As discussed in Chapter 1, `ssq1` and `sis1` are examples of trace-driven discrete-event simulation programs. By definition, a trace-driven simulation relies on input data from an external source to supply recorded realizations of naturally occurring stochastic processes. Total reliance on such external data limits the applicability of a discrete-event simulation program, naturally inhibiting the user's ability to do "what if" studies. Given this limitation, a general discrete-event simulation objective is to develop methods for using a random number generator to convert a trace-driven discrete-event simulation program to a discrete-event simulation program that is not dependent on external data. This chapter provides several examples.

3.1.1 SINGLE-SERVER SERVICE NODE

Relative to the single-server service node model from Chapter 1, two stochastic assumptions are needed to free program `ssq1` from its reliance on external data. One assumption relates to the arrival times, the other assumption relates to the service times. We consider the service times first, using a *Uniform*(a, b) random variate model.

Example 3.1.1 Suppose that time is measured in minutes in a single-server service node model and all that is known about the service time is that it is random with possible values between 1.0 and 2.0. That is, although we know the range of possible values, we are otherwise in such a state of ignorance about the stochastic behavior of this server that we are unwilling to say some service times (between 1.0 and 2.0) are more likely than others. In this case we have modeled service time as a *Uniform*$(1.0, 2.0)$ random variable. Accordingly, random variate service times, say s, can be generated via the assignment

```
s = Uniform(1.0, 2.0);
```

Exponential Random Variates

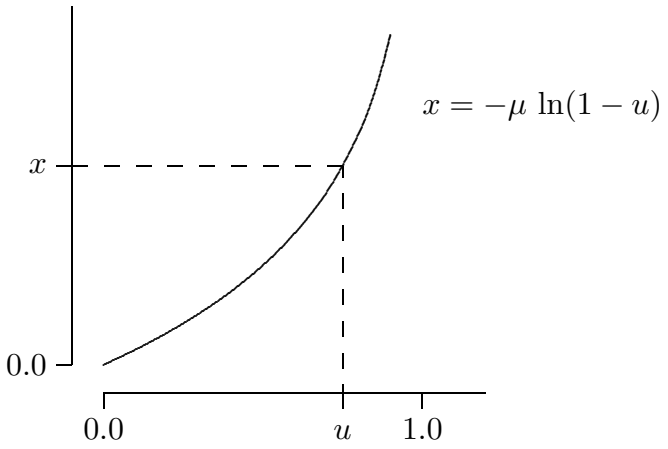
A *Uniform*(a, b) random variable has the property that all values between a and b are equally likely. In most applications this is an unrealistic assumption; instead, some values will be more likely than others. Specifically, there are many discrete-event simulation applications that require a continuous random variate, say x, that can take on any positive value but in such a way that small values of x are more likely than large values.

To generate such a random variate we need a *nonlinear* transformation that maps values of the random number u between 0.0 and 1.0 to values of x between 0.0 and ∞ and does so by "stretching" large values of u much more so than small values. Although a variety of such nonlinear transformation are possible, for example $x = u/(1 - u)$, perhaps the most common is $x = -\mu \ln(1 - u)$ where $\mu > 0$ is a parameter that controls the rate of stretching and $\ln(\cdot)$ is the natural logarithm (base e).* As explained in Chapter 7, this transformation generates what is known as an *Exponential*(μ) random variate.

* The function `log`(x) in the ANSI C library `<math.h>` represents the mathematical natural log function $\ln(x)$. In the equation $x = -\mu \ln(1 - u)$ do not confuse the positive, real-valued parameter μ with the *Uniform*$(0, 1)$ random variate u.

It is often appropriate to generate interarrivals as *Exponential*(μ) random variates, as we will do later in Example 3.1.2. In some important theoretical situations, service times are also modeled this way (see Section 8.5). The geometry associated with the *Exponential*(μ) random variate transformation $x = -\mu \ln(1-u)$ is illustrated in Figure 3.1.1.

Figure 3.1.1. Exponential variate generation geometry.



By inspection we see that the transformation is monotone increasing and, for any value of the parameter $\mu > 0$, the interval $0 < u < 1$ is mapped one-to-one and onto the interval $0 < x < \infty$. That is,

$$\begin{aligned} 0 < u < 1 &\iff 0 < (1-u) < 1 \\ &\iff -\infty < \ln(1-u) < 0 \\ &\iff 0 < -\mu \ln(1-u) < \infty \\ &\iff 0 < x < \infty. \end{aligned}$$

Definition 3.1.1 This ANSI C function generates an *Exponential*(μ) random variate*

```
double Exponential(double μ)                          /* use μ > 0.0 */
{
   return (-μ * log(1.0 - Random()));
}
```

The statistical significance of the parameter μ is that if repeated calls to the function `Exponential`(μ) are used to generate a random variate sample x_1, x_2, ..., x_n then, in the limit as $n \to \infty$, the sample mean (average) of this sample will converge to μ. In the same sense, repeated calls to the function `Uniform`(a, b) will produce a sample whose mean converges to $(a+b)/2$ and repeated calls to the function `Equilikely`(a, b) will also produce a sample whose mean converges to $(a+b)/2$.

* The ANSI C standard says that `log(0.0)` may produce "a range error". This possibility is naturally avoided in the function `Exponential` because the largest possible value of `Random` is less than 1.0. See Exercise 3.1.3.

Example 3.1.2 In a single-server service node simulation, to generate a sequence of random variate *arrival* times $a_1, a_2, a_3, \ldots, a_n$ with an average interarrival time that will converge to μ as $n \to \infty$ it is common to generate *Exponential*(μ) *interarrival* times (see Definition 1.2.3) and then (with $a_0 = 0$) create the arrival times by the assignment

a_i `=` a_{i-1} `+ Exponential(`μ`);` $\qquad i = 1, 2, 3, \ldots, n$

As discussed in Chapter 7, this use of an *Exponential*(μ) random variate corresponds naturally to the idea of jobs arriving *at random* with an arrival rate that will converge to $1/\mu$ as $n \to \infty$. Similarly, as in Example 3.1.1, to generate a random variate sequence of service times $s_1, s_2, s_3, \ldots, s_n$ equally likely to lie anywhere between a and b (with $0 \le a < b$) and with an average service time that converges to $(a + b)/2$, the assignment

s_i `= Uniform(`a`,` b`);` $\qquad i = 1, 2, 3, \ldots, n$

can be used. The average service time $(a + b)/2$ corresponds to a service rate of $2/(a + b)$.

Program ssq2

Program `ssq2` is based on the two stochastic modeling assumptions in Example 3.1.2.* Program `ssq2` is an extension of program `ssq1` in that the arrival times and service times are generated randomly (rather than relying on a trace-driven input) and a complete set of first-order statistics $\bar{r}$, $\bar{w}$, $\bar{d}$, $\bar{s}$, $\bar{l}$, $\bar{q}$, $\bar{x}$ is generated. Note that each time the function `GetArrival()` is called the static variable `arrival`, which represents an *arrival* time, is incremented by a call to the function `Exponential(2.0)`, which generates an *interarrival* time.

Because program `ssq2` generates stochastic data as needed there is essentially no restriction on the number of jobs that can be processed. Therefore, the program can be used to study the *steady-state* behavior of a single-server service node. That is, by experimenting with an increasing number of jobs processed, one can investigate whether or not the service node statistics will converge to constant values, *independent* of the choice of the `rng` initial seed and the initial state of the service node. Steady-state behavior — can it be achieved and, if so, how many jobs does it take to do so — is an important issue that will be explored briefly in this chapter and in more detail in Chapter 8.

Program `ssq2` can also be used to study the *transient* behavior of a single-server service node. The idea in this case is to fix the number of jobs processed at some finite value and run (replicate) the program repeatedly with the initial state of the service node fixed, changing *only* the `rng` initial seed from run to run. In this case replication will produce a natural variation in the service node statistics consistent with the fact that for a fixed number of jobs, the service node statistics are *not* independent of the initial seed or the initial state of the service node. Transient behavior, and its relation to steady-state behavior, will be considered in Chapter 8.

* In the jargon of queuing theory (see Section 8.5), program `ssq2` simulates what is known as an $M/G/1$ queue (see Kleinrock, 1975, 1976, or Gross and Harris, 1985).

Steady-State Statistics

Example 3.1.3 If the *Exponential*(μ) interarrival time parameter is set to $\mu = 2.0$ so that the steady-state arrival rate is $1/\mu = 0.5$ and if the *Uniform*(a, b) service time parameters are set to $a = 1.0$ and $b = 2.0$ respectively so that the steady-state service rate is $2/(a + b) \cong 0.67$, then to $d.dd$ precision the theoretical steady-state statistics generated from an analytic model (exact, not estimates by simulation) program `ssq2` will produce are (see Gross and Harris, 1985)

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	3.83	2.33	1.50	1.92	1.17	0.75

Therefore, although the server is only busy 75% of the time ($\bar{x} = 0.75$), "on average" there are approximately two jobs ($\bar{l} = 23/12 \cong 1.92$) in the service node and a job can expect to spend more time ($\bar{d} = 23/6 - 3/2 \cong 2.33$ time units) in the queue than in service ($\bar{s} = 1.50$ time units). As illustrated in Figure 3.1.2 for the average wait, the number of jobs that must be pushed through the service node to achieve these steady-state statistics is large. To produce this figure, program `ssq2` was modified to print the accumulated average wait every 20 jobs. The results are presented for three choices of the `rng` initial seed.* The solid, horizontal line at height $23/6 \cong 3.83$ represents the steady-state value of $\bar{w}$.

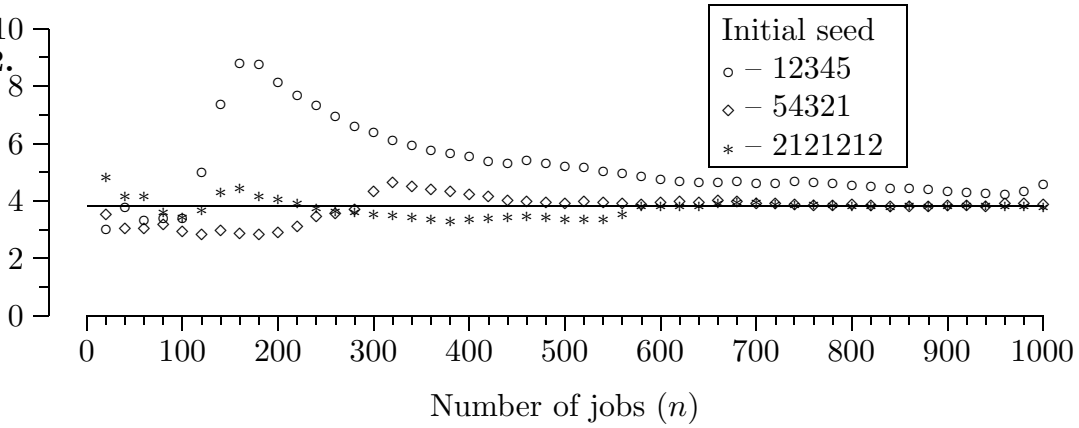


Figure 3.1.2. Average wait times.

In Example 3.1.3 convergence of $\bar{w}$ to the steady-state value 3.83 is slow, erratic, and very dependent on the random variate sequence of stochastic arrival and service times, as manifested by the choice of initial seed. Note, for example, the dramatic rise in average wait beginning at about job 100 associated with the `rng` initial seed 12345. You are encouraged to add diagnostic printing and additional statistics gathering to program `ssq2` to better understand what combination of chance events occurred to produce this rise.

The *Uniform*(a, b) service time model in Example 3.1.3 may not be realistic. Service times seldom "cut off" beyond a minimum and maximum value. More detail will be given in subsequent chapters on how to select realistic distributions for input models.

* There is nothing special about these three initial seeds. Do not fall into the common trap of thinking that some `rng` initial seeds are necessarily better than others.

Example 3.1.3 shows that the stochastic character of the arrival times and service times, as manifested by the choice of `rng` initial seed, has a significant effect on the transition-to-steady-state behavior of a single-server service node. This example also illustrates the use of the library `rng` to conduct *controlled* "what if" experiments. Studies like this are the stuff of discrete-event simulation. Additional examples of this kind of experimentation, and the selection of values of μ, a, and b, are presented in later chapters.

Geometric Random Variates

As discussed in Chapter 2, an *Equilikely*(a, b) random variate is the discrete analog of a continuous *Uniform*(a, b) random variate. Consistent with this characterization, one way to generate an *Equilikely*(a, b) random variate is to generate a *Uniform*$(a, b + 1)$ random variate instead (note that the upper limit is $b + 1$, not b) and then convert (cast) the resulting floating-point result to an integer. That is, if a and b are integers with $a < b$ and if x is a *Uniform*$(a, b + 1)$ random variate then $\lfloor x \rfloor$ is an *Equilikely*(a, b) random variate.

Given the analogy between *Uniform*(a, b) and *Equilikely*(a, b) random variates, it is reasonable to expect that there is a discrete analog to a continuous *Exponential*(μ) random variate; and there is. Specifically, if x is an *Exponential*(μ) random variate, let y be the *discrete* random variate defined by $y = \lfloor x \rfloor$. For a better understanding of this discrete random variate, let $p = \Pr(y \neq 0)$ denote the probability that y is not zero. Since x is generated as $x = -\mu \ln(1 - u)$ with u a *Uniform*$(0, 1)$ random variate, $y = \lfloor x \rfloor$ will not be zero if and only if $x \geq 1$. Equivalently

$$\begin{aligned} x \geq 1 &\iff -\mu \ln(1 - u) \geq 1 \\ &\iff \ln(1 - u) \leq -1/\mu \\ &\iff 1 - u \leq \exp(-1/\mu) \end{aligned}$$

where $\exp(-1/\mu)$ is $e^{-1/\mu}$, and so $y \neq 0$ if and only if $1 - u \leq \exp(-1/\mu)$. Like u, $1 - u$ is also a *Uniform*$(0, 1)$ random variate. Moreover, for any $0 < p < 1$, the condition $1 - u \leq p$ is true with probability p (see Section 7.1). Therefore, $p = \Pr(y \neq 0) = \exp(-1/\mu)$.

If x is an *Exponential*(μ) random variate and if $y = \lfloor x \rfloor$ with $p = \exp(-1/\mu)$, then it is conventional to call y a *Geometric*(p) random variable (see Chapter 6). Moreover, it is conventional to use p rather than $\mu = -1/\ln(p)$ to define y directly by the equation

$$y = \lfloor \ln(1 - u)/\ln(p) \rfloor.$$

Definition 3.1.2 This ANSI C function generates a *Geometric*(p) random variate*

```
long Geometric(double p)                          /* use 0.0 < p < 1.0 */
{
   return ((long) (log(1.0 - Random()) / log(p)));
}
```

* Note that `log(0.0)` is avoided in this function because the largest possible value returned by `Random` is less than 1.0.

In addition to its significance as $\Pr(y \neq 0)$, the parameter p is also related to the mean of a *Geometric*(p) sample. Specifically, if repeated calls to the function `Geometric(p)` are used to generate a random variate sample $y_1, y_2, \ldots, y_n$ then, in the limit as $n \to \infty$, the mean of this sample will converge to $p/(1-p)$. Note that if p is close to 0.0 then the mean will be close to 0.0. At the other extreme, if p is close to 1.0 then the mean will be large.

In the following example, a *Geometric*(p) random variate is used as part of a *composite* service time model. In this example the parameter p has been adjusted to make the average service time match that of the *Uniform*(1.0, 2.0) server in program `ssq2`.

Example 3.1.4 Usually one has sufficient information to argue that a *Uniform*(a, b) random variate service time model is not appropriate; instead, a more sophisticated model is justified. Consider a hypothetical server that, as in program `ssq2`, processes a stream of jobs arriving, at random, with a steady-state arrival rate of 0.5 jobs per minute. The service requirement associated with each arriving job has two stochastic components:

- the *number* of service tasks is one plus a *Geometric*(0.9) random variate;
- the *time* (in minutes) per task is, independently for each task, a *Uniform*(0.1, 0.2) random variate.

In this case, program `ssq2` would need to be modified by including the function `Geometric` from Definition 3.1.2 and changing the function `GetService` to the following.

```
double GetService(void)
{
   long   k;
   double sum   = 0.0;
   long   tasks = 1 + Geometric(0.9);

   for (k = 0; k < tasks; k++)
     sum += Uniform(0.1, 0.2);
   return (sum);
}
```

With this modification, the population steady-state statistics from the analytic model to *d.dd* precision that will be produced are:

$\bar{r}$	$\bar{w}$	$\bar{d}$	$\bar{s}$	$\bar{l}$	$\bar{q}$	$\bar{x}$
2.00	5.77	4.27	1.50	2.89	2.14	0.75

Program `ssq2` will produce results that converge to these values for a very long run. When compared with the steady-state results in Example 3.1.3, note that although the arrival rate $1/\bar{r} = 0.50$, the service rate $1/\bar{s} = 0.67$, and the utilization $\bar{x} = 0.75$ are the same, the other four statistics are significantly larger than in Example 3.1.3. This difference illustrates the sensitivity of the performance measures to the service time distribution. This highlights the importance of using an accurate service time model. See Exercise 3.1.5.