Д/з №10

№1

a) $\int_{-\infty}^{+\infty} dx \exp(-x^2) H_{2n}(xy)$

$$H_{2n}(xy) = 2xy H_{2n-1}(xy) - 2(2n-1) H_{2n-2}(xy)$$

Пусть $I_n = \int_{-\infty}^{+\infty} dx \exp(-x^2) H_{2n}(xy)$

Тогда $I_n = y \int_{-\infty}^{+\infty} 2x e^{-x^2} H_{2n-1}(xy) dx - 2(2n-1) I_{n-1} \underset{\text{инт. по частям}}{=}$

$$= \underbrace{-y e^{-x^2} H_{2n-1}(xy) \Big|_{-\infty}^{+\infty}}_{=0} + y \int_{-\infty}^{+\infty} e^{-x^2} \frac{d}{dx} H_{2n-1}(xy) - 2(2n-1) I_{n-1} =$$

$$\frac{d}{dx} H_{2n-1}(xy) = 2(2n-1) H_{2n-2}(xy) \cdot y$$

$$I_n = y^2 2(2n-1) I_{n-1} - 2(2n-1) I_{n-1} = 2(y^2-1)(2n-1) I_{n-1}$$

$$H_0(xy) = 1 \qquad I_0 = \int_{-\infty}^{+\infty} dx \exp[-x^2] = \sqrt{\pi}$$

$$I_n = 2^n (y^2-1)^n (2n-1)!! \sqrt{\pi}$$

б) $\int_{-\infty}^{+\infty} dx\, x \exp(-x^2) H_{2n-1}(xy) = I_n$

$$\cancel{H_{2n-1}(xy) = 2x H_{2n-2}(xy) - 2(n-1) H_{2n-3}(xy)}$$

$$\cancel{I_n = \int_{-\infty}^{+\infty} dx\, 2x^2 \exp(-x^2) H_{2n-2}(xy) - 2(n-1) I_{n-1}}$$

~~б)~~

Рассмотрим

$$\cancel{\frac{d^2}{dy^2}\left(\int_{-\infty}^{+\infty} dx \exp(-x^2) H_{2n}(xy)\right) = \frac{d}{dy}\left(\int_{-\infty}^{+\infty} dx\, 2\cdot 2n x \exp(-x^2) H_{2n-1}(xy)\right)}$$

$$I_n = \int_{-\infty}^{+\infty} dx\, x \exp(-x^2) H_{2n-1}(xy)$$

Рассмотрим

$$\frac{d}{dy}\left(\int_{-\infty}^{+\infty} dx \exp(-x^2) H_{2n}(xy)\right) = 4n \int_{-\infty}^{+\infty} dx\, x \exp(-x^2) H_{2n-1}(xy)$$

$$\Rightarrow I_n = \frac{d}{dy}\left(2^n (y^2-1)^n (2n-1)!!\sqrt{\pi}\right) =$$

$$= 2^n (2n-1)!!\sqrt{\pi}\cdot n\cdot 2y(y^2-1)^{n-1} =$$

$$= \underline{2^{n+1}\cdot n(2n-1)!!\, y(y^2-1)^{n-1}\sqrt{\pi}}$$

№2 a)

$$I_n = \int_{-\infty}^{+\infty} dx\, e^{-x^2} \sinh(\beta x) H_{2n+1}(x)$$

$$H_{2n+1}(x) = (-1)^{2n+1}\exp(x^2)\frac{d^{2n+1}}{dx^{2n+1}}\exp(-x^2)$$

$$I_n = \int_{-\infty}^{+\infty} (-1)^{2n+1}\sinh(\beta x)\frac{d^{2n+1}}{dx^{2n+1}}\left(\exp(-x^2)\right) =$$ инт. по частям $2n+1$ раз

$$= (-1)^{4n+2}\beta^{2n+1}\int_{-\infty}^{+\infty}\cosh(\beta x)\exp(-x^2)dx =$$

$$= \beta^{2n+1}\sqrt{\pi}\, e^{\frac{\beta^2}{4}}$$

~~№3~~

б) $\int_{-\infty}^{+\infty} dx\, e^{-x^2} \cosh(\beta x) H_{2n}(x) = \int_{-\infty}^{+\infty} \cosh(\beta x) \frac{d^{2n}}{dx^{2n}} \exp(-x^2)\, dx =$

$=$ по частям $2n$ раз $= \beta^{2n} \int_{-\infty}^{+\infty} \cosh(\beta x) \exp(-x^2)\, dx = \beta^{2n} \sqrt{\pi}\, e^{\frac{\beta^2}{4}}$

№3 $\int_{-\infty}^{+\infty} dx \exp[-(x-y)^2] H_n(x) = (-1)^n \int_{-\infty}^{+\infty} dx \exp[-(x-y)^2] \exp(x^2) \frac{d^n}{dx^n}(\exp(-x^2))$

$= (-1)^n \exp(-y^2) \int_{-\infty}^{+\infty} \exp(2xy) \frac{d^n}{dx^n} \exp(-x^2)\, dx =$ инт. по частям n $=$

$= (-1)^{2n} \exp(-y^2) (2y)^n \int_{-\infty}^{+\infty} \exp(2xy - x^2)\, dx =$

$= (2y)^n \int_{-\infty}^{+\infty} \exp(-(x-y)^2)\, dx = (2y)^n \sqrt{\pi}$

№4

$I_n(y) = \int_{-\infty}^{+\infty} dx\, e^{-x^2} x^n H_n(xy)$

Рассмотрим

$I_{n+1}(y) = \int_{-\infty}^{+\infty} dx \exp(-x^2) x^{n+1} H_{n+1}(xy) =$

$= \frac{1}{2} \int_{-\infty}^{+\infty} dx \left(\frac{d}{dx}(\exp(-x^2))\right) x^n H_{n+1}(xy) =$ по частям

$= \frac{1}{2} \int_{-\infty}^{+\infty} \exp(-x^2) \left(n x^{n-1} H_{n+1}(xy) + x^n \frac{d}{dx} H_{n+1}(xy)\right) dx =$

$= \frac{1}{2} \int_{-\infty}^{+\infty} \exp(-x^2) dx \left(n x^{n-1} \left(2x H_n(x) - 2n H_{n-1}(x)\right) + x^n \cdot 2(n+1) y H_n(xy)\right)$

$\cancel{= \frac{1}{2} \int_{-\infty}^{+\infty} dx \exp(-x^2) \left(2n x^n H_n(x) + 2n H_{n-1}(x) x^{n-1} (xy - n)\right)}$

$$= \frac{1}{2}\int_{-\infty}^{+\infty} \exp(-x^2)\,dx \left(2H_n(xy)\,x^n(\cancel{y\,n} + yn + y) - 2n^2 x^{n-1} H_{n-1}(xy)\right)$$

$$= \frac{1}{\cancel{2}} \int_{-\infty}^{+\infty} dx \exp(-x^2) H_n(xy)\, x^n \cdot y(2n+1) - \int_{-\infty}^{+\infty} \exp(-x^2) H_{n-1} x^{n-1} n^2 dx$$

$$= y(2n+1)\, I_n(y) - n^2 I_{n-1}(y)$$

Пусть $I_n(y) = n!\, g_n(y)$

$$(n+1)!\, g_{n+1}(y) = y(2n+1)\, n!\, g_n(y) - n^2 (n-1)!\, g_n(y)$$

$$\cancel{(n+1)n}(n+1)\, g_{n+1}(y) = y(2n+1)\, g_n(y) - n\, g_n(y)$$

Если ~~$g_{n+1}(y)$~~ рассмотреть, ~~$g_{n+1}(y) = P_{n+1}$~~

$$g_0 = I_0 = \int_{-\infty}^{+\infty} \exp(-x^2)\,dx = \sqrt{\pi} = \sqrt{\pi}\, P_0(y)$$

$$g_1 = I_1 = 2y\int_{-\infty}^{+\infty} x^2 \exp(-x^2)\,dx = \sqrt{\pi}\, y = \sqrt{\pi}\, P_1(y)$$

тогда методом мат. индукции

показываем что $g_{n+1} = \sqrt{\pi}\, P_{n+1}$

и тогда $I_n(y) = n!\sqrt{\pi}\, P_n(y)$