Д/з №9

№1

$$G(z;\cos\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} P_n(\cos\theta)$$

$$P_n(\cos\theta) = \int_0^{\pi} \frac{d\varphi}{\pi} (\cos\theta + i\sin\theta\cdot\cos\varphi)^n$$

$$G(z;\cos\theta) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_0^{\pi} \frac{d\varphi}{\pi} (\cos\theta + i\sin\theta\cdot\cos\varphi)^n =$$

$$= \int_0^{\pi} \frac{d\varphi}{\pi} \sum_{n=0}^{\infty} \frac{(z(\cos\theta + i\sin\theta\cos\varphi))^n}{n!} = \int_0^{\pi} \frac{d\varphi}{\pi} e^{z\cos\theta}\cdot e^{iz\sin\theta\cos\varphi} =$$

$$= \frac{e^{z\cos\theta}}{\pi} \int_0^{\pi} d\varphi\, e^{iz\sin\theta} \underset{\alpha=\varphi-\frac{\pi}{2}}{=} \frac{e^{z\cos\theta}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{iz\sin\theta\cdot\cos(\alpha+\frac{\pi}{2})} d\alpha =$$

$$= \frac{e^{z\cos\theta}}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{iz\sin\theta\cdot\sin\alpha} d\alpha \;\textcircled{=}$$

$$\boxed{\begin{array}{l} \text{на} \\ \int_0^{\frac{\pi}{2}} e^{ik\sin\alpha} = \int_{\frac{\pi}{2}}^{\pi} e^{ik\cdot\sin\alpha} \\ \int_{-\frac{\pi}{2}}^{0} \dots = \int_{-\pi}^{-\frac{\pi}{2}} \end{array}}$$

$$= \frac{e^{z\cos\theta}}{2\pi} \int_{-\pi}^{\pi} e^{iz\sin\theta\cdot\sin\alpha} d\alpha$$

$$= e^{z\cos\theta} J_0(z\sin\theta)$$

N2

$$\int_{-1}^{1} P_{n-1}(x) P_{n+1}(x) x^2 dx = \frac{1}{(2n-1)(2n+3)} \int_{-1}^{1} \left(n P_n + (n-1) P_{n-2} \right) x$$

$$x \left((n+2) P_{n+2} + (n+1) P_n \right) = \frac{n(n+2)}{(2n-1)(2n+3} \int_{-1}^{1} P_n^2 dx =$$

$$= \frac{2n(n+2)}{(2n-1)(2n+1)(2n+3)}$$

N3

$$f(x) = \operatorname{arctanh}(x)$$

$$\operatorname{arctanh}(x) = \sum_{n=0}^{\infty} P_n(x) \cdot f_n$$

$$f_n = \frac{2n+1}{2} \int_{-1}^{1} P_n(x) \operatorname{arctanh}(x) dx$$

$$\frac{d}{dx}\left[(1-x^2) \frac{d}{dx} P_n \right] = -n(n+1) P_n$$

$$f_n = - \frac{2n+1}{2n(n+1)} \int_{-1}^{1} \left((1-x) \frac{d}{dx} P_n \right)' \operatorname{arctanh}(x) dx =$$

$$= - \frac{2n+1}{2n(n+1)} \left(\cancelto{0}{(1-x^2) P_n' \operatorname{arctanh}(x) \Big|_{-1}^{1}} - \int_{-1}^{1} (1-x^2) P_n' \frac{1}{1-x^2} dx \right) =$$

$$= \frac{2n+1}{2n(n+1)} \int_{-1}^{1} P_n' dx = \frac{2n+1}{2n(n+1)} P_n(x) \Big|_{-1}^{1}$$

$$= \frac{2n+1}{2n(n+1)} \left(1 - (-1)^n \right) = \frac{2n+1}{2n(n+1)} \left[n \text{ - чётное} \right]$$

$$f_n = \frac{2n+1}{2n(n+1)} \left[n \text{ - чётное} \right]$$

$$\operatorname{arctanh}(x) = \sum_{n=0}^{+\infty} P_n(x)\,\frac{2n+1}{2n(n+1)}\ [n\text{-чётное}] =$$

$$= \sum_{k=0}^{+\infty} P_{2k}(x)\,\frac{4k+1}{4k(2k+1)}$$

№4

$$f(x) = x^k = \sum_{n=0}^{N}{}' f_n P_n(x)$$

$$f_n = \frac{\int_{-1}^{1} x^k P_n(x)\,dx}{\int_{-1}^{1} P_n^2(x)\,dx} = \cancel{\frac{2}{2n+1}}\ \frac{2n+1}{2}\int_{-1}^{1} x^k P_n(x)\,dx$$

$$= \left(n+\frac{1}{2}\right)\int_{-1}^{1} x^k P_n(x)\,dx = \left(n+\frac{1}{2}\right)\frac{1}{2^n n!}\int_{-1}^{1} x^k \frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{\left(n+\frac{1}{2}\right)}{2^n n!}\left(\left. \cancelto{0}{x^k \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n}\right|_{-1}^{1} - k\int_{-1}^{1} x^{k-1}\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\,dx\right.$$

т.к $\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n$ - будет содержать член (x^2-1)

$$= \begin{cases} \frac{\left(n+\frac{1}{2}\right)}{2^n n!}(-1)^k (k)_n \int_{-1}^{1} x^{k-n}(x^2-1)^n & \text{при } ~~k\le n~~\ n\le k \\ 0 & n>k \end{cases} \Rightarrow N = k$$

далее $\int_{-1}^{1} x^{k-n}\underset{\substack{\uparrow \\ \text{чётная}}}{(x^2-1)^n}\,dx = \begin{cases} \int_{-1}^{1} x^{k-n}(x^2-1)^n\,dx & k-n \text{ чётное} \\ 0 & k-n - \text{нечётное} \end{cases}$

обозначим $k-n=2m$

$$\int_{-1}^{1} x^{2m}(x^2-1)^n\,dx = 2\int_{0}^{1} x^{2m}(x^2-1)^n\,dx =$$

$$= \int_0^1 u^{m-\frac{1}{2}}(u-1)^n\,du = (-1)^n \int_0^1 u^{m-\frac{1}{2}}(1-u)^n\,du$$

$$= (-1)^n B\left(m+\frac{1}{2};\, n+1\right) = (-1)^n B\left(\frac{k-n+1}{2};\, n+1\right) =$$

$$= (-1)^n \frac{\Gamma\left(\frac{k-n}{2}+\frac{1}{2}\right)\Gamma(n+1)}{\Gamma\left(\frac{k+n+3}{2}\right)} = \cancel{\frac{(-1)^n n!\,(k-n)!}{4^n\left(\frac{k-n}{2}\right)!}}$$

$$\cancel{= (-1)^n n! \frac{\Gamma\left(\frac{k-n-2}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{k+n}{2}+\frac{1}{2}\right)} = (-1)^n n! \frac{(k-n-2)!}{4^{\frac{k-n-2}{2}}\left(\frac{k-n-2}{2}\right)!}}$$

$$= (-1)^n \cdot n! \frac{\Gamma\left(\frac{k-n}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{k+n}{2}+\frac{1}{2}\right)} \textcircled{=}$$

$$\Gamma\left(\frac{k-n}{2}+\frac{1}{2}\right) = \frac{(k-n-1)!!}{2^{\frac{k-n}{2}}}\sqrt{\pi}$$

$$\Gamma\left(\frac{k-n+2}{2}+\frac{1}{2}\right) = \frac{(k-n+1)!!}{2^{\frac{k-n+2}{2}}}\sqrt{\pi}$$

$$\textcircled{=}\ (-1)^n \cdot n! \frac{(k-n-1)!!}{(k-n+1)!!}\, 2^{\frac{k-n}{2}+1-\frac{k-n}{2}} =$$

$$= \frac{2(-1)^n n!}{(k-n+1)}$$

$$\Rightarrow f_n = \frac{(n+\frac{1}{2})}{2^n \cancel{n!}} (-1)^k (k)_n \cdot \frac{2(-1)^n \cancel{n!}}{k-n+1} =$$

$k-n = 2m$

$k+n = 2m+2n$

$$= \frac{(2n+1)\,(k)_n}{2^n(k-n+1)} (-1)^{\overset{\overset{1}{\parallel}}{k+n}} = \frac{(2n+1)\,k(k-1)\dots(k-n+1)}{2^n(k-n+1)} =$$

$$= \frac{2n+1}{2^n}(k)_{n-1}$$ fur $k-n=2m$

$k-2m=n$

$n_{max} = k$

$m = 0$

$n_{min} = 0$

$m = [\frac{k}{2}]$

$$f_m = \frac{(2k-4m+1)}{2^{k-2m}} (k)_{k-2m\mp 1}$$

$$\Rightarrow \underbrace{x^k = \sum_{m=0}^{[\frac{k}{2}]} \frac{(2k-4m+1)}{2^{k-2m}} (k)_{k-2m-1} P_{k-2m}(x)}$$