac{x^{10} + 1}{10^x + 1}\,dx$.

10. Find $\sum\limits_{n=0}^{\infty} \dfrac{\sin(nx)}{3^n}$ if $\sin(x) = \frac{1}{3}$ and $0 \leq x \leq \frac{\pi}{2}$.