Nonperturbative Renormalization in Lattice QCD (with open boundary conditions)

Simon Bürger

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Overview

1. Intro: Why Renormalization
2. Our method
3. Open boundaries

Why Renormalization? (practical)

- Naive calculations often yield infinite results:
- Loop integrals perturbation theory: $\int \frac{d^4 k}{(2\pi)^4} \frac{1}{(k+p)^2 k^2} = \infty$
- Pathintegral is usually not mathematically well-defined.

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Regularization schemes

- Dimensional Regularization: $\int \frac{d^4k}{(2\pi)^4} \rightarrow \mu^{2\epsilon} \int \frac{d^{4-2\epsilon}k}{(2\pi)^{4-2\epsilon}}$
- Lattice Regularization: $\int d^4x \rightarrow a^4 \sum_x$

Parameter Renormalization

- Idea: Make parameters of the Lagrangian depend on cutoff
- Coupling: $g_{\text{bare}} = Z_\alpha^{1/2}(a)\ g_R$
- Masses: $m_{\text{bare}} = Z_m(a)\ m_R$

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- On the lattice:
  - Set the lattice spacing to match $r_0 = 0.5\text{fm}$
  - Tune κ_l,κ_s to match $m_\pi = m_{\pi,\text{phys}}$ and $m_K = m_{K,\text{phys}}$
  - $\Rightarrow$ Observables from LQCD alone need no further renormalization (e.g. hadron masses)

Operator Renormalization

Physical observables are finite now, but general operators are not. Example:

- Quark propagator $S_{\alpha\beta}(p) = \frac{1}{V}\sum_{x,y} e^{-ip(x-y)}\langle u(x)\overline{u}(y)\rangle$
- Quark-Bilinears:
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Note: These Z will cancel in physical observables
$\Rightarrow$ Precise definition is somewhat arbitrary

Renormalization schemes by example (continuum perturbation)

- $O_{\{\mu\nu\}} = \frac{1}{2}\overline{u}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right) d$
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- $\Gamma^{\text{RI'-MOM}} \approx \Gamma_0$ at $p^2 = \mu^2$

Quark-bilinear operators

- Currents $O_\Gamma = \overline{u}\Gamma d$, with $\Gamma \in \{\mathbb{1}, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}, \gamma_5\}$
- Ops with derivative, e.g.:
  $O_{v_{2,a}} = O_{\{14\}} = \overline{u}\left(\gamma_1 \overleftrightarrow{D}_4 + \gamma_4 \overleftrightarrow{D}_1\right) d$

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Rome-Southampton approach

- Quark field: $\frac{1}{12}\,\mathrm{tr}(S_R(p)^{-1}S_0(p)) \stackrel{!}{=} 1$
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- Take free lattice operators as Born term, i.e.
  $S_0(p) = \frac{-i}{\sum_\mu \sin^2(p_\mu)} \sum_\mu \gamma_\mu \sin(p_\mu)$

Numerical method

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$$= \frac{1}{V} \sum_{z,z'} \gamma_5 \hat{S}(z,p)^{+} \gamma_5 J(z,z') \hat{S}(z',p)$$
- $\hat{S}(z,p) = \sum_x S(z,x) e^{ipx}$ computed by solving the lattice Dirac equation
$$\sum_z M(y,z) \hat{S}(z,p) = e^{ipx}$$

Numerical method (cont.)

- Fix Landau gauge on the lattice my maximizing average Link $F[U] = \frac{1}{4V} \sum_{x,\mu} \mathrm{Re}\,\mathrm{tr}\, U_\mu(x)$

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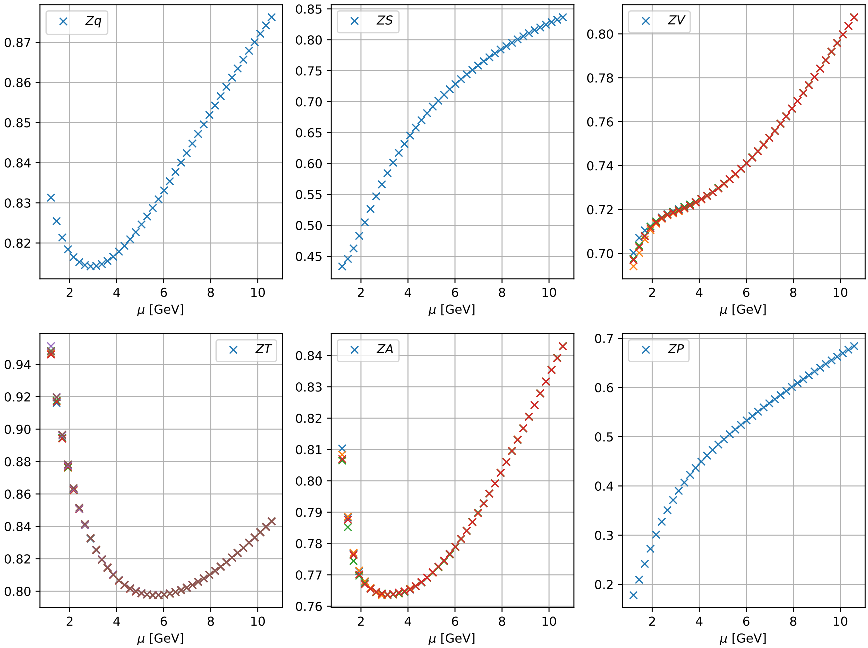
- In the end: Chiral extrapolation $Z(m_\pi) = z_0 + z_1(r_0 m_\pi)^2$



Zq
ZS
ZV
ZT
ZA
ZP
μ [GeV]

Conversion between schemes

- We use Lattice regulator and RI'-(S)MOM, most other physicists use dim. regulator and $\overline{MS}$
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- $Z^{\overline{\text{MS}}}(\mu, a) = Z^{\overline{\text{MS}}}_{\text{RI'-MOM}}(\mu) Z^{\text{RI'-MOM}}_{\text{bare}}(\mu, a)$
- $Z^{\text{RGI}}(a) = \Delta Z^{\overline{\text{MS}}}(\mu) Z^{\overline{\text{MS}}}_{\text{RI'-MOM}}(\mu) Z^{\text{RI'-MOM}}_{\text{bare}}(\mu, a)$

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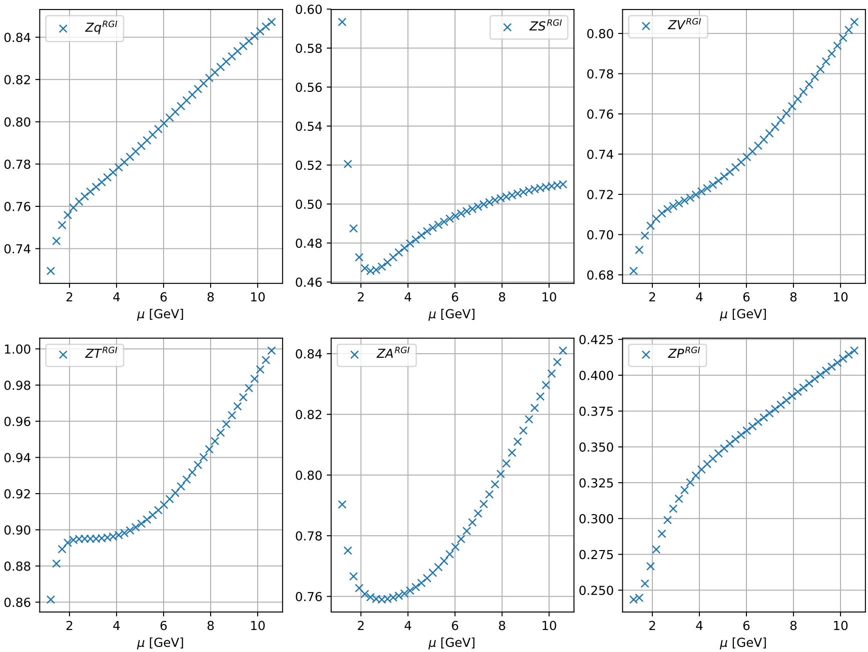
- Window problem: $\Lambda^{\overline{\mathrm{MS}}}_{\mathrm{QCD}} \overset{!}{\ll} \mu \overset{!}{\ll} a^{-1}$



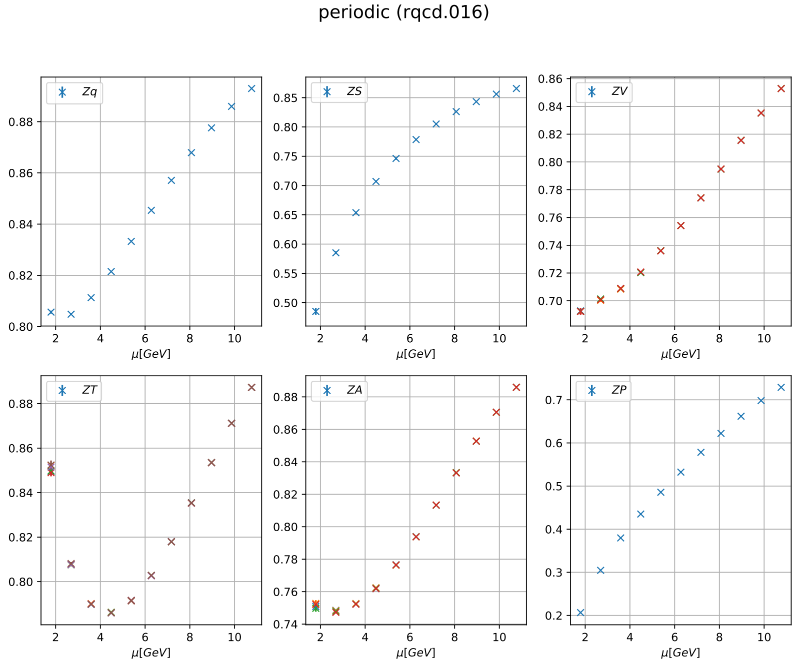
Zq^{RGI}
ZS^{RGI}
ZV^{RGI}
ZT^{RGI}
ZA^{RGI}
ZP^{RGI}
μ [GeV]

Open boundary conditions (naive approach)

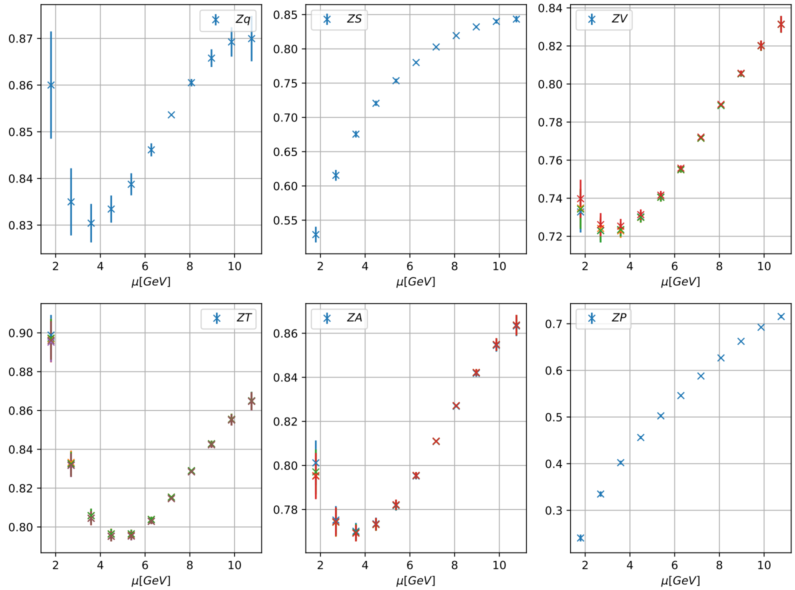
- Restrict source and operator to the bulk of the lattice

$$\sum_z M(y,z)\hat{S}(z,p) = \begin{cases} e^{ipx} & \text{bulk} \\ 0 & \text{boundary} \end{cases}$$

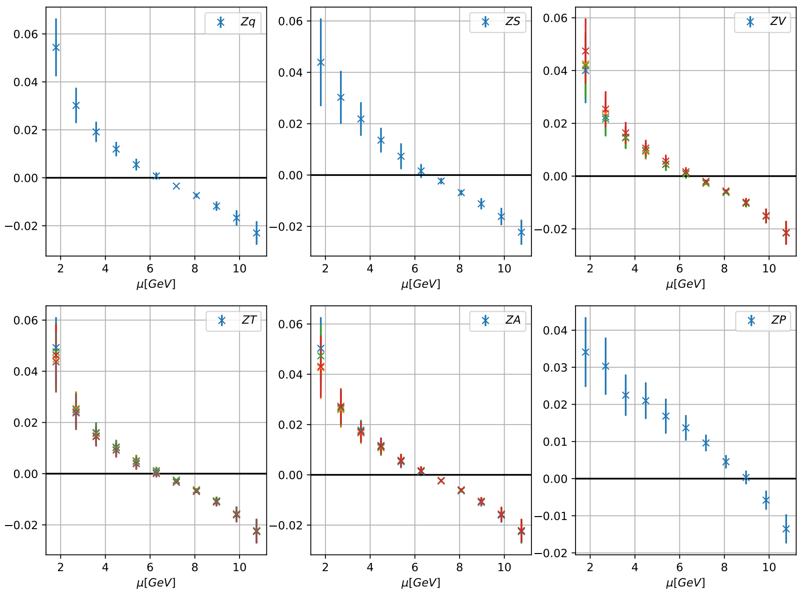
$$G(p) = \frac{1}{V_{\text{bulk}}} \sum_{z,z' \text{ in bulk}} \gamma_5 \hat{S}(z,p)^{+} \gamma_5 J(z,z') \hat{S}(z',p)$$


periodic (rqcd.016)
Zq
ZS
ZV
ZT
ZA
ZP
μ[GeV]

open (H101) (boundary = 2*32)



open - periodic (boundary = 2*32)



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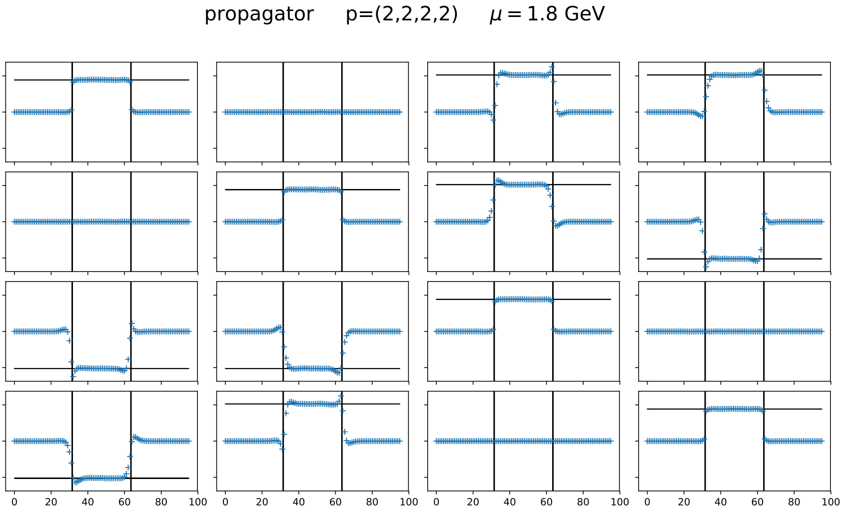
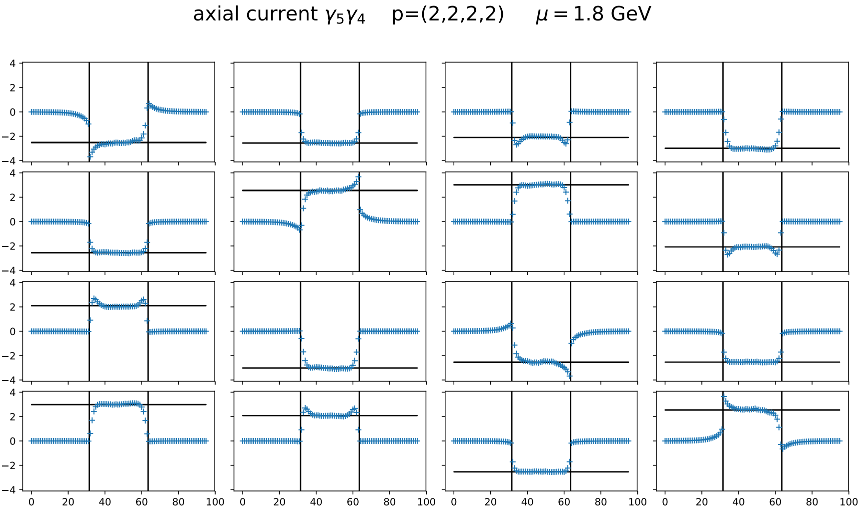
  - Only compute ratios of renormalization factors
- Better idea: actually Look at contribution of individual timeslices


propagator p=(2,2,2,2) $\mu = 1.8$ GeV
2
0
−2
0
20
40
60
80
100



axial current $\gamma_5\gamma_4$ p=(2,2,2,2) $\mu = 1.8$ GeV
4
2
0
−2
−4
0
20
40
60
80
100

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- Restrict source only mildly (boundary $= 2 \cdot 4$)
- Restrict averaging more severely (boundary $= 2 \cdot 32$)

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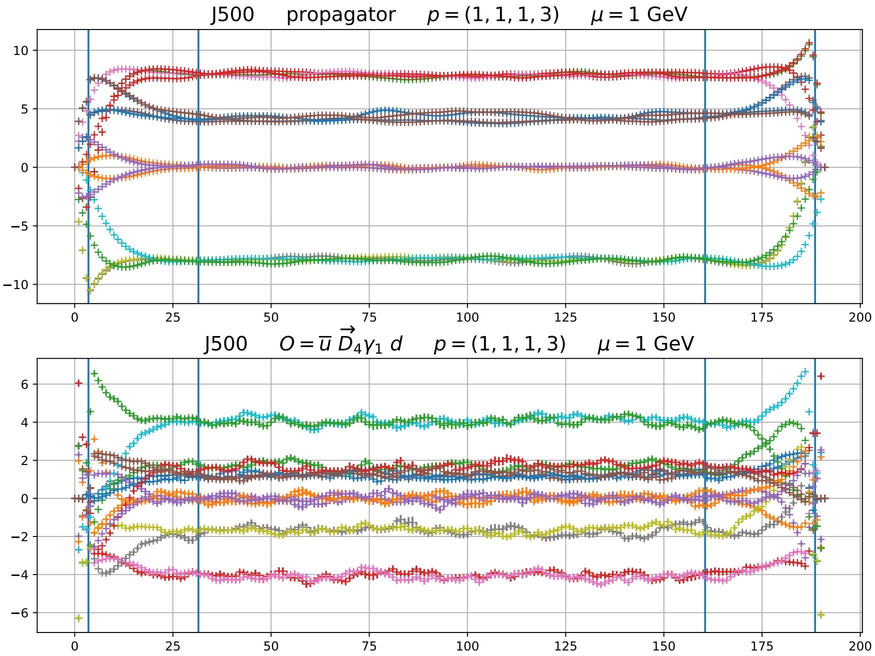
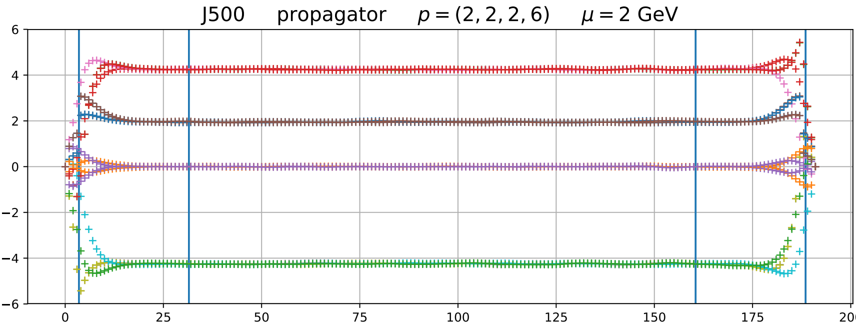
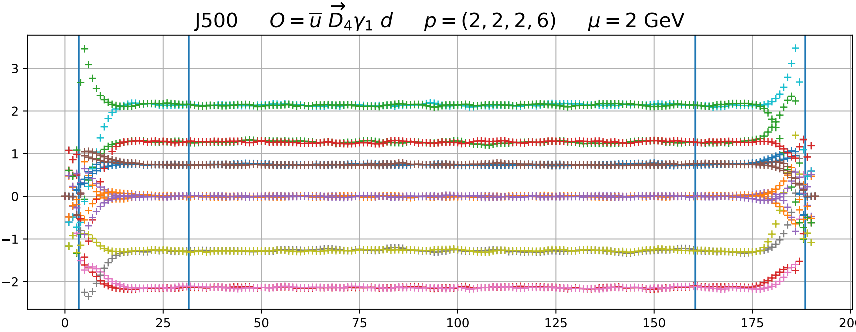
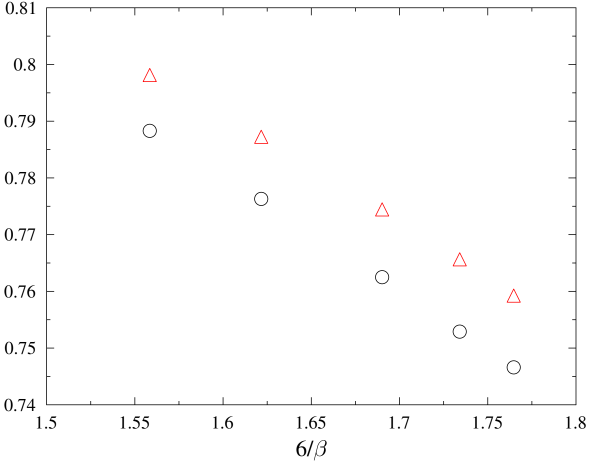
- Also: fractional momenta in time-direction instead of twisted boundaries


J500 propagator $p = (1, 1, 1, 3)$ $\mu = 1$ GeV
10
5
0
−5
−10
0
25
50
75
100
125
150
175
200
J500 $O = \overline{u}\ \overrightarrow{D}_4 \gamma_1\ d$ $p = (1, 1, 1, 3)$ $\mu = 1$ GeV
6
4
2
0
−2
−4
−6
0
25
50
75
100
125
150
175
200


J500 propagator $p = (2, 2, 2, 6)$ $\mu = 2$ GeV
6
4
2
0
−2
−4
−6
0
25
50
75
100
125
150
175
200


J500 $O = \overline{u}\, \overrightarrow{D}_4 \gamma_1\, d$ $p = (2, 2, 2, 6)$ $\mu = 2$ GeV
3
2
1
0
−1
−2
0
25
50
75
100
125
150
175
200



Z_A (circle = MOM, triangle = SMOM)

Summary

- We now have renormalization for large β / small *a* using open boundary conditions
- As a bonus: explicit visualization of boundary effects