CHAPTER

8

Calculus of Variations

1. Introduction

We know that a function $y = f(x)$ may have a maxima or a minima at $x = a$ if $\frac{dy}{dx} = 0$ at $x = a$.

We also know that a function of two or more variables may have a maxima or a minima if all its partial derivatives of the first order are zero.

Here we consider a different problem : If we are given the definite integral

$$I = \int_{x_1}^{x_2} F\left(x, y, \frac{dy}{dx}\right) dx \quad \text{.................... (A)}$$

where F is a function of $x, y, dy/dx$, we have to find the function $y = f(x)$ for which the above definite integral may be **maximum or minimum.** The value is also called an **extremum or stationary value.**

The integral of the above type (A) is called a **functional.** It gives different functions for different arbitrary constants that enter into the solution while integrating. It is also denoted by

$$I[y(x)] = \int_{x_1}^{x_2} F(x, y, y')dx$$

In this chapter we shall learn how to solve such a problem.

(a) An Example

Suppose $A(x_1, y_1)$ and $B(x_2, y_2)$ are two given points in a plane. It is required to find a curve passing through A, B such that if the curve is rotated about the x-axis it will generate the surface whose area is minimum.

If $y = f(x)$ is any curve passing through A, B we know that the surface of the solid generated by revolving the curve $y = f(x)$, between $A(x_1, y_1)$ and $B(x_2, y_2)$ about the x-axis is given by

$$S = 2\pi \int_{x_1}^{x_2} y \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \cdot dx$$

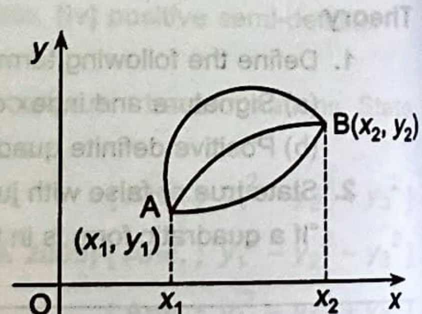


Fig. 8.1

For different curves $y = f(x)$, we will have different values of S. We want to find that particular curve which will give minimum value of S.

(b) Neighbouring Curve

We shall denote the required curve by $y(x)$. To obtain the condition for the required curve $y(x)$, we consider a neighbouring curve in the form

$$Y(x) = y(x) + \epsilon \cdot \eta(x)$$

where $\eta(x)$ is an arbitrary function and ϵ is a parameter. We consider this curve $y = \eta(x)$ passing through A and B, so that

$$\eta(x_1) = 0 \quad \text{and} \quad \eta(x_2) = 0$$

We have shown in the adjoining figure the required curve $y(x)$, the curve $\eta(x)$ and one neighbouring curve $Y(x)$ for some value of ϵ. For different values of ϵ, we will get different neighbouring curves.

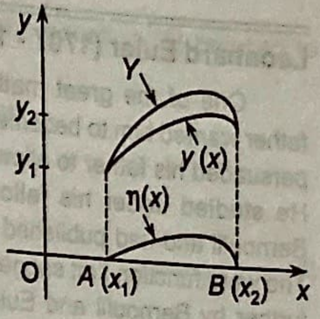


Fig. 8.2

2. Euler's Differential Equation

If $y(x)$ is a curve joining the two points (x_1, y_1) and (x_2, y_2), if $Y(x) = y(x) + \epsilon \cdot \eta(x)$ where $\eta(x_1) = 0, \eta(x_2) = 0$ is a neighbouring curve, and if $y(x)$ makes $I = \int_{x_1}^{x_2} F(x, y, y')dx$ an extremum then

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0} \qquad \text{.............. (1)}$$

This equation is known as **Euler's Equation** or **Euler-Lagrange Equation.**

Proof : We put $y = Y(x) = y(x) + \epsilon \cdot \eta(x)$ and $y' = y'(x) + \epsilon \cdot \eta'(x)$ in $I = \int_{x_1}^{x_2} F(x, y, y')dx$. The value of the integral along the neighbouring curve, *i.e.* for $Y(x)$ is

$$I(\epsilon) = \int_{x_1}^{x_2} F\{x, y(x) + \epsilon \cdot \eta(x), y'(x) + \epsilon \cdot \eta'(x)\}dx \qquad \text{.............. (2)}$$

But this is a function of ϵ and has a maxima or a minima for the curve $Y = y(x)$ if $\frac{dI}{d\epsilon} = 0$ at $\epsilon = 0$.

Denoting the integrand in (2) by F_ϵ to avoid confusion and differentiating it under the integral sign w.r.t ϵ,

$$\frac{dI(\epsilon)}{d\epsilon} = \int_{x_1}^{x_2} \left[\frac{\partial F_\epsilon}{\partial y} \cdot \eta(x) + \frac{\partial F_\epsilon}{\partial y'} \cdot \eta'(x)\right]dx \qquad \left[\because \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}\right]$$

At $\epsilon = 0$ we should have $\frac{dI(\epsilon)}{d\epsilon} = 0$.

(Putting back F for F_ϵ)

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} \cdot \eta(x) + \frac{\partial F}{\partial y'} \cdot \eta'(x)\right]dx = 0$$

Integrating the second term w.r.t. x by parts, we get

$$\int_{x_1}^{x_2} \frac{\partial F}{\partial y} \cdot \eta(x)\,dx + \left[\frac{\partial F}{\partial y'} \cdot \eta(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x) \cdot \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)dx = 0$$

Since, $\eta(x_1) = 0$ and $\eta(x_2) = 0$, the middle term is zero and we get

$$\int_{x_1}^{x_2} \left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]\eta(x)\,dx = 0 \qquad \text{.............. (3)}$$

Since $\eta(x)$ is an arbitrary function of x, the equation (3) implies that the integrand is also zero.

$$\therefore \quad \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Leonhard Euler (1707 - 1783)

One of the great mathematicians of Switzerland. His father wanted him to become a pastor (a priest). But Bernoulli persuaded his father to allow his son to pursue mathematics. He studied under his fellow countryman, mathematician Bernoulli and had published his first paper when he was 18. The word function first suggested by Leibnitz was generalised further by Bernoulli and Euler. Euler is supposed to be the most prolific mathematical writer in history. He has written a number of text books which are known for his clarity, detail and completeness. Although he had lost his eye-sight for the last 17 years of his life, he did not allow his work to be hampered because all the formulae from trigonometry and analysis (and many poems including the entire Latin epic-Aeneid) were on the tip of his tongue.

Corollary 1 : If F does not contain y explicitly then Euler's equation reduces to

$$\boxed{\frac{\partial F}{\partial y'} = c} \qquad \text{.............. (4)}$$

Proof : Since $\dfrac{\partial F}{\partial y} = 0$, from (1) we have $\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) = 0 \quad \therefore \quad \dfrac{\partial F}{\partial y'} = c$.

(a) Another Form of Euler's Equation

The above stated Euler's equation can also be written as

$$\boxed{\frac{\partial F}{\partial x} - \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = 0} \qquad \text{.............. (5)}$$

This is **second form** of Euler's equation.

Proof : Since F is a function of x, y, y', we have

$$\frac{dF}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial F}{\partial y'}\cdot\frac{dy'}{dx} = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}y' + \frac{\partial F}{\partial y'}\cdot y''$$

Also $\dfrac{d}{dx}\left[y'\cdot\dfrac{\partial F}{\partial y'}\right] = y'\cdot\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) + \dfrac{\partial F}{\partial y'}\cdot\dfrac{dy'}{dx} = y'\dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y'}\right) + \dfrac{\partial F}{\partial y'}\cdot y''$

By subtraction, we get,

$$\therefore \quad \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = \frac{\partial F}{\partial x} + y'\left[\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)\right]$$

But by Euler's equation (1), above the second term on the r.h.s. is zero

$$\therefore \quad \frac{\partial F}{\partial x} - \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = 0 \text{ as required.}$$

Corollary 2 : If F does not contain x explicitly then Euler's equation (5) reduces to

$$\boxed{F - y'\frac{\partial F}{\partial y'} = c} \qquad \text{.............. (6)}$$

Proof : If F does not contain x explicitly then $\frac{\partial F}{\partial x} = 0$ and the equation (5) reduces to $\frac{d}{dx}\left(F - y'\frac{\partial F}{\partial y}\right) = 0$. *i.e.*, $F - y'\frac{\partial F}{\partial y'} = c$, where c is in arbitrary constant.

Corollary 3 : If F does not contain y' then

$$\boxed{\frac{\partial F}{\partial y} = 0} \quad \text{.................. (7)}$$

Proof : If F is independent of y', then $\frac{\partial F}{\partial y'} = 0$.

Putting $\frac{\partial F}{\partial y'} = 0$ in the Euler's equation (1), we get $\frac{\partial F}{\partial y} = 0$.

(b) Third Form of Euler's Equation

The Euler's equation can also be written as

$$\boxed{\frac{\partial F}{\partial y} - \frac{\partial^2 F}{\partial x \partial y'} - y'\frac{\partial^2 F}{\partial y \partial y'} - y''\frac{\partial^2 F}{\partial y'^2} = 0} \quad \text{.................. (8)}$$

Proof : Since $\frac{\partial F}{\partial y'}$ also is a function of x, y, y'.

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y'}\right)\frac{dx}{dx} + \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y'}\right)\frac{dy}{dx} + \frac{\partial}{\partial y'}\left(\frac{\partial F}{\partial y'}\right)\cdot\frac{dy'}{dx}$$

$$= \frac{\partial^2 F}{\partial x \partial y'} + \frac{\partial^2 F}{\partial y \partial y'}\cdot\frac{dy}{dx} + \frac{\partial^2 F}{\partial y'^2}\cdot\frac{dy'}{dx}$$

$$= \frac{\partial^2 F}{\partial x \partial y'} + y'\frac{\partial^2 F}{\partial y \partial y'} + y''\frac{\partial^2 F}{\partial y'^2}$$

Putting this value of $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)$ in the Euler's equation (1), we get the required result given above.

Corollary 4 : If F is independent of x and y both, then

$$\boxed{y = ax + b} \quad \text{.................. (9)}$$

Proof : Since F is independent of x and y, $\frac{\partial F}{\partial x} = 0$ and $\frac{\partial F}{\partial y} = 0$.

This means $\frac{\partial^2 F}{\partial x \partial y'} = 0$ and $\frac{\partial^2 F}{\partial y \partial y'} = 0$.

Putting these values in (8), the first three terms are zero and, we get

$$y''\frac{\partial^2 F}{\partial y'^2} = 0 \qquad \therefore\ y'' = 0 \qquad \therefore\ \frac{d^2y}{dx^2} = 0 \qquad \therefore\ \frac{dy}{dx} = a, \text{ constant.}$$

Integrating again, $y = ax + b$.

Extremal : Any function which satisfies the Euler's equation is called an Extremal.

Important Note ....

It should be clearly understood that the function

$$\boxed{y = f(x)}$$

which extremises

$$\boxed{I = \int_{x_1}^{x_2} F(x, y, y')\,dx}$$

is obtained by solving the differential equation

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

known as Euler's equation.

Type I : When F foes not contains y explicity

Example 1 : Find the extremals of $\int_{x_1}^{x_2} \frac{y'^2}{x^2}\,dx$.

Sol. : We have $F = \frac{y'^2}{x^2}$.

Since F does not contain y explicitly, we shall use the formula (4), which is easier to apply.

Now, $\frac{\partial F}{\partial y'} = \frac{2y'}{x^2}$

But by (4), $\frac{\partial F}{\partial y'} = c \quad \therefore \quad \frac{2y'}{x^2} = c \quad \therefore \quad y' = \frac{c}{2}x^2 \quad \therefore \quad \frac{dy}{dx} = \frac{c}{2}x^2$

By integration, we get $y = \frac{c}{6}x^3 + c'$.

Taking the constants, suitably $y = c_1 x^3 + c_2$.

Example 2 : Find the extremals of the functional $\int_{x_2}^{x_1} \frac{y'^2}{x^3}\,dx = 0$.

Sol. : We have $F = \frac{y'^2}{x^3}$.

Since F does not contain y explicitly, we shall use the formula (4) which is easier to apply.

Now, $\frac{\partial F}{\partial y'} = \frac{2y'}{x^3}$

But by (4), $\frac{\partial F}{\partial y'} = c \quad \therefore \quad \frac{2y'}{x^3} = c. \quad \therefore \quad y' = \frac{c}{2}x^3 \quad \therefore \quad \frac{dy}{dx} = \frac{c}{2}x^3.$

By integration, we get $\quad y = \frac{c}{8}x^4 + c'$

Taking the constants suitably, $\quad y = c_1 x^4 + c_2$.

Example 3 : Find the extremals of $\int_{x_1}^{x_2} (1 + x^2 y')\,y'\,dx$. **(M.U. 2017)**

Sol. : We have $\quad F = (1 + x^2 y')\,y'$

Since F does not contain y explicitly, we shall use (4) which is easier to apply.

Now, $F = y' + x^2 y'^2$ $\quad \therefore \dfrac{\partial F}{\partial y'} = 1 + 2x^2 y'$

But by (4), $\dfrac{\partial F}{\partial y'} = c$ $\quad \therefore 1 + 2x^2 y' = c'$

$\therefore 2x^2 y' = c' - 1 = c_1$ $\quad \therefore y' = \dfrac{c_1}{2x^2} = \dfrac{c}{x^2}$

By integration, we get $y = -\dfrac{c}{x} + c_2$.

Changing the constants suitably $y = \dfrac{c_1}{x} + c_2$.

Example 4 : Solve the Euler's equation for $\int_{x_1}^{x_2} (x + y')y'dx$. (M.U. 2016, 19)

Sol. : We have $F = xy' + y'^2$

Since F does not contain y explicitly, we shall use (4), which is easier to apply.

Now, $F = xy' + y'^2$ $\quad \therefore \dfrac{\partial F}{\partial y'} = x + 2y'$

But by (4), $\dfrac{\partial F}{\partial y'} = c$ $\quad \therefore x + 2y' = c.$

$\therefore 2\dfrac{dy}{dx} = c - x$ $\quad \therefore \dfrac{dy}{dx} = \dfrac{c}{2} - \dfrac{x}{2}$ $\quad \therefore dy = \left(\dfrac{c}{2} - \dfrac{x}{2}\right)dx$

By integration, we get $\quad y = \dfrac{c}{2}x - \dfrac{1}{2}\left(\dfrac{x^2}{2}\right) + c_2$

Taking the arbitrary constant suitably, $y = -\dfrac{x^2}{4} + c_1 x + c_2$.

Example 5 : Find the extremals of $\int_{x_1}^{x_2} \dfrac{\sqrt{1+y'^2}}{x} dx$. (M.U. 2015)

Sol. : We have $F = \dfrac{\sqrt{1+y'^2}}{x}$.

Since F does not contain y explicitly, we shall use (4) which is easier to apply.

Now, $F = \dfrac{\sqrt{1+y'^2}}{x}$ $\quad \therefore \dfrac{\partial F}{\partial y'} = \dfrac{1}{x} \cdot \dfrac{y'}{\sqrt{1+y'^2}}$

Now, by (4), $\dfrac{\partial F}{\partial y'} = c$ $\quad \therefore \dfrac{y'}{x\sqrt{1+y'^2}} = c$ $\quad \therefore y' = cx\sqrt{1+y'^2}$

Squaring we get

$$y'^2 = c^2x^2(1+y'^2) = c^2x^2 + c^2x^2y'^2 \quad \therefore (1-c^2x^2)y'^2 = c^2x^2$$

$$\therefore y'^2 = \frac{c^2x^2}{1-c^2x^2} \quad \therefore \frac{dy}{dx} = \frac{cx}{\sqrt{1-c^2x^2}} = -\frac{1}{c} \cdot \frac{-c^2x}{\sqrt{1-c^2x^2}}$$

(The terms are adjusted for integration.)

By integration, $y = -\dfrac{\sqrt{1-c^2x^2}}{c} + c'$.

Type II : When F foes not contains x explicity

Example 1 : Find the extremal of $\int_{x_1}^{x_2} \dfrac{1+y^2}{y'^2}\,dx$.

Sol. : We have $F = \dfrac{1+y^2}{y'^2}$.

Since F does not contain x explicitly, we shall use the formula (6) which is easier to apply.

Now, $\dfrac{\partial F}{\partial y'} = (1+y^2)\left(-\dfrac{2}{y'^3}\right)$.

But by (6), $F - y'\dfrac{\partial F}{\partial y'} = c$

$\therefore \dfrac{1+y^2}{y'^2} + \dfrac{2(1+y^2)}{y'^2} = c \qquad \therefore \dfrac{3(1+y^2)}{y'^2} = c$

$\therefore \dfrac{y'^2}{3(1+y^2)} = \dfrac{1}{c} \qquad \therefore y' = \sqrt{\dfrac{3}{c}(1+y^2)}$

$\therefore \dfrac{dy}{dx} = c_1\sqrt{1+y^2}$ where $\sqrt{\dfrac{3}{c}} = c_1$. $\quad \therefore \dfrac{dy}{\sqrt{1+y^2}} = c_1\,dx$.

By integration,

$\therefore \sin h^{-1} y = c_1 x + c_2 \qquad \therefore y = \sin h\,(c_1 x + c_2)$

(See page 3-35 of Engineering Mathematics - I by the same author.)

Example 2 : Find the extremals of $\int_0^{\pi}(y'^2 - y^2)\,dx$ given that when $x = 0$, $y = 0$ and when $x = \pi$, $y = 0$.

Sol. : We have $F = y'^2 - y^2$.

Since F does not contain x explicitly we shall use the formula (6) which is easier to apply.

Now, $\dfrac{\partial F}{\partial y'} = 2y'$.

But by (6), $F - y'\cdot\dfrac{\partial F}{\partial y'} = c_1 \qquad \therefore y'^2 - y^2 - 2y'^2 = c_1$

$\therefore y'^2 + y^2 = c^2$ say $\qquad \therefore \dfrac{dy}{dx} = \sqrt{c^2 - y^2} \qquad \therefore \dfrac{dy}{\sqrt{c^2-y^2}} = dx$

By integration,

$\therefore \sin^{-1}\dfrac{y}{c} = x + c' \qquad \therefore y = c\sin(x + c')$

But $y = 0$ when $x = 0 \qquad \therefore c' = 0 \qquad \therefore y = c\sin x$.

But $y = 0$ when $x = \pi$, hence the general solution will be $y = a_n \sin nx$ where $n = 1, 2, 3, \ldots$

Example 3 : Find the extremal of $\int_0^{3\pi/2} (y^2 - y'^2)\,dx$ given $y(0) = 0,\ y\left(\frac{3\pi}{2}\right) = 1$.

Sol. : We have $F = y^2 - y'^2$

Since F is independent of x explicitly, we shall use the formula (6) which is easier to apply.

Now, $\frac{\partial F}{\partial y'} = -2y'$. But by (6), $F - y'\frac{\partial F}{\partial y'} = c$

$\therefore\ y^2 - y'^2 + y' \cdot 2y' = c \quad \therefore\ y^2 + y'^2 = c^2 \quad \therefore\ y'^2 = c^2 - y^2$

$\therefore\ \frac{dy}{dx} = \sqrt{c^2 - y^2} \quad \therefore\ \frac{dy}{\sqrt{c^2 - y^2}} = dx$

By integration, $\sin^{-1}\frac{y}{c} = x + c' \quad \therefore\ y = c\sin(x + c')$

When $x = 0,\ y = 0 \quad \therefore\ c' = 0$

When $x = \frac{3\pi}{2},\ y = 1 \quad \therefore\ 1 = c \quad \therefore\ y = \sin x$.

Type III : When F contains x, y, y'

Example 1 : Find the curve on which the functional $\int_0^1 (y'^2 + 12xy)\,dx$ with $y(0) = 0$ and $y(1) = 1$ is extremal. **(M.U. 2015, 16)**

Sol. : We have $F = y'^2 + 12xy$.

Since, F contains x, y, y', we shall use (1), page 8-2.

Now, $\frac{\partial F}{\partial y} = 12x,\ \frac{\partial F}{\partial y'} = 2y'$

Hence, the Euler's equation (1), becomes

$$12x - \frac{d}{dx}(2y') = 0 \quad \therefore\ 12x - 2\frac{d^2y}{dx^2} = 0 \quad \therefore\ \frac{d^2y}{dx^2} = 6x$$

Integrating we get $\frac{dy}{dx} = 3x^2 + c_1$.

Integrating again, we get, $y = x^3 + c_1 x + c_2$.

But by data when $x = 0,\ y = 0 \quad \therefore\ c_2 = 0. \quad \therefore\ y = x^3 + c_1 x$.

Again by data when $x = 1,\ y = 1 \quad \therefore\ 1 = 1 + c_1 \quad \therefore\ c_1 = 0$

Hence, the required curve is $y = x^3$.

Example 2 : Find the extremal of the function $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dy$ with $y(0) = 0$, $y(\pi/2) = 0$. **(M.U. 2016, 18)**

Sol. : We have $F = y'^2 - y^2 + 2xy$

$\therefore\ \frac{\partial F}{\partial y} = -2y + 2x$ .... (1) and $\frac{\partial F}{\partial y'} = 2y'$ .... (2)

Putting these values in the Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, we get

$$-2y + 2x - \frac{d}{dx}(2y') = 0 \quad \therefore\ \frac{d^2y}{dx^2} + y = x \qquad \text{.............. (3)}$$

The A.E. is $D^2+1=0$ $\therefore$ $D=i,-i$

$\therefore$ The C.F. is $y=c_1\cos x+c_2\sin x$.

$$\therefore \text{ P.I. } = \frac{1}{D^2+1}\cdot x = \frac{1}{1+D^2}(x)$$

$$= (1+D^2)^{-1}\cdot x = (1-D^2+D^4-\ldots)\,x = x$$

$\therefore$ The complete solution is $y=c_1\cos x+c_2\sin x+x$.

When $x=0, y=0$ $\therefore$ $0=c_1$

When $x=\frac{\pi}{2}$, $y=0$ $\therefore$ $0=c_2+\frac{\pi}{2}$ $\therefore$ $c_2=-\frac{\pi}{2}$

$\therefore$ $y=x-\frac{\pi}{2}\sin x$.

Example 3 : Find the extremal of $\int_{x_1}^{x_2}[y^2-y'^2-2y\cosh x]\,dx$. **(M.U. 2014, 16)**

Sol. : We have $F=y^2-y'^2-2y\cosh x$

$$\therefore \frac{\partial F}{\partial y}=2y-2\cosh x \quad \ldots\ldots\ldots\ldots (1) \quad \text{and} \quad \frac{\partial F}{\partial y'}=-2y' \quad \ldots\ldots\ldots\ldots (2)$$

Putting these values in Euler's equation $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$, we get

$$2y-2\cosh x-\frac{d}{dx}(-2y')=0 \qquad \therefore \quad 2y-2\cosh x+2\frac{d^2y}{dx^2}=0$$

$$\therefore \quad \frac{d^2y}{dx^2}+y=\cosh x \qquad \ldots\ldots\ldots\ldots (3)$$

We have to solve the equation (3).

Now, (3) can be written as $(D^2+1)\,y=\cosh x$ .... (4)

The A.E. equation is $D^2+1=0$ $\therefore$ $D=+i,-i$.

$\therefore$ C.F. is $y=c_1\cos x+c_2\sin x$.

$$\therefore \text{ P.I. } = \frac{1}{D^2+1}\cosh x = \frac{1}{D^2+1}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{1}{2}\left[\frac{1}{D^2+1}e^x+\frac{1}{D^2+1}e^{-x}\right]$$

$$= \frac{1}{2}\left[\frac{e^x}{2}+\frac{e^{-x}}{2}\right] = \frac{1}{2}\left(\frac{e^x+e^{-x}}{2}\right) = \frac{1}{2}\cosh x.$$

$\therefore$ The complete solution of (3) is $y=c_1\cos x+c_2\sin x+\frac{1}{2}\cosh x$.

Example 4 : Find the extremal of $\int_0^1 (xy+y^2-2y^2y')\,dx$. **(M.U. 2018)**

Sol. : We have $F=xy+y^2-2y^2y'$

$$\therefore \frac{\partial F}{\partial y}=x+2y-4yy' \quad \text{and} \quad \frac{\partial F}{\partial y'}=-2y^2$$

Putting these values in the Euler's equation $\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)=0$, we get

$$x + 2y - 4y\,y' - \frac{d}{dx}(-2y^2) = 0$$

$$\therefore \quad x + 2y - 4yy' + 4yy' = 0 \qquad \therefore \; x + 2y = 0 \qquad \therefore \; y = -\frac{x}{2}.$$

This is the required extremal.

Example 5 : Test for an extremal the function

$$I[y(x)] = \int_0^1 (xy + y^2 - 2y^2y')\,dx \text{ with } y(0) = 1 \text{ and } y(1) = 2.$$

Sol. : Proceeding as in the example 4 above, the extremal is $x + 2y = 0$.

Now, when $x = 0, y = 0$ and when $x = 1, y = -\frac{1}{2}$.

Thus, the extremal does not satisfy the given conditions.

Hence, there is no extremal with the given boundary conditions.

Example 6 : Solve the variational problem

$$\delta \int_1^2 [x^2 y'^2 + 2y(x + y)]\,dx = 0 \text{ given } y(1) = y(2) = 0.$$

Sol. : The given problem is equivalent to saying, find the extremal of the function

$$\int_1^2 [x^2 y'^2 + 2y(x + y)]\,dx = 0 \text{ with } y(1) = y(2) = 0.$$

We have $\quad F = x^2 y'^2 + 2y(x + y)$

$$\therefore \quad \frac{\partial F}{\partial y} = 2x + 4y \quad \text{and} \quad \frac{\partial F}{\partial y'} = 2x^2 y'$$

Putting these values in Euler's equation $\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$, we get

$$2x + 4y - \frac{d}{dx}(2x^2 y') = 0 \qquad \therefore \; 2x + 4y - (4xy' + 2x^2 y'') = 0$$

$$\therefore \quad x^2 y'' + 2xy' = 2y + x \qquad \therefore \; x^2 y'' + 2xy' - 2y = x \qquad \text{.............. (1)}$$

This is Cauchy's homogeneous linear differential equation.

Putting $x = e^z$, we get $z = \log x$.

$$\therefore \quad \frac{dz}{dx} = \frac{1}{x} \qquad \therefore \; \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\cdot\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\cdot\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx} = -\frac{1}{x^2}\cdot\frac{dy}{dz} + \frac{1}{x^2}\cdot\frac{d^2y}{dz^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

Putting these values in equation (1), we get

$$x^2\cdot\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) + 2x\cdot\frac{1}{x}\cdot\frac{dy}{dz} - 2y = e^z$$

$$\therefore \quad D^2 - D + 2D - 2y = e^z \qquad \therefore \; (D^2 + D - 2)\,y = e^z$$

A.E. is $D^2 + D - 2 = 0$ $\therefore$ $(D+2)(D-1) = 0$ $\therefore$ $D = 1, -2$

$\therefore$ C.F. is $y = c_1 e^z + c_2 e^{-2z}$.

$$\text{P.I.} = \frac{1}{D^2 + D - 2} \cdot e^z = \frac{1}{(D+2)(D-1)} \cdot e^z = z \cdot \frac{1}{D+2} \cdot e^z = \frac{z}{3} \cdot e^z.$$

$\therefore$ The complete solution is $y = c_1 e^z + c_2 e^{-2z} + \frac{z}{3} e^z$.

Putting $e^z = x$ and $z = \log x$, the solution is $y = c_1 x + \frac{c_2}{x^2} + \frac{x}{3} \log x$.

But by data when $x = 1, y = 0$ $\therefore$ $0 = c_1 + c_2$ .... (2)

Also when $x = 2, y = 0$ $\therefore$ $0 = 2c_1 + \frac{c_2}{4} + \frac{2}{3} \log 2$ .... (3)

Putting $c_2 = -c_1$ in (3), we get

$$2c_1 - \frac{c_1}{4} + \frac{2}{3} \log 2 = 0 \quad \therefore \quad \frac{2}{3} \log 2 = -\frac{7c_1}{4}$$

$$\therefore \quad c_1 = -\frac{8}{21} \log 2 \quad \therefore \quad c_1 = -c_2 = -\frac{8}{21} \log 2$$

$\therefore$ The complete solution is

$$y = -\frac{8}{21} \log 2 \cdot x + \frac{8}{21} \log 2 \cdot \frac{1}{x^2} + \frac{x}{3} \log x$$

$$= \frac{1}{21}\left[8 \log 2 \left(\frac{1}{x^2} - x\right) + 7x \log x\right]$$

EXERCISE - I

Find the extremals of the following :

1. $\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{y} dx$
2. $\int_{x_1}^{x_2} \sqrt{y(1+y'^2)} \cdot dx$
3. $\int_{x_1}^{x_2} y\sqrt{1+y'^2} \cdot dx$
4. $\int_{x_1}^{x_2} (y'^2 + 2yy' - 16y^2) dx$
5. $\int_{x_1}^{x_2} (y^2 + y'^2 - 2y \sin x) dx$
6. $\int_{x_1}^{x_2} (y^2 + y'^2 + 2y e^x) dx$

[Ans. : (1) $(x + c_1)^2 + y^2 = c_2$, (2) $y - c_1 = \frac{1}{4} \cdot \frac{(x + c_2)^2}{c_1}$, (3) $y = c_1 \cos h\left(\frac{x + c_2}{c_1}\right)$

(4) $y = c_1 \sin(4x + c_2)$, (5) $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} \sin x$. (6) $y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} x e^x$.

3. Geodesics

Definition : A curve on a surface along which the distance between any two points is minimum is called a geodesic.

Example : Geodesics on a plane are straight lines.

Sol. : See Ex. 6, page S-133.

4. Isoperimetric Problems (Constraints)

In some problems we are required to make the given integral

$$\int_{x_1}^{x_2} F(x,\, y,\, y')\,dx \qquad \text{.............. (10)}$$

maximum or minimum under a condition that another integral, say

$$\int_{x_1}^{x_2} G(x,\, y,\, y')\,dx \qquad \text{.............. (11)}$$

is equal to a constant.

For example, we may want to find a curve which will enclose maximum area when its perimeter is constant. Such problems with constraints are also sometimes called **Isoperimetric problems.**

5. Lagrange's Method

The problems with constraints can be solved by using the method of Lagrange's Multipliers.

We multiply the constraint integral (11) by λ and add the result to the given integral (10). Thus, we get

$$\int_{x_1}^{x_2} (F+\lambda G)\,dx \qquad \text{.............. (12)}$$

where λ is the Lagrange's multiplier.

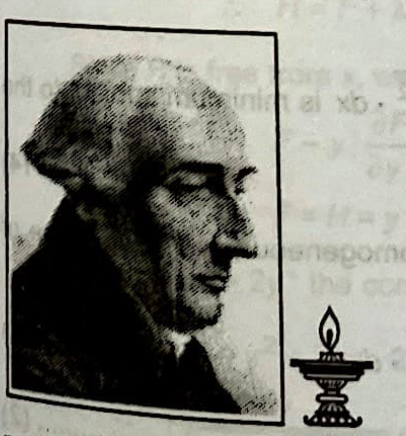
Now, the integral (11) must be an extremum.

To solve this problem we use Euler's equation (1) or (4)

$$\boxed{\frac{\partial H}{\partial y}-\frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right)=0} \quad \text{where } H = F+\lambda G.$$

or equivalently. $\boxed{H - y'\dfrac{\partial H}{\partial y'} = c}$ if H does not contain x explicitly.

Joseph Louis Lagrange (1736 - 1813)



A French-Italian mathematician who developed a new branch of mathematics called the **calculus of variations.** His father wanted him to be a lawyer but he was attracted to mathematics after reading a memoir by the great astronomer Halley. At the age of 16 he began to study mathematics on his own and at the age of 19 he was appointed as a professor. By the age of 25, he was regarded by many as the greatest living mathematician. In 1766, Fredrick the great wrote Lagrange that "the greatest king in Europe would like to have the greatest mathematician of Europe" at his court. Langrange accepted this invitation and remained in Berlin for the next 20 years. He made significant contributions to analysis, analytic mechanics, calculus, probability and number theory. He is known for his memoir on analysis of mechanics. He was buried in Pantheon - a temple in Rome build in 27B.C. where famous persons are buried.

Example 1 : Show that the extremal of the isoperimetric problem $I[y(x)] = \int_{x_1}^{x_2} y'^2\, dx$ su to the condition $\int_{x_1}^{x_2} y\, dx = k$ is a parabola. **(M.U. 2017**

Sol. : We have to find $y = f(x)$ such that $\int_{x_1}^{x_2} F\, dx = \int_{x_1}^{x_2} y'^2\, dx$ ....

is minimum subject to the condition $\int_{x_1}^{x_2} G\, dx = \int_{x_1}^{x_2} y\, dx = k$ ....

To use Lagrange's equation, we multiply (2) by λ and add it to (1).

$$\therefore \quad H = F + \lambda G = \int_{x_1}^{x_2} (y'^2 + \lambda y)\, dx \quad ....$$

Since the integrand is free from x, we use (6), page 8-3.

$$\therefore \quad F - y' \frac{\partial F}{\partial y'} = c \quad ....$$

where, $\quad F = H = y'^2 + \lambda y$ ....

Hence, from (4) using (5), we get

$$y'^2 + \lambda y - y' \cdot 2y' = c \qquad \therefore\ -y'^2 + \lambda y = c \qquad \therefore\ y'^2 - \lambda y = -c = c_1$$

$$\therefore\ y' = \sqrt{c_1 + \lambda y} \qquad \therefore\ \frac{dy}{\sqrt{c_1 + \lambda y}} = dx$$

Integrating, we get

$$\frac{2}{\lambda}\sqrt{c_1 + \lambda y} = x + c_2 \qquad \therefore\ \sqrt{c_1 + \lambda y} = \frac{\lambda}{2}(x + c_2)$$

$$\therefore\ (c_1 + \lambda y) = \left(\frac{\lambda}{2}\right)^2 (x + c_2)^2 \qquad \therefore\ \lambda y = \frac{\lambda^2}{4}x^2 + \frac{\lambda^2}{2}c_2 x + \frac{\lambda^2}{4}c_2^2 - c_1$$

$$\therefore\ y = \frac{\lambda}{4}x^2 + \frac{c_2 \lambda}{2}x + \left(\frac{\lambda}{4}c_2^2 - \frac{c_1}{\lambda}\right) = \frac{\lambda}{4}x^2 + c'x + c''$$

This is a parabola.

Example 2 : Find the curve $y = f(x)$ for which $\int_{x_1}^{x_2} y\sqrt{1 + y'^2} \cdot dx$ is minimum subject to th constraint $\int_{x_1}^{x_2} \sqrt{1 + y'^2} \cdot dx = l$. **(M.U. 201**

OR Find the shape of an absolutely flexible, inextensible homogeneous and heavy rope given length l suspended at points $A(x_1, y_1)$ and $B(x_2, y_2)$.

Sol. : We have to find $y = f(x)$ such that $\int_{x_1}^{x_2} F\, dx = \int_{x_1}^{x_2} y\sqrt{1 + y'^2} \cdot dx$ .... (

is minimum subject to the condition $\int_{x_1}^{x_2} G\, dx = \int_{x_1}^{x_2} \sqrt{1 + y'^2} \cdot dx = l$ .... (2

To use Lagrange's method, we multiply (2) by λ and add it to (1)

$$\therefore \quad H = F + \lambda G = \int_{x_1}^{x_2} (y + \lambda)\sqrt{1 + y'^2} \cdot dx \quad .... (3$$

Since the integrand is free from x, we use (6), page 8-3.

$$\therefore \quad F - y'\frac{\partial F}{\partial y'} = c \quad \text{.............. (4)}$$

where, $F = H = (y+\lambda)\sqrt{1+y'^2}$. .... (5)

Hence, from (4) using (5), we get

$$(y+\lambda)\sqrt{1+y'^2} - y' \cdot \frac{(y+\lambda)y'}{\sqrt{1+y'^2}} = c$$

$$\therefore \quad \frac{(y+\lambda)(1+y'^2)-(y+\lambda)\cdot y'^2}{\sqrt{1+y'^2}} = c \qquad \therefore \quad \frac{y+\lambda}{\sqrt{1+y'^2}} = c$$

$$\therefore \quad c^2(1+y'^2) = (y+\lambda)^2 \qquad \therefore \quad y'^2 = \frac{(y+\lambda)^2 - c^2}{c^2}$$

$$\therefore \quad \frac{c}{\sqrt{(y+\lambda)^2 - c^2}} \cdot dy = dx.$$

Integrating, we get $\quad c\cosh^{-1}\left(\frac{y+\lambda}{c}\right) = x + c'. \qquad \therefore \quad \cosh^{-1}\left(\frac{y+\lambda}{c}\right) = \frac{x+c'}{c}$

$\therefore \quad y = c\cosh\left(\frac{x+c'}{c}\right) - \lambda$ is the required curve.

Example 3 : Find the curve $y = f(x)$ for which $\int_0^{\pi}(y'^2 - y^2)\,dx$ is extremum if $\int_0^{\pi} y\,dx = 1$.

(M.U. 2016)

Sol. : We have to find $y = f(x)$ such that $\quad \int_{x_1}^{x_2} F\,dx = \int_0^{\pi}(y'^2 - y^2)\,dx$ .... (1)

is extremum with the condition $\quad \int_{x_1}^{x_2} G\,dx = \int_0^{\pi} y\,dx = 1$ .... (2)

To use Lagranges's method, multiply (2) by λ and add it to (1)

$$\therefore \quad H = F + \lambda G = \int_{x_1}^{x_2}(y'^2 - y^2 + \lambda y)\,dx \quad \text{.............. (3)}$$

Since H is free from x, we use (6) of page 8-3

$$\therefore \quad F - y'\frac{\partial F}{\partial y'} = c \quad \text{.............. (4)}$$

where, $F = H = y'^2 - y^2 + \lambda y$.

Since, $\frac{\partial F}{\partial y'} = 2y'$ the condition (4) becomes

$$y'^2 - y^2 + \lambda y - 2y'^2 = c \qquad \therefore \quad y'^2 = -c - y^2 + \lambda y$$

$$\therefore \quad \frac{dy}{\sqrt{-c - y^2 + \lambda y}} = dx$$

Completing the square on $-y^2 + \lambda y$, we get

$$\therefore \quad \frac{dy}{\sqrt{\left(\frac{\lambda^2}{4} - c\right) - \left(y - \frac{\lambda}{2}\right)^2}} = dx$$

By integration

$$\sin^{-1}\frac{y-(\lambda/2)}{\sqrt{(\lambda^2/4)-c}} = x + c' \qquad \therefore \; y - \left(\frac{\lambda}{2}\right) = \sqrt{\frac{\lambda^2}{4} - c} \cdot \sin(x + c')$$

$$\therefore \quad y = \left(\frac{\lambda}{2}\right) + \sqrt{\frac{\lambda^2}{4} - c} \cdot \sin(x + c') \qquad \text{.............. (5)}$$

Example 4 : Find the curve C of given length l which encloses a maximum area.

(**Alternatively :** (I) Find the plane curve of fixed perimeter and maximum area. **(M.U. 2014, 15)**

(II) Show that a closed curve C of given fixed length (perimeter) which encloses maximum area is a circle.) **(M.U. 2016)**

Sol. : Area enclosed by a simple closed curve C is given by (By Green's Theorem)

$$A = \frac{1}{2}\int_C (x\,dy - y\,dx) = \frac{1}{2}\int_C (xy' - y)\,dx \qquad \text{.............. (1)}$$

We also know that the length of an arc is given by

$$s = \int_C \sqrt{1 + y'^2} \cdot dx = l \text{ (given)} \qquad \text{.............. (2)}$$

We have to maximise (1) subject to (2).

To use Lagrange's method of multiplier, we multiply (2) by λ and add it to (1)

$$\therefore \quad H = \int_C \left[\frac{1}{2}(xy' - y) + \lambda\sqrt{1 + y'^2}\right] dx \qquad \text{.............. (3)}$$

But for maxima or minima the condition is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \qquad \text{.............. (4)}$$

where, $\quad F = H = \frac{1}{2}(xy' - y) + \lambda\sqrt{1 + y'^2}$.

Now, $\quad \frac{\partial F}{\partial y} = -\frac{1}{2}$ and $\frac{\partial F}{\partial y'} = \frac{1}{2}x + \lambda\frac{y'}{\sqrt{1 + y'^2}}$.

Hence, from (4), we get

$$-\frac{1}{2} - \frac{d}{dx}\left[\frac{1}{2} \cdot x + \lambda\frac{y'}{\sqrt{1 + y'^2}}\right] = 0 \quad \therefore \quad -\frac{1}{2} - \frac{1}{2} - \frac{d}{dx}\left[\lambda \cdot \frac{y'}{\sqrt{1 + y'^2}}\right] = 0$$

$$\therefore \quad \frac{d}{dx}\left[\lambda \cdot \frac{y'}{\sqrt{1 + y'^2}}\right] = -1$$

By integration, we get $\quad \dfrac{\lambda y'}{\sqrt{1 + y'^2}} = -x + c_1$

We solve this equation for y' as follows.

$$\therefore \quad \lambda^2 y'^2 = (1 + y'^2)(x - c_1)^2 \qquad \therefore \quad \lambda^2 y'^2 = (x - c_1)^2 + y'^2 (x - c_1)^2$$

$$\therefore \quad y'^2 [\lambda^2 - (x - c_1)^2] = (x - c_1)^2 \qquad \therefore \quad y' = \frac{dy}{dx} = \frac{(x - c_1)}{\pm\sqrt{\lambda^2 - (x - c_1)^2}}$$

By integration $y = \pm\sqrt{\lambda^2 - (x - c_1)^2} + c_2 \quad \therefore \quad y - c_2 = \pm\sqrt{\lambda^2 - (x - c_1)^2}$

$\therefore \quad (x - c_1)^2 + (y - c_2)^2 = \lambda^2$ which is a circle.

Example 5 : Find the solid of revolution which for a given surface area has maximum volume.

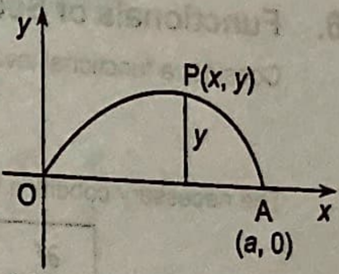


Fig. 8.3

Sol. : Let the equation of the rotating curve be $y = f(x)$.

When the curve rotates about the axis, the surface area of the solid is given by

$$S = \int_0^a 2\pi y \, ds = \int_0^a 2\pi y \sqrt{1 + y'^2} \cdot dx$$

The volume of the solid so obtained is given by

$$V = \int_0^a \pi y^2 \, dx$$

We have to maximise V subject to S = a constant.

Now, taking $\quad F = \pi y^2 \quad$ .... (1) $\quad$ and $\quad G = 2\pi y \sqrt{1 + y'^2}$ .... (2)

we get $\quad H = F + \lambda G = \pi y^2 + 2\pi\lambda y \sqrt{1 + y'^2}$ .... (3)

Now, H has to satisfy Euler's equation.

But it does not contain x explicitly. Therefore, we shall use (6), page 8-3.

$$\therefore \quad H - y' \cdot \frac{\partial H}{\partial y'} = c.$$

Hence, $\pi y^2 + 2\pi\lambda y\sqrt{1 + y'^2} - y' \cdot 2\pi\lambda y \dfrac{y'}{\sqrt{1 + y'^2}} = c.$

$\therefore \quad \pi y^2 + 2\pi\lambda y \cdot \dfrac{(1 + y'^2) - y'^2}{\sqrt{1 + y'^2}} = c \qquad \therefore \quad \pi y^2 + \dfrac{2\pi\lambda y}{\sqrt{1 + y'^2}} = c$ .... (4)

But the curve passes through O (and A) for which $y = 0$. Hence, from (4) we get $c = 0$.

$\therefore \quad \pi y^2 + \dfrac{2\pi\lambda y}{\sqrt{1 + y'^2}} = 0 \qquad \therefore \quad y + \dfrac{2\lambda}{\sqrt{1 + y'^2}} = 0$

$\therefore \quad y = -\dfrac{2\lambda}{\sqrt{1 + y'^2}} \qquad \therefore \quad \sqrt{1 + y'^2} = -\dfrac{2\lambda}{y}$

$\therefore \quad 1 + y'^2 = \dfrac{4\lambda^2}{y^2} \qquad \therefore \quad y'^2 = \dfrac{4\lambda^2}{y^2} - 1 = \dfrac{4\lambda^2 - y^2}{y^2}$

$\therefore \quad y' = \dfrac{dy}{dx} = \dfrac{\sqrt{4\lambda^2 - y^2}}{y} \qquad \therefore \quad \dfrac{y}{\sqrt{4\lambda^2 - y^2}} dy = dx$

By integrating (put $4\lambda^2 - y^2 = t^2$) $\qquad -\sqrt{4\lambda^2 - y^2} = x - c$ .... (5)

$\therefore \quad x = c - \sqrt{4\lambda^2 - y^2}$

But when $x = 0, y = 0 \qquad \therefore \quad c = 2\lambda.$

$\therefore \quad -\sqrt{4\lambda^2 - y^2} = (x - 2\lambda)$

$\therefore\ (x-2\lambda)^2 = 4\lambda^2 - y^2 \qquad \therefore\ (x-2\lambda)^2 + y^2 = 4\lambda^2$

But this is a circle. Hence, the solid of revolution is a sphere.

6. Functionals of Second Order Derivatives

Consider a functional involving higher order derivatives

$$\int_{x_1}^{x_2} f(x,\ y,\ y',\ y'')\,dx \qquad \text{.............. (1)}$$

The necessary condition for this functional to be extremum is

$$\boxed{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0} \qquad \text{.............. (2)}$$

Proof : Putting $y + \alpha\,\eta\,(x)$ for y in (1) where α is a parameter and $\eta\,(x)$ is a differentiable function, we get,

$$\therefore\ \int_{x_1}^{x_2} f[x,\ y+\alpha\eta(x),\ \ y'+\alpha\eta'(x),\ \ y''+\alpha\eta''(x)]\,dx$$

Let the boundary conditions be $y\,(x_1) = y_1,\ y\,(x_2) = y_2,\ y'(x_1) = y_1'$ and $y'(x_2) = y_2'$.

Let at the end points $\eta\,(x_1) = \eta\,(x_2) = 0$ and $\eta'\,(x_1) = \eta'\,(x_2) = 0$.

Let us write

$$\int_{x_1}^{x_2} f[x,\ y+\alpha\eta(x),\ y'+\alpha\eta'(x),\ y''+\alpha\eta''(x)]\,dx = \int_{x_1}^{x_2} F\,dx = I \qquad \text{.............. (3)}$$

For extreme value of (2),

$$\frac{dI}{d\alpha} = 0 \qquad \therefore\ \frac{dI}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial F}{\partial \alpha} = 0$$

Differentiating (3) under integral sign w.r.t. α,

$$\int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\cdot\eta(x) + \frac{\partial f}{\partial y'}\eta'(x) + \frac{\partial f}{\partial y''}\cdot\eta''(x)\right]dx = \frac{dI}{d\alpha}$$

But $\quad \dfrac{dI}{d\alpha} = 0$

$$\therefore\ \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\eta'(x) + \frac{\partial f}{\partial y''}\eta''(x)\right]dx = 0$$

$$\therefore\ \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\,dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\eta'(x)\,dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y''}\eta''(x)\,dx = 0$$

Integrating by parts w.r.t. x, the second term once and the third term twice, we get

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y}\eta(x)\,dx + \left[\frac{\partial f}{\partial y'}\eta(x) - \int_{x_1}^{x_2} \eta(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx\right]_{x_1}^{x_2}$$

$$+\left[\frac{\partial f}{\partial y''}\eta'(x) - \frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)\eta(x) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\int_{x_1}^{x_2} \eta(x)\,dx\right]_{x_1}^{x_2} = 0$$

But $\eta\,(x_1) = 0,\ \eta\,(x_2) = 0,\ \eta'(x_1) = 0,\ \eta'(x_2) = 0$.

$$\therefore \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)\right]\cdot \eta(x)\,dx = 0$$

$$\therefore \frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y''}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0.$$ This is the required condition.

In general $I=\int_{x_1}^{x_2} f(x, y, y', y'', \ldots, y_n)$ will be extremum if

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)-\frac{d^3}{dx^3}\left(\frac{\partial f}{\partial y'''}\right)+\ldots+(-1)\frac{d^n}{dx^n}\left(\frac{\partial f}{\partial y^n}\right)=0.$$

Example 1 : Find the extremal of $\int_{x_0}^{x_1}(y''^2-y^2+x^2)\,dx$.

Sol. : We have $f=y''^2-y^2+x^2$.

$$\therefore \frac{\partial f}{\partial y}=-2y,\quad \frac{\partial f}{\partial y'}=0,\quad \frac{\partial f}{\partial y''}=2y''$$

Hence, the equation $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0$ becomes

$$-2y-\frac{d}{dx}(0)+\frac{d^2}{dx^2}(2y'')=0 \qquad \therefore -2y+\frac{d^2}{dx^2}\left(2\frac{d^2y}{dx^2}\right)=0$$

$$\therefore -2y+2\frac{d^4y}{dx^4}=0 \qquad \therefore \frac{d^4y}{dx^4}-y=0.$$

This is a linear differential equation of the fourth order.

Its A.E. is $D^4-1=0$ $\quad\therefore (D^2-1)(D^2+1)=0$ $\quad\therefore D=\pm 1, \pm i.$

Hence, its C.F. *i.e.*, its solution is

$$y=c_1 e^{x}+c_2 e^{-x}+c_3\cos x+c_4\sin x.$$

Example 2 : Find the extremal of $\int_{x_0}^{x_1}(16y^2-y''^2+x^2)\,dx$. **(M.U. 2014, 16)**

Sol. : We have $f=16y^2-y''^2+x^2$

$$\therefore \frac{\partial f}{\partial y}=32y,\quad \frac{\partial f}{\partial y'}=0,\quad \frac{\partial f}{\partial y''}=-2y''$$

Hence, the equation $\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right)=0$ becomes

$$32y-0+\frac{d^2}{dx^2}\left(-2\frac{d^2y}{dx^2}\right)=0 \qquad \therefore 32y-2\frac{d^4y}{dx^4}=0 \qquad \therefore \frac{d^4y}{dx^4}-16y=0$$

This is a linear differential equation of the fourth order.

Its A.E. is $D^4-16=0$ $\quad\therefore (D^2-4)(D^2+4)=0$ $\quad\therefore D=-2, 2, 2i, -2i.$

Hence, its C.F. *i.e.*, the solution is

$$y=c_1 e^{2x}+c_2 e^{-2x}+c_3\cos 2x+c_4\sin 2x.$$

Example 3 : Find the extremal of $\int_{x_0}^{x_1} (2xy - y''^2)\,dx$. **(M.U. 2015, 18)**

Sol. : We have $f = 2xy - y''^2$

$$\therefore \quad \frac{\partial f}{\partial y} = 2x, \quad \frac{\partial f}{\partial y'} = 0, \quad \frac{\partial f}{\partial y''} = -2y''$$

Hence, the equation $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}\right) = 0$ becomes

$$2x - 0 + \frac{d^2}{dx^2}(-2y'') = 0 \quad \therefore \quad 2x - 2\frac{d^2}{dx^2}\left(\frac{d^2y}{dx^2}\right) = 0 \quad \therefore \quad \frac{d^4y}{dx^4} = x$$

This is a linear differential equation of the fourth order.

Its A.E. is $D^4 = 0$. $\therefore$ $D = 0, 0, 0, 0$.

$\therefore$ The C.F. is $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3$.

And P.I., $y = \frac{1}{D^4} x$

$$\therefore \quad y = \frac{1}{D^3}\int x\,dx = \frac{1}{D^3}\cdot\frac{x^2}{2} = \frac{1}{D^2}\int\frac{x^2}{2}dx$$

$$= \frac{1}{D^2}\cdot\frac{x^3}{3\cdot 2} = \frac{1}{D}\int\frac{x^3}{3\cdot 2}dx = \frac{1}{D}\cdot\frac{x^4}{4\cdot 3\cdot 2}$$

$$= \int\frac{x^4}{4\cdot 3\cdot 2}dx = \frac{x^5}{5\cdot 4\cdot 3\cdot 2} = \frac{x^5}{5!}.$$

Hence, the solution is $y = c_1 + c_2 x + c_3 x^2 + c_4 x^3 + \frac{x^5}{5!}$.

EXERCISE - II

1. Find the extremals of $\int_{x_0}^{x_1} (2xy + y'''^2)\,dx$.

[**Ans. :** $y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_4 x^5 + \frac{x^7}{7!}$.]

7. Rayleigh-Ritz Method

Rayleigh-Ritz method consists of assuming a trial solution.

$$\overline{y}(x) = y_0(x) + c_1\phi_1(x) + c_2\phi_2(x) \quad \text{to} \quad I = \int_a^b F(x, y, y')\,dx$$

and finding the constants, $c_1, c_2, \ldots$ by putting $\overline{y}$ in I.

Method

1. Assume a trial solution

$$\overline{y}(x) = y_0(x) + c_1\Phi_1(x) + c_2\Phi_2(x) + \ldots$$

2. Using $\overline{y}(x)$ in $\int_a^b F(x, y, y')\,dx$ and the given conditions, we find the constants $c_1, c_2, \ldots$

Thus, we get the required solution.

Example 1 : Solve by Rayleigh-Ritz method the boundary value problem

$$I = \int_0^1 (2xy - y^2 - y'^2)\,dx \text{ given } y(0) = 0 \text{ and } y(1) = 0.$$

(M.U. 2014, 19)

Sol. : We have to extremise the integral

$$I = \int_0^1 F(x, y, y')\,dx \quad \text{where, } F = 2xy - y^2 - y'^2.$$

Now, we assume the trial solution of (1) to be

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2 \qquad \text{.............. (1)}$$

But by data $y(0) = 0 \quad \therefore c_0 = 0$

and $y(1) = 0 \quad \therefore c_1 + c_2 = 0 \quad \therefore c_2 = -c_1$ .... (2)

$$\therefore \bar{y}(x) = c_1 x - c_1 x^2 = c_1 x(1-x)$$

$$\therefore \bar{y}'(x) = c_1 - 2c_1 x = c_1(1-2x) \qquad \text{.............. (3)}$$

Putting these values in $I = \int_0^1 (2x\bar{y} - \bar{y}^2 - \bar{y}'^2)\,dx$, we get

$$I = \int_0^1 [2x\, c_1 x(1-x) - c_1^2 x^2 (1-x)^2 - c_1^2 (1-2x)^2]\,dx$$

$$= c_1 \int_0^1 \left\{2(x^2 - x^3) - c_1(x^2 - 2x^3 + x^4 + 1 - 4x + 4x^2)\right\} dx$$

$$= c_1 \int_0^1 \left[2(x^2 - x^3) - c_1(1 - 4x + 5x^2 - 2x^3 + x^4)\right] dx$$

$$= c_1 \left[2\left(\frac{x^3}{3} - \frac{x^4}{4}\right) - c_1\left(x - 2x^2 + \frac{5x^3}{3} - \frac{x^4}{2} + \frac{x^5}{5}\right)\right]_0^1$$

$$= c_1 \left[\left(\frac{2}{3} - \frac{1}{2}\right) - c_1\left(1 - 2 + \frac{5}{3} - \frac{1}{2} + \frac{1}{5}\right)\right]$$

$$= c_1 \left[\frac{1}{6} - c_1\left(\frac{-30 + 50 - 15 + 6}{30}\right)\right] = c_1\left[\frac{1}{6} - c_1\left(\frac{56 - 45}{30}\right)\right]$$

$$\therefore I = c_1\left[\frac{1}{6} - \frac{11}{30} c_1\right] = \frac{c_1}{6} - \frac{11}{30} c_1^2$$

Its stationary values are given by

$$\frac{d\bar{I}}{dc_1} = 0 \quad \therefore \frac{1}{6} - \frac{11}{15} c_1 = 0 \quad \therefore c_1 = \frac{15}{11 \times 6} = \frac{5}{22}.$$

Hence, from (3), the approximate solution is $\bar{y} = \frac{5}{22} x(1-x)$.

Example 2 : Using Rayleigh-Ritz method, solve the boundary value problem

$$I = \int_0^1 (2xy + y^2 - y'^2)\,dx; \quad 0 \le x \le 1, \text{ given } y(0) = y(1) = 0.$$

(M.U. 2015, 16, 18)

Sol. : We have to extremise $I = \int_0^1 F(x, y, y')\,dx$ .... (1)

where, $F = 2xy + y^2 - y'^2$ .... (2)

Now, assume the trial solution $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ .... (3)

By data $\bar{y}(0) = 0 \;\therefore\; c_0 = 0$; $\bar{y}(1) = 0 \;\therefore\; 0 = c_1 + c_2 \;\therefore\; c_2 = -c_1$

$$\therefore\; \bar{y}(x) = c_1 x - c_1 x^2 = c_1 x(1-x) \qquad \text{.............. (4)}$$

$$\bar{y}'(x) = c_1 - 2c_1 x = c_1(1-2x)$$

Putting these values in $I = \int_0^1 (2xy + y^2 - y'^2)\,dx$, we get

$$I = \int_0^1 \left\{2x[c_1 x(1-x)] + c_1^2 x^2 (1-x)^2 - c_1^2(1-2x)^2\right\} dx$$

$$= c_1 \int_0^1 \left\{2(x^2 - x^3) + c_1[x^2 - 2x^3 + x^4 - (1 - 4x + 4x^2)]\right\} dx$$

$$= c_1 \int_0^1 [2(x^2 - x^3) + c_1(-1 + 4x - 3x^2 - 2x^3 + x^4)]\,dx$$

$$= c_1\left[2\left(\frac{x^3}{3} - \frac{x^4}{4}\right) + c_1\left(-x + 2x^2 - x^3 - \frac{x^4}{2} + \frac{x^5}{5}\right)\right]_0^1$$

$$= c_1\left[2\left(\frac{1}{3} - \frac{1}{4}\right) + c_1\left(-1 + 2 - 1 - \frac{1}{2} + \frac{1}{5}\right)\right]$$

$$\therefore\; I = c_1\left(\frac{1}{6} - \frac{3}{10}c_1\right) = \frac{c_1}{6} - \frac{3}{10}c_1^2.$$

Its stationary values are given by

$$\frac{dI}{dc_1} = 0 \quad \therefore\; \frac{1}{6} - \frac{3}{5}c_1 = 0 \quad \therefore\; c_1 = \frac{1}{6}\cdot\frac{5}{3} = \frac{5}{18}.$$

Hence, from (4), the approximate solution is $\bar{y}(x) = \frac{5}{18}x(1-x)$.

Example 3 : Using Rayleigh-Ritz method, solve the boundary value problem

$I = \int_0^1 \left(xy + \frac{1}{2}y'^2\right) dx$; $0 \le x \le 1$ given $y(0) = 0$ and also $y(1) = 0$. **(M.U. 2014, 16, 18)**

Sol. : We have to extremise $\quad I = \int_0^1 \left(xy + \frac{1}{2}y'^2\right) dx \quad$ .... (1)

where $\quad F = xy + \frac{1}{2}y'^2 \quad$ .... (2)

Assume the trial solution $\quad \bar{y}(x) = c_0 + c_1 x + c_2 x^2 \quad$ .... (3)

By data $\bar{y}(0) = 0 \;\therefore\; c_0 = 0$ and $\bar{y}(1) = 0 \;\therefore\; c_1 + c_2 = 0 \;\therefore\; c_2 = -c_1$

$$\therefore\; \bar{y}(x) = c_1 x - c_1 x^2 = c_1(x - x^2) \qquad \text{.............. (4)}$$

$$\therefore\; \bar{y}'(x) = c_1(1-2x)$$

Putting these values in (1) *i.e.*, in $I = \int_0^1 \left(xy + \frac{1}{2}y'^2\right) dx$, we get

$$I = \int_0^1 \left\{x[c_1(x - x^2)] + \frac{1}{2}c_1^2(1-2x)^2\right\} dx$$

$$= \int_0^1 \left[c_1(x^2 - x^3) + \frac{1}{2}c_1^2(1 - 4x + 4x^2)\right] dx$$

$$= c_1\left[\int_0^1 x^2\,dx - \int_0^1 x^3\,dx\right] + \frac{1}{2}c_1^2\left[\int_0^1 1\,dx - 4\int_0^1 x\,dx + 4\int_0^1 x^2\,dx\right]$$

$$= c_1\left\{\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1\right\} + \frac{1}{2}c_1^2\left\{[x]_0^1 - 4\left[\frac{x^2}{2}\right]_0^1 + 4\left[\frac{x^3}{3}\right]_0^1\right\}$$

$$= c_1\left[\frac{1}{3} - \frac{1}{4}\right] + \frac{1}{2}c_1^2\left[1 - 2 + \frac{4}{3}\right]$$

$$\therefore\ I = c_1\left(\frac{1}{12}\right) + \frac{1}{2}c_1^2\left(\frac{1}{3}\right) = \frac{c_1}{12} + \frac{c_1^2}{6}$$

Its stationary values are given by $\dfrac{dI}{dc_1} = 0 \quad \therefore\ \dfrac{1}{12} + \dfrac{2c_1}{6} = 0 \quad \therefore\ c_1 = -\dfrac{1}{4}.$

Putting this value in (4), we get $\bar{y}(x) = -\dfrac{1}{4}(x - x^2) = \dfrac{1}{4}x(x-1).$

EXERCISE - III

Using Rayleigh-Ritz method, find an approximate solution for the extremals of the following.

1. $\int_0^1 (y'^2 - y^2 - 2xy)\,dx$ with $y(0) = 0$ and $y(1) = 0.$ [Ans. : $\bar{y}(x) = \dfrac{1}{4}x(1-x)$]

2. $\int_0^1 (y'^2 - 4y^2 + 2x^2y)\,dx$ with $y(0) = 0$ and $y(1) = 0.$ [Ans. : $\bar{y}(x) = \dfrac{5}{18}x(1-x)$]

3. $\int_0^1 (y'^2 - 2y - 2xy)\,dx$ with $y(0) = 2$, $y(1) = 1$. (M.U. 2015) [Ans. : $\bar{y}(x) = 2 + \dfrac{3}{8}x - \dfrac{11}{8}x^2$]

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