* Laurent's Series Expansion:

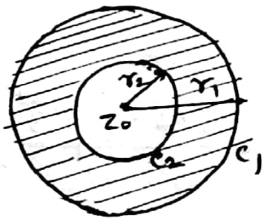
let C_1 and C_2 be two circle of radii r_1 and r_2 with centre z_0

let $f(z)$ be analytic on C_1, C_2 and between C_1 and C_2 then Laurent's series Expansion of $f(z)$ is

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where,
$$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w - z_0)^{n+1}} dw$$

and
$$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w - z_0)^{-n+1}} dw$$



* some Important power series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots, \quad |z| < \infty$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad |z| < \infty$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad |z| < \infty$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad |z| < \infty$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad |z| < \infty$$

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad |z| < 1$$

Example ① find Laurent's series for $f(z) = \frac{e^{3z}}{(z-1)^3}$ about $z=1$

Solution: Given: $f(z) = \frac{e^{3z}}{(z-1)^3}$

since, we want expansion around $z=1$, therefore we have to obtain Laurent's series in the power of $(z-1)$.

$$\therefore f(z) = \frac{e^{3z}}{(z-1)^3} = \frac{e^{3(z-1)+3}}{(z-1)^3}$$

$$= \frac{e^{3} \cdot e^{3(z-1)}}{(z-1)^3}$$

$$= \frac{e^{3}}{(z-1)^3}\left[e^{3(z-1)} \right]$$

$$= \frac{e^{3}}{(z-1)^3}\left[1 + 3(z-1) + \frac{3^2(z-1)^2}{2!} + \frac{3^3(z-1)^3}{3!} + \frac{3^4(z-1)^4}{4!} + \cdots \right]$$

$$\left(\because e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots \right)$$

$$= e^{3}\left[\frac{1}{(z-1)^3} + \frac{3(z-1)}{(z-1)^3} + \frac{3^2(z-1)^2}{2!\,(z-1)^3} + \frac{3^3(z-1)^3}{3!\,(z-1)^3} + \frac{3^4(z-1)^4}{4!\,(z-1)^3} + \cdots \right]$$

$$= e^{3}\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{2!(z-1)} + \frac{3^3}{3!} + \frac{3^4}{4!}\cdot(z-1) + \cdots \right]$$

i.e

$$\boxed{f(z) = e^{3}\left[\frac{1}{(z-1)^3} + \frac{3}{(z-1)^2} + \frac{3^2}{2!\,(z-1)} + \frac{3^3}{3!} + \frac{3^4}{4!}\cdot(z-1) + \cdots \right]}$$

Example② Find Laurent's series which represent the function $f(z) = \frac{1}{z(z+1)(z-2)}$ when

i) $|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$

solution: Given: $f(z) = \frac{1}{z(z+1)(z-2)}$

consider, $\frac{1}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$

$$\Rightarrow \frac{1}{z(z+1)(z-2)} = \frac{A(z+1)(z-2) + Bz(z-2) + Cz(z+1)}{z(z+1)(z-2)}$$

$$\Rightarrow A(z+1)(z-2) + Bz\cdot(z-2) + Cz(z+1) = 1$$

if $z=0$ then $A(0+1)(0-2) + B(0) + C(0) = 1 \Rightarrow A = -\frac{1}{2}$

if $z=-1$ then $A(0) + B(-1)(-1-2) + C(0) = 1 \Rightarrow B = \frac{1}{3}$

if $z=2$ then $A(0) + B(0) + C(2)(2+1) = 1 \Rightarrow C = \frac{1}{6}$

$$\therefore f(z) = \frac{1}{z(z+1)(z-2)} = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

case i) When $0 < |z| < 1$

$$\therefore f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

$$\Rightarrow f(z) = -\frac{1}{2z} + \frac{1}{3(1+z)} - \frac{1}{12[1-(\frac{z}{2})]}$$

since, $|z|<1$ i.e. $|z|<1<2 \Rightarrow |z|<2$

$\therefore |z|<2 \Rightarrow \left|\frac{z}{2}\right| < 1$

therefore,

$$f(z) = -\frac{1}{2z} + \frac{1}{3}[1+z]^{-1} - \frac{1}{12}\left[1-\left(\frac{z}{2}\right)\right]^{-1}$$

$$\Rightarrow f(z) = -\frac{1}{2z} + \frac{1}{3}[1 - z + z^2 - z^3 + z^4 - \cdots]$$
$$- \frac{1}{12}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$\left(\because [1-z]^{-1} = 1+z+z^2+\cdots \text{ and } [1+z]^{-1} = 1-z+z^2-z^3+\cdots \right)$$

case ii) when $1 < |z| < 2$

i.e $1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1$ and

$|z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{3z\left[1+\left(\frac{1}{z}\right)\right]} - \frac{1}{12\left[1-\left(\frac{z}{2}\right)\right]}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1+\left(\frac{1}{z}\right)\right]^{-1} - \frac{1}{12}\left[1-\left(\frac{z}{2}\right)\right]^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1-\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$
$$-\frac{1}{12}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$

case iii) when $|z| > 2$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)}$$

$$= -\frac{1}{2z} + \frac{1}{3z\left(1+\frac{1}{z}\right)} + \frac{1}{6z\left(1-\frac{2}{z}\right)}$$

Since, $|z| > 2$ i.e. $|z| > 2 > 1 \Rightarrow |z| > 1$

$\therefore$ $1 < |z| \Rightarrow \left|\frac{1}{z}\right| < 1$ and

$2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3z}\left[1+\left(\frac{1}{z}\right)\right]^{-1} + \frac{1}{6z}\left[1-\left(\frac{2}{z}\right)\right]^{-1}$$

$$\Rightarrow \quad f(z) = -\frac{1}{2z} + \frac{1}{3z}\left[1-\frac{1}{z}+\left(\frac{1}{z}\right)^2-\left(\frac{1}{z}\right)^3+\cdots\right]$$
$$+\frac{1}{6z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots\right]$$