Synthesis of RLC Circuits

INTRODUCTION

In the study of electrical networks, broadly there are two topics: 'Network Analysis' and 'Network Synthesis'. Any network consists of excitation, response and network function. In network analysis, network and excitation are given, whereas the response has to be determined. In network synthesis, excitation and response are given, and the network has to be determined. Thus, in network synthesis we are concerned with the realisation of a network for a given excitation-response characteristic. Also, there is one major difference between analysis and synthesis. In analysis, there is a unique solution to the problem. But in synthesis, the solution is not unique and many networks can be realised.

The first step in synthesis procedure is to determine whether the network function can be realised as a physical passive network. There are two main considerations; causality and stability. By *causality* we mean that a voltage cannot appear at any port before a current is applied or vice-versa. In other words, the response of the network must be zero for $t < 0$. For the network to be stable, the network function cannot have poles in the right half of the s-plane. Similarly, a network function cannot have multiple poles on the $j\omega$ axis.

HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when s is real.

(b) The roots of $P(s)$ have real parts which are zero or negative.

Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.

2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.

(a) $s^4 + 4s^3 + 3s + 2$

(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

Solution (a) In the given polynomial, the term s^2 is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

Test whether the polynomial $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 3s\,)\,s^4 + 5s^2 + 4\,(\,s$$

$$\underline{s^4 + 3s^2}$$

$$2s^2 + 4\Big)\,s^3 + 3s\left(\frac{1}{2}s\right.$$

$$\underline{s^3 + 2s}$$

$$s)\,2s^2 + 4\,(2s$$

$$\underline{2s^2}$$

$$4\Big)\,s\left(\frac{1}{4}s\right.$$

$$\underline{s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Test whether the polynomial $P(s) = s^3 + 4s^2 + 5s + 2$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)\ n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$4s^2 + 2 \Big) s^3 + 5s \Big(\frac{1}{4}s$$

$$\underline{s^3 + \frac{2}{4}s}$$

$$\frac{9}{2}s \Big) 4s^2 + 2 \Big(\frac{8}{9}s$$

$$\underline{4s^2}$$

$$2 \Big) \frac{9}{2}s \Big(\frac{9}{4}s$$

$$\underline{\frac{9}{2}s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Test whether the polynomial $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 6s^2 + 8$

Odd part of $P(s) = n(s) = 7s^3 + 21s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
7s^3 + 21s \,\big)\, s^4 + 6s^2 + 8 \,\big(\, \frac{1}{7}s \\
\qquad\qquad\quad \underline{s^4 + 3s^2} \\
\qquad\qquad\qquad 3s^2 + 8 \,\big)\, 7s^3 + 21s \,\big(\, \frac{7}{3}s \\
\qquad\qquad\qquad\qquad\qquad \underline{7s^3 + \frac{56}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\quad \frac{7}{3}s \,\big)\, 3s^2 + 8 \,\big(\, \frac{9}{7}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{3s^2} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 8 \,\big)\, \frac{7}{3}s \,\big(\, \frac{7}{24}s \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{\frac{7}{3}s} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Test whether the polynomial P(s) is Hurwitz.

$$P(s) = s^5 + s^3 + s$$

Solution Since the given polynomial contains odd functions only, it is not possible to perform continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 3s^2 + 1$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$5s^4+3s^2+1\,\overline{)\,s^5+s^3+s\,}\left(\frac{1}{5}s\right.$$

$$\underline{s^5+\frac{3}{5}s^3+\frac{1}{5}s}$$

$$\frac{2}{5}s^3+\frac{4}{5}s\,\overline{)\,5s^4+3s^2+1\,}\left(\frac{25}{2}s\right.$$

$$\underline{5s^4+10s^2}$$

$$-7s^2+1\,\overline{)\,\frac{2}{5}s^3+\frac{4}{5}s\,}\left(-\frac{2}{35}s\right.$$

$$\underline{\frac{2}{5}s^3-\frac{2}{35}s}$$

$$\frac{26}{35}s\,\overline{)\,-7s^2+1\,}\left(-\frac{245}{26}s\right.$$

$$\underline{-7s^2}$$

$$1\,\overline{)\,\frac{26}{35}s\,}\left(\frac{26}{35}s\right.$$

$$\underline{\frac{26}{35}s}$$

$$0$$

Since the third and fourth quotient terms are negative, $P(s)$ is not Hurwitz.

There is another method to test a Hurwitz polynomial. In this method, we construct the Routh–Hurwitz array for the required polynomial.

Let $\quad P(s) = a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0$

The Routh–Hurwitz array is given by,

$$
\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_n & b_{n-1} & b_{n-2} & \ldots \\
s^{n-3} & c_n & c_{n-1} & & \ldots \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
s^1 & . & & & \\
s^0 & . & & &
\end{array}
$$

The coefficients of s^n and s^{n-1} rows are directly written from the given equation.

where
$$b_n = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-2} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$c_n = \frac{b_n a_{n-3} - a_{n-1} b_{n-1}}{b_n}$$

$$c_{n-1} = \frac{b_n a_{n-5} - a_{n-1} b_{n-2}}{b_n}$$

Hence, for polynomial $P(s)$ to be Hurwitz, there should not be any sign change in the first column of the array.

Test whether $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 8 \\ s^3 & 7 & 21 & \\ s^2 & 3 & 8 & \\ s^1 & \frac{7}{3} & 0 & \\ s^0 & 8 & & \end{array}$$

Since all the elements in the first column are positive, the polynomial $P(s)$ is Hurwitz.

Determine whether $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is Hurwitz.*

Solution The Routh array is given by,

$$
\begin{array}{c|ccc}
s^4 & 1 & 2 & 2 \\
s^3 & 1 & 3 & \\
s^2 & -1 & 2 & \\
s^1 & 5 & 0 & \\
s^0 & 2 & &
\end{array}
$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Test whether the polynomial $P(s) = s^8 + 5s^6 + 2s^4 + 3s^2 + 1$ is Hurwitz.

Solution The given polynomial contains even functions only.

$$P'(s) = 8s^7 + 30s^5 + 8s^3 + 6s$$

The Routh array is given by,

$$\begin{array}{c|ccccc}
s^8 & 1 & 5 & 2 & 3 & 1 \\
s^7 & 8 & 30 & 8 & 6 & 0 \\
s^6 & 1.25 & 1 & 2.25 & 1 & \\
s^5 & 23.6 & -6.4 & -0.4 & 0 & \\
s^4 & 1.33 & 2.27 & 1 & & \\
s^3 & -46.6 & -18.14 & 0 & & \\
s^2 & 1.75 & 1 & & & \\
s^1 & 8.49 & & & & \\
s^0 & 1 & & & &
\end{array}$$

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Test whether $P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ is Hurwitz.

Solution The Routh array is given by,

$$
\begin{array}{c|ccc}
s^5 & 1 & 45 & 44 \\
s^4 & 12 & 60 & 48 \\
s^3 & 40 & 40 & \\
s^2 & 48 & 48 & \\
s^1 & 0 & 0 & \\
s^0 & & &
\end{array}
$$

Notes: When all the elements in any one row is zero, the following steps are followed:

(i) Write an auxiliary equation with the help of the coefficients of the row just above the row of zeros.

(ii) Differentiate the auxiliary equation and replace its coefficient in the row of zeros.

(iii) Proceed for the Routh test.

Auxiliary equation,

$$A(s) = 48s^2 + 48$$
$$A'(s) = 96s$$

$$\begin{array}{c|ccc}
s^5 & 1 & 45 & 44 \\
s^4 & 12 & 60 & 48 \\
s^3 & 40 & 40 & \\
s^2 & 48 & 48 & \\
s^1 & 96 & 0 & \\
s^0 & 48 & &
\end{array}$$

Since there is no sign change in the first column of the array, the polynomial $P(s)$ is Hurwitz.

Check whether $P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ is Hurwitz.

Solution The Routh array is given by,

$$
\begin{array}{c|cccc}
s^6 & 2 & 13 & 56 & 25 \\
s^5 & 1 & 6 & 25 & \\
s^4 & 1 & 6 & 25 & \\
s^3 & 0 & 0 & 0 & \\
s^2 & & & & \\
s^1 & & & & \\
s^0 & & & &
\end{array}
$$

$$A(s) = s^4 + 6s^2 + 25$$
$$A'(s) = 4s^3 + 12s$$

Now, the Routh array will be given by,

$$
\begin{array}{c|cccc}
s^6 & 2 & 13 & 56 & 25 \\
s^5 & 1 & 6 & 25 & \\
s^4 & 1 & 6 & 25 & \\
s^3 & 4 & 12 & & \\
s^2 & 3 & 25 & & \\
s^1 & -21.3 & & & \\
s^0 & 25 & & &
\end{array}
$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Determine the range of values of 'a' so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & a & 3 \\ s^3 & 1 & 2 & \\ s^2 & a-2 & 3 & \\ s^1 & \dfrac{2a-7}{a-2} & & \\ s^0 & 3 & & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive, i.e.,

$$a - 2 > 0$$

$$a > 2$$

and

$$\frac{2a-7}{a-2} > 0$$

$$a > \frac{7}{2}$$

Hence, $P(s)$ will be Hurwitz when $a > \frac{7}{2}$.

Determine the range of values of k so that the polynomial $P(s) = s^3 + 3s^2 + 2s + k$ is Hurwitz.

Solution The Routh array is given by,

$$
\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 3 & k \\
s^1 & \dfrac{6-k}{3} & 0 \\
s^0 & k &
\end{array}
$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive,

i.e., $$\frac{6-k}{3} > 0$$

$$6 - k > 0$$

i.e., $k < 6$ and $k > 0$

Hence, $P(s)$ will be Hurwitz for $0 < k < 6$.

POSITIVE REAL FUNCTIONS

A function $F(s)$ is positive real if the following conditions are satisfied:

(a) $F(s)$ is real for real s.

(b) The real part of $F(s)$ is greater than or equal to zero when the real part of s is greater than or equal to zero, i.e.,

$$\operatorname{Re} F(s) \geq 0 \qquad \text{for } \operatorname{Re}(s) \geq 0$$

Properties of Positive Real Functions

1. If $F(s)$ is positive real then $\dfrac{1}{F(s)}$ is also positive real.
2. The sum of two positive real functions is positive real.
3. The poles and zeros of a positive real function cannot have positive real parts, i.e., they cannot be in the right half of the s plane.
4. Only simple poles with real positive residues can exist on the $j\omega$-axis.
5. The poles and zeros of a positive real function are real or occur in conjugate pairs.
6. The highest powers of the numerator and denominator polynomials may differ at most by unity. This condition prevents the possibility of multiple poles and zeros at $s = \infty$.
7. The lowest powers of the denominator and numerator polynomials may differ by at most unity. Hence, a positive real function has neither multiple poles nor zeros at the origin.

Necessary and Sufficient Conditions for Positive Real Functions

The necessary and sufficient conditions for a function with real coefficients $F(s)$ to be positive real are the following:

1. $F(s)$ must have no poles and zeros in the right half of the s-plane.
2. The poles of $F(s)$ on the $j\omega$-axis must be simple and the residues evaluated at these poles must be real and positive.
3. $\mathrm{Re}\, F(j\omega) \geq 0$ for all ω.

Testing of the Above Conditions Condition (1) requires that we test the numerator and denominator of $F(s)$ for roots in the right half of the s-plane, i.e., we must determine whether the numerator and denominator of $F(s)$ are Hurwitz. This is done through a continued fraction expansion of the odd to even or even to odd parts of the numerator and denominator.

Condition (2) is tested by making a partial-fraction expansion of $F(s)$ and checking whether the residues of the poles on the $j\omega$-axis are positive and real. Thus, if $F(s)$ has a pair of poles at $s = \pm j\omega_0$, a partial-fraction expansion gives terms of the form

$$\frac{K_1}{s-j\omega_0}+\frac{K_1^*}{s+j\omega_0}$$

Since residues of complex conjugate poles are themselves conjugate, $K_1 = K_1^*$ and should be positive and real.

Condition (3) requires that Re $F(j\omega)$ must be positive and real for all ω.

Now, to compute Re $F(j\omega)$ from $F(s)$, the numerator and denominator polynomials are separated into even and odd parts.

$$F(s)=\frac{m_1(s)+n_1(s)}{m_2(s)+n_2(s)}=\frac{m_1+n_1}{m_2+n_2}$$

Multiplying $N(s)$ and $D(s)$ by m_2-n_2,

$$F(s)=\frac{m_1+n_1}{m_2+n_2}\frac{m_2-n_2}{m_2-n_2}=\frac{m_1m_2-n_1n_2}{m_2^2-n_2^2}+\frac{m_2n_1-m_1n_2}{m_2^2-n_2^2}$$

But the product of two even functions or odd functions is itself an even function, while the product of an even and odd function is odd.

$$\text{Ev } F(s) = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$\text{Od } F(s) = \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

Now, substituting $s = j\omega$ in the even polynomial gives the real part of $F(s)$ and substituting $s = j\omega$ into the odd polynomial gives imaginary part of $F(s)$.

$$\text{Ev } F(s)\big|_{s=j\omega} = \text{Re } F(j\omega)$$

$$\text{Od } F(s)\big|_{s=j\omega} = j \,\text{Im } F(j\omega)$$

We have to test $\text{Re } F(j\omega) \geq 0$ for all ω.
The denominator of $\text{Re } F(j\omega)$ is always a positive quantity because

$$m_2^2 - n_2^2\Big|_{s=j\omega} \geq 0$$

Hence, the condition that Ev $F(j\omega)$ should be positive requires

$$m_1 m_2 - n_1 n_2\big|_{s=j\omega} = A(\omega^2)$$

should be positive and real for all $\omega \geq 0$.

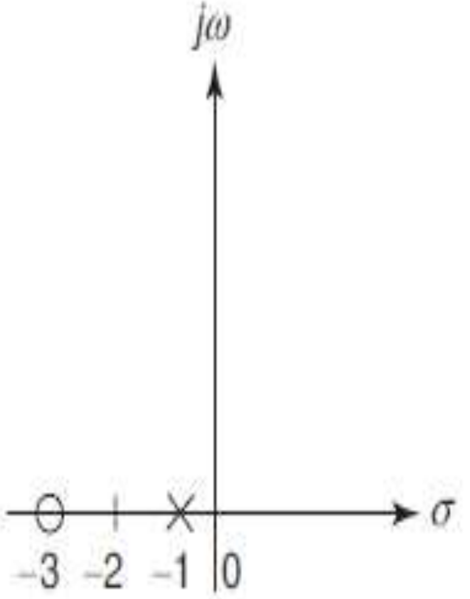
Test whether $F(s)=\dfrac{s+3}{s+1}$ *is a positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s+3}{s+1}$

The function $F(s)$ has pole at $s=-1$ and zero at $s=-3$ as shown in Fig. 10.1.

Thus, pole and zero are in the left half of the s-plane.



(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 3$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (3)(1) - (s)(s) \big|_{s=j\omega} = 3 - s^2 \big|_{s=j\omega} = 3 + \omega^2$$

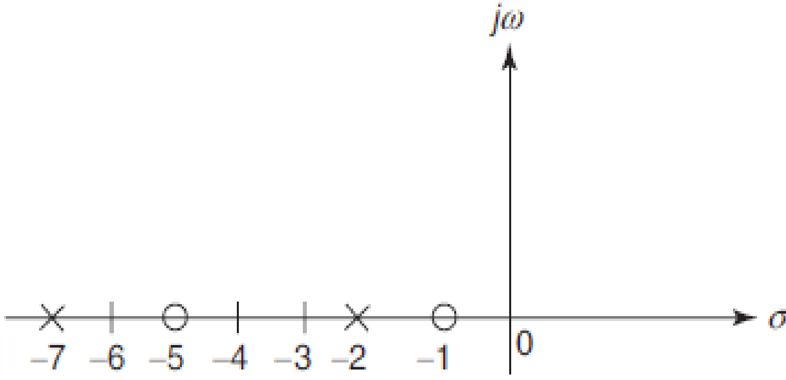
$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s)=\dfrac{s^2+6s+5}{s^2+9s+14}$ *is positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s^2+6s+5}{s^2+9s+14}=\dfrac{(s+5)(s+1)}{(s+7)(s+2)}$



The function $F(s)$ has poles at $s=-7$ and $s=-2$ and zeros at $s=-5$ an $s=-1$ as shown in Fig. 10.2.

Thus, all the poles and zeros are in the left half of the s plane.

(b) Since there is no pole on the $j\omega$ axis, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 5$

Odd part of $N(s) = n_1 = 6s$

Even part of $D(s) = m_2 = s^2 + 14$

Odd part of $D(s) = n_2 = 9s$

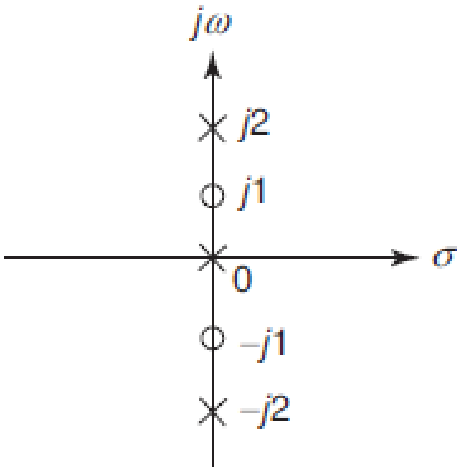
$$A(\omega^2) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (s^2+5)(s^2+14) - (6s)(9s) \big|_{s=j\omega} = s^4 - 35s^2 + 70 \big|_{s=j\omega} = \omega^4 + 35\omega^2 + 70$$

$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s) = \dfrac{s^2+1}{s^3+4s}$ *is positive real function.*

Solution



(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+1}{s^3+4s} = \dfrac{(s+j1)(s-j1)}{s(s+j2)(s-j2)}$

The function $F(s)$ has poles at $s = 0$, $s = -j2$ and $s = j2$ and zeros at $s = -j1$ and $s = j1$ as shown in Fig. 10.4.

Thus, all the poles and zeros are on the $j\omega$ axis.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{s^2+1}{s^3+4s} = \frac{s^2+1}{s(s^2+4)}$$

By partial-fraction expansion,

$$F(s) = \frac{K_1}{s} + \frac{K_2}{s+j2} + \frac{K_2^*}{s-j2}$$

The constants K_1, K_2 and K_2^* are called residues.

$$K_1 = s\,F(s)\,|_{s=0} = \left.\frac{s^2+1}{s^2+4}\right|_{s=0} = \frac{1}{4}$$

$$K_2 = (s+j2)F(s)\,|_{s=-j2} = \left.\frac{s^2+1}{s(s-j2)}\right|_{s=-j2} = \frac{-4+1}{(-j2)(-j2-j2)} = \frac{3}{8}$$

$$K_2^* = K_2 = \frac{3}{8}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = s^2 + 1$

Odd part of $N(s) = n_1 = 0$

Even part of $D(s) = m_2 = 0$

Odd part of $D(s) = n_2 = s^3 + 4s$

$$A\left(\omega^2\right) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (s^2 + 1)(0) - (0)(s^3 + 4s)\big|_{s=j\omega} = 0$$

$A(\omega^2)$ is zero for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s)=\dfrac{2s^3+2s^2+3s+2}{s^2+1}$ *is positive real function.*

Solution

(a) $$F(s)=\frac{N(s)}{D(s)}=\frac{2s^3+2s^2+3s+2}{s^2+1}=\frac{2s^3+2s^2+3s+2}{(s+j1)(s-j1)}$$

Since numerator polynomial cannot be easily factorized, we will prove whether $N(s)$ is Hurwitz.

Even part of $N(s)=m(s)=2s^2+2$

Odd part of $N(s)=n(s)=2s^3+3s$

By continued fraction expansion,

$$
\begin{array}{l}
2s^2+2\,\big)\,2s^3+3s\,\big(\,s \\
\qquad\quad\;\; \underline{2s^3+2s} \\
\qquad\qquad\qquad s\,\big)\,2s^2+2\,\big(\,2s \\
\qquad\qquad\qquad\quad\;\; \underline{2s^2} \\
\qquad\qquad\qquad\qquad\quad 2\,\big)\,s\,\big(\,\frac{1}{2}s \\
\qquad\qquad\qquad\qquad\qquad\;\; \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\;\; 0
\end{array}
$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that zeros are in the left half of the s plane.

The function $F(s)$ has poles at $s=-j1$ and $s=j1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$$

As the degree of the numerator is greater than that of the denominator, division is first carried out before partial-fraction expansion.

$$\begin{array}{r|l} s^2+1 & 2s^3 + 2s^2 + 3s + 2 \;\Big(\; 2s+2 \\ & \underline{2s^3 \qquad\quad + 2s} \\ & \qquad 2s^2 + s + 2 \\ & \qquad \underline{2s^2 \qquad + 2} \\ & \qquad\qquad\quad s \end{array}$$

$$F(s) = 2s + 2 + \frac{s}{s^2 + 1}$$

By partial-fraction expansion,

$$F(s) = 2s + 2 + \frac{K_1}{s + j1} + \frac{K_1^*}{s - j1}$$

$$K_1 = (s + j1)F(s)\big|_{s=-j1} = \frac{-j1}{-j1 - j1} = \frac{1}{2}$$

$$K_1^* = K_1 = \frac{1}{2}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = 2s^2 + 2$

Odd part of $N(s) = n_1 = 2s^3 + 3s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 0$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (2s^2 + 2)(s^2 + 1) - (2s^3 + 3s)(0)\big|_{s=j\omega} = 2s^4 + 4s^2 + 2\ \big|_{s=j\omega} = 2(\omega^4 - 2\omega^2 + 1)$$

$$= 2(\omega^2 - 1)^2$$

$A(\omega^2) \geq 0$ for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s)=\dfrac{s^2+s+6}{s^2+s+1}$ *is a positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s^2+s+6}{s^2+s+1}=\dfrac{\left(s+\frac{1}{2}+j\frac{\sqrt{23}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{23}}{2}\right)}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$

The function $F(s)$ has zeros at $s=-\frac{1}{2}\pm j\frac{\sqrt{23}}{2}$ and poles at $s=-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s)=m_1=s^2+6$

Odd part of $N(s)=n_1=s$

Even part of $D(s)=m_2=s^2+1$

Odd part of $D(s)=n_2=s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2+6)(s^2+1) - (s)(s)\,|_{s=j\omega} = s^4 + 6s^2 + 6\,|_{s=j\omega} = \omega^4 - 6\omega^2 + 6$$

For $\omega = 2$, $A(\omega^2) = 16 - 24 + 6 = -2$

This condition is not satisfied.

Hence, the function $F(s)$ is not positve real.