* Examples on Line Integral :

→ If the Given contour is straight line or parabola:

Example 5: Evaluate the integral $\int_0^{1+i} (x-y+ix^2)\, dz$

i) along the line from $z=0$ to $z=1+i$

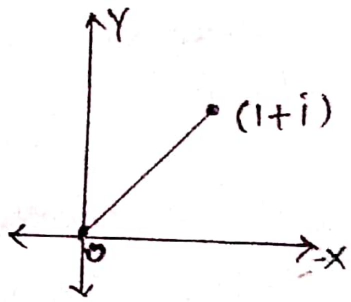
ii) along the real axis from $z=0$ to $z=1$

iii) along the parabola $y^2 = x$

Solution:

i) given: C : straight line from $z=0$ to $z=i$

$\therefore$ Equation of line from $z=0$ to $z=1+i$



is
$$\frac{y-y_0}{y_1-y_0} = \frac{x-x_0}{x_1-x_0}$$

$$\Rightarrow \frac{y-0}{1-0} = \frac{x-0}{1-0}$$

$$\Rightarrow y = x \quad , \quad 0 \le x \le 1 \quad , \quad 0 \le y \le 1$$

Now, put $x = t$

$$\Rightarrow y = t$$

$\therefore$ $dx = dt$ and $dy = dt$

$\therefore$ $dz = dx + i\,dy = dt + i\,dt = (1+i)\,dt$

$\therefore$ If $x=0$ then $t=0$

if $x=1$ then $t=1$

$\therefore$ t varies from $t=0$ to $t=1$

Now
$$\int_0^{1+i} (x-y+ix^2)\,dz = \int_{t=0}^{1} (t-t+it^2)(1+i)\,dt$$

$$= \int_0^1 i(1+i)\, t^2 \, dt$$

$$= i(1+i)\left[\frac{t^3}{3}\right]_0^1$$

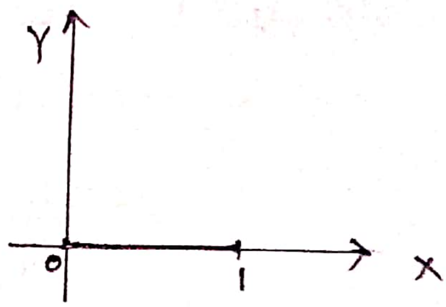
$$= i(1+i)\left[\frac{1^3}{3} - \frac{0^3}{3}\right]$$

$$= \frac{i(1+i)}{3}$$

$$= \frac{1}{3}(i-1)$$

ii) Given: C : real axis from $z=0$ to $z=1$

$\therefore$ equation of real axis from $z=0$ to $z=1$ is



$$y = 0 \quad , \quad 0 \le x \le 1$$

$$\therefore \quad dy = 0$$

$$\therefore \quad dz = dx + i\,dy = dx + i0 = dx$$

Now $\displaystyle\int_0^{1+i} (x - y + ix^2)\,dz = \int_0^1 (x - 0 + ix^2)\,dx$

$$= \int_0^1 (x + ix^2)\,dx$$

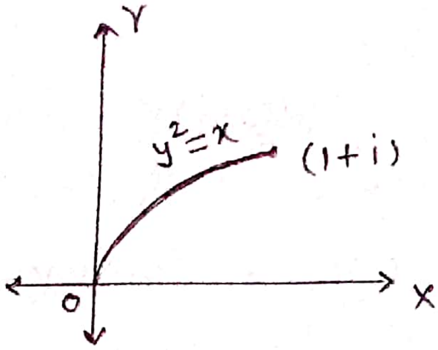
$$= \left[\frac{x^2}{2} + \frac{ix^3}{3}\right]_0^1$$

$$= \left[\left(\frac{1^2}{2} + \frac{i(1)^3}{3}\right) - \left(\frac{0^2}{2} + \frac{i(0)^3}{3}\right)\right]$$

$$= \frac{1}{2} + \frac{i}{3}$$

$$= \frac{1}{2} + i\frac{1}{3}$$

iii) Given: C : Parabola $y^2 = x$



put $y = t$

$\Rightarrow x = t^2$

$\therefore dy = dt$

and $dx = 2t\, dt$

$\therefore dz = dx + i\,dy = 2t\,dt + i\,dt$

$\Rightarrow dz = (2t+i)\,dt$

if $y = 0 \Rightarrow t = 0$

if $y = 1 \Rightarrow t = 1$

$\therefore$ t varies from $t=0$ to $t=1$

$$\therefore \int_0^{1+i} (x - y + i x^2)\, dz = \int_0^1 \left(t^2 - t + i(t^2)^2\right)(2t+i)\, dt$$

$$= \int_0^1 (t^2 - t + i t^4)(2t+i)\, dt$$

$$= \int_0^1 (2t^3 + i t^2 - 2t^2 - i t + 2i t^5 - t^4)\, dt$$

$$= \left[2\frac{t^4}{4} + i\frac{t^3}{3} - 2\frac{t^3}{3} - i\frac{t^2}{2} + 2i\frac{t^6}{6} - \frac{t^5}{5} \right]_0^1$$

$$= \left[\frac{t^4}{2} + i\frac{t^3}{3} - 2\frac{t^3}{3} - i\frac{t^2}{2} + i\frac{t^6}{3} - \frac{t^5}{5} \right]_0^1$$

$$= \left[\frac{1}{2} + \frac{i}{3} - \frac{2}{3} - \frac{i}{2} + \frac{i}{3} - \frac{1}{5} - 0 \right]$$

$$= -\frac{11}{30} + \frac{i}{6}$$

$$= -\frac{11}{30} + i\frac{1}{6}$$

Example 2. Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along the straight line joining $(1-i)$ to $(2+i)$

Solution:

Given: C : straight line from $(1-i)$ to $(2+i)$

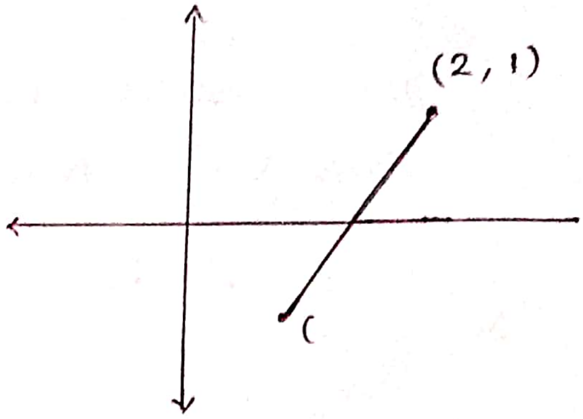
$\therefore$ Equation of line from $(1,-1)$ to $(2,1)$

is $$\frac{y-y_0}{y_1-y_0} = \frac{x-x_0}{x_1-x_0}$$

$$\Rightarrow \frac{y-(-1)}{1-(-1)} = \frac{x-1}{2-1}$$

$$\Rightarrow \frac{y+1}{2} = \frac{x-1}{1}.$$

$$\Rightarrow y = 2x-3$$



put $x = t$

$\Rightarrow y = 2t-3$

$\therefore dx = dt$ and $dy = 2dt$

$\therefore dz = dx + idy = dt + 2idt = (1+2i)\,dt$

if $x = 1$ then $t = 1$

if $x = 2$ then $t = 2$

$\therefore t$ varies from $t=1$ to $t=2$

Now, $$\int_{1-i}^{2+i} (2x+iy+1)\,dz = \int_1^2 (2t+i(2t-3)+1)(1+2i)\,dt$$

$$= (1+2i)\int_1^2 (2t+2it-3i+1)\,dt$$

$$= (1+2i)\left[2\frac{t^2}{2} + 2i\frac{t^2}{2} - 3it + t\right]_1^2$$

$$= (1+2i)\left[t^2+it^2-3it+t\right]_1^2$$

$$= (1+2i)\left[(2)^2+i(2)^2-3i(2)+2 - (1+i-3i+1)\right]$$

$$= (1+2i)\left[4+4i-6i+2-2+2i\right]$$

$$= (1+2i)\cdot 4$$

$$\therefore \int_{1-i}^{2+i} (2x+iy+1)\,dz = 4(1+2i)$$

<u>Example 6.</u> Integrate the function $f(z) = 2x+iy+1$ from $A(1,-1)$ to $B(2,1)$ Along the curve $x = t+1$, $y = 2t^2-1$

<u>Solution:</u> Given: $f(z) = 2x+iy+1$

and C : $x = t+1$, $y = 2t^2-1$ from $A(1,-1)$ to $B(2,1)$

$$\therefore \quad dx = dt \quad , \quad dy = 4t\,dt$$

$$dz = dx + i\,dy = dt + 4it\,dy = (1+4it)\,dt$$

if $x = 1$ then $t = x-1 = 1-1 = 0$

if $x = 2$ then $t = x-1 = 2-1 = 1$

$\therefore$ t varies from $t=0$ to $t=1$

Now, $$\int_C f(z)\,dz = \int_C (2x+iy+1)\,dz$$

$$= \int_0^1 \left[2(t+1) + i(2t^2-1) + 1\right](1+4it)\,dt$$

$$= \int_0^1 (2t+2+2it^2-i+1)(1+4it)\,dt$$

$$= \int_0^1 (2t+2+2it^2-i+1+8it^2+8it-8t^3+4t + 4it)\,dt$$

$$= \int_0^1 (-8t^3 + 10it^2 + 12it + 6t + 3 - i)\, dt$$

$$= \left[-8\frac{t^4}{4} + 10i\frac{t^3}{3} + 12i\frac{t^2}{2} + 6\frac{t^2}{2} + 3t - it \right]_0^1$$

$$= \left[-2t^4 + \frac{10i}{3}t^3 + 6it^2 + 3t^2 + 3t - it \right]_0^1$$

$$= \left[-2 + \frac{10i}{3} + 6i + 3 + 3 - i \right]$$

$$= 4 + \frac{25}{3} i$$

$$\therefore \int_C f(z)\, dz = 4 + \frac{25}{3} i$$

Homework:

Example 8. Evaluate $\int_0^{1+i} z^2\, dz$ along

i) the line $y = x$

ii) the parabola $x = y^2$

iii) Is the line integral independent of the path?

(Hint: for iii) if value of Integral using i) and ii) are same then path independent otherwise, dependent)

$\boxed{\text{Ans: } \frac{2}{3}(i-1)}$

Example 9. Evaluate $\int f(z)\, dz$ along the parabola $y = 2x^2$ from $z = 0$ to $z = 3 + 18i$ where $f(z) = x^2 - 2iy$

$\boxed{\text{Ans: } 333 + 45i}$