TAMS Tournament 2013: Calculus

1. Evaluate the following integrals.

   (a) $\int \ln(x^2)\, dx$

   (b) $\int \ln(1+x^2)\, dx$

   (c) $\int \cos(\ln(x))\, dx$

2. Given a function $f(t)$, define the *Laplace transform* as

   $$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = F(s)$$

   if the integral converges. $\mathcal{L}$ turns a function of t into a function of s.

   (a) Find $\mathcal{L}\{1\}$.

   (b) Find $\mathcal{L}\{\sin(t)\}$.

   (c) Show that

   $$\int_0^\infty \mathcal{L}\{f(t)\}ds = \int_0^\infty \frac{f(t)}{t}dt.$$

   (d) Evaluate $\displaystyle\int_{-\infty}^\infty \frac{\sin(x)}{x}dx$.

3. Let $y(x)$ be a differentiable function and consider the expression $y' + y/x$.

   (a) Find a function $\mu(x)$ such that

   $$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)[y'(x) + y(x)/x].$$

   (b) Solve the differential equation (find all functions $y(x)$ such that) $y' + y/x = 0$.

   (c) Solve the differential equation $y' + y/x = 3x$.

   (d) Determine the general solution to the *first order linear differential equation* $y' + p(x)y = q(x)$.

4. The cat rides a unicycle on level ground at constant velocity v. The wheel has radius r . He marks a certain point on the wheel such that the point touches the ground at time $t = 0$ and does not touch the ground again until time $t = t_1$.

   (a) Let C be the curve traced out by the point on the time interval $[0, t_1]$. Find an equation to describe C.

   (b) Find the area of the region bounded by C and the ground.

   (c) What is the name of the cat?

5. Let $f : S^1 \to \mathbb{R}$ be continuous, where $S^1 = \{(x, y) : x^2 + y^2 = 1\}$. Show that there exists $z \in S^1$ such that $f(z) = f(-z)$.

6. Show that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniquely determined by its values on the rationals.

7. Consider the sequence $\{a_n\}$ defined by $a_n = \cos(n\sqrt{2})$.

   (a) Determine the convergence of $\{a_n\}$.

   (b) Prove your answer. If $\{a_n\}$ converges, find the limit.