Module No2

Time Domain Analysis of Continuous And Discrete Signals and Systems

2.1 Properties of Linear Time Invariant (LTI) systems, Impulse and Step Response
2.2 Use of Convolution Integral and Convolution Sum and Correlation for Analysis of LTI Systems
2.3 Properties of Convolution Integral/Sum

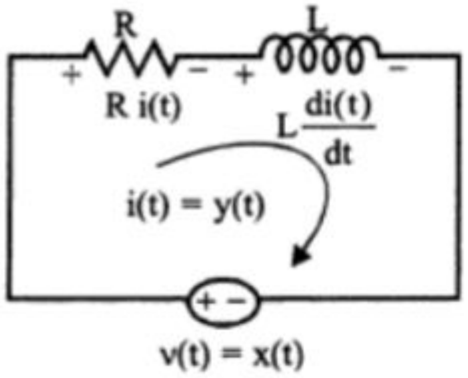
2.6.1 Mathematical Equation Governing LTI Continuous Time System

The electric heaters, motors, generators, etc., are examples of electrical continuous time systems. The continuous time systems that operate on electrical energy can be modelled by three basic elements Resistor(R), Inductor(L) and Capacitor(C). The models constructed using these fundamental elements are called ***electric circuits.***

In electric circuits the inputs and outputs are either voltage signals or current signals. The continuous time voltage signal is denoted by $v(t)$ and current signal by $i(t)$.

The basic RL, RC, and RLC circuits and their time domain KVL (Kirchoff's Voltage Law) equations are shown in fig 2.38, fig 2.39 and fig 2.40 respectively. From these circuits it can be observed that the equations governing the continuous time systems are differential equations.

Also, it can be shown that all continuous time systems like Mechanical systems, Thermal systems, Hydraulic systems, etc., are all governed by differential equations.



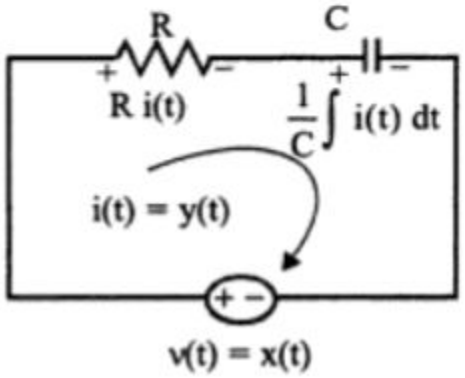
$$R\,i(t) + L\frac{di(t)}{dt} = v(t)$$

$$\Big\downarrow \text{Replace } i(t) \text{ by } y(t) \text{ and } v(t) \text{ by } x(t)$$

$$Ry(t) + L\frac{dy(t)}{dt} = x(t)$$

$$\therefore \frac{dy(t)}{dt} + \frac{R}{L}y(t) = \frac{1}{L}x(t)$$

***Fig 2.38 :** RL circuit and the mathematical equation governing RL circuit.*



$$R\,i(t) + \frac{1}{C}\int i(t)\,dt = v(t)$$

$$\Big\downarrow \text{differentiate}$$

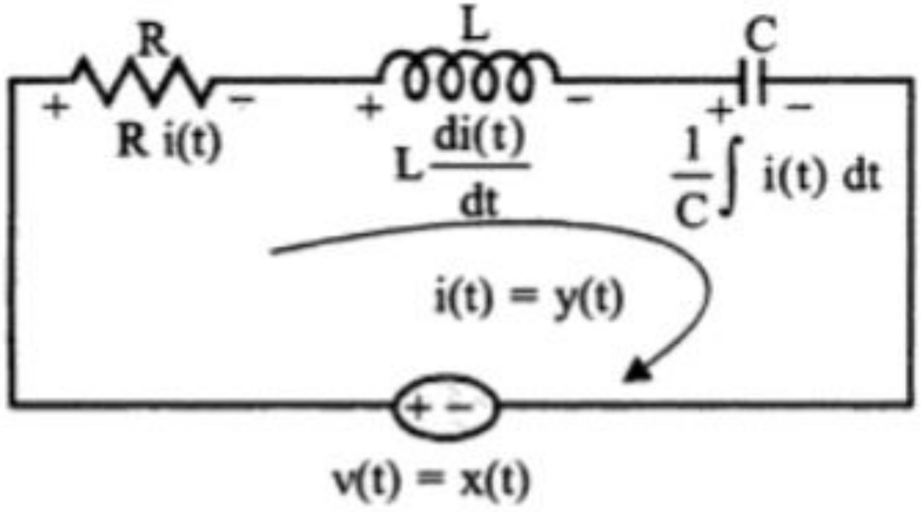
$$R\frac{di(t)}{dt} + \frac{1}{C}i(t) = \frac{dv(t)}{dt}$$

$$\Big\downarrow \text{Replace } i(t) \text{ by } y(t) \text{ and } v(t) \text{ by } x(t)$$

$$R\frac{dy(t)}{dt} + \frac{1}{C}y(t) = \frac{dx(t)}{dt}$$

$$\therefore \frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{R}\frac{dx(t)}{dt}$$

***Fig 2.39 :** RC circuit and the mathematical equation governing RC circuit.*



$$R\,i(t) + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt = v(t)$$

$$\Big\downarrow\ \text{differentiate}$$

$$R\frac{di(t)}{dt} + L\frac{d^2i(t)}{dt^2} + \frac{1}{C}\,i(t) = \frac{dv(t)}{dt}$$

$$\Big\downarrow\ \text{Replace } i(t) \text{ by } y(t) \text{ and } v(t) \text{ by } x(t)$$

$$R\frac{dy(t)}{dt} + L\frac{d^2y(t)}{dt^2} + \frac{1}{C}\,y(t) = \frac{dx(t)}{dt}$$

$$\therefore\ \frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}\,y(t) = \frac{1}{L}\frac{dx(t)}{dt}$$

***Fig 2.40 :** RLC circuit and the mathematical equation governing RLC circuit.*

In general, the input-output relation of an LTI (Linear Time Invariant) continuous time system is represented by a constant coefficient differential equation shown below (equation (2.18)).

$$a_0 \frac{d^N}{dt^N} y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}} y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}} y(t) + \ldots\ldots. + a_{N-1} \frac{d}{dt} y(t) + a_N\, y(t) = b_0 \frac{d^M}{dt^M} x(t)$$

$$+ b_1 \frac{d^{M-1}}{dt^{M-1}} x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}} x(t) + \ldots\ldots. + b_{M-1} \frac{d}{dt} x(t) + b_M x(t) \quad .....(2.13)$$

where, N = Order of the system, $M \leq N$, and $a_0 = 1$.

The solution of the above differential equation is the response y(t) of the system, for the input x(t).

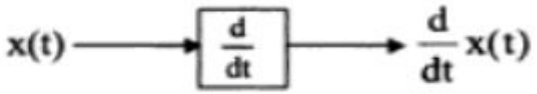
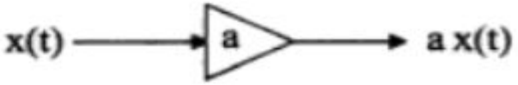
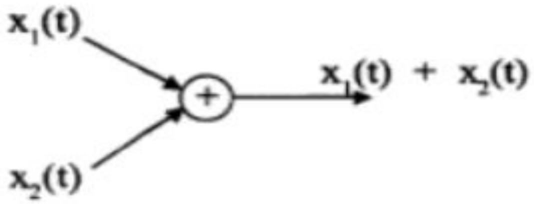
***Note :** A system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship do not change with time.*

Block Diagram

A ***block diagram*** of a system is a pictorial representation of the functions performed by the system. The block diagram of a system is constructed using the mathematical equation governing the system.

The basic elements of a block diagram are Differentiator, Integrator, Constant Multiplier and Signal Adder. The symbols used for the basic elements and their input-ouput relation are listed in table 2.2.

Table 2.2 : Basic Elements of Block Diagram

Description	Elements of block diagram
Differentiator	
Integrator (with zero initial condition)	
Constant Multiplier	
Signal Adder	

Impulse Signal

The impulse signal is a special signal which can be derived as follows.

Consider a pulse signal, $P_\Delta(t)$ with height A/Δ and width Δ as shown in fig 2.28. Now, the pulse signal, $P_\Delta(t)$ can be defined as,

$$P_\Delta(t) = \frac{A}{\Delta} \quad ; \ 0 \le t \le \Delta$$
$$= 0 \quad ; \ t > \Delta$$

The area of the pulse signal for any value of t is given by,

$$\text{Area} = \text{Height} \times \text{Width} = \frac{A}{\Delta} \times \Delta = A$$

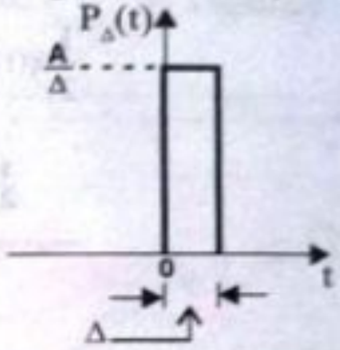


Fig 2.28 : Pulse signal.

In the signal, $P_\Delta(t)$ if the width Δ is reduced, then the height A/Δ increases, but the area of the pulse remains same as A. When the width Δ tends to zero, the height A/Δ tends to infinity. This limiting value of the pulse signal is called ***impulse signal***, $\delta(t)$. Even when the width Δ tends to zero, the area of the pulse remains as A.

$$\therefore \ \text{Impulse Signal, } \delta(t) = \underset{\Delta \to 0}{\mathrm{Lt}} \ P_\Delta(t) = \underset{\Delta \to 0}{\mathrm{Lt}} \ \frac{A}{\Delta} \quad ; \ t = 0$$
$$= 0 \quad ; \ t \ne 0$$

$$\therefore \ \text{Impulse Signal, } \delta(t) = \infty \ ; \ t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\,dt = A$$
$$= 0 \ ; \ t \ne 0$$

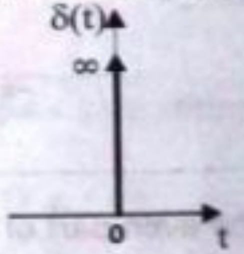


Fig 2.29 : Impulse signal (or Unit impulse signal).

Impulse Signal

Definition of impulse signal : The impulse signal is a signal with infinite magnitude and zero duration, but with an area of A. Mathematically, an impulse signal is defined as,

$$\text{Impulse Signal, } \delta(t) = \infty \; ; \; t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\,dt = A$$
$$= 0 \; ; \; t \neq 0$$

Definition of unit impulse signal : The unit impulse signal is a signal with infinite magnitude and zero duration, but with unit area. Mathematically, a unit impulse signal is defined as,

$$\text{Unit Impulse Signal, } \delta(t) = \infty \; ; \; t = 0 \qquad \text{and} \qquad \int_{-\infty}^{+\infty} \delta(t)\,dt = 1$$
$$= 0 \; ; \; t \neq 0$$

Properties of Impulse Signal

Property - 1: $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$

Property - 2: $\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = x(0)$

Proof :

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t)\,dt = \int_{-\infty}^{+\infty} x(0)\,\delta(t)\,dt$$

Since $\delta(t)$ is nonzero only at $t=0$, $x(t)$ is replaced by $x(0)$.

$$= x(0)\int_{-\infty}^{+\infty} \delta(t)\,dt = x(0)\times 1 = x(0)$$

Since $x(0)$ is constant, it is taken outside integration.

Using property-1

Property - 3: $\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_o)\,dt = x(t_o)$

Proof :

$$\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_o)\,dt = \int_{-\infty}^{+\infty} x(t_o)\,\delta(t-t_o)\,dt$$

Since $\delta(t_o)$ is nonzero only at $t=t_o$, $x(t)$ is replaced by $x(t_o)$.

$$= x(t_o)\int_{-\infty}^{+\infty} \delta(t-t_o)\,dt = x(t_o)\times 1 = x(t_o)$$

Since $x(t_o)$ is constant, it is taken outside integration.

Using property-1

Property - 4: $\int_{-\infty}^{+\infty} x(\lambda)\,\delta(t-\lambda)\,d\lambda = x(t)$

Proof :

Consider the property-3 of impulse signal. $\int_{-\infty}^{+\infty} x(t)\,\delta(t-t_o)\,dt = x(t_o)$

On substituting $t=\lambda$ in the above equation we get, $\int_{-\infty}^{+\infty} x(\lambda)\,\delta(\lambda-t_o)\,d\lambda = x(t_o)$

On substituting $t_o = t$ in the above equation we get, $\int_{-\infty}^{+\infty} x(\lambda)\,\delta(\lambda-t)\,d\lambda = x(t)$

Since impulse signal is even, $\delta(\lambda-t) = \delta(t-\lambda)$. Therefore the above equation is written as shown below.

$$\int_{-\infty}^{+\infty} x(\lambda)\,\delta(t-\lambda)\,d\lambda = x(t)$$

$$\delta(t) \longrightarrow \boxed{\text{LTI System}} \longrightarrow h(t)$$

Unit Impulse signal — Unit Impulse Response

$h(t)$ is fixed for given LTI system

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, d\tau$$

$$\Big\downarrow \text{LT}$$

$$Y(s) = \frac{X(s)}{s}$$

$$\frac{Y(s)}{X(s)} = \frac{1}{s} = H(s) \quad \text{Transfer function of LTI system}$$

$$\Big\downarrow \text{Inverse Laplace}$$

$$\boxed{h(t) = u(t)}$$

Impulse response of the system is $u(t)$

$$h(t) \xrightarrow{LT} H(s) = \frac{Y(s)}{X(s)}\Bigg|_{\text{Initial conditions are zero}}$$

$$Y(s) = X(s) \cdot H(s) = H(s)\,X(s)$$

$$Y(s) \xrightarrow{\text{Inv LT}} \boxed{y(t) = h(t) * x(t)}$$

convolution

Convolution: It is an important mathematical tool which is used to calculate the output of LTI system when the Impulse response and the input is given.

$$y(t) = h(t) * x(t)$$

Convolution

- A convolution is an integral that expresses the amount of overlap of one function when it is shifted over another function.

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda}$$

2.9 Convolution of Continuous Time Signals

The ***convolution*** of two continuous time signals $x_1(t)$ and $x_2(t)$ is defined as,

$$\boxed{x_3(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda} \quad \text{....(2.21)}$$

where, $x_3(t)$ is the signal obtained by convolving $x_1(t)$ and $x_2(t)$,

and λ is a dummy variable used for integration.

The convolution relation of equation (2.21) can be symbolically expressed as,

$$x_3(t) = x_1(t) * x_2(t) \quad \text{..... (2.22)}$$

where the symbol $*$ indicates convolution operation.

2.9.1 Response of LTI Continuous Time System Using Convolution

In an LTI continuous time system, the response y(t) of the system for an arbitrary input x(t) is given by convolution of input x(t) with impulse response h(t) of the system. It is expressed as,

$$\boxed{y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda} \qquad \text{....(2.23)}$$

where the symbol $*$ represents convolution operation.

In an LTI system, if the input x(t) is a unit step signal, then the response is called a ***unit step response***.

2.9.2 Properties of Convolution

The convolution of continuous time signals will satisfy the following properties.

Commutative property : $x_1(t) * x_2(t) = x_2(t) * x_1(t)$

Associative property : $[\, x_1(t) * x_2(t)\,] * x_3(t) = x_1(t) * [\, x_2(t) * x_3(t)\,]$

Distributive property : $x_1(t) * [\, x_2(t) + x_3(t)\,] = [\, x_1(t) * x_2(t)\,] + [\, x_1(t) * x_3(t)\,]$

Proof of Commutative Property :

Consider two continuous time signals, $x_1(t)$ and $x_2(t)$.

By Commutative property we can write,

$$\underbrace{x_1(t) * x_2(t)}_{\text{(LHS)}} = \underbrace{x_2(t) * x_1(t)}_{\text{(RHS)}}$$

$$LHS = x_1(t) * x_2(t)$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, x_2(t-m)\, dm \qquad .....(2.29)$$

where m is a dummy variable used for convolution operation.

Let, $t - m = p$; when $m = -\infty$, $p = t - m = t + \infty = +\infty$

$\therefore m = t - p$; when $m = +\infty$, $p = t - m = t - \infty = -\infty$

$dm = -dp$

On replacing m by $(t-p)$ and $(t-m)$ by p in equation (2.29) we get,

$$LHS = -\int_{p=+\infty}^{-\infty} x_1(t-p)\, x_2(p)\, dp = \int_{p=-\infty}^{+\infty} x_2(p)\, x_1(t-p)\, dp$$

Here p is a dummy variable used for convolution operation

$$= x_2(t) * x_1(t)$$

$$= RHS$$

Proof of Associative Property :

Consider three continuous time signals $x_1(t)$, $x_2(t)$ and $x_3(t)$. By Associative property we can write,

$$\underbrace{[x_1(t) * x_2(t)] * x_3(t)}_{\text{LHS}} = \underbrace{x_1(t) * [x_2(t) * x_3(t)]}_{\text{RHS}}$$

Let, $\quad y_1(t) = x_1(t) * x_2(t)$ .....(2.30)

Let us replace t by p.

$$\therefore y_1(p) = x_1(p) * x_2(p)$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, dm \qquad .....(2.31)$$

Let, $\quad y_2(t) = x_2(t) * x_3(t)$ .....(2.32)

$$\therefore y_2(t) = \int_{q=-\infty}^{+\infty} x_2(q)\, x_3(t-q)\, dq$$

$$\therefore y_2(t-m) = \int_{q=-\infty}^{+\infty} x_2(q)\, x_3(t-q-m)\, dq \qquad .....(2.33)$$

where p, m and q are dummy variables used for convolution operation.

$$\text{LHS} = [x_1(t) * x_2(t)] * x_3(t)$$

$$= y_1(t) * x_3(t)$$

Using equation (2.30)

$$= \int_{p=-\infty}^{+\infty} y_1(p)\, x_3(t-p)\, dp$$

$$= \int_{p=-\infty}^{+\infty} \int_{m=-\infty}^{+\infty} x_1(m)\, x_2(p-m)\, x_3(t-p)\, dm\, dp$$

Using equation (2.31)

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, dm \int_{p=-\infty}^{+\infty} x_2(p-m)\, x_3(t-p)\, dp \qquad .....(2.34)$$

Let, $p - m = q$ | when $p = -\infty$, $q = p - m = -\infty - m = -\infty$

$\therefore p = q + m$ | when $p = +\infty$, $q = p - m = +\infty - m = +\infty$

$dp = dq$

On replacing $(p - m)$ by q and p by $(q + m)$ in the equation (2.34) we get,

$$\text{LHS} = \int_{m=-\infty}^{+\infty} x_1(m)\, dm \int_{q=-\infty}^{+\infty} x_2(q)\, x_3(t - q - m)\, dq$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, y_2(t - m)\, dm \qquad \boxed{\text{Using equation (2.33)}}$$

$$= x_1(t) * y_2(t)$$

$$= x_1(t) * [x_2(t) * x_3(t)] \qquad \boxed{\text{Using equation (2.32)}}$$

$$= \text{RHS}$$

Proof of Distributive Property :

Consider three continuous time signals $x_1(t)$, $x_2(t)$ and $x_3(t)$. By distributive property we can write,

$$\underbrace{x_1(t) * [x_2(t) + x_3(t)]}_{\text{LHS}} = \underbrace{[x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]}_{\text{RHS}}$$

$$\text{LHS} = x_1(t) * [x_2(t) + x_3(t)]$$

$$= x_1(t) * x_4(t) \qquad \boxed{x_4(t) = x_2(t) + x_3(t)}$$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\; x_4(t-m)\, dm$$

m is dummy variable used for integration

$$= \int_{m=-\infty}^{+\infty} x_1(m)\, [x_2(t-m) + x_3(t-m)]\, dm$$

if, $x_4(t) = x_2(t) + x_3(t)$, then $x_4(t-m) = x_2(t-m) + x_3(t-m)$

$$= \int_{m=-\infty}^{+\infty} x_1(m)\; x_2(t-m)\, dm + \int_{m=-\infty}^{+\infty} x_1(m)\; x_3(t-m)\, dm$$

$$= [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)]$$

$$= \text{RHS}$$

2.9.3 Interconnections of Continuous Time Systems

Smaller continuous time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are ***cascade connection*** and ***parallel connection***. The cascade and parallel connections of two continuous time systems with impulse responses $h_1(t)$ and $h_2(t)$ are shown in fig 2.43.

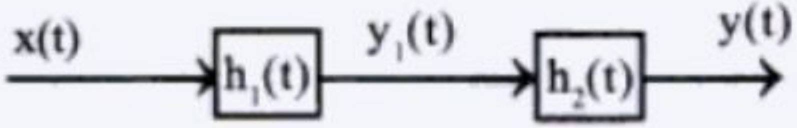


Fig 2.43a : *Cascade connection.*

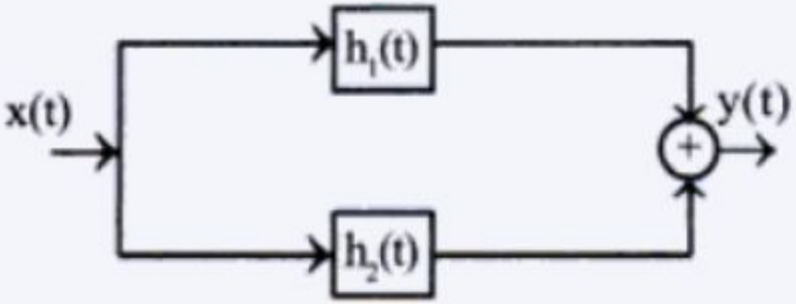


Fig 2.43b : *Parallel connection.*

Fig 2.43 : *Interconnection of continuous time systems.*

Cascade Connected Continuous Time Systems

Two cascade connected continuous time systems with impulse response $h_1(t)$ and $h_2(t)$ can be replaced by a single equivalent continuous time system whose impulse response is given by convolution of individual impulse responses.

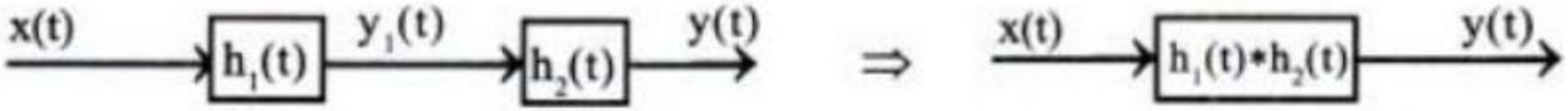


Fig 2.44 : Cascade connected continuous time systems and their equivalent.

Proof :

With reference to fig 2.44 we can write,

$$y_1(t) = x(t) * h_1(t) \qquad .....(2.35)$$

$$y(t) = y_1(t) * h_2(t) \qquad .....(2.36)$$

Using equation (2.35) the equation (2.36) can be written as,

$$y(t) = [\,x(t) * h_1(t)\,] * h_2(t)$$

$$= x(t) * [\,h_1(t) * h_2(t)\,] \qquad \boxed{\text{Using associative property}}$$

$$= x(t) * h(t) \qquad .....(2.37)$$

where, $h(t) = h_1(t) * h_2(t)$

From equation (2.37) we can say that the overall impulse response of two cascaded continuous time systems is given by convolution of individual impulse responses.

Parallel Connected Continuous Time Systems

Two parallel connected continuous time systems with impulse responses $h_1(t)$ and $h_2(t)$ can be replaced by a single equivalent continuous time system whose impulse response is given by the sum of individual impulse responses.

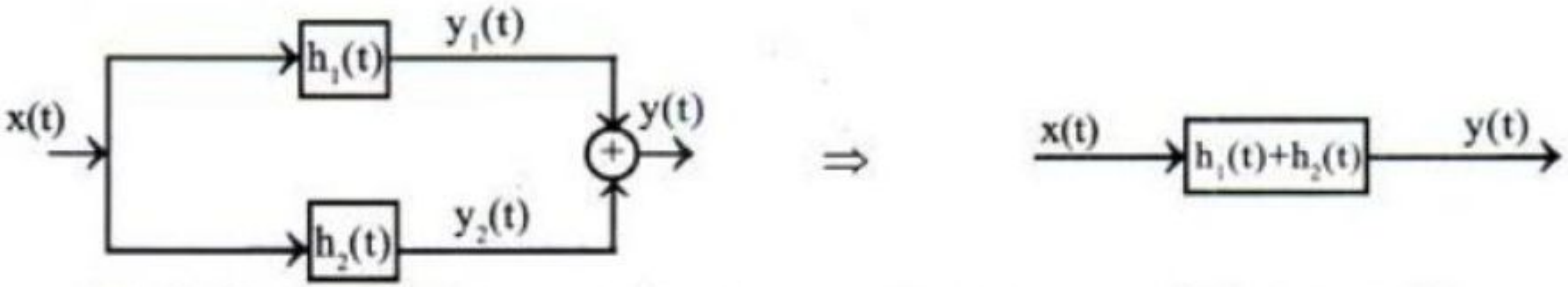


Fig 2.45 : Parallel connected continuous time systems and their equivalent.

Proof:

With reference to fig 2.45 we can write,

$$y_1(t) = x(t) * h_1(t) \quad .....(2.38)$$

$$y_2(t) = x(t) * h_2(t) \quad .....(2.39)$$

$$y(t) = y_1(t) + y_2(t) \quad .....(2.40)$$

On substituting for $y_1(t)$ and $y_2(t)$ from equations (2.38) and (2.39) in equation (2.40) we get,

$$y(t) = [\, x(t) * h_1(t) \,] + [\, x(t) * h_2(t) \,] \quad .....(2.41)$$

By using distributive property of convolution, the equation (2.41) can be written as shown below.

$$y(t) = x(t) * [\, h_1(t) + h_2(t) \,]$$

$$= x(t) * h(t) \quad .....(2.42)$$

where, $h(t) = h_1(t) + h_2(t)$

From equation (2.42) we can say that the overall impulse response of two parallel connected continuous time systems is given by the sum of individual impulse responses.

2.9.4 Procedure to Perform Convolution

The ***convolution*** of two continuous time signals $x_1(t)$ and $x_2(t)$ is defined as,

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

where, $x_3(t)$ is the signal obtained by convolving $x_1(t)$ and $x_2(t)$,
λ is a dummy variable used for integration,
$*$ indicates convolution operation.

The computation of $x_3(t)$ using the above convolution equation for any value of t involves the following operations,

1. ***Change of time index :*** The time index t in signals $x_1(t)$ and $x_2(t)$ is changed to λ to get $x_1(\lambda)$ and $x_2(\lambda)$.
2. ***Folding*** : The signal $x_2(\lambda)$ is folded to get $x_2(-\lambda)$.
3. ***Shifting*** : The signal $x_2(-\lambda)$ is shifted by t units of time to get $x_2(t-\lambda)$.

4. Multiplication : The signals $x_1(\lambda)$ and $x_2(t-\lambda)$ are multiplied to get a product signal.

5. Integration : The product signal is integrated to get $x_3(t)$. Let the product signal is nonzero in the interval $\lambda=\lambda_1$ to $\lambda=\lambda_2$, Now the signal $x_3(t)$ is given by,

$$x_3(t) = \int_{\lambda=\lambda_1}^{\lambda=\lambda_2} x_1(\lambda)\ x_2(t-\lambda)\ d\lambda$$

If both $x_1(t)$ and $x_2(t)$ are defined for $t > 0$, (i.e., both $x_1(t)$ and $x_2(t)$ are causal) then the product signal is nonzero in the interval $\lambda=0$ to $\lambda=t$, Now the signal $x_3(t)$ is given by,

$$x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\ x_2(t-\lambda)\ d\lambda$$

2.9.5 Unit Step Response Using Convolution

In general the response y(t) of a system is given by convolution of input x(t) and impulse response h(t) of the system.

$$y(t) = x(t) * h(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda$$

Let the input x(t) be unit step input u(t), and the corresponding response be s(t). Now the unit step response s(t) is given by,

$$\text{Unit Step Response, } s(t) = u(t) * h(t)$$

$$= h(t) * u(t) \qquad \boxed{\text{Using Commutative property}}$$

$$= \int_{\lambda=-\infty}^{\lambda=+\infty} h(\lambda)\, u(t-\lambda)\, d\lambda$$

In the above convolution operation, $u(\lambda) = 1$ for $\lambda > 0$,

$u(-\lambda) = 1$ for $\lambda < 0$,

$u(t-\lambda) = 1$ for $\lambda < t$, and $u(t-\lambda) = 0$ for $\lambda > t$.

Therefore the unit step response s(t) is given by,

$$\textbf{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\, d\lambda$$

Example 2.20

Perform convolution of the following causal signals.

a) $x_1(t) = 2\,u(t)$, $x_2(t) = u(t)$

b) $x_1(t) = e^{-2t}\,u(t)$, $x_2(t) = e^{-5t}\,u(t)$

c) $x_1(t) = t\,u(t)$, $x_2(t) = e^{-5t}\,u(t)$

d) $x_1(t) = \cos t\,u(t)$, $x_2(t) = t\,u(t)$

Solution

a) Given that, $x_1(t) = 2\,u(t) = 2 \quad ; \; t \geq 0$

$$x_2(t) = u(t) = 1 \quad ; \; t \geq 0$$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

Since $x_1(t)$ and $x_2(t)$ are causal, the limits of integration is 0 to t.

$$x_3(t) = x_1(t) * x_2(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{\lambda=0}^{\lambda=t} 2 \times 1 \;\; d\lambda = 2\int_{\lambda=0}^{\lambda=t} d\lambda$$

$$= 2[\lambda]_0^t = 2\,[t-0] = 2t \quad ; \text{ for } t \geq 0 = 2t\,u(t)$$

b) Given that, $x_1(t) = e^{-2t}\,u(t) = e^{-2t}$; $t \ge 0$

$x_2(t) = e^{-5t}\,u(t) = e^{-5t}$; $t \ge 0$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

Since $x_1(t)$ and $x_2(t)$ are causal, the limits of integration is 0 to t.

$$x_3(t) = x_1(t) * x_2(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{\lambda=0}^{\lambda=t} e^{-2\lambda}\, e^{-5(t-\lambda)}\, d\lambda = \int_{\lambda=0}^{\lambda=t} e^{-2\lambda}\, e^{-5t}\, e^{5\lambda} d\lambda$$

$$= e^{-5t} \int_{\lambda=0}^{\lambda=t} e^{-2\lambda+5\lambda}\, d\lambda = e^{-5t} \int_{\lambda=0}^{\lambda=t} e^{3\lambda}\, d\lambda = e^{-5t}\left[\frac{e^{3\lambda}}{3}\right]_0^t = e^{-5t}\left[\frac{e^{3t}}{3} - \frac{e^0}{3}\right]$$

$$= \frac{e^{-5t}}{3}\left(e^{3t} - 1\right) = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right);\ \text{for } t \ge 0 = \frac{1}{3}\left(e^{-2t} - e^{-5t}\right) u(t)$$

c) **Given that,** $x_1(t) = t\,u(t) = t \quad ; \quad t \geq 0$

$$x_2(t) = e^{-5t}\,u(t) = e^{-5t} \quad ; \quad t \geq 0$$

Let, $\quad x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = x_1(t) * x_2(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Since $x_1(t)$ and $x_2(t)$ are causal, the limits of integration is 0 to t.

$$= \int_{\lambda=0}^{\lambda=t} \lambda\, e^{-5(t-\lambda)}\, d\lambda = \int_{\lambda=0}^{\lambda=t} \lambda\, e^{-5t}\, e^{5\lambda} d\lambda$$

$$= e^{-5t} \int_{\lambda=0}^{\lambda=t} \lambda\, e^{5\lambda}\, d\lambda = e^{-5t}\left[\lambda\, \frac{e^{5\lambda}}{5} - \int\left[1 \times \frac{e^{5\lambda}}{5}\right] d\lambda\right]_0^t$$

$\int uv = u\int v - \int\left[du \int v\right]$	
$u = \lambda$	$v = e^{5\lambda}$

$$= e^{-5t}\left[\lambda\, \frac{e^{5\lambda}}{5} - \frac{e^{5\lambda}}{25}\right]_0^t \quad = e^{-5t}\left[t\,\frac{e^{5t}}{5} - \frac{e^{5t}}{25} - 0 \times \frac{e^0}{5} + \frac{e^0}{25}\right]$$

$$= \frac{e^{-5t}}{25}\left(5te^{5t} - e^{5t} + 1\right); \text{ for } t \geq 0 = \frac{1}{25}\left(e^{-5t} + 5t - 1\right) u(t)$$

Example 2.21

Determine the unit step response of the following systems whose impulse responses are given below.

a) $h(t) = 3t\,u(t)$ b) $h(t) = e^{-5t}\,u(t)$ c) $h(t) = u(t+2)$

d) $h(t) = u(t-2)$ e) $h(t) = u(t+2) + u(t-2)$

a) Given that, $h(t) = 3t\,u(t) = 3t \quad ; \; t \geq 0$

$$\text{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\; d\lambda = \int_{\lambda=0}^{\lambda=t} 3\lambda\; d\lambda = 3\int_{\lambda=0}^{\lambda=t} \lambda\; d\lambda$$

$$= 3\left[\frac{\lambda^2}{2}\right]_0^t = 3\left[\frac{t^2}{2} - \frac{0}{2}\right] = \frac{3}{2}t^2\,;\; \text{for } t \geq 0 = \frac{3}{2}t^2\,u(t)$$

---

b) Given that, $h(t) = e^{-5t}\,u(t) = e^{-5t} \quad ; \; t \geq 0$

$$\text{Unit Step Response, } s(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\; d\lambda = \int_{\lambda=0}^{\lambda=t} e^{-5\lambda}\; d\lambda = \left[\frac{e^{-5\lambda}}{-5}\right]_0^t$$

$$= \left[\frac{e^{-5t}}{-5} - \frac{e^0}{-5}\right] = \frac{1}{5}(1 - e^{-5t});\; \text{for } t \geq 0 = \frac{1}{5}(1 - e^{-5t})\,u(t)$$

e) Given that, $h(t) = u(t+2) + u(t-2)$

Let, $h(t) = h_1(t) + h_2(t)$

where, $h_1(t) = u(t+2) = 1 \quad ; \quad t \geq -2$

$h_2(t) = u(t-2) = 1 \quad ; \quad t \geq 2$

Unit step response, $s(t) = h(t) * u(t) = [\, h_1(t) + h_2(t)\,] * u(t)$

$= [\, h_1(t) * u(t)\,] + [\, h_2(t) * u(t)\,] = s_1(t) + s_2(t)$

where, $s_1(t) = h_1(t) * u(t)$

$s_2(t) = h_2(t) * u(t)$

$$s_1(t) = \int_{\lambda=-\infty}^{\lambda=t} h_1(\lambda)\; d\lambda = \int_{\lambda=-2}^{\lambda=t} d\lambda = [\lambda]_{-2}^{t} = [t+2] = t+2 \quad ;\text{for } t > -2$$

$$s_2(t) = \int_{\lambda=-\infty}^{\lambda=t} h(\lambda)\; d\lambda = \int_{\lambda=2}^{\lambda=t} d\lambda = \int_{\lambda=2}^{\lambda=t} d\lambda = [\lambda]_{2}^{t} = [t-2] = t-2 \quad ;\text{for } t > 2$$

Now, Unit step response, $s(t) = s_1(t) \quad ; \text{ for } t = -2 \text{ to } 2$

$= s_1(t) + s_2(t) \quad ; \text{ for } t > 2$

$\therefore$ Unit step response, $s(t) = t+2 \quad ; \text{ for } t = -2 \text{ to } 2$

$= t+2 + t-2 = 2t \quad ; \text{ for } t > 2$

Graphical Convolution on CT signals

EX.1

Perform convolution of the following signals, by graphical method.

a) $x_1(t) = e^{-2t}\,u(t)$, $x_2(t) = t\,u(t)$ b) $x_1(t) = e^{-at}$; $0 \le t \le T$, $x_2(t) = 1$; $0 \le t \le 2T$

Solution

a) Given that, $x_1(t) = e^{-3t}\,u(t) = e^{-3t}$; $t \ge 0$

$x_2(t) = t\,u(t) = t$; $t \ge 0$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Let us change the time index t, in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

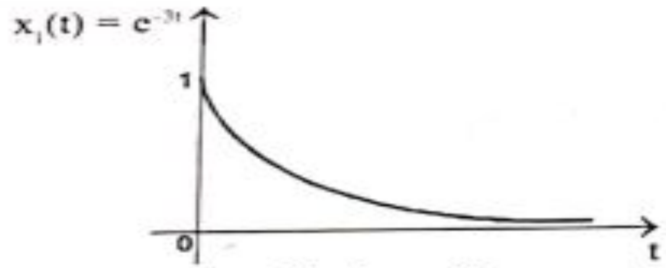
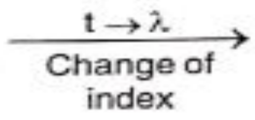


Fig 1 : $x_1(t)$.



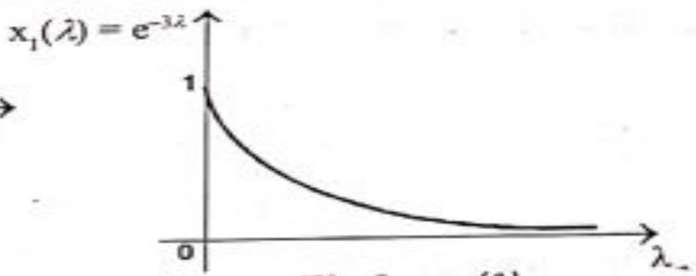


Fig 2 : $x_1(\lambda)$.

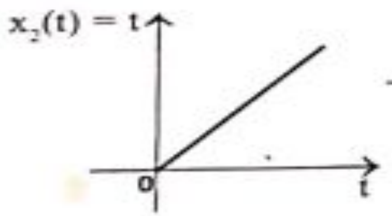
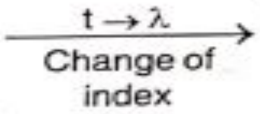


Fig 3 : $x_2(t)$.



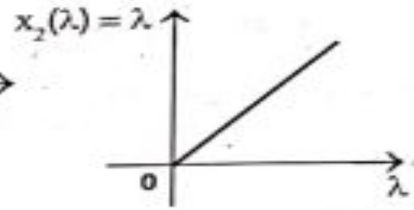
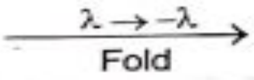


Fig 4 : $x_2(\lambda)$.



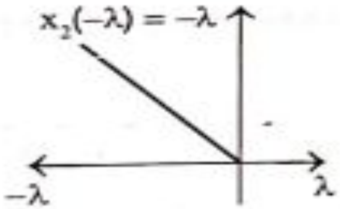


Fig 5 : $x_2(-\lambda)$.

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t-\lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

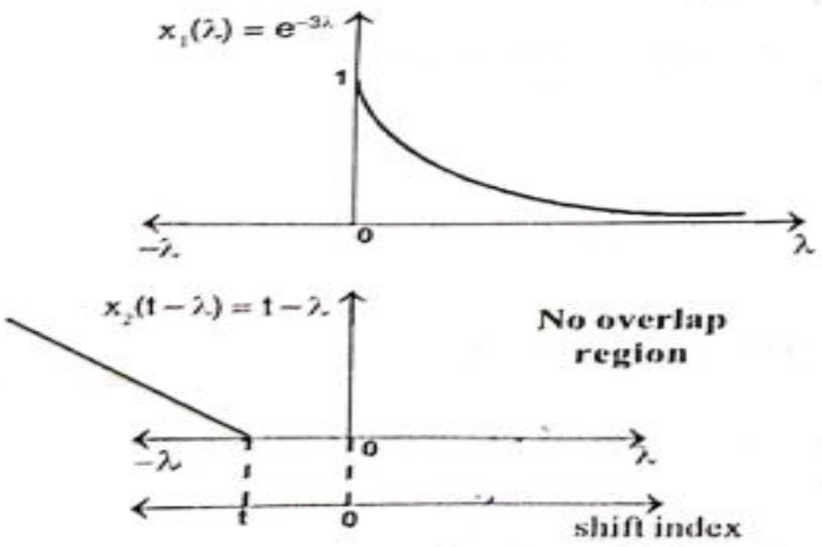


Fig 6 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t < 0$.

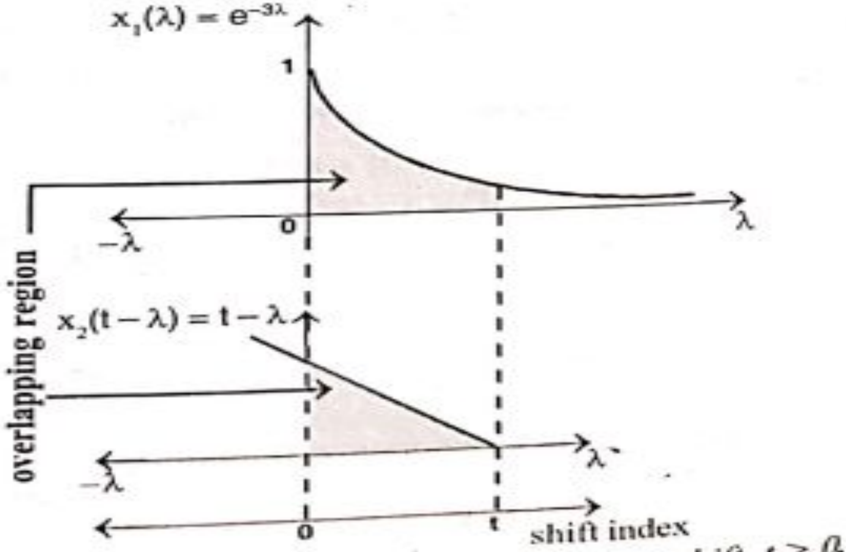


Fig 7 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t > 0$.

From fig 6 and fig 7, it is observed that the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is non-zero only for time shift $t > 0$.

For any time shift $t > 0$, the non-zero product exists in the overlapping region shown in fig 7. Here the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence integration of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore\ x_3(\lambda) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda \quad ; \quad t \geq 0$$

$$= \int_0^t e^{-3\lambda}\,(t-\lambda)\, d\lambda = \int_0^t t\, e^{-3\lambda}\, d\lambda - \int_0^t \lambda\, e^{-3\lambda}\, d\lambda$$

$$\boxed{\begin{array}{c|c} \multicolumn{2}{c}{\int uv = u\int v - \int\left[du \int v\right]} \\ \hline u = \lambda & v = e^{-3\lambda} \end{array}}$$

$$= t\int_0^t e^{-3\lambda}\, d\lambda - \left[\lambda\,\frac{e^{-3\lambda}}{-3} - \int 1 \times \frac{e^{-3\lambda}}{-3}\, d\lambda\right]_0^t = t\left[\frac{e^{-3\lambda}}{-3}\right]_0^t - \left[\lambda\,\frac{e^{-3\lambda}}{-3} - \frac{e^{-3\lambda}}{9}\right]_0^t$$

$$= t\left[\frac{e^{-3t}}{-3} - \frac{e^{0}}{-3}\right] - \left[t\,\frac{e^{-3t}}{-3} - \frac{e^{-3t}}{9} - 0 + \frac{e^{0}}{9}\right] = -\frac{t\,e^{-3t}}{3} + \frac{t}{3} + \frac{t\,e^{-3t}}{3} + \frac{e^{-3t}}{9} - \frac{1}{9}$$

$$= \frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9} \quad ; \quad t \geq 0$$

$$= \left(\frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9}\right) u(t) = \frac{1}{9}\left(e^{-3t} + 3t - 1\right) u(t)$$

EX.2

Given that, $x_1(t) = e^{-at}$; $0 \le t \le T$

$x_2(t) = 1$; $0 \le t \le 2T$

Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Let us change the time index t in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

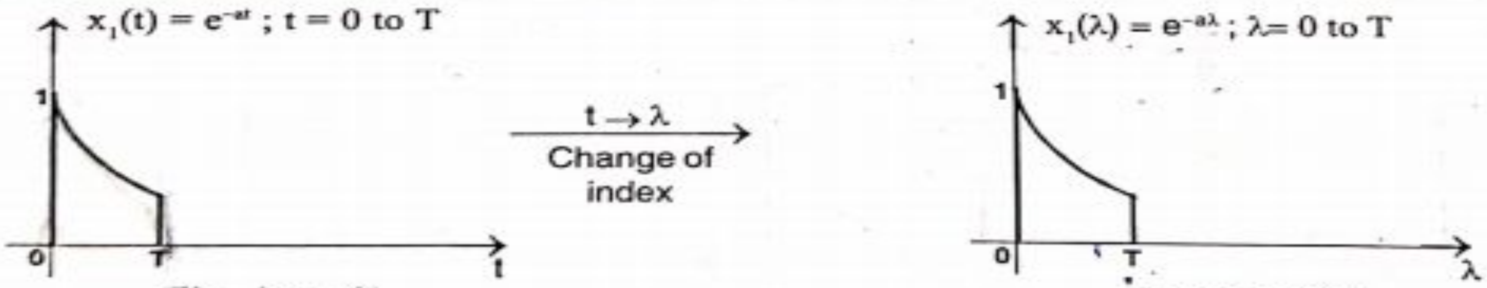


Fig 1 : $x_1(t)$. *Fig 2 :* $x_1(\lambda)$.

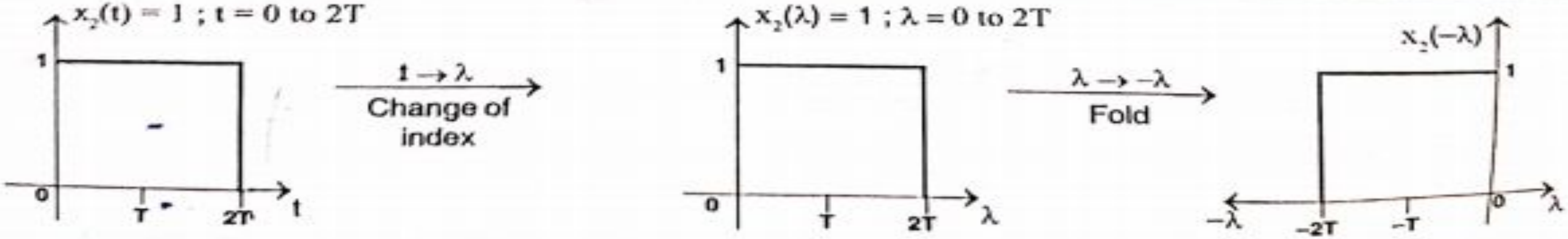


Fig 3 : $x_2(t)$. *Fig 4 :* $x_2(\lambda)$. *Fig 5 :* $x_2(-\lambda)$.

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t-\lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

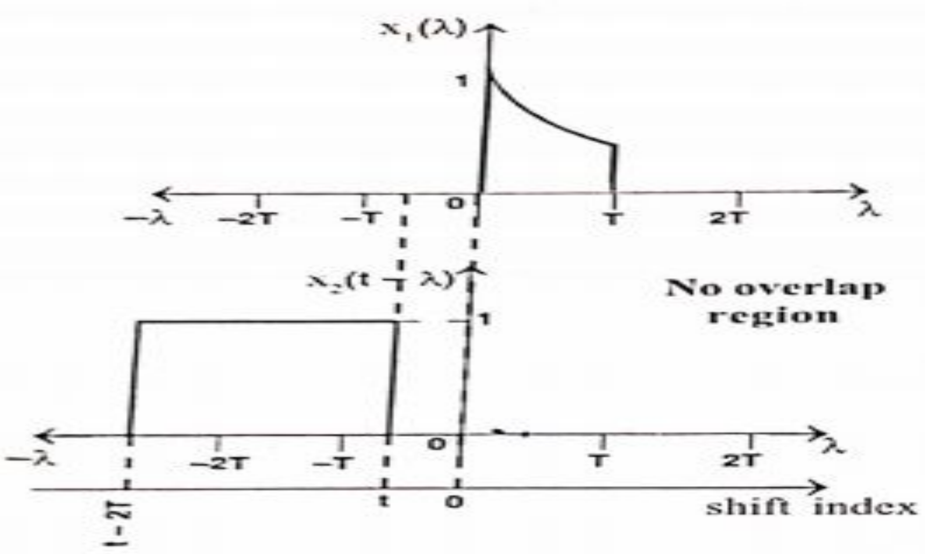


Fig 6 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t < 0$.

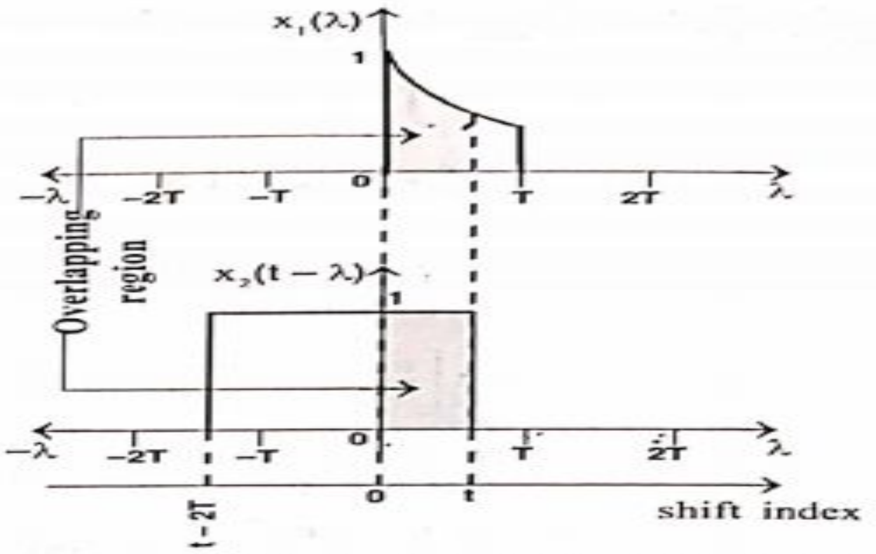


Fig 7 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 0$ to T.

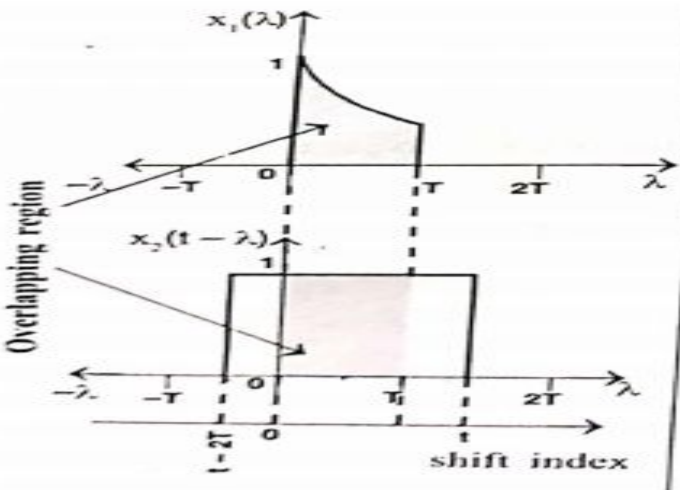


Fig 8 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = T$ to $2T$.

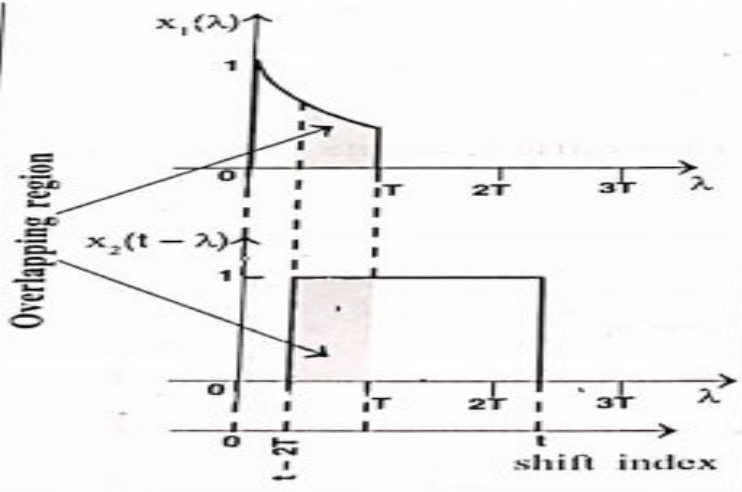


Fig 9 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 2T$ to $3T$.

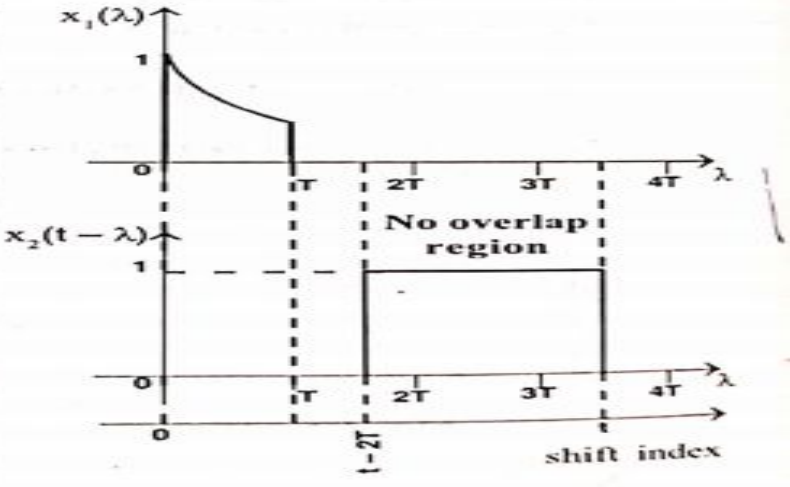


Fig 10 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t > 3T$.

For $t = 0$ to T : In this interval, the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^t e^{-a\lambda} \times 1\, d\lambda = \left[\frac{e^{-a\lambda}}{-a}\right]_0^t$$

$$= \frac{e^{-at}}{-a} - \frac{e^0}{-a} = -\frac{1}{a}e^{-at} + \frac{1}{a} = \frac{1}{a}(1 - e^{-at})$$

For $t = T$ to $2T$: In this interval, the overlapping region is $\lambda = 0$ to $\lambda = T$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = T$.

$$\therefore x_3(t) = \int_{\lambda=0}^{\lambda=T} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^T e^{-a\lambda} \times 1\, d\lambda = \left[\frac{e^{-a\lambda}}{-a}\right]_0^T$$

$$= \frac{e^{-aT}}{-a} - \frac{e^0}{-a} = -\frac{1}{a}e^{-aT} + \frac{1}{a} = \frac{1}{a}(1 - e^{-aT})$$

For $t = 2T$ to $3T$: In this interval, the overlapping region is $\lambda = t-2T$ to $\lambda = T$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = t-2T$ to $\lambda = T$.

$$\therefore x_3(t) = \int_{\lambda=t-2T}^{\lambda=T} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{t-2T}^T e^{-a\lambda} \times 1\, d\lambda = \left[\frac{e^{-a\lambda}}{-a}\right]_{t-2T}^T$$

$$= \frac{e^{-aT}}{-a} - \frac{e^{-a(t-2T)}}{-a} = -\frac{1}{a}e^{-aT} + \frac{1}{a}e^{-a(t-2T)} = \frac{1}{a}(e^{-a(t-2T)} - e^{-aT})$$

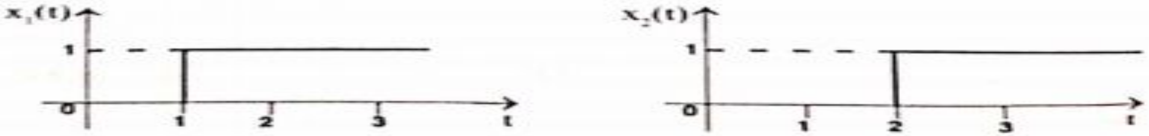
$$\begin{aligned}
\therefore x_3(t) &= 0 && ; \quad t < 0 \\
&= \frac{1}{a}(1 - e^{-at}) && ; \quad 0 < t < T \\
&= \frac{1}{a}(1 - e^{-aT}) && ; \quad T < t < 2T \\
&= \frac{1}{a}(e^{-a(t-2T)} - e^{-aT}) && ; \quad 2T < t < 3T \\
&= 0 && ; \quad t > 3T
\end{aligned}$$

EX.3

Perform convolution of the following signals, by graphical method and sketch the resultant signal.

a)

b)



a) Let, $x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Let us change the time index t in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

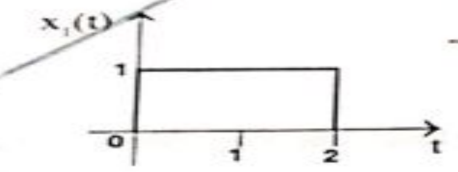


Fig 1 : $x_1(t)$.

$t \rightarrow \lambda$ Change of index

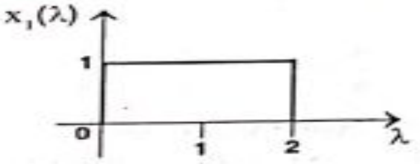


Fig 2 : $x_1(\lambda)$.

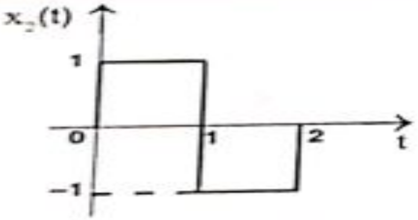


Fig 3 : $x_2(t)$.

$t \rightarrow \lambda$ Change of index

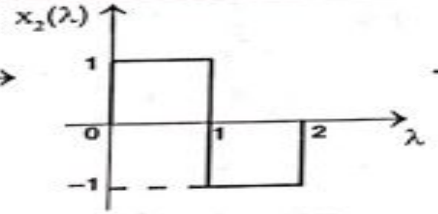


Fig 4 : $x_2(\lambda)$.

$\lambda \rightarrow -\lambda$ Fold

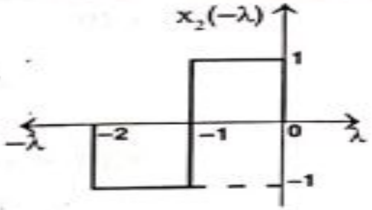


Fig 5 : $x_2(-\lambda)$.

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t-\lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

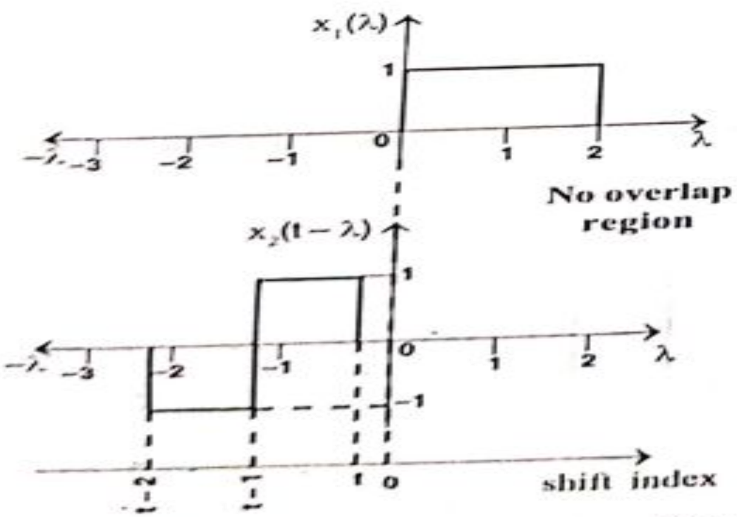


Fig 6 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t < 0$.

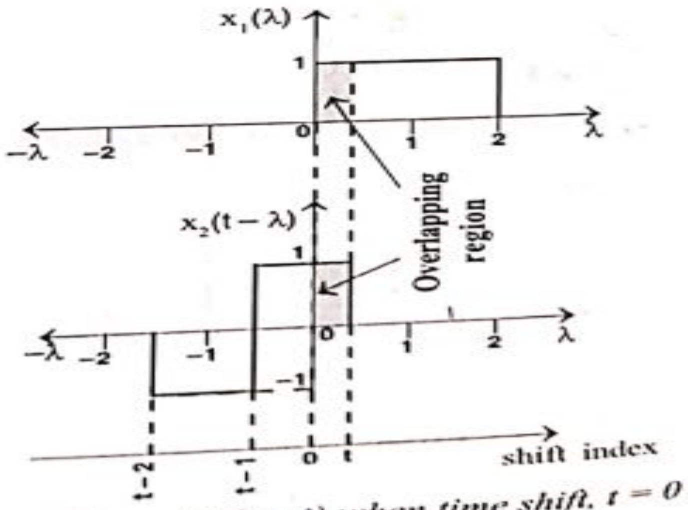


Fig 7 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 0$ to 1.

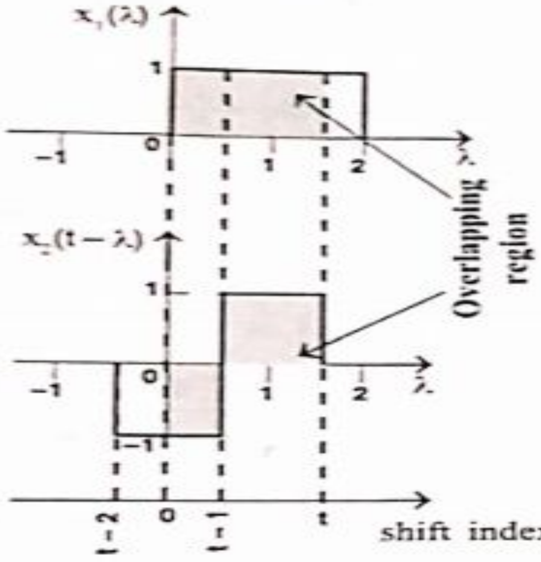


Fig 8 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 1$ to 2.

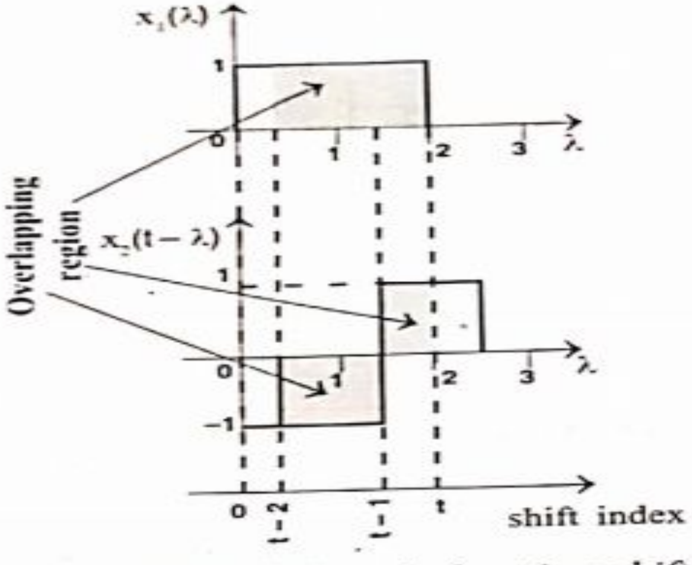


Fig 9 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 2$ to 3.

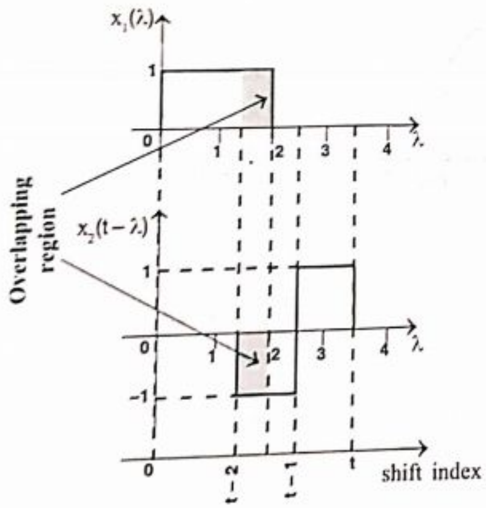


Fig 10 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t = 3$ to 4.

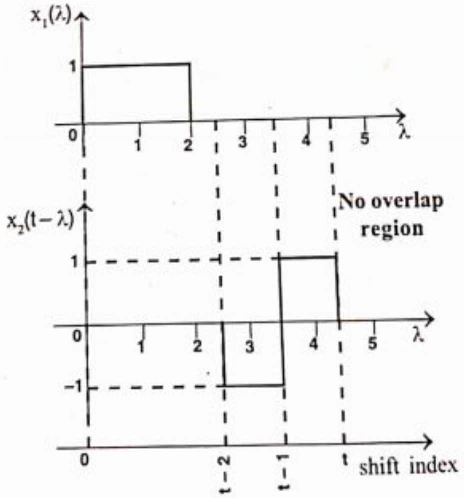


Fig 11 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t > 4$.

From fig 6 to fig 11, it is observed that the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is non-zero only for time shift $t = 0$ to 4. For any time shift in the range $t = 0$ to 4, the non-zero product exists in the overlapping regions shown in fig 7 to fig 10.

For $t = 0$ to 1 : In this interval, the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^t 1 \times 1\, d\lambda = [\lambda]_0^t = t$$

For $t = 1$ to 2 : In this interval, the overlapping region is $\lambda = 0$ to $\lambda = t$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = 0$ to $\lambda = t$.

$$\therefore x_3(t) = \int_{\lambda=0}^{\lambda=t} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^{t-1} 1 \times (-1)\, d\lambda + \int_{t-1}^{t} 1 \times 1\, d\lambda = [-\lambda]_0^{t-1} + [\lambda]_{t-1}^{t}$$

$$= -(t-1) + 0 + t - (t-1) = -t + 1 + t - t + 1 = 2 - t$$

For $t = 2$ to 3 : In this interval, the overlapping region is $\lambda = t-2$ to $\lambda = 2$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = t-2$ to $\lambda = 2$.

$$\therefore x_3(t) = \int_{\lambda=t-2}^{\lambda=2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{t-2}^{t-1} 1 \times (-1)\, d\lambda + \int_{t-1}^{2} 1 \times 1\, d\lambda = [-\lambda]_{t-2}^{t-1} + [\lambda]_{t-1}^{2}$$

$$= -(t-1) + (t-2) + 2 - (t-1) = -t + 1 + t - 2 + 2 - t + 1 = 2 - t$$

For $t = 3$ to 4 : In this interval, the overlapping region is $\lambda = t-2$ to $\lambda = 2$. Hence the integration of the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is performed from $\lambda = t-2$ to $\lambda = 2$.

$$\therefore x_3(t) = \int_{\lambda=t-2}^{\lambda=2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_{t-2}^{2} 1 \times (-1)\, d\lambda = [-\lambda]_{t-2}^{2}$$

$$= -2 + (t-2) = -2 + t - 2 = t - 4$$

The result of the convolution of $x_1(t)$ with $x_2(t)$ is given below and the resultant waveform is shown in fig 12.

$$x_1(t) * x_2(t) = x_3(t) = \begin{cases} 0 & ; t < 0 \\ t & ; 0 < t < 1 \\ 2 - t & ; 1 < t < 3 \\ t - 4 & ; 3 < t < 4 \\ 0 & ; t > 4 \end{cases}$$

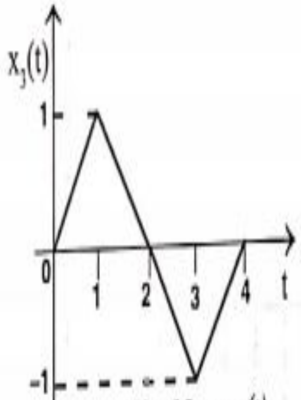


Fig 12 : $x_3(t)$.

At $t = 0$, $x_3(0) = 0$

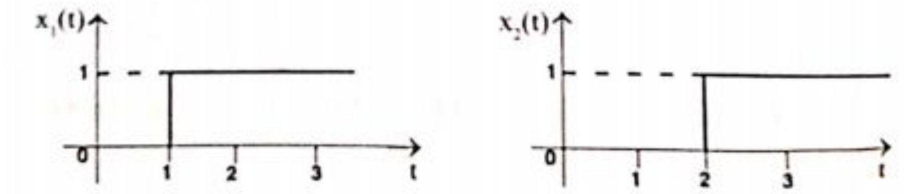
At $t = 1$, $x_3(1) = t = 1$

At $t = 2$, $x_3(2) = 2 - t = 2 - 2 = 0$

At $t = 3$, $x_3(3) = 2 - t = 2 - 3 = -1$

At $t = 4$, $x_3(4) = t - 4 = 4 - 4 = 0$

EX.4



Let, $\quad x_3(t) = x_1(t) * x_2(t)$

By definition of convolution,

$$x_3(t) = \int_{\lambda=-\infty}^{\lambda=+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

Let us change the time index t in $x_1(t)$ and $x_2(t)$ to λ, to get $x_1(\lambda)$ and $x_2(\lambda)$, and then fold $x_2(\lambda)$ to get $x_2(-\lambda)$ graphically as shown below.

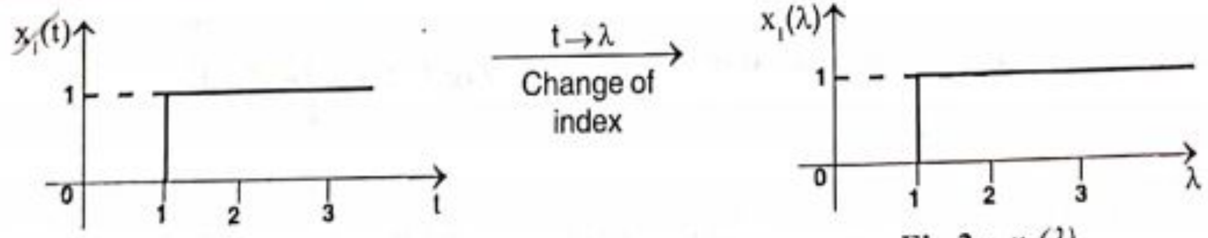


Fig 1 : $x_1(t)$.

Fig 2 : $x_1(\lambda)$.

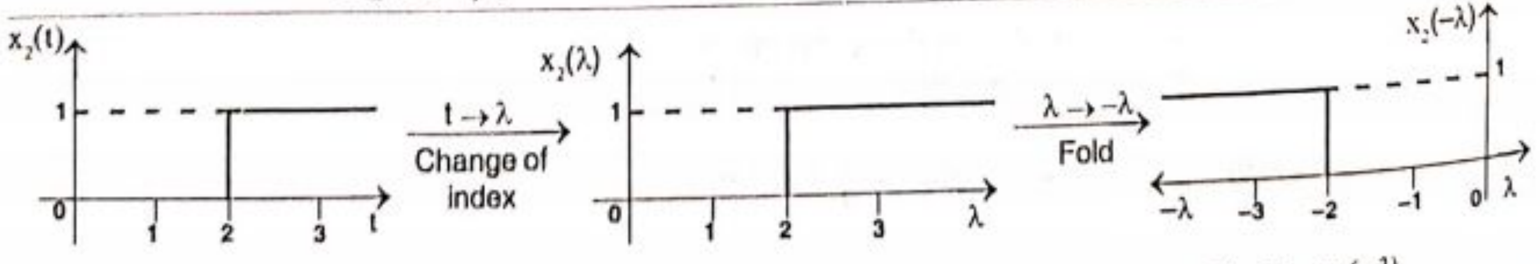


Fig 3 : $x_2(t)$.

Fig 4 : $x_2(\lambda)$.

Fig 5 : $x_2(-\lambda)$.

Let us shift $x_2(-\lambda)$ by t units of time to get $x_2(t-\lambda)$ and then multiply with $x_1(\lambda)$ as shown below.

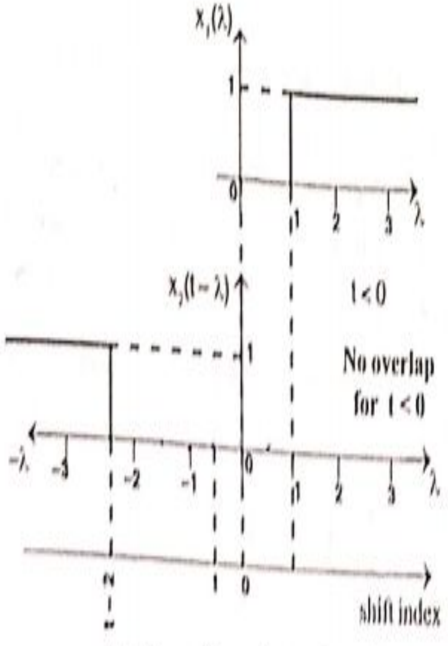


Fig 6 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t < 0$.

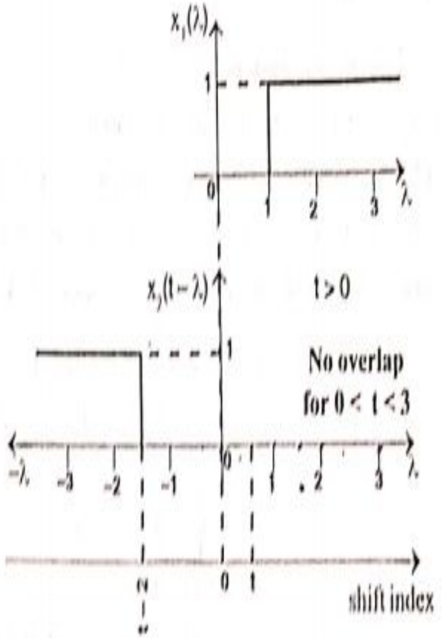


Fig 7 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t > 0$.

From fig 6, fig 7 and fig 8, it is observed that the product of $x_1(\lambda)$ and $x_2(t-\lambda)$ is non-zero only for time shift $t \geq 3$. For any time shift $t > 3$, the non-zero product exists in the overlapping region shown in fig 8.

For t > 3 :

$$x_3(t) = \int_{\lambda=1}^{\lambda=t-2} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda$$

$$= \int_{1}^{t-2} 1 \times 1\, d\lambda = [\lambda]_1^{t-2}$$

$$= t - 2 - 1$$

$$= t - 3 \quad ; t \geq 3$$

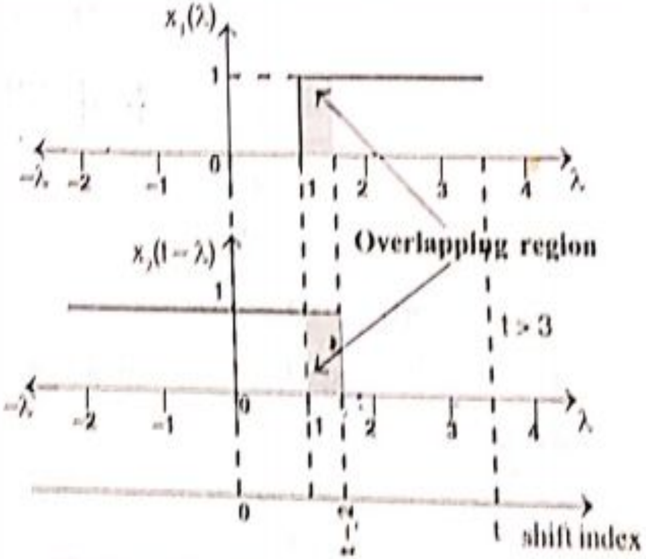


Fig 8 : $x_1(\lambda)$ and $x_2(t-\lambda)$ when time shift, $t > 3$.

The result of the convolution of $x_1(t)$ with $x_2(t)$ is given below and the resultant waveform is shown in fig 9.

$$x_1(t) * x_2(t) = x_3(t) = t - 3 \quad ; t \geq 3$$

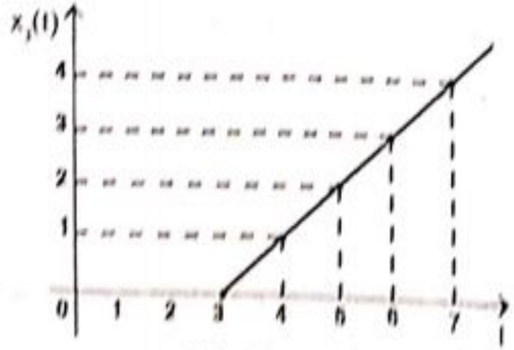


Fig 9 : $x_3(t)$.

At $t = 3$, $x_3(t) = t - 3 = 3 - 3 = 0$
At $t = 4$, $x_3(t) = t - 3 = 4 - 3 = 1$
At $t = 5$, $x_3(t) = t - 3 = 5 - 3 = 2$
At $t = 6$, $x_3(t) = t - 3 = 6 - 3 = 3$
At $t = 7$, $x_3(t) = t - 3 = 7 - 3 = 4$

6.6 Discrete Time System

A ***discrete time system*** is a device or algorithm that operates on a discrete time signal, called the input or excitation, according to some well defined rule, to produce another discrete time signal called the output or the response of the system. We can say that the input signal x(n) is transformed by the system into a signal y(n), and the transformation can be expressed mathematically as shown in equation (6.13). The diagrammatic representation of discrete time system is shown in fig 6.17.

$$\text{Response,} \quad y(n) = \mathcal{H}\{x(n)\} \qquad \text{.....(6.13)}$$

where, $\mathcal{H}$ denotes the transformation (also called an operator).

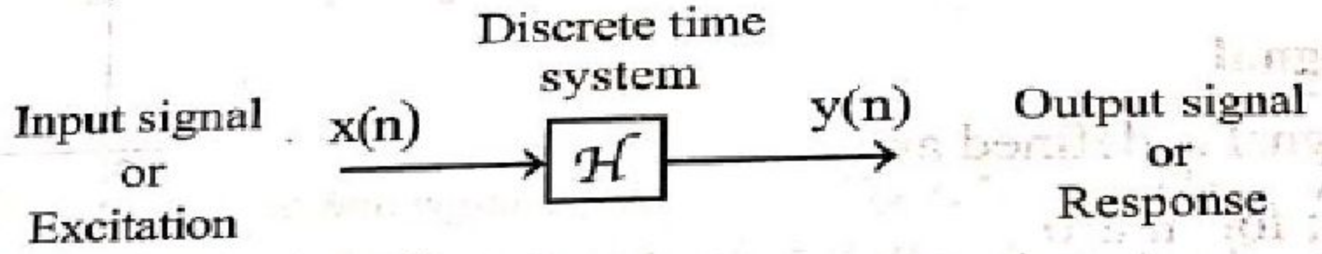


Fig 6.17 : Representation of discrete time system.

A discrete time system is linear if it obeys the principle of superposition and it is time invariant if its input-output relationship do not change with time. When a discrete time system satisfies the properties of linearity and time invariance then it is called an ***LTI system*** (Linear Time Invariant system).

Impulse Response

When the input to a discrete time system is a unit impulse $\delta(n)$ then the output is called an *impulse response* of the system and is denoted by $h(n)$.

$$\therefore \text{Impulse Response, } h(n) = \mathcal{H}\{\delta(n)\} \quad \text{.....(6.14)}$$

$$\xrightarrow{\delta(n)} \boxed{\mathcal{H}} \xrightarrow{h(n)}$$

***Fig 6.18 :** Discrete time system with impulse input.*

6.6.1 Mathematical Equation Governing Discrete Time System

The mathematical equation governing the discrete time system can be developed as shown below.

The response of a discrete time system at any time instant depends on the present input, past inputs and past outputs.

Let us consider the response at $n = 0$. Let us assume a relaxed system and so at $n = 0$, there is no past input or output. Therefore the response at $n = 0$, is a function of present input alone.

$$\text{i.e., } y(0) = F[x(0)]$$

Let us consider the response at $n = 1$. Now the present input is $x(1)$, the past input is $x(0)$ and past output is $y(0)$. Therefore the response at $n = 1$, is a function of $x(1)$, $x(0)$, $y(0)$.

$$\text{i.e., } y(1) = F[y(0), x(1), x(0)]$$

Let us consider the response at $n = 2$. Now the present input is $x(2)$, the past inputs are $x(1)$ and $x(0)$, and past outputs are $y(1)$ and $y(0)$. Therefore the response at $n = 2$, is a function of $x(2)$, $x(1)$, $x(0)$, $y(1)$, $y(0)$.

$$\text{i.e., } y(2) = F[y(1), y(0), x(2), x(1), x(0)]$$

Similarly, at $n = 3$, $y(3) = F[y(2), y(1), y(0), x(3), x(2), x(1), x(0)]$

at $n = 4$, $y(4) = F[y(3), y(2), y(1), y(0), x(4), x(3), x(2), x(1),$ and so on.

In general, at any time instant n,

$$y(n) = F[y(n-1), y(n-2), y(n-3), \ldots y(1), y(0), x(n), x(n-1), x(n-2), x(n-3) \ldots x(1), x(0)] \quad \ldots(6.15)$$

For an LTI system, the response y(n) can be expressed as a weighted summation of dependent terms. Therefore the equation (6.15) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \ldots + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + \ldots \quad \ldots(6.16)$$

where, $a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$ are constants.

Note : Negative constants are inserted for output signals, because output signals are feedback from output to input. Positive constants are inserted for input signals, because input signals are feed forward from input to output.

signals are feed for ...

Practically, the response y(n) at any time instant n, may depend on N number of past outputs, present input and M number of past inputs where $M \le N$. Hence the equation (6.16) can be written as,

$$y(n) = -a_1\, y(n-1) - a_2\, y(n-2) - a_3\, y(n-3) - \dots\dots - a_N\, y(n-N)$$
$$+ b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + b_3\, x(n-3) + \dots\dots + b_M\, x(n-M)$$

$$\therefore \quad y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m) \qquad \dots\dots(6.17)$$

The equation (6.17) is a constant coefficient ***difference equation***, governing the input-output relation of an LTI discrete time system.

In equation (6.17) the value of "N" gives the ***order*** of the system.

If N = 1, the discrete time system is called 1st order system

If N = 2, the discrete time system is called 2nd order system

If N = 3, the discrete time system is called 3rd order system , and so on.

The general difference equation governing 1st order discrete time LTI system is,

$$y(n) = -a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1)$$

The general difference equation governing 2nd order discrete time LTI system is,

$$y(n) = -a_2\, y(n-2) - a_1\, y(n-1) + b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2)$$

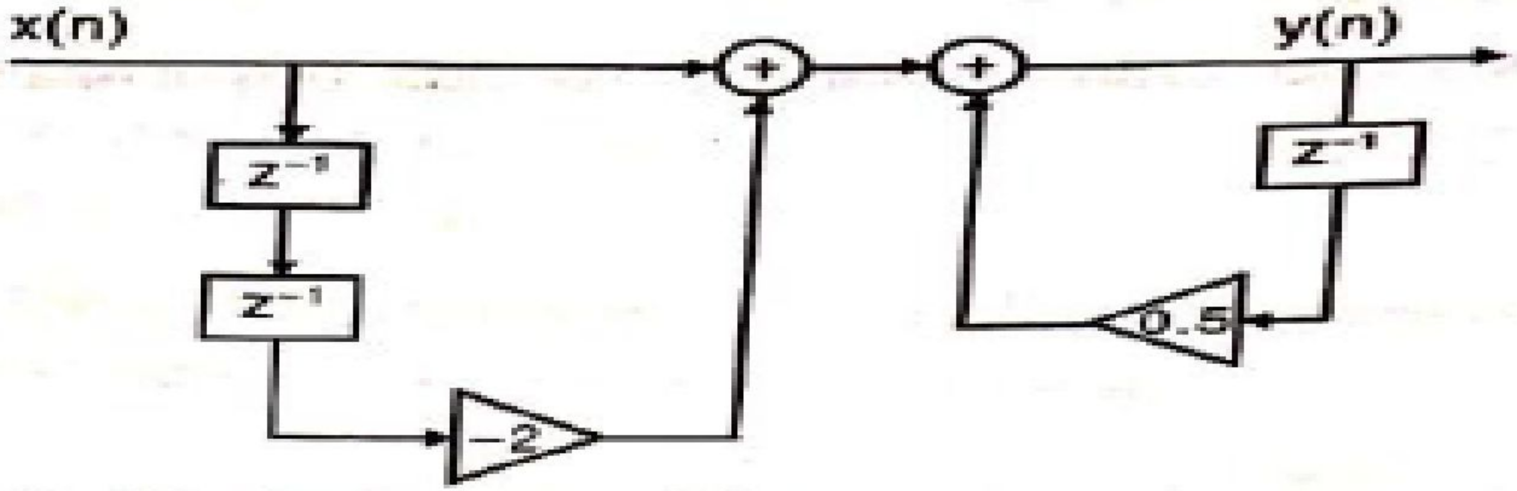
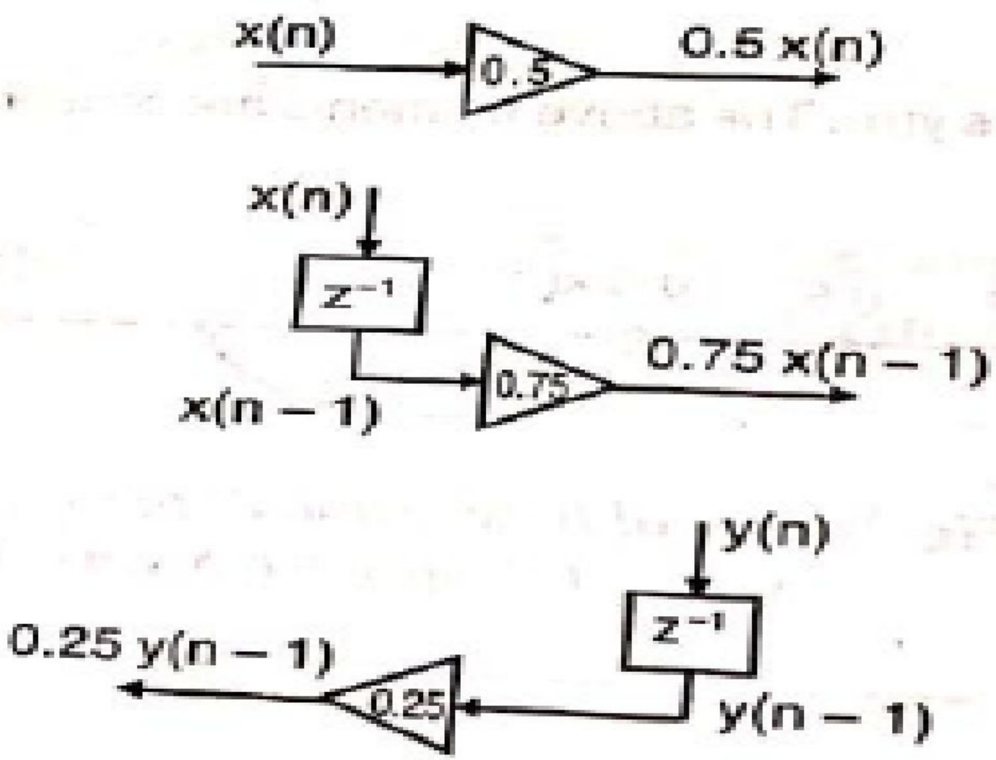


Fig 3 : Block diagram of the system described by the equation $y(n) = 0.5\, y(n-1) + x(n) - 2\, x(n-2)$.

c) **Given that, $y(n) = 0.25\, y(n-1) + 0.5\, x(n) + 0.75\, x(n-1)$**

The individual terms of the given equation are $0.25\, y(n-1)$, $0.5\, x(n)$ and $0.75\, x(n-1)$. They are represented by basic elements as shown below.

<u>Block diagram representation</u>



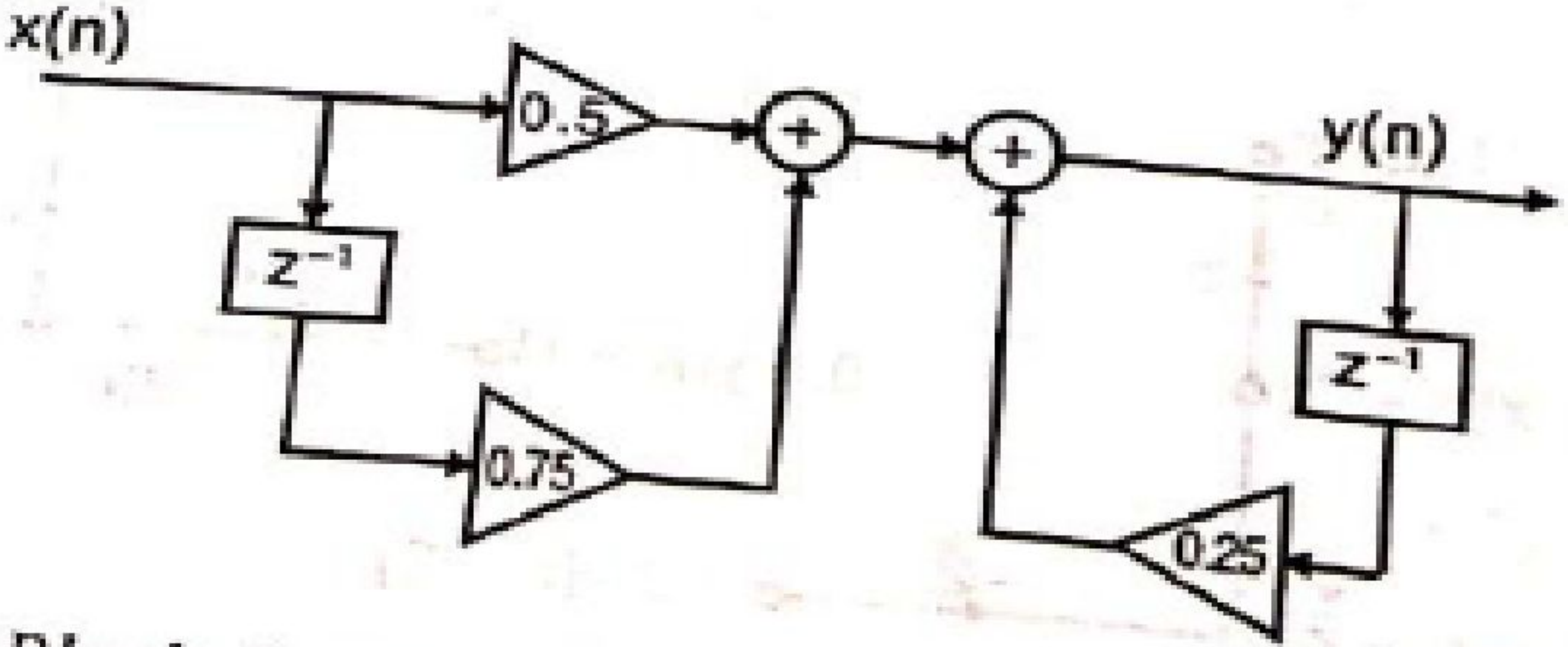


Fig 5 : *Block diagram of the system described by the equation*
$y(n) = 0.25\,y(n-1) + 0.5\,x(n) + 0.75\,x(n-1)$.

6.7 Response of LTI Discrete Time System in Time Domain

The general equation governing an LTI discrete time system is,

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\therefore\quad y(n) + \sum_{m=1}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m)$$

$$\text{(or)}\quad \sum_{m=0}^{N} a_m\, y(n-m) = \sum_{m=0}^{M} b_m\, x(n-m) \quad \text{with } a_o = 1 \qquad \text{.....(6.18)}$$

The solution of the difference equation (6.18) is the ***response*** y(n) of LTI system, which consists of two parts. In mathematics, the two parts of the solution y(n) are homogeneous solution $y_h(n)$ and particular solution $y_p(n)$.

6.9 Discrete or Linear Convolution

The ***Discrete*** or ***Linear convolution*** of two discrete time sequences $x_1(n)$ and $x_2(n)$ is defined as,

$$\boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)} \quad \text{or} \quad \boxed{x_3(n) = \sum_{m=-\infty}^{+\infty} x_2(m)\, x_1(n-m)} \qquad \text{.....(6.29)}$$

where, $x_3(n)$ is the sequence obtained by convolving $x_1(n)$ and $x_2(n)$

m is a dummy variable

If the sequence $x_1(n)$ has N_1 samples and sequence $x_2(n)$ has N_2 samples then the output sequence $x_3(n)$ will be a finite duration sequence consisting of "N_1+N_2-1" samples. The convolution results in a nonperiodic sequence. Hence this convolution is also called ***aperiodic convolution.***

The convolution relation of equation (6.29) can be symbolically expressed as

$$x_3(n) = x_1(n) * x_2(n) = x_2(n) * x_1(n) \qquad \text{..... (6.30)}$$

where, the symbol $*$ indicates convolution operation.

Procedure For Evaluating Linear Convolution

Let, $x_1(n)$ = Discrete time sequence with N_1 samples

$x_2(n)$ = Discrete time sequence with N_2 samples

Now, the convolution of $x_1(n)$ and $x_2(n)$ will produce a sequence $x_3(n)$ consisting of N_1+N_2-1 samples. Each sample of $x_3(n)$ can be computed using the equation (6.29). The value of $x_3(n)$ at $n = q$ is obtained by replacing n by q, in equation (6.29).

$$\therefore x_3(q) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(q-m) \qquad .....(6.31)$$

The evaluation of equation (6.31) to determine the value of $x_3(n)$ at $n = q$, involves the following five steps.

1. **Change of index** : Change the index n in the sequences $x_1(n)$ and $x_2(n)$, to get the sequences $x_1(m)$ and $x_2(m)$.
2. **Folding** : Fold $x_2(m)$ about $m = 0$, to obtain $x_2(-m)$.
3. **Shifting** : Shift $x_2(-m)$ by q to the right if q is positive, shift $x_2(-m)$ by q to the left if q is negative to obtain $x_2(q-m)$.
4. **Multiplication** : Multiply $x_1(m)$ by $x_2(q-m)$ to get a product sequence. Let the product sequence be $v_q(m)$. Now, $v_q(m) = x_1(m) \times x_2(q-m)$.
5. **Summation** : Sum all the values of the product sequence $v_q(m)$ to obtain the value of $x_3(n)$ at $n = q$. [i.e., $x_3(q)$].

The above procedure will give the value $x_3(n)$ at a single time instant say $n = q$. In general, we are interested in evaluating the values of the seqence $x_3(n)$ over all the time instants in the range $-\infty < n < \infty$. Hence the steps 3, 4 and 5 given above must be repeated, for all possible time shifts in the range $-\infty < n < \infty$.

In the convolution of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x_1(n)$ starts at $n = n_1$ and $x_2(n)$ starts at $n = n_2$ then, the initial value of n for $x_3(n)$ is "$n = n_1 + n_2$". The value of $x_1(n)$ for $n < n_1$ and the value of $x_2(n)$ for $n < n_2$ are then assumed to be zero. The final value of n for $x_3(n)$ is "$n = (n_1 + n_2) + (N_1 + N_2 - 2)$".

6.9.1 Representation of Discrete Time Signal as Summation of Impulses

A discrete time signal can be expressed as summation of impulses and this concept will be useful to prove that the response of discrete time LTI system can be determined using discrete convolution.

Let, $x(n)$ = Discrete time signal

$\delta(n)$ = Unit impulse signal

$\delta(n-m)$ = Delayed impulse signal

We know that, $\delta(n) = 1$; at $n = 0$

$= 0$; when $n \neq 0$

and, $\delta(n-m) = 1$; at $n = m$

$= 0$; when $n \neq m$

If we multiply the signal $x(n)$ with the delayed impulse $\delta(n-m)$ then the product is non-zero only at $n=m$ and zero for all other values of n. Also at $n=m$, the value of product signal is m^{th} sample $x(m)$ of the signal $x(n)$.

$$\therefore\ x(n)\ \delta(n-m) = x(m)$$

Each multiplication of the signal $x(n)$ by an unit impulse at some delay m, in essence picks out the single value $x(m)$ of the signal $x(n)$ at $n = m$, where the unit impulse is non-zero. Consequently if we repeat this multiplication for all possible delays in the range $-\infty < m < \infty$ and add all the product sequences, the result will be a sequence that is equal to the sequence $x(n)$.

For example, $x(n)\ \delta(n-(-2)) = x(-2)$

$$x(n)\ \delta(n-(-1)) = x(-1)$$

$$x(n)\ \delta(n) = x(0)$$

$$x(n)\ \delta(n-1) = x(1)$$

$$x(n)\ \delta(n-2) = x(2)$$

From the above products we can say that each sample of $x(n)$ can be expressed as a product of the sample and delayed impulse, as shown below.

$$\therefore\ x(-2) = x(-2)\ \delta(n-(-2))$$

$$x(-1) = x(-1)\ \delta(-(-1))$$

$$x(0) = x(0)\ \delta(n)$$

$$x(1) = x(1)\ \delta(n-1)$$

$$x(2) = x(2)\ \delta(n-2)$$

$$\therefore x(n) = \cdots + x(-2) + x(-1) + x(0) + x(1) + x(2) + \cdots$$

$$= \cdots + x(-2)\ \delta(n-(-2)) + x(-1)\ \delta(n-(-1)) + x(0)\ \delta(n) + x(1)\ \delta(n-1) + x(2)\ \delta(n-2) + \cdots$$

$$= \sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m) \qquad \text{....(6.32)}$$

In equation (6.32) each product $x(m)\ \delta(n-m)$ is an impulse and the summation of impulses gives the sequence $x(n)$.

6.9.2 Response of LTI Discrete Time System Using Discrete Convolution

In an LTI system, the response y(n) of the system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system. It is expressed as,

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \qquad \text{.....(6.33)}$$

where, the symbol * represents convolution operation.

Proof :

Let y(n) be the response of system $\mathcal{H}$ for an input x(n)

$$\therefore\ y(n) = \mathcal{H}\{x(n)\} \qquad \text{.....(6.34)}$$

From equation (6.32) we know that the signal x(n) can be expressed as a summation of impulses,

$$\text{i.e., } x(n) = \sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m) \qquad \text{.....(6.35)}$$

where, $\delta(n-m)$ is the delayed unit impulse signal.

From equation (6.34) and (6.35) we get,

$$y(n) = \mathcal{H}\left\{\sum_{m=-\infty}^{+\infty} x(m)\,\delta(n-m)\right\} \qquad \text{.....(6.36)}$$

The system $\mathcal{H}$ is a function of n and not a function of m. Hence by linearity property the equation (6.36) can be written as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\,\mathcal{H}\{\delta(n-m)\} \qquad \text{.....(6.37)}$$

Let the response of the LTI system to the unit impulse input $\delta(n)$ be denoted by h(n).

$$\therefore\ h(n) = \mathcal{H}\{\delta(n)\}$$

Then by time invariance property the response of the system to the delayed unit impulse input $\delta(n-m)$ is given by

$$h(n-m) = \mathcal{H}\{\delta(n-m)\} \qquad \text{.....(6.38)}$$

Using equation (6.38), the equation (6.37) can be expressed as,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\,h(n-m)$$

The above equation represents the convolution of input x(n) with the impulse response h(n) to yield the output y(n). Hence it is proved that the response y(n) of LTI discrete time system for an arbitrary input x(n) is given by convolution of input x(n) with impulse response h(n) of the system.

6.9.3 Properties of Linear Convolution

The Discrete convolution will satisfy the following properties.

Commutative property : $x_1(n) * x_2(n) = x_2(n) * x_1(n)$

Associative property : $[x_1(n) * x_2(n)] * x_3(n) = x_1(n) * [x_2(n) * x_3(n)]$

Distributive property : $x_1(n) * [x_2(n) + x_3(n)] = [x_1(n) * x_2(n)] + [x_1(n) * x_3(n)]$

6.9.4 Interconnections of Discrete Time Systems

Smaller discrete time systems may be interconnected to form larger systems. Two possible basic ways of interconnection are ***cascade connection*** and ***parallel connection***. The cascade and parallel connections of two discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ are shown in fig 6.21.

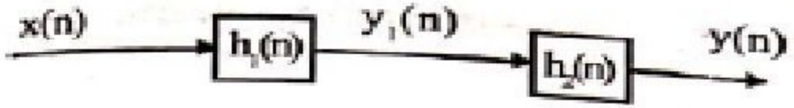


Fig 6.21a : Cascade connection.

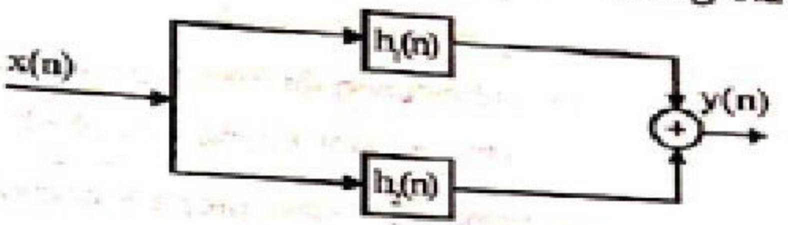


Fig 6.21b : Parallel connection.

Fig 6.21 : Interconnection of discrete time systems.

Cascade Connected Discrete Time System

Two cascade connected discrete time systems with impulse response $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by convolution of individual impulse responses.

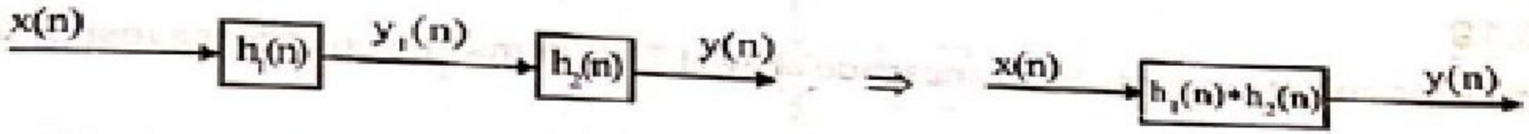


Fig 6.22 : Cascade connected discrete time systems and their equivalent

Parallel Connected Discrete Time Systems

Two parallel connected discrete time systems with impulse responses $h_1(n)$ and $h_2(n)$ can be replaced by a single equivalent discrete time system whose impulse response is given by sum of individual impulse responses.

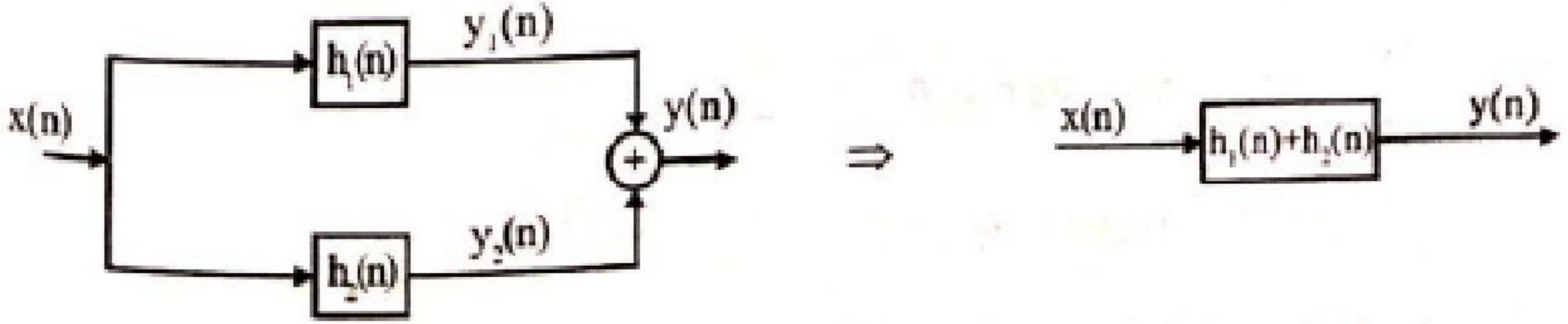


Fig 6.23 : Parallel connected discrete time systems and their equivalent.

Example 6.19

Determine the impulse reponse for the cascade of two LTI systems having impulse responses,

$$h_1(n) = \left(\frac{1}{2}\right)^n u(n) \text{ and } h_2(n) = \left(\frac{1}{4}\right)^n u(n).$$

Solution

Let h(n) be the impulse response of the cascade system. Now h(n) is given by convolution of $h_1(n)$ and $h_2(n)$.

$$\therefore h(n) = h_1(n) * h_2(n)$$

$$= \sum_{m=-\infty}^{+\infty} h_1(m)\, h_2(n-m) \quad \text{where, } m \text{ is a dummy variable used for convolution operation}$$

The product $h_1(m)\, h_2(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above quation is changed to $m = 0$ to n.

The product $h_1(m)\,h_2(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above quation is changed to $m = 0$ to n.

$$\therefore\ h(n) = \sum_{m=0}^{n} h_1(m)\,h_2(n-m) = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{4}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{4}\right)^n \left(\frac{1}{4}\right)^{-m} = \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m 4^m$$

$$= \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} \left(\frac{4}{2}\right)^m$$

$$= \left(\frac{1}{4}\right)^n \sum_{m=0}^{n} 2^m$$

Finite geometric series sum formula

$$\sum_{n=0}^{N} C^n = \frac{C^{N+1}-1}{C-1}$$

$$= \left(\frac{1}{4}\right)^n \left(\frac{2^{n+1}-1}{2-1}\right)$$

Using finite geometric series sum formula

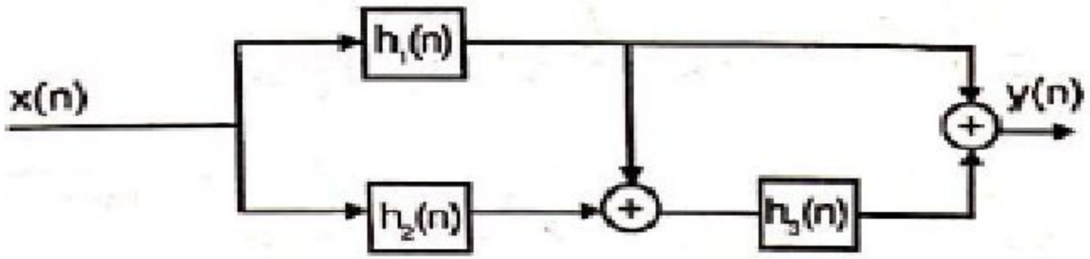
$$= \left(\frac{1}{4}\right)^n (2^{n+1}-1)\ ;\ \text{for } n \ge 0$$

$$= \left(\frac{1}{4}\right)^n (2^{n+1}-1)\,u(n)\ ;\ \text{for all } n$$

Example 6.20

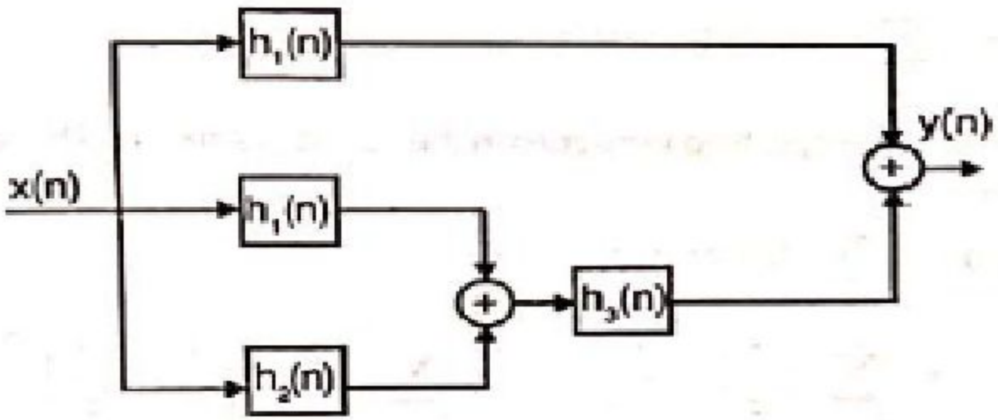
Determine the overall impulse response of the interconnected discrete time system shown below,

where, $h_1(n) = \left(\frac{1}{3}\right)^n u(n)$, $h_2(n) = \left(\frac{1}{2}\right)^n u(n)$ and $h_3(n) = \left(\frac{1}{5}\right)^n u(n)$.

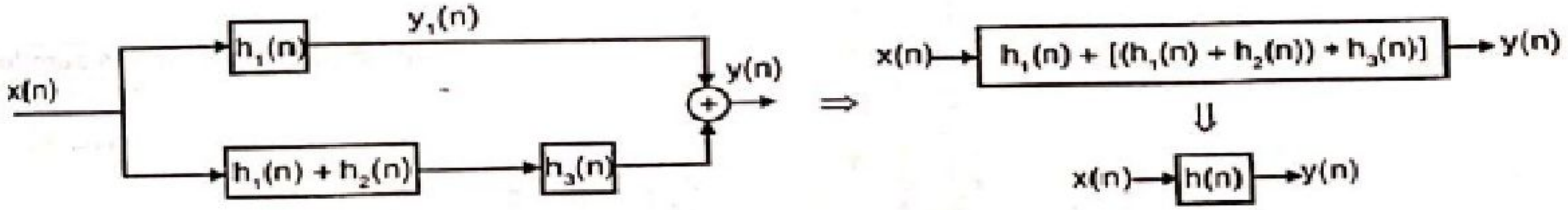


Solution

The given system can be redrawn as shown below.



The above system can be reduced to single equivalent system as shown below.



Here, $h(n) = h_1(n) + [(h_1(n) + h_2(n)) * h_3(n)]$

$= h_1(n) + [h_1(n) * h_3(n)] + [h_2(n) * h_3(n)]$ $\boxed{\text{Using distributive property}}$

Let us evaluate the convolution of $h_1(n)$ and $h_3(n)$.

$$h_1(n) * h_3(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_3(n-m)$$

The product of $h_1(m)\, h_3(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above equation can be changed to $m = 0$ to n.

$$\therefore\ h_1(n) * h_3(n) = \sum_{m=0}^{n} h_1(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{3}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^{n+1} - 1}{\frac{5}{3} - 1}$$

Using finite geometric series sum formula

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{3}\right)^n \frac{5}{3} - 1}{\frac{5-3}{3}} = \left(\frac{1}{5}\right)^n \left[\frac{3}{2}\left(\frac{5}{3}\right)^n \frac{5}{3} - \frac{3}{2}\right]$$

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1} - 1}{C - 1}$$

$$= \frac{5}{2}\left(\frac{1}{5}\right)^n \left(\frac{5}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n = \frac{5}{2}\left(\frac{1}{3}\right)^n - \frac{3}{2}\left(\frac{1}{5}\right)^n \text{ ; for } n \geq 0$$

$$= \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) \text{ ; for all } n$$

Let us evaluate the convolution of $h_2(n)$ and $h_3(n)$.

$$h_2(n) * h_3(n) = \sum_{m=-\infty}^{+\infty} h_2(m)\, h_3(n-m)$$

The product of $h_2(m)$ and $h_3(n-m)$ will be non-zero in the range $0 \le m \le n$. Therefore the summation index in the above equation can be change to m = 0 to n.

$$\therefore\ h_2(n) * h_3(n) = \sum_{m=0}^{n} h_2(m)\, h_3(n-m)$$

$$= \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^{n-m} = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-m}$$

Finite geometric series sum formula

$$\sum_{m=0}^{N} C^m = \frac{C^{N+1}-1}{C-1}$$

$$= \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{1}{2}\right)^m 5^m = \left(\frac{1}{5}\right)^n \sum_{m=0}^{n} \left(\frac{5}{2}\right)^m$$

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^{n+1} - 1}{\frac{5}{2} - 1}$$

Using finite geometric series sum formula

$$= \left(\frac{1}{5}\right)^n \frac{\left(\frac{5}{2}\right)^n \frac{5}{2} - 1}{\frac{5-2}{2}} = \left(\frac{1}{5}\right)^n \left[\frac{2}{3}\left(\frac{5}{2}\right)^n \frac{5}{2} - \frac{2}{3}\right]$$

$$= \frac{5}{3}\left(\frac{1}{5}\right)^n \left(\frac{5}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n = \frac{5}{3}\left(\frac{1}{2}\right)^n - \frac{2}{3}\left(\frac{1}{5}\right)^n \text{ for } n \ge 0$$

$$= \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n) \text{ for all } n$$

Now, the overall impulse response h(n) is given by,

$$h(n) = h_1(n) + [h_1(n) * h_3(n)] + [h_2(n) * h_3(n)]$$

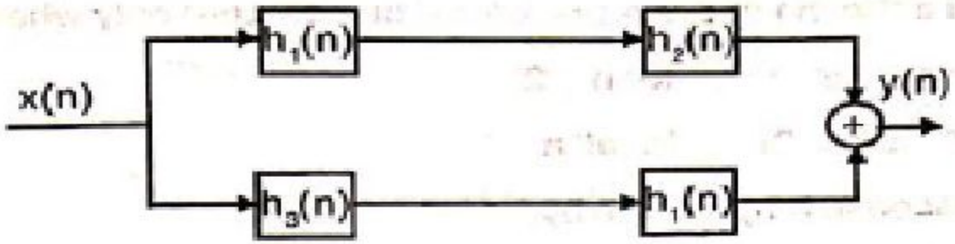
$$= \left(\frac{1}{3}\right)^n u(n) + \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{5}\right)^n u(n)$$

$$= \left(1 + \frac{5}{2}\right)\left(\frac{1}{3}\right)^n u(n) - \left(\frac{3}{2} + \frac{2}{3}\right)\left(\frac{1}{5}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{2}\right)^n u(n)$$

$$= \left[\frac{7}{2}\left(\frac{1}{3}\right)^n - \frac{13}{6}\left(\frac{1}{5}\right)^n + \frac{5}{3}\left(\frac{1}{2}\right)^n\right] u(n)$$

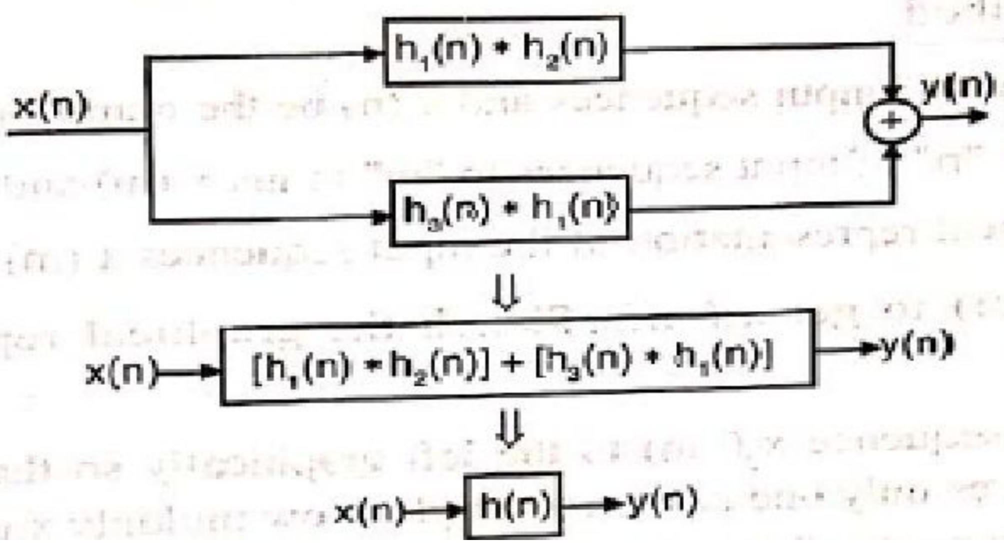
Example 6.21

Find the overall impulse response of the interconnected system shown below. Given that $h_1(n) = a^n u(n)$, $h_2(n) = \delta(n-1)$, $h_3(n) = \delta(n-2)$.



Solution

The given system can be reduced to single equivalent system as shown below.



Here, $h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$

Let us evaluate the convolution of $h_1(n)$ and $h_2(n)$.

$$h_1(n) * h_2(n) = \sum_{m=-\infty}^{\infty} h_1(m)\, h_2(n-m)$$

$$= \sum_{m=-\infty}^{\infty} h_2(m)\, h_1(n-m) \qquad \boxed{\text{Using commutative property}}$$

$$= \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{n}\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-1)\, a^{-m}$$

The product of $\delta(m-1)$ and a^{-m} in the above equation will be non-zero only when $m = 1$.

$$\therefore h_1(n) * h_2(n) = a^n\, a^{-1} = a^{n-1} \quad ; \text{ for } n \geq 1$$

$$= a^{n-1}\, u(n-1) \quad ; \text{ for all } n.$$

Let us evaluate the convolution of $h_3(n)$ and $h_1(n)$.

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} h_3(m)\, h_1(n-m)$$

$$h_3(n) * h_1(n) = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{(n-m)} = \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^n\, a^{-m}$$

$$= a^n \sum_{m=-\infty}^{\infty} \delta(m-2)\, a^{-m}$$

The product of $\delta(m-2)$ and a^{-m} in the above equation will be non-zero only when $m = 2$.

$$\therefore h_1(n) * h_2(n) = a^n\, a^{-2} = a^{n-2} \quad ; \quad \text{for } n \geq 2$$

$$= a^{n-2} u(n-2) \quad ; \quad \text{for all } n$$

Now, the overall impulse response h(n) is given by,

$$h(n) = [h_1(n) * h_2(n)] + [h_3(n) * h_1(n)]$$

$$= a^{(n-1)} u(n-1) + a^{(n-2)} u(n-2)$$

Methods of performing Linear Convolution

Graphical Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence.

1. Change the index "n" of input sequences to "m" to get $x_1(m)$ and $x_2(m)$.
2. Sketch the graphical representation of the input sequences $x_1(m)$ and $x_2(m)$.
3. Let us fold $x_2(m)$ to get $x_2(-m)$. Sketch the graphical representation of the folded sequence $x_2(-m)$.
4. Shift the folded sequence $x_2(-m)$ to the left graphically so that the product of $x_1(m)$ and shifted $x_2(-m)$ gives only one non-zero sample. Now multiply $x_1(m)$ and shifted $x_2(-m)$ to get a product sequence, and then sum-up the samples of product sequence, which is the first sample of output sequence.
5. To get the next sample of output sequence, shift $x_2(-m)$ of previous step to one position right and multiply the shifted sequence with $x_1(m)$ to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
6. To get subsequent samples of output sequence, the step-5 is repeated until we get a non-zero product sequence.

Method -2: Tabular Method

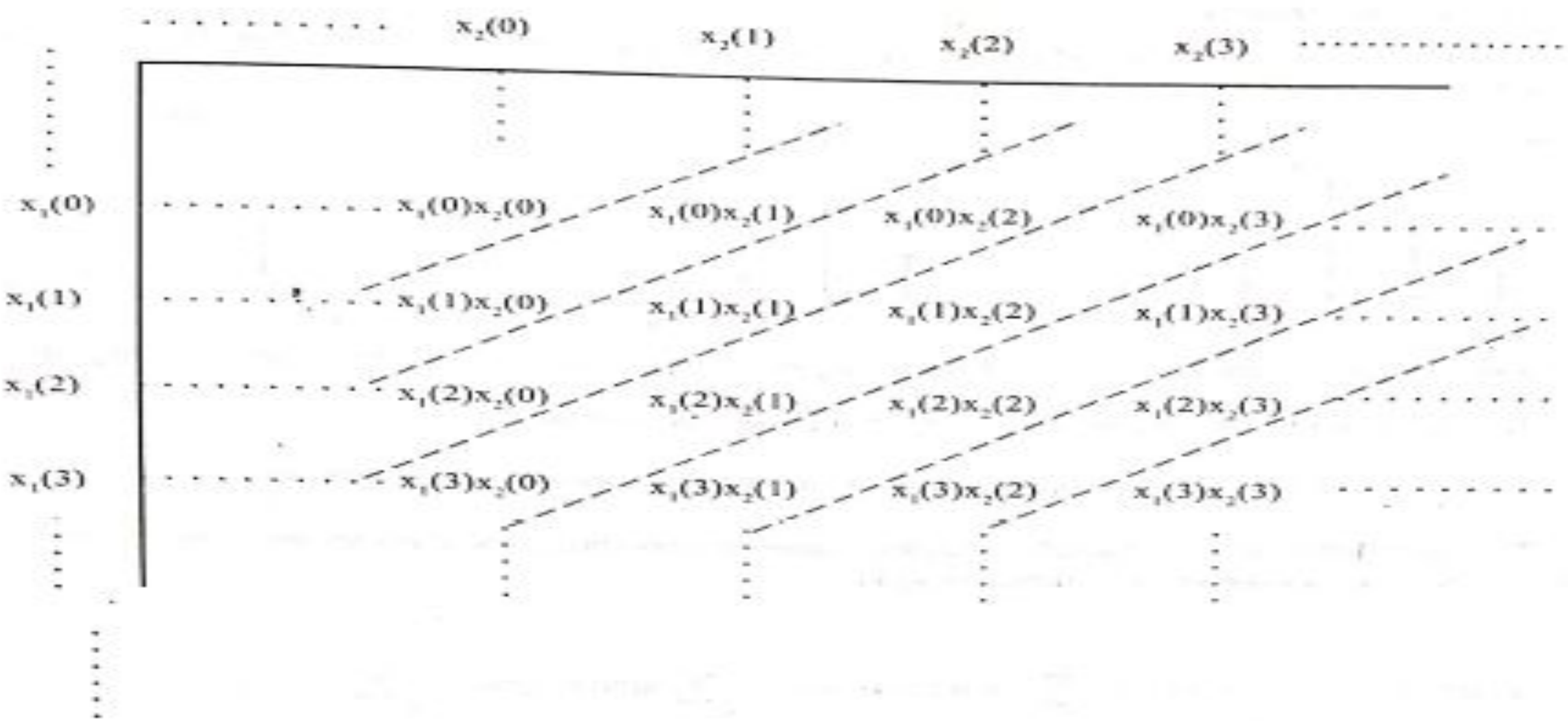
The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence, folded and shifted sequence is represented by a row in a table.

Method -3: Matrix Method

Let $x_1(n)$ and $x_2(n)$ be the input sequences and $x_3(n)$ be the output sequence. In matrix method one of the sequences is represented as a row and the other as a column as shown below.

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $x_3(n)$. (The sum of the diagonal elements are shown below for reference).



$$x_3(0) = \ldots\ldots + x_1(0)\,x_2(0) + \ldots\ldots$$

$$x_3(1) = \ldots\ldots + x_1(1)\,x_2(0) + x_1(0)\,x_2(1) + \ldots\ldots$$

$$x_3(2) = \ldots\ldots + x_1(2)\,x_2(0) + x_1(1)\,x_2(1) + x_1(0)\,x_2(2) + \ldots\ldots$$

$$x_3(3) = \ldots\ldots + x_1(3)\,x_2(0) + x_1(2)\,x_2(1) + x_1(1)\,x_2(2) + x_1(0)\,x_2(3) + \ldots\ldots$$

$$\therefore\ x_3(n) = \{\ldots\ldots\ x_3(0),\ x_3(1),\ x_3(2),\ x_3(3),\ \ldots\ldots\}$$

EX.1

Determine the response of the LTI system whose input $x(n)$ and impulse response $h(n)$ are given by, $x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$ and $h(n) = \{1, \underset{\uparrow}{2}, 1, -1\}$

Solution

The response $y(n)$ of the system is given by convolution of $x(n)$ and $h(n)$.

$$y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m)$$

In this example the convolution operation is performed by three methods.

The input sequence starts at $n = 0$ and the impulse response sequence starts at $n = -1$. Therefore the output sequence starts at $n = 0 + (-1) = -1$.

The input and impulse response consists of 4 samples, so the output consists of $4 + 4 - 1 = 7$ samples.

Method 1 : Graphical Method

The graphical representation of $x(n)$ and $h(n)$ after replacing n by m are shown below. The sequence $h(m)$ is folded with respect to $m = 0$ to obtain $h(-m)$.

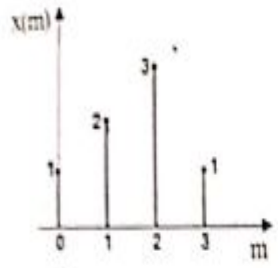


Fig 1 : Input sequence.

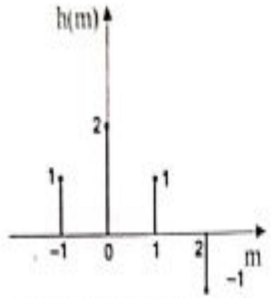


Fig 2 : Impulse response.

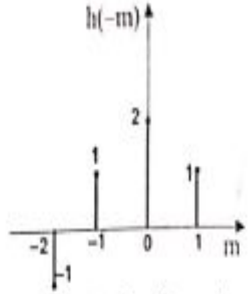


Fig 3 : Folded impulse response.

The samples of $y(n)$ are computed using the convolution formula.

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m) \;;\; \text{where } h_n(m) = h(n-m)$$

The computation of each sample using the above equation are graphically shown in fig 4 to fig 10. The graphical representation of output sequence is shown in fig 11.

When $n = -1$; $y(-1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(-1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_{-1}(m) = \sum_{m=-\infty}^{+\infty} v_{-1}(m)$

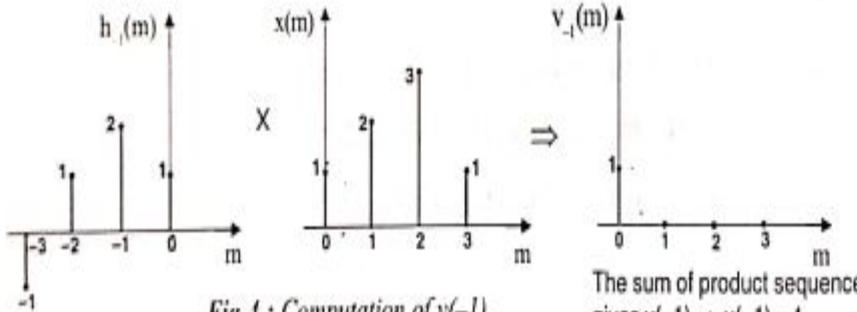


Fig 4 : Computation of y(−1).

The sum of product sequence $v_{-1}(m)$ gives $y(-1)$. $\therefore y(-1) = 1$

When $n = 0$; $y(0) = \sum_{m=-\infty}^{+\infty} x(m)\, h(0-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_0(m) = \sum_{m=-\infty}^{+\infty} v_0(m)$

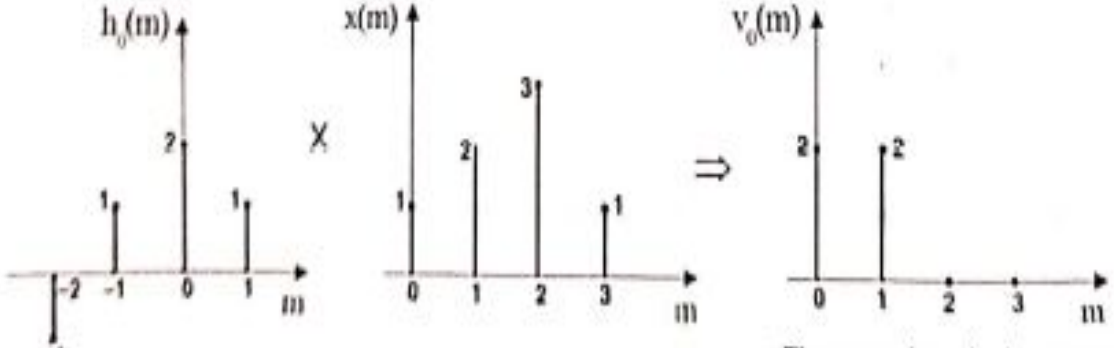


Fig 5 : Computation of y(0).

The sum of product sequence $v_0(m)$ gives $y(0)$. $\therefore$ $y(0) = 2 + 2 = 4$

When $n = 1$; $y(1) = \sum_{m=-\infty}^{+\infty} x(m)\, h(1-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_1(m) = \sum_{m=-\infty}^{+\infty} v_1(m)$

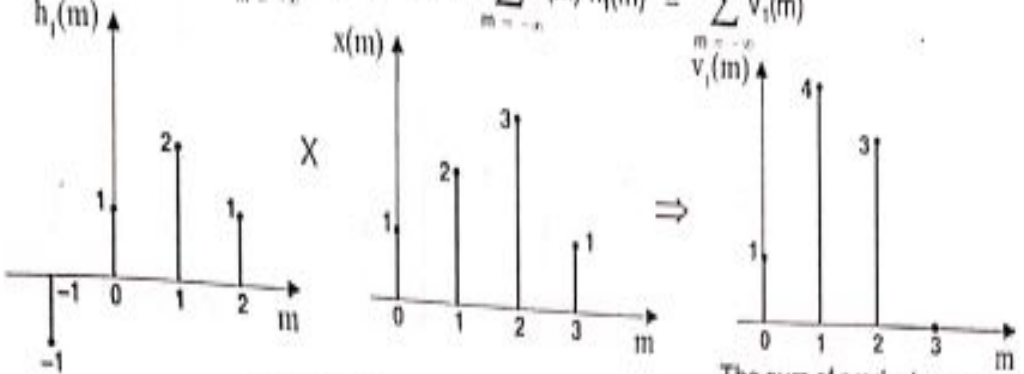


Fig 6 : Computation of y(1).

The sum of product sequence $v_1(m)$ gives $y(1)$. $\therefore$ $y(1) = 1 + 4 + 3 = 8$

When $n = 2$; $y(2) = \sum_{m=-\infty}^{+\infty} x(m)\, h(2-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_2(m) = \sum_{m=-\infty}^{+\infty} v_2(m)$

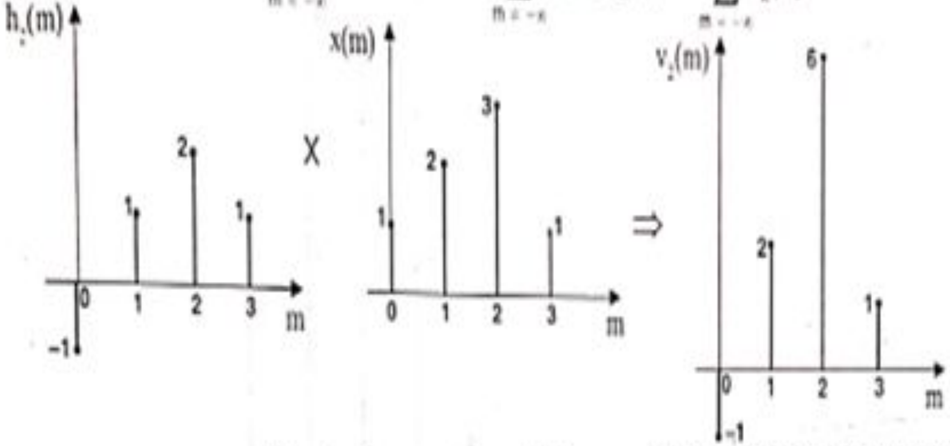


Fig 7 : Computation of y(2).

The sum of product sequence $v_2(m)$ gives $y(2)$. $\therefore$ $y(2) = -1 + 2 + 6 + 1 = 8$

When $n = 3$; $y(3) = \sum_{m=-\infty}^{+\infty} x(m)\, h(3-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_3(m) = \sum_{m=-\infty}^{+\infty} v_3(m)$

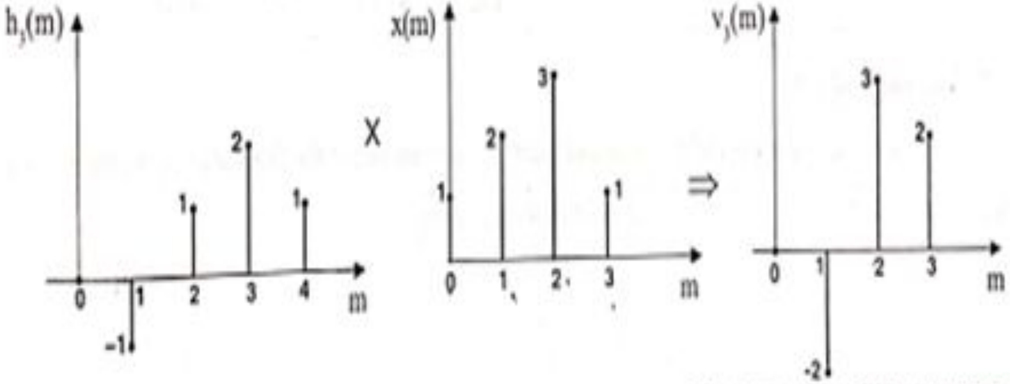


Fig 8 : Computation of y(3).

The sum of product sequence $v_3(m)$ gives $y(3)$. $\therefore$ $y(3) = -2 + 3 + 2 = 3$

When $n = 4$; $y(4) = \sum_{m=-\infty}^{+\infty} x(m)\, h(4-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_4(m) = \sum_{m=-\infty}^{+\infty} v_4(m)$

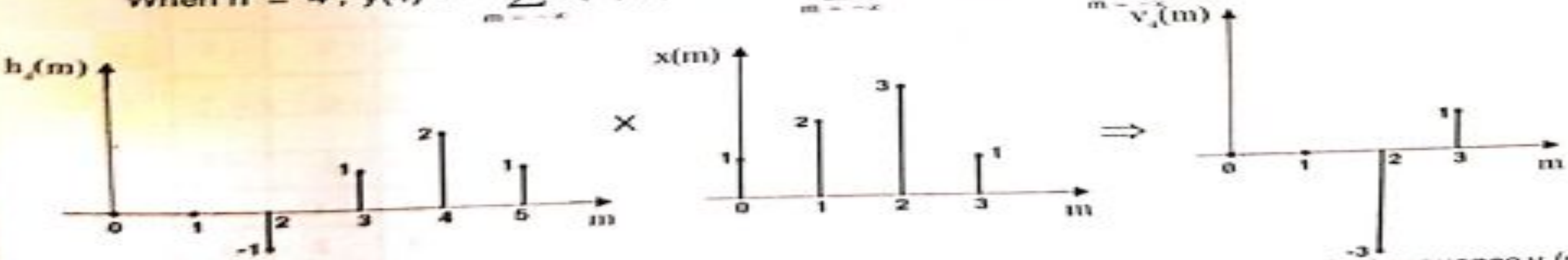
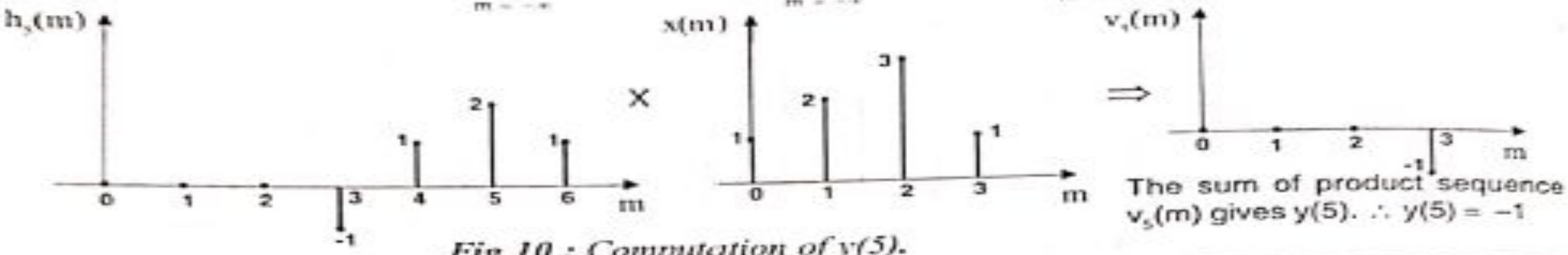


Fig 9 : Computation of y(4).

The sum of product sequence $v_4(m)$ gives $y(4)$. $\therefore\ y(4) = -3 + 1 = -2$

When $n = 5$; $y(5) = \sum_{m=-\infty}^{+\infty} x(m)\, h(5-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_5(m) = \sum_{m=-\infty}^{+\infty} v_5(m)$



The sum of product sequence $v_5(m)$ gives $y(5)$. $\therefore\ y(5) = -1$

Fig 10 : Computation of y(5).

The output sequence, $y(n) = \{1,\ \underset{\uparrow}{4},\ 8,\ 8,\ 3,\ -2,\ -1\}$

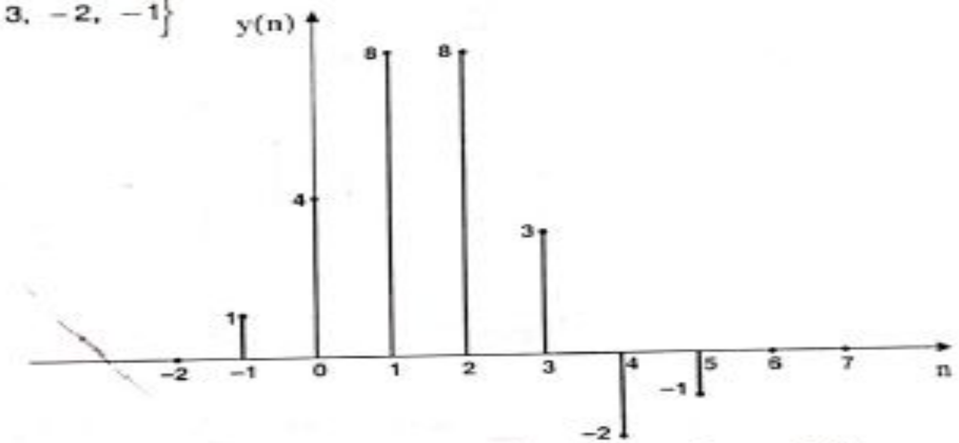


Fig 11 : Graphical representation of y(n).

Method - 2 : Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

Note : The unfilled boxes in the table are considered as zeros.

m	-3	-2	-1	0	1	2	3	4	5	6
$x(m)$				1	2	3	1			
$h(m)$			1	2	1	-1				
$h(-m)$		-1	1	2	1					
$h(-1-m) = h_{-1}(m)$	-1	1	2	1						
$h(0-m) = h_0(m)$		-1	1	2	1					
$h(1-m) = h_1(m)$			-1	1	2	1				
$h(2-m) = h_2(m)$				-1	1	2	1			
$h(3-m) = h_3(m)$					-1	1	2	1		
$h(4-m) = h_4(m)$						-1	1	2	1	
$h(5-m) = h_5(m)$							-1	1	2	1

Each sample of y(n) is computed using the convolution formula,

$$y(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{+\infty} x(m)\, h_n(m), \quad \text{where } h_n(m) = h(n-m)$$

To determine a sample of y(n) at n = q, multiply the sequence x(m) and $h_q(m)$ to get a product sequence (i.e., multiply the corresponding elements of the row x(m) and $h_q(m)$). The sum of all the samples of the product sequence gives y(q).

When n = −1 ; $y(-1) = \sum_{m=-3}^{3} x(m)\, h_{-1}(m)$ $\boxed{\because \text{ The product is valid only for } m = -3 \text{ to } +3}$

$$= x(-3)\, h_{-1}(-3) + x(-2)h_{-1}(-2) + x(-1)h_{-1}(-1) + x(0)\, h_{-1}(0) + x(1)\, h_{-1}(1) + x(2)\, h_{-1}(2) + x(3)\, h_{-1}(3)$$

$$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1$$

The samples of y(n) for other values of n are calculated as shown for n = −1.

When n = 0 ; $y(0) = \sum_{m=-2}^{3} x(m)\, h_0(m) = 0 + 0 + 2 + 2 + 0 + 0 = 4$

When n = 1 ; $y(1) = \sum_{m=-1}^{3} x(m)\, h_1(m) = 0 + 1 + 4 + 3 + 0 = 8$

When n = 2 ; $y(2) = \sum_{m=0}^{3} x(m)\, h_2(m) = -1 + 2 + 6 + 1 = 8$

When n = 3 ; $y(3) = \sum_{m=0}^{4} x(m)\, h_3(m) = 0 - 2 + 3 + 2 + 0 = 3$

When n = 4 ; $y(4) = \sum_{m=0}^{5} x(m)\, h_4(m) = 0 + 0 - 3 + 1 + 0 + 0 = -2$

When n = 5 ; $y(5) = \sum_{m=0}^{6} x(m)\, h_5(m) = 0 + 0 + 0 - 1 + 0 + 0 + 0 = -1$

The output sequence, $y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

Method - 3 : Matrix Method

The input sequence x(n) is arranged as a column and the impulse response is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with the column element. The sum of the diagonal elements gives the samples of y(n).

$x(n) \downarrow$ \ $h(n) \rightarrow$	1	2	1	−1
1	1×1	1×2	1×1	$1 \times (-1)$
2	2×1	2×2	2×1	$2 \times (-1)$
3	3×1	3×2	3×1	$3 \times (-1)$
1	1×1	1×2	1×1	$1 \times (-1)$

$\Rightarrow$

$x(n) \downarrow$ \ $h(n) \rightarrow$	1	2	1	1
1	1	2	1	−1
2	2	4	2	−2
3	3	6	3	−3
1	1	2	1	−1

$y(-1) = 1$

$y(0) = 2 + 2 = 4$

$y(1) = 3 + 4 + 1 = 8$

$y(2) = 1 + 6 + 2 + (-1) = 8$

$y(3) = 2 + 3 + (-2) = 3$

$y(4) = 1 + (-3) = -2$

$y(5) = -1$

$\therefore\ y(n) = \{1, \underset{\uparrow}{4}, 8, 8, 3, -2, -1\}$

EX.2

Determine the output y(n) of a relaxed LTI system with impulse response,

$$h(n) = a^n u(n) \; ; \text{ where } |a| < 1 \text{ and}$$

When input is a unit step sequence, i.e., $x(n) = u(n)$.

Solution

The graphical representation of x(n) and h(n) after replacing n by m are shown below. Also the sequence x(m) is folded to get x(−m)

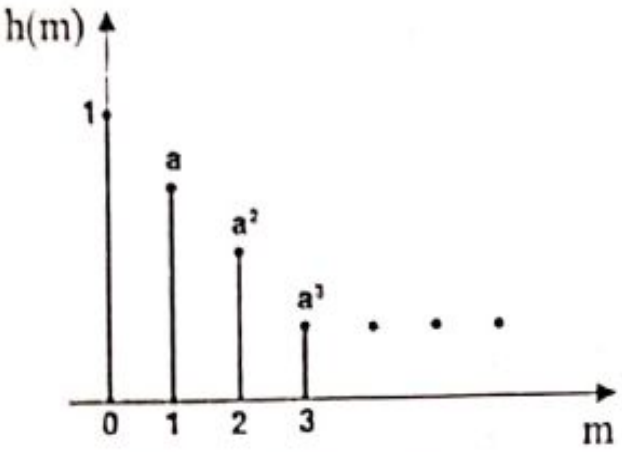


Fig 1 : Impulse response.

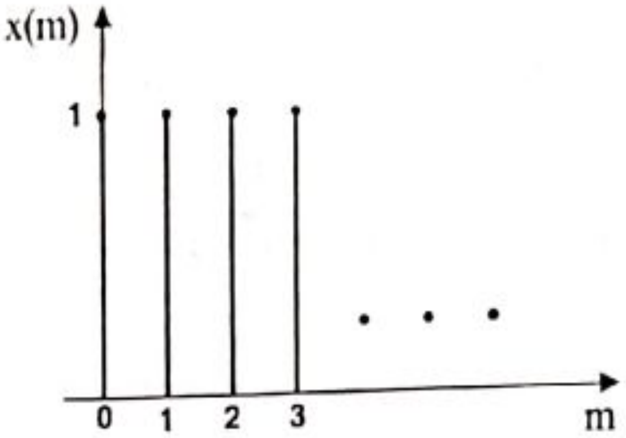


Fig 2 : Input sequence.

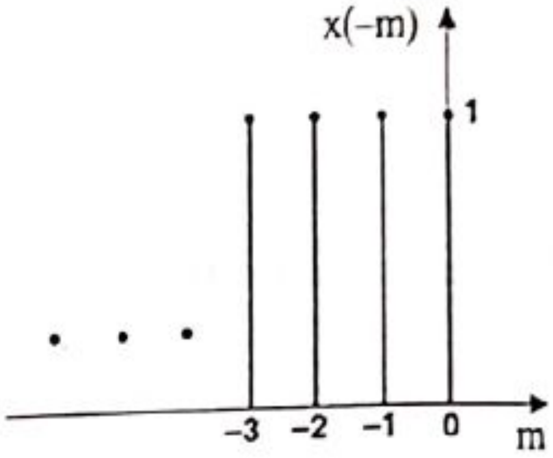


Fig 3 : Folded input sequence.

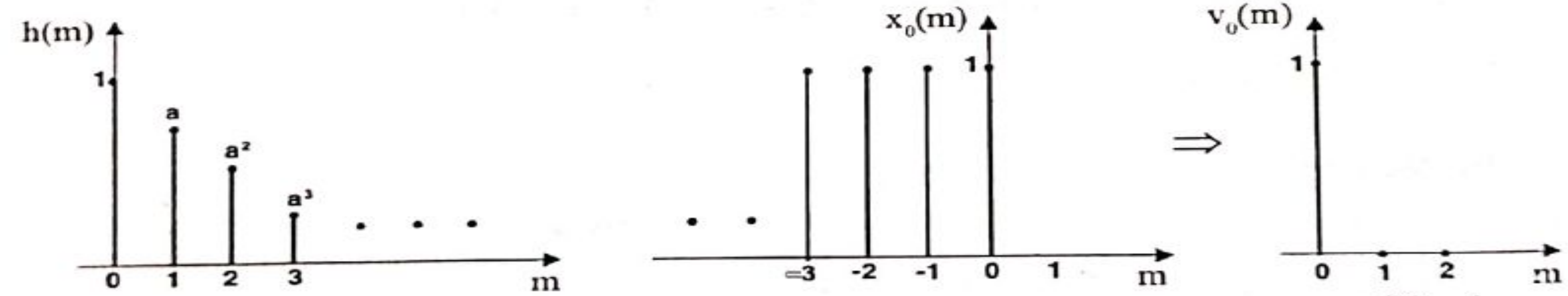
Here both h(m) and x(m) are infinite duration sequences starting at $n = 0$. Hence the output sequence y(n) will also be an infinite duration sequence starting at $n = 0$

By convolution formula,

$$y(n) = \sum_{m=-\infty}^{\infty} h(m)\, x(n-m) = \sum_{m=0}^{\infty} h(m)\, x_n(m) \; ; \quad \text{where } x_n(m) = x(n-m)$$

The computation of some samples of y(n) using the above equation are graphically shown below.

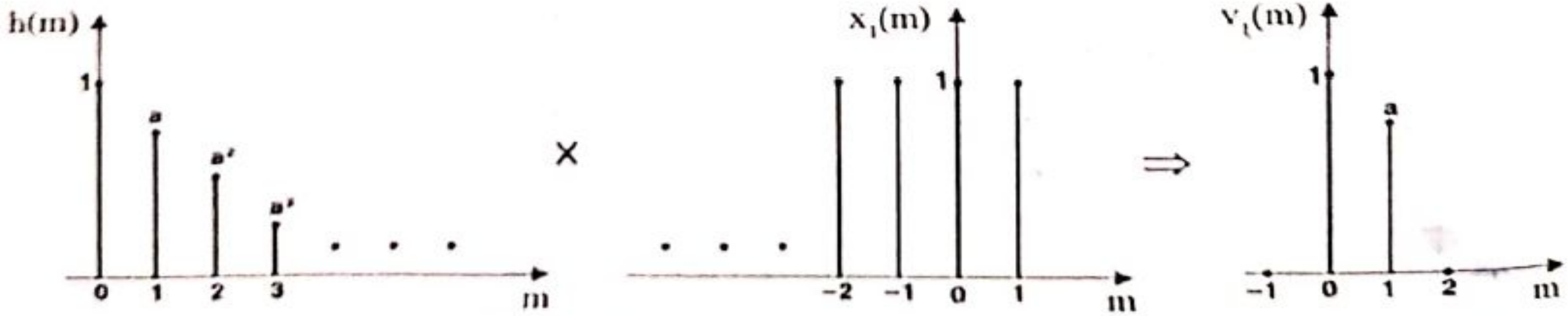
$$\text{When } n = 0\ ;\ y(0) = \sum_{m=0}^{\infty} h(m)\, x(m) = \sum_{m=0}^{\infty} h(m)\, x_0(m) = \sum_{m=0}^{\infty} v_0(m)$$



y(0) = 1

Fig 4 : Computation of y(0).

$$\text{When } n = 1\ ;\ y(1) = \sum_{m=0}^{\infty} h(m)\, x(1-m) = \sum_{m=0}^{\infty} h(m)\, x_1(m) = \sum_{m=0}^{\infty} v_1(m)$$



y(1) = 1 + a

Fig 5 : Computation of y(1).

When n = 2 ; $y(2) = \sum_{m=0}^{\infty} h(m)\, x(2-m) = \sum_{m=0}^{\infty} h(m)\, x_2(m) = \sum_{m=0}^{\infty} v_2(m)$

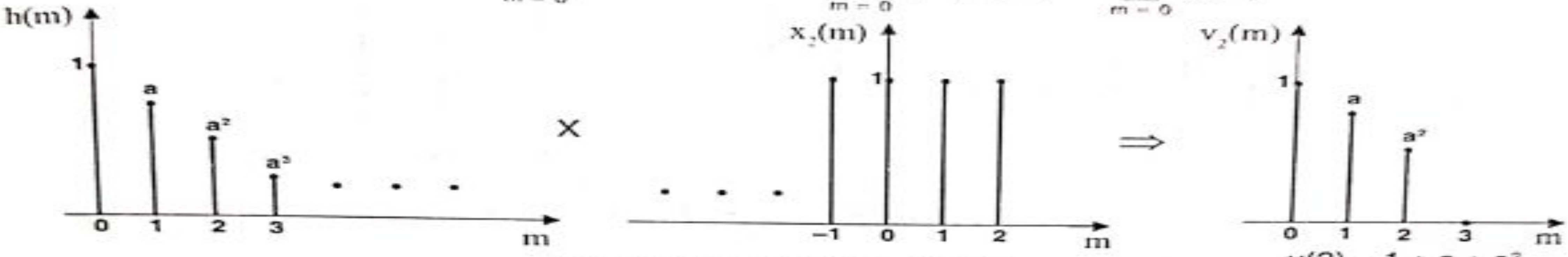


Fig 6 : Computation of y(2).

Solving similarly for other values of n, we can write y(n) for any value of n as shown below.

$$y(n) = 1 + a + a^2 + \ldots\ldots + a^n = \sum_{p=0}^{n} a^p \quad ; \text{ for } n \geq 0$$

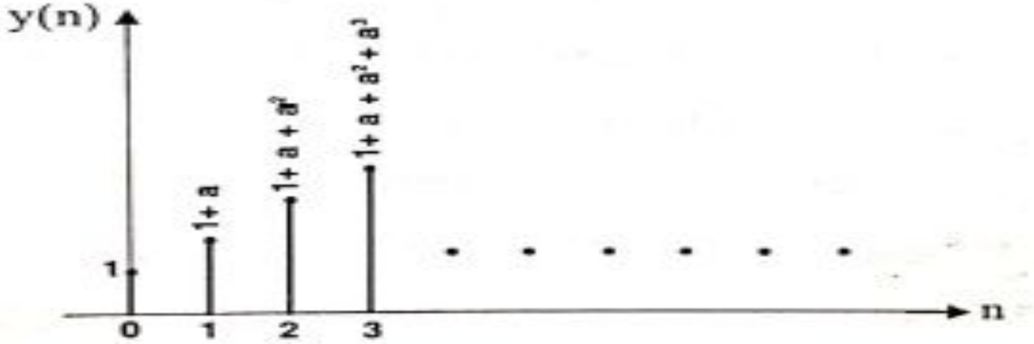


Fig 7 : Graphical representation of y(n).

EX.3

Find the linear and circular convolution of the sequences, $x(n) = \{\underset{\uparrow}{1}, 0.5\}$ and $h(n) = \{\underset{\uparrow}{0.5}, 1\}$.

Solution

Linear Convolution by Tabular Array

Let, $y(n) = x(n) * h(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)$; where m is a dummy variable for convolution.

Since both x(n) and h(n) starts at n = 0, the output sequence y(n) will also start at n = 0.

The length of y(n) is $2 + 2 - 1 = 3$.

Let us change the index n to m in x(n) and h(n). The sequences x(m) and h(m) are represented in the tabular arr as shown below.

Note : *The unfilled boxes in the table are considered as zeros.*

m	−1	0	1	2
x(m)		1	0.5	
h(m)		0.5	1	
$h(-m) = h_0(m)$	1	0.5		
$h(1-m) = h_1(m)$		1	0.5	
$h(2-m) = h_2(m)$			1	0.5

Each sample of y(n) is given by the relation,

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m) = \sum_{m=-\infty}^{\infty} x(m)\, h_n(m) \quad ; \quad \text{where } h_n(m) = h(n-m)$$

When $n = 0$; $y(0) = \sum_{m=-\infty}^{\infty} x(m)\, h(-m) = \sum_{m=-1}^{1} x(m)\, h_0(m)$

$$= x(-1)\, h_0(-1) + x(0)\, h_0(0) + x(1)\, h_0(1) = 0 + 0.5 + 0 = 0.5$$

When $n = 1$; $y(1) = \sum_{m=-\infty}^{\infty} x(m)\, h(1-m) = \sum_{m=0}^{1} x(m)\, h_1(m) = 1 + 0.25 = 1.25$

When $n = 2$; $y(2) = \sum_{m=-\infty}^{\infty} x(m)\, h(2-m) = \sum_{m=0}^{2} x(m)\, h_2(m) = 0 + 0.5 + 0 = 0.5$

$$\therefore y(n) = \{\underset{\uparrow}{0.5}, 1.25, 0.5\}$$

Correlation, Crosscorrelation and and Autocorrelation

6.13 Correlation, Crosscorrelation and Autocorrelation

The *correlation* of two discrete time sequences x(n) and y(n) is defined as,

$$\boxed{r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)} \qquad .....(6.65)$$

where $r_{xy}(m)$ is the correlation sequence obtained by correlation of x(n) and y(n) and m is the variable used for time shift. The correlation of two different sequences is called ***crosscorrelation*** and the correlation of a sequence with itself is called ***autocorrelation***. Hence autocorrelation of a discrete time sequence is defined as,

$$\boxed{r_{xx}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, x(n-m)} \qquad .....(6.66)$$

If the sequence x(n) has N_1 samples and sequence y(n) has N_2 samples then the crosscorrelation sequence $r_{xy}(m)$ will be a finite duration sequence consisting of $N_1 + N_2 - 1$ samples. If the sequence x(n) has N samples, then the autocorrelation sequence $r_{xx}(m)$ will be a finite duration sequence consisting of $2N - 1$ samples.

In the equation (6.65) the sequence x(n) is unshifted and the sequence y(n) is shifted by m units of time for correlation operation. The same results can be obtained if the sequence y(n) is unshifted and the sequence x(n) is shifted opposite to that of earlier case by m units of time, hence the crosscorrelation operation can also be expressed as,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n+m)\, y(n) \qquad \text{....(6.67)}$$

6.13.1 Procedure for Evaluating Correlation

Let, $x(n)$ = Discrete time sequence with N_1 samples

$y(n)$ = Discrete time sequence with N_2 samples

Now the correlation of $x(n)$ and $y(n)$ will produce a sequence $r_{xy}(m)$ consisting of N_1+N_2-1 samples. Each sample of $r_{xy}(m)$ can be computed using the equation (6.65). The value of $r_{xy}(m)$ at $m = q$ is obtained by replacing m by q, in equation (6.65).

$$\therefore \; r_{xy}(q) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-q) \qquad .....(6.68)$$

The evaluation of equation (6.68) to determine the value of $r_{xy}(m)$ at $m = q$ involves the following three steps.

1. **Shifting** : Shift $y(n)$ by q times to the right if q is positive, shift $y(n)$ by q times to the left if q is negative to obtain $y(n-q)$.

2. **Multiplication** : Multiply $x(n)$ by $y(n-q)$ to get a product sequence. Let the product sequence be $v_q(n)$. Now, $v_q(n) = x(n) \times y(n-q)$.

3. **Summation** : Sum all the values of the product sequence $v_q(n)$ to obtain the value of $r_{xy}(m)$ at $m = q$. [i.e., $r_{xy}(q)$].

The above procedure will give the value $r_{xy}(m)$ at a single time instant say $m = q$. In general we are interested in evaluating the values of the sequence $r_{xy}(m)$ over all the time instants in the range $-\infty < m < \infty$. Hence the steps 1, 2 and 3 given above must be repeated, for all possible time shifts in the range $-\infty < m < \infty$.

In the correlation of finite duration sequences it is possible to predict the start and end of the resultant sequence. If $x(n)$ is N-point sequence and starts at $n = n_1$ and if $y(n)$ is N_2-point sequence and starts at $n = n_2$ then, the initial value of $m = m_i$ for $r_{xy}(m)$ is $\mathbf{m_i = n_1 - (n_2 + N_2 - 1)}$. The value of $x(n)$ for $n < n_1$ and the value of $y(n)$ for $n < n_2$ are then assumed to be zero. The final value of $m = m_f$ for $r_{xy}(m)$ is $\mathbf{m_f = m_i + (N_1+N_2-2)}$.

The correlation operation involves all the steps in convolution operation except the folding. Hence it can be proved that the convolution of x(n) and folded sequence y(−n) will generate the crosscorrelation sequence $r_{xy}(m)$.

i.e., $r_{xy}(m) = x(n) * y(-n)$ .....(6.69)

The procedure given above can be used for computing autocorrelation of x(n). For computing autocorrelation using equation (6.68) replace y(n − q) by x(n − q). Similarly when equation (6.69) is used, replace y(−n) by x(−n).

The autocorrelation of N-point sequence x(n) will give 2N−1 point autocorrelation sequence. If x(n) starts at $n = n_x$ then initial value of $m = m_i$ for $r_{xx}(m)$ is $m_i = -(N-1)$. The final value of $m = m_f$ for $r_{xx}(m)$ is $m_f = m_i + (2N-2)$.

Properties of Correlation

1. The crosscorrelation sequence $r_{xy}(m)$ is simply folded version of $r_{yx}(m)$.

   i.e., $r_{xy}(m) = r_{yx}(-m)$

   Similarly for autocorrelation sequence,

   $r_{xx}(m) = r_{xx}(-m)$

   Hence autocorrelation is an even function.

2. The crosscorrelation sequence satisfies the condition,

   $$|r_{xy}(m)| \leq \sqrt{r_{xx}(0)\, r_{yy}(0)} = \sqrt{E_x\, E_y}$$

   where, E_x and E_y are energy of x(n) and y(n) respectively.

   On applying the above condition to autocorrelation sequence we get,

   $$|r_{xx}(m)| \leq r_{xx}(0) = E_x$$

   From the above equations we infer that the crosscorrelation sequence and autocorrelation sequences attain their respective maximum values at zero shift/lag.

3. Using the maximum value of crosscorrelation sequence, the normalized crosscorrelation sequence is defined as,

   $$\rho_{xy}(m) \leq \frac{r_{xy}(m)}{\sqrt{r_{xx}(0)\, r_{yy}(0)}}$$

   Using the maximum value of autocorrelation sequence, the normalized autocorrelation sequence is defined as,

   $$\rho_{xx}(m) \leq \frac{r_{xx}(m)}{r_{xx}(0)}$$

Methods of Computing Correlation

Method -1: Graphical Method

Let x(n) and y(n) be the input sequences and $r_{xy}(m)$ be the output sequence.

1. Sketch the graphical representation of the input sequences x(n) and y(n).
2. Shift the sequence y(n) to the left graphically so that the product of x(n) and shifted y(n) gives only one non-zero sample. Now multiply x(n) and shifted y(n) to get a product sequence, and then sum-up the samples of product sequence, which is the first sample of output sequence.
3. To get the next sample of output sequence, shift y(n) of previous step to one position right and multiply the shifted sequence with x(n) to get a product sequence. Now the sum of the samples of product sequence gives the second sample of output sequence.
4. To get subsequent samples of output sequence, the step-3 is repeated until we get a non-zero product sequence.

Method -2: Tabular Method

The tabular method is same as that of graphical method, except that the tabular representation of the sequences are employed instead of graphical representation. In tabular method, every input sequence and shifted sequence is represented on a row in a table.

Method -3: Matrix Method

Let x(n) and y(n) be the input sequences and $r_{xy}(m)$ be the output sequence. We know that the convolution of x(n) and folded sequence y(–n) will generate the crosscorrelation sequence $r_{xy}(m)$. Hence fold y(n) to get y(–n), and compute convolution of x(n) and y(–n) by matrix method.

In matrix method one of the sequence is represented as a row and the other as a column as shown below.

$$
\begin{array}{c|ccccc}
 & \cdots & y(0) & y(-1) & y(-2) & y(-3) \;\cdots \\
\hline
\vdots & & \vdots & \vdots & \vdots & \vdots \\
x(0) & \cdots & x(0)y(0) & x(0)y(-1) & x(0)y(-2) & x(0)y(-3) \;\cdots \\
x(1) & \cdots & x(1)y(0) & x(1)y(-1) & x(1)y(-2) & x(1)y(-3) \;\cdots \\
x(2) & \cdots & x(2)y(0) & x(2)y(-1) & x(2)y(-2) & x(2)y(-3) \;\cdots \\
x(3) & \cdots & x(3)y(0) & x(3)y(-1) & x(3)y(-2) & x(3)y(-3) \;\cdots \\
\vdots & & \vdots & \vdots & \vdots & \vdots
\end{array}
$$

Multiply each column element with row elements and fill up the matrix array.

Now the sum of the diagonal elements gives the samples of output sequence $r_{xy}(m)$. (The sum of the diagonal elements are shown below for reference).

$$
\begin{aligned}
&\vdots \\
r_{xy}(0) &= \ldots + x(0)\,y(0) + \ldots \\
r_{xy}(1) &= \ldots + x(1)\,y(0) + x(0)\,y(-1) + \ldots \\
r_{xy}(2) &= \ldots + x(2)\,y(0) + x(1)\,y(-1) + x(0)\,y(-2) + \ldots \\
r_{xy}(3) &= \ldots + x(3)\,y(0) + x(2)\,y(-1) + x(1)\,y(-2) + x(0)\,y(-3) + \ldots \\
&\vdots
\end{aligned}
$$

Example 6.32

Perform crosscorrelation of the sequences, $x(n) = \{1, 1, 2, 2\}$ and $y(n) = \{1, 3, 1\}$.

Solution

Let $r_{xy}(m)$ be the crosscorrelation sequence obtained by crosscorrelation of x(n) and y(n).

The crosscorrelation sequence $r_{xy}(m)$ is given by,

$$r_{xy} = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)$$

The x(n) starts at n = 0 and has 4 samples.

$$\therefore\ n_1 = 0,\ N_1 = 4$$

The y(n) starts at n = 0 and has 3 samples.

$$\therefore\ n_2 = 0,\ N_2 = 3$$

Now, $r_{xy}(m)$ will have $N_1 + N_2 - 1 = 4 + 3 - 1 = 6$ samples.

The initial value of $m = m_i = n_1 - (n_2 + N_2 - 1)$

$$= 0 - (0 + 3 - 1) = -2$$

The final value of $m = m_f = m_i + (N_1 + N_2 - 2)$

$$= -2 + (4 + 3 - 2) = 3$$

In this example the correlation operation is performed by three methods.

Method-1 : Graphical Method

The graphical representation of x(n) and y(n) are shown below.

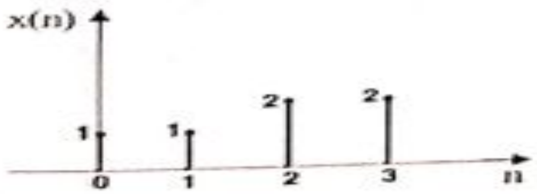


Fig 1.

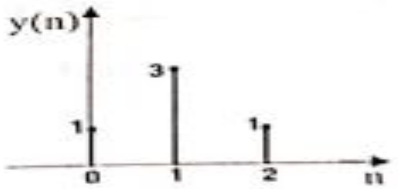


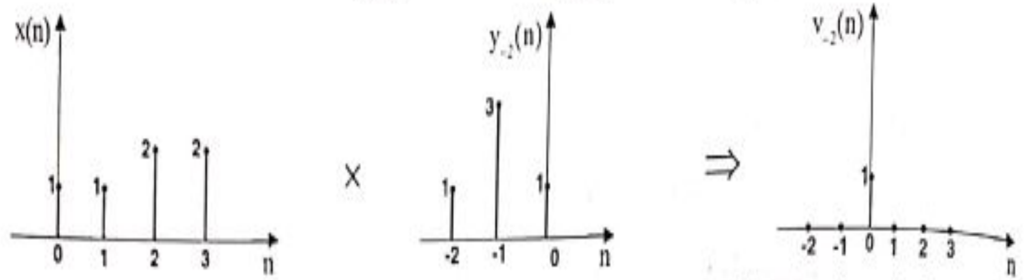
Fig 2.

The 6 samples of $r_{xy}(m)$ are computed using the equation,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, y_m(n)\ ;\ \text{where } y_m(n) = y(n-m)$$

The computation of each sample of $r_{xy}(n)$ using the above equation are graphically shown in fig 3 to fig 8. The graphical representation of output sequence is shown in fig 9.

When m = −2 ; $r_{xy}(-2) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-(-2)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_{-2}(n) = \sum_{n=-\infty}^{+\infty} v_{-2}(n)$

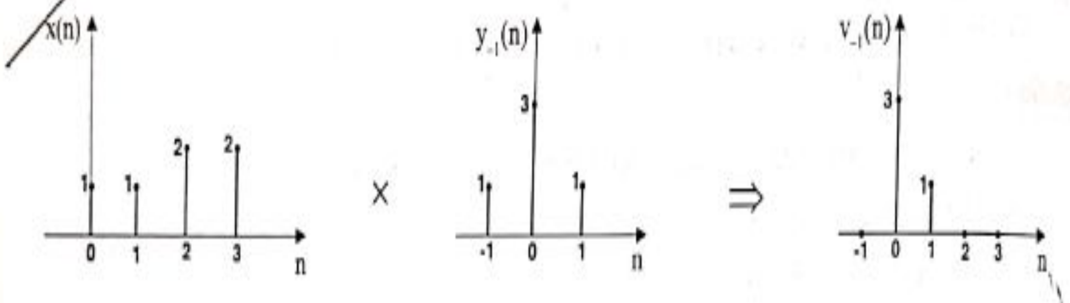


The sum of product sequence $v_{-2}(n)$ gives $r_{xy}(-2)$

$\therefore r_{xy}(-2) = 0 + 0 + 1 + 0 + 0 + 0 = 1$

Fig 3 : Computation of $r_{xy}(-2)$.

When m = −1 ; $r_{xy}(-1) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-(-1)) = \sum_{n=-\infty}^{+\infty} x(n)\, y_{-1}(n) = \sum_{n=-\infty}^{+\infty} v_{-1}(n)$

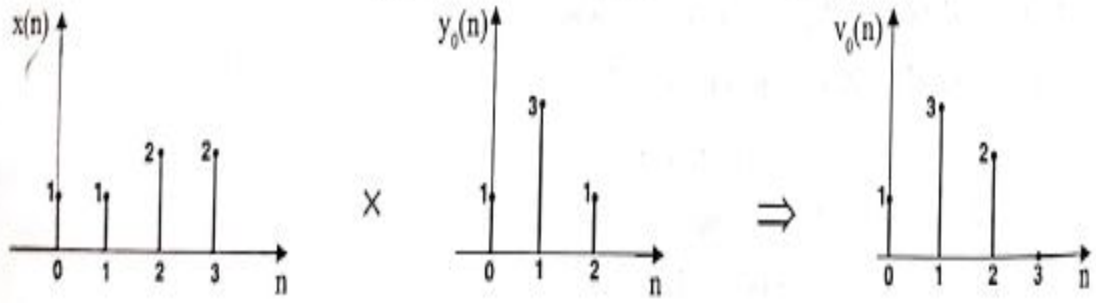


The sum of product sequence $v_{-1}(n)$ gives $r_{xy}(-1)$

$\therefore r_{xy}(-1) = 0 + 3 + 1 + 0 + 0 = 4$

Fig 4 : Computation of $r_{xy}(-1)$.

When m = 0 ; $r_{xy}(0) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n) = \sum_{n=-\infty}^{+\infty} x(n)\, y_0(n) = \sum_{n=-\infty}^{+\infty} v_0(n)$

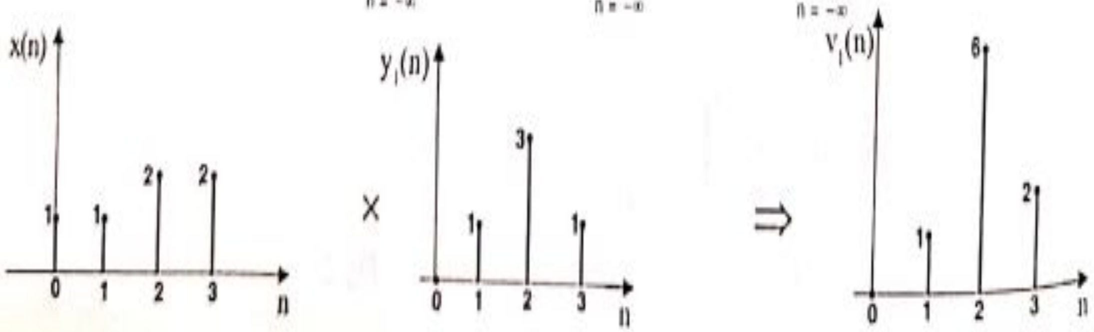


The sum of product sequence $v_0(n)$ gives $r_{xy}(0)$

$\therefore r_{xy}(0) = 1 + 3 + 2 + 0 = 6$

Fig 5 : Computation of $r_{xy}(0)$.

When m = 1 ; $r_{xy}(1) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-1) = \sum_{n=-\infty}^{+\infty} x(n)\, y_1(n) = \sum_{n=-\infty}^{+\infty} v_1(n)$

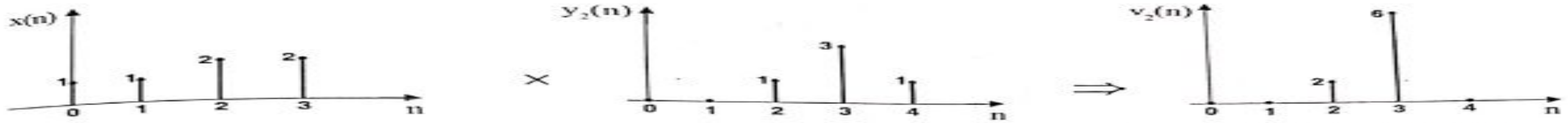


The sum of product sequence $v_1(n)$ gives $r_{xy}(1)$

$\therefore r_{xy}(1) = 0 + 1 + 6 + 2 = 9$

Fig 6 : Computation of $r_{xy}(1)$.

When $m = 2$; $r_{xy}(2) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-2) = \sum_{n=-\infty}^{+\infty} x(n)\, y_2(n) = \sum_{n=-\infty}^{+\infty} v_2(n)$

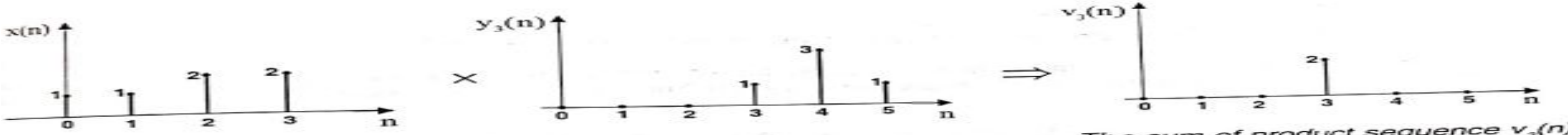


The sum of product sequence $v_2(n)$ gives $r_{xy}(2)$

$\therefore\ r_{xy}(2) = 0 + 0 + 2 + 6 + 0 = 8$

Fig 7 : Computation of $r_{xy}(2)$.

When $m = 3$; $r_{xy}(3) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-3) = \sum_{n=-\infty}^{+\infty} x(n)\, y_3(n) = \sum_{n=-\infty}^{+\infty} v_3(n)$



The sum of product sequence $v_3(n)$ gives $r_{xy}(3)$

$\therefore\ r_{xy}(3) = 0 + 0 + 0 + 2 + 0 + 0 = 2$

Fig 8 : Computation of $r_{xy}(3)$.

The crosscorrelation sequence, $r_{xy}(m) = \{1, 4, \underset{\uparrow}{6}, 9, 8, 2\}$

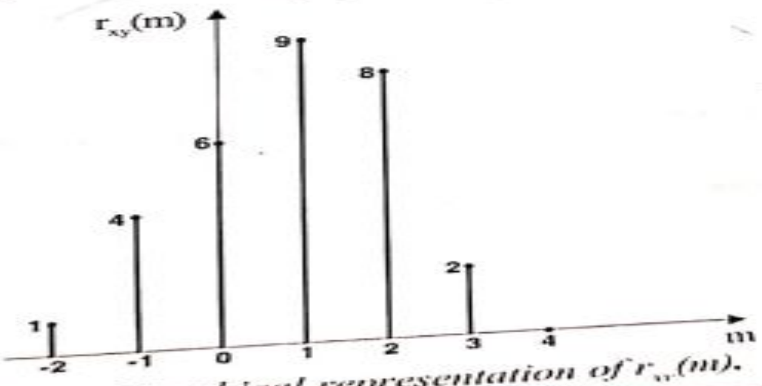


Fig 9 : Graphical representation of $r_{xy}(m)$.

Method - 2: Tabular Method

The given sequences and the shifted sequences can be represented in the tabular array as shown below.

n	−2	−1	0	1	2	3	4	5
$x(n)$			1	1	2	2		
$y(n)$			1	3	1			
$y(n-(-2)) = y_{-2}(n)$	1	3	1					
$y(n-(-1)) = y_{-1}(n)$		1	3	1				
$y(n) = y_0(n)$			1	3	1			
$y(n-1) = y_1(n)$				1	3	1		
$y(n-2) = y_2(n)$					1	3	1	
$y(n-3) = y_3(n)$						1	3	1

***Note:** The unfilled boxes in the table are considered as zeros.*

Each sample of $r_{xy}(m)$ is given by,

$$r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) = \sum_{n=-\infty}^{+\infty} x(n)\, y_m(n)\ ;\ \text{where } y_m(n) = y(n-m)$$

To determine a sample of $r_{xy}(m)$ at $m = q$, multiply the sequence $x(n)$ and $y_q(n)$ to get a product sequence (i.e., multiply the corresponding elements of the row $x(n)$ and $y_q(n)$). The sum of all the samples of the product sequence gives $r_{xy}(q)$.

$$\text{When } m = -2\ ;\ r_{xy}(-2) = \sum_{n=-2}^{3} x(n)\, y_{-2}(n) = 0 + 0 + 1 + 0 + 0 + 0 = 1$$

$$\text{When } m = -1\ ;\ r_{xy}(-1) = \sum_{n=-1}^{3} x(n)\, y_{-1}(n) = 0 + 3 + 1 + 0 + 0 = 4$$

$$\text{When } m = 0\ ;\ r_{xy}(0) = \sum_{n=0}^{3} x(n)\, y_0(n) = 1 + 3 + 2 + 0 = 6$$

$$\text{When } m = 1\ ;\ r_{xy}(1) = \sum_{n=0}^{3} x(n)\, y_1(n) = 1 + 6 + 2 + 0 = 9$$

$$\text{When } m = 2\ ;\ r_{xy}(2) = \sum_{n=0}^{4} x(n)\, y_2(n) = 0 + 0 + 2 + 6 + 0 = 8$$

$$\text{When } m = 3\ ;\ r_{xy}(3) = \sum_{n=0}^{5} x(n)\, y_3(n) = 0 + 0 + 0 + 2 + 0 + 0 = 2$$

$$\therefore \text{Crosscorrelation sequence, } r_{xy}(m) = \{1,\ 4,\ \underset{\uparrow}{6},\ 9,\ 8,\ 2\}$$

Method - 3: Matrix Method

Given that, $x(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$

$$y(n) = \{\underset{\uparrow}{1}, 3, 1\}$$

$$y(-n) = \{1, 3, \underset{\uparrow}{1}\}$$

The sequence $x(n)$ is arranged as a column and the folded sequence $y(-n)$ is arranged as a row as shown below. The elements of the two dimensional array are obtained by multiplying the corresponding row element with column element. The sum of the diagonal elements gives the samples of the crosscorrelation sequence, $r_{xy}(m)$.

$$\begin{array}{c|ccc}
{}_{x(n)\downarrow}\ {}^{y(-n)\rightarrow} & 1 & 3 & 1 \\
\hline
1 & 1\times 1 & 1\times 3 & 1\times 1 \\
1 & 1\times 1 & 1\times 3 & 1\times 1 \\
2 & 2\times 1 & 2\times 3 & 2\times 1 \\
2 & 2\times 1 & 2\times 3 & 2\times 1
\end{array}
\quad \Rightarrow \quad
\begin{array}{c|ccc}
{}_{x(n)\downarrow}\ {}^{y(-n)\rightarrow} & 1 & 3 & 1 \\
\hline
1 & 1 & 3 & 1 \\
1 & 1 & 3 & 1 \\
2 & 2 & 6 & 2 \\
2 & 2 & 6 & 2
\end{array}$$

$r_{xy}(-2) = 1$; $r_{xy}(-1) = 1 + 3 = 4$; $r_{xy}(0) = 2 + 3 + 1 = 6$

$r_{xy}(1) = 2 + 6 + 1 = 9$; $r_{xy}(2) = 6 + 2 = 8$; $r_{xy}(3) = 2$

$\therefore r_{xy}(m) = \{1, 4, \underset{\uparrow}{6}, 9, 8, 2\}$.