CHAPTER

7

Quadratic Forms

1. Introduction

In this chapter, we shall learn what we mean by a quadratic form and how a quadratic form is transformed through congruent transformations and also through orthogonal transformations, to sum of squares.

2. Definition

(a) ***A homogeneous polynomial of second degree in n variables is called a quadratic form.***

For example,

1. $ax^2 + by^2 + 2hxy$ is a quadratic form in two variables.
2. $ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gzx$ is a quadratic form in three variables.
3. $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + 2a_{12}x_1x_2$
$$+2a_{13}x_1x_3 + 2a_{14}x_1x_4 + 2a_{23}x_2x_3 + 2a_{24}x_2x_4 + 2a_{34}x_3x_4$$
is a quadratic form in four variables.

(b) **Theorem :** ***Every quadratic form can be expressed in matrix notation as X'AX where X is a column matrix, X' is its transpose i.e. a row matrix and A is a square symmetric matrix of the same order as X.***

We accept the theorem without proof.

The above quadratic forms can be expressed in matrix notation as

(i) $\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ **(ii)** $\begin{bmatrix} x & y & z \end{bmatrix}\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(iii) $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Example : Express the following quadratic forms in matrix notation.

(i) $x^2 - 6xy + 2y^2$

(ii) $2x^2 + 3y^2 - 5z^2 - 2xy + 4xz - 6yz$

(iii) $x_1^2 + 2x_2^2 - 3x_3^2 + x_4^2 - 2x_1x_2 + 4x_1x_3 - 2x_1x_4 + 4x_2x_3 - 6x_2x_4 + 8x_3x_4$.

Sol. : The matrix notation is

(i) $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ (ii) $\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -3 \\ 2 & -3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

(iii) $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & -1 \\ -1 & 2 & 2 & -3 \\ 2 & 2 & -3 & 4 \\ -1 & -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

(c) Matrix of a quadratic form

Definition : ***If a quadratic form is expressed in matrix notation as X ' AX where A is a square symmetric matrix, then A is called the matrix of the quadratic form.***

Example : Write down the matrix of each of the following quadratic forms

(i) $x^2 - 2y^2 + 3z^2 - 2xy - 6xz + 10zy$

(ii) $2x_1^2 - 3x_2^2 + 4x_3^2 + x_4^2 - 2x_1x_2 + 3x_1x_3 - 4x_1x_4 - 5x_2x_3 + 6x_2x_4 + x_3x_4$.

Sol. : The matrix of the quadratic form is

(i) $A = \begin{bmatrix} 1 & -1 & -3 \\ -1 & -2 & 5 \\ -3 & 5 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -1 & 3/2 & -2 \\ -1 & -3 & -5/2 & 3 \\ 3/2 & -5/2 & 4 & 1/2 \\ -2 & 3 & 1/2 & 1 \end{bmatrix}$

EXERCISE - I

1. Write down the matrix corresponding to each of the following quadratic forms.

(i) $x^2 - 2y^2 + 3z^2 - 4xy + xz - 2yz$

(ii) $x_1^2 + x_2^2 - 3x_3^2 + 2x_1x_2 - 4x_1x_3 + 6x_2x_3$

(iii) $x_1^2 - x_2^2 + 2x_3^2 - 2x_4^2 - 5x_1x_2 + 6x_1x_3 - 4x_1x_4 + x_2x_3 - 3x_2x_4 + 5x_3x_4$

(iv) $2x_1^2 - 3x_2^2 + 4x_3^2 - x_4^2 - x_1x_2 + 2x_1x_3 - 3x_1x_4 + 2x_2x_3 - 4x_2x_4 + 6x_3x_4$

(v) $x_1^2 - 2x_2^2 + 4x_3^2 - 4x_4^2 - 2x_1x_2 + 3x_1x_4 + 4x_2x_3 - 5x_3x_4$

[**Ans. :** (i) $\begin{bmatrix} 1 & -2 & 1/2 \\ -2 & -2 & -1 \\ 1/2 & -1 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 3 \\ -2 & 3 & -3 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & -5/2 & 3 & -2 \\ -5/2 & -1 & 1/2 & -3/2 \\ 3 & 1/2 & 2 & 5/2 \\ -2 & -3/2 & 5/2 & -2 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & -1/2 & 1 & -3/2 \\ -1/2 & -3 & 1 & -2 \\ 1 & 1 & 4 & 3 \\ -3/2 & -2 & 3 & -1 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & -1 & 0 & 3/2 \\ -1 & -2 & 2 & 0 \\ 0 & 2 & 4 & -5/2 \\ 3/2 & 0 & -5/2 & -4 \end{bmatrix}$

2. Write down the matrix of each of the following quadratic forms and verify that it can be written as $X'AX$.

(i) $x_1^2 - 18x_1x_2 + 5x_2^2$ (ii) $2x_1^2 - 3x_2^2 + 4x_3^2 - 2x_1x_2 + 4x_2x_3$

(iii) $x_1^2 - x_3^2 + 4x_2x_3$

3. Linear Transformation

Consider two sets of variables $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$. Let these variables be related by the following linear equations

$$y_1 = a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$y_n = a_{n1}x_1 + x_{n2}x_2 + \ldots + a_{nn}x_n$$

The above n equations can be expressed as a single matrix equation.

$$Y = AX$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ \ldots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \ldots \\ \ldots \\ y_n \end{bmatrix}$$

The transformation from y's to x's by the above equations is called ***a linear transformation***. The linear transformation is ***singular or non-singular*** according as the matrix A is singular or non-singular.

Example 1 : Write the matrices corresponding to the following linear transformations.

(i) $y_1 = x_1 - 3x_2 + 2x_3$
$y_2 = 2x_2 - 3x_3$
$y_3 = 3x_1 + 5x_3$

(ii) $y_1 = 2x_1 - x_2 - x_3$
$y_2 = 3x_3$
$y_3 = x_1 + x_2$

Sol. : Do it.

Example 2 : Write the linear transformations corresponding to the following matrices.

(i) $\begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & -3 \\ 1 & -2 & 3 \\ 3 & 1 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & 0 \\ 2 & 0 & 3 \end{bmatrix}$

Sol. : Do it.

Example 3 : For the linear transformation $X = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$, find the co-ordinates (y_1, y_2, y_3) in Y corresponding to $(1, 2, -1)$ in X.

Sol. : Since $X = AY$, when $X = [1, 2, -1]'$.

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This can be written as

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

This is a non-homogeneous system of equations and can be solved by the method studied earlier.

By R_{13} $\begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$ By $\begin{matrix} R_2 - R_1 \\ R_3 - 2R_1 \end{matrix}$ $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}$

By $R_3 - R_2$ $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$

Now, $A \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and $[A, B] \to \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$\therefore$ The rank of A = the rank of $[A, B] = 3$.

$\therefore$ The equations are consistent. Further, since the rank of the coefficient matrix A is equal to the number of unknowns the system has unique solution. Now, the equations can be written as,

$$y_1 + 0y_2 - 2y_3 = -1, \quad y_2 + 4y_3 = 3, \quad y_3 = 0 \qquad \therefore \quad y_2 = 3 \text{ and } y_1 = -1$$

$\therefore$ $(1, 2, -1)$ in X corresponds to $(-1, 3, 0)$ in Y.

Example 4 : Express each of the following transformations

$$x_1 = 2y_1 - 3y_2 \,;\; x_2 = 4y_1 + y_2 \quad \text{and} \quad y_1 = z_1 - 2z_2 \,;\; y_2 = 2y_1 + 3y_2$$

in the matrix form and find the composite transformation which expresses x_1, x_2 in terms of z_1, z_2.

Sol. : The two transformations can be expressed as

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad i.e. \quad X = AY$$

and $$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad i.e. \quad Y = BZ$$

The resulting composite transformation is

$$X = A(BZ) = ABZ$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} -4 & -13 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\therefore \; x_1 = -4z_1 - 13z_2, \; x_2 = 6z_1 - 5z_2$$

Example 5 : Find a linear transformation $Y = AX$ which carries $X_1 = (2, 2)'$ and $X_2 = (4, -1)'$ to $Y_1 = (3, 2)'$ and $Y_2 = (2, 3)'$ respectively.

Sol. : Let the transformation matrix be $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

$\therefore$ $Y = AX$ gives $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Now, $Y_1 = AX_1$ gives $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

$\therefore$ $3 = 2a_1 + 2b_1,\ 2 = 2a_2 + 2b_2$ .... (i)

Further, $Y_2 = AX_2$ gives $\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$

$\therefore$ $2 = 4a_1 - b_1,\ 3 = 4a_2 - b_2$ .... (ii)

Solving $3 = 2a_1 + 2b_1$, and $4a_1 - b_1 = 2$, we get $a_1 = \dfrac{7}{10}$ and $b_1 = \dfrac{4}{5}$.

Solving $1 = a_2 + b_2$ and $3 = 4a_2 - b_2$, we get $a_2 = \dfrac{4}{5}$ and $b_2 = \dfrac{1}{5}$.

$\therefore$ The required transformation matrix is $A = \begin{bmatrix} 7/10 & 4/5 \\ 4/5 & 1/5 \end{bmatrix}$.

Example 6 : Find the linear transformation $Y = AX$ which carries,

$X_1 = (1, 0, 1)',\ X_2 = (1, -1, 1)',\ X_3 = (1, 2, -1)'$ on to

$Y_1 = (2, 3, -1)',\ Y_2 = (3, 0, -2)',\ Y_3 = (-2, 7, 1)'$.

Sol. : Let the transformation matrix be $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$\therefore$ $Y = AX$ gives $\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Now, $Y_1 = AX_1$ gives $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ $2 = a_1 + c_1,\ 3 = a_2 + c_2,\ -1 = a_3 + c_3$ .... (i)

Further, $Y_2 = AX_2$ gives $\begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\therefore$ $3 = a_1 - b_1 + c_1,\ 0 = a_2 - b_2 + c_2,\ -2 = a_3 - b_3 + c_3$ .... (ii)

Again, $Y_3 = AX_3$ gives $\begin{bmatrix} -2 \\ 7 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

$\therefore\ -2 = a_1 + 2b_1 - c_1,\ 7 = a_2 + 2b_2 - c_2,\ 1 = a_3 + 2b_3 - c_3$ .... (iii)

Putting the values from (i) in (ii), we get,

$3 = 2 - b_1 \quad \therefore\ b_1 = -1,\ 0 = 3 - b_2 \quad \therefore\ b_2 = 3$

$-2 = -1 - b_3 \quad \therefore\ b_3 = 1$

Subtracting (ii) from (iii),

$-5 = 3b_1 - 2c_1.$ But $b_1 = -1$ $\therefore\ c_1 = 1$

$7 = 3b_2 - 2c_2.$ But $b_2 = 3$ $\therefore\ c_2 = 1$

$3 = 3b_3 - 2c_3.$ But $b_3 = 1$ $\therefore\ c_3 = 0$

Putting the values of c_1, c_2, c_3 in (i), we get, $a_1 = 1,\ a_2 = 2,\ a_3 = -1$.

Hence, the transformation matrix is $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 3 & 1 \\ -1 & 1 & 0 \end{bmatrix}$.

EXERCISE - II

1. If $y_1 = x_1 \cos\alpha - x_2 \sin\alpha$ and $y_2 = x_1 \sin\alpha + x_2 \cos\alpha$, show that the transformation matrix A is orthogonal and find A^{-1}.

   $\left(\textbf{Ans. :} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}\right)$

2. Given the transformation $Y = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ find the coordinates (x_1, x_2, x_3) corresponding to (2, 0, 5) in Y.

   $\left[\textbf{Ans. :} \left(\frac{13}{5}, \frac{11}{5}, -\frac{7}{5}\right)\right]$

3. Given the transformation $Y = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, find the coordinates (x_1, x_2, x_3) corresponding to (9, 6, 2) in Y.

   [**Ans. :** (1, 2, 3)']

4. Given the transformation $Y = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, find the co-ordinates (x_1, x_2, x_3) in X corresponding to (9, 52, 0) in Y.

   [**Ans. :** (1, 3, 5)']

5. Express the following transformations

   $x_1 = 3y_1 + 2y_2 \qquad y_1 = z_1 + 2z_2$

   $x_2 = -y_1 + 4y_2 \quad$ and $\quad y_2 = 0 + 3z_2$

   in matrix form and find the composite transformation which will express x_1, x_2 in terms of z_1, z_2.

   [**Ans. :** $x_1 = 3z_1 + 12z_2,\ x_2 = -z_1 + 10z_2$]

6. Express the following transformations

$$x_1 = y_1 + 3y_2,\ x_2 = -y_1 + 2y_2 + y_3,\ x_3 = 2y_3$$

and $y_1 = 2z_1 + 3z_2 + 4z_3,\ y_2 = z_1 + 2z_2 + 3z_3,\ y_3 = -z_1 + z_2 + 2z_3.$

in matrix form and find the composite transformation which expresses x_1, x_2, x_3 in terms of z_1, z_2, z_3.
[**Ans. :** $x_1 = 5z_1 + 9z_2 + 13z_3,\ x_2 = -z_1 + 2z_2 + 4z_3,\ x_3 = -2z_1 + 2z_2 + 4z_3.$]

7. Express the following transformation

$$x_1 = y_1 - 2y_2 + 3y_3,\ x_2 = 2y_1 + 3y_2 - y_3,\ x_3 = -3y_1 + y_2 + 2y_3.$$

and $y_1 = z_1 + 2z_3,\ y_2 = z_2 + 2z_3,\ y_3 = z_1 + 2z_2$

in matrix form and find the composite transformation which expresses x_1, x_2, x_3 in terms of z_1, z_2, z_3.
[**Ans. :** $x_1 = 4z_1 + 4z_2 - 2z_3,\ x_2 = z_1 + z_2 + 10z_3,\ x_3 = -z_1 + 5z_2 - 4z_3.$]

8. Find a linear transformation $Y = AX$ which carries $X_1 = (1, 2)'$ and $X_2 = (2, 3)'$ to $Y_1 = (5, 11)'$ and $Y_2 = (8, 18)'$ respectively.

[**Ans. :** $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$]

9. Find a linear operator in two dimensional space which maps the vectors (1, 1) and (3, – 2) on to the vectors (2, 1) and (1, 2) respectively.

[**Ans. :** $\begin{bmatrix} 1 & 1 \\ 4/5 & 1/5 \end{bmatrix}$]

10. Find the linear transformation $Y = AX$ which carries $X_1 = (1, 1, -1)', X_2 = (1, -1, 1)', X_3 = (-1, 1, 1)'$ on to $Y_1 = (2, 1, 3)'$ and $Y_2 = (2, 3, 1)'$. $Y_3 = (4, 1, 3)'$.

[**Ans. :** $\begin{bmatrix} 2 & 3 & 3 \\ 2 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix}$]

11. Find a linear transformation which maps the vectors (1, 2) and (2, 1) of two dimensional space on to vectors (7, 0, – 8) and (5, 3, 2) of three dimensional space.

[**Ans. :** $\begin{bmatrix} 1 & 3 \\ 2 & -1 \\ 4 & -6 \end{bmatrix}$]

4. Linear Transformation of a Quadratic Form

Consider a quadratic form $X'AX$ (where A is symmetric) and a non-singular linear transformation given by $X = PY$.

Now,
$$\begin{aligned} X'AX &= (PY)'\,APY = (Y'P')\,APY \\ &= Y'(P'AP)\,Y \\ &= Y'BY \quad \text{where } B = P'AP \end{aligned}$$

The quadratic form $Y'BY$ is called a *linear transform* of the quadratic form $X'AX$ under $X = PY$.

Further,
$$\begin{aligned} B' &= (P'AP)' = P'A'P'' \\ &= P'A'P \qquad [\because A \text{ is symmetric}] \\ &= B \end{aligned}$$

Hence, the matrix B is also symmetric.

Example : Obtain the linear transform of the quadratic form

$$2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_3$$

under the linear transformation and interpret your result.

$$x_1 = y_1 - y_2 + 2y_3,\ x_2 = 2y_2 + 2y_3,\ x_3 = 3y_3.$$

Sol. : The matrix of the given quadratic form is $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$

The matrix of the given transformation is $P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

The matrix of the transform is

$$B = P'AP = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The resulting linear transform is

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

This means the given quadratic form

$2x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 4x_1x_3 - 4x_2x_3$ under the linear transformation

$x_1 = y_1 - y_2 + 2y_3,\ x_2 = 2y_2 + 2y_3,\ x_3 = 3y_3$ transforms to the quadratic form

$2y_1^2 + 6y_2^2 + 3y_3^2$.

EXERCISE - III

1. Obtain the transform of the quadratic form $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$ under the linear transformation $x_1 = y_1 + \frac{1}{3}y_2 - \frac{2}{7}y_3,\ x_2 = y_2 + \frac{1}{7}y_3,\ x_3 = y_3$ and interpret the result. [**Ans. :** $6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2$]

2. Obtain the transform of the quadratic form $x_1^2 + 2x_2^2 - 7x_3^2 - 4x_1x_2 + 8x_1x_3$ under the linear transformation $x_1 = y_1 + 2y_2 + 4y_3,\ x_2 = y_2 + 4y_3,\ x_3 = y_3$ and interpret the result. [**Ans. :** $y_1^2 - 2y_2^2 + 9y_3^2$]

5. Congruence of a Square Matrix

Definition

A square matrix B of order n is said to be **congruent** to another square matrix A of the same order n if there exists a non-singular matrix P such that $B = P'AP$.

For example, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is congruent to $A = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}$ because there exists a non-singular matrix $P = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$ such that $B = P'AP$ as seen on the previous page.

6. Congruent Transformations

Effecting a row transformation on P and taking its transpose P' means effecting a column transformation on P. Since, further if $B = P'AP$, B is congruent to A, row and column transformations on a square matrix are congruent transformations.

The following three operations are called **congruent transformations**.

(i) Interchange of i-th and j-th rows and interchange of i-th and j-th columns simultaneously. These are denoted by

$$R_i \to R_j, \quad C_i \to C_j$$

(ii) Multiplying the i-th row and j-th column simultaneously by a non-zero element k. This is denoted by

$$R_i \to kR_i, \quad C_j \to kC_j$$

(iii) Adding the k-th multiple of i-th row to j-th row and adding the k-th multiple of i-th column to j-th column simultaneously. This is denoted by

$$R_i \to R_i + kR_j, \quad C_i \to C_i + kC_j.$$

Congruence of Quadratic Forms

Definition : Two quadratic forms $X'AX$ and $Y'BY$ are said to be **congruent** if A and B are congruent *i.e.* if there exists a non-singular square matrix P such that $B = P'AP$.

Theorem : *If $X'AX$ is a quadratic form where A is a matrix of rank r then there exists a non-singular linear transformation $X = PY$ which transforms the given quadratic form to a "sum of r square terms".*

$$b_1y_1^2 + b_2y_2^2 + \ldots.. + b_ry_r^2$$

Unfortunately we shall accept this theorem without proof.

Reduction of Quadrtic Form to Diagonal Form

Example 1 : Reduce the matrix of the quadratic form

$$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$$

to the diagonal form through congruent transformations and interpret your result.

(M.U. 1990, 92, 2001, 11)

Sol. : The quadratic form can be written in matrix notation as

$$[x_1\ x_2\ x_3]\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

i.e. $X'AX$ where $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

We now write $A = IAI$ and reduce A on l.h.s. to a diagonal matrix by applying congruent transformations.

1. We apply row transformations to prefactor I and column transformations to post-factor I of r.h.s. It may be noted that it is enough to apply row transformations on the prefactor I and write its transpose as the transformed post-factor.

2. We apply the same row transformations and column transformations to the matrix A on l.h.s. It may be noted that it is enough to apply row transformation only to A. For this, use row transformations on R_2 and R_3 with the help of R_1 such that the first element in the second and third rows are zero. The same column transformations will make the second and third elements of the first row zero without affecting the remaining elements in the second and third rows. This is so because A is a symmetric matrix.

3. In the same way bring zeros in other non-diagonal places.

Now,
$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 + \frac{1}{3}R_1$, $R_3 - \frac{1}{3}R_1$, $C_2 + \frac{1}{3}C_1$, $C_3 - \frac{1}{3}C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + \frac{1}{7}R_2$, $C_3 + \frac{1}{7}C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{2}{7} & \frac{1}{7} & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $B = P'AP$ where B is a diagonal matrix.

This means the quadratic form $X'AX = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$ will be transformed to $Y'BY = 6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2$ by the transformation $X = PY$.

i.e., $x_1 = y_1 + \frac{1}{3}y_2 - \frac{2}{7}y_3$, $x_2 = y_2 + \frac{1}{7}y_3$, $x_3 = y_3$.

Example 2 : Reduce the following quadratic form

$$6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 18x_1x_3 + 4x_2x_3$$

to diagonal form through congruent transformations.

Sol. : The quadratic form can be written as

$$[x_1\ x_2\ x_3]\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \therefore\ A = \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix}$$

We write $A = IAI$

$$\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{By}\ \begin{matrix} R_2 - \frac{1}{3}R_1 \\ R_3 - \frac{3}{2}R_1 \\ C_2 - \frac{1}{3}C_1 \\ C_3 - \frac{3}{2}C_1 \end{matrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & -1 \\ 0 & -1 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{3}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{By}\ \begin{matrix} R_3 + \frac{3}{7}R_2 \\ C_3 + \frac{3}{7}C_2 \end{matrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{1}{14} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{23}{14} & \frac{3}{7} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{1}{3} & -\frac{23}{14} \\ 0 & 1 & \frac{3}{7} \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $B = P'AP$ where B is a diagonal matrix.

This means the quadratic form $X'AX = 6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 18x_1x_3 + 4x_2x_3$ will be transformed to $Y'BY = 6y_1^2 + \frac{7}{3}y_2^2 + \frac{1}{14}y_3^2$ by the transformation $X = PY$.

i.e., $\quad x_1 = y_1 - \frac{1}{3}y_2 - \frac{23}{14}y_3,\quad x_2 = y_2 + \frac{3}{7}y_3,\quad x_3 = y_3.$

Example 3 : Reduce the symmetric matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ to the diagonal form using congruent transformations and interpret in terms of quadratic form. **(M.U. 2004)**

Sol. : We first write $\quad A = IAI$

$$\therefore \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\begin{matrix} R_1+R_2 \\ C_1+C_2 \end{matrix}$ $\begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} R_3-R_1 \\ C_3-C_1 \\ R_2-(1/2)R_1 \\ C_2-(1/2)C_1 \end{matrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ -1/2 & 1/2 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/2 & -1 \\ 1 & 1/2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

Thus, $B = P'AP$

This means the quadratic form $X'AX = 0x_1^2 + 0x_2^2 + 0x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$ will be transformed to

$$Y'BY = 2y_1^2 - \frac{1}{2}y_2^2 - 2y_3^2$$

by the transformation $X = PY$. .... (1)

i.e., $x_1 = y_1 - \frac{1}{2}y_2 - y_3,\quad x_2 = y_1 + \frac{1}{2}y_2 - y_3,\quad x_3 = y_3.$

EXERCISE - IV

Reduce the following quadratic forms to diagonal form through congruent transformations.

1. $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$ [Ans. : $u^2 - 2v^2 - 4w^2$]
2. $x^2 + 2y^2 - 7z^2 - 4xy + 8xz$ [Ans. : $u^2 - 2v^2 + 9w^2$]
3. $x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$ [Ans. : $u^2 + v^2 + \frac{1}{2}w^2$]
4. $y^2 + 2xy - 2yz + 4zx$ [Ans. : $u^2 - v^2 + 8w^2$]
5. $xy + yz + zx$ **(M.U. 2015)** (Hint : First use $R_1 + 2R_2$, $C_1 + 2C_2$) [Ans. : $2u^2 - \frac{1}{8}v^2 - w^2$]
6. $2xy + 4xz + 6yz$ [Ans. : $2u^2 - \frac{1}{2}v^2 - 12w^2$]
7. $x^2 + 2y^2 - 3z^2 + 5w^2 - 4xy + 8yz + 2yw - 2zx$ [Ans. : $p^2 - 2q^2 - 2r^2 - 6s^2$]

7. Canonical Form or Normal Form

Before we give the definition of Canonical Form and some more definitions we state the relevant following theorem.

Theorem : ***If A is a square symmetric non-zero matrix of rank r then there exists a real non-singular matrix P such that***

$$P'AP = \text{diag}\,[\,+1, +1, \ldots, +1, -1, -1, \ldots, -1, 0, 0, \ldots, 0\,]$$

We accept this theorem without proof but we shall give some explanation.

The theorem states that if A is a square symmetric matrix of rank r and of order n, then there exists a matrix P such that $P'AP$ is diagonal matrix of order n and rank r in which some diagonal elements are + 1 and some diagonal elements are – 1 and $(n - r)$ diagonal elements are zero. There are r non-zero (+ 1 or – 1) diagonal elements.

We can interpret this theorem in terms of quadratic form as :

If $X'AX$ is a quadratic form of rank r, then there exists a real non-singular linear transformation $X = PY$ which transforms $X'AX$ to

$$Y'BY = y_1^2 + y_2^2 + \ldots\ldots + y_s^2 - y_{s+1}^2 - y_{s+2}^2 - \ldots\ldots - y_r^2 \qquad \text{where } B = P'AP.$$

8. Rank, Index, Signature

(i) If a quadratic form by non-singular linear congruent transformations $X = PY$ is expressed as the sum or difference of squares of new r variables then the new expression is called the **Canonical form or the Normal form** of the given quadratic form, where r is the rank of the matrix of the form.

Thus, a quadratic form

$$a_{11}x_1^2 + a_{22}x_2^2 + \ldots\ldots + a_{nn}x_n^2 + 2a_{12}x_1x_2 + \ldots\ldots + 2a_{1n}x_1x_n + \ldots\ldots$$

can be expressed as $\quad y_1^2 + y_2^2 + \ldots\ldots + y_s^2 - y_{s+1}^2 - y_{s+2}^2 - \ldots\ldots - y_r^2$

where coefficients of the squares are either +1 or –1 or 0. The new form (of sum or difference of squares) is called the **canonical form.** If there are r terms in all (positive or negative) then r is the **rank** of the matrix.

(ii) The numbers of the positive squares is called the **index** of the quadratic form.

(iii) The difference between the number of positive squares and the number of negative squares is called the **signature** of the quadratic form.

Since, there are in all r terms and s of them are positive there will be $r - s$ negative terms and hence, the difference between them = $s - (r - s)$.

$$\therefore \qquad \boxed{\textbf{Signature} = 2s - r.}$$

Corollary : If the rank of a matrix is even then its signature is also even. **(M.U. 2001)**

Sol. : Since signature = $2s - r$ if r even then clearly $2s - r$ is also even.

9. Sylvester's Law of Inertia

For **all** real non-singular matrices P of rank r such that

$$P'AP = \text{diag. } [\, 1, 1, \ldots., 1, -1, -1, \ldots\ldots, -1, 0, 0, \ldots\ldots, 0 \,]$$

the number s of 1 and the number $r - s$ of –1 are constants.

This is known as Sylvester's law of inertia.

By definition signature = $2s - r$ and since s and r are constants, the sylvester's law of inertia can be stated as -

"The **signature** of a real quadratic form **is invariant** for all normal reductions".

10. Value Classes (Classification of Quadratic Forms)

A symmetric matrix A and the corresponding quadratic form $X'AX$ are classified into five classes

(i) **Positive Definite** (ii) **Negative Definite**

(iii) **Positive Semidefinite** (iv) **Negative Semidefinite**

(v) **Indefinite**

(1) Positive Definite Form

If all the eigenvalues of a symmetric matrix A **are positive** then the matrix A and the quadratic form $X'AX$ is called **positive definite.**

When such a quadratic form is reduced to the sum of squares then the corresponding diagonal matrix has **all the elements positive**. In other words,

$$D = \text{diag}\,[\,d_1^2, d_2^2, \ldots., d_n^2\,]$$

(2) Negative Definite Form

If all the eigenvalues of a symmetric matrix A are **negative**, then the matrix A and the corresponding quadratic form $X'AX$ is called **negative definite.**

When such a matrix is reduced to sum of squares then the corresponding diagonal matrix has **all the elements negative**. In other words,

$$D = \text{diag}\,[-d_1^2, -d_2^2, \ldots., -d_n^n\,]$$

(3) Positive Semidefinite

If all the eigenvalues of a symmetric matrix A are **non-negative** (*i.e.* some are positive and some are zero) then the matrix A and the quadratic form $X'AX$ is called **positive semidefinite.**

When such a quadratic form is reduced to the sum of squares then the corresponding diagonal matrix has some elements zero and some positive elements. In other words,

$$D = \text{diag}\,[\,d_1^2, d_2^2, \ldots., d_r^2, 0, 0, \ldots., 0\,]$$

where r is the rank of A.

(4) Negative Semidefinite

If all the eigenvalues of a square matrix A are **non-positive** (*i.e.* some are negative and some are zero) then the matrix A and the quadratic form $X'AX$ is called **negative semidefinite.**

When such a quadratic form is reduced to the sum of squares then the corresponding diagonal matrix has some elements zero and some negative elements. In other words,

$$D = \text{diag}\,[-d_1^2, -d_2^2, -d_r^2, 0, 0, \ldots., 0\,]$$

where r is the rank of A.

(5) Indefinite Form

In any other case the symmetric matrix A and the quadratic form $X'AX$ is called **indefinite.**

Some eigenvalues of such a matrix are positive, some are negative and some may be zero (or may not be zero). When such a quadratic form is reduced to the sum of squares, then the corresponding diagonal matrix has some elements positive, some elements negative including some elements zero. In other words,

$$D = \text{diag}\,[\,d_1^2, d_2^2, \ldots., -d_k^2, -d_{k+1}^2, 0, 0, \ldots., 0\,]$$

In short a quadratic form in n variables is

Positive definite if $y_1^2 + y_2^2 + \ldots.. + y_r^2 + \ldots.. + y_n^2$

Negative definite if $-y_1^2 - y_2^2 - \ldots.. - y_r^2 - \ldots.. - y_n^2$

Positive semi-definite if $y_1^2 + y_2^2 + \ldots.. + y_r^2 \;(r < n)$

Negative semi-definite if $-y_1^2 - y_2^2 - \ldots.. - y_r^2 \;(r < n)$

Indefinite in any other case.

Note ....

The above five classes of a quadratic form are mutually exclusive in the sense that a given quadratic form belongs to one and only one value class. These classes are called **value classes** of a quadratic form.

Example 1 : Determine whether the quadratic form $4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3$ is positive definite or not.

Sol. : The matrix of the above quadratic form is

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

We shall obtain its eigenvalues. Its characteristic equation is

$$\begin{vmatrix} 4-\lambda & 2 & 2 \\ 2 & 4-\lambda & 2 \\ 2 & 2 & 4-\lambda \end{vmatrix} = 0$$

$\therefore \quad (4-\lambda)[(4-\lambda)(4-\lambda)-4] - 2[2(4-\lambda)-4] + 2[4-2(4-\lambda)] = 0$

$\therefore \quad (4-\lambda)[16-8\lambda+\lambda^2-4] - 2[8-2\lambda-4] + 2[4-8+2\lambda] = 0$

$\therefore \quad (4-\lambda)(\lambda^2-8\lambda+12) - 2(4-2\lambda) + 2[-4+2\lambda] = 0$

$\therefore \quad (4-\lambda)(\lambda-6)(\lambda-2) + 4(\lambda-2) + 4(\lambda-2) = 0$

$\therefore \quad (\lambda-2)[(4-\lambda)(\lambda-6)+8] = 0 \qquad \therefore \quad (\lambda-2)[10\lambda-\lambda^2-24+8] = 0$

$\therefore \quad (\lambda-2)(-\lambda^2+10\lambda+16) = 0 \qquad \therefore \quad (\lambda-2)(\lambda^2-10\lambda+16) = 0$

$\therefore \quad (\lambda-2)(\lambda-8)(\lambda-2) = 0 \qquad \therefore \quad \lambda = 2, 2, 8.$

Since all the eigenvalues are positive, the quadratic form is positive definite.

Example 2 : Show that the quadratic form $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ is positive definite.

Sol. : The matrix of the quadratic form is

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

In Engg. Maths. - III by the same author, Ex. 2, on page 5-18, we have obtained the eigenvalues of this matrix A.

$$\lambda = 2, 3, 6.$$

Since all the eigenvalues are positive, the quadratic form is positive definite.

11. Reduction to Canonical Form Under Congruent Transformations

We have already learnt (Example 1 page 7-9 and Example 2 page 7-11) how to reduce a given quadratic form to sum of squares. In this form the matrix of the given form is reduced to a diagonal matrix. To reduce a given quadratic form to canonical form we first reduce the matrix of the form to diagonal form as before. We then use congruent transformations $\frac{1}{\sqrt{k_1}}R_1$, $\frac{1}{\sqrt{k_1}}C_1$ etc. and reduce the diagonal elements to +1 or −1 thus, getting the canonical form.

Example 1 : Reduce the quadratic form $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ to normal form through congruent transformations. Also find its rank, signature, index and value class.

(M.U. 2003, 09, 10)

Sol. : The quadratic form can be written as

$$[x_1\ x_2\ x_3]\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \therefore\ A = \begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix}$$

We write $A = IAI$

$$\begin{bmatrix} 2 & 6 & -2 \\ 6 & 1 & -4 \\ -2 & -4 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\begin{matrix} R_2 - 3R_1 \\ C_2 - 3C_1 \\ R_3 + R_1 \\ C_3 + C_1 \end{matrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 2 \\ 0 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} R_3 + \frac{2}{17}R_2 \\ C_3 + \frac{2}{17}C_2 \end{matrix}$ $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -17 & 0 \\ 0 & 0 & \frac{81}{17} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ \frac{11}{17} & \frac{2}{17} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3 & \frac{11}{17} \\ 0 & 1 & \frac{2}{7} \\ 0 & 0 & 1 \end{bmatrix}$

By $\frac{1}{\sqrt{2}}R_1, \frac{1}{\sqrt{2}}C_1\ ;\ \frac{1}{\sqrt{17}}R_2, \frac{1}{\sqrt{17}}C_2\ ;\ \frac{\sqrt{17}}{9}R_3, \frac{\sqrt{17}}{9}C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{3}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0 \\ \frac{11}{9\sqrt{17}} & \frac{2}{9\sqrt{17}} & \frac{\sqrt{17}}{9} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{3}{\sqrt{17}} & \frac{11}{9\sqrt{17}} \\ 0 & \frac{1}{\sqrt{17}} & \frac{2}{9\sqrt{17}} \\ 0 & 0 & \frac{\sqrt{17}}{9} \end{bmatrix}$$

Thus, $B = P'AP$ where B is a diagonal matrix.

This means the quadratic form $X'AX = 2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ will be transformed to $Y'BY = y_1^2 - y_2^2 + y_3^2$ by the transformation $X = PY$.

i.e., $x_1 = \frac{1}{\sqrt{2}}y_1 - \frac{3}{\sqrt{17}}y_2 + \frac{11}{9\sqrt{7}}y_3;\ x_2 = \frac{1}{\sqrt{17}}y_2 + \frac{2}{9\sqrt{17}}y_3;\ x_3 = \frac{\sqrt{17}}{9}y_3.$

$\therefore$ The rank = 3. Index = 2.

Signature = difference between positive squares and negative squares

= 2 − 1 = 1

Since some diagonal elements are positive, some are negative, the value class is indefinite.

Example 2 : Reduce the quadratic form $x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$ to canonical form. Also find its rank, index and signature.

Sol. : The matrix of the form is $A = \begin{bmatrix} 1 & -1 & 1/2 \\ -1 & 2 & -1 \\ 1/2 & -1 & 2 \end{bmatrix}$

We write $A = I\,A\,I$ $\quad \therefore \begin{bmatrix} 1 & -1 & 1/2 \\ -1 & 2 & -1 \\ 1/2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} R_2 + R_1 \\ C_2 + C_1 \\ R_3 - (1/2)R_1 \\ C_3 - (1/2)C_1 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1/2 \\ 0 & -1/2 & 7/4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} R_3 + (1/2)R_2 \\ C_3 + (1/2)C_2 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} \dfrac{\sqrt{2}}{3} R_3 \\ \dfrac{\sqrt{2}}{3} C_3 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1/\sqrt{6} & \sqrt{2/3} \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1/\sqrt{6} \\ 0 & 0 & \sqrt{2/3} \end{bmatrix}$

The linear transform $X = PY$

$$x = u + v \;;\; y = v + \frac{1}{\sqrt{6}} w \;;\; z = \sqrt{\frac{2}{3}}\, w$$

transforms the given quadratic form to, $u^2 + v^2 + w^2$.

The rank = 3. Index = 3.

Signature = The difference between positive squares and negative squares = 3 – 0 = 3.

Since all diagonal elements are positive, the form is positive definite.

Example 3 : Reduce the following quadratic form to canonical form and find its rank and signatures. Also write linear transformation which brings about the normal reduction.

$$21x_1^2 + 11x_2^2 + 2x_3^2 - 30x_1x_2 + 12x_1x_3 - 8x_2x_3$$

Also find a non-zero set of values of x_1, x_2, x_3 which makes the form zero. **(M.U. 2009)**

Sol. : (i) The matrix of the quadratic form is

$$A = \begin{bmatrix} 21 & -15 & 6 \\ -15 & 11 & -4 \\ 6 & -4 & 2 \end{bmatrix}$$

We first write $A = I\,A\,I$

$$\begin{bmatrix} 21 & -15 & 6 \\ -15 & 11 & -4 \\ 6 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\begin{matrix} R_2+\frac{5}{7}R_1 \\ C_2+\frac{5}{7}C_1 \\ R_3-\frac{2}{7}R_1 \\ C_3-\frac{2}{7}C_1 \end{matrix}$ $\begin{bmatrix} 21 & 0 & 0 \\ 0 & \frac{2}{7} & \frac{2}{7} \\ 0 & \frac{2}{7} & \frac{2}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{7} & 1 & 0 \\ -\frac{2}{7} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{5}{7} & -\frac{2}{7} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} R_3-R_2 \\ C_3-C_2 \end{matrix}$ $\begin{bmatrix} 21 & 0 & 0 \\ 0 & \frac{2}{7} & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{7} & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{5}{7} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

By $\frac{1}{\sqrt{21}}R_1, \frac{1}{\sqrt{21}}C_1, \sqrt{\frac{7}{2}}R_2, \sqrt{\frac{7}{2}}C_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{21}} & 0 & 0 \\ \frac{5}{\sqrt{14}} & \sqrt{\frac{7}{2}} & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{21}} & \frac{5}{\sqrt{14}} & -1 \\ 0 & \sqrt{\frac{7}{2}} & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

The linear transform $X = PY$.

$$x_1 = \frac{1}{\sqrt{21}}y_1 + \frac{5}{\sqrt{14}}y_2 - y_3 \ ; \ x_2 = \sqrt{\frac{7}{2}}y_2 - y_3 \ ; \ x_3 = y_3$$

transforms the given quadratic form to $y_1^2 + y_2^2$. The rank = 2

Signature = Difference between positive squares and negative squares
= $2 - 0 = 2$.

Since some diagonal elements are positive some are zero, the form is positive semi-definite.

If we put $y_1 = 0$, $y_2 = 0$ and $y_3 = 1$, $y_1^2 + y_2^2 = 0$ and for these values of y_1, y_2, y_3, we get $x_1 = -1$, $x_2 = -1$, $x_3 = 1$. And the given quadratic form will be zero.

(For these values of x_1, x_2, x_3 the given quadratic form will be zero because
$21 + 11 + 2 - 30 - 12 + 8 = 0$.)

Example 4 : Reduce the following quadratic form to canonical form and hence find its rank, index, signature and value class where $Q = 3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$.

(M.U. 2016)

Sol. : The quadratic form can be written as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \therefore \ A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

We write $A = IAI$

$$\therefore \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_2 + (1/3)R_1 \\ C_2 + (1/3)C_1 \\ R_3 - (1/3)R_1 \\ C_3 - (1/3)C_1 \end{array} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 14/3 & -2/3 \\ 0 & -2/3 & 8/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{array}{l} R_3 + (1/7)R_2 \\ C_3 + (1/7)C_2 \end{array} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 14/3 & 0 \\ 0 & 0 & 18/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{3}}R_1,\ \frac{1}{\sqrt{3}}C_1,\ \sqrt{\frac{3}{14}}R_2,\ \sqrt{\frac{3}{14}}C_2,\ \sqrt{\frac{7}{18}}R_3,\ \sqrt{\frac{7}{18}}C_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{42} & \sqrt{3/14} & 0 \\ -\sqrt{2/63} & 1/\sqrt{126} & \sqrt{7/18} \end{bmatrix} A \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{42} & -\sqrt{2/63} \\ 0 & \sqrt{3/14} & 1/\sqrt{126} \\ 0 & 0 & \sqrt{7/18} \end{bmatrix}$$

Thus, $B = P'AP$ where B is the diagonal matrix.

This means the quadratic form $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_3x_1$ will be transformed to $y_1^2 + y_2^2 + y_3^2$ by the transformation $X = PY$.

i.e.,
$$x_1 = \frac{1}{\sqrt{3}}y_1 + \frac{1}{\sqrt{42}}y_2 - \sqrt{\frac{2}{63}}y_3$$

$$x_2 = \sqrt{\frac{3}{14}}y_2 + \frac{1}{\sqrt{126}}y_3$$

$$x_3 = \sqrt{\frac{7}{18}}y_3$$

The rank = 3, The index = 3, Signature = 3.

The form is positive definite.

Example 5 : Reduce the quadratic form $2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6zx$ to canonical form and hence find its rank, index, signature and value class. **(M.U. 2016, 19)**

Sol. : The quadratic form can be written as

$$[x \ \ y \ \ z]\begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \therefore A = \begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix}$$

We write $A = IAI$

$$\therefore \begin{bmatrix} 2 & -1 & 3 \\ -1 & -2 & -4 \\ 3 & -4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x^2 - 2y^2 + 3z^2 - 4yz +$

$$R_2 + (1/2)R_1,\ R_3 - (3/2)R_1,\ C_2 + (1/2)C_1,\ C_3 - (3/2)C_1 \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & -5/2 \\ 0 & -5/2 & -5/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 - R_2,\ C_3 - C_2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & -5/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/2 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}R_1,\ \frac{1}{\sqrt{2}}C_1,\ \sqrt{\frac{2}{5}}R_2,\ \sqrt{\frac{2}{5}}C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{10} & \sqrt{2/5} & 0 \\ -2 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{10} & -2 \\ 0 & \sqrt{2/5} & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The linear transform, $X = PY$.

$$x = \frac{1}{\sqrt{2}}u + \frac{1}{\sqrt{10}}v - 2w$$

$$y = -\sqrt{\frac{2}{5}}u - w$$

$$z = w$$

transforms the given quadratic form to $u^2 - v^2$.

The rank = 2, Index = 1, Signature = 1.

The form is positive semi-definite.

Example 6 : Reduce the quadratic form $5x_1^2 + 26x_2^2 + 10x_3^2 + 6x_1x_2 + 4x_2x_3 + 14x_3x_1$ to normal form. Show that the quadratic form is positive semi-definite and find non zero set of values of x_1, x_2, x_3 which will make the quadratic form zero. **(M.U. 1992, 98)**

Sol. : The quadratic form can be written in matrix form as

$$[x_1\ x_2\ x_3] \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \therefore A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}$$

We write $A = IAI$

$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - (3/5)R_1,\ R_3 - (7/5)R_1,\ C_2 - (3/5)C_1,\ C_3 - (7/5)C_1$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & -11/5 \\ 0 & -11/5 & 1/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -7/5 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -7/5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 + \frac{1}{11}R_2,\ C_3 + \frac{1}{11}C_2$

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 121/5 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3/5 & 1 & 0 \\ -16/11 & 1/11 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -3/5 & -16/11 \\ 0 & 1 & 1/11 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\frac{1}{\sqrt{5}}R_1, \frac{1}{\sqrt{5}}C_1, \frac{\sqrt{5}}{11}R_2, \frac{\sqrt{5}}{11}C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 0 & 0 \\ 3\sqrt{5}/11 & \sqrt{5}/11 & 0 \\ -16/11 & 1/11 & 1 \end{bmatrix} A \begin{bmatrix} 1/\sqrt{5} & 3\sqrt{5}/11 & -16/11 \\ 0 & \sqrt{5}/11 & -11/11 \\ 0 & 0 & 1 \end{bmatrix}$$

∴ The linear transform $X = PY$

$$x_1 = \frac{1}{\sqrt{5}}y_1 + \frac{3\sqrt{5}}{11}y_2 - \frac{16}{11}y_3$$

$$x_2 = \frac{\sqrt{5}}{11}y_2 + \frac{1}{11}y_3$$

$$x_3 = y_3$$

transforms the given quadratic form to $y_1^2 + y_2^2$.

The rank = 2

Signature = Difference between positive square and negative squares
= 2 – 0 = 2

Since some diagonal elements are positive and some are zero the form is positive semi-definite.

If we put $y_1 = 0$, $y_2 = 0$ and if y_3 takes any value, the $Y'BY$ $(y_1^2 + y_2^2)$ will reduce to zero. If we put $y_3 = 11$, $y_1 = 0$, $y_2 = 0$, then $x_1 = -16$, $x_2 = 1$, $x_3 = 11$.

Hence, $x_1 = -16$, $x_2 = 1$, $x_3 = 11$ will reduce the quadratic form to zero. (Verify it).

Example 7 : Reduce the following quadratic form to normal form and interpret your result.

$$3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3$$

Find two sets of values of x_1, x_2, x_3 which make the form positive and negative respectively.

Sol. : The quadratic form can be expressed in matrix form as

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \therefore \; A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

We write $A = IAI$

$$\therefore \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_2 - (2/3)R_1$, $R_3 + (1/3)R_1$, $C_2 - (2/3)C_1$, $C_3 + (1/3)C_1$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2/3 & 11/3 \\ 0 & 11/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2/3 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

By $R_3 - (11/2)R_2$, $C_3 - (11/2)C_2$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2/3 & 0 \\ 0 & 0 & -117/6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 4 & -11/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2/3 & 4 \\ 0 & 1 & -11/2 \\ 0 & 0 & 1 \end{bmatrix}$$

By $\begin{matrix} 1/\sqrt{3}\,R_1,\ 1/\sqrt{3}\,C_1 \\ \sqrt{3/2}\,R_2,\ \sqrt{3/2}\,C_2 \\ \sqrt{6/117}\,R_3,\ \sqrt{6/117}\,C_3 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ -\sqrt{\frac{2}{3}} & \sqrt{\frac{3}{2}} & 0 \\ \frac{4\sqrt{6}}{\sqrt{117}} & \frac{-11\sqrt{6}}{2\sqrt{117}} & \frac{\sqrt{6}}{\sqrt{117}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & \frac{4\sqrt{6}}{\sqrt{117}} \\ 0 & \sqrt{\frac{3}{2}} & \frac{-11\sqrt{6}}{2\sqrt{117}} \\ 0 & 0 & \frac{\sqrt{6}}{\sqrt{117}} \end{bmatrix}$$

This means the quadratic form

$$3x_1^2 + 2x_2^2 + x_3^2 + 4x_1x_2 - 2x_1x_3 + 6x_2x_3$$

will be reduced to

$$y_1^2 + y_2^2 - y_3^2 \quad \text{.................... (1)}$$

by the transformation $X = PY$

i.e., $x_1 = \frac{1}{\sqrt{3}} y_1 - \sqrt{\frac{2}{3}} y_2 + \frac{4\sqrt{6}}{117} y_3$

$$x_2 = \sqrt{\frac{3}{2}} y_2 - \frac{11\sqrt{6}}{2\sqrt{117}} y_3,\ x_3 = \sqrt{\frac{6}{117}} y_3 \quad \text{.................... (2)}$$

If we put $y_1 = 0$, $y_2 = 0$ and $y_3 = 1$, the equation (1) we be negative and if we put $y_1 = 0$, $y_3 = 0$ and $y_2 = 1$ the equation (1) we be positive.

Putting these value of y_1, y_2, y_3 in (2), we get

$$x_1 = \frac{4\sqrt{6}}{117},\ x_2 = \frac{-11\sqrt{6}}{2\sqrt{117}},\ x_3 = \sqrt{\frac{6}{117}} \quad \text{and} \quad x_1 = -\sqrt{\frac{2}{3}},\ x_2 = \sqrt{\frac{3}{2}},\ x_3 = 0$$

will make the form negative and positive.

Example 8 : Show that the form $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 + 2x_1x_2 - 2x_3x_1$ is indefinite and find two sets of values of x_1, x_2, x_3 which will make the form positive and negative.

(M.U. 1998, 2003)

Sol. : The matrix of the form is

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

We write $A = IAI$ $\quad \therefore \quad \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} R_2 - R_1 \\ R_3 + R_1 \\ C_2 - C_1 \\ C_3 + C_1 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $\begin{matrix} R_3 - 2R_2 \\ C_3 - 2C_2 \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Since we are not asked to obtain the normal form, we shall not carry out further operations.

$\therefore$ The transformation

$$\therefore \quad \begin{aligned} x_1 &= y_1 - y_2 + 3y_3 \\ x_2 &= \quad\;\; y_2 - 2y_3 \\ x_3 &= \qquad\quad y_3 \end{aligned} \qquad \text{.............. (A)}$$

transforms the given quadratic form to

$$y_1 + y_2 - 2y_3^2 \qquad \text{.............. (B)}$$

If we put $y_1 = 0$, $y_2 = 1$, $y_3 = 0$, then the form (B) will be positive and if we put $y_1 = 0$, $y_2 = 0$, $y_3 = 1$, the form (B) will be negative.

Putting these values in (A), we get

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0 \qquad \text{and} \qquad x_1 = 3,\ x_2 = -2,\ x_3 = 1.$$

These two sets of values of x_1, x_2, x_3 will make the given form positive and negative. (Verify this.)

EXERCISE - V

1. Determine the nature of the following quadratic forms.

(i) $2x_1^2 + 3x_2^2 + 2x_3^2 + 4x_1x_2$ (ii) $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$

(iii) $6x_1^2 + 49x_2^2 + 51x_3^2 - 82x_2x_3 + 20x_3x_1 - 4x_1x_2$

(iv) $x_1^2 + x_2^2 + x_3^2 - 2x_1x_2$ (v) $x_1^2 - x_2^2 + 4x_3^2 + 2x_1x_2 + 2x_2x_3 + 6x_3x_1$

[**Ans. :** (i) positive definite. (ii) indefinite. (iii) positive definite. (iv) positive semi-definite. (v) positive semi-definite.]

2. Reduce the following quadratic forms to canonical form by congruent transformations. State also form of the quadratic form.

(i) $y^2 + 2xy - 2yz + 4zx$ [**Ans. :** $y_1^2 - y_2^2 + y_3^2$]

(ii) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$ (M.U. 2005) [**Ans. :** $y_1^2 - y_2^2 - y_3^2$]

(iii) $x^2 + 2y^2 - 7z^2 - 4xy + 8xz$ [**Ans. :** $y_1^2 - y_2^2 + y_3^2$]

(iv) $x^2 + 2y^2 + 2z^2 - 2xy - 2yz + zx$ [**Ans. :** $y_1^2 + y_2^2 + y_3^2$]

(v) $xy + yz + zx$ (**Hint :** First use $R_1 + 2R_2$, $C_1 + 2C_2$) [**Ans. :** $y_1^2 - y_2^2 - y_3^2$]

(vi) $2xy + 4xz + 6yz$ [**Ans. :** $y_1^2 - y_2^2 - y_3^2$]

3. Reduce each of the following quadratic forms to canonical form. Also write linear transformation which brings about normal reduction. State its rank, signature, index and value class.

(i) $x^2 - 2y^2 + 3z^2 - 4yz + 6zx$ **(M.U. 1999, 2003)**

(ii) $2x_1^2 + x_2^2 - 3x_3^2 + 12x_1x_2 - 8x_2x_3 - 4x_3x_1$ **(M.U. 2005, 10)**

(iii) $11x^2 + y^2 + 6z^2 + 6xy + 4yz + 16zx$

(iv) $2x^2 - 2y^2 + 2z^2 - 2xy - 8yz + 6zx$

(v) $x^2 + 3y^2 + 8z^2 + 4t^2 + 4xy + 6xz - 4xt + 12yz - 8yt - 12zt$

(vi) $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_1x_3 + 2x_2x_3$ **(M.U. 1998, 2003)**

(vii) $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$ **(M.U. 2015)**

[**Ans. :** (i) $u^2 - v^2 - w^2$, rank = 3, index = 1, signature = –1, indefinite

(ii) $y_1^2 - y_2^2 - y_3^3$, rank = 3, index = 1, signature = –1, indefinite.

(iii) $u^2 + v^2$ rank = 2, index = 2, signature = 2, positive semi-definite

(iv) $u^2 - v^2$, rank = 2, index = 1, signature = 0, indefinite.

(v) $u^2 - v^2 - w^2$, rank = 3, index = 1, signature = –1, indefinite.

(vi) $u^2 + v^2 - 2w^2$, rank = 3, signature = 1, index = 2, indefinite.

(vii) $8y_1^2 + 2y_2^2 + 2y_3^2$, rank = 3, signature = 3, index = 3, indefinite.]

4. Reduce the quadratic form $x^2 - 3y^2 - 2xy + 4xz - 12yz$ to canonical form and hence, find its rank, index and signature.
Also find non-zero values of x, y, z which will make the quadratic form positive. **(M.U. 2001)**

[**Ans. :** $u^2 - v^2$. Rank = 2, Index = 1, Signature = 0, $x_1 = -2$, $x_2 = -1$, $x_3 = 1$.]

5. Reduce the quadratic form $x^2 - 2y^2 + 10z^2 - 10xy + 4xz - 2yz$ to canonical form and find its rank, index, signature and class-value. **(M.U. 1996, 2011)**

[**Ans. :** $u^2 - v^2 + w^2$, rank = 3, index = 2, signature = 1, indefinite]

EXERCISE - VI

Theory

1. Define the following terms.
   (a) Signature and index of a real quadratic form **(M.U. 2003)**
   (b) Positive definite quadratic form. **(M.U. 2003)**
2. State true or false with justification.
   "If a quadratic form is in three variables, then its signature cannot be zero". **(M.U. 2002)**

[**Ans. :** False. See in above Exercise-V, Ex. 3(iv)]

☆☆☆