Д/З № 8

№1 $I(m, k)$ $\quad m, k \in \mathbb{N}_0$

$$\int_0^{\infty} \frac{J_{m+k}(x)}{x^k} = -\frac{1}{k-1} \cancel{\frac{J_{m+k}(x)}{x^{k-1}}}\Bigg|_0^{\infty} + \frac{1}{k-1}\int_0^{+\infty} \frac{1}{x^{k-1}} J'_{m+k}(x)\,dx$$

$$= \frac{1}{k-1}\int_0^{+\infty}\left(J_{m+k-1} - \frac{m}{x} J_{m+k}\right)\frac{1}{x^{k-1}}\,dx =$$

$$= \frac{1}{k-1}\underbrace{\int_0^{+\infty} \frac{J_{m+k-1}}{x^{k-1}}}_{I(m,k-1)} - \frac{m+k}{k-1}\int_0^{+\infty}\frac{J_{m+k}}{x^k}$$

$$I(m,k) = \frac{1}{k-1} I(m,k-1) - \frac{k+m}{k-1} I(m;k)$$

$$I(m,k)\left(\frac{m+2k-1}{k-1}\right) = \frac{1}{k-1} I(m,k-1)$$

$$I(m,k) = \frac{1}{m+2k-1} I(m,k-1)$$

$$I(m,0) = \int_0^{\infty} J_m(x) = 1 \quad \forall m$$

$$I(m;1) = \frac{1}{m+2k-1} I(m,0) = \frac{1}{m+1}$$

$$I(m;2) = \frac{1}{(m+3)(m+1)}$$

$$I(m;k) = \frac{1}{(m+2k-1)\cdot(m+2k-3)\cdot\ldots\cdot(m+1)} = \begin{cases} \dfrac{(m-1)!!}{(m+2k-1)!!} & m>0 \\[2ex] \dfrac{1}{(2k-1)!!} & m=0 \end{cases}$$

№2

$$f(x) = \int_0^x J_0(y)\, J_0(x-y)\, dy$$

$$k(t) = J_0(t) \;;\; \varphi(t) = J_0(t)$$

$$\tilde f(p) = \frac{\tilde\varphi(p)}{\tilde K(p)}$$

$$\tilde\varphi(p) = \tilde k(p)$$

$$\cancel{\tilde f(p)} \quad \tilde f(p) = 1$$

$$f(x) = \int_{c-i\infty}^{c+i\infty} \frac{\exp(px)}{2\pi i}\, dp = -\frac{1}{2\pi} \int_{-i\infty}^{+\infty} \exp\left(\frac{u-c}{i} x\right)$$

$$\phi(x) = \int_0^x J_0(y)\, J_0(x-y)\, dy$$

$$f(t) = J_0(t) \qquad \tilde f(p) = \frac{\tilde\varphi(p)}{\tilde K(p)}$$

$$k(t) = J_0(t) \qquad \tilde\varphi(p) = \tilde f(p)\, \tilde k(p)$$

$$\int_0^{+\infty} \exp(-pt)\, J_0(t)\, dt = \int_0^{+\infty} \frac{\exp(-pt)}{2\pi} dt \int_{-\pi}^{\pi} \exp(it\sin\theta)\, d\theta =$$

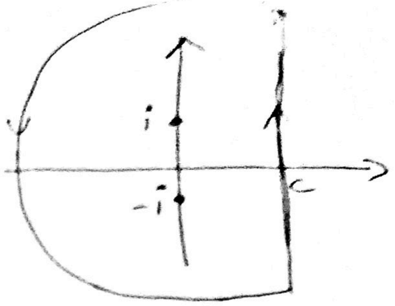
$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta\, \frac{(-1)}{p - i\sin\theta}\, e^{-t(p - i\sin\theta)}\Big|_0^{+\infty} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{p - i\sin\theta}\, d\theta =$$

$$= \left| \begin{array}{l} \alpha = e^{i\theta} \\ \sin\theta = \left(\alpha - \frac{1}{\alpha}\right) \end{array} \right| = \frac{1}{2\pi i} \oint_{r=1} \frac{d\alpha}{\alpha\left(p - \frac{1}{2}\left(\alpha - \frac{1}{\alpha}\right)\right)} =$$

$$= -\frac{1}{2\pi i} \oint \frac{2\, d\alpha}{\alpha^2 - 2p\alpha - 1} = -\frac{1}{\pi i} \oint \frac{d\alpha}{\alpha^2 - 2p\alpha - 1} = -2\, \mathrm{res}\left[p - \sqrt{p^2+1}\right]$$

$$= \frac{-2}{-2\sqrt{p^2+1}} = \frac{1}{\sqrt{p^2+1}}$$

$$\Phi(p) = \frac{1}{\sqrt{p^2+1}} \cdot \frac{1}{\sqrt{p^2+1}} = \frac{1}{p^2+1}$$



$$\phi(x) = \int\limits_{c-i\infty}^{c+i\infty} \frac{dp}{2\pi i} \exp(pt) \cdot \frac{1}{p^2+1}$$

$$= \operatorname{Res}(p=i) + \operatorname{Res}(p=-i) =$$

$$= \frac{\exp(i\cancel{x})}{2i} + \frac{\exp(-i\cancel{x})}{-2i} = \sin(\cancel{x})$$

$$\Rightarrow \int\limits_0^x J_0(y) J_0(x-y)\, dy = \sin(x)$$

<u>№3</u> Доказать $\int\limits_0^{+\infty} J_n(kx) J_n(qx)\, x\, dx = \frac{\delta(k-q)}{k}$

доказать исходя

из $\exp(iz\sin\varphi) = \sum\limits_{-\infty}^{+\infty} J_n(z) \exp(in\varphi)$

Распишем:

$$\exp(i\vec{k}\vec{r}) \cdot \exp(-i\vec{q}\vec{r}) = \exp\left[ikr \cdot \cos\left(\tfrac{\pi}{2}-\varphi\right)\right] \exp\left[-iqr \cos\left(\tfrac{\pi}{2}-\varphi+\theta\right)\right]$$

$$= \exp(ikr \cdot \sin\varphi) \exp\left(-iqr \sin(\varphi-\theta)\right) = \exp(ikr \cdot \sin\varphi) \cdot \exp(iqr \sin(\theta-\varphi))$$

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} J_n(kr) \cdot J_m(qr) \exp(in\varphi - im\varphi + im\theta)$$

$$\frac{1}{(2\pi)^2} \iint d\varphi\, dr\, \exp\left(i(\vec{k}-\vec{q})\vec{r}\right) r = \delta(\vec{k}-\vec{r}) = \frac{\delta(k-q)\,\delta(\theta)}{k}$$

С другой стороны, т.к. $\int\limits_{-\pi}^{\pi} \exp[i(n-k)\varphi]\, d\varphi = 2\pi\, \delta_{nk}$

$$\frac{1}{(2\pi)^2} \int\limits_0^{\infty} dr\, r \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{+\infty} J_n(kr) J_m(qr) e^{im\theta} \cdot 2\pi\, \delta_{nm} =$$

$$= \int\limits_0^{\infty} dr\, r \sum_{n=-\infty}^{+\infty} J_n(kr) J_n(qr) e^{in\theta} \cdot \frac{2\pi}{(2\pi)^2}$$

$$\cancel{\int_{-\pi}^{\pi} d\Theta \, \frac{\delta(k-q)}{k} \, i^{n} \delta(\theta) = \int_{-\pi}^{\pi} d\theta \int_0^{+\infty} dr\, r\, J_n(kr) J(qr)\, e^{in\theta}}$$

$$\cancel{\frac{\delta(k-q)}{k} = \int_{-\pi}^{\pi} d\theta\, e^{in\theta}}$$

$$\int_{-\pi}^{\pi} d\Theta \, \frac{\delta(k-q)}{k} \, \delta\Theta\, e^{-ij\theta} = \frac{2\pi}{(2\pi)^2}\int_{-\pi}^{\pi} d\Theta \int_0^{+\infty} dr\, r \sum_{n=-\infty}^{+\infty} J_n(kr)\, J_n(qr)\, e^{i(n-j)\theta}$$

$$\Rightarrow \frac{\delta(k-q)}{k} = \frac{1}{2\pi}\int_0^{+\infty} dr\, r \sum_{n=-\infty}^{\infty} J_n(kr)\, J_n(qr) \underbrace{\int_{-\pi}^{\pi} d\theta\, e^{i(n-j)\theta}}_{2\pi \delta_{nj}} =$$

$$\boxed{\frac{\delta(k-q)}{k} = \int_0^{+\infty} J_j(kr)\, J_j(qr)\, r\, dr}$$

N 4 $\quad f(r) = \exp(-p^2 r^2)$

$$\tilde{f}(q) = \int_0^{+\infty} r\, J_0(qr) \exp(-p^2 r^2)\, dr$$

$$J_0(qr) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} \exp(iqr\sin\theta)\, d\theta$$

$$\tilde{f}(q) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \int_0^{+\infty} r \exp(iqr\sin\theta - p^2 r^2)\, dr = \left| \begin{array}{l} r\cdot\sin\theta = y \\ r\cdot\cos\theta = x \end{array} \right|$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\, dy \exp\left(iqy - (x^2+y^2) r^2\right) =$$

~~Рассмотрим ф-цию: $F(q,k) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx\,dy\, \exp(-p^2(x^2+y^2) + ixk + iqy)$~~

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} dy \exp(-r^2 y^2 + iqy) \int_{-\infty}^{\infty} dx \exp(-x^2 r^2) =$$

$$= \frac{\sqrt{\pi}}{2\pi r} \int_{-\infty}^{+\infty} dy \exp\left(-(r^2 y^2 - iqy)\right) = \frac{\sqrt{\pi}}{2r^2} \cdot e^{-\frac{q^2}{4r^2}} = \frac{e^{-\frac{q^2}{4r^2}}}{2r^2}$$