$2.45 ACADEMIC PAPERBACKS

Noneuclidean Geometry

H. Meschkowski

NONEUCLIDEAN GEOMETRY

by Herbert Meschkowski
THE FREE UNIVERSITY OF BERLIN
BERLIN, GERMANY

TRANSLATED BY

A. Shenitzer
DEPARTMENT OF MATHEMATICS
ADELPHI UNIVERSITY
GARDEN CITY, LONG ISLAND
NEW YORK

ACADEMIC PRESS, New York and London

First published in the German language under the title Nichteuklidische Geometrie and copyrighted in 1961 *by* FRIEDR. VIEWEG & SOHN, *Braunschweig, Germany.*

ACADEMIC PRESS INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS INC. (LONDON) LTD.
Berkeley Square House, London W.1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 64-24662

First Printing, 1964
Second Printing, 1965

PRINTED IN THE UNITED STATES OF AMERICA

Preface

Noneuclidean geometry is of great importance in the study of the foundations of mathematics. Noneuclidean geometry can be presented in various ways. One way is to follow the historical development and show how the fruitless efforts to produce a proof of the parallel postulate led to the idea of constructing a geometry with more than one parallel. However, a presentation of this geometry based on a set of axioms (in the manner of Bolyai or Lobachevski) and not involving the use of a model makes great demand on the reader's ability to follow abstract reasoning. This is due to the fact that the assertions of non-euclidean geometry cannot be illustrated by means of euclidean points and lines.

Understanding is made easier by early introduction of one of the models of Felix Klein or Poincaré. We decided to use the Poincaré model because the measurement of angles in this model is particularly simple (euclidean). Ordinarily one defines the hyperbolic plane of the Poincaré model as the upper complex half plane. In our presentation real numbers alone are used. The result is a version of noneuclidean geometry which a beginner can hope to follow. Our version of hyperbolic trigonometry must also be classified as elementary.

Those who wish to pursue the theory more deeply or who wish to give thought to the consequences which the existence of noneuclidean geometry has for knowledge in general are invited to consult the bibliography at the end of the book. Numbers in square brackets appearing in the text refer to appropriate items.

In this second edition we corrected a few misprints and supplemented the bibliography.

Berlin HERBERT MESCHKOWSKI

CHAPTER I

On Proofs and Definitions

The discoverers of noneuclidean geometry fared somewhat like the biblical king Saul. Saul was looking for some donkeys and found a kingdom. The mathematicians wanted merely to pick a hole in old Euclid and show that one of his postulates which he thought was not deducible from the others is, in fact, so deducible. In this they failed. But they found a new world, a geometry in which there are infinitely many lines parallel to a given line and passing through a given point; in which the sum of the angles in a triangle is less than two right angles; and which is nevertheless free of contradictions.

This book is an introduction to this "noneuclidean" geometry.

We must first give some thought to the structure of geometry and its methods of proof. The best way to gain an appreciation of the issues involved is to see on what foundations a proof rests.

Take the familiar theorem of Pythagoras which states that the square on the hypotenuse of a right triangle is equal to the sum of the squares on its two legs.

Why is this so? Certainly, there is nothing intuitively obvious about this assertion. Consequently, this theorem must be proved, i.e., deduced from "already known" facts.

There are over 80 proofs of this theorem. One of the best known proofs runs as follows: Let a and b denote the legs of a right triangle and c its hypotenuse. Construct the square

$ABCD$ (Fig. 1) with side $a + b$. On the sides of this square, choose four points E, F, G, H, such that

$$AE = BF = CG = DH = a; \quad EB = FC = GD = HA = b.$$

Then

$$EF = FG = GH = HE = c.$$

To see this observe that the triangles AEH, BFE, CGF, DHG agree with the given triangle on two sides and the enclosed (right) angle.

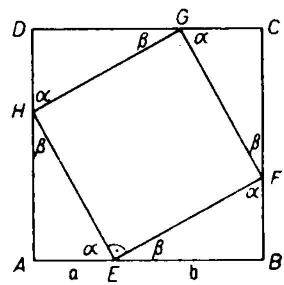

FIG. 1.

But why must two triangles which agree on two sides and the enclosed angle agree on all sides and angles? Can this "first congruence theorem" be proved?

If you think back to your early training in geometry you will most likely be unable to recall a proof of this theorem. This is not due to poor memory. In high school the congruence theorems are made plausible by reflection arguments and by pointing to the possibility of constructing a triangle given three pieces of information.

The fact remains that the theorem in question is usually not proved.

The student is taught the elements of geometry at an age when he can not appreciate such questions. He relies on intuition and it is proper that present-day mathematical training should take such psychological laws into consideration.

This does not free us from the obligation to consider, at a later time, the foundations of all mathematics. By ignoring this question we fail to exhaust the educational potential of mathematics.

In this connection we ask whether the congruence theorems, which are familiar tools of proof, are themselves provable or not.

Let us return to the theorem of Pythagoras. To prove it we note that the quadrilateral *EFGH* is a square. The equality of its sides is already established. To prove that its angles are right angles, observe that if α and β are the angles formed by the hypotenuse and the legs of our triangle, then

$$\alpha + \beta = R$$

and

$$\alpha + \beta + \angle HEF = 2R,$$

which implies that $\angle HEF = R$.†

Question: Why do we claim that $\alpha + \beta = R$?

Answer: Because the sum of the angles in a triangle equals two right angles. Why is the sum of the angles in a triangle equal to two right angles? This we usually prove using the theorem on the equality of the alternate interior angles in the case of parallel lines.

But why are alternate angles equal in the case of parallel lines?

In proving this assertion we make use of the equality of vertical angles and of corresponding angles.

Why are corresponding angles equal in the case of parallel lines?

At this point proofs usually cease. One can make this last assertion plausible by sliding one of the corresponding angles into the other along the transversal. However, this is no proof.

† To complete the proof we could argue as follows: The area of the big square is the sum of the areas of the four right triangles and the small square. Hence

$$(a + b)^2 = c^2 + 4 \cdot ab/2,$$
$$a^2 + b^2 = c^2.$$

Does there exist a "known fact" from which one can deduce the theorem on corresponding angles?

At the moment the answer to this question is not very important. The answer, if there were one, would give rise to another question. But the game of asking questions and supplying proofs cannot continue indefinitely. We must have a foundation on which to build. This is the lesson which reflection on the Pythagorean theorem teaches us.

So far this foundation was supplied by intuition. The congruence theorems and the theorem on corresponding angles belong to the intuitive foundations of high school geometry.

Where is the boundary between "intuition" and proof? Pascal [2] set up the following "rules for proofs":

1. One should not attempt to prove statements so obvious that nothing more obvious exists with which to prove them.
2. One should prove all theorems which are not quite clear and in the proofs one should use only very obvious axioms or theorems which are accepted or proved.

It is natural to ask: What is "so obvious that nothing more obvious exists," and what theorems are "not quite clear"?

The theorem about the equality of the base angles in an isosceles triangle is usually proved using congruence considerations. However, this theorem could also be regarded as "self-evident." On the other hand, there are many examples which show that intuitition can deceive and lead to false conclusions.

It is therefore not expedient to simply make "intuition" the foundation of geometry. A flawless structure must be based on a number of propositions accepted without proof.

Such fundamental propositions are called axioms.

Clearly, such "axioms" must be chosen with great care. There must not be too few of them; otherwise some geometric problems may not admit of a solution. This "sufficiency" of a system of axioms is referred to as its completeness. Also, mathematicians insist,† that such a system must not contain

† For esthetic reasons. (Tr.)

axioms which can be derived from the remaining axioms of the system, or briefly, that the system be "independent." Finally, it is clear that a system of axioms must be consistent, i.e., free of contradictions.

Thus consistency, independence, and completeness are three important characteristics of a usable system of axioms of geometry. †

The construction of such a system has been referred to as the "big cleanup" in geometry. This is not entirely unjustified since it is only in this way that we can have clarity and order in the edifice of geometry. There are those who dislike cleanups at home and fear that this type of activity in geometry may be, if not equally cheerless, then at least somewhat boring.

Whoever thinks so is greatly mistaken. Concern with foundations leads to the most interesting mathematical and philosophical questions.

The Greek mathematician Euclid (325 B.C.) was the first to attempt the construction of such a system. His "Elements" are the first remarkable attempt to build all geometry beginning with a carefully thought through system of definitions, postulates, and axioms. There is no book in the history of mankind which has retained a position of prominence for as long a time as this work of Euclid. Up until the last century Euclid's "Elements" (in modified but not always improved form [4]) was the textbook of geometry in the high schools.

Euclid begins with definitions such as:

> "A point is that which has no part."
> "A straight line is a line which lies evenly with the points on itself."

Such definitions were rightly criticized. For we are not told what a point is but rather what it is not. Again, the definition of a straight line is not clear.

† Consistency is an essential feature of a system of axioms. Independence is desirable but not essential. Completeness, in one sense or another, is essential in *some* systems of axioms, but definitely *not* required in all systems of axioms investigated by mathematicians. (Tr.)

But criticism must be constructive. Many later mathematicians tried to improve on *Euclid.* Consider, for example, the following definitions [17]:

> "A point is that which has position but no magnitude."
> "A straight line represents the shortest distance between two points."

But here use is made of concepts (position, magnitude, distance) which are hardly easier to define than point and straight line. After centuries of vain effort it has been realized that one must abandon definitions of the kind attempted by Euclid.

Already Pascal [2] says in his "1. Rule for definitions":

> "One should not attempt to define things which are of themselves so well known that no clearer concepts exist with which to explain them."

We encountered this thought previously in Pascal's rules pertaining to proofs. There must be a foundation on which to build. Point and straight line are the most elementary notions of geometry. One must not attempt to reduce them to simpler notions. We shall show in the next chapter how modern geometry handles this issue.

A few additional comments on Euclid's "Elements" are in order.

Euclid's fundamental propositions are divided into "postulates" and "axioms." The first group of fundamental propositions contains requirements pertaining to the possibility of effecting certain constructions, e.g.,

> "A straight line can be drawn from any point to any point."

The axioms are assertions about magnitudes, e.g.,

> "Two magnitudes equal to a third are equal to one another."

Modern mathematics ignores the distinction between postulates and axicms.

The famous parallel postulate will be discussed extensively in Chapter 3. Apart from this exception we refrain from an extensive discussion of the euclidean system. Instead, we present in the next chapter a modern axiomatization of geometry.

However, a few words of critical appreciation of Euclid's work are in order.

Euclid succeeded in basing his development of geometry on a system. This system is not complete. Thus, for example, statements concerning position are based on intuition rather than on postulates and axioms of his system. One relevant example:

A ray passing through a vertex of a triangle and lying in its interior intersects the side opposite to the vertex in question.

The author of a modern work on the evolution of the concept of space [34] finds that these shortcomings and the inadequacy of Euclid's definitions justify the following harsh critical judgment:

> "This fact obviously vitiates any claim of the work to be a logical system."

Nevertheless, it must be acknowledged that the "Elements" are a work of genius. One fact in particular deserves our appreciation. This fact is Euclid's clear recognition of the impossibility of deriving the proposition on parallels (cf. Chapter 3) from the remaining postulates and axioms. Also, the theory of proportions (for incommensurable segments) is a very beautiful creation.

In 1899 the German mathematician David Hilbert (1862–1943) published his "Foundations of Geometry" in which he took into account the various objections directed against the work of Euclid. Later Baldus [24–26] constructed a system of axioms which are an interesting modification of the formulations of Hilbert. Again, attempts have been made [22, 27–29] to include the notion of reflection, so frequently used in elementary geometric training, in the axiomatic foundation. However, these very interesting efforts strike us as not yet fully mature. Therefore our purpose will be served best if we adhere to the Hilbert system.

CHAPTER 2

Hilbert's System of Axioms

As is to be expected Hilbert does not define points and lines. He begins [11] with:

Explanation I. Let us consider three distinct systems of things. The things composing the first system, we will call points and designate them by the letters $A, B, C, \ldots$; those of the second, we will call straight lines and designate them by the letters $a, b, c, \ldots$; and those of the third system, we will call planes and designate them by the Greek letters $\alpha, \beta, \gamma, \ldots$. The points are called the elements of linear geometry; the points and straight lines, the elements of plane geometry; and the points, lines, and planes, the elements of the geometry of space or the elements of space.

We think of these points, straight lines, and planes as having certain mutual relations, which we indicate by means of such words as "are situated," "between," "parallel," "congruent," "continuous," etc. The complete and exact description of these relations follows as a consequence of the axioms of geometry.

Nothing is said about the nature of the things in the three systems. While "the axioms express certain fundamental facts of our intuition," the unavoidable use of undefined terms leaves us free to assign to the words "point," "line," "plane" any meaning consistent with the axioms to be stated. In the sequel we shall give examples of such admissible interpretations.

The axioms restrict the possible interpretations of the undefined terms. One could therefore say that the axioms are "implicit definitions" of the undefined terms.

We wish to emphasize the following point. While the axioms to be stated are essentially intended as an unambiguous description of the intuitive foundations of our geometry, their formal character opens up possibilities for interpretation which will make us reflect on the very nature of geometry.

We now state Hilbert's axioms for plane geometry. These axioms are divided into the following five groups:

I. Axioms of connection
II. Axioms of order
III. Axioms of congruence
IV. Axioms of continuity
V. Axiom of parallels.

Following Baldus [24] we put the continuity axioms before the axiom of parallels and make slight changes in the second continuity axiom and in some of the congruence axioms as stated in [11].

I. Axioms of Connection

I, 1. Two distinct points A and B always determine a line a.
I, 2. Any two distinct points of a line determine that line uniquely.
I, 3. Every line has at least two points.

These axioms imply:

Theorem 1. Two distinct lines in a plane† have either one point or no point in common.

Indeed, if the lines in question had two points in common then, in view of I, 2, they would coincide.

It is clear that the above three axioms express properties of

† Since we are studying plane geometry it goes without saying that our points and lines lie in a plane. This being so we shall occasionally leave out reference to a plane.

what we conceive as points and lines. However, other "things" may also satisfy these axioms. Thus:

Example 1. Let "pseudopoint" denote a point in the interior of a fixed circle, and "pseudoline" a chord of that circle exclusive of its end points (Fig. 2). This means that the "pseudoplane" is the interior of our circle.

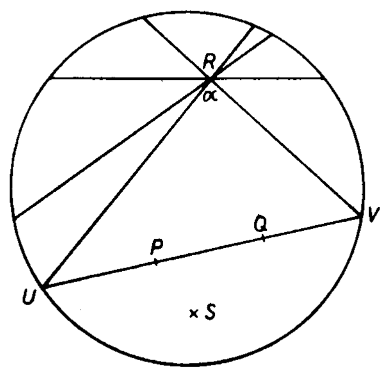

FIG. 2.

The pseudopoints and pseudolines are readily seen to satisfy the three axioms of connection. To this extent we are free to identify them with the things referred to by Hilbert as points and lines.

It follows that Theorem I holds in our "pseudogeometry."

This is not the only possible realization of the axioms in the first group. For another realization consider:

Example 2. Take as a "pseudoplane" a fixed spherical surface. Define a "pseudopoint" as a pair of antipodal points and a "pseudoline" as a great circle.

It is easy to see that the axioms of connection hold in this "pseudogeometry" (Fig. 3).

It is natural to ask why a "pseudopoint" is taken to be a pair of antipodal points rather than a single point. The reason is clear. If two antipodal points were distinct pseudopoints, then infinitely many pseudolines (great circles) would pass through two pseudopoints and this would violate the axioms of connection.

We now turn to the next group of axioms.

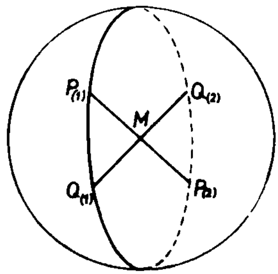

FIG. 3.

II. Axioms of Order

Explanation 2. The points of a line have a certain relation described by means of the word "between." The relevant axioms are:

II, 1. If A, B, C are points of a straight line and B lies between A and C, then B lies also between C and A. In symbols, ABC or CBA.

II, 2. If A and C are two points of a straight line, then there exists at least one point B lying between A and C and at least one point D so situated that C lies between A and D.

II, 3. Of any three points situated on a straight line, there is always one and only one which lies between the other two.

Definition 3. We will call the system of two points A and B, lying upon a straight line, a segment and denote it by AB or BA. The points lying between A and B are called the points of the segment AB or the points lying within the segment AB. All other points of the straight line are referred to as the points lying outside the segment AB. The points A and B are called the end points of the segment† AB.

† In the sequel "segment" frequently stands for the collection of its points and end points. (Tr.)

II, 4. Let A, B, C be three points not lying in the same straight line and let a be a straight line lying in the plane ABC and not passing through any of the points A, B, C. Then, if the straight line a passes through a point of the segment AB, it will also pass through either a point of the segment BC or a point of the segment AC.

Using the "triangle axiom" II, 4 (Fig. 4) we can now prove theorems such as that mentioned on p. 7 concerning a ray issuing from a vertex of a triangle and lying in its interior.

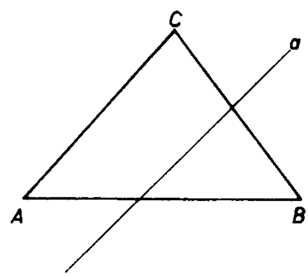

FIG. 4.

This and similar intuitively "obvious" theorems are based on Axiom II, 4. This is one aspect of the superiority of Hilbert's system over Euclid's.

Space limitations preclude giving proofs of the various theorems.

Do the models of Examples 1 and 2 above satisfy the axioms of order? The answer is "yes" for the model in Example 1 and "no" for the model in Example 2. Indeed, given three pairs of antipodal points on a circle it is not possible to say that one pair lies between the other two.

The axioms of order enable us to define the notions of ray, halfplane, and angle.

Definition 4. Let O, A, B be three points on a line. Then AOB or not AOB. In the first case we say that A and B lie on different sides of O. In the second case we say that A and

B lie on the same side of O. The points on the same side of O are referred to as a ray issuing from O.

One can prove that a point of a line divides the line into two rays.

Definition 5. Let g be a line and A a point not on g. The set of points B such that the segment AB contains no point of g and the point A are said to form a halfplane determined by the line g (Fig. 5).

Using the "triangle axiom" II, 4 one can prove that a line divides a plane into two halfplanes.

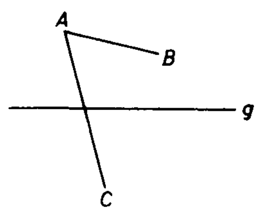

FIG. 5.

Definition 6. A pair of rays h, k issuing from a point O (but not constituting a line) is called, without reference to order, an angle and is denoted as $\angle(h, k)$ or $\angle(k, h)$. h and k are called the sides of the angle and O is called its vertex (Fig. 6).

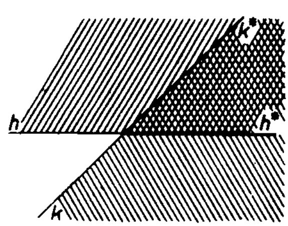

FIG. 6.

III. Axioms of Congruence

Explanation 7. There is a relation connecting segments and angles referred to as "congruence." The relevant axioms are:

III, 1. Consider a segment AB and a point A'. On every line passing through the point A' it is possible to find at least two points B'_1 and B'_2 such that A' is between B'_1 and B'_2 and such that the segment AB is congruent to each of the segments $A'B'_1$ and $A'B'_2$. In symbols, $AB \equiv A'B'_1$, $AB \equiv A'B'_2$.

We say "at least two points" rather than "exactly two points" because we do not want to postulate provable propositions. Axiom III, 1 and other congruence axioms imply that there are "exactly two points."

III, 2. $A'B' \equiv AB$ and $A''B'' \equiv AB$ imply $A'B' \equiv A''B''$.

These axioms imply that $AB \equiv AB$.

III, 3. If B is a point of the segment AC and B' is a point of the segment $A'C'$ and $AB \equiv A'B'$, and $BC \equiv B'C'$, then $AC \equiv A'C'$.

III, 4. Let $\angle(h, k)$ be a given angle and let α' be one of the two halfplanes defined by the line a'. Let h' be a ray on a'. Then there is exactly one ray k' such that $\angle(h, k) \equiv \angle(h', k')$ and such that a point of k' is in the halfplane α' (Fig. 7).

III, 5. $\angle(h, k) \equiv \angle(h, k)$. Any angle is congruent to itself.

III, 6. Let ABC and $A'B'C'$ be two triangles such that

$$AB \equiv A'B',$$
$$AC \equiv A'C',$$
$$\angle BAC \equiv \angle B'A'C'.$$

Then

$$\angle ABC \equiv \angle A'B'C', \qquad \text{and} \qquad \angle ACB \equiv \angle A'C'B'.$$

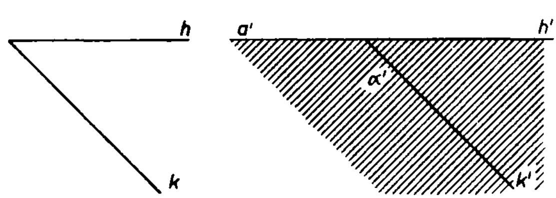

FIG. 7.

Observe that congruence was introduced formally. We have not been told when two segments are to be viewed as congruent. In particular, no numbers have been used as measures of segments and angles. Hence we have no means of effecting comparisons using numbers. For the time being we note that the six axioms of congruence hold for figures which, intuitively speaking, can be made to coincide.

Axiom III, 6 is similar to the first congruence theorem. Indeed, III, 6 states that: If two sides of a triangle and the angle formed by these sides are congruent, respectively, to two sides of another triangle and the angle formed by those sides, then the angles of these triangles are congruent in pairs.

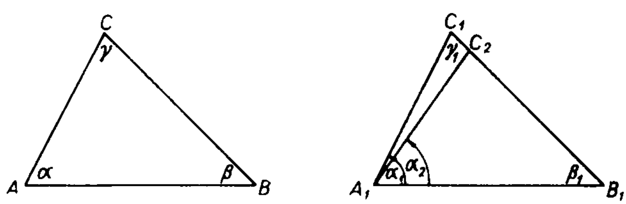

FIG. 8.

If we could prove the congruence of the remaining pair of sides we would have the first congruence theorem. Thus (Fig. 8), let

$$AB \equiv A_1B_1, \qquad AC \equiv A_1C_1, \qquad \alpha \equiv \alpha_1.$$

Then III, 6 implies that

$$\beta \equiv \beta_1, \qquad \gamma \equiv \gamma_1.$$

If BC were not congruent to B_1C_1, then, by III, 1, we could assert the existence of a point C_2 on the ray containing the point C_1 and belonging to the line B_1C_1 such that $BC \equiv B_1C_2$. In view of $AB \equiv A_1B_1$, $BC \equiv B_1C_2$, and $\beta \equiv \beta_1$, it follows, by III, 6, that

$$\angle C_2A_1B_1 \equiv \alpha.$$

Since $\angle \alpha_1 \equiv \angle \alpha$, III, 4 implies that $C_1 = C_2$. This means that

$$BC \equiv B_1C_1.$$

Thus:

Theorem 2. If the assumptions of III, 6 hold for two triangles, then these triangles are congruent† (first congruence theorem).

The remaining congruence theorems can also be proved. Using the axioms stated so far it is possible to establish a considerable part of plane geometry. Still missing, however, are connections between geometric figures and numbers as well as propositions whose proofs require the use of the axiom of parallels. The proposition on the sum of the angles of a triangle belongs here.

Before we can state some of the important theorems which are now provable we must introduce the concept of a right angle.

What is a right angle? We cannot say that this is a 90° angle since division into degrees assumes the possibility of measurement which requires additional axioms. We must therefore try a different approach.

Definition 8. An angle congruent to its supplement‡ is called a right angle.

It is now possible to prove that:

Theorem 3. There exist right angles.

Theorem 4. All right angles are congruent.

If these theorems strike the reader as trivial he must not forget that all he is allowed to use in proving them is the stated axioms and the "things" (points, lines, planes) introduced above.

A few more theorems which can now be proved are:

Theorem 5. From a point not on a line one can drop exactly one perpendicular to that line.

† Two triangles are congruent if their sides and angles are congruent in pairs.

‡ We do not define the notions of supplement, vertical angle, perpendicular, etc., since they are used here in much the same sense as in high school study of geometry.

Theorem 6. A segment has exactly one midpoint. An angle has exactly one bisector.

Theorem 7. Vertical angles are congruent.

Theorem 8. If two lines are cut by a transversal and the alternate interior angles are congruent, then the lines do not intersect.

Define two lines in a plane to be parallel if they have no points in common. Then we can assert that

Theorem 9. Through a given point not on a given line one can always pass a parallel line.

For proof join the given point P to a point Q on the given line g. There results an angle α with vertex Q (Fig. 9). Let g_1 be the side of α on g.

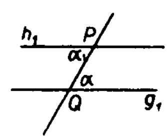

FIG. 9.

Axiom III, 4 implies that there is exactly one ray h_1 emanating from P which forms with PQ an angle α_1 congruent to α and which lies in the halfplane determined by PQ which does not contain g_1.

Since α and α_1 are congruent alternate interior angles, it follows from Theorem 8 that the lines g and h (h is the line determined by the ray h_1) do not intersect.

Thus there is *at least* one line through P parallel to g. This is the only admissible formulation since we cannot rule out the existence of other parallel lines through P.

This proposition on parallels is the most important result thus far. Since it is a direct consequence of Theorem 8 we shall prove Theorem 8 also.

Thus (Fig. 10) let g_3 be the line which forms congruent alternate interior angles with the lines g_1 and g_2. These angles are denoted by α_1 and α_2, and their vertices by A_1 and A_2. Let B be the midpoint of A_1A_2 (cf. Theorem 6).

Let C_1 and C_2 be the feet of the perpendiculars from B to g_1 and g_2. Then the triangles A_1BC_1 and A_2BC_2 are congruent (saa) and we have

$$\angle C_1BA_1 \equiv \angle C_2BA_2.$$

This implies that the points C_1, B, C_2 are collinear.

The latter assertion is not obvious. To prove it extend C_1B beyond B. The extended ray and g_3 form the angle vertical to $\angle C_1BA_1$. Theorem 7 states that vertical angles are congruent. Since, as just proved, $\angle C_1BA_1 \equiv \angle C_2BA_2$, BC_2 must belong to the extension of BC_1 beyond B. This follows from the uniqueness assertion in Axiom III, 4.

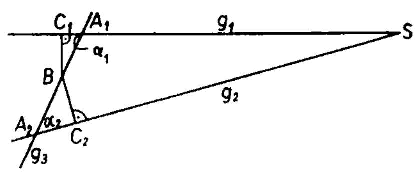

FIG. 10.

The line determined by C_1, B, C_2 is perpendicular to g_1 and g_2. If g_1 and g_2 intersected in a point S, then g_1 and g_2 would be two perpendiculars from the point S to the line C_1C_2. This contradicts Theorem 5. Hence the lines g_1 and g_2 do not intersect, as asserted.

The axioms of congruence enable us to define the notions of "smaller" and "greater" for segments and angles without the use of numbers.

Definition 9. The segment AB is said to be smaller than the segment CD if there exists an interior point E of CD such that $AB \equiv CE$.

The angle (g, h) is said to be smaller than the angle (k, l) if there exists a ray m emanating from the vertex of $\angle(k, l)$ and lying in its interior such that $\angle(g, h) \equiv \angle(k, m)$.

The well-known size relations between sides and angles in a triangle can now be deduced from our three groups of axioms.

IV. Axioms of Continuity

IV, 1. Let A, B, A_1 be three collinear points with A_1 between A and B. Construct the points A_2, A_3, A_4, ... so that A_1 is between A and A_2, A_2 is between A_1 and A_3, A_3 is between A_2 and A_4, etc., and so that the segments

$$AA_1, \quad A_1A_2, \quad A_2A_3, \quad A_3A_4, \quad \ldots$$

are equal (i.e., congruent). Then the sequence of points A_2, A_3, A_4, ... contains a point A_n such that B lies between A and A_n (Fig. 11) (axiom of Archimedes).

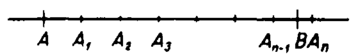

FIG. 11.

This axiom is the basis of measurement in the following sense. Let AB be a given segment. Using an arbitrary unit AA_1 and laying it off a sufficient number of times we are certain to reach the age when the end point A_n falls "beyond" the end point B of the segment AB. Obviously, one cannot always expect A_{n-1} to coincide with B. In other words this axiom does not enable us to associate with a segment a number as a measure of its "length." For this we need the additional axiom IV, 2.

IV, 2. Let AB be a segment and let A_n and B_n be two sequences of interior points of AB with the following properties:

(a) The segment A_nB_n lies in the interior of the segment $A_{n-1}B_{n-1}$.

(b) There is no segment whose end points belong to all the segments A_nB_n.

Then there is a unique point X common to all the segments A_nB_n (Fig. 12).

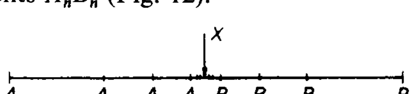

FIG. 12.

This "nested intervals axiom" enables us to associate with every segment a real number as the measure of its length, and so supplies the basis for "analytic geometry." It will not be examined in detail here.

We add that Axioms IV, 1 and IV, 2 enable us to measure angles. In particular we have the following theorem which we will come across later:

Theorem 10. Let $\angle(g, h)$ be a given angle and k a ray emanating from its vertex and lying in its interior. Let k_1 be the bisector of $\angle(g, h)$, k_2 the bisector of $\angle(k_1, h), \ldots, k_n$ the bisector of $\angle(k_{n-1}, h)$. Then there exists *a* ray k_N lying in the interior of $\angle(k, h)$ (Fig. 13).

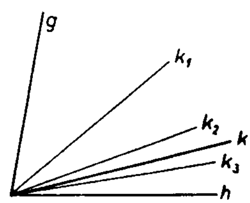

FIG. 13.

V. The Axiom of Parallels

V. Let a be a line and A a point not on a. Then there is exactly one line (in the plane α determined by a and A) which passes through A and does not intersect a. This line is called the parallel to a through A.

Only now can we prove that the sum of the angles in a triangle is equal to two right angles and that for parallel lines the alternate interior angles are equal. The latter theorem is the converse of Theorem 8.

CHAPTER 3

From the History of the Parallel Postulate

The history of the axiom of parallels affords the best approach to noneuclidean geometry. Familiarity with the Hilbert system of axioms will make it easier for us to understand the problem. However, we must begin with Euclid. Euclid's fifth postulate states:

(P) If a straight line falling on two straight lines makes the interior angles on the same side together less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are together less than two right angles."

The word "parallel" is defined by Euclid as follows (def. 23):

> Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction.

For centuries mathematicians were convinced that the fifth postulate could be deduced from the others. If so, it should not, strictly speaking, be included among the postulates.

There were arguments which supported this point of view. Thus it was well known that the converse of the parallel postulate was a provable proposition. This converse can be formulated as follows:

(C) Consider a line falling on two intersecting lines. Then the sum of exactly one pair of interior angles on the same side is less than two right angles.

Proposition (C), i.e., the converse of the parallel postulate, is equivalent to Theorem 8 of the preceding chapter. Indeed, if Theorem 8 were false, then we could have $\alpha_1 \equiv \beta$ (cf. Fig. 14) or, what amounts to the same thing, $\alpha + \beta = 2R$ and g_1 could intersect g_2. But this would contradict (C) which asserts that if g_1 and g_2 intersect, then $\alpha + \beta < 2R$. Thus (C) implies Theorem 8. It is easy to prove that Theorem 8 implies (C).

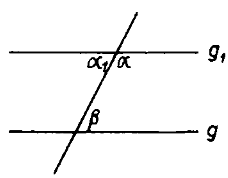

FIG. 14.

While mathematicians of past ages were not familiar with Hilbert's system, they could deduce Theorem 8 or (C) from Euclid's axioms and postulates without the use of the parallel postulate, (In proving Theorem 8 we did not make use of Axiom V.)

It was thought impossible for a geometric theorem not to be provable if its converse was provable. Indeed, in the case of most theorems of elementary geometry proving the converse of a theorem does not require the use of additional axioms. For example, the theorem on the congruence of base angles in an isosceles triangle and its converse (if two angles in a triangle are congruent then the sides opposite to these angles are congruent) are both provable by means of the congruence axioms. It is therefore understandable why mathematicians tried so eagerly for 2000 years to prove the axiom of parallels.

In this they failed.

In many cases the "proofs" involved the use of an intuitive fact equivalent to the parallel postulate. For example, it is easy to prove the parallel postulate if we take it for granted that the sum of the angles in a triangle is equal to two right angles.

Other attempts at proof made use of the assertion that the length of the perpendicular from a point on a line to a parallel line is independent of that point. However, parallel lines are defined as nonintersecting lines (cf. above) and this does not automatically mean that the lengths in question are equal.

Of course, it is possible to prove this last assertion if we make use of the theorem on the alternate interior angles for parallel lines. But we would then be guilty of circular reasoning, since the theorem just mentioned cannot be proved without the use of the parallel postulate.

In 1763 G. S. Kluegel, a student of Kestner, wrote a dissertation [3] in which he brought together and criticized all significant attempts to prove the parallel postulate contributed by mathematicians over the 2000 years between the publication of the "Elements" and his own time. He found, correctly, that all 28 "proofs" were false.

Most of these attempts are of historical interest only. Some, however, pointed the way to new insights.

The first significant contribution is due to the Jesuit priest Saccheri (1667–1733). In his extensive work on parallels Saccheri studied a quadrilateral $ABCD$ in which $AC \equiv BD$ and the angles at A and B are right angles (Fig. 15).

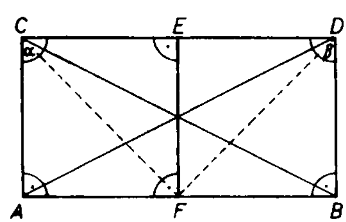

FIG. 15.

If we do not wish to make use of the parallel postulate, then it is not possible to prove that the angles ACD and BCD are right angles, i.e., that $ABDC$ is a rectangle. However, we can use congruence considerations to prove that $\angle ACD \equiv \angle BDC$; namely,

$$\triangle ABC \equiv \triangle ABD \qquad \text{(sas)},$$

so that

$$AD \equiv BC.$$

Congruence of the diagonals implies

$$\triangle ACD \equiv \triangle BCD \qquad \text{(sss)}.$$

Hence

$$\alpha \equiv \beta.$$

Now we ask: Are the congruent angles α and β in the Saccheri quadrilateral acute, obtuse, or right angles?

First we establish a connection between this question and the sum of the angles in a triangle.

Let ABC (Fig. 16) be a triangle; let D, E be the midpoints of

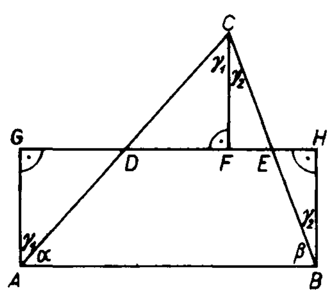

FIG. 16.

AC and BC; let F, G, H be the feet of the perpendiculars from C, A, B on to the line DE. Then

$$FC \equiv AG \equiv BH.$$

This follows from the congruence of the triangles DFC and ADG (saa) and the congruence of the triangles EFC and BEH.

The quadrilateral $ABHG$ is thus a Saccheri quadrilateral with right angles at G and H.

It follows that the the angles at A and B are congruent:

$$\alpha + \gamma_1 \equiv \beta + \gamma_2.$$

Since

$$\alpha + \beta + (\gamma_1 + \gamma_2) \equiv W$$

is the sum of the angles in the triangle ABC, it follows that each of the angles at A and B is equal to $W/2$.

"The hypothesis of the acute angle"—$W/2 < R$—implies that the sum of the angles in a triangle is $< 2R$.

"The hypothesis of the obtuse angle"—$W/2 > R$—implies that the sum of the angles in a triangle is $> 2R$.

"The hypothesis of the right angle"—$W/2 \equiv R$—implies that the sum of the angles in a triangle is $\equiv 2R$.

Saccheri attempted to eliminate the first two hypotheses by showing that they led to contradictions. If proved, this would imply that the sum of the angles in a triangle is equal to two right angles. But this would imply (cf. Theorem 12, below) the parallel postulate.

Saccheri succeeded in eliminating "the hypothesis of the obtuse angle." His proof is clumsy but correct. On the other hand, his proof of the inadmissibility of "the hypothesis of the acute angle" is not acceptable.

To sum up: Saccheri succeeded in showing that the sum of the angles in a triangle does not exceed two right angles.

Saccheri failed to show that the assumption

$$\alpha + \beta + \gamma < 2R$$

leads to a contradiction.

We shall see that there are deep reasons for these outcomes.

Instead of reproducing the clumsy argument of Saccheri we present Legendre's (1752–1833) version of the theorem on the sum of the angles in a triangle.

Thus, let α, β, γ be the angles of a given triangle ABC. Lay off the segment AB n times in succession on a line (Fig. 17) and construct over each of these segments a triangle $A_v A_{v+1} B_v$ congruent to the given triangle (all of the contructed triangles are to lie in the same plane).

We claim that $\gamma_1 \geqq \gamma$. If we have $\gamma_1 < \gamma$, then

$$A_v A_{v+1} > B_v B_{v+1}. \tag{1}$$

This conclusion can easily be deduced from the axioms of congruence. Further, the polygonal line $A_1B_1 + B_1B_2 + \cdots + B_nA_{n+1}$ is greater than the segment A_1A_{n+1}:

$$\begin{aligned} A_1B_1 + n\cdot B_1B_2 + B_nA_{n+1} &> n\cdot A_1A_2, \\ A_1B_1 + B_1A_2 &> n\cdot(A_1A_2 - B_1B_2). \end{aligned} \tag{2}$$

In view of (1) the difference $A_1A_2 - B_1B_2$ is equal to some segment ε. Hence (2) can be written in the form

$$n\cdot\varepsilon < A_1B_1 + B_1A_2. \tag{3}$$

Since n can be taken arbitrarily large, (3) contradicts Axiom IV, 1 (axiom of Archimedes).

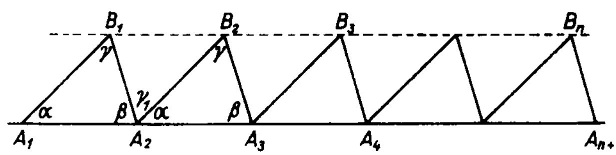

FIG. 17.

The assumption $\gamma_1 < \gamma$ is thus ruled out. But $\gamma_1 \geqslant \gamma$ implies $\alpha + \beta + \gamma \leqslant \alpha + \beta + \gamma_1 \equiv 2R$. This proves:

Theorem 11. The sum of the angles in a triangle is at most equal to two right angles.

Note that Theorem 11 was proved using the axioms in group I–III and the first continuity axiom (axiom of Archimedes) in group IV.

There is a very close connection between the sum of the angles in a triangle and the axiom of parallels. As noted before one can prove that:

Theorem 12. If the sum of the angles in a triangle is equal to two right angles, then the axiom of parallels holds.

We give a brief proof of this assertion.

Let A be the given point and g the given line (Fig. 18); B the foot of the perpendicular from A to g; C another point on g; h the perpendicular to AB at A. By Theorem 5, g and h cannot intersect. We shall show presently that h is the only parallel to g at A.

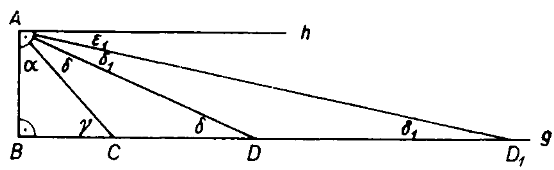

FIG. 18.

Thus let α and γ be the angles in the triangle ABC with vertices A and C. Then, in view of our assumption,†

$$\alpha + \gamma = R.$$

Choose D on g so that $AC \equiv CD$ and so that C lies between B and D. Then $\triangle ACD$ is isosceles and

$$\delta = \angle ADC = \gamma/2.$$

Again, let D_1 be a point on the extension of BD beyond D such that $AD \equiv DD_1$. Then

$$\delta_1 = \angle AD_1B = \delta/2 = \gamma/4.$$

Since $\angle D_1AB = R - \delta_1$, it follows that $\varepsilon_1 = \angle(h, AD_1) = \delta_1 = \gamma/4$.

By continuing this procedure we obtain segments AD_2, $AD_3, \ldots$ which form with $D_2B, D_3B, \ldots$ angles $\gamma/8, \gamma/16, \ldots$. The corresponding angles $\varepsilon_2, \varepsilon_3, \ldots$ are likewise equal to $\gamma/8, \gamma/16, \ldots$.

Now assume that $k \neq h$ is a ray emanating from A which does not intersect g. Let φ be the angle between k and h. If n is large enough, then the angle $\varepsilon_n = \gamma/2^{n+1}$ between the ray AD_n

† We shall sometimes refer to congruent angles as "equal". Equal angles have the same radian measure. Thus $\varphi = \psi$ is an equality between numbers.

and h will be smaller than φ. But then the ray k will lie in the interior of the angle BAD_n (Fig. 19) and, by the "triangle axiom" II, 4, will intersect BD_n and therefore the line g. A similar argument holds for the halfplane determined by AB and not containing the point C. Thus h is seen to be the only line through A parallel to g. This proves the parallel postulate under the assumption that the sum of the angles in a triangle is equal to two right angles.

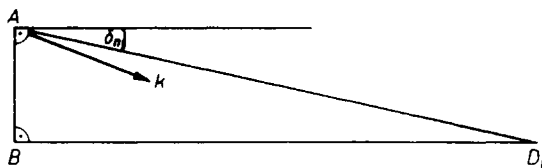

FIG. 19.

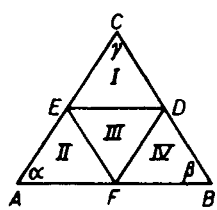

FIG. 20.

What follows is a quick "proof" that the sum of the angles in a triangle is equal to two right angles.

Thus, let D, E, F in Fig. 20 be the midpoints of the sides of a triangle, and I, II, III, IV the triangles determined by the segments DE, EF, DF. Let W be the (unknown) sum of the angles in a triangle. Then it is easy to see that

$$4W = (\alpha + \beta + \gamma) + 3\cdot 2R.$$

It follows that

$$3W = 6R, \qquad W = 2R.$$

According to Theorem 12 this implies the parallel postulate. Where is the error?

The careful reader will note that the only thing which has been proved is the following interesting assertion:

Theorem 13. If the sum of the angles in a triangle is the same for all triangles, then it is equal to two right angles.

To resume our narrative. As we saw, Legendre proved that the sum of the angles in a triangle could not exceed two right angles. It is understandable that he thought he was close to the solution of a problem which had kept mathematicians busy for more than 2000 years.

In 1823 Legendre thought that he had proved the parallel postulate. He reasoned as follows:

If $\alpha + \beta + \gamma < 2R$, then there exists an angle δ such that

$$\alpha + \beta + \gamma = 2R - \delta.$$

δ is called the "defect" of the triangle. The following is clear (Fig. 21): If we divide a triangle by means of a transversal

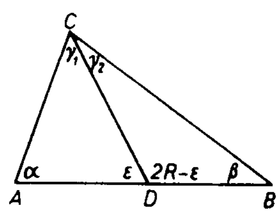

FIG. 21.

through a vertex into two triangles ADC and BDC, then the defect of the whole triangle is the sum of the defects of its parts. Indeed, by consulting Fig. 21 we see that

$$2R - (\alpha + \beta + \gamma) = 2R - (\alpha + \varepsilon + \gamma_1) + 2R - (\beta + \gamma_2 + 2R - \varepsilon).$$

This result remains true if the triangle is divided into more than two "subtriangles."

The additivity of the defect is at the heart of Legendre's argument. He reflects the triangle ABC about AB and draws through the image D of C a line which meets the rays CA and CB in E and F. The defect of the triangle CEF is at least equal to the sum 2δ of the defects of the triangles ACB and BAD (Fig. 22). Another reflection applied to $\triangle CEF$ yields a triangle with defect $\geqslant 4\delta$, etc. After n steps we obtain a triangle with defect $2^n \cdot \delta$. For n large enough $2^n \cdot \delta$ would exceed $2R$, which is impossible. Hence $\delta = 0$ and, in view of Theorem 12, the parallel postulate follows.

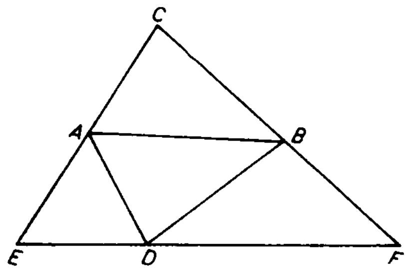

FIG. 22.

The reader is invited to study the argument carefully and detect the flaw in it.

The proof uses as an "axiom" the proposition that it is always possible to draw a line through a point (D) in the interior of an angle which intersects both sides of the angle. Legendre was fully aware of the fact that he was making use of this proposition as an "axiom."

One is likely to regard this proposition as being intuitively obvious. However, in trying to decide whether the available axioms imply this "obvious" result, we get into difficulties.

It is certainly not immediately apparent that this result is equivalent to the parallel postulate and we can hardly blame Legendre for thinking that it was independent of the parallel postulate.

To see that Legendre was mistaken we need only consult our pseudogeometry of Example 1. It is only fair to point out that this technique of testing independence of axioms was unknown to Legendre.

It is obvious that in the case of the point S, say (Fig. 2), there exists no "pseudoline" which intersects *both* sides of the angle α.

Legendre's axiom is seen to be another equivalent of the parallel postulate. Thus we end up with an interesting fact and an incorrect proof of the parallel postulate.

We cannot give here a more detailed history of the attempts to prove the parallel postulate and must leave many such interesting attempts unmentioned.

After 2000 years of vain effort many mathematicians began to show signs of skepticism. They began to doubt that a proof was at all possible. It should be noted that the persistence displayed by so many mathematicians was due to the fact that they viewed the unproved parallel postulate as a shocking flaw in geometry.

This statement is well illustrated by the following excerpt from a letter which the Hungarian mathematician Wolfgang Bolyai [21] wrote to his son, Johann:

> "It is unbelievable that this stubborn darkness, this eternal eclipse, this flaw in geometry, this eternal cloud on virgin truth can be endured."

At the same time the father is horrified by the thought that his son is attracted by the problem of parallels. Wolfgang Bolyai writes:

> "You must not attempt this approach to parallels. I know this way to its very end. I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone .... I thought I would sacrifice myself for the sake of the truth. I was ready to become a martyr who would remove the flaw from geometry and return it purified to mankind. I accomplished monstrous, enormous labors; my creations are far better than those of others and yet I have not achieved complete satisfaction. For here it is true that si paullum a summo discessit, vergit ad imum. I turned back when I saw that no man can reach the bottom of this night. I turned back unconsoled, pitying myself and all mankind."

And yet again:

> "I admit that I expect little from the deviation of your lines. It seems to me that I have been in these regions; that I have traveled past all reefs of this infernal Dead Sea and have always come back with broken mast and torn sail. The ruin of my disposition and my fall date back to this time. I thoughtlessly risked my life and happiness—aut Caesar aut nihil."

But this resignation of experienced mathematicians was not to be the last chapter in the exciting story of our geometric theorem. Owing in no small measure to the work of the younger Bolyai (1831) the issue took a surprising turn.

Most attempts to prove the parallel postulate employed the indirect method of proof: One argued that if the parallel theorem is false, then the sum of the angles in a triangle is less than two right angles. Then the triangle has a "defect." Then ... .

This mode of argument was expected to produce a contradiction. But none could be found. The thought suggested itself that it might after all be possible to construct a geometry in which the negation of the parallel theorem rather than the parallel theorem holds. After all, it is conceivable that the sum of the angles in a triangle is only approximately equal to two right angles.

Gauss followed up this thought. He used a transit to measure the sum of the angles of a huge triangle and found no significant deviation from two right angles. It was nevertheless possible to conceive of a geometry in which the parallel postulate did not hold and in which the sum of the angles in a triangle was less than two right angles.

Gauss was perhaps the first to have a clear conception of such a "noneuclidean" geometry. The thought struck him as being so revolutionary that he would not make it public. In 1829 he wrote to Bessel:

> "It may take very long before I make public my investigations on this issue; in fact this may not

> happen in my lifetime for I fear the scream of the dullards if I made my views explicit."

In a few places in his letters Gauss expresses the conviction that there exists a closed and consistent geometry in which the euclidean parallel postulate does not hold.

The first published accounts dealing with such a geometry are due to the Hungarian Johann Bolyai (1802–1860) and the Russian Lobachevski (1793–1856).

In 1823 Johann Bolyai could tell his father, who had tried so hard to make him give up his interest in the problem, that he was succeeding:

> "I am resolved to publish a work on parallels as soon as I can put it in order, complete it, and the opportunity arises. I have not yet made the discovery but the path which I have followed is almost certain to lead me to my goal, provided this goal is possible. I do not yet have it but I have found things so magnificent that I was astounded. It would be an eternal pity if these things were lost as you, my dear father, are bound to admit when you see them. All I can say now is that I have created a new and different world out of nothing. All that I have sent you thus far is like a house of cards compared with a tower."

His father advised him to publish his results as soon as possible. Johann Bolyai's comment follows:

> "He advised me that, if I was really successful, I should speedily make a public announcement and that for two reasons. One reason is that the idea might easily pass to someone else who would then publish it. Another reason—and one that seems valid enough—is that when the time is ripe for certain things, these things appear in different places in the manner of violets coming to light in early spring. And since scientific striving is like a war of which one does not know when it will be replaced by peace one

> must, if possible, win; for here preeminence comes to him who is first."

The advice was sound for, when Johann Bolyai published his work in 1831, he had been anticipated by Lobachevski who had presented his treatise to the physical-mathematical division of the University of Kazan on February 2, 1826.

Priority arguments are not important. The time was ripe for this particular insight and it is safe to say that Gauss, Bolyai, and Lobachevski independently drew the same conclusion from the impossibility of proving the parallel postulate.

Their contemporaries paid almost no attention to these new ideas. The view that euclidean geometry was the only possible geometry was so firmly rooted in thought that the new notion was hardly recognized. Although its discoverers developed the new geometry to the point of working out its trigonometry, this geometry lacked intuitive appeal and so was almost incomprehensible. A few decades passed before significant numbers of mathematicians took notice of the works of Bolyai and Lobachevski. The new geometry gained intuitive appeal as a result of the "models" constructed by Felix Klein and Poincaré.

In our introduction we shall make use of the easy-to-follow Poincaré model.

We precede our exposition with a chapter containing results of elementary geometry which may not be known to some of our readers.

CHAPTER 4

Lemmas

I. Pencil of Circles

Definition 10. The power of a point P with respect to a circle $\mathscr{K}$ is the square of PQ_1 (or PQ_2), where Q_1 and Q_2 are the points at which the tangents from P touch $\mathscr{K}$.

The power of a point is positive for points outside $\mathscr{K}$, zero for points on $\mathscr{K}$, and undefined for points in $\mathscr{K}$.

Definition 11. The radical axis of two circles $\mathscr{K}_1$ and $\mathscr{K}_2$ is the locus of all points which have the same power relative to $\mathscr{K}_1$ and $\mathscr{K}_2$.

Theorem 14. The radical axis of two intersecting circles is the line joining their points of intersection.† The radical axis of two tangent circles is their common tangent. The radical axis of two circles without common points is the perpendicular from a point of equal power (with respect to both circles) to the line joining the centers of the circles (Figs. 23 and 24).

Consider first the case of circles $\mathscr{K}_1$ and $\mathscr{K}_2$ which intersect in points A and B. Let C be a point on the line AB exterior to $\mathscr{K}_1$ and $\mathscr{K}_2$. By the secant-tangent theorem we have (Fig. 23)

$$CA \cdot CB = CQ^2 = CR^2.$$

We leave it to the reader to prove Theorem 14 in the case of tangent circles.

†Minus the segment in the interior of these circles. (Tr.)

If $\mathscr{K}_1$ and $\mathscr{K}_2$ are exterior to each other, then they have common outer tangents (Fig. 24). Let A, B be the points at which such a common outer tangent touches $\mathscr{K}_1$ and $\mathscr{K}_2$ and let C be the midpoint of AB.

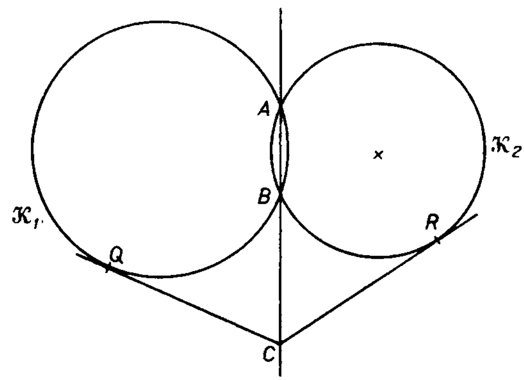

FIG. 23.

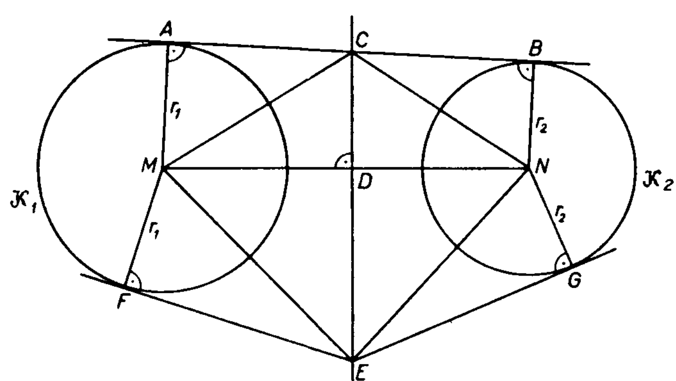

FIG. 24.

C has equal power relative to $\mathscr{K}_1$ and $\mathscr{K}_2$. Let D be the foot of the perpendicular from C to MN and let E be any point on that perpendicular. We must show that E has equal power relative to $\mathscr{K}_1$ and $\mathscr{K}_2$. Let r_1 and r_2 be the radii of $\mathscr{K}_1$ and

$\mathscr{K}_2$ and let F and G be points at which tangents from E touch $\mathscr{K}_1$ and $\mathscr{K}_2$. Repeated application of the theorem of Pythagoras yields the equalities

$$\begin{aligned}
EF^2 = -{r_1}^2 + EM^2 &= -{r_1}^2 + ED^2 + MD^2 \\
&= -{r_1}^2 + ED^2 \\
&\quad - CD^2 + MC^2 \\
&= -{r_1}^2 + ED^2 - CD^2 + {r_1}^2 + AC^2 \\
&= ED^2 - CD^2 + AC^2.
\end{aligned}$$

Similarly (consult Fig. 24),

$$EG^2 = ED^2 - CD^2 + BC^2.$$

Since $AC^2 = BC^2$, it follows that E has the same power relative to $\mathscr{K}_1$ and $\mathscr{K}_2$. Thus CD is the radical axis of $\mathscr{K}_1$ and $\mathscr{K}_2$.

If $\mathscr{K}_2$ is interior to $\mathscr{K}_1$, but not concentric with it, then we can find the radical axis of the two circles as follows: Let $\mathscr{K}_3$ be a circle intersecting $\mathscr{K}_1$ and $\mathscr{K}_2$. Then the radical axis of $\mathscr{K}_1$ and $\mathscr{K}_3$ intersects the radical axis of $\mathscr{K}_2$ and $\mathscr{K}_3$ in some point P. The perpendicular from P to the line joining the centers of $\mathscr{K}_1$ and $\mathscr{K}_2$ is the required radical axis of $\mathscr{K}_1$ and $\mathscr{K}_2$. The proof is similar to that given in the case when $\mathscr{K}_1$ and $\mathscr{K}_2$ are exterior to each other.

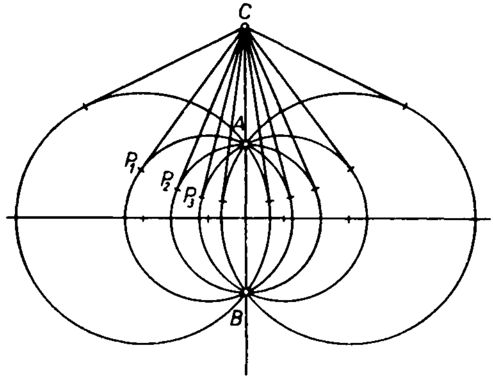

FIG. 25.

Definition 12. The set of circles $\mathscr{K}_e$ passing through two fixed points A and B is called an elliptic pencil.

According to Theorem 14 the line AB is the common radical axis for all circles of the pencil. Thus the tangents from a point C on the radical axis to the circles of the pencil have equal length (Fig. 25), $CP_1 = CP_2 = CP_3 = \cdots$. This means that the points P_r of tangency lie on a circle orthogonal† to all the circles in $\mathscr{K}_e$. This fact justifies:

Definition 13. The set $\mathscr{K}_h$ of all circles with centers on the radical axis p_e of $\mathscr{K}_e$ which are orthogonal to the circles of $\mathscr{K}_e$ is called a hyperbolic pencil (Fig. 26).

The common radical axis of the circles in $\mathscr{K}_h$ is the perpendicular bisector of AB (Fig. 26). The proof of this assertion is left to the reader.

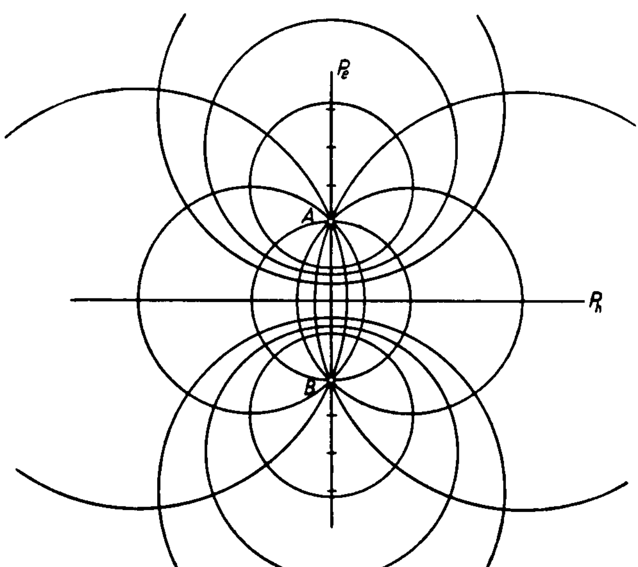

FIG. 26.

† Two intersecting circles are said to be orthogonal if their tangents at a point of intersection are perpendicular.

II. Inversion

Let $\mathscr{K}$ be a circle with radius r and center M. Let $P \neq M$ be a point in the plane. Then there is exactly one point Q on the ray MP such that

$$MP \cdot MQ = r^2. \tag{1}$$

The equality (1) defines a one-to-one transformation of the points of the plane other than M. This transformation associates with points in the interior of $\mathscr{K}$ (minus the point M) points in its exterior and conversely, and it leaves the points of $\mathscr{K}$ fixed (Fig. 27).

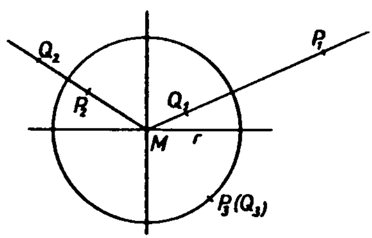

FIG. 27.

The transformation just described is called inversion or reflection in the circle $\mathscr{K}$. Q is called the reflection of P and P is called the reflection of Q (in the circle $\mathscr{K}$).

Figure 28 shows how to construct the reflection Q of a point P (outside $\mathscr{K}$). Draw a semicircle with MP as diameter. If R is the point of intersection of this semicircle and the circle

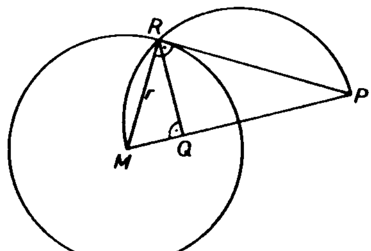

FIG. 28.

$\mathscr{K}$, then Q is the foot of the perpendicular from R to MP. This follows from the fact that MPR is a right triangle and, by Euclid's theorem,

$$MP \cdot MQ = r^2.$$

By consulting Fig. 28 the reader will readily see how to find the reflection P of a preassigned point Q (in the interior of $\mathscr{K}$).

We shall now compute the coordinates of Q given the coordinates of P.

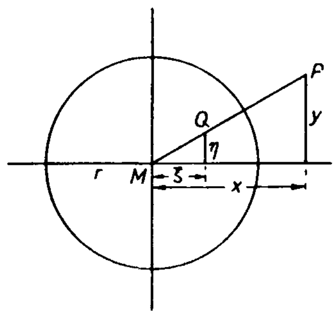

FIG. 29.

Let P have coordinates x, y. Let Q have coordinates ξ, η. Then (Fig. 29)

$$\frac{x}{y} = \frac{\xi}{\eta}. \tag{2}$$

Since $MP \cdot MQ = r^2$, we have

$$(\xi^2 + \eta^2)(x^2 + y^2) = r^4. \tag{3}$$

Substituting the value of η from (2) in (3) we get

$$\xi^2 + \xi^2 \cdot \frac{y^2}{x^2} = \frac{r^4}{x^2 + y^2},$$

that is,

$$\left.\begin{aligned} \xi &= \frac{xr^2}{x^2 + y^2}, \\ \eta &= \frac{yr^2}{x^2 + y^2}. \end{aligned}\right\} \tag{4}$$

Obviously, one can express x and y in terms of ξ and η:

$$x = \frac{\xi r^2}{\xi^2 + \eta^2},$$
$$y = \frac{\eta r^2}{\xi^2 + \eta^2}. \tag{5}$$

Formulas (4) and (5) are fully symmetric. Hence, if we substitute for x and y in (4) the coordinates of an interior point, then ξ and η are the coordinates of its reflection (in the exterior).

From Fig. 30 we can read off the proof of the following theorem.

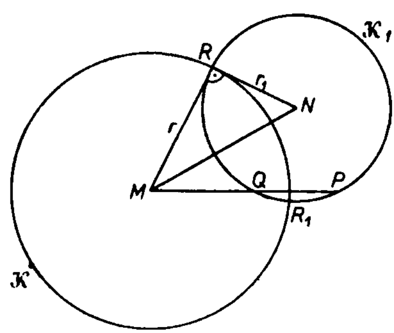

FIG. 30.

Theorem 15. Reflection takes orthogonal circles into themselves. More specifically, if $\mathcal{K}_1$ and $\mathcal{K}$ are orthogonal circles, then the reflection of $\mathcal{K}_1$ in $\mathcal{K}$ coincides with $\mathcal{K}_1$ and the reflection of $\mathcal{K}$ in $\mathcal{K}_1$ coincides with $\mathcal{K}$.

Consider Fig. 30. The radii r (of $\mathcal{K}$) and r_1 (of $\mathcal{K}_1$) are perpendicular. RN is tangent to $\mathcal{K}$ and RM is tangent to $\mathcal{K}_1$ with center N and radius r_1. Thus $\mathcal{K}$ and $\mathcal{K}_1$ are orthogonal.

Now if P and Q are the points in which a ray from M intersects $\mathcal{K}_1$, then $MP \cdot MQ = r^2$. Hence P and Q are reflections of one another in $\mathscr{K}$. This means that reflection in $\mathscr{K}$ takes a point in $\mathscr{K}_1$ into a point in $\mathscr{K}_1$. Briefly, the reflection of $\mathscr{K}_1$ in $\mathscr{K}$ coincides with $\mathscr{K}_1$. A similar argument shows that the reflection of $\mathscr{K}$ in $\mathscr{K}_1$ coincides with $\mathscr{K}$.

We shall now study the reflection of any circle or line in a circle $\mathscr{K}$.

Consider the equation

$$p(x^2 + y^2) + ax + by + c = 0. \tag{6}$$

If $p \neq 0$, then (6) is the equation of a circle. If $p = 0$, then (6) represents a line.

To obtain the reflection of the circle (or line) (6) in $\mathscr{K}$ (with center at the origin and radius r) we put the transformation formulas (5) in (6). The result is

$$\frac{r^4 p \cdot (\xi^2 + \eta^2)}{(\xi^2 + \eta^2)^2} + \frac{a\xi r^2}{\xi^2 + \eta^2} + \frac{br^2\eta}{\xi^2 + \eta^2} + c = 0.$$

Multiplication by $\xi^2 + \eta^2$ and division by r^2 yield

$$pr^2 + a\xi + b\eta + \frac{c}{r^2}(\xi^2 + \eta^2) = 0. \tag{7}$$

Equation (7) is again the equation of a circle or a line. More precisely, we have the following possibilities:

$p = 0$ *and* $c \neq 0$. Then (6) represents a line which does not pass through the origin and the image (7) is a circle passing through the origin.

$p \neq 0$ *and* $c = 0$. Then the original object is a circle passing through the origin and the image is a line not passing through the origin.

$p = c = 0$. Then both (6) and (7) represent lines passing through the origin.

The following table is a summary of our findings.

The reader is invited to construct examples illustrating these findings.

Original object	Image	Conditions
Circle not through M	Circle not through M	$p \neq 0; c \neq 0$
Circle through M	Line not through M	$p \neq 0; c = 0$
Line not through M	Circle through M	$p = 0; c \neq 0$
Line through M	Line through M	$p = 0; c = 0$

Figures 32 and 33 show examples of reflections of lines in $\mathscr{K}$.

Theorem 16. Reflection in a circle preserves angles.

This means that if two circles (or lines) form an angle α, then their image circles (or image lines) form the same angle.

Thus let $\mathscr{K}_1$ and $\mathscr{K}_2$ be two circles which intersect at P and form an angle α. Their tangents t_1 and t_2 at P form the same angle. The images of $\mathscr{K}_1$ and t_1 are two tangent circles (why?). One of the images may be a line (when does this happen?).

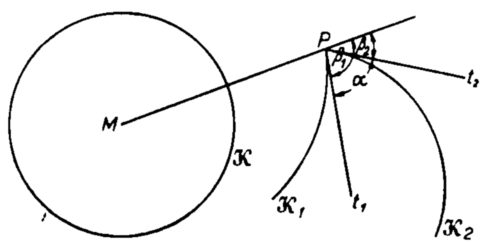

FIG. 31.

We see that in order to prove our theorem it suffices to prove that reflection preserves the angle between two intersecting lines. In fact, it suffices to prove that reflection preserves the angle between *every* line through P and the ray PM. For if (Fig. 31) the image angles of β_1 and β_2 are preserved then so is their difference

$$\beta_1 - \beta_2 = \alpha.$$

We consider first the case when P (Fig. 32) is a point in the interior of $\mathscr{K}$ and g is a line through P which intersects $\mathscr{K}$ in A and B. According to the table above the image of g is a "circle through M." Since A and B are not affected by re-reflection in $\mathcal{K}$, the image circle $\mathcal{K}_1$ is uniquely determined by the points A, B, M. The center M_1 of the image circle lies on the perpendicular bisector of AB. Hence the line perpendicular to MM_1 at M is tangent to the image circle. This line forms the same angle with MP as the line AB:

$$\alpha = \beta.$$

Continuing, let Q denote the second point of intersection of the image circle with the line MP. Then Q is the reflection of P in $\mathcal{K}$. The angle β_1 at Q is equal to the angle β (both are

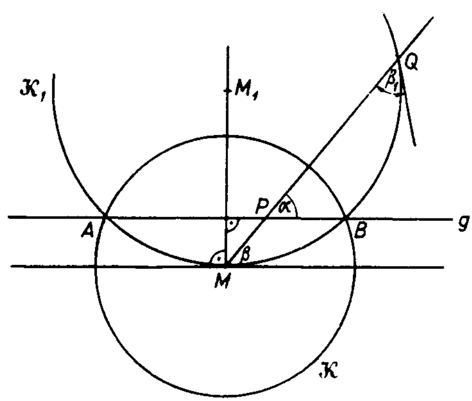

FIG. 32.

angles between a chord and tangents at the endpoints of the chord) which, in turn, is equal to α. Hence

$$\beta_1 = \alpha.$$

But this is what we wished to prove: reflection in $\mathcal{K}$ takes PQ into QP and PB into the circular arc $\overset{\frown}{QB}$. The angles between the original rays and their images are the same. More is true: the "orientation" of the angles is reversed in the sense that rotation from PB to PQ is counterclockwise whereas the rotation from the tangent to $\overset{\frown}{QB}$ to QP is clockwise.

A similar argument (we leave it to the reader) takes care of the case when g intersects $\mathscr{K}$ but P is outside $\mathscr{K}$. In the remaining case—P is outside $\mathscr{K}$ and g does not intersect $\mathscr{K}$ (Fig. 33)—we reason as follows: Let R be the foot of the perpendicular from M to g and let S be the reflection of R. According to our table, the image of g is a circle through M. This circle must pass through S and must be symmetric with respect to the line RM.

As before, we see that

$$\alpha = \beta, \qquad \beta = \beta_1$$

i.e., $\alpha = \beta_1$, so that in this case inversion also preserves the angle.

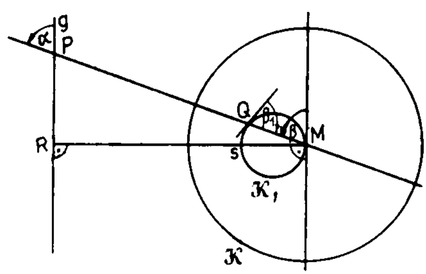

FIG. 33.

III. Cross Ratio

We now introduce an important concept of projective geometry which is probably known to many readers.

Definition 14. The cross ratio of four collinear points A, B, C, D is the quotient

$$(ABCD) = \frac{AC \cdot BD}{BC \cdot AD} = \frac{AC}{BC} : \frac{AD}{BD}. \tag{8}$$

Here AC, etc., denote *directed* segments (Fig. 34). For example, if C is between A and B and B is between C and D (Fig. 34a) then AC, AD, BD are positive and BC negative. Thus in this case the cross ratio is negative.

If C and D divide AB internally and externally in the same ratio, i.e., if

$$\left|\frac{AC}{CB}\right| = \left|\frac{AD}{DB}\right|,$$

then the cross ratio $(ABCD)$ is seen to have the value -1.

In later applications we shall find it useful to know what effect moving the points C and D has on the cross ratio (A and B are held fixed).

We already mentioned the fact that in the case depicted in Fig. 34a the cross ratio is negative.

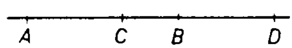

FIG. 34a.

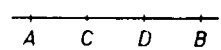

FIG. 34b.

If B and D coincide the cross ratio is 0. If C and D lie between A and B, the cross ratio is positive; its value is <1 if C is between A and D.

If C approaches B, or D approaches A, then the absolute value of the cross ratio increases indefinitely.

The definition of the cross ratio implies the following result:

Theorem 16a. Interchanging A and B changes the cross ratio $(ABCD)$ to its reciprocal. Interchanging A and B as well as C and D leaves $(ABCD)$ unchanged. Briefly,

$$\delta = (ABCD) = (BADC);$$

$$1/\delta = (BACD).$$

CHAPTER 5

The Poincaré Model

After these preliminaries we propose to develop the basic relations of noneuclidean geometry with the aid of a "model." We shall find it helpful to return to the considerations which arose in connection with our discussion of Hilbert's fundamental assumptions concerning the "three systems of things" (p. 8).

Since "point," "line," "plane" are undefined terms, we are free to assign to them any meanings which accord with some or all of the axioms. In this way we create a model of some or all of the axioms. We followed this idea in Example I (p. 10) when we developed a model of a pseudogeometry which accorded with a number of groups of Hilbert's axioms but obviously violated the parallel axiom. The model of noneuclidean geometry of example I is due to Felix Klein (1849–1925). We propose to investigate a model of noneuclidean geometry which is in some respects simpler than that of F. Klein. This latter model is due to the French mathematician Poincaré (1854–1912). (Henri Poincaré was a cousin of the famous politician.)

We introduce a rectangular coordinate system in the plane, designate the halfplane $y > 0$ as the "hyperbolic plane" and its points as "hyperbolic points." Thus the points on the x-axis do not belong to our "h-plane."

As "hyperbolic lines" we designate semicircles in the upper halfplane orthogonal to the x-axis and rays in the upper halfplane perpendicular to the x-axis.

Thus g_1, g_2, g_3, g_4 in Fig. 35 are h-lines and g, while a "true" line, is not an h-line since it is not perpendicular to the x-axis.

A and B are h-points of the h-line g_3. U and V are not h-points since they do not belong to our model. We shall sometimes refer to points on the x-axis as "boundary points."

We may identify the h-points and h-lines with the "things" in Hilbert's system to the extent to which they satisfy its axioms. If we retain the prefix "h-" it is to avoid confusion with euclidean elements.

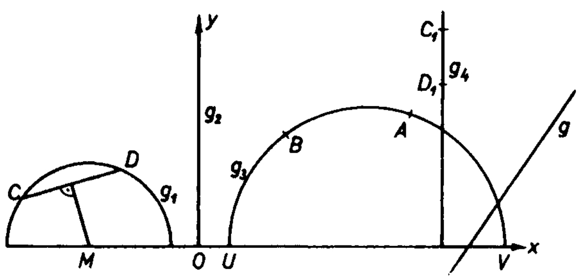

FIG. 35.

When dealing with euclidean points, lines, segments, semicircles, etc., we shall sometimes prefix them with the letter e.

It is easy to prove that the h-points and h-lines satisfy Hilbert's axioms of connection and order. We leave the proofs to the reader. By way of example we prove that two h-points determine a unique h-line.

Thus if the e-line determined by the points C, D (Fig. 35) is perpendicular to the x-axis, then its ray in the halfplane $y > 0$ is the h-line determined by the points C, D. If the e-segment CD is not perpendicular to the x-axis, then its euclidean perpendicular bisector intersects the x-axis. Denote the point of intersection by M. Clearly, there is exactly one e-semicircle with center M passing through C and D and orthogonal to the x-axis. Thus there is exactly one h-line (g_1 in Fig. 35) passing through C and D.

Before discussing the axioms in group IV we shall show quickly that the parallel axiom (V) does not hold in our model. In fact, all the *h*-lines through the *h*-points P in Fig. 36 are "parallel" to the *h*-line g; indeed, they have no point in common with g. In particular, this is true of h_1 and h_2, for U and V do not belong to our geometry. h_1 and h_2 are called bounding parallels to g and P and the remaining parallels are called hyperparallels to g at P.†

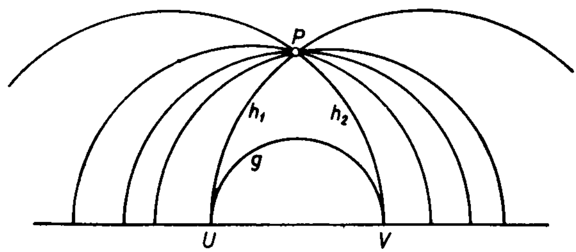

FIG. 36.

This is a very significant result. In the sequel we shall give an appropriate definition of congruence and verify that the axioms in groups III and IV hold in our model. If we assume this for the moment, then we have a proof that the parallel axiom cannot be deduced from the remaining axioms. Euclid was right and generations of mathematicians were on the wrong track. Indeed, a correct proof of the axiom of parallels must be based on the axioms in groups I–IV only. Since, as will be shown in the sequel, these axioms hold in our model, every deduction from them must likewise hold in our model. But it is not possible to prove that through a point P in our model there passes only one line parallel to a given line, for, as we have just shown, the number of such lines is infinite. Thus the parallel axiom turns out to be independent of the other axioms.

This then is the peculiar solution of a problem which defied the best efforts of mathematicians for over 2000 years.

† If g is an *e*-ray then one of the bounding parallels to g at P is the *e*-ray through P. For a general definition of these concepts see footnote on p. 63. (Tr.)

The above argument is an example of how one demonstrates the independence of an axiom. We have already pointed out (cf. p. 5) that it is desirable for a system of axioms to be independent, i.e., to have the property that no axiom in it can be derived from the others. While limitations of space do not allow us to prove the independence of the system of axioms used in this book we wish to point out that proofs of independence follow the method just presented in the case of the parallel axiom. Thus, for example, in proving Axiom III, 6 independent of the other congruence axioms Hilbert defined a relation of "pseudocongruence" which satisfied Axioms III, 1–5, but not III, 6.

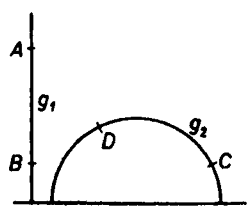

FIG. 37.

We shall now define "congruence" in our model. We do this by introducing a suitable definition of "length" and by defining two segments as being congruent if they have the same hyperbolic length.

This is easier said than done. The "ordinary" notion of length if useless for our purposes.

Indeed, if Axiom III, 1 is to hold then, given a segment AB (Fig. 37), a line g_2 ,and a point C on g_2, it must be possible to find on the rays determined by C on g_2 two points D_1 and D_2 such that

$$AB \equiv CD_1, \qquad AB \equiv CD_2.$$

If we define segments as congruent when their "ordinary" lengths are the same, then Axiom III, 1 is not always satisfied. Thus, in Fig. 37 we could find a point D_1 on g_2 such that the arc CD_1 has the same length as the segment AB but it is not possible to find on the other side of the h-line g_2 an h-point D_2

with the corresponding property. The only point that could qualify for the role of D_2 is a point on the euclidean circle determined by the "line" g_2 which is in the lower halfplane. But such a point does not belong to our mode. We repeat: Ordinary length cannot be used to define an acceptable notion of congruence in our model.

Definition 15. Let AB be an h-segment on an e-circle which meets the x-axis in the e-points U and V. Let A', B' be the projections of A, B on the x-axis (Fig. 38). Then the hyperbolic length of AB is defined as

$$L_B^A = \tfrac{1}{2}\log(A'B'UV). \tag{1}$$

If A_1B_1 lies on the e-ray through the e-point U, on the x-axis, then its hyperbolic length is defined as

$$L_{B_1}^{A_1} = \log\frac{A_1U_1}{B_1U_1}. \tag{2}$$

Two hyperbolic segments are said to be congruent if the absolute values of their lengths are the same.

Angles are measured in the euclidean way.†

Thus the measure of the angle formed by two h-rays is the radian measure of the angle formed by their tangents.

† We wish to point out to those familiar with hyperbolic geometry that the present notion of hyperbolic length differs only formally from the commonly accepted one. Indeed, the usual definition of hyperbolic length is

$$L_B^A = \log(ABUV), \tag{3}$$

where A, B, U, V are complex numbers. If ϑ_1 and ϑ_2 are the angles AMV and BMV, then in place of (3) we can write

$$L_B^A = \log\frac{\tan(\vartheta_2/2)}{\tan(\vartheta_1/2)}.$$

Since

$$\tan\frac{\vartheta_\nu}{2} = \sqrt{\frac{1-\cos\vartheta_\nu}{1+\cos\vartheta_\nu}},$$

(1) follows.

The advantage of our formulation is methodological: it avoids the use of complex numbers.

If B' lies between U and A', then $\delta = (A'B'UV) > 1$ and $\log \delta > 0$ (cf. pp. 45, 46).

If A and B coincide, then $\delta = 1$ and the h-lengths of AB is 0, as is to be expected. As A moves along the h-line toward V, δ and $\log \delta$ increase indefinitely. Hence the point V is "infinitely distant." If A lies between U and B, then $0 < \delta < 1$ and the h-length is negative. As A moves along the h-line toward U, δ decreases indefinitely, $\lim_{A'U} \delta = 0$. Hence

$$\lim_{A \to U} L_A^B = -\infty.$$

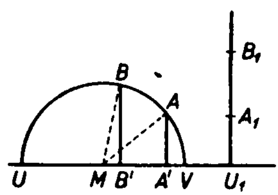

FIG. 38.

We shall now show that the axioms in group III hold in the h-plane.

To prepare for this task we study the behavior of the h-length of a segment under various geometric transformations. It is clear from the definition that δ is unchanged under a translation which takes an h-line into an h-line, i.e., under a translation parallel to the x-axis. But this means that such transformations preserve h-length.

Another transformation which preserves h-length is a similarity transformation with center M on the x-axis (Fig. 39). Clearly, the h-segment AB and its image $A'B'$ have the same h-length. This follows from the fact that the segments

$$A'U, \quad B'U, \ldots$$

and

$$A_1'U_1', \quad B_1'U_1', \ldots$$

differ only by a proportionality factor which is canceled in the process of forming the cross ratio.

Equation (2) (in Definition 15) implies the analogous statement for a segment on an e-ray under a similarity transformation with center U (Fig. 39).

Finally, h-length is preserved in absolute value under reflection in an h-line. To show this we show that δ is unchanged under reflection of a segment in a circle. Thus, choose a coordinate system with origin at the center O of the circle of inversion (radius r) (Fig. 40). Let M be the center of the euclidean circle (coordinates m, O, radius ρ) containing the hyperbolic segment AB. The h-length of AB is

$$L_B^A = \tfrac{1}{2}\log\delta = \tfrac{1}{2}\log(A'B'UV). \tag{4}$$

We shall show that the cross ratio δ of the reflections of A', B', U, V is equal to δ.

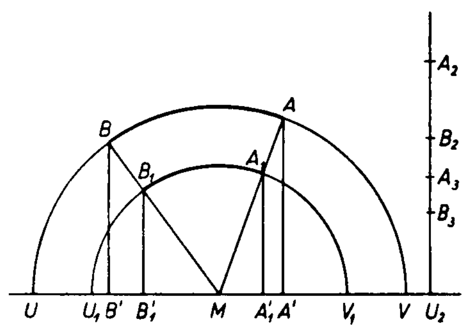

FIG. 39.

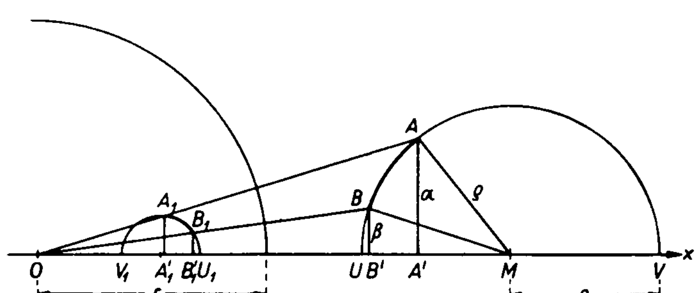

FIG. 40.

Let U, V be the points of intersection of the image circle with the x-axis. Here U_1 is the reflection of U and V_1 is the reflection of V. Let A_1', B_1' be the projections of A_1, B_1. If (a, α) and (b, β) are the coordinates of A and B, then

$$\alpha^2 = \rho^2 - (m - a)^2.$$

Hence

$$a^2 + \alpha^2 = \rho^2 + 2am - m^2. \tag{5}$$

Similarly

$$b^2 + \beta^2 = \rho^2 + 2bm - m^2. \tag{6}$$

Using formulas (4) (p. 40) we can compute the coordinates of the image points. The results, with $a^2 + \alpha^2$ and $b^2 + \beta^2$ replaced by the appropriate expressions in (5) and (6), are give in the following table.

Original object		Image	
Point	x-coordinate	Point	x-coordinate
U	$m - \rho$	U_1	$r^2/(m - \rho)$
V	$m + \rho$	V_1	$r^2/(m + \rho)$
A'	a	A_1'	$r^2a/(\rho^2 + 2am - m^2)$
B'	b	B_1'	$r^2b/(\rho^2 + 2bm - m^2)$

The cross ratio of the image is

$$\delta_1 = (A_1'B_1'U_1V_1)$$

and its value is

$$\delta_1 = \frac{\left(\dfrac{a}{\rho^2 + 2am - m^2} - \dfrac{1}{m - \rho}\right)\left(\dfrac{b}{\rho^2 + 2bm - m^2} - \dfrac{1}{m + \rho}\right)}{\left(\dfrac{a}{\rho^2 + 2am - m^2} - \dfrac{1}{m + \rho}\right)\left(\dfrac{b}{\rho^2 + 2bm - m^2} - \dfrac{1}{m - \rho}\right)}. \tag{7}$$

Removing the denominator in (7) we get

$$\delta_1 = \frac{(-a\rho - \rho^2 - am + m^2)(b\rho - \rho^2 - bm + m^2)}{(a\rho - \rho^2 - am + m^2)(-b\rho - \rho^2 - bm + m^2)}. \tag{8}$$

Dividing the first factor of the numerator and the second factor in the denominator by $-(m+\rho)$ and the other two factors by $\rho - m$ we get

$$\delta_1 = \frac{(a-m+\rho)(b-m-\rho)}{(a-m-\rho)(b-m+\rho)}. \tag{9}$$

By inserting in (4) the coordinates of the points A', B', U, V as listed in the table above we find $\delta_1 = \delta$.

Now Theorem 16a implies

$$\begin{aligned} \tfrac{1}{2}\log(A'B'UV) &= \tfrac{1}{2}\log(A_1{}'B_1{}'U_1V_1) \\ &= \tfrac{1}{2}\log(B_1{}'A_1{}'V_1U_1) = -\tfrac{1}{2}\log(A_1{}'B_1{}'V_1U_1). \end{aligned} \tag{10}$$

In view of the definition of h-length we have

$$L_B^A = L_{A_1}^{B_1} = -L_{B_1}^{A_1}.$$

Thus upon reflection of the segment AB in a circle we obtained a segment A_1B_1 such that the absolute value of the length of AB is the same as the absolute value of the length of A_1B_1. But then, by Definition 15, the two segments are congruent.

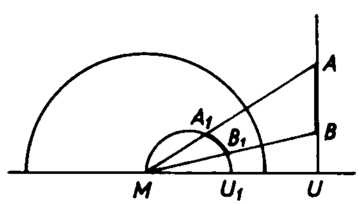

FIG. 41.

If AB lies on an e-ray (Fig. 41), then reflection in a circle again yields an image A_1B_1 whose h-length is the negative of the h-length of the original segment. The proof is easier than in the case just treated and we leave it to the reader.

To sum up:

Theorem 17. The h-length of a segment is unchanged by a parallel translation (parallel to the x-axis) and a similarity transformation (with center on the x-axis) and it changes its sign under a reflection in an h-line.

Before we check the validity of the congruence axioms (group III) in our model we must establish the existence of a perpendicular bisector of a segment AB. Thus let C (Fig. 42) be the point of intersection of the e-line AB with the x-axis; CD the tangent to the e-semicircle containing AB; h the semicircle

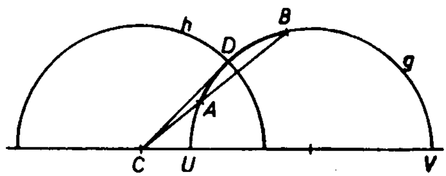

FIG. 42.

with center C and radius CD (in the halfplane $y > 0$). Then h is an h-line perpendicular to g at D. According to Theorem 15, A and B are reflections of each other in h. Since D is its own reflection in h, we have by Theorem 17: $AD \equiv BD$. Hence the h-line h is the perpendicular bisector of AB.

We are now in a position to verify Aixom III, 1; namely, let AB be a given h-segment and C a point on a given h-line g (Fig. 43). By means of a suitable translation and similarity

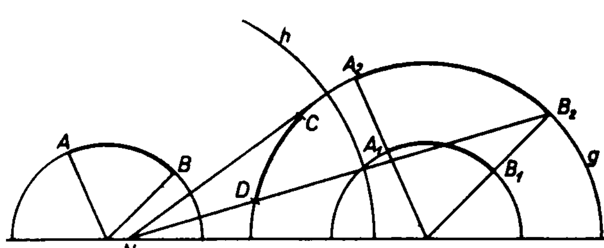

FIG. 43.

transformation we can find a segment A_2B_2 on g congruent to AB. This is clear from Fig. 43. Upon reflection of A_2B_2 in the perpendicular bisector h of A_2C we obtain the segment CD congruent to AB. Reflection of CD in the perpendicular to g

at C yields the segment CE congruent to AB.† The points C, D, E correspond to the points A', B_1', B_2' of Axiom III, 1.

We leave it to the reader to show how the argument must be altered when the carrier of the segment AB is an e-ray.

The validity of Axioms III, 2 and III, 5 in our model follows from Definition 15. The validity of Axiom III, 4 in our model can be read off from Fig. 44. Thus, let α be a given angle which

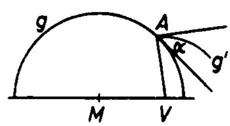

FIG. 44.

is to be laid off at a given line g so that its vertex is at a given point A of g. It is clear that in each of the h-halfplanes determined by g there is exactly one h-ray g' emanating from A which yields an angle congruent to α. It remains to verify Axioms III, 3 and III, 6.

If B is an interior point of the segment AC, then the projection B' lies between A' and C'. The definition of cross ratio implies

$$(ABUV)\cdot(BCUV) = (ACUV).$$

But then

$$\log(ABUV) + \log(BCUV) = \log(ACUV),$$

so that

$$L_A^B + L_B^C = L_A^C. \tag{8}$$

It follows readily that Axiom III, 3 holds in our model.

Next we verify Axiom III, 6. Let ABC and $A_1B_1C_1$ be two triangles which agree on two sides and the enclosed angle. By a translation, similarity transformation, and at most two reflections we can take the points A, B, C into the points A_1, B_1, C_1. The proof resembles that used above to verify Axiom III, 1 and is left to the reader. Since all the required

† We have not included this reflection in Fig. 43 to avoid cluttering up the drawing.

transformations take segments and angles into congruent segments and angles, it follows that the triangles ABC and $A_1B_1C_1$ agree on all sides and angles, i.e., their sides and angles are h-congruent in pairs. Q.E.D.

We have seen that the congruence axioms hold in the Poincaré model. This means that all theorems based on the axioms in the first three groups hold in this model. This applies, in particular, to Theorems 5 through 9.

The two continuity axioms also hold in our model. Axiom IV, 2 carries over to h-segments without difficulty. That every segment is measurable in the sense of Axiom IV, 1 is apparent from the following argument: Let ε be the h-length of AA_1 (Axiom IV, 1) and λ the h-length of AB. Let

$$\lambda = n\varepsilon + \vartheta\varepsilon \qquad (\vartheta < 1,\ n \text{ integer}).$$

If we lay off the segments AA_1 on the h-line AB $n + 1$ times beginning at A, then by (8), B lies between A_n and A_{n+1}.

This completes the verification of axioms in groups I–IV in the Poincaré model.

CHAPTER 6

Elementary Theorems of Hyperbolic Geometry

Except for the parallel axiom all axioms of Hilbert hold in the Poincaré model. In place of Axiom V we can put:

Va. The noneuclidean axiom of parallels. Through a point not on a line there are at least two parallel lines.

The geometry based on the axioms in groups I–IV and Axiom Va is called hyperbolic. It is this geometry which was first investigated by Bolyai and Lobachevski. It is also known as *the* noneuclidean geometry. However, prevalent usage refers to this geometry as hyperbolic and to any geometry which deviates from euclidean geometry as "noneuclidean." Hence hyperbolic geometry is one of many noneuclidean geometries.

Hyperbolic geometry is a particularly important noneuclidean geometry. It shares with euclidean geometry all of "absolute geometry," i.e., the part of euclidean geometry based on the axioms in groups I through IV. As already noted many well-known theorems of elementary geometry belong to absolute geometry.

The Poincaré model shows that the concept of hyperbolic geometry is meaningful. Another model of hyperbolic geometry is the model of Example 1 (p. 10) due to F. Klein. This model has the disadvantage that angles in it can not be measured the euclidean way. Hence our preference for the Poincaré model.

In this chapter we propose to prove a number of elementary theorems of hyperbolic geometry.

Theorem 18. In hyperbolic geometry the sum of the angles in a triangle is less than two right angles.

In fact, by Theorem 11 the sum of the angles in a triangle is less than or equal to two right angles. If this sum were equal to two right angles in *all* triangles, then, in view of Theorem 12, the parallel axiom V would hold in our geometry. Since this is not the case, there must be at least one triangle with angle sum less than two right angles. But then we claim that the angle sum must be less than two right angles in *every* triangle. To justify this assertion we make use of the concept of defect used in the proof of Theorem 13. Thus if we divide a triangle by means of a transversal into two triangles, then the defect

$$\delta \mathrel{\overset{\cdot}{=}} \pi - (\alpha + \beta + \gamma)$$

of the whole triangle is the sum of the defects of the triangular parts. The same is true of a division of a triangle into more than two triangular parts. Now if ABC is a triangle with angle sum less than two right angles, i.e., with positive defect, then we can divide it into two triangular parts of which at least one has positive defect. By further subdivision of this triangle we can obtain triangles with arbitrarily small sides and positive defect.

Continuing, let $A_1B_1C_1$ be an arbitrary triangle. It is possible to choose a "subtriangle" in ABC above which has positive defect and for which there exists a congruent subtriangle in $A_1B_1C_1$ (Fig. 45; here the subtriangles in question

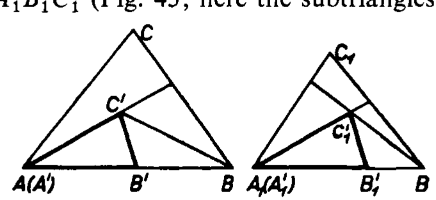

FIG. 45.

are $A'B'C'$ and $A_1'B_1'C_1'$). The latter must have positive defect. Since defect is additive, it follows that the defect of the triangle $A_1B_1C_1$ must be positive, Q.E.D.

The reader is invited to draw h-triangles in the model and measure their angle sum.

We observe that there exist in the Poincaré model triangles with arbitrarily small angle sum. Figure 46 shows some

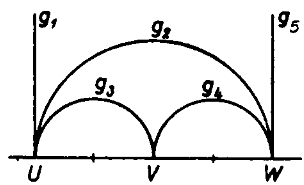

FIG. 46.

"limiting triangles" with angle sum 0, i.e., with defect π. Since the h-lines g_ν $(\nu = 1, 2, 3, 4, 5)$ do not intersect in h-points, limiting triangles are not "real" triangles with "real" vertices. However, it is possible to use a continuity argument to show that the angle sum of an h-triangle with vertices sufficiently close to U, V, W is arbitrarily small.

Theorem 19. Two hyperparallels (cf. p. 49) in the Poincaré model have exactly one common perpendicular.

Before proving this result we wish to point out that such a theorem does not hold in euclidean geometry where a perpendicular to a line is also perpendicular to any of its parallels.

Now we prove Theorem 19. Let g_1 and g_2 be two hyperparallels and let p be the associated radical axis (in the sense of euclidean geometry; cf. Chapter 4, Lemmas). Let O be the point of intersection of the radical axis with the x-axis (Fig. 47). The e-tangents from O to g_1 and g_2 (OA and OB) are of equal length, and the h-line h_1 through A and B is a common h-perpendicular of g_1 and g_2. We claim that h_1 is the only such common perpendicular.

This is easily proved. Indeed, let C, D be the points of intersection of the e-circle h_1 with the x-axis. The e-circles orthogonal to the e-circles g_1 and g_2 are the circles of the elliptic pencil determined by C, D. Of these only h_1 is an h-line since the centers of the other circles in the pencil are not on the x-axis. This proves Theorem 19 apart from special cases when an h-line is an e-ray. This part of the proof is left to the reader.

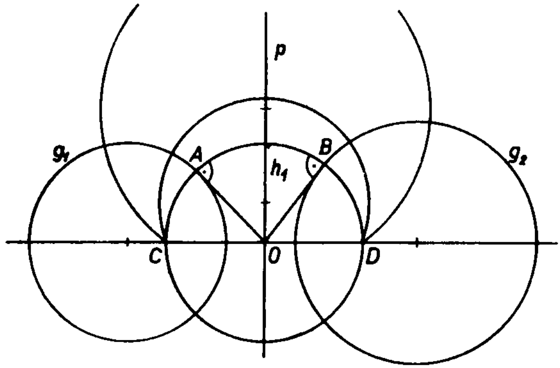

FIG. 47.

The statements and proofs of Theorems 18 and 19 exhibit an important difference. Theorem 18 asserts a property of hyperbolic geometry. Theorem 19 asserts a property of the Poincaré model. In proving Theorem 18 we draw on the axioms without using the model. In proving Theorem 19 we made use of the fact that the h-lines in the Poincaré model lie on euclidean circles (and lines). This enabled us to apply information from euclidean geometry on properties of radical axes, etc. However, we must bear in mind that the special nature of h-lines in the Poincaré model is a characteristic of that model and is not part of the axioms of hyperbolic geometry.

It follows that Theorem 19, while valid in the Poincaré model, need not, to begin with, be valid in hyperbolic geometry in general (i.e., in *every* model of its axioms). It so happens that two hyperparallels† always have a unique common perpendicular but this we have not proved.

It is easy to prove that in hyperbolic geometry two hyperparallels have *at most* one common perpendicular. In fact, the existence of two common perpendiculars would imply the existence of a quadrilateral with four right angles. But this is impossible since the angle sum in a triangle is less than two right angles (Theorem 18) and the angles sum in a quadrilateral is less than four right angles. Again, if g is a line and if we erect a perpendicular h to g at some point of g and then erect a perpendicular k to h at some point of h, then, by Theorem 8

† The definition of hyperparallels given on p. 49 is tied to the Poincaré model and of no use in general hyperbolic geometry; hence the need for a different approach.

Definition. Let a and b be two parallel lines. From a point P on a drop a perpendicular to b, which intersects b at Q. PQ divides the plane into a "right" halfplane and a "left" halfplane. Let c be any parallel to b passing through P, $c \neq a$. Let a' be the right ray of a (i.e., the ray on a emanating from P and lying in the right halfplane) and c' the right ray of c and suppose that $\angle(PQ, a') < \angle(PQ, c')$ for all c'. Then we say that a is a right bounding parallel to b at P.

The definition of a left bounding parallel to b at P is formulated in a similar manner.

Theorem. There is exactly one right bounding parallel to b at P and exactly one left bounding parallel to b at P.

Theorem. The property of being a right (left) bounding parallel to b at P is independent of P.

This means that if P and P' are points on a and a is the right (left) bounding parallel to b at P, then a is also the right (left) bounding parallel to b at P'. We may therefore speak of a being a right (left) bounding parallel to b.

Theorem. The relation of being a right (left) bounding parallel is symmetric, i.e., if a is a right (left) bounding parallel to b, then b is a right (left) bounding parallel to a.

Hence we may speak of a and b being right (left) bounding parallels.

Definition. a and b are called hyperparallels if they are parallels but are neither right nor left bounding parallels. (Tr.)

(a theorem of absolute geometry), g and k have no common point. However, this does not prove that it is always possible to find a common perpendicular to two given hyperparallels. That this is the case will be shown in the next chapter.

Theorem 20. Let g be an h-line in the Poincaré model with "boundary points" U and V. Then the locus of points equidistant from g is a euclidean circular arc passing through U and V (Fig. 48).

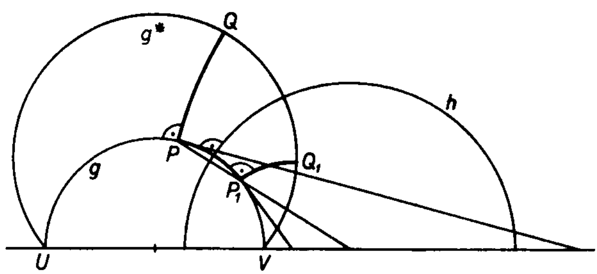

FIG. 48.

Before proving Theorem 20 we wish to make a relevant comment. We recall that a tacit assumption frequently employed in efforts to prove the parallel postulate was that the distance between two parallel lines was constant. We pointed out that this assumption was not part of the definition of parallel (i.e., nonintersecting) lines and so required proof. We are about to see, literally, that the points equidistant from a line need not lie on a line.

We now prove Theorem 20. Let g^* be an arbitrary euclidean arc through U and V. Let the perpendiculars to the h-line g at points P and P_1 intersect g^* in Q and Q_1. We wish to show that PQ and P_1Q_1 are h-congruent. To this end we draw the perpendicular bisector h of PP_1. h is orthogonal to the elliptic pencil determined by the points U, V. Under reflection in h, g, g^* and the x-axis go over into themselves (Theorem 15). P and P_1 are reflections of each other in h. The same is true of Q and Q_1. It follows that the segments PQ and P_1Q_1 are h-congruent (Theorem 17).

Thus the euclidean circular arc g^* is the locus of points which are δ $(=PQ)$ away from g and lie in the same h-halfplane as Q. Reflection of g^* in g yields another circular arc g^{**} with similar property.

These "equidistants" are not h-lines. Since the theorems of absolute geometry hold in our geometry, it follows that it is impossible to prove from the axioms of absolute geometry, i.e., without the use of the axiom of parallels, that parallel lines are necessarily equidistant lines. We emphasize that such conclusion must be based on the axiom of parallels.

Figure 48 makes it clear that h-congruent segments can differ radically in their euclidean length. The euclidean length of the perpendiculars from g to g^* decreases as we approach a boundary point (U or V).

We leave it to the reader to investigate the locus of points equidistant from a line which happens to be an e-ray.

We shall now try to transfer the notion of circle to the Poincaré model. We note that one of the properties of a circle is that it forms right angles with all of its radii. We shall try to find a curve in the h-plane which has the corresponding property.

Figure 49 makes it clear that if we reflect the h-lines through an h-point M_h in the x-axis, the result, in euclidean terms, is the elliptic pencil determined by M_h and its reflection M_h' in the x-axis. The curves orthogonal to this elliptic pencil are the circles of the corresponding hyperbolic pencil. We shall therefore designate these (euclidean) circles as circles in the Poincaré model. The euclidean center M_e (Fig. 49) of such a circle does not coincide with the point M_h which we designate as the hyperbolic center of the circle $\mathscr{K}$. This name is appropriate since $\mathscr{K}$ makes right angles with all the h-lines through M_h and, furthermore, the points of $\mathscr{K}$ are equidistant (in the sense of hyperbolic distance) from M_h. The latter assertion requires proof. One way to produce a proof is to compute the lengths in question. A much simpler way is the following: To show that $M_hA_1 \equiv M_hA_2$ we use reflection in the bisector (not shown in Fig. 49) of the angle $A_1M_hA_2$. Under this reflection

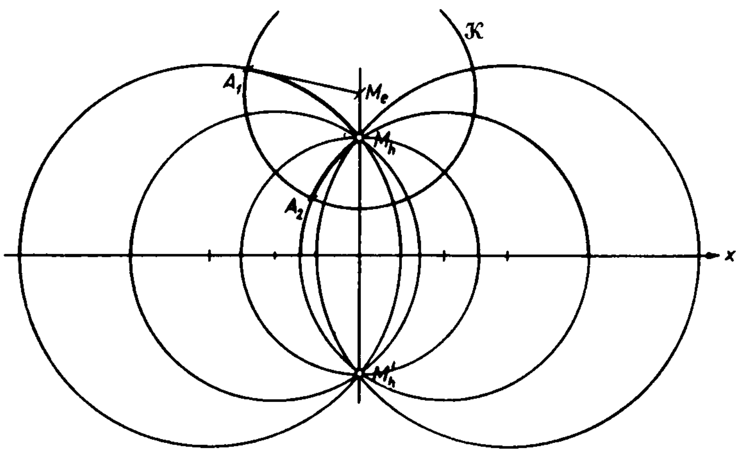

FIG. 49.

the orthogonal circle $\mathcal{K}$ goes over into itself, the point M_h remains fixed, and the h-line through M_hA_1 goes over into the h-line through M_hA_2. It follows that A_1 goes over into A_2 and we have $M_hA_1 \equiv M_hA_2$. Hence:

Theorem 21. In the Poincaré model the locus of points equidistant (in the sense of hyperbolic distance) from a point M_h is a euclidean circle. However, the euclidean center M_e and the hyperbolic center M_h do not coincide.

Theorem 22. In the Poincaré model there exist triangles which have no circumscribed circle.

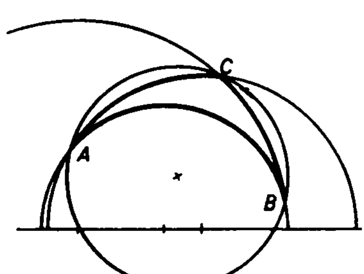

FIG. 50.

This can be immediately seen from Fig. 50. The h-circle through three points must coincide with the e-circle through these points. With proper choice of the points A, B, C the e-circle circumscribed on the e-triangle ABC will not lie completely in the upper halfplane and will thus not be an h-circle.

Theorem 23. In hyperbolic geometry two triangles with pairwise congruent angles are congruent (sixth congruence theorem).

The usual congruence theorems are proved without the use of the parallel axiom and are thus theorems of absolute geometry. We note that in high school study of geometry the cases saa and asa are lumped together. Indeed, if one makes use of the theorem on the sum of the angles in a triangle, then the case saa reduces to the case asa. However, if we do not make use of the axiom of parallels or, equivalently, the theorem on the angle sum in a triangle, then these cases must be dealt with separately. In this way we obtain five congruence theorems in absolute geometry.

Theorem 23 is a sixth congruence theorem valid in hyperbolic geometry but not in euclidean geometry. This theorem states that there are no similar triangles in hyperbolic geometry since triangles with (pairwise) equal angles are, as we are about to prove, congruent. This implies the important fact that theorems on proportions have no place in hyperbolic geometry. Thus the study of similarity is restricted to euclidean geometry.

Our proof of Theorem 23 will apply to hyperbolic geometry in general, i.e., to any realization of the axioms in groups I–IV and Va. This does not prevent us from drawing the figures needed in the proof using the Poincaré model.

Thus let ABC and $A'B'C'$ be two triangles with pairwise congruent angles. If AC were not congruent to $A'C'$, then we could find a point A_1 on the line AC such that $A_1C \equiv A'C'$. Assume, for definiteness, that $A_1C < AC$. Further let $CB_1 \equiv C'B'$ (Fig. 51). B_1 cannot coincide with B. For if $B_1 = B$, then the congruence $\triangle A'B'C' \equiv \triangle A_1B_1C$ (first congruence theorem) would imply $\triangle A'B'C' \equiv \triangle A_1BC$. Hence $\angle A_1BC \equiv \angle A'B'C' \equiv \angle ABC$, or $\angle A_1BC \equiv \angle ABC$. Since $A_1 \neq A$ and both are on the same side of BC, this would contradict Axiom III, 4.

Suppose $CB_1 > CB$ (Fig. 51). Then, by the triangle Axiom II, 4, the segment A_1B_1 must intersect the segment AB in an interior point D.

Since the angles α and α_1 (Fig. 51) are congruent corresponding angles, this contradicts Theorem 8, which is a theorem of absolute geometry.

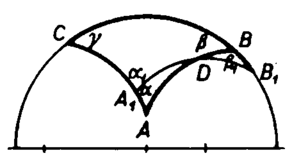

FIG. 51.

Suppose now that $CB_1 < CB$. Recalling that the defect δ of a triangle is the sum of the defects of its subtriangles, we have

$$\delta(ABC) = \delta(A_1B_1C) + \delta(A_1B_1A) + \delta(ABB_1).$$

Hence

$$\delta(ABC) > \delta(A_1B_1C).$$

Since $\triangle A'B'C' \equiv \triangle A_1B_1C$, this would mean that $\delta(ABC) > \delta(A'B'C')$, contrary to the assumption that the triangles ABC and $A'B'C'$ have pairwise congruent angles.

Thus the assumption $A_1 \neq A$ is false and the triangles ABC and $A'B'C'$ must be congruent, Q.E.D.

We see that the theory of similarity cannot be transferred to hyperbolic geometry. We investigate next the possibility of transferring the theory of area to hyperbolic geometry.

There is no use starting with the notion of a square since we cannot have a quadrilateral with four right angles. This makes it clear that in developing a theory of area we must follow an altogether different path.

In euclidean geometry the area function F considered on triangles has the following properties:

1. Congruent triangles have the same area.
2. If we divide a triangle ABC by means of transversals into a finite number of subtriangles $A_v B_v C_v$, then

$$F(ABC) = \sum F(A_v B_v C_v).$$

We know of a function in hyperbolic geometry which has these very properties. This function is the defect of a triangle:

$$\delta(ABC) = \pi - (\alpha + \beta + \gamma).$$

In addition to properties 1 and 2 above the defect has the following property: if two triangles have the same defect, then it is possible to subdivide them into a finite number of pairwise congruent subtriangles. For proof see [22].

These facts justify the following definition.

Definition 16. By the *area of a triangle* with angles α, β, γ we mean the defect

$$F(ABC) = \delta(ABC) = \pi - (\alpha + \beta + \gamma).$$

In euclidean geometry two triangles with equal base and height have equal area. As it stands, this theorem is false in hyperbolic geometry. However, a modified version of this theorem is true in hyperbolic geometry. Thus, let ABC be a triangle with sides a, b, c. Let D and E be the midpoints of the sides a and b. Call the line DE the midline corresponding to c (Fig. 16). Then we have the following theorem of euclidean geometry.

Theorem 24. Two triangles with the same base and midline have equal areas.

The proof is immediate, since the distance from the midline to the base is half the height.

Theorem 24 holds in hyperbolic geometry. In proving this result we shall make use of Fig. 16 but suggest that the reader draw the appropriate figure in the Poincaré model.

As was shown on p. 24 the quadrilateral $ABHG$ is a Saccheri quadrilateral with right angles at G and H. The angles GAB and HBA are congruent,

$$\angle GAB \equiv \angle HBA \equiv W/2.$$

Since all triangles with the same base and midline yield the same Saccheri quadrilateral with the same angle GAB, they must all have the same angle sum, and so the same defect, Q.E.D.

CHAPTER 7

Constructions

The issue of constructions in the Poincaré model has arisen already in the preceding chapters. Thus, e.g., on p. 56 we described a procedure for constructing the perpendicular bisector of a segment. This procedure enables us also to construct the midpoint of a segment.

We now list construction problems which we solved or can readily solve with the information at our disposal.

1. At a given point P on an h-line lay off a segment congruent to a given h-segment AB (p. 56).
2. Construct a common perpendicular to two hyperparallels (p. 61).
3. Double a given segment (cf. Fig. 42; this time we are given B, D and are required to find A).
4. Construct an h-circle with given hyperbolic center P and passing through a given point Q.
5. Given a circle in the h-plane find its h-center. (The solution appears in Fig. 52. M_h lies on all h-lines perpendicular to the given circle; hence, in particular on the lines g_1 and g_2.)
6. Halve a given angle.
7. Drop a perpendicular from a given point P onto a given h-line g.

One possible solution of problem 7 is shown in Fig. 53. g^* is the e-circle through U, V, P (center N). PT is perpendicular to NP. The e-circle with center T and radius TP yields an h-line

(h) perpendicular to the x-axis. Hence h is orthogonal to all the e-circles in the elliptic pencil determined by U, V and so, in particular, to g. Thus h is the required perpendicular.

An even simpler construction consists in obtaining the reflection P_1 of P in g. The required perpendicular is the h-line determined by P and P_1.

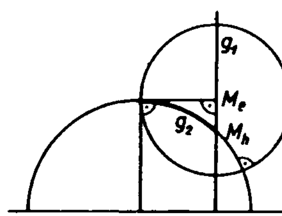

FIG. 52.

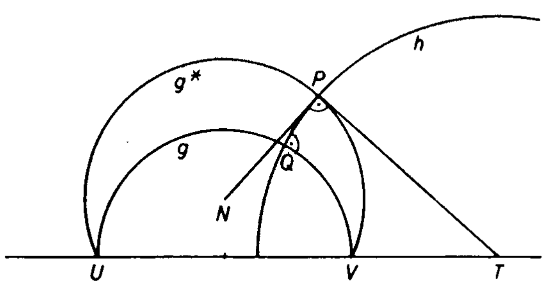

FIG. 53.

We could add to the number of elementary constructions which we are now in a position to carry out. Instead, we prefer to ask a critical question which may have occurred to the reader.

In these constructions we have used aids not given by hyperbolic geometry. Thus, for example, when halving a segment we use of a point C on the x-axis (Fig. 42). C is not an h-point. Again, in the process of constructing an h-circle we employed euclidean tangents. These are not h-lines. To repeat: in our constructions we have used aids not supplied by the geometry under investigation.

Let us make one thing clear. What we must do is not so much rule out various aids but rather form a clear picture of the use we make of such aids. For example, the customary tools of euclidean geometry are compass and ruler. However, it is possible to restrict construction possibilities by using ruler only or to widen them by using compass and insertion ruler. (Then it is possible, among others, to trisect an arbitrary angle.)

So far we have worked in the hyperbolic plane (of the Poincaré model) using "euclidean compass and euclidean ruler" and allowing the use of points not in the h-plane. In this way we quickly gained insight into the laws of hyperbolic geometry.

However, we are justified in asking the following question: what constructions are feasible with "hyperbolic compass and hyperbolic ruler"? Must we then use only points in the h-plane?

First let us clarify the terms "hyperbolic compass" and "hyperbolic ruler." A ruler serves to draw lines and a compass serves to draw circles. In "ruler and compass" constructions the construction elements are lines and circles. If the setting is to be the h-plane this means that it is possible (a) to draw an h-circle with given center and radius; and (b) to draw an h-line through two given points in the h-plane.

It is immaterial how the drawing of circles and lines is accomplished. We shall continue to draw an h-line through two h-points using a euclidean compass and placing its point at a point which does not belong to our model, namely, a point on the x-axis. We also use our former technique in drawing h-circles.

But—and this is what counts—a problem will be regarded as solved only if the required points can be shown to be intersections of h-lines and h-circles. The technical aspects of drawing the necessary h-lines and h-circles are irrelevant.

An example will clarify the matter. At one point we considered the problem of halving a segment (Fig. 42). It is clear that the procedure used on that occasion does not meet our present requirements. However, this problem can be handled with h-ruler and h-compass in much the same way in which the corresponding problem is handled in euclidean geometry.

Thus to bisect an h-segment AB draw two h-circles with centers A and B. Let C and D be the points of intersection of these circles. Then the h-line CD is the (perpendicular) bisector of the segment AB. We suggest that the reader carry out this construction in the Poincaré model. This procedure is technically more complicated than the procedure used in Fig. 42 but it uses only allowable aids in the sense just described.

The procedure used in solving the problem in h-geometry was identical with that used in solving the same problem in e-geometry except, of course, that in the first case we use h-segments and h-circles. This is true for all problems which belong to absolute geometry (laying off of segments and angles, halving an angle, dropping a perpendicular, etc.).

There exist problems characteristic of hyperbolic geometry and such that the restriction to "h-compass and h-ruler" makes for extra difficulties. Examples of such problems are:

(a) Construction of the common perpendicular to two hyperparallels.

The reader can readily see that the solution of this problem given above is not admissible from our present point of view.

(b) Construction of a triangle with prescribed angles.

Theorem 23 suggests that such a construction is possible.

(c) The construction of the "angle of parallelism."

By this we mean the following: Let P be a point, g an h-line, PQ the perpendicular from P to g (Fig. 54). Let g_1, g_2 be the bounding parallels through P relative to g. Then

$$\angle(g_1, PQ) \equiv \angle(g_2, PQ).$$

~~Indeed,~~ reflection in PQ takes one of the angles into the other. The angle $\alpha \equiv \angle(g_1, PQ)$ is called the angle of parallelism associated with the distance PQ. Observe that the simple construction of the angle of parallelism shown in Fig. 54 is objectionable because of the use of the "nonhyperbolic" points U and V in the determination of the lines g_1 and g_2.

While an admissible construction is possible (cf. e.g., [17], this reference also contains a solution of the construction problem (b)) it cannot be presented in our brief account. However, we shall present an admissible construction of the common perpendicular of two hyperparallels.

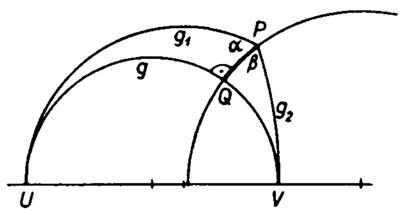

FIG. 54.

Such a construction based solely on the axioms of hyperbolic geometry and thus independent of a particular model is due to Hilbert. It makes use of a property of the Saccheri quadrilateral which we have not mentioned thus far, namely, that the line joining the midpoints E and F of the sides CD and AB of the Saccheri quadrilateral is perpendicular to these sides (Fig. 15). This is easy to prove using congruence considerations. Thus,

$$\triangle AFC \equiv \triangle BDF \qquad \text{(first congruence theorem)}$$

implies

$$CF \equiv DF.$$

This, in turn, implies

$$\triangle CEF \equiv \triangle DEF \qquad \text{(third congruence theorem).}$$

Hence

$$\angle CEF \equiv \angle DEF = R.$$

A similar argument shows that the angle at F is a right angle.

Thus we see that in order to construct the common perpendicular to two hyperparallels we must construct a suitable Saccheri quadrilateral with vertices on the given hyperparallels.

This construction is depicted in Fig. 55. While the setting is that of the Poincaré model, no special properties of this model are used; the construction can be repeated in any model of hyperbolic geometry. Thus, let a and b be the given hyperparallels;† A, C two points on a; B, D the feet of perpendiculars from A and C onto b. If $AB \equiv CD$, then $ABCD$ is already a Saccheri quadrilateral and the line joining the midpoints of AC and BD is the required common perpendicular.

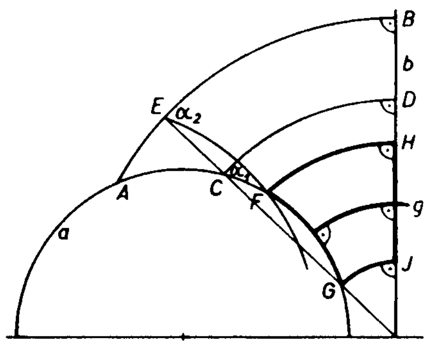

FIG. 55.

Otherwise, let $AB > CD$. Then there is a point E on AB such that

$$EB \equiv CD.$$

Let α_1 be the angle between CD and the ray on a which does not contain A. Lay off at the ray EB an angle $\alpha_2 \equiv \alpha_1$. Let F be the point of intersection of a with the side of α_2 other than EB.‡ Now lay off on a the segment $CG \equiv EF$ and drop the perpendiculars FH and GJ from F and G to b. Using congruence of triangles we see readily that the quadrilaterals $EBFH$ and $CDGJ$ are congruent. Hence

$$FH \equiv GJ.$$

Thus the quadrilateral $FGJH$ is a Saccheri quadrilateral and the line joining the midpoints of FG and HJ is the required common perpendicular g.

† The fact that b is an e-ray is without special significance.

‡ The existence of the point F requires proof. Cf. e.g. [32] or [11].

CHAPTER 8

Trigonometry

In order to deduce the relations between the sides and angles of a triangle in the h-plane we must make use of the so-called hyperbolic functions, which are related in the manner of the trigonometric functions. We define

$$\begin{aligned} \sinh t &= \frac{e^t - e^{-t}}{2} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots \\ \cosh t &= \frac{e^t + e^{-t}}{2} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \end{aligned} \tag{1}$$

In words: hyperbolic sine, etc.

$\sinh t$ and $\cosh t$ are connected by the relation

$$\cosh^2 t - \sinh^2 t = 1 \tag{2}$$

(cf. $\sin^2 t + \cos^2 t = 1$). This relation follows immediately from the definition (1). We also define

$$\begin{aligned} \tanh t &= \frac{\sinh t}{\cosh t} = \frac{e^{2t} - 1}{e^{2t} + 1} \\ \coth t &= \frac{1}{\tanh t}. \end{aligned} \tag{3}$$

We shall now rewrite the defining equation for h-length. We assume $BC = a$ in Fig. 56 to be a given h-segment.

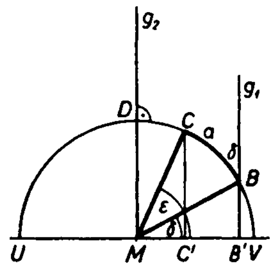

FIG. 56.

We denote the h-length of a by $\mathscr{A}$. Then

$$\mathscr{A} = L_B^C = \tfrac{1}{2}\log(B'C'UV) = \tfrac{1}{2}\log\frac{(1-\cos\varepsilon)(1+\cos\delta)}{(1+\cos\varepsilon)(1-\cos\delta)}.$$

Here $\varepsilon = \angle CMV$, $\delta = \angle BMV$.

Hence

$$e^{2\mathscr{A}} = \frac{(1-\cos\varepsilon)(1+\cos\delta)}{(1+\cos\varepsilon)(1-\cos\delta)}$$

and, in view of (3),

$$\tanh\mathscr{A} = \frac{e^{2\mathscr{A}}-1}{e^{2\mathscr{A}}+1} = \frac{\cos\delta-\cos\varepsilon}{1-\cos\delta\cos\varepsilon}. \tag{4}$$

In particular, if $\varepsilon = \pi/2$, then $C = D$ (Fig. 56) and

$$\tanh\mathscr{A} = \cos\delta. \tag{5}$$

In this case $\mathscr{A}$ is the hyperbolic length of BD and δ is the angle of parallelism (cf. p. 74) of BD with respect to g_2 (Fig. 56). This is so because BD is perpendicular to g_2, g_1 is one of the bounding parallels† to g_2 at B, and the angle between BC and g_1 is δ.

We shall now derive relations between the sides and angles in a right triangle. In the process we shall make use of the "principle of special position" which consists in using reflection, etc., to obtain a triangle congruent to the initial

† Cf. footnote, p. 63. (Tr.)

triangle but located more conveniently than the initial triangle. This procedure is justified by the fact that reflections, parallel translations, and similitudes do not change angles and (with the possible exception of sign) lengths.

Thus let $A_2B_2C_2$ be a right triangle with right angle at C (Fig. 57). Let U, V be the "boundary points" of the h-line

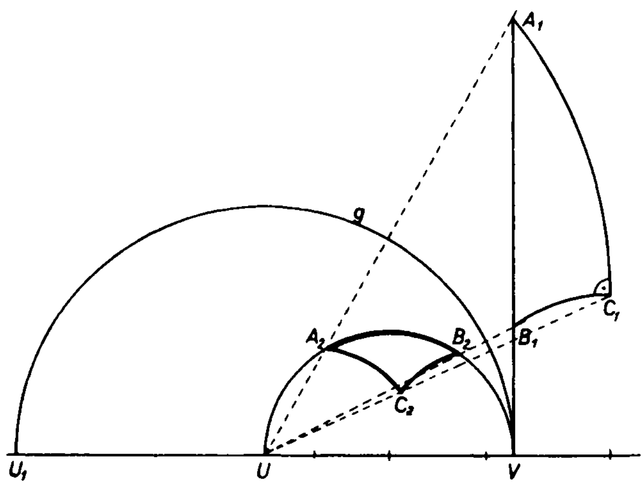

FIG. 57.

A_2B_2. Let g be the h-line with boundary points U_1, V, with $U_1U = UV$. Observe that inversion of a euclidean circle passing through the center of inversion U yields a euclidean line. Thus the reflection of the h-line A_2B_2 in g is the e-ray through V perpendicular to the x-axis. It follows that the reflection of the triangle $A_2B_2C_2$ in g is the triangle $A_1B_1C_1$ whose hypotenuse A_1B_1 lies on the e-ray through V perpendicular to the x-axis.

By applying a suitable parallel translation and similitude we can map the triangle $A_1B_1C_1$ on a congruent triangle ABC with hypotenuse AB on the y-axis and with vertex B at the point with rectangular coordinates (0, 1). The coordinates of A are $(0, \eta)$ (Fig. 58).

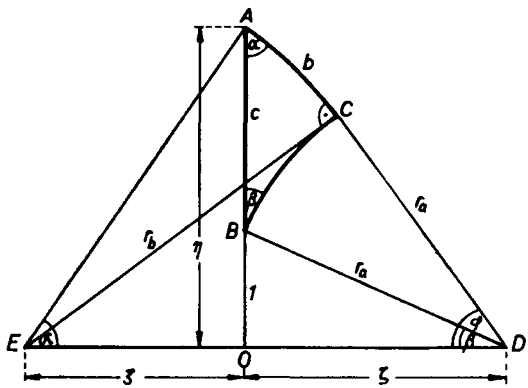

FIG. 58.

We shall now deduce relations between the sides and angles of the triangle ABC.

Let r_a and r_b be the (euclidean) radii of the e-circles containing the sides $a = BC$ and $b = AC$ and let $D(\zeta, 0)$, $E(\xi, 0)$ be their centers. Since $r_b{}^2 = \xi^2 + \eta^2$ and $r_a{}^2 = \xi^2 + 1$, application of Pythagoras' theorem to the triangle ECD gives

$$\xi^2 + \eta^2 = (\zeta + \xi)^2 - (\zeta^2 + 1),$$

or

$$\eta^2 + 1 = 2\xi\zeta. \tag{6}$$

In view of the definition on p. 51 the h-length of the hypotenuse c is

$$\mathscr{C} = \log \frac{\eta}{1} = \log \eta.$$

Hence

$$\cosh \mathscr{C} = \frac{e^{\mathscr{C}} + e^{-\mathscr{C}}}{2} = \frac{\eta^2 + 1}{2\eta}.$$

Since the (acute) angles α and β of our triangle are equal to the angles AEO and BDO in that order, we have

$$\cot \alpha = \xi/\eta, \qquad \cot \beta = \zeta/1 = \zeta,$$

and, in view of (6),

$$\boxed{\cosh \mathscr{C} = \cot \alpha \cot \beta.} \tag{7}$$

This is the first relation in our trigonometry of the h-plane (Poincaré model).

To obtain another relation we rewrite $\tanh \mathscr{C}$ with the aid of (6):

$$\tanh \mathscr{C} = \frac{e^{2\mathscr{C}} - 1}{e^{2\mathscr{C}} + 1} = \frac{\eta^2 + 1 - 2}{\eta^2 + 1} = \frac{\xi\zeta - 1}{\xi\zeta}. \tag{8}$$

Now according to (4) we can express $\tanh \mathscr{A}$ in terms of the angles with vertex at D (Fig. 58):

$$\tanh \mathscr{A} = \frac{\cos(\pi - \varphi) - \cos(\pi - \beta)}{1 - \cos(\pi - \varphi)\cos(\pi - \beta)} = \frac{\cos \beta - \cos \varphi}{1 - \cos \beta \cos \varphi}.$$

On the other hand (Fig. 58)

$$\cos \beta = \frac{\zeta}{\sqrt{1 + \zeta^2}}; \qquad \cos \varphi = \frac{\sqrt{1 + \zeta^2}}{\xi + \zeta}.$$

Thus

$$\tanh \mathscr{A} = \frac{\xi\zeta - 1}{\xi\sqrt{1 + \zeta^2}}.$$

Comparison with (8) yields

$$\boxed{\tanh \mathscr{A} = \tanh \mathscr{C} \cdot \cos \beta.} \tag{9}$$

Interchanging $\mathscr{A}$ with $\mathscr{B}$ and α with β we get

$$\boxed{\tanh \mathscr{B} = \tanh \mathscr{C} \cdot \cos \alpha.} \tag{10}$$

Formulas (7), (9), and (10) contain all of the trigonometry of the Poincaré model. All other formulas, including formulas for arbitrary triangles, can be deduced from them.

Thus (7) and (9) imply

$$\cosh^2 \mathscr{C} = \frac{\cot^2 \alpha}{\tan^2 \beta} = \frac{\cot^2 \alpha \cdot \tanh^2 \mathscr{A}}{\tanh^2 \mathscr{C} - \tanh^2 \mathscr{A}},$$

and further

$$\sinh^2 \mathscr{C} - \cosh^2 \mathscr{C} \cdot \tanh^2 \mathscr{A} = \cot^2 \alpha \cdot \tanh^2 \mathscr{A}.$$

In view of (2)

$$\sinh^2 \mathscr{C} - \tanh^2 \mathscr{A} - \tanh^2 \mathscr{A}_1 \sinh^2 \mathscr{C} = \cot^2 \alpha \cdot \tanh^2 \mathscr{A}.$$

From this relation we can compute $\sin^2 \alpha = 1/(\cot^2 \alpha + 1)$:

$$\sin^2 \alpha = \frac{\tanh^2 \mathscr{A}}{\sinh^2 \mathscr{C} - \tanh^2 \mathscr{A} \cdot \sinh^2 \mathscr{C}} = \frac{\tanh^2 \mathscr{A}}{\sinh^2 \mathscr{C}(1 - \tanh^2 \mathscr{A})}$$
$$= \frac{\sinh^2 \mathscr{A}}{\sinh^2 \mathscr{C}}.$$

Hence we find

$$\boxed{\sin \alpha = \frac{\sinh \mathscr{A}}{\sinh \mathscr{C}}.} \tag{11}$$

and, by interchanging α and β and $\mathscr{A}$ and $\mathscr{B}$:

$$\boxed{\sin \beta = \frac{\sinh \mathscr{B}}{\sinh \mathscr{C}}.} \tag{12}$$

Equations (11) and (7) yield

$$\sin^2 \alpha = \frac{\sinh^2 \mathscr{A}}{\cosh^2 \mathscr{C} - 1} = \frac{\sinh^2 \mathscr{A}}{\cot^2 \alpha \cot^2 \beta - 1},$$
$$\cos^2 \alpha \cot^2 \beta - \sin^2 \alpha = \sinh^2 \mathscr{A}$$
$$\cos^2 \alpha \cot^2 \beta + \cos^2 \alpha = \sinh^2 \mathscr{A} + 1 = \cosh^2 \mathscr{A}$$

$$\boxed{\cos \alpha = \sin \beta \cdot \cosh \mathscr{A}} \tag{13}$$

and, similarly,

$$\boxed{\cos \beta = \sin \alpha \cdot \cosh \mathscr{B}.} \tag{14}$$

Equations (10) and (11) imply

$$\tanh^2 \mathscr{B} = \tanh^2 \mathscr{C}(1 - \sin^2 \alpha) = \tanh^2 \mathscr{C} \frac{\sinh^2 \mathscr{C} - \sinh^2 \mathscr{A}}{\sinh^2 \mathscr{C}}.$$

Using (2) we get

$$\cosh^2 \mathscr{C}(1 - \tanh^2 \mathscr{B}) = \cosh^2 \mathscr{A}$$

and

$$\boxed{\cosh \mathscr{C} = \cosh \mathscr{A} \cdot \cosh \mathscr{B}.} \tag{15}$$

Finally, (10) and (15), with the aid of (2), give

$$\tanh^2 \mathscr{B} = \cos^2 \alpha \cdot \frac{\cosh^2 \mathscr{A} \cosh^2 \mathscr{B} - 1}{\cosh^2 \mathscr{A} \cdot \cosh^2 \mathscr{B}}$$

$$\cosh^2 \mathscr{A} \cdot \sinh^2 \mathscr{B}(1 - \cos^2 \alpha) = \cos^2 \alpha(\cosh^2 \mathscr{A} - 1),$$

$$\boxed{\tan \alpha \cdot \sinh \mathscr{B} = \tanh \mathscr{A}} \tag{16}$$

and similarly

$$\boxed{\tan \beta \cdot \sinh \mathscr{A} = \tanh \mathscr{B}.} \tag{17}$$

Using these formulas we can compute all the elements of the triangle ABC (i.e., $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, α, β) given any two of them.

To begin with these formulas hold for the triangle ABC. To show that they apply without modification to the original triangle $A_2B_2C_2$ we reason as follows: The lengths of the sides of these two triangles differ only in sign (cf. Theorem 17). On the other hand the formulas (7) and (9)–(17) are not affected by a change in sign of the quantities $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$. Indeed, formulas (7), (13), (14), and (15) involve only the even function cosh [$\cosh \mathscr{A} = \cosh(-\mathscr{A})$; cf. (1)], and in the remaining formulas the odd functions sinh [$\sinh(-\mathscr{A}) = \sinh \mathscr{A}$; cf. (1)] and tanh [$\tanh(-\mathscr{A}) = -\tanh \mathscr{A}$; cf. (3)] appear in pairs.

To sum up: Our formulas hold for every right triangle in the h-plane (Poincaré model).

The reader will be struck by the similarity of these formulas and the formulas of spherical trigonometry. In fact, all of our formulas can be obtained from the Neper formulas by replacing in the case of *sides* the trigonometric functions sin, cos, ... by sinh, cosh, ... There is a deep reason for this connection but we shall not comment on it in this book.

From formulas (7) we can read off the already established fact that the sum of the angles in a triangle in h-geometry is less than two right angles. Indeed, by (7),

$$\cosh \mathscr{C} = \cot \alpha \cot \beta > 1.$$

[That $\cosh \mathscr{C} > 1$ for $\mathscr{C} \neq 0$ can be read off from (1).] Hence

$$\tan(\pi/2 - \alpha) > \tan \beta.$$

Since $\tan \alpha$ is a monotonic function, it follows that

$$\frac{\pi}{2} - \alpha > \beta, \qquad \alpha + \beta + \frac{\pi}{2} < \pi.$$

On the other hand for small triangles $\cosh \mathscr{C}$ differs little from 1, as can be seen from the fact that

$$\lim_{\mathscr{C} \to 0} \frac{\tan[(\pi/2) - \alpha]}{\tan \beta} = 1.$$

Using a continuity argument we can assert that the sum of the angles in a right triangle will differ from π by an arbitrarily small amount provided the h-length $\mathscr{C}$ of the hypotenuse is chosen sufficiently small.

It is now easy to derive the trigonometric formulas for a triangle other than a right triangle.†

By dropping the perpendicular from C to the opposite side we get two right triangles to which we can apply formula (11):

$$\sin \alpha = \frac{\sinh \mathscr{H}_c}{\sinh \mathscr{B}}, \qquad \sin \beta = \frac{\sinh \mathscr{H}_c}{\sinh \mathscr{A}}.$$

This yields the law of sines for the Poincaré model

$$\boxed{\sin \alpha : \sin \beta = \sinh \mathscr{A} : \sinh \mathscr{B}.} \tag{18}$$

Next let $\mathscr{P}$ and $\mathscr{Q}$ be the lengths of the segments into which the height h_c divides the side c. Then formula (15) can be applied to both subtriangles and we get

$$\cosh \mathscr{A} = \cosh \mathscr{H}_c \cosh \mathscr{Q} \tag{19}$$

$$\cosh \mathscr{B} = \cosh \mathscr{H}_c \cosh(\mathscr{C} - \mathscr{Q}). \tag{20}$$

Now the addition theorem for hyperbolic functions‡ implies

$$\cosh \mathscr{B} = \cosh \mathscr{H}_c(\cosh \mathscr{C} \cdot \cosh \mathscr{Q} - \sinh \mathscr{C} \cdot \sinh \mathscr{Q}).$$

† We assume that the angles of the triangle ABC are acute. We leave it to the reader to settle the remaining case.

‡ This relation follows easily from the defining equation (1).

Division by (19) yields

$$\frac{\cosh \mathscr{B}}{\cosh \mathscr{A}} = \cosh \mathscr{C} - \sinh \mathscr{C} \cdot \tanh \mathscr{Q}.$$

In view of (9) we can write

$$\tanh \mathscr{Q} = \tanh \mathscr{A} \cdot \cos \beta$$

Hence

$$\boxed{\begin{aligned} \cosh \mathscr{B} &= \cosh \mathscr{A} \cdot \cosh \mathscr{C} - \sinh \mathscr{A} \sinh \mathscr{C} \cos \beta \\ \cosh \mathscr{C} &= \cosh \mathscr{A} \cdot \cosh \mathscr{B} - \sinh \mathscr{A} \sinh \mathscr{B} \cos \gamma \\ \cosh \mathscr{A} &= \cosh \mathscr{B} \cdot \cosh \mathscr{C} - \sinh \mathscr{B} \sinh \mathscr{C} \cos \alpha. \end{aligned}} \tag{21}$$

This is the law of cosines for hyperbolic geometry (Poincaré model). [The last two formulas in (21) are obtained from the first formula by cyclic permutation.]

It is possible to derive from our formulas a "law of cosines for angles" which enables us to compute the sides of a triangle given its angles.

However the computations involved are rather tedious and we shall therefore content ourselves with stating the result:

$$\boxed{\begin{aligned} \cosh \mathscr{B} &= \frac{\cos \alpha \cos \gamma + \cos \beta}{\sin \alpha \sin \gamma} \\ \cosh \mathscr{C} &= \frac{\cos \alpha \cos \beta + \cos \gamma}{\sin \alpha \sin \beta} \\ \cosh \mathscr{A} &= \frac{\cos \beta \cos \gamma + \cos \alpha}{\sin \beta \sin \gamma}. \end{aligned}} \tag{22}$$

This shows again that two triangles are congruent if they have the same angles. The length of the sides is then determined (to within sign!) by (22).

Our derivation of the trigonometric formulas of hyperbolic geometry involved the use of the Poincaré model. However, these relations can be derived directly from the axioms of hyperbolic geometry. This was first done by Bolyai and

Lobachevski. Needless to say, the latter approach is much more difficult than our own.

A relatively simple approach to hyperbolic trigonometry not employing a model is due to O. Perron [30].

Using trigonometry it is possible, among others, to compute the area F of a circle. If $\mathscr{K}$ is the h-length of the radius, then (cf. e.g., [26])

$$F = 4\pi \sinh^2(\mathscr{K}/2).$$

We shall not derive this result. Instead, we shall use trigonometry to solve another problem in the theory of area.

The area of a triangle is determined by its angles. Hence it must be possible to express this area in terms of its sides. In particular, we are interested in a hyperbolic counterpart of the formula of euclidean geometry

$$F = \tfrac{1}{2}\cdot g \cdot h.$$

Theorem 24 states that triangles with the same base and common midline have the same area. Such triangles determine the same Saccheri quadrilateral (Fig. 16). This quadrilateral is uniquely determined by its base and the common perpendicular (*EF* in Fig. 15). It must therefore be possible to express the area of the triangle in terms of these two quantities.

Let $ABHG$ in Fig. 59 be the Saccheri quadrilateral associated

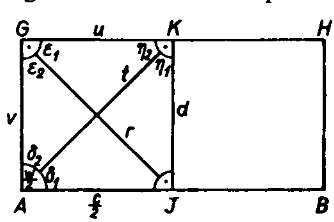

FIG. 59.

with our triangle and let J, K be the midpoints of AB, GH. Put $AG = v$, $GK = u$, $AK = t$, $GJ = r$. Denote the angles as follows:

$$\angle KAJ = \delta_1, \quad \angle KAG = \delta_2, \quad \angle KGJ = \varepsilon_1, \quad \angle AGJ = \varepsilon_2,$$
$$\angle JKA = \eta_1, \quad \angle GKA = \eta_2.$$

Then

$$\varepsilon_1 + \varepsilon_2 = \eta_1 + \eta_2 = \pi/2; \qquad \delta_1 + \delta_2 = W/2. \tag{23}$$

Here W is the sum of the angles in the triangle ABC (cf. p. 25). Applying the law of sines to the triangle AJG we get†

$$\sin\frac{W}{2} = \frac{\sin\varepsilon_2 \cdot \sinh\mathscr{R}}{\sinh\mathscr{C}/2}. \tag{24}$$

In view of (23) and (9) we have for the right triangle GJK

$$\sin\varepsilon_2 = \cos\varepsilon_1 = \frac{\tanh\mathscr{U}}{\tanh\mathscr{R}}.$$

Substituting in (24) we obtain

$$\sin\frac{W}{2} = \frac{\tanh\mathscr{U} \cdot \cosh\mathscr{R}}{\sinh\mathscr{C}/2}. \tag{25}$$

Now we apply (15) to the triangle GJK:

$$\cosh\mathscr{R} = \cosh\mathscr{U} \cdot \cosh\mathscr{D}.$$

Hence

$$\sin\frac{W}{2} = \frac{\sinh\mathscr{U} \cdot \cosh\mathscr{D}}{\sinh\mathscr{C}/2}.$$

We must still eliminate $\mathscr{U}$ from the latter relation. Since $\sin\eta_1 = \cos\eta_2$, (9) and (11) imply

$$\frac{\sinh\mathscr{C}/2}{\sinh\mathscr{T}} = \frac{\tanh\mathscr{U}}{\tanh\mathscr{T}}.$$

Thus

$$\tanh\mathscr{U} = \frac{\sinh\mathscr{C}/2}{\cosh\mathscr{T}}.$$

Now, in view of (15),

$$\cosh\mathscr{T} = \cosh\mathscr{C}/2 \cdot \cosh\mathscr{D}.$$

Hence

$$\tanh\mathscr{U} = \frac{\tanh\mathscr{C}/2}{\cosh\mathscr{D}}.$$

† As before, script capitals denote the h-length of segments.

But then

$$\sinh \mathscr{U} = \frac{\tanh \mathscr{C}/2}{\cosh \mathscr{D}\sqrt{1 - \dfrac{\tanh^2 \mathscr{C}/2}{\cosh^2 \mathscr{D}}}}.$$

The required area is

$$F = \pi - W$$

so that

$$\cos F/2 = \sin W/2.$$

Replacing $\sinh \mathscr{U}$ in the expression for $\sin W/2$ above we get

$$\cos \frac{F}{2} = \frac{\cosh \mathscr{D}}{\sqrt{1 + \cosh^2 \mathscr{C}/2 \cdot \sinh^2 \mathscr{D}}}.$$

Using (2) we get

$$\boxed{\tan F/2 = \tanh \mathscr{D} \cdot \sinh \mathscr{C}/2.} \qquad (26)$$

This is the required formula. For small $\mathscr{D}$ and $\mathscr{C}$

$$F \sim \mathscr{D} \cdot \mathscr{C}.$$

In euclidean geometry we have

$$F = d \cdot c = \tfrac{1}{2} \cdot c \cdot h_c.$$

Another expression for the area of a triangle in terms of its sides is the Heron triangle formula for hyperbolic geometry:

$$\sin \frac{F}{2} = \frac{\sqrt{\sinh \mathscr{S} \cdot \sinh(\mathscr{S} - \mathscr{A}) \cdot \sinh(\mathscr{S} - \mathscr{B}) \cdot \sinh(\mathscr{S} - \mathscr{C})}}{2 \cdot \cosh \mathscr{A}/2 \cdot \cosh \mathscr{B}/2 \cdot \cosh \mathscr{C}/2},$$

$$\mathscr{S} = \tfrac{1}{2}(\mathscr{A} + \mathscr{B} + \mathscr{C}). \qquad (27)$$

The proof of (27) is already found in Liebmann [14], 1st ed., 1905.

CHAPTER 9

Elliptic Geometry

Replacement of the parallel axiom V by the assertion Va (p. 59) yields the system of axioms of hyperbolic geometry. It is natural to ask what other assertion can be put in place of the parallel axiom. One possibility is

Vb. Any two lines in the plane intersect.

We know that this assertion conflicts with the other axioms. That this is so follows from Theorem 9 of absolute geometry. However, it is possible to obtain a system of axioms for a geometry without parallels by suitably modifying the axioms of order.

This can be best made clear by means of a model. Example 2 (p. 10) affords us this possibility. Since any two great circles intersect, the geometry of that example cannot have any parallels. The geometry of this model is called elliptic. Thus an elliptic point (el-point) is a pair of antipodal points on a sphere of arbitrary radius and an el-line is a great circle on that sphere.

We pointed out earlier that the order relations in this model are necessarily different from those in euclidean and hyperbolic geometry. We shall not enter into a discussion of the modified axioms of order. Instead we shall use our model to illustrate certain characteristic aspects of this geometry.

Theorem 5 fails to hold in elliptic geometry. In fact, given an el-line g it is always possible to associate with it an el-point PP_1 (P and P_1 are antipodal euclidean points) such that *every* el-line (great circle) through PP_1 is perpendicular to g (Fig. 60). PP_1 is obtained as the intersection of the sphere with the normal to the plane of g passing through the center of the sphere. The el-point PP_1 is usually called the "pole" of the el-line g, and the el-line g is called the polar of the point PP_1.

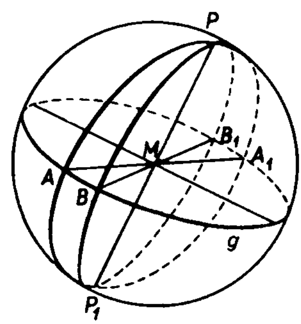

FIG. 60.

Angles in the el-plane are measured the euclidean way. This means that the angle between two el-lines intersecting in PP_1 is equal to the angle (measured in radians) formed by the euclidean tangents to the great circles (el-lines) at P or P_1.

We shall now define distance between two el-points AA_1 and BB_1. Thus let g be the el-line determined by these points and let PP_1 be the pole of g (Fig. 60). The perpendiculars to g at AA_1 and BB_1 form at the pole PP_1 two adjacent supplementary angles of which one is at most $\pi/2$. The radian measure of this angle is defined as the distance between AA_1 and BB_1. Hence the distance between two points in the el-plane is at most $\pi/2$.

It is known from spherical geometry that the sum of the angles in a spherical triangle (from our point of view this is an el-triangle) is greater than two right angles.

We define the area of a triangle with angles α, β, γ as the "spherical excess"†

$$\delta = \alpha + \beta + \gamma - \pi.$$

Compared with our model of hyperbolic geometry our model of elliptic geometry has the disadvantage of not lying in the (euclidean) plane. This can be remedied by projecting the sphere stereographically on the plane. This projection effects a one-to-one mapping of the sphere on a plane which preserves angles. A detailed explanation follows.

Consider a sphere of unit radius. On the sphere choose a great circle. We shall refer to this circle as the "equator" and to its pole NS (Fig. 61) as the "north and south pole." The

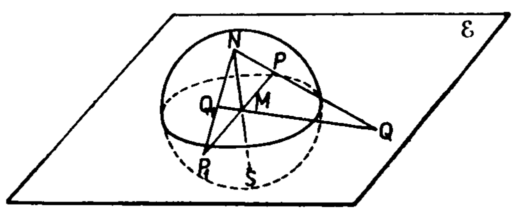

FIG. 61.

plane of the equator is denoted by $\mathscr{E}$. Now consider the set of (euclidean) lines through the (euclidean) point N which are not parallel to the equatorial plane $\mathscr{E}$. Each of these lines intersects the sphere in a point P (other than N) and the plane $\mathscr{E}$ in a point Q (Fig. 61). The correspondence $P \leftrightarrow Q$ is a one-to-one correspondence between the points on the sphere other than N and the points in the plane $\mathscr{E}$. This correspondence is called a "stereographic projection."

Under the stereographic projection points on the equator correspond to themselves, points inside the equator correspond to points in the "southern" hemisphere, and points outside the equator correspond to the points in the "northern" hemisphere exclusive of N.

† We leave out the proof of the fact that this definition is meaningful and that area as defined has the usual properties (cf. p. 69).

The stereographic projection has two remarkable properties:

1. It maps circles on the sphere on lines or circles in the plane.

2. It preserves angles, i.e., the angle between two intersecting circles on the sphere is the same as the angle between the lines or circles in the plane which correspond to them under the sterographic projections.

The proof of these two facts are given in many books on geometry (e.g., [17]) and will not be given here.

We shall use stereographic projection to transfer our model of elliptic geometry into the plane.

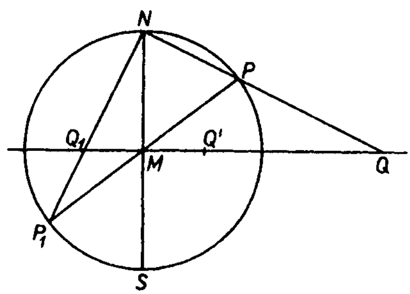

FIG. 62.

Consider great circles on the sphere (i.e., the el-lines). They are characterized by the fact that they intersect the equator in a pair of antipodal points. Since stereographic projection into the plane keeps equator points fixed, the images of great circles must be circles (or lines) passing through endpoints of a diameter of the unit circle (in the plane).

Next we consider the effect of stereographic projection on el-points (i.e., pairs of antipodal points on the sphere).

Let P, P_1 be a pair of antipodal points on the sphere (Fig. 62) and let Q, Q_1 be their images under stereographic projection. The angle P_1NP is a right angle. Hence

$$MQ \cdot MQ_1 = MN^2 = 1.$$

We are familiar with this relation from the study of inversion.

However, in our case Q and Q_1 are on different collinear rays emanating from M whereas in the case of reflection in the circle, Q and its image are on the same ray emanating from M. To obtain Q_1 from Q one must first reflect Q in the circle and then reflect the image Q' of Q in the point M (Fig. 62). In other words, if Q is a point in the $\mathscr{E}$ plane and P the corresponding point on the sphere then, to obtain the image Q_1 (in the $\mathscr{E}$ plane) of the point P_1 antipodal to P we must first reflect Q in the unit circle and then reflect the resulting image in the center of the unit circle. We shall call the images Q and Q_1 of antipodal points on the sphere "opposite points."

By now we are ready to use stereographic projection to construct a plane model of elliptic geometry.

To avoid confusion with the el-points and el-lines, etc., of the spherical model, the points and lines of the plane model will be referred to as El-points and El-lines, etc.

Definition 17. An El-point is a pair of "opposite points" relative to the unit circle.† An El-line is a euclidean circle (or euclidean line) passing through the endpoints of a diameter of the unit circle.

In Fig. 63, g is an El-line. A euclidean line through M intersects g in a pair of opposite points, i.e., in an El-point. Indeed (Fig. 63), by the theorem on chords,

$$MQ \cdot MQ_1 = MR \cdot MR_1 = 1.$$

Thus Q and Q_1 are images of antipodal points on the sphere.

With every el-line (great circle on the sphere) there is associated a pole (cf. p. 90). It can be obtained as the intersection of perpendiculars to the el-line at two of its points.

Using stereographic projection we can transfer the pole-polar relation to the plane model of elliptic geometry.

Thus, let g be a given El-line which intersects the unit circle $\mathscr{K}$ in the El-point RR_1. To find the pole of this El-line we must draw perpendiculars to g at two points of g (Fig. 64). It is simplest to draw the perpendicular to g passing through the

† The origin is regarded as an El-point!

center M of the circle $\mathscr{K}$ and the perpendicular to g passing through RR_1. The (El-) point of intersection PP_1 of these two perpendiculars is the required pole of g. Its characteristic property is that all El-lines through PP_1 (e.g., h_1 and h_2 in Fig. 64) are perpendicular to g.

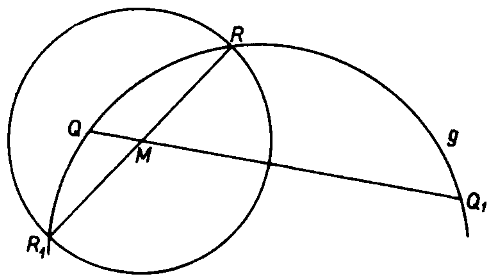

FIG. 63.

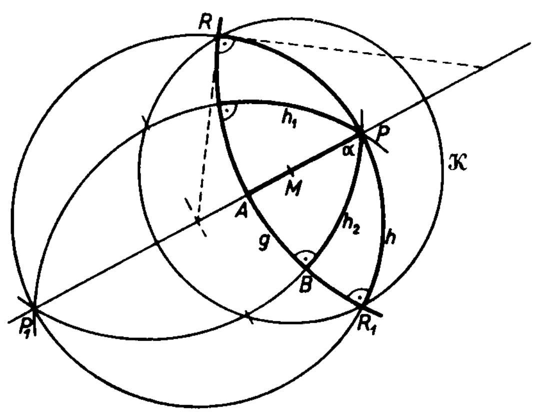

FIG. 64.

We leave it to the reader to solve the converse problem: Find the polar of a given point PP_1, i.e., find the El-line for which the given point PP_1 is a pole.

The distance between two points of the el-model (sphere!) was defined as the angle (or, rather, the smaller one of the two angles) formed at the pole of the line determined by the points in question by perpendiculars at these points (cf. p. 90). Since stereographic projection preserves angles, this definition carries over to our plane model of elliptic geometry.

Thus the distance of the points AA_1 and BB_1 in Fig. 64† is the radian measure of the angle α between the euclidean (and El-) line MP and the El-line h.

Figure 64 makes it clear that in the language of euclidean geometry the El-lines through PP_1 form an elliptic pencil determined by the euclidean points P and P_1.

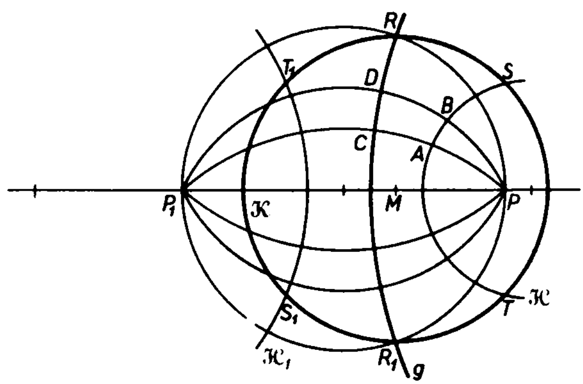

FIG. 65.

This insight enables us to obtain the circles of El-geometry. Thus, observe that all (euclidean) circles of the associated hyperbolic pencil are orthogonal to the elliptic pencil. In Fig. 64 we see only one of the circles of the hyperbolic pencil, namely, the El-line g. Clearly, this is the only circle of the hyperbolic pencil which qualifies as an El-line.

In Fig. 65 PP_1 is again the pole of the El-line g. $\mathscr{H}$ is another circle of the hyperbolic pencil orthogonal to the El-lines comprising the elliptic pencil.

Let S and T be the (euclidean) points in which $\mathscr{H}$ intersects the unit circle $\mathscr{K}$. The opposite points of the points on $\mathscr{H}$

† The opposite points A_1, B_1 of A, B have been omitted from the drawing.

belong to a (euclidean) circle $\mathscr{H}_1$. $\mathscr{H}_1$ passes through the points T_1 and S_1 such that T_1T and S_1S are diameters of $\mathscr{K}$ and ($\mathscr{H}_1$) is orthogonal to the elliptic pencil determined by the points P and P_1.†

It is reasonable to call the pair of circles ($\mathscr{H}$, $\mathscr{H}_1$) an El-circle. (We must have a *pair* since *e*-points are not El-points.)

We leave it to the reader to prove that an El-circle is the locus of points equidistant from its center PP_1.‡

The following is another result stated without proof: Every El-circle about PP_1 is an equidistant‡ of the polar of PP_1.

In the Poincaré model we treated a number of construction problems. These problems have their counterpart in El-geometry both with regard to formulation and the manner of solution. We shall not go into the details here.

† This is clear if one recalls that the El-model is the stereographic image of the el-model on the sphere.

‡ In this case an equidistant is a curve whose points are at the same distance from the polar of PP_1.

CHAPTER 10

Epilog

It was our initial modest aim to make the intuitive foundations of geometry into a system of axioms. We soon realized that it was impossible to begin with definitions of the fundamental geometric concepts of point, line, and plane in the manner of Euclid. Absence of explicit definitions of the basic concepts gave the system of axioms a formal character. The "things" of Explanation 1 could be interpreted in various ways. Further, we saw that it was possible to replace the parallel axiom with a proposition which contradicts our intuition. The resulting hyperbolic geometry, from the logical point of view, is as consistent as the euclidean geometry we are familiar with.

By taking full advantage of the freedom to interpret the "things" point, line, plane (Explanation 1), within the bounds determined by the axioms, it is possible to construct models of hyperbolic geometry.

The brief introduction to elliptic geometry given in the last chapter shows that replacement of Axiom V by Va is not the only possibility of constructing a geometry which deviates from the usual one.

We wish to mention that modern physics describes astronomical laws by means of a "noneuclidean" geometry which remotely resembles the elliptic geometry of Chapter 9.

Serious objections have been raised against putting various "noneuclidean" geometries on par with euclidean geometry.

Those familiar with the nature of axiomatic systems do not deny that from a *logical* point of view all of these geometries are equally valid. However, the fact remains that the straight line of our "pure intuitition" is always "straight" and infinitely long. Even though the physicist may find it expedient to use some "noneuclidean" geometry for the description of the physical universe, the straight line of Kant's "pure intuition" is something different than the bent light ray of the physicist or the orthogonal circle of the Poincaré model. These are the arguments of those who maintain that euclidean geometry will always occupy a special position among all possible geometries.

We do not propose to debate this issue of the theory of knowledge. It arose in a natural way out of concern with the foundations of geometry. The interested reader may wish to consult Chapter VIII of [36] which also includes an extensive bibliography.

References

[1] *Euclid:* "The 13 Books of Euclid's Elements," 3 vols. Dover, 1956.
[2] *Pascal:* "Vermächtnis eines grossen Herzens. Die kleineren Schriften." Übertragen und herausgegeben von Wolfgang Rüttenauer. Leipzig, 1938.
[3] *Klügel:* "Conatum Praecipuorum Theoriam Parallelarum Demonstrandeum Recensio." Göttingen, 1763.
[4] *Michelsen:* "Euclides Elemente, für den gegenwärtigen Zustand der Mathematik bearbeitet, erweitert, fortgesetzt." Berlin, 1791.
[5] *Kaestner:* "Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspektive." Göttingen, 1800.
[6] *Lobatschewsky:* Géométrie imaginaire. *J. reine angew. Math.* **17** (1837).
[7] *Lobatschewsky:* "Geometrische Untersuchungen zur Theorie der Parallelen." Berlin, 1840.
[8] *Engel* and *Stäckel:* "Urkunden zur Geschichte der nichteuklidischen Geometrie." Vol. 1: *N. I. Lobatschewsky*, Zwei geometrische Abhandlungen. Leipzig, 1899. Vol. 2: *W.* and *J. Bolyai*, Geometrische Untersuchungen. Leipzig, 1913.
[9] *Stäckel:* "Briefwechsel zwischen Gauss und Bolyai." Leipzig, 1899.
[10] *Engel* and *Stäckel:* "Die Theorie der Parallelen von Euklid bis auf Gauss." Leipzig, 1895.
[11] *D. Hilbert:* "The Foundations of Geometry." Open Court Publ. Co., 1902.
[12] *D. Poincaré:* "The Foundations of Science." Science Press, 1946.
[13] *F. Klein:* "Elementary Mathematics from an Advanced Standpoint," 2 vols. Dover, 1945.
[14] *Liebmann:* "Nichteuklidische Geometrie. Berlin and Leipzig, 1923.
[15] *Simon:* "Nichteuklidische Geometrie in elementarer Behandlung." Bearbeitet und herausgegeben von Dr. K. Fladt. Leipzig and Berlin, 1925.
[16] *Zacharias:* Elementare nichteuklidische Geometrie. *In* "Encyklopädie der mathematischen Wissenschaften." Leipzig, 1914–1931.
[17] *Zacharias:* "Elementargeometrie der Ebene und des Raumes." Berlin, 1930.
[18] *Zacharias:* "Das Parallelenproblem und seine Lösung," 2nd ed. Leipzig, 1951.

[19] *Hessenberg:* "Grundlagen der Geometrie." Berlin, 1930.
[20] *K. Fladt:* "Elementargeometrie," Vols. 1, 2, 3. Leipzig, 1928, 1931.
[21] *Kopecny:* Das berühmte fünfte Postulat des Euklid, bewiesen durch seine eigenen Forderungen und Lehrsätze... . Bratislava, 1933.
[22] *Thomsen:* "Grundlagen der Elementargeometrie." Leipzig, 1933.
[23] *Podehl* and *Reidemeister:* "Eine Begründung der ebenen elliptischen Geometrie." Leipzig, 1934.
[24] *Baldus:* Zur Axoimatik der Geometrie, V (Vereinfachung des Hilbertschen Axiomensystems der Euklidischen Geometrie). *Sitzungsber. Bayrischen Akad. Wiss. München* 1937).
[25] *Baldus:* Über nicht beweisbare und doch entbehrliche Axiome. *Math. Z.* **44** (1938).
[26] *Baldus:* "Nichteuklidische Geometrie," 2nd ed. Berlin, 1944.
[27] *Bachmann:* Eine Begründung der absoluten Geometrie in der Ebene. *Math. Ann.* **113** (1937).
[28] *Bachmann:* Zur Begründung der Geometrie aus dem Spiegelungsbegriff. *Math. Ann.* **123** (1951).
[29] *A. Schmidt:* Die Dualität von Inzidenz und Senkrechtstehen in der absoluten Geometrie. *Math. Ann.* **118** (1941–1943).
[30] *Perron:* Neuer Aufbau der nichteuklidischen (hyperbolischen) Trigonometrie. *Math. Ann.* **119** (1944).
[31] *Hamel:* Was ist Geometrie: *Math. Nachr.* **4**, 502–511 (1950).
[32] *Verriest:* "Introduction à la géométrie non euclidienne par la méthode élémentaire." Paris, 1951.
[33] *Meschkowski:* Die Ableitung der trigonometrischen Formeln im Poincaréschen Modell der hyperbolischen Geometrie. *Elem. Math.* **7**, No. 6 (1952).
[34] *E. T. Whittaker:* "From Euclid to Eddington." Cambridge Univ. Press, 1949.
[35] *Bachmann:* "Aufbau der Geometrie aus dem Spiegelungsbegriff." Berlin-Göttingen-Heidelberg, 1959.
[36] *Meschkowski:* "Wandlungen des mathematischen Denkens," 2nd ed. Braunschweig, 1960.
[37] *Behnke et al.:* "Grundzüge der Mathematik." Vol. II: Geometrie. Göttingen, 1960.
[38] *Meschkowski:* "Ungelöste und unlösbare Probleme der Geometrie." Braunschweig, 1960.
[39] *Efimow:* "Höhere Geometrie." Berlin, 1960.

A few books relevant to the issues raised in this book are:

1. *L. M. Blumenthal:* "A Modern View of Geometry." Freeman, 1961.
2. *H. Eves:* "A Survey of Geometry." Allyn and Bacon, 1963.
3. *H. Eves* and *C. V. Newsom:* "The Foundations and Fundamental Concepts of Mathematics. Rinehart, 1960.

4. *H. Buseman* and *P. J. Kelly:* "Projective Geometry and Projective Metrics." Academic Press, 1963.
5. *H. S. M. Coxeter:* "Introduction to Geometry." Wiley, 1961.
6. *S. Kulczycki:* "Non-Euclidean Geometry." Pergamon Press, 1961.
7. *E. Moise:* "Elementary Geometry from an Advanced Standpoint." Addison-Wesely, 1963.
8. *B. V. Kutuzov:* "Studies in Mathematics." Vol. IV: Geometry. SMSG, 1960.
9. *W. Prenowitz* and *M. Jordan:* "Basic Concepts of Geometry." Ginn (Blaisdell), 1964.

Subject Index

Absolute geometry, 59
angle between two el-rays, 90
between two h-rays, 51
of parallelism, 74
area of an el-triangle, 91
of an h-triangle, 69
axiom of parallels, 20
axioms, 4
of congruence, 13
of connection, 9
of continuity, 19
of order, 11

Bolyai, 31, 32, 33, 34
boundary points (of the Poincaré model), 48
bounding parallel, 49, 63

completeness of a system of axioms, 4
consistency of a system of axioms, 5
cross ratio, 45

defect, 60

e-(point, line, plane, etc.), 48
el-(point, line, plane, etc.), 89
El-(point, line, plane, etc.), 93
elliptic geometry, 89
elliptic pencil, 38
equidistant, 65
Euclid, 5, 6, 7, 49

Gauss, 32, 33, 34

h-(point, line, plane, etc.), 47
Hilbert, 7, 8
history of the parallel postulate, 21 ff
hyperbolic compass, 73
hyperbolic geometry, 59
hyperbolic pencil, 38
hyperbolic ruler, 73
hyperparallel, 49, 63

independence of a system of axioms, 5
inversion, 39

Klein, 34, 47

Legendre, 25, 29, 30, 31
length of an h-segment, 51
of an el-segment, 90
Lobachevski, 33, 34

model of a system of axioms, 47
of hyperbolic geometry in the plane, 47
of elliptic geometry in the plane, 93
of elliptic geometry on a sphere, 89

noneuclidean geometry, 32

orthogonal circles, 38

Pascal, 4, 6
Poincaré, 34, 47

Poincaré model of hyperbolic geometry, 47
polar of an el-point, 90
pole of an el-line, 90
power of a point with respect to a circle, 35

radical axis of two circles, 35

reflection in a circle, 39

Saccheri, 23
Saccheri quadrilateral, 23
stereographic projection, 91

undefined terms, 8