Module No. 3

Part-2

Fourier Analysis of Discrete time Signals and Systems

Fourier Transform of Discrete Time Signals (Discrete Time Fourier Transform)

Development of Discrete Time Fourier Transform From Discrete Time Fourier Series

Let $\tilde{x}(n)$ be a periodic sequence with period N. If the period N tends to infinity then the periodic sequence $\tilde{x}(n)$ will become a nonperiodic sequence x(n).

$$\therefore x(n) = \underset{N\to\infty}{\text{Lt}}\ \tilde{x}(n)$$

Let c_k be Fourier coefficients of $\tilde{x}(n)$.

$$\therefore c_k = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}(n)\, e^{\frac{-j2\pi kn}{N}} \qquad \Rightarrow \qquad Nc_k = \sum_{n=0}^{N-1}\tilde{x}(n)\, e^{\frac{-j2\pi kn}{N}}$$

Since $\tilde{x}(n)$ is periodic, for even values of N, the summation index in the above equation can be changed from to $n = -\left(\frac{N}{2}-1\right)$ to $+\frac{N}{2}$. (For odd values of N, the summation index is $n = -\frac{N}{2}$ to $+\frac{N}{2}$).

$$\therefore Nc_k = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\, e^{\frac{-j2\pi kn}{N}} = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}}\tilde{x}(n)\, e^{-j\omega_k n} \qquad .....(8.3)$$

$$\text{where,}\quad \omega_k = \frac{2\pi k}{N}$$

Let us define Nc_k as a function of $e^{j\omega_k}$.

$$\therefore\ X(e^{j\omega_k}) = Nc_k \qquad .....(8.4)$$

Now, using equation (8.3), the equation (8.4) can be expressed as shown below.

$$X(e^{j\omega_k}) = \sum_{n=-\left(\frac{N}{2}-1\right)}^{+\frac{N}{2}} \tilde{x}(n)\, e^{-j\omega_k n} \qquad .....(8.5)$$

Let, $N \rightarrow \infty$, in equation (8.5).

Now, $\tilde{x}(n) \rightarrow x(n)$, $\omega_k \rightarrow \omega$, and the summation index become $-\infty$ to $+\infty$.

Therefore, the equation (8.5) can be written as shown below.

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(8.6)$$

The equation (8.6) is called Fourier transform of x(n), which is used to represent nonperiodic discrete time signal (as a function of frequency,ω) in frequency domain.

Consider the Fourier series representation of $\tilde{x}(n)$ given below.

$$\tilde{x}(n) = \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}}$$

Let us multiply and divide the above equation by $N/2\pi$.

$$\tilde{x}(n) = \frac{N}{2\pi} \times \frac{2\pi}{N} \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}} \quad = \frac{N}{2\pi} \sum_{k=0}^{N-1} c_k \, e^{\frac{j2\pi kn}{N}} \frac{2\pi}{N}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} Nc_k \, e^{\frac{j2\pi kn}{N}} \frac{2\pi}{N}$$

$$\boxed{\omega_k = \frac{2\pi k}{N}}$$

$$\boxed{\text{Using equation (8.4).}}$$

$$= \frac{1}{2\pi} \sum_{k=0}^{N-1} X(e^{j\omega_k}) \, e^{j\omega_k n} \frac{2\pi}{N} \quad .....(8.7)$$

Let, $N \rightarrow \infty$, in equation (8.7).

Now, $\tilde{x}(n) \rightarrow x(n)$, $\omega_k \rightarrow \omega$, $2\pi/N \rightarrow d\omega$, and summation becomes integral with limits 0 to 2π.

Therefore, the equation (8.7) can be written as shown below.

$$x(n) = \frac{1}{2\pi} \int_0^{2\pi} X(e^{j\omega}) \, e^{j\omega n} \, d\omega \quad .....(8.8)$$

The equation (8.8) is called inverse Fourier transform of x(n), which is used to extract the discrete time signal from its frequency domain representation.

Since equation (8.6) extracts the frequency components of discrete time signal, the transformation using equation (8.6) is also called ***analysis*** of discrete time signal x(n). Since equation (8.8) integrates or combines the frequency components of discrete time signal, the inverse transformation using equation (8.8) is also called ***synthesis*** of discrete time signal x(n).

Definition of Discrete Time Fourier Transform

The Fourier transform (FT) of discrete-time signals is called ***Discrete Time Fourier Transform*** (i.e., DTFT). But for convenience the DTFT is also referred as FT in this book.

Let, $x(n)$ = Discrete time signal

$X(e^{j\omega})$ = Fourier transform of $x(n)$

The Fourier transform of a finite energy discrete time signal, x(n) is defined as,

$$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}}$$

Symbolically the Fourier transform of x(n) is denoted as,

$\mathcal{F}\{x(n)\}$

where, $\mathcal{F}$ is the operator that represents Fourier transform.

$$\boxed{\therefore\ X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}}$$

The Fourier transform of a signal is said to exist if it can be expressed in a valid functional form. Since the computation of Fourier transform involves summing infinite number of terms, the Fourier transform exists only for the signals that are absolutely summable, i.e., given a signal $x(n)$, the $X(e^{j\omega})$ exists only when,

$$\sum_{n=-\infty}^{+\infty} |x(n)| < \infty$$

Frequency Spectrum of Discrete Time Signal

The Fourier transform $X(e^{j\omega})$ of a signal x(n) represents the frequency content of x(n). We can say that, by taking Fourier transform, the signal x(n) is decomposed into its frequency components. Hence $X(e^{j\omega})$ is also called ***frequency spectrum*** of discrete time signal or ***signal spectrum***.

Magnitude and Phase Spectrum

The $X(e^{j\omega})$ is a complex valued function of ω, and so it can be expressed in rectangular form as,

$$X(e^{j\omega}) = X_r(e^{j\omega}) + jX_i(e^{j\omega})$$

where, $X_r(e^{j\omega})$ = Real part of $X(e^{j\omega})$

$X_i(e^{j\omega})$ = Imaginary part of $X(e^{j\omega})$

The polar form of $X(e^{j\omega})$ is,

$$X(e^{j\omega}) = |X(e^{j\omega})| \angle X(e^{j\omega})$$

where, $|X(e^{j\omega})|$ = Magnitude spectrum

$\angle X(e^{j\omega})$ = Phase spectrum

The ***magnitude spectrum*** is defined as,

$$|X(e^{j\omega})|^2 = X(e^{j\omega})\ X^*(e^{j\omega})$$

$$= [X_r(e^{j\omega}) + jX_i(e^{j\omega})]\,[X_r(e^{j\omega}) - jX_i(e^{j\omega})]$$

where, $X^*(e^{j\omega})$ is complex conjugate of $X(e^{j\omega})$

Alternatively, $|X(e^{j\omega})|^2 = X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})$

$$\text{or } |X(e^{j\omega})| = \sqrt{X_r^2(e^{j\omega}) + X_i^2(e^{j\omega})}$$

The ***phase spectrum*** is defined as,

$$\angle X(e^{j\omega}) = \mathrm{Arg}[X(e^{j\omega})] = \tan^{-1}\left[\frac{X_i(e^{j\omega})}{X_r(e^{j\omega})}\right]$$

Inverse Discrete Time Fourier Transform

Let, $x(n)$ = Discrete time signal

$X(e^{j\omega})$ = Fourier transform of $x(n)$

The ***inverse discrete time Fourier transform*** of $X(e^{j\omega})$ is defined as,

$$\boxed{x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \quad ; \text{for } n = -\infty \text{ to } +\infty} \qquad .....(8.9)$$

Symbolically the inverse Fourier transform can be expressed as, $\mathcal{F}^{-1}\{X(e^{j\omega})\}$, where, $\mathcal{F}^{-1}$ is the operator that represents the inverse Fourier transform.

$$\therefore\ x(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})\, e^{j\omega n}\, d\omega \ ; \text{ for } n = -\infty \text{ to } +\infty$$

Since $X(e^{j\omega})$ is periodic with period 2π, the limits of integral in the above definition of inverse Fourier transform can be either "$-\pi$ to $+\pi$", or "0 to 2π", or "any interval of 2π".

We also refer to x(n) and $X(e^{j\omega})$ as a Fourier transform pair and this relation is expressed as,

$$\boxed{x(n) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\longleftrightarrow}} X(e^{j\omega})}$$

Comparison of Fourier Transform of Discrete and Continuous Time Signals

1. The Fourier transform of a continuous time signal consists of a spectrum with a frequency range $-\infty$ to $+\infty$. But the Fourier transform of a discrete time signal is unique in the frequency range $-\pi$ to $+\pi$ (or equivalently 0 to 2π). Also Fourier transform of discrete time signal is periodic with period 2π. Hence the frequency range for any discrete-time signal is limited to $-\pi$ to π (or 0 to 2π) and any frequency outside this interval has an equivalent frequency within this interval.

2. Since the continuous time signal is continuous in time the Fourier transform of continuous time signal involves integration but the Fourier transform of discrete time signal involves summation because the signal is discrete.

Properties of Discrete Time Fourier Transform

1. Linearity property

The linearity property of Fourier transform states that the Fourier transform of a linear weighted combination of two or more signals is equal to the similar linear weighted combination of the Fourier transform of the individual signals.

Let $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$ then by linearity property

$$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$; where a_1 and a_2 are constants.

Proof :

By the definition of Fourier transform,

$$X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(8.11)$$

$$X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} \qquad .....(8.12)$$

$$\mathcal{F}\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n) + a_2 x_2(n)\right] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} \left[a_1 x_1(n)\, e^{-j\omega n} + a_2 x_2(n)]\, e^{-j\omega n}\right]$$

$$= \sum_{n=-\infty}^{+\infty} a_1 x_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{+\infty} a_2 x_2(n)\, e^{-j\omega n}$$

$$= a_1 \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} + a_2 \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n}$$

$$= a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$$

Using equations (8.11) and (8.12)

2. Periodicity

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $X(e^{j\omega})$ is periodic with period 2π.

$$\therefore X(e^{j(\omega + 2\pi m)}) = X(e^{j\omega}) \;;\; \text{where m is an integer}$$

Proof :

$$X(e^{j(\omega + 2\pi m)}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega+2\pi m)n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\, e^{-j2\pi mn}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = X(e^{j\omega})$$

Since m and n are integers, $e^{-j2\pi mn} = 1$

3. Time shifting or Fourier transform of delayed signal

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{x(n-m)\} = e^{-j\omega m}\ X(e^{j\omega})$

Also $\mathcal{F}\{x(n + m)\} = e^{j\omega m} X(e^{j\omega})$

This relation means that if a signal is shifted in time domain by m samples, its magnitude spectrum remains unchanged. However, the phase spectrum is changed by an amount $-\omega m$. This result can be explained if we recall that the frequency content of a signal depends only on its shape. Mathematically, we can say that delaying by m units in time domain is equivalent to multiplying the spectrum by $e^{-j\omega m}$ in the frequency domain.

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(8.13)$$

$$\mathcal{F}\{x(n-m)\} = \sum_{n=-\infty}^{+\infty} x(n-m)\, e^{-j\omega n}$$

$\boxed{\begin{array}{l}\text{Let, } n-m = p, \quad \therefore n = p+m \\ \text{when } n \to -\infty, \quad p \to -\infty \\ \text{when } n \to +\infty, \quad p \to +\infty\end{array}}$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega(m+p)}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega m}\; e^{-j\omega p}$$

$$= e^{-j\omega m} \sum_{p=-\infty}^{+\infty} x(p)\, e^{-j\omega p} = e^{-j\omega m} \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$\boxed{\text{Let, } p \to n}$

$$= e^{-j\omega m}\, X(e^{j\omega})$$

$\boxed{\text{Using equation (8.13)}}$

4. Time reversal

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{x(-n)\} = X(e^{-j\omega})$

This means that if a signal is folded about the origin in time, its magnitude spectrum remains unchanged and the phase spectrum undergoes a change in sign (phase reversal).

Proof :

By the definition of Fourier transform,

$$\mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(8.14)$$

$$\mathcal{F}\{x(-n)\} = \sum_{n=-\infty}^{+\infty} x(-n)\, e^{-j\omega n}$$

Let, $p = -n$
when $n \rightarrow -\infty$, $p \rightarrow +\infty$
when $n \rightarrow +\infty$, $p \rightarrow -\infty$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, e^{j\omega p}$$

$$= \sum_{p=-\infty}^{+\infty} x(p)\, (e^{-j\omega})^{-p} \qquad .....(8.15)$$

$$= X(e^{-j\omega})$$

The equation (8.15) is similar to the form of equation (8.14)

5. *Conjugation*

If $\mathcal{F}\{x(n)\} = X(e^{j\omega})$

then $\mathcal{F}\{x^*(n)\} = X^*(e^{-j\omega})$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}$$

$$\mathcal{F}\{x^*(n)\} = \sum_{n=-\infty}^{+\infty} x^*(n)\, e^{-j\omega n}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, (e^{-j\omega})^{-n}\right]^*$$

$$= \left[X(e^{-j\omega})\right]^*$$

$$= X^*(e^{-j\omega})$$

6. Frequency shifting

Let $\mathcal{F}\{x(n)\} = X(e^{j\omega})$, then $\mathcal{F}\{e^{j\omega_0 n}\, x(n)\} = X(e^{j(\omega - \omega_0)})$

According to this property, multiplication of a sequence x(n) by $e^{j\omega_0 n}$ is equivalent to a frequency translation of the spectrum $X(e^{j\omega})$ by ω_0.

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad .....(8.16)$$

$$\therefore\ \mathcal{F}\{e^{j\omega_0 n}\, x(n)\} = \sum_{n=-\infty}^{+\infty} e^{j\omega_0 n}\, x(n)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j(\omega - \omega_0)n} \qquad .....(8.17)$$

$$= X(e^{j(\omega - \omega_0)})$$

The equation (8.17) is similar to the form of equation (8.16)

7. Fourier transform of the product of two signals

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$

$\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

Now, $$\mathcal{F}\left\{x_1(n)\, x_2(n)\right\} = \frac{1}{2\pi}\int_{-\pi}^{-\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda \qquad \text{.....(8.18)}$$

The equation (8.18) is convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$

This relation is the dual of time domain convolution. In other words, the Fourier transform of the product of two discrete time signals is equivalent to the convolution of their Fourier transform. On the other hand, the Fourier transform of the convolution of two discrete time signals is equivalent to the product of their Fourier transform.

Proof :

Let, $x_2(n)\, x_1(n) = x_3(n)$

Now, $\mathcal{F}\{x_2(n)\, x_1(n)\} = \mathcal{F}\{x_3(n)\} = \sum_{n=-\infty}^{+\infty} x_3(n)\, e^{-j\omega n}$

$$= \sum_{n=-\infty}^{+\infty} x_2(n)\, x_1(n)\, e^{-j\omega n} \qquad .....(8.19)$$

By the definition of inverse Fourier transform we get,

$$x_1(n) = \frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad \boxed{\text{Let, } \omega = \lambda}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda \qquad .....(8.20)$$

On substituting for $x_1(n)$ from equation (8.20) in equation (8.19) we get,

$$\mathcal{F}\{x_1(n)\, x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n) \left[\frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, e^{j\lambda n}\, d\lambda\right] e^{-j\omega n}$$

On interchanging the order of summation and integration in the above equation we get,

$$\mathcal{F}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\left[\sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j(\omega-\lambda)n}\right] X_1(e^{j\lambda})\, d\lambda$$

The term in the paranthesis in the above equation is similar to the definition of fourier transform of $x_2(n)$ but at a frequency argument of $(\omega - \lambda)$

$$\therefore\ \mathcal{F}\{x_1(n)\, x_2(n)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_2(e^{j(\omega-\lambda)})\, X_1(e^{j\lambda})\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega-\lambda)})\, d\lambda$$

8. Differentiation in frequency domain

If $\quad \mathcal{F}\{x(n)\} = X(e^{j\omega})$

then $\mathcal{F}\{n\,x(n)\} = j\dfrac{d}{d\omega}X(e^{j\omega})$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n} \qquad .....(8.21)$$

$$\mathcal{F}\{n\,x(n)\} = \sum_{n=-\infty}^{+\infty} n\,x(n)\,e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} n\,x(n)\,j(-j)\,e^{-j\omega n} \qquad \boxed{\text{Multiply by j and } -j}$$

$$= j\sum_{n=-\infty}^{+\infty} x(n)\,[(-jn)\,e^{-j\omega n}]$$

$$= j\sum_{n=-\infty}^{+\infty} x(n)\left[\frac{d}{d\omega}e^{-j\omega n}\right] \qquad \boxed{\frac{d}{d\omega}e^{-j\omega n} = -jn\,e^{-j\omega n}}$$

$$= j\frac{d}{d\omega}\left[\sum_{n=-\infty}^{+\infty} x(n)\,e^{-j\omega n}\right] \qquad \boxed{\text{Interchanging summation and differentiation}}$$

$$= j\frac{d}{d\omega}X(e^{j\omega}) \qquad \boxed{\text{Using equation (8.21)}}$$

9. Convolution theorem

If $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$

and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

then $\mathcal{F}\{x_1(n) * x_2(n)\} = X_1(e^{j\omega})\, X_2(e^{j\omega})$

$$\text{where, } x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m) \qquad \text{....(8.22)}$$

The Fourier transform of the convolution of $x_1(n)$ and $x_2(n)$ is equal to the product of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$. It means that if we convolve two signals in time domain, it is equivalent to multiplying their spectra in frequency domain.

Proof :

By the definition of Fourier transform,

$$X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\} = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(8.23)$$

$$X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\} = \sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n} \qquad .....(8.24)$$

$$\mathcal{F}\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left[x_1(n) * x_2(n)\right] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\right] e^{-j\omega n}$$

Using equation (8.22)

$$= \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\, e^{-j\omega n}\, e^{-j\omega m}\, e^{j\omega m}$$

Multiply by $e^{-j\omega m}$ and $e^{j\omega m}$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, e^{-j\omega m} \sum_{n=-\infty}^{+\infty} x_2(n-m)\, e^{-j\omega(n-m)}$$

Let, $n - m = p$
when $n \to -\infty$, $p \to -\infty$
when $n \to +\infty$, $p \to +\infty$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, e^{-j\omega m} \sum_{p=-\infty}^{+\infty} x_2(p)\, e^{-j\omega p}$$

$$= \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n}\right]\left[\sum_{n=-\infty}^{+\infty} x_2(n)\, e^{-j\omega n}\right]$$

Let $m = n$, in first summation
Let $p = n$, in second summation

$$= X_1(e^{j\omega})\, X_2(e^{j\omega})$$

Using equations (8.23) and (8.24)

10. Correlation

If $\mathcal{F}\{x(n)\} = X(e^{j\omega})$ and $\mathcal{F}\{y(n)\} = Y(e^{j\omega})$

then $\mathcal{F}\{r_{xy}(m)\} = X(e^{j\omega})\, Y(e^{-j\omega})$

$$\text{where, } r_{xy}(m) = \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m) \qquad \text{.....(8.25)}$$

Proof :

By the definition of Fourier transform,

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \qquad \text{.....(8.26)}$$

$$Y(e^{j\omega}) = \mathcal{F}\{y(n)\} = \sum_{n=-\infty}^{+\infty} y(n)\, e^{-j\omega n} \qquad \text{.....(8.27)}$$

$$\mathcal{F}\{r_{xy}(m)\} = \sum_{m=-\infty}^{+\infty} r_{xy}(m)\, e^{-j\omega m}$$

$$= \sum_{m=-\infty}^{+\infty} \left[\sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\right] e^{-j\omega m}$$

Using equation (8.25)

$$= \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} x(n)\, y(n-m)\, e^{-j\omega m}\, e^{-j\omega n}\, e^{j\omega n}$$

Multiply by $e^{-j\omega n}$ and $e^{j\omega n}$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \sum_{m=-\infty}^{+\infty} y(n-m)\, e^{j\omega (n-m)}$$

$$= \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} \sum_{p=-\infty}^{+\infty} y(p)\, e^{j\omega p}$$

Let, $n - m = p$ $\quad \therefore m = n - p$
when $m \to -\infty$, $\quad p \to +\infty$,
when $m \to +\infty$, $\quad p \to -\infty$.

$$= \left[\sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n}\right] \left[\sum_{p=-\infty}^{+\infty} y(p)\, (e^{-j\omega})^{-p}\right]$$

Using equations (8.26) and (8.27)

$$= X(e^{j\omega})\, Y(e^{-j\omega})$$

11. Parseval's relation

If $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

then the Parseval's relation states that,

$$\sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\; d\omega \qquad .....(8.28)$$

When $x_1(n) = x_2(n) = x(n)$, then Parseval's relation can be written as,

$$\sum_{n=-\infty}^{+\infty} |x(n)|^2 = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} |X(e^{j\omega})|^2\; d\omega$$

The above equation is also called energy density spectrum of the signal x(n).

Proof :

Let, $\mathcal{F}\{x_1(n)\} = X_1(e^{j\omega})$ and $\mathcal{F}\{x_2(n)\} = X_2(e^{j\omega})$

Now, by definition of Fourier transform,

$$\mathcal{F}\left\{x_1(n)\right\} = X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} \qquad .....(8.29)$$

Now, by definition of inverse Fourier transform,

$$x_2(n) = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_2(e^{j\omega})\, e^{j\omega n}\, d\omega \qquad .....(8.30)$$

Consider left hand side of Parseval's relation [equation (8.28)],

$$\frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega$$

In the above expression, Let us substitute for $X_1(e^{j\omega})$ from equation (8.29),

$$\therefore\ \frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_1(e^{j\omega})\, X_2^*(e^{j\omega})\, d\omega = \frac{1}{2\pi j} \int_{-\pi}^{+\pi} \left[\sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n}\right] X_2^*(e^{j\omega})\, d\omega$$

$$= \sum_{n=-\infty}^{+\infty} x_1(n) \left[\frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_2^*(e^{j\omega})\, e^{-j\omega n}\, d\omega\right]$$

Interchanging summation and integration

$$= \sum_{n=-\infty}^{+\infty} x_1(n) \left[\frac{1}{2\pi j} \int_{-\pi}^{+\pi} X_2(e^{j\omega})\, e^{j\omega n}\, d\omega\right]^*$$

$$= \sum_{n=-\infty}^{+\infty} x_1(n)\, x_2^*(n)$$

Using equation (8.30)

Properties of Discrete Time Fourier Transform

Note : $X(e^{j\omega}) = \mathcal{F}\{x(n)\}$; $X_1(e^{j\omega}) = \mathcal{F}\{x_1(n)\}$; $X_2(e^{j\omega}) = \mathcal{F}\{x_2(n)\}$; $Y(e^{j\omega}) = \mathcal{F}\{y(n)\}$

Property	Discrete time signal	Fourier transform
Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(e^{j\omega}) + a_2 X_2(e^{j\omega})$
Periodicity	$x(n)$	$X(e^{j\omega + 2\pi m}) = X(e^{j\omega})$
Time shifting	$x(n - m)$	$e^{-j\omega m} X(e^{j\omega})$
Time reversal	$x(-n)$	$X(e^{-j\omega})$
Conjugation	$x^*(n)$	$X^*(e^{-j\omega})$
Frequency shifting	$e^{j\omega_0 n} x(n)$	$X(e^{j(\omega - \omega_0)})$
Multiplication	$x_1(n)\, x_2(n)$	$\frac{1}{2\pi} \int_{-\pi}^{+\pi} X_1(e^{j\lambda})\, X_2(e^{j(\omega - \lambda)})\, d\lambda$

Differentiation in frequency domain	$n\,x(n)$	$j\,\dfrac{dX(e^{j\omega})}{d\omega}$
Convolution	$x_1(n) * x_2(n) = \sum_{m=-\infty}^{+\infty} x_1(m)\,x_2(n-m)$	$X_1(e^{j\omega})\,X_2(e^{j\omega})$
Correlation	$r_{xy}(m) = \sum_{m=-\infty}^{+\infty} x(n)\,y(n-m)$	$X(e^{j\omega})\,Y(e^{-j\omega})$
Symmetry of real signals	x(n) is real	$X(e^{j\omega}) = X^*(e^{-j\omega})$ $\mathrm{Re}\{X(e^{j\omega})\} = \mathrm{Re}\{X(e^{-j\omega})\}$ $\mathrm{Im}\{X(e^{j\omega})\} = -\mathrm{Im}\{X(e^{-j\omega})\}$ $\lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert,\ \angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$
Symmetry of real and even signal	x(n) is real and even	$X(e^{j\omega})$ is real and even
Symmetry of real and odd signal	x(n) is real and odd	$X(e^{j\omega})$ is imaginary and odd
Parseval's relation	$\sum_{n=-\infty}^{+\infty} x_1(n)\,x_2^*(n)$	$\dfrac{1}{2\pi}\int_{-\pi}^{+\pi} X_1(e^{j\omega})\,X_2^*(e^{j\omega})\,d\omega$
Parseval's relation	Energy in time domain, $E = \sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2$	Energy in frequency domain, $E = \dfrac{1}{2\pi}\int_{-\pi}^{+\pi} \lvert X(e^{j\omega})\rvert^2\,d\omega$
	$\sum_{n=-\infty}^{+\infty} \lvert x(n)\rvert^2 = \dfrac{1}{2\pi}\int_{-\pi}^{\pi} \lvert X(e^{j\omega})\rvert^2\,d\omega$	

Discrete Time Fourier Transform of Periodic Discrete Time Signals

The Fourier transform of any periodic discrete time signal can be obtained from the knowledge of Fourier transform of periodic discrete time signal $e^{j\omega_0 n}$, with period N.

In chapter-4, section 4.12, it is observed that the Fourier transform of continuous time periodic signal is a train of impulses. Similarly, the Fourier transform of discrete time periodic signal is also a train of impulses, but the impulse train should be periodic. Therefore, the Fourier transform of $e^{j\omega_0 n}$ will be in the form of periodic impulse train with period 2π as shown in equation (8.31).

$$\text{Let,} \quad g(n) = e^{j\omega_0 n}$$

$$\therefore G(e^{j\omega}) = \mathcal{F}\{g(n)\} = \mathcal{F}\{e^{j\omega_0 n}\} = \sum_{m=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi m) \quad .....(8.31)$$

$$\text{where,} \quad \omega_0 = \frac{2\pi}{N} = \text{Fundamental frequency of } g(n).$$

In equation (8.31), $\delta(\omega)$ is an impulse function of ω and ω_0 lie in the range $-\pi$ to $+\pi$.

The equation (8.31) can be proved by taking inverse Fourier transform of $G(e^{j\omega})$ as shown below.

Proof :

$$G(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi m)$$

By the definition of inverse Fourier transform,

$$g(n) = \mathcal{F}^{-1}\left\{G(e^{j\omega})\right\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} G(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{m=-\infty}^{+\infty} 2\pi\, \delta(\omega - \omega_0 - 2\pi m)\, e^{j\omega n}\, d\omega$$

$$= \int_{-\pi}^{+\pi} \delta(\omega - \omega_0)\, e^{j\omega n}\, d\omega = e^{j\omega n}\Big|_{\omega=\omega_0} = e^{j\omega_0 n}$$

Note : Here the integral limit is $-\pi$ to $+\pi$, and in this range there is only one impulse located at ω_0.

Consider the Fourier series representation of periodic discrete time signal x(n), shown below.

$$x(n) = \sum_{k=0}^{N-1} c_k \, e^{j\omega_k n}$$

$$\text{where,} \quad c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}} \quad ; \text{ for } k = 0, 1, 2, \ldots.., (N-1) \qquad \ldots..(8.32)$$

$$\omega_k = \frac{2\pi k}{N}$$

On comparing g(n) and x(n), we can say that the Fourier transform of x(n) can be obtained from its Fourier series representation, as shown below.

$$X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \mathcal{F}\left\{\sum_{k=0}^{N-1} c_k \, e^{j\omega_k n}\right\} = \sum_{k=-\infty}^{+\infty} c_k \, 2\pi \, \delta(\omega - \omega_k) \qquad \ldots..(8.33)$$

The equation (8.33) can be used to compute Fourier transform of any periodic discrete time signal x(n), and the Fourier transform consists of train of impulses located at the harmonic frequencies of x(n).

Some Common Discrete Time Fourier Transform Pairs

x(t)	x(n)	$X(e^{j\omega})$	
		with positive power of $e^{j\omega}$	with negative power of $e^{j\omega}$
	$\delta(n)$	1	1
	$\delta(n-n_0)$	$\dfrac{1}{e^{j\omega n_0}}$	$e^{-j\omega n_0}$
	$u(n)$	$\dfrac{e^{j\omega}}{e^{j\omega}-1}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$	$\dfrac{1}{1-e^{-j\omega}}+\sum_{m=-\infty}^{+\infty}\pi\,\delta(\omega-2\pi m)$
	$a^n\,u(n)$	$\dfrac{e^{j\omega}}{e^{j\omega}-a}$	$\dfrac{1}{1-a e^{-j\omega}}$
	$n\,a^n\,u(n)$	$\dfrac{a e^{j\omega}}{(e^{j\omega}-a)^2}$	$\dfrac{a e^{-j\omega}}{(1-a e^{-j\omega})^2}$
	$n^2\,a^n\,u(n)$	$\dfrac{a e^{j\omega}(e^{j\omega}+a)}{(e^{j\omega}-a)^3}$	$\dfrac{a e^{-j\omega}(1+a e^{-j\omega})}{(1-a e^{-j\omega})^3}$

x(t)	x(n)	$X(e^{j\omega})$ with positive power of $e^{j\omega}$	$X(e^{j\omega})$ with negative power of $e^{j\omega}$
	1	$2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - 2\pi m)$	
	$a^{\|n\|}$	$\dfrac{1-a^2}{1-2a\cos\omega + a^2}$	
	$\sum_{m=-\infty}^{+\infty} \delta(n - mN)$	$\dfrac{2\pi}{N} \sum_{m=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi m}{N}\right)$	
$e^{j\Omega_0 t}$	$e^{j\Omega_0 nt} = e^{j\omega_0 n}$ where, $\omega_0 = \Omega_0 T$	$2\pi \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi m)$	
$\sin\Omega_0 t$	$\sin\Omega_0 nT = \sin\omega_0 n$ where, $\omega_0 = \Omega_0 T$	$\dfrac{\pi}{j} \sum_{m=-\infty}^{+\infty} \left[\delta(\omega - \omega_0 - 2\pi m) - \delta(\omega + \omega_0 - 2\pi m)\right]$	
$\cos\Omega_0 t$	$\cos\Omega_0 nT = \cos\omega_0 n$ where, $\omega_0 = \Omega_0 T$	$\pi \sum_{m=-\infty}^{+\infty} \left[\delta(\omega - \omega_0 - 2\pi m) + \delta(\omega + \omega_0 - 2\pi m)\right]$	

Analysis of LTI Discrete Time System Using Discrete Time Fourier Transform

Transfer Function of LTI Discrete Time System in Frequency Domain

The ratio of Fourier transform of output and the Fourier transform of input is called ***transfer function*** of LTI discrete time system in frequency domain.

Let, $x(n)$ = Input to the discrete time system

$y(n)$ = Output of the discrete time system

$\therefore X(e^{j\omega})$ = Fourier transform of $x(n)$

$Y(e^{j\omega})$ = Fourier transform of $y(n)$

$$\text{Now, Transfer function} = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad .....(8.34)$$

The transfer function of an LTI discrete time system in frequency domain can be obtained from the difference equation governing the input-output relation of the LTI discrete time system given below, (refer chapter-6, equation (6.17)).

$$y(n) = -\sum_{m=1}^{N} a_m\, y(n-m) + \sum_{m=0}^{M} b_m\, x(n-m)$$

On taking Fourier transform of above equation and rearraging the resultant equation as a ratio of $Y(e^{j\omega})$ and $X(e^{j\omega})$, the transfer function of LTI discrete time system in frequency domain is obtained.

Impulse Response and Transfer Function

Let, x(n) = Input of an LTI discrete time system

y(n) = Output / Response of the LTI discrete time system for the input x(n)

h(n) = Impulse response (i.e., response for impulse input)

Now, the response y(n) of the discrete time system is given by convolution of input and impulse response, (Refer chapter-6, equation (6.33)).

$$\therefore\ y(n) = x(n) * h(n) = \sum_{m=-\infty}^{+\infty} x(m)\, h(n-m) \qquad .....(8.35)$$

Let, $\mathcal{F}\{y(n)\} = Y(e^{j\omega})$; $\mathcal{F}\{x(n)\} = X(e^{j\omega})$; $\mathcal{F}\{h(n)\} = H(e^{j\omega})$

Now by convolution theorem of Fourier transform,

$$\mathcal{F}\{x(n) * h(n)\} = X(e^{j\omega})\, H(e^{j\omega}) \qquad .....(8.36)$$

Using equation (8.35), the equation (8.36) can be written as,

$$\mathcal{F}\{y(n)\} = X(e^{j\omega})\ H(e^{j\omega})$$

$$\therefore Y(e^{j\omega}) = X(e^{j\omega})\ H(e^{j\omega})$$

$$\boxed{\therefore\ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}} \qquad .....(8.37)$$

From equations (8.34) and (8.37) we can say that the ***transfer function*** of a discrete time system in frequency domain is also given by discrete time Fourier transform of impulse response.

Response of LTI Discrete Time System using Discrete Time Fourier Transform

Consider the transfer function of LTI discrete time system.

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

Now, response in frequency domain, $Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega})$ .....(8.38)

On taking inverse Fourier transform of equation (8.38) we get,

$$y(n) = \mathcal{F}^{-1}\{X(e^{j\omega})\, H(e^{j\omega})\} \qquad .....(8.39)$$

From the equation (8.39) we can say that the output y(n) is given by the inverse Fourier transform of the product of $X(e^{j\omega})$ and $H(e^{j\omega})$.

Since the transfer function is defined with zero initial conditions, the response obtained by using equation (8.39) is the forced response or steady state response of discrete time system.

Frequency Response of LTI Discrete Time System

The output y(n) of LTI system is given by convolution of h(n) and x(n).

$$y(n) = x(n) * h(n) = h(n) * x(n) = \sum_{m=-\infty}^{+\infty} h(m)\, x(n-m) \qquad .....(8.40)$$

Consider a special class of input (sinusoidal input),

$$\boxed{Ae^{j\omega n} = A(\cos\omega n + j\sin\omega n)}$$

$$x(n) = A\, e^{j\omega n}\ ;\quad -\infty < n < \infty \qquad .....(8.41)$$

where, $A = \text{Amplitude}$

ω = Arbitrary frequency in the interval $-\pi$ to $+\pi$.

$$\therefore\ x(n-m) = Ae^{j\omega(n-m)} \qquad .....(8.42)$$

On substituting for x(n–m) from equation (8.42) in equation (8.40) we get,

$$y(n) = \sum_{m=-\infty}^{+\infty} h(m)\, A\, e^{j\omega(n-m)} = \sum_{m=-\infty}^{+\infty} h(m)\, A\, e^{j\omega n}\, e^{-j\omega m}$$

$$= A\, e^{j\omega n} \sum_{m=-\infty}^{+\infty} h(m)\, e^{-j\omega m} \qquad .....(8.43)$$

By the definition of Fourier transform,

$\boxed{\text{Replace n by m.}}$

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{+\infty} h(n)\, e^{-j\omega n} = \sum_{m=-\infty}^{+\infty} h(m)\, e^{-j\omega m} \qquad .....(8.44)$$

Using equations (8.41) and (8.44), the equation (8.43) can be written as,

$$y(n) = x(n)\, H(e^{j\omega}) \qquad .....(8.45)$$

From equation (8.45) we can say that if a complex sinusoidal signal is given as input signal to an LTI system, then the output is a sinusoidal of the same frequency modified by $H(e^{j\omega})$. Hence $H(e^{j\omega})$ is called the ***frequency response*** of the system.

The $H(e^{j\omega})$ produces a change in the amplitude and phase of the input signal. An LTI system is characterized in the frequency domain by its frequency response. The function $H(e^{j\omega})$ is a complex quantity and so it can be expressed as magnitude function and phase function.

$$\therefore H(e^{j\omega}) = |H(e^{j\omega})| \angle H(e^{j\omega})$$

where, $|H(e^{j\omega})|$ = Magnitude function

$\angle H(e^{j\omega})$ = Phase function

The sketch of magnitude function and phase function with respect to ω will give the frequency response graphically.

$$\text{Let, } H(e^{j\omega}) = H_r(e^{j\omega}) + jH_i(e^{j\omega})$$

where, $H_r(e^{j\omega})$ = Real part of $H(e^{j\omega})$

$H_i(e^{j\omega})$ = Imaginary part of $H(e^{j\omega})$

The ***magnitude function*** is defined as,

$$|H(e^{j\omega})|^2 = H(e^{j\omega})\, H^*(e^{j\omega}) = [H_r(e^{j\omega}) + jH_i(e^{j\omega})]\,[H_r(e^{j\omega}) - jH_i(e^{j\omega})]$$

where, $H^*(e^{j\omega})$ is complex conjugate of $H(e^{j\omega})$

$$\therefore\ |H(e^{j\omega})|^2 = H_r^2(e^{j\omega}) + H_i^2(e^{j\omega}) \quad \Rightarrow \quad |H(e^{j\omega})| = \sqrt{H_r^2(e^{j\omega}) + H_i^2(e^{j\omega})}$$

The ***phase function*** is defined as,

$$\angle H(e^{j\omega}) = \mathrm{Arg}[H(e^{j\omega})] = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

From equation (8.37) we can say that the frequency response $H(e^{j\omega})$ of an LTI system is same as transfer function in frequency domain and so, the frequency response is also given by the ratio of Fourier transform of output to Fourier transform of input.

$$\text{i.e., Frequency response, } H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \qquad .....(8.46)$$

Properties of Frequency Response

1. The frequency response is periodic function of ω with a period of 2π.
2. If $h(n)$ is real, then the magnitude of $H(e^{j\omega})$ is symmetric and phase of $H(e^{j\omega})$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
3. If $h(n)$ is complex, then the real part of $H(e^{j\omega})$ is symmetric and the imaginary part of $H(e^{j\omega})$ is antisymmetric over the interval $0 \le \omega \le 2\pi$.
4. The impulse response $h(n)$ is discrete, whereas the frequency response $H(e^{j\omega})$ is continuous function of ω.

Example 8.3

Find the Fourier transform of x(n), where $x(n) = 1 \;;\; 0 \le n \le 4$

$= 0 \;;\;$ otherwise

Solution

By the definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{4} x(n)\, e^{-j\omega n} = \frac{1 - e^{-j5\omega}}{1 - e^{-j\omega}}$$

Using finite geometric series sum formula,

$$\sum_{n=0}^{N-1} C^n = \frac{1 - C^N}{1 - C}$$

$$= \frac{1 - e^{\frac{-j5\omega}{2}}\, e^{\frac{-j5\omega}{2}}}{1 - e^{\frac{-j\omega}{2}}\, e^{\frac{-j\omega}{2}}} = \frac{\left(e^{\frac{j5\omega}{2}} - e^{\frac{-j5\omega}{2}}\right) e^{\frac{-j5\omega}{2}}}{\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right) e^{\frac{-j\omega}{2}}}$$

$$= \left(\frac{2j \sin\frac{5\omega}{2}}{2j \sin\frac{\omega}{2}}\right) e^{\frac{-j5\omega}{2} + \frac{j\omega}{2}} = \frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}}\, e^{\frac{-j4\omega}{2}} = \frac{\sin\frac{5\omega}{2}}{\sin\frac{\omega}{2}}\, e^{-j2\omega}$$

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Example 8.6

Compute the Fourier transform and sketch the magnitude and phase function of causal three sample sequence given by,

$$x(n) = \frac{1}{3}\ ;\ 0 \le n \le 2$$

$$= 0 \quad ;\ \text{else}$$

Solution

Let, $X(e^{j\omega})$ be Fourier transform of x(n).

Now by definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = \sum_{n=0}^{2} x(n)\, e^{-j\omega n}$$

$$= x(0)\, e^{0} + x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} = \frac{1}{3} + \frac{1}{3} e^{-j\omega} + \frac{1}{3} e^{-j2\omega}$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

$$= \frac{1}{3} + \frac{1}{3}(\cos\omega - j\sin\omega) + \frac{1}{3}(\cos 2\omega - j\sin 2\omega)$$

$$= \frac{1}{3}(1 + \cos\omega + \cos 2\omega) - j\frac{1}{3}(\sin\omega + \sin 2\omega)$$

The $X(e^{j\omega})$ is evaluated for various values of ω and tabulated in table 1. The magnitude and phase of $X(e^{j\omega})$ for various values of ω are also listed in table 1. Using the values listed in table 1, the magnitude and phase function are sketched as shown in fig 1 and fig 2 respectively.

Table 1 : Frequency Response of the System

ω	$X(e^{j\omega})$	$\lvert X(e^{j\omega})\rvert$	$\angle X(e^{j\omega})$ in rad
0	$1 + j0 \quad = 1\angle 0$	1	0
$\frac{\pi}{8}$	$0.877 - j0.363 \quad = 0.949\angle{-0.392} = 0.949\angle{-0.125\pi}$	0.949	-0.125π
$\frac{2\pi}{8}$	$0.569 - j0.569 \quad = 0.805\angle{-0.785} = 0.805\angle{-0.25\pi}$	0.805	-0.25π
$\frac{3\pi}{8}$	$0.225 - j0.544 \quad = 0.587\angle{-1.179} = 0.587\angle{-0.375\pi}$	0.587	-0.375π
$\frac{4\pi}{8} = \frac{\pi}{2}$	$0 - j0.333 \quad = 0.333\angle{-1.571} = 0.333\angle{-0.5\pi}$	0.333	-0.5π
$\frac{5\pi}{8}$	$-0.03 - j0.072 \quad = 0.078\angle{-1.966} = 0.078\angle{-0.625\pi}$	0.078	-0.625π
$\frac{6\pi}{8}$	$0.098 - j0.098 \quad = 0.139\angle{-0.785} = 0.139\angle{-0.25\pi}$	0.139	-0.25π
$\frac{7\pi}{8}$	$0.261 + j0.108 \quad = 0.282\angle 0.392 = 0.282\angle 0.125\pi$	0.282	0.125π

$\frac{8\pi}{8} = \pi$	$0.333 + j0 \quad = 0.333 \angle 0 \quad = 0.333 \angle 0$	0.333	0
$\frac{9\pi}{8}$	$0.261 - j0.108 = 0.282 \angle 0.392 = 0.282 \angle -0.125\pi$	0.282	-0.125π
$\frac{10\pi}{8}$	$0.098 + j0.098 = 0.139 \angle 0.785 = 0.139 \angle 0.25\pi$	0.139	0.25π
$\frac{11\pi}{8}$	$-0.03 + j0.072 = 0.078 \angle 1.966 = 0.078 \angle 0.625\pi$	0.078	0.625π
$\frac{12\pi}{8} = \frac{3\pi}{2}$	$0 + j0.333 \quad = 0.333 \angle 1.571 = 0.333 \angle 0.5\pi$	0.333	0.5π
$\frac{13\pi}{8}$	$0.225 + j0.544 = 0.589 \angle 1.179 = 0.589 \angle 0.375\pi$	0.589	0.375π
$\frac{14\pi}{8}$	$0.569 + j0.569 = 0.805 \angle 0.785 = 0.805 \angle 0.25\pi$	0.805	0.25π
$\frac{15\pi}{8}$	$0.877 + j0.363 = 0.949 \angle 0.392 = 0.949 \angle 0.125\pi$	0.949	0.125π
$\frac{16\pi}{8} = 2\pi$	$1 + j0 \quad = 1 \angle 0$	1	0

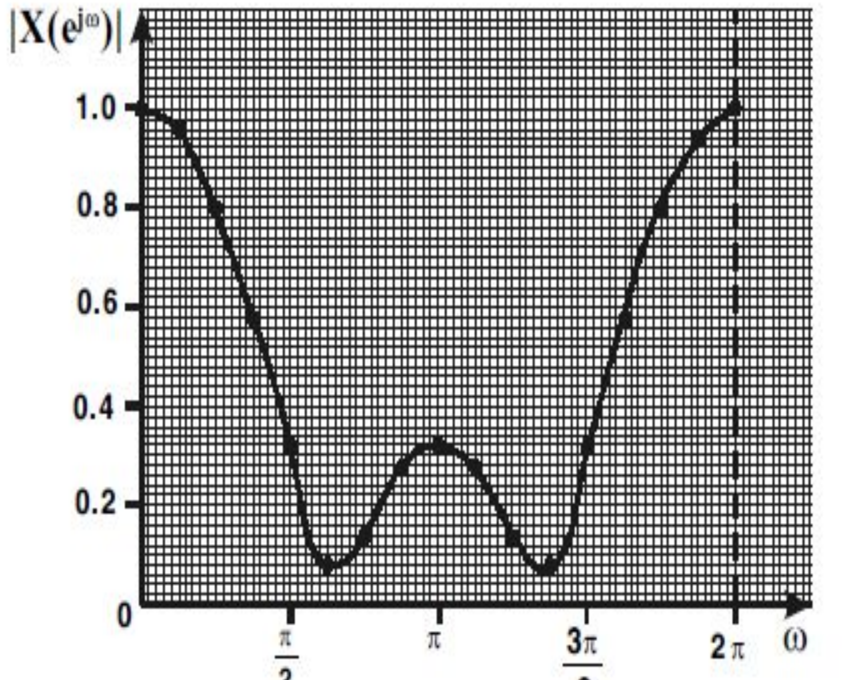

Fig 1 : *Magnitude of $X(e^{j\omega})$.*

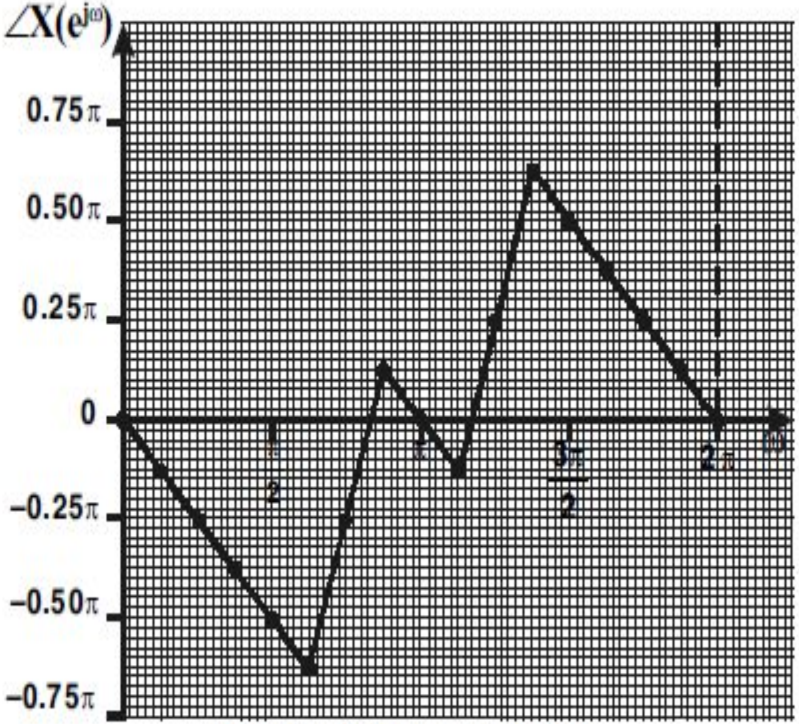

Fig 2 : *Phase of $X(e^{j\omega})$.*

Example 8.7

Find the convolution of the sequences, $x_1(n) = x_2(n) = \{1, \underset{\uparrow}{1}, 1\}$

Solution

$$X_1(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_1(n)\, e^{-j\omega n} = \sum_{n=-1}^{+1} x_1(n) e^{-j\omega n} = e^{j\omega} + 1 + e^{-j\omega}$$

Since, $x_1(n) = x_2(n)$, $\quad X_2(e^{j\omega}) = X_1(e^{j\omega}) = e^{j\omega} + 1 + e^{-j\omega}$

Let, $x(n) = x_1(n) * x_2(n)$, and $X(e^{j\omega}) = \mathcal{F}\{x(n)\} = \mathcal{F}\{x_1(n) * x_2(n)\}$

By convolution property of Fourier transform.

$$\mathcal{F}\{x_1(n) * x_2(n)\} = X_1(e^{j\omega})\, X_2(e^{j\omega})$$

$$\therefore\ X(e^{j\omega}) = X_1(e^{j\omega})\, X_2(e^{j\omega}) = (e^{j\omega} + 1 + e^{-j\omega})\,(e^{j\omega} + 1 + e^{-j\omega})$$

$$= e^{j2\omega} + e^{j\omega} + 1 + e^{j\omega} + 1 + e^{-j\omega} + 1 + e^{-j\omega} + e^{-j2\omega}$$

$$= e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega} \qquad .....(1)$$

By definition of Fourier transform,

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x(n)\, e^{-j\omega n} = ...\ x(-2)\, e^{j2\omega} + x(-1)\, e^{j\omega} + x(0)$$

$$+ x(1)\, e^{-j\omega} + x(2)\, e^{-j2\omega} + ... \qquad .....(2)$$

On comparing the coefficient of $e^{j\omega n}$ in the two equations of $X(e^{j\omega})$ [equations (1) and (2)] we get,

$$x(n) = \{1, 2, \underset{\uparrow}{3}, 2, 1\}$$

Example 8.13

If $H(e^{j\omega}) = 1 \quad ; \quad \omega \leq \omega_0$

$= 0 \quad ; \quad |\omega_0| < \omega \leq \pi,$ Find the impulse response h(n).

Solution

The impulse response h(n) can be obtained by taking inverse Fourier transform of $H(e^{j\omega})$.

By definition of inverse Fourier transform,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}\, d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_0}^{\omega_0}$$

$$= \frac{1}{j2\pi n}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] = \frac{1}{\pi n}\left[\frac{e^{j\omega_0 n} - e^{-j\omega_0 n}}{2j}\right] = \frac{\sin \omega_0 n}{\pi n} \quad \text{except when } n = 0$$

When $n = 0$; h(n) can be evaluated using L' Hospitals rule.

L' Hospitals rule

$$\underset{\theta \to 0}{\text{Lt}} \frac{\sin A\theta}{\theta} = A$$

When $n = 0$; $h(n) = \underset{n \to 0}{\text{Lt}} \dfrac{\sin\omega_0 n}{\pi n} = \dfrac{1}{\pi}\underset{n \to 0}{\text{Lt}} \dfrac{\sin\omega_0 n}{n} = \dfrac{1}{\pi}\omega_0 = \dfrac{\omega_0}{\pi}$

$\therefore$ Impulse response, $h(n) = \dfrac{\omega_0}{\pi}$, when $n = 0$

$= \dfrac{\sin \omega_0 n}{\pi n}$, when $n \neq 0$

Example 8.17

Determine the impulse response and frequency response of the LTI system defined by, $y(n) = x(n) + b\,y(n-1)$.

Solution

a) Impulse Response

The impulse response h(n) is given by inverse $\mathcal{Z}$-transform of H(z), where, $H(z) = \dfrac{Y(z)}{X(z)}$.

Given that, $y(n) = x(n) + b\,y(n-1)$. .....(1)

On taking $\mathcal{Z}$-transform of equation (1) we get,

$$Y(z) = X(z) + b\,z^{-1}\,Y(z) \quad \Rightarrow \quad Y(z) - b\,z^{-1}\,Y(z) = X(z) \quad \Rightarrow \quad Y(z)\,(1 - b\,z^{-1}) = X(z)$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - b\,z^{-1}} \qquad .....(2)$$

On taking inverse $\mathcal{Z}$-transform of equation (2) we get,

$$h(n) = \mathcal{Z}^{-1}\{H(z)\} = b^n\,u(n)$$

The impulse response, $h(n) = b^n\,u(n)$, for all n.

b) Frequency Response

The frequency response $H(e^{j\omega})$ is obtained by evaluating $H(z)$ when $z = e^{j\omega}$.

$$\therefore \text{ Frequency response, } H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}} = \left.\frac{1}{1 - bz^{-1}}\right|_{z = e^{j\omega}} = \frac{1}{1 - be^{-j\omega}}$$

The magnitude function of $H(e^{j\omega})$ is defined as,

$$\left|H(e^{j\omega})\right| = \sqrt{H(e^{j\omega})\, H^*(e^{j\omega})}\text{ , where } H^*(e^{j\omega}) = \text{Conjugate of } H(e^{j\omega}).$$

$$\therefore \text{ Magnitude function, } \left|H(e^{j\omega})\right| = \left[\frac{1}{1 - b\,e^{-j\omega}} \times \frac{1}{1 - b\,e^{j\omega}}\right]^{\frac{1}{2}} = \left[\frac{1}{1 - b\,e^{j\omega} - b\,e^{-j\omega} + b^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{1 + b^2 - b\left(e^{j\omega} + e^{-j\omega}\right)}\right]^{\frac{1}{2}} \quad \frac{1}{\left(1 + b^2 - 2b\cos\omega\right)^{\frac{1}{2}}} \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\text{The phase function, } \angle H(e^{j\omega}) = \tan^{-1}\left[\frac{H_i(e^{j\omega})}{H_r(e^{j\omega})}\right]$$

where, $H_i(e^{j\omega})$ = Imaginary part of $H(e^{j\omega})$ and $H_r(e^{j\omega})$ = Real part of $H(e^{j\omega})$

To separate the real parts and imaginary parts of $H(e^{j\omega})$, multiply the numerator and denominator by the complex conjugate of the denominator.

$$\therefore\ H(e^{j\omega}) = \frac{1}{1 - be^{-j\omega}} \times \frac{1 - b\,e^{j\omega}}{1 - b\,e^{j\omega}} = \frac{1 - b\,e^{j\omega}}{1 - b\,e^{j\omega} - b\,e^{-j\omega} + b^2}$$

$$= \frac{1 - b(\cos\omega + j\sin\omega)}{1 + b^2 - b\left(e^{j\omega} + e^{-j\omega}\right)} = \frac{1 - b\cos\omega - jb\sin\omega}{1 + b^2 - 2b\cos\omega}$$

$$= \frac{1 - b\cos\omega}{1 + b^2 - 2b\cos\omega} - j\frac{b\sin\omega}{1 + b^2 - 2b\cos\omega}$$

$$\therefore\ H_i(e^{j\omega}) = \frac{-b\sin\omega}{1 + b^2 - 2b\cos\omega} \text{ and } H_r(e^{j\omega}) = \frac{1 - b\cos\omega}{1 + b^2 - 2b\cos\omega}$$

Phase function, $\angle H(e^{j\omega}) = \tan^{-1}\dfrac{H_i(e^{j\omega})}{H_r(e^{j\omega})} = \tan^{-1}\left[\dfrac{-b\sin\omega}{1 - b\cos\omega}\right]$