* Vector space :-

Let V be the non-empty set and F be the field then V is said to be a vector space if it satisfy following axioms for any $u, v, w \in V$ and for any $k, l \in F$

① closure axioms

C_1 : $u + v \in V$

C_2 : $ku \in V$

② Addition axioms

A_1 : $u + v = v + u$

A_2 : $(u + v) + w = u + (v + w)$

A_3 : There exist an element 0 in V such that $0 + u = u$

(Existance of additive identity)

A_4 : for any $u \in V$, there exist $-u \in V$ such that $u + (-u) = 0$

(Existance of additive inverse)

③ scalar Multiplication axioms:

M_1 : $k(u + v) = ku + kv$

M_2 . $(k + l)u = ku + lu$

M_3 : $(kl)u = k(lu)$

M_4 : $1 \cdot u = u$

Examples on vector space :

Ex. ① Check whether the set of all pairs of real numbers of the form $(1, x)$ with operations $(1, y) + (1, y') = (1, y+y')$ and $k(1, y) = (1, ky)$

Solution: Let $u = (1, x)$, $v = (1, y)$, $w = (1, z)$ be any element and $k, l \in F$ be any elements

① C_1: $u + v = (1, x) + (1, y) = (1, x+y) \in V$

C_2: $ku = k(1, x) = (1, ky) \in V$

$\therefore$ V is closed under addition and multiplication.

② A_1: $u + v = (1, x) + (1, y) = (1, x+y)$

$= (1, y+x)$

$= (1, y) + (1, x)$

$= v + u$

A_2: $(u+v) + w = [(1, x) + (1, y)] + (1, z)$

$= (1, x+y) + (1, z)$

$= (1, x+y+z)$ —— ⓐ

$u + (v+w) = (1, x) + [(1, y) + (1, z)]$

$= (1, x) + (1, y+z)$

$= (1, x+y+z)$ —— ⓑ

$\therefore (u + v + w) = u + (v + w)$ ($\because$ from ⓐ & ⓑ)

A_3: $u + 0 = (1, x) + (1, 0)$

$= (1, x+0) = (1, x) = u$

$\therefore 0 = (1, 0)$ is additive identity.

A_4: $u + (-u) = (1, x) + (1, -x) = (1, x + (-x)) = (1, 0)$

$\therefore -u = (1, -x)$ is additive inverse of u

③ M_1 : $k(u+v) = k[(1,x)+(1,y)] = k(1, x+y)$

$= (1, k(x+y)) = (1, kx+ky)$

$= (1, kx) + (1, ky) = k(1,x) + k(1,y)$

$= ku + kv$

M_2 : $(k+l)u = (k+l)(1,x) = (1, (k+l)x)$

$= (1, kx+lx) = (1,kx) + (1,lx)$

$= k(1,x) + l(1,x)$

$= ku + lu$

M_3 : $(kl)u = kl(1,x)$

$= (1\ klx) = k(1, lx)$

$= k[l(1,x)]$

$= k[lu]$

M_4 : $1.u = 1\cdot(1,x) = (1, 1\cdot x)$

$= (1,x) = u$

therefore, By definition of vector space

Given set of vectors V is a vector space

Homework :

Ex ② Let V be the set of positive real numbers with addition and scalar multiplication defined as $x+y = xy$ and $cx = x^c$, show that V is a vector space

Ex ③ Let $V = R^2$ and define addition and scaler multiplication as

$u = (u_1, u_2)$, $v = (v_1, v_2)$ then $u+v = (u_1+v_1, u_2+v_2)$

and $ku = (ku_1, u_2)$

Check whether V is vector space or not ?

* **Subspace:**

Let V be the vector space and W be the subset of V then W is said to be subspace of V if W itself is a vector space.

Theorem: (Necessary and sufficient conditions for subspace)

If W is non-empty subset of vector space V then W is subspace of V if

1. for any $u, v \in W$, $u+v \in W$
2. for any scaler k and a vector $u \in W$, $ku \in W$

Example 1: show that $W = \{ (x,y) \mid x = 3y \}$ is a subspace of $\mathbb{R}^2$

Solution: let $u = (x_1, y_1)$, $v = (x_2, y_2) \in W$ be any elements and k be any scaler

$\Longrightarrow \quad x_1 = 3y_1$ and $x_2 = 3y_2$

① $u+v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Note that $x_1 + x_2 = 3y_1 + 3y_2 = 3(y_1 + y_2)$

$\Rightarrow \quad u+v \in W$

② $ku = k(x_1, y_1) = (kx_1, ky_1)$

Note that $kx_1 = k(3y_1) = 3(ky_1) =$

$\Rightarrow \quad ku \in W$

$\therefore$ By Necessary and sufficient condition of subspace W is a subspace of $\mathbb{R}^2$

Example ② Show that a lines passing through origin in R^2 is a Subspace of R^2

Solution: Note that the lines passing through origin is of the form $y = mx$

$\therefore W = \{ (x, y) \in R^2 \mid y = mx \}$, m is fixed

To show: W is a subspace of R^2

let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ be any element in W and k be the scalar.

$\Rightarrow$ $y_1 = mx_1$ and $y_2 = mx_2$

① $u + v = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$

Clearly, $y_1 + y_2 = mx_1 + mx_2 = m(x_1 + x_2)$

$\Rightarrow$ $y_1 + y_2 = m(x_1 + x_2)$

$\Rightarrow$ $u + v \in W$

② $ku = k(x_1, y_1) = (kx_1, ky_1)$

and $ky_1 = k(mx_1) = m(kx_1)$

i.e. $ky_1 = m(kx_1)$

$\Rightarrow$ $ku \in W$.

$\therefore$ By Necessary and sufficient condition of subspace,

W is Subspace of R^2

$\Rightarrow$ A lines passing through origin in R^2 is a subspace of R^2