Ex ⑤ Find the extremal of the function

$$\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\, dy \quad \text{with} \quad y(0) = 0,\ y\left(\frac{\pi}{2}\right) = 0$$

solution: here, $F = y'^2 - y^2 + 2xy$

clearly, F contains x, y, y'

∴ The Euler Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{——— ①}$$

since, $F = y'^2 - y^2 + 2xy$

$$\Rightarrow \frac{\partial F}{\partial y} = -2y + 2x \quad \text{and} \quad \frac{\partial F}{\partial y'} = 2y'$$

equation ① becomes

$$-2y + 2x - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow -2y + 2x - 2y'' = 0$$

$$\Rightarrow y'' + y - x = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = x$$

The Auxillary equation is

$$D^2 + 1 = 0$$

$$\Rightarrow D = i, -i$$

∴ The C.F is $y_c = c_1 \cos x + c_2 \sin x$

Now, $P.I = y_p = \frac{1}{f(D)} x = \frac{1}{D^2+1} x$

$$= \frac{1}{1+D^2} x = [1 - D^2 + D^4 - D^6 + \cdots] x$$

$$= [x - D^2 x + D^4 x - \cdots]$$

$\therefore\ y_p = x - 0 + 0 \cdots = x$

$\therefore$ The complete solution is

$$y = y_c + y_p$$

$$\Rightarrow \quad y(x) = c_1 \cos x + c_2 \sin x + x$$

But $y(0) = 0$ and $y(\frac{\pi}{2}) = 0$

$$0 = c_1 \cos 0 + c_2 \sin 0 + 0 \Rightarrow c_1 = 0$$

$$0 = c_1 \cos \frac{\pi}{2} + c_2 \sin \frac{\pi}{2} + \frac{\pi}{2} \Rightarrow c_2 + \frac{\pi}{2} = 0$$

$$\Rightarrow c_2 = -\frac{\pi}{2}$$

$\therefore$ The required solution is

$$y(x) = -\frac{\pi}{2} \sin x + x$$

i.e $\boxed{y(x) = x - \frac{\pi}{2} \sin x}$

HW:

① find the extremals of $\int_{x_1}^{x_2} \frac{\sqrt{1+y'^2}}{x} dx$

$\left(\text{Ans: } y = -\frac{\sqrt{1-c^2x^2}}{c} + c_1\right)$

② Using the relation that the length of the arc between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $s = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,.\,dx$ show that the shortest smooth plane curve between two points on a plane is straight line.

③ find the extremal of $\int_{x_1}^{x_2} [y^2 - y'^2 - 2y\cosh x]\, dx$

(Ans: $y = c_1 \cos x + c_2 \sin x + \frac{1}{2} \cosh x$]

* Isoperimetric problem :

suppose we required to make the given integral

$$\int_{x_1}^{x_2} F(x, y, y')\, dx$$

maximum or minimum under a condition that another integral, say $\int_{x_1}^{x_2} G(x, y, y')\, dx$

is equal to a constant

such types of problem are called as isoperimetric problem

* Lagrange's method:

let λ be the Lagranges mult pl

and $H = F + \lambda G$

therefore, the functional $\int_{x_1}^{x_2} H(x, y, y')\, dx$

can be solve using Eulers equation

$$\frac{\partial H}{\partial y} - \frac{d}{dx}\left(\frac{\partial H}{\partial y'}\right) = 0$$

or $$\frac{\partial H}{\partial x} - \frac{d}{dx}\left[F - y'\frac{\partial H}{\partial y'}\right] = 0$$

Ex ① show that the extremal of the isoperimetric problem $I[y(x)] = \int_{x_1}^{x_2} y'^2 \, dx$ subject to the condition $\int_{x_1}^{x_2} y \, dx = k$ is a parabola.

solution: here, $F = y'^2$ and $G = y$

$\therefore \quad H = F + \lambda G = y'^2 + \lambda y$

where λ is Lagranges multiplies

Now consider the functional,

$$\int_{x_1}^{x_2} H(x, y, y') \, dx = \int_{x_1}^{x_2} (y'^2 + \lambda y) \, dx$$

clearly, H does not contain x explicitely.

$\therefore$ The Euler Lagranges equation is

$$H - y' \frac{\partial H}{\partial y'} = c \qquad \text{——— ①}$$

here, $H = y'^2 + \lambda y \Rightarrow \frac{\partial H}{\partial y'} = 2y'$

$\therefore$ equation ① becomes

$$y'^2 + \lambda y - y'(2y') = c$$

$$\Rightarrow \quad y'^2 + \lambda y - 2y'^2 = c$$

$$\Rightarrow \quad -y'^2 + \lambda y = c$$

$$\Rightarrow \quad y'^2 - \lambda y = -c = c_1$$

$$\Rightarrow \quad y'^2 = c_1 + \lambda y$$

$$\Rightarrow \quad y' = \sqrt{c_1 + \lambda y}$$

i.e. $$\frac{dy}{dx} = \sqrt{c_1+\lambda y}$$

$$\Rightarrow \quad \frac{1}{\sqrt{c_1+\lambda y}}\, dy = dx$$

Integrating both side we get

$$\int \frac{1}{\sqrt{c_1+\lambda y}}\, dy = \int dx + c_2$$

$$\int (c_1+\lambda y)^{-\frac{1}{2}}\, dy = \int dx + c_2$$

$$\Rightarrow \quad \frac{(c_1+\lambda y)^{\frac{1}{2}}}{\frac{1}{2}\lambda} = x + c_2$$

$$\Rightarrow \quad \frac{2}{\lambda}(c_1+\lambda y)^{\frac{1}{2}} = x + c_2$$

$$\Rightarrow \quad \frac{4}{\lambda^2}(c_1+\lambda y) = (x+c_2)^2$$

$$\Rightarrow \quad c_1+\lambda y = \frac{\lambda^2}{4}(x+c_2)^2$$

$$\Rightarrow \quad c_1+\lambda y = \frac{\lambda^2}{4}(x^2+2xc_2+c_2^2)$$

$$\Rightarrow \quad c_1+\lambda y = \frac{\lambda^2 x^2}{4} + \frac{\lambda^2}{2}xc_2 + \frac{\lambda^2 c_2^2}{4}$$

$$\Rightarrow \quad \lambda y = \frac{\lambda^2}{4}x^2 + \frac{\lambda^2}{2}xc_2 + \frac{\lambda^2}{4}c_2^2 - c_1$$

$$\Rightarrow \quad y(x) = \frac{\lambda}{4}x^2 + \frac{\lambda c_2}{2}x + \left(\frac{\lambda}{4}c_2^2 - \frac{c_1}{\lambda}\right)$$

$$\Rightarrow \quad \boxed{y(x) = \frac{\lambda}{4}x^2 + c_3 x + c_4}$$

is required equation of parabola

Ex ② Find the curve $y = f(x)$ for which $\int_0^{\pi} (y'^2 - y^2)\,dx$ is extremum if $\int_0^{\pi} y\,dx = 1$

Solution: here, $F = y'^2 - y^2$ and $G = y$

$\therefore \quad H = F + \lambda G = (y'^2 - y^2) + \lambda y$

where, λ is lagranges multiplier

Now consider the extremal $\int_{x_1}^{x_2} H(x, y, y')\,dx$

clearly, H does not contains x variable

$\therefore$ The Eulers lagranges equation is

$$H - y' \frac{\partial H}{\partial y'} = c \quad \text{——— ①}$$

here, $H = y'^2 - y^2 + \lambda y \Rightarrow \frac{\partial H}{\partial y'} = 2y'$

$\therefore$ equation ① becomes

$$y'^2 - y^2 + \lambda y - y'(2y') = c$$

$$\Rightarrow \quad -y'^2 - y^2 + \lambda y = c$$

$$\Rightarrow \quad y'^2 + y^2 - \lambda y = -c = c_1$$

$$\Rightarrow \quad y'^2 = c_1 - y^2 + \lambda y$$

$$\Rightarrow \quad y' = \sqrt{c_1 - y^2 + \lambda y}$$

i.e $$\frac{dy}{dx} = \sqrt{c_1 - y^2 + \lambda y}$$

$$\Rightarrow \quad \frac{dy}{\sqrt{c_1 - y^2 + \lambda y}} = dx$$

$$\Rightarrow \quad \frac{dy}{\sqrt{c_1 - \left(y^2 - \lambda y + \frac{\lambda^2}{4}\right) + \frac{\lambda^2}{4}}} = dx$$

$$\Rightarrow \quad \frac{dy}{\sqrt{\left(c_1 + \frac{\lambda^2}{4}\right) - \left(y - \frac{\lambda}{2}\right)^2}} = dx$$

Integrating both side, we get

$$\int \frac{1}{\sqrt{\left(c_1 + \frac{\lambda^2}{4}\right) - \left(y - \frac{\lambda}{2}\right)^2}}\, dy = \int dx + c_2$$

$$\Rightarrow \quad \sin^{-1}\left(\frac{\left(y - \frac{\lambda}{2}\right)}{\sqrt{c_1 + \frac{\lambda^2}{4}}}\right) = x + c_2$$

$$\Rightarrow \quad \frac{y - \frac{\lambda}{2}}{\sqrt{c_1 + \frac{\lambda^2}{4}}} = \sin(x + c_2)$$

$$\Rightarrow \quad y - \frac{\lambda}{2} = \sqrt{c_1 + \frac{\lambda^2}{4}} \cdot \sin(x + c_2)$$

$$\Rightarrow \quad y(x) = \frac{\lambda}{2} + \sqrt{\frac{\lambda^2}{4} + c_1} \cdot \sin(x + c_2)$$

is required solution

H.W: find the curve $y = f(x)$ for which

$\int_{x_1}^{x_2} y\sqrt{1 + y'^2}\, dx$ is minimum subject to the

constraint $\int_{x_1}^{x_2} \sqrt{1 + y'^2} \cdot dx = l$

(Ans: $y = c_1 \cosh\left(\frac{x + c_2}{c_1}\right) - \lambda$)