* Function involving higher order derivative :

* Rayleigh - Ritz Method :-

consider, the functional $I = \int_{x_1}^{x_2} F(x, y, y')\, dx$

with $y(a) = b_1$ and $y'(a) = b_2$

Working Rule:

Step 1. Consider, $\bar{y}(x) = c_0 + c_1 x + c_2 x^2$ ——— ①

Step 2 use initial condition $y(a) = b_1$ and $y'(a) = b_2$ and find $c_0, c_1, c_2$ and put in ①

Step 3 Now use new $\bar{y}(x)$ and find $I$ by usual integration we get $I = \phi(c_i)$

Step 4 use $I = \phi(c_i)$ & find $c_i$ (using stationary method) and put in equation ① we get the Required solution

EX ① solve by Rayleigh - Ritz method the boundary value problem.

$$I = \int_0^1 (2xy - y^2 - y'^2)\, dx \quad \text{given } y(0) = 0, \quad y(1) = 0$$

Solution: here, $F = 2xy - y^2 - y'^2$

Now we Assume the trial solution

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2 \quad \text{———} \quad ①$$

since, $y(0) = 0$ and $y(1) = 0$

$$0 = c_0 + 0 + 0 \Rightarrow \boxed{c_0 = 0}$$

and $\quad 0 = c_0 + c_1(1) + c_2(1)^2$

$\Rightarrow \quad 0 = c_1 + c_2 \qquad (\because c_0 = 0)$

$\Rightarrow \quad c_1 = -c_2$

$\therefore$ equation ① becomes

$$\bar{y}(x) = c_1 x - c_1 x^2 = c_1(x - x^2)$$

$$\Rightarrow \bar{y}'(x) = c_1 - 2c_1 x = c_1(1-2x)$$

But $\quad I = \int_0^1 (2xy - y^2 - y'^2)\,dx$

$$= \int_0^1 \left[2x\left(c_1(x-x^2)\right) - [c_1(x-x^2)]^2 - [c_1(1-2x)]^2\right] dx$$

$$= \int_0^1 \left[c_1(2x^2 - 2x^3) - c_1^2(x^2 - 2x^3 + x^4) - c_1^2(1 - 4x + 4x^2)\right] dx$$

$$= c_1\left[\frac{2x^3}{3} - \frac{2x^4}{4}\right]_0^1 - c_1^2\left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5}\right]_0^1 - c_1^2\left[x - \frac{4x^2}{2} + \frac{4x^3}{3}\right]_0^1$$

$$= c_1\left[\left(\frac{2}{3} - \frac{2}{4}\right) - (0-0)\right] - c_1^2\left[\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) - (0-0+0)\right]$$
$$- c_1^2\left[\left(1 - \frac{4}{2} + \frac{4}{3}\right) - (0-0+0)\right]$$

$$I = \frac{c_1}{6} - \frac{11}{30}c_1^2$$

* for the stationary value:

$$\frac{dI}{dc_1} = 0 \quad \Rightarrow \quad \frac{1}{6} - 2c_1\frac{11}{30} = 0$$

$$\Rightarrow \quad c_1 = \frac{5}{22}$$

$\therefore$ equation ① becomes

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2 = \frac{5}{22}x - \frac{5}{22}x^2$$

$\Rightarrow \quad \boxed{\bar{y}(x) = \frac{5}{22}x(1-x)}$ is Required solution

Ex ② Using Rayleigh - Ritz method, solve the boundary value problem

$$I = \int_0^1 \left(xy + \tfrac{1}{2} y'^2\right) dx \;; \quad 0 \le x \le 1$$

and $\quad y(0) = 0 \quad , \quad y(1) = 0$

solution: here $\quad F = xy + \frac{1}{2} y'^2 \quad$ and $\; y(0) = 0 \; , \; y(0) = 0$

we Assume the trial solution

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2 \quad \text{——— ①}$$

if $\; y(0) = 0 \quad$ Then

$$0 = c_0 + c_1(0) + c_2(0) \quad \Rightarrow \quad \boxed{c_0 = 0}$$

if $\; y(1) = 0 \quad$ then

$$0 = 0 + c_1(1) + c_2(1) \quad \Rightarrow \quad \boxed{c_2 = -c_1}$$

$\therefore$ equation ① becomes

$$\bar{y}(x) = c_1 x - c_1 x^2 = c_1(x - x^2)$$

$$\bar{y}'(x) = c_1 - 2c_1 x = c_1(1 - 2x)$$

But $\quad I = \int_0^1 \left(x\bar{y} + \frac{1}{2} \bar{y}'^2\right) dx$

$$= \int_0^1 \left[x\left[c_1(x - x^2)\right] + \tfrac{1}{2}\left[c_1(1-2x)\right]^2\right] dx$$

$$= \int_0^1 \left[c_1(x^2 - x^3) + \tfrac{1}{2} c_1^2 (1 - 4x + 4x^2)\right] dx$$

$$= c_1\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 + \frac{1}{2} c_1^2 \left[x - \frac{4x^2}{2} + 4\frac{x^3}{3}\right]_0^1$$

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$$= c_1\left[\left(\frac{1}{3}-\frac{1}{4}\right)-(0-0)\right]+\frac{1}{2}c_1^2\left[\left(1-\frac{4}{2}+\frac{4}{3}\right)-(0-0+0)\right].$$

$$= \frac{c_1}{12}+\frac{1}{2}c_1^2\left(\frac{1}{3}\right)$$

$$\Rightarrow I = \frac{c_1}{12}+\frac{1}{6}c_1^2$$

$\therefore$ for stationary value

$$\frac{dI}{dc_1} = 0$$

$$\Rightarrow \quad \frac{1}{12}+\frac{1}{6}2c_1 = 0$$

$$\Rightarrow \quad c_1 = -\frac{1}{4}$$

$\therefore$ equation (1) becomes

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2$$

$$= c_1 x - c_1 x^2$$

$$= -\frac{1}{4}x+\frac{1}{4}x^2$$

$$= \frac{1}{4}x(x-1)$$

$\therefore$ $\boxed{\bar{y}(x) = \frac{1}{4}x(x-1)}$ is Required solution

H.W

Ex 3. Using Rayleigh - Ritz method, solve the boundary value problem

$$I = \int_0^1 (2xy + y^2 - y'^2)\,dx \; ; \quad 0 \le x \le 1$$

given $y(0) = y(1) = 0$

Ans: $\bar{y}(x) = \frac{5}{18}x(1-x)$