* Baye's Theorem:-

statement: let $A_1, A_2, \cdots, A_n$ be the partition of the sample space S. and let B be any other event of S such that $P(A_i) \neq 0$ for every $i = 1, 2, \cdots n$ and $P(B) \neq 0$

then
$$P(A_i / B) = \frac{P(A_i) \cdot P(B/A_i)}{\sum_{i=1}^{n} P(A_i) \cdot P(B/A_i)}$$

that is
$$P(A_i / B) = \frac{P_i \, P_i'}{P_1 P_1' + P_2 P_2' + \cdots + P_n P_n'}$$

where, $P_i = P(A_i)$ and $P_i' = P(B/A_i)$

Proof: we know that for any two events A, B of S,
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

therefore,
$$P(A_i | B) = \frac{P(A_i \cap B)}{P(B)} \quad \text{——— i>}$$

and
$$P(B/A_i) = \frac{P(B \cap A_i)}{P(A_i)} \quad \text{——— ii>}$$

from ii> we can write,

$$P(A_i \cap B) = P(B \cap A_i) = P(A_i) \cdot P(B/A_i) \quad \text{——— iii>}$$

Now from i> and iii>

$$P(A_i/B) = \frac{P(A_i)\cdot P(B|A_i)}{P(B)} \quad \text{—— iv)}$$

But we know that

$$B = (B\cap A_1) \cup (B\cap A_2) \cup \cdots \cup (B\cap A_n)$$

$$\Rightarrow P(B) = P(B\cap A_1) + P(B\cap A_2) + \cdots + P(B\cap A_n)$$

$$\left(\because P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + \cdots + P(A_n)\right)$$

$$\Rightarrow P(B) = P(A_1)\cdot P(B/A_1) + P(A_2)\cdot P(B/A_2) + \cdots + P(A_n)\,P(B/A_n)$$

$$(\because \text{using iii)})$$

$$\Rightarrow P(B) = \sum_{i=1}^{n} P(A_i)\,P(B/A_i)$$

$\therefore$ equation iv) becomes

$$P(A_i/B) = \frac{P(A_i)\cdot P(B/A_i)}{\sum_{i=1}^{n} P(A_i)\cdot P(B/A_i)}$$

i.e. $$P(A_i/B) = \frac{P_i\, P_i'}{P_1 P_1' + P_2 P_2' + \cdots + P_n P_n'}$$

where, $P_i = P(A_i)$, $P_i' = P(B/A_i)$

Hence the proof.

Ex ① A bag contains 7 red and 3 black balls and another bag contains 4 red and 5 black balls. One ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. If this ball happens to be red, find the probability that a black ball was transferred.

Solution: Note that we transferring one ball from the first bag to the second bag.

* Total number of balls in first bag are 10

$\therefore$ P_1 = probability of transferring black ball $= \frac{3}{10}$

P_1' = Probability of drawing a red ball $= \frac{4}{10}$

P_2 = Probability of transferring red ball $= \frac{7}{10}$

P_2' = probability of drawing red ball $= \frac{5}{10}$

By Baye's theorem,

$\therefore$

$$\text{Required probability} = \frac{P_1 P_1'}{P_1 P_1' + P_2 P_2'}$$

$$= \frac{\left(\frac{3}{10}\right)\cdot\left(\frac{4}{10}\right)}{\left(\frac{3}{10}\right)\left(\frac{4}{10}\right)+\left(\frac{7}{10}\right)\left(\frac{5}{10}\right)} = \frac{12}{47}$$

$$= \frac{12}{47}$$

Ex ② A bag contains five balls, the colours of which are not known. Two balls were drawn from the bag and they were found to be white. What is the probability that all balls are white?

Solutions: Given: two drawn balls are white

$\therefore$ the bag may contains 2 white or 3 white or 4 white or 5 white balls.

let these events be denoted by A_1, A_2, A_3, A_4 respectively.

Now we Assume $P(A_1) = P(A_2) = P(A_3) = P(A_4) = \frac{1}{4}$

i.e $P_1 = P_2 = P_3 = P_4 = \frac{1}{4}$

Note that two balls out of 5 can be drawn in ${}^5C_2$ ways.

$\therefore$ $P_1' = p$ (drawing two balls when two balls are white)

$$= \frac{{}^2C_2}{{}^5C_2} = \frac{\frac{2!}{(2-2)!\ 2!}}{\frac{5!}{(5-2)!\cdot 2!}} = \frac{1}{1} \times \frac{3! \times 2!}{5!} = \frac{2}{20}$$

$P_2' = p$ (drawing two white balls when 3 balls are white)

$$= \frac{{}^3C_2}{{}^5C_2} = \frac{\frac{3!}{(3-2)!\ 2!}}{\frac{5!}{(5-2)!\cdot 2!}} = \frac{6}{20}$$

$P_3' = p$ (drawing 2 white balls when 4 balls are white)

$$= \frac{{}^4C_2}{{}^5C_2} = \frac{12}{20}$$

$P_4' = P$ (drawing 2 white balls when 5 balls are white)

$$= \frac{{}^5C_2}{{}^5C_2} = \frac{20}{20}$$

$\therefore$ By Baye's theorem,

$$\text{Required probability} = \frac{P_4 P_4'}{P_1 P_1' + P_2 P_2' + P_3 P_3' + P_4 P_4'}$$

$$= \frac{\left(\frac{1}{4}\right)\left(\frac{20}{20}\right)}{\left(\frac{1}{4}\right)\left(\frac{2}{20}\right) + \left(\frac{1}{4}\right)\left(\frac{6}{20}\right) + \left(\frac{1}{4}\right)\left(\frac{12}{20}\right) + \left(\frac{1}{4}\right)\left(\frac{20}{20}\right)}$$

$$= \frac{20}{2 + 6 + 12 + 20}$$

$$= \frac{1}{2}.$$

Ex ③. There are in a bag three true coins and one false coin with head on both sides. A coin is chosen at random and tossed four times. If head occurs all the four times, what is the probability that the false coin was chosen and used?

Solution: P_1 = selecting true coin $= \frac{{}^3C_1}{4} = \frac{3}{4}$

P_2 = selecting false coin $= \frac{{}^1C_1}{4} = \frac{1}{4}$

$p_1' = p(\text{getting all four heads with true coin})$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$$

$p_2' = p(\text{getting all four heads with false coin})$

$$= 1 \cdot 1 \cdot 1 \cdot 1 = 1$$

$\therefore$ The Required probability

$$= \frac{p_2 \, p_2'}{p_1 \, p_1' + p_2 \, p_2'}$$

$$= \frac{\left(\frac{1}{4}\right) \cdot 1}{\left(\frac{3}{4}\right)\left(\frac{1}{16}\right) + \left(\frac{1}{4}\right) \cdot 1}$$

$$= \frac{16}{19}$$

Homework:

Ex 9. A coin is tossed. If it turns up heads two balls are drawn from urn A otherwise two balls are drawn from urn B. urn B contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. What is the probability that urn A was used, given that balls drawn are black?

$\boxed{\text{Ans: } \frac{1}{8}}$

* Random Variables:

The variable which is associated with the outcomes of the sample space of the random experiment is called random variable

for example:

In a throw of three coins

$\boxed{\text{Sample space}}$ 'S' = { HHH, HHT, HTH, HTT, THH, THT, TTH, TTT }

$\therefore n(S) = 8$

$\boxed{\text{Random experiment}}$: Tossing of three coins

$\boxed{\text{Random variable (r.v)}}$ X = number of heads

X = 0, 1, 2, 3

$\boxed{\text{probability distribution}}$

X	0	1	2	3
$P(X=x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

* probability Distribution:-

let X be the random variable and x_i be the values of X then the set of pairs $\{x_i, P(x_i)\}$ is called as probability distribution, where $P(x_i)$ is probability of x_i