Kevin Chen
CS 5800
Professor Jonathan Mwaura
Problem set 1

Problem # 1

a. $f(n)=n^2+2n+3$ $g(n)=n^2$ (Need to find constants c & n_0 such that $0 \le n^2+2n+3 \le cn^2$ for $n \ge 1$)

$n^2+2n+3 \le cn^2$

↓

Subtract $3n^2+2n+3$ from both sides:

$0 \le (c-1)n^2-2n-3$

↓

I'll choose $c=2$, therefore:

$0 \le n^2-2n-3$

↓

Factor right side:

$0 \le (n-3)\cdot(n+1)$ → This inequality holds for $n \ge 3$.

↓

Therefore for $c=2$ and $n_0=3$, we have $0 \le n^2+2n+3 \le 2n^2$ for all $n \ge 3$. Thus $f(n)=O(n^2)$.

For $f(n) \ne O(n)$ proof:

If $f(n)=O(n)$, there should be constants c and n_0 such that $0 \le n^2+2n+3 \le cn$ for all $n \ge n_0$.
Now, looking at the terms using n^2 and n, it's clear that n^2+2n+3 grows faster than cn for any constant if c and n grow very large. Therefore, $f(n)$ is not in $O(n)$.

Thus: $f(n)=O(n^2)$, but $f(n) \ne O(n)$.

b. $f(n)=n\log n+100n$ and $g(n)=n\log n$ (need to find c and n_0 such that $0 \le n\log n+100n \le c \cdot n\log n$ for all $n \ge n_0$.
Assuming $n>0$, divide inequality by $n\log n$:

$0 \le 1+100/\log n \le c$

↓

See the behavior of $100/\log n$: as n becomes larger, the denominator grows slower than numerator, and this expression will approach infinity.
Therefore I'll choose $c=101$ and $2n_0=2$ to satisfy the inequality for $n \ge n_0$. This would imply that $f(n)=O(n\log n)$.
Now, to prove $f(n) \ne O(n)$, if $f(n)=O(n)$, there should be constants 'c' and 'n_0' such that $0 \le n\log n+100n \le c \cdot n$ for all $n \ge n_0$.
If we analyze the terms involving $n\log n$ and n, however, you can see that $n\log n+100n$ grows faster than $c \cdot n$ for any constant c when n becomes large. Thus, $f(n)$ is NOT $O(n)$.
We can conclude that $f(n)=O(n\log n)$ and $f(n) \ne O(n)$.

c) Need to find constants $c1$, $c2$, $n1$, $n2$

$0 \le 2n^2+4 \le c1(4n^2+2)$ for all $n \ge n1$, to prove $f(n)=O(g(n))$

$0 \le 4n^2+2 \le c2(2n^2+4)$ for all $n \ge n2$, to prove $g(n)=O(f(n))$

Prove $f(n) = O(g(n))$:

Find c1 and n1 that satisfy the condition given $f(n) = 2n^2+4$, and $g(n) = 4n^2+2$.

$0 \le 2n^2+4 \le c1(4n^2+2)$ for all $n \ge n1$.

$2n^2+4 \le c1(4n^2+2)$

$2n^2+4 \le 4c1n^2+2c1$

$4-2c1 \le (4c1-2)n^2$

I'll assume $c1 \ge 2$ since c1 needs to be a positive constant. We can now choose c1 and n1 such that $4-2c1 \le 0$ for all $n \ge n1$.

I'll set $c1=2$, and $n1=1$.

So, $2n^2+4 \le 2(4n^2+2)$ for all $n \ge 1$. This would prove $f(n) = O(g(n))$.

To prove $g(n) = O(f(n))$, I need to find c2 and n2 such that $0 \le 4n^2+2 \le c2(2n^2+4)$ for all $n \ge n2$.

$4n^2+2 \le c2(2n^2+4)$

$4n^2+2 \le 2c2-4c2$

$2c2-4 \ge 0$ (for simplicity, I assume $c2 \ge 2$)

I'll set $c2=2$ and $n2=1$.

So, $4n^2+2 \le 2(2n^2+4)$ for all $n \ge 1$. This proves $g(n) = O(f(n))$.

To conclude: $f(n) = O(g(n))$ and $g(n) = O(f(n))$ for $f(n) = 2n^2+4$ and $g(n) = 4n^2+2$.

d. To prove this, suppose the following two functions;

1. $f(n) = e^n$
2. $g(n) = n$

Prove: $f(n) = O(g(n))$

This statement means c and n_0 constants are positive, which means that $0 \le e^n \le cn$ for all $n \ge n_0$. It's proved by showing that e^n grows at a slower rate compared to n, meaning e^n is asymptomatically dominated by n.

Consider $\lim_{n\to\infty} \frac{e^n}{n}$. Apply L'Hopital's rule, for any positive constant c, there exists a large n_0 such that $0 \le e^n \le cn$ for all $n \ge n_0$. This proves $f(n) = O(g(n))$.

Disprove: $g(n) = O(f(n))$

This states that constants c and n_0 are positive such that $0 \le n \le ce^n$ for all $n \ge n_0$. However this is not true.

Consider $\lim_{n\to\infty} \frac{n}{e^n}$. This approaches 0, meaning that n is dominated by e^n. For any positive constant c, the n value is $n > ce^n$. Thus $g(n) \ge O(f(n))$. This would disprove $g(n) = O(f(n))$.

$f(n) = O(g(n))$ is true, and $g(n) = O(f(n))$ is NOT true.

Problem #2

a) Find positive constants such that $0 \le 2n+1 \le c \cdot 2n$ for all $n \ge n_0$

Analyze inequality:

$0 \le 2n+1 \le c \cdot 2^n$

$\downarrow$

Simplify middle term:

$0 \le 2n+1 \le c \cdot 2^{n-1}$

$\downarrow$

I'll choose $c=1$ and $n_0=1$:

$$0 \le 2n+1 \le 2 \cdot 2^{n-1}$$

$\downarrow$

This holds for all $n \ge 1$, therefore $2n+1 = O(2^n)$.

b) I will disprove, show that for constants c and n_0, there exists $n \ge n_0$ such that $2^{2n} > c \cdot 2^{n^2}$

Consider this limit: $\lim_{n\to\infty} 2^{2n}/2^{n^2}$

This equals 0, meaning 2^{2n} is dominated by 2^{n^2}. So $2^{2n} = O(2^{n^2})$.

c) I'll compare the growth rates of the function: $\log(\log^k n)$ grows slower than $\log^k(\log n)$. $\log^k n$ is an iterated function, which is very slow. So $\log(\log^k n)$ is slower in comparison to $\log^k(\log n)$, thus $g(n) = \log^k(\log n)$ is asmptotically larger.

d) Upper bound (O): Find constants c and n_0 such that:

$0 \le \log_3 n \le c \cdot \log_2 n$ for all $n \ge n_0$.

Next, analyze inequality:

$0 \le \log_3 n \le c \cdot \log_2 n$

$0 \le \log_3 n / \log_2 n \le c$

$\downarrow$

I'll choose $c=1$, $n_0=1$ since $\log_3 n / \log_2 n \ge 0$ for $n \ge 1$

$0 \le \log_3 n \le \log_2 n$

$\downarrow$

This holds for all $n \ge 1$, so $f(n) = O(g(n))$.

Lower bound (Ω): Find constants c and n_0 such that $0 \le c \cdot \log_2 n \le \log_3 n$ for all $n \ge n_0$

Analyze inequality:

$0 \le c \cdot \log_2 n \le \log_3 n$

$0 \le c \cdot \log_3 n / \log_3 2 \le \log_3 n$

$\downarrow$

I'll chose $c = \frac{1}{\log_3 2}$ and $n_0 = 1$, since $\log_3 n / \log_3 2 \ge 0$ for $n \ge 1$:

$0 \le \log_2 n / \log_3 2 \le \log_3 n$

$\downarrow$

This holds for all $n \ge 1$, so $f(n) = \Omega(g(n))$.

Tight bound (Θ): Since both $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$ hold, we can deduce $f(n) = \Theta(g(n))$

Therefore the relationship between $f(n) = \log_3 n$ and $g(n) = \log_2 n$ is given by $f(n) = \Theta(g(n))$