Example ③ Expand $\dfrac{z^2-1}{z^2+5z+6}$ around $z=0$

Solution: let $f(z) = \dfrac{z^2-1}{z^2+5z+6}$

here, degree of the numerator is not less than degree of denominator

$\therefore$ we divide the numerator by denominator.

$$\therefore \quad f(z) = 1 - \frac{5z+7}{z^2+5z+6}$$

$$\begin{array}{r} 1 \\ z^2-1 \overline{\big) z^2+5z+6} \\ \underset{-\ \ \ +}{z^2-1} \\ \hline 5z+7 \end{array}$$

$$= 1 - \frac{5z+7}{(z+3)(z+2)}$$

$$= 1 + \frac{(-5z-7)}{(z+3)(z+2)}$$

Now let $\dfrac{-5z-7}{(z+3)(z+2)} = \dfrac{A}{z+3} + \dfrac{B}{z+2}$

$$\Rightarrow \quad \frac{-5z-7}{(z+3)(z+2)} = \frac{A(z+2) + B(z+3)}{(z+3)(z+2)}$$

$$\Rightarrow \quad A(z+2) + B(z+3) = -5z-7$$

if $z=-3$ then $A(-3+2) + B(0) = -5(-3)-7$

$\Rightarrow \quad A = -8$

if $z=-2$ then $A(0) + B(-2+3) = -5(-2)-7$

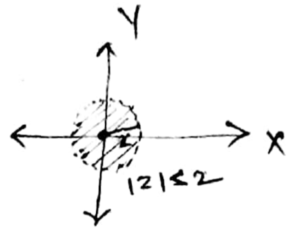
$\Rightarrow \quad B = 3$

$$\therefore \quad \frac{-5z-7}{(z+3)(z+2)} = \frac{-8}{z+3} + \frac{3}{z+2}$$

$$\therefore \quad f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

case i) when $|z| < 2$

$$\therefore \quad f(z) = 1 - \frac{8}{3\left[1+\left(\frac{z}{3}\right)\right]} + \frac{3}{2\left[1+\left(\frac{z}{2}\right)\right]}$$



clearly, $|z| < 2 < 3$

$$\therefore \quad |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1 \quad \text{and}$$

$$|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{8}{3}\left[1+\left(\frac{z}{3}\right)\right]^{-1} + \frac{3}{2}\left[1+\left(\frac{z}{2}\right)\right]^{-1}$$

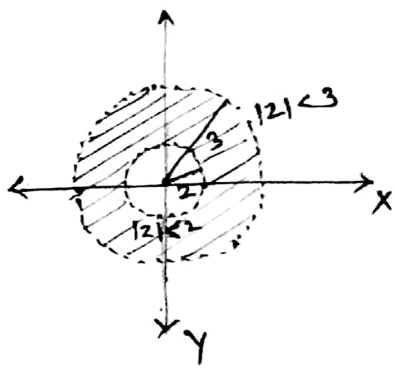
$$= 1 - \frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\cdots\right] + \frac{3}{2}\left[1-\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2-\cdots\right]$$

$$\left(\because (1+z)^{-1} = 1-z+z^2-z^3+\cdots\right)$$

case ii) when $2 < |z| < 3$

i.e. $2 < |z| \Rightarrow \left|\frac{2}{z}\right| < 1$

and $|z| < 3 \Rightarrow \left|\frac{z}{3}\right| < 1$



$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

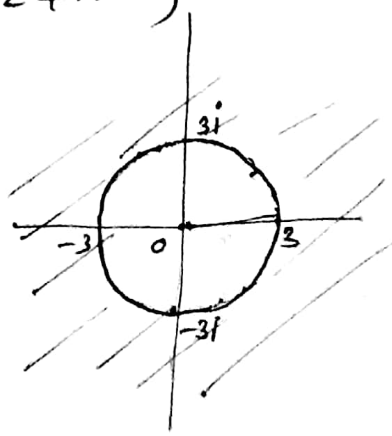
$$= 1 - \frac{8}{3\left[1+\left(\frac{z}{3}\right)\right]} + \frac{3}{z\left[1+\left(\frac{2}{z}\right)\right]}$$

$$= 1 - \frac{8}{3}\left[1+\left(\frac{z}{3}\right)\right]^{-1} + \frac{3}{z}\left[1+\frac{2}{z}\right]^{-1}$$

$$= 1 - \frac{8}{3}\left[1-\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2-\cdots\right] + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2-\cdots\right]$$

$$\left(\because (1+z)^{-1} = 1-z+z^2-z^3+\cdots\right)$$

case iii) when $|z| > 3$



$$f(z) = 1 - \frac{8}{z+3} + \frac{3}{z+2}$$

since, $|z| > 3 > 2 \Rightarrow |z| > 2$

$$\therefore \quad |z| > 3 \Rightarrow \left|\frac{3}{z}\right| < 1$$

$$\text{and} \quad |z| > 2 \Rightarrow \left|\frac{2}{z}\right| < 1$$

$$\therefore \quad f(z) = 1 - \frac{8}{z\left(1+\frac{3}{z}\right)} + \frac{3}{z\left(1+\frac{2}{z}\right)}$$

$$= 1 - \frac{8}{z}\left[1+\left(\frac{3}{z}\right)\right]^{-1} + \frac{3}{z}\left[1+\left(\frac{2}{z}\right)\right]^{-1}$$

$$= 1 - \frac{8}{z}\left[1-\left(\frac{3}{z}\right)+\left(\frac{3}{z}\right)^2 - \cdots\right] + \frac{3}{z}\left[1-\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2 - \cdots\right]$$

Example ④ Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region

i) $1 < |z-1| < 2$ ii) $1 < |z-3| < 2$ iii) $|z| < 1$

Solution: Given: $f(z) = \frac{1}{(z-1)(z-2)}$

Consider,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$$\Rightarrow \quad A(z-2) + B(z-1) = 1$$

if $z = 1 \Rightarrow A(1-2) + B(0) = 1 \Rightarrow A = -1$

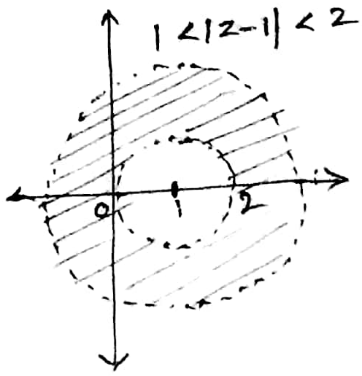
if $z = 2 \Rightarrow A(0) + B(2-1) = 1 \Rightarrow B = 1$

$$\therefore \quad \frac{1}{(z-1)(z-2)} = -\frac{1}{z-1} + \frac{1}{z-2}$$

i) when $1 < |z-1| < 2$

$\therefore \quad 1 < |z-1| \Rightarrow \left|\frac{1}{z-1}\right| < 1$

and $|z-1| < 2 \Rightarrow \left|\frac{z-1}{2}\right| < 1$



$$f(z) = -\frac{1}{z-1} + \frac{1}{(z-1)-1}$$

$$= -\frac{1}{z-1} - \frac{1}{1-(z-1)}$$

$$= -\frac{1}{z-1} - [1-(z-1)]^{-1}$$

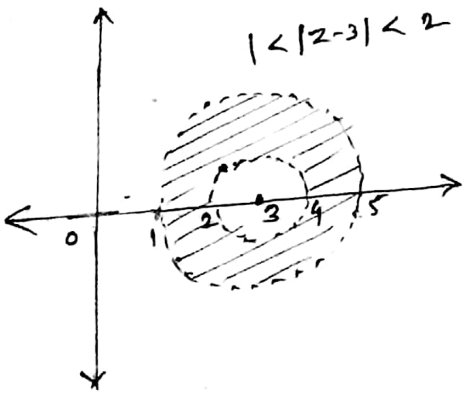
$$\therefore \quad f(z) = -\frac{1}{z-1} - \left[1 + (z-1) + (z-1)^2 + (z+1)^3 + \cdots\right]$$

$$\left(\because (1-z)^{-1} = 1 + z + z^2 + z^3 + \cdots\right)$$

ii) When $1 < |z-3| < 2$

$$\therefore \quad 1 < |z-3| \Rightarrow \left|\frac{1}{z-3}\right| < 1$$

$$\text{and} \quad |z-3| < 2 \Rightarrow \left|\frac{z-3}{2}\right| < 1$$



$$\therefore \quad f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= -\frac{1}{(z-3)+2} + \frac{1}{(z-3)+1}$$

$$= -\frac{1}{2\left[1 + \left(\frac{z-3}{2}\right)\right]} + \frac{1}{(z-3)\left[1 + \left(\frac{1}{z-3}\right)\right]}$$

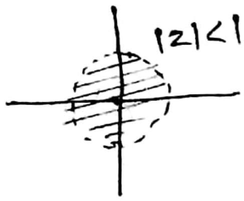
$$= \frac{-1}{2}\left[1 + \left(\frac{z-3}{2}\right)\right]^{-1} + \frac{1}{(z-3)}\left[1 + \left(\frac{1}{z-3}\right)\right]^{-1}$$

$$= -\frac{1}{2}\left[1 - \left(\frac{z-3}{2}\right) + \left(\frac{z-3}{2}\right)^2 - \left(\frac{z-3}{2}\right)^3 + \cdots\right]$$

$$+ \frac{1}{(z-3)}\left[1 - \left(\frac{1}{z-3}\right) + \left(\frac{1}{z-3}\right)^2 - \left(\frac{1}{z-3}\right)^3 + \cdots\right]$$

iii) when $|z| < 1 \Rightarrow |z| < 1 < 2 \Rightarrow |z| < 2 \Rightarrow \left|\frac{z}{2}\right| < 1$

$$\therefore \quad f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$



$$= [1-z]^{-1} + \frac{1}{-2\left[1 - \frac{z}{2}\right]}$$

$$= [1-z]^{-1} - \frac{1}{2}\left[1 - \frac{z}{2}\right]^{-1}$$

$$= \left[1 + z + z^2 + z^3 + \cdots\right] - \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \cdots\right]$$

Homework :

Ex. ① obtain Taylor's or Laurent's series of the function $f(z) = \dfrac{1}{z^2 - 3z + 2}$

when i) $|z| < 1$ ii) $|z| < 2$

Ex ② Expand $f(z) = \dfrac{z^2 - 1}{z^2 + 5z + 6}$ around $z = 1$

Ex. ③ Find all possible Laurent's expansion of the function. $f(z) = \dfrac{7z - 2}{z(z - \text{[illegible]})(z+1)}$ about $z = -1$