LSZ Reduction in QFT and Lattice Systems

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Overview

1. Framework of Lattice Scattering
2. LSZ reduction formula
3. Path Integrals and Semi-Classical approximations

Quasi-local Algebra

- Lattice $\Gamma = \mathbb{Z}^d$ with local observable algebras $\mathcal{A}(x)$
- Local Algebras $\mathcal{A}(\Lambda) = \otimes_{x\in\Lambda}\mathcal{A}(x)$ (for finite $\Lambda \subset \Gamma$)
- Quasilocal algebra $\mathcal{A} = \overline{\bigcup_\Lambda \mathcal{A}(\Lambda)}$

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Almost-Local observables

An observable $A \in \mathcal{A}$ is *almost local*, if there are local observables $A_n \in \mathcal{A}(X_n)$ such that

$$\|A - A_n\| \in O((\operatorname{diam} X_n)^{-\infty})$$

Local Interactions

- For *finite* Λ, we assume an interaction

$$\Phi(\Lambda) \in \mathcal{A}(\Lambda)$$

- Local Hamiltonians

$$H_\Lambda = \sum_{X \subset \Lambda} \Phi(X)$$

- Local time evolution

$$\tau_t^\Lambda(A) = e^{itH_\Lambda} A e^{-itH_\Lambda}$$

Global dynamics and Lieb-Robinson

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- But if the interaction is “sufficiently small”, we have for any $A \in \mathcal{A}(X)$ and $B \in \mathcal{A}(Y)$:

$$\left\| [\tau_t^\Lambda(A), B] \right\| \leqslant const \cdot e^{-\lambda(d(X,Y) - V_\lambda t)}$$

- This is used to prove existence of global time evolution

$$\tau_t(A) = \lim_{\Lambda\to\Gamma} \tau_t^\Lambda(A)$$

- Notation: $A(t,x) = \tau_t(\tau_x(A))$

Vacuum Representation

- Fix a translation invariant state ω
- GNS representation $(\mathcal{H}, \pi, \Omega)$
- Assume translation group $U^{(d+1)}(x, t)$ is strongly continuous,
- and the generator H is positive with isolated eigenvalue 0.

EM Spectrum

$$U^{(d+1)}(x,t) = \int_{\mathbb{R}\times\hat{\Gamma}} e^{iEt-ipx}\, dP(E,p)$$
$$SpU = \operatorname{supp} dP$$

where $\hat{\Gamma} = T^d$ is the dual lattice.

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Mass Shell

- Dispersion relation $\omega : \hat{\Gamma} \to \mathbb{R}$ such that
- $h = \operatorname{graph}(\omega) \subset SpU$
- One-particle space $\mathcal{H}_\omega = P(h)\mathcal{H}$

Arveson Spectrum

For any observable $A \in \mathcal{A}$, we define

$$Sp_A\tau = \operatorname{supp} \check{A}$$
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EM Transfer Relation

For any $\psi \in P(\Delta)\mathcal{H}$ and $A \in \mathcal{A}$ it is

$$\pi(A)\psi \in P(\overline{\Delta + Sp_A\tau})\mathcal{H}$$

Wave Packets

- Given a smooth function $\hat{g} \in C^\infty(\hat{\Gamma})$, we define
- the wave packets

$$g_t(x) = (2\pi)^{-d/2} \int_{\hat{\Gamma}} dp \; e^{-i\omega(p)t+ipx} \hat{g}(p)$$

$$f_t(x) = i(2\pi)^{-d/2} \int_{\hat{\Gamma}} \frac{dp}{2\omega(p)} e^{-i\omega(p)t+ipx} \hat{g}(p)$$

- and the velocity-support $V(g) = \{\nabla\omega(p) \mid p \in \operatorname{supp} \hat{g}\}$.

Creation Operators

Let $B^* \in \mathcal{A}_{a-loc}$ with compact Arveson spectrum and $Sp_{B^*}\tau \cap SpU \subset h$. Then

$$B_t^*(g_t) = (2\pi)^{-d/2} \sum_{x \in \Gamma} B^*(t,x) g_t(x)$$

is a *creation operator* because

$$B_t^*(g_t)\Omega \in \mathcal{H}_\omega$$

(by the EM transfer relation)

Asymptotic States

- $\psi_i = B^*_{i,t}(g_{i,t})\Omega$ one-particle states with $V(g_i) \cap V(g_j) = \emptyset$ for $i \neq j$
- $\psi_1 \times_{out} \ldots \times_{out} \psi_n = \lim_{t\to\infty} B^*_{1,t}(g_{1,t}) \ldots B^*_{n,t}(g_{n,t})\Omega$

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S-Matrix

$$\begin{aligned} &S(\psi_1, \ldots, \psi_n; \psi'_1, \ldots, \psi'_{n'}) \\ &\qquad = \left\langle \psi_1 \times_{out} \ldots \times_{out} \psi_n \middle| \psi'_1 \times_{in} \ldots \times_{in} \psi'_{n'} \right\rangle \end{aligned}$$

Lattice Gradient

- $\tilde{\nabla} : L^2(\Gamma) \to L^2(\Gamma)$
  $\tilde{\nabla} g = i\mathcal{F}^{-1} p \mathcal{F} g$
- $\mathcal{F} : L^2(\Gamma) \to L^2(\hat{\Gamma})$ Fourier transform
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Properties

- $\omega^2(-i\tilde{\nabla})$ is a (bounded) self-adjoint operator on $L^2(\Gamma)$
- Klein-Gordon equation $(\partial_t^2 + \omega^2(-i\tilde{\nabla}))g_t(x) = 0$ for any wave-packet g.

LSZ reduction for lattice systems

$$S_c(\psi_1, \psi_2; \psi_1', \psi_2') = \int dt_1 dt_2 \int dt_1' dt_2' \sum_{\substack{x_1, x_2 \in \Gamma \\ x_1', x_2' \in \Gamma}} \overline{f_1(t_1, x_1) f_2(t_2, x_2)} f_1'(t_1', x_1') f_2'(t_2', x_2')$$
$$K_1 K_2 K_1' K_2' \langle \Omega | T(B_1(t_1, x_1) B_2(t_2, x_2) B_1'^{*}(t_1', x_1') B_2'^{*}(t_2', x_2')) | \Omega \rangle_T$$

where

- $\psi_i = B^{*}_{i,t}(g_{i,t})\Omega$ and $\psi_i' = B'^{*}_{i,t}(g'_{i,t})\Omega$, are one-particle states.
- $K_i = (\partial^2_{t_i} + \omega^2(-i\tilde{\nabla}_i))$ are Klein-Gordon operators

Connected part of S-Matrix

$$S(A;A') = \sum_k \sum_{\substack{\{A_j\} \\ \{A'_j\}}} \prod_{i=1}^{k} S_c(A_i;A'_i)$$

where $A = A_1 \sqcup \ldots \sqcup A_k$ and $A' = A'_1 \sqcup \ldots \sqcup A'_k$

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Truncated Expectation Values

$$\langle\Omega|T(B_1(x_1)\ldots B_n(x_n))|\Omega\rangle = \sum_k \sum_{\{A_j\}} \prod_{i=1}^k \langle\Omega|T(B_{A_i})|\Omega\rangle_T$$

where $\{1,\ldots n\} = A_1 \sqcup \ldots \sqcup A_k$.

Spin System

- Fix a spin $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$
- local Hilbert space $\mathcal{H}_x = \mathbb{C}^{2s+1} = span\{|-s\rangle_x, \ldots, |s\rangle_x\}$
- ladder operators

$$\begin{aligned} S_x^3|m\rangle_x &= m|m\rangle_x \\ S_x^{\pm}|m\rangle_x &= const \cdot |m \pm 1\rangle_x \\ S_x^{\pm}|\pm s\rangle_x &= 0 \end{aligned}$$

Coherent States

For $z \in \mathbb{C}^\Gamma$, define

$$\left|z_x\right\rangle_x = e^{z_x S_x^-} \left|s\right\rangle_x$$
$$\left|z\right\rangle = \otimes_x \left|z_x\right\rangle_x$$

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Properties

- Scalar product: $\langle w|z\rangle = \prod_x (1 + \overline{w_x} z_x)^{2s}$
- Partition of Unity: $\mathbb{1} = \prod_x \int_{\mathbb{C}} \frac{d\mu(z_x)}{(1+|z_x|^2)^{2s}} |z\rangle\langle z|$
- with measure $d\mu(z_x) = \frac{2s+1}{(1+|z_x|^2)^2} \frac{d^2 z_x}{\pi}$

Symbol

For any operator O we define the *symbol* o as

$$o : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

$$o(w, z) = \frac{\langle w|O|z\rangle}{\langle w|z\rangle}$$

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Examples

$$s_x^1(w, z) = s \frac{z_r + \overline{w_x}}{1 + \overline{w_r} z_x}$$
$$s_x^2(w, z) = \frac{s}{i} \frac{z_x - \overline{w_x}}{1 + \overline{w_x} z_x}$$
$$s_x^3(w, z) = s \frac{1 - \overline{w_x} z_x}{1 + \overline{w_x} z_x}$$

Path Integral for Time Evolution

$$\langle z^f | e^{-itH} | z^i \rangle = \int_{\substack{z(0)=z^i \\ z(t)=z^f}} D\mu(z) \exp\left(-i \int_0^t d\tau L(z(\tau), \dot{z}(\tau)) \right)$$

with

$$L(z, \dot{z}) = h(z, z) + is \sum_{x \in \Gamma} \frac{\overline{\dot{z}_x} z_x - \dot{z}_x \overline{z_x}}{1 + |z_x|^2}$$

and

$$\int_{\substack{z(0)=z^i \\ z(t)=z^f}} D\mu(z)(\cdot) = const \cdot \lim_{n \to \infty} \int \prod_{k=1}^{n-1} d\mu(z_x^k)(\cdot)$$

using the substitution $z(\frac{tk}{n}) = z^k$

Path Integral for Time-Ordered Products

$$\langle z^f, t^f | T(O_n(t_n) \dots O_1(t_1)) | z^i, t^i \rangle$$
$$= \int D\mu(z)\; o_1(\overline{z(t_1)}, z(t_1)) \dots o_n(\overline{z(t_n)}, z(t_n)) \times e^{-i \int d\tau L(z, \dot{z})}$$

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Semi-Classical Evaluation

- Euler-Lagrange equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{z}_x} - \frac{\partial L}{\partial z_x}$$

- leads to equation of motion

$$0 = \frac{\partial h}{\partial z_x} + \frac{2is\dot{\overline{z}}_x}{(1 + |z_x|^2)^2}$$

Ising model ($s = 1/2$)

- $H = -\frac{1}{2}\sum_x \left(\sigma_x^{(3)} - 1\right) - \epsilon \sum_{|x-y|=1} \sigma_x^{(1)}\sigma_y^{(1)}$

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- e.o.m.: $0 = i\dot{\overline{z_x}} - \overline{z_x} - 2\epsilon(1 - \overline{z_x}^2) \sum_{\substack{y \\ |x-y|=1}} \frac{z_y + \overline{z_y}}{1 + |z_y|^2}$

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Free solution ($\epsilon = 0$)

- $0 = i\dot{\overline{z_x}} - \overline{z_x} \Rightarrow z(t) = const \cdot e^{-it}$
- direct solution: $e^{-itH}|z\rangle = |e^{-it}z\rangle$

Problems with Perturbation theory

- "free" $\epsilon = 0$ theory is problematic:
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Outlook

Possible future directions:

- Use lattice LSZ formula to prove $S \neq \mathbb{1}$ in a concrete model.
- Derive rigorous path-integral expressions.
- Solve the semi-classical equations of motion.