$$= \frac{1}{9} \frac{1}{[1 - \frac{1}{9}]} + \frac{4}{3}$$

$$= \frac{1}{8} + \frac{4}{3}$$

$$\boxed{E = \frac{35}{24}}$$

$$P = \frac{(2N+1)}{(2N+1)} = 1$$

$$\boxed{P = 1 \text{ and } E = \infty}$$

(b) $x[n] = \left[\frac{1}{2}\right]^n u[n]$

$$E = \underset{N\to\infty}{Lt} \sum_{0}^{N} \left(\frac{1}{2}\right)^{2n}$$

$$= \underset{N\to\infty}{Lt} \sum_{0}^{N} \left(\frac{1}{4}\right)^{n}$$

$$= \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$\boxed{E = \frac{4}{3} \text{ and } P = 0}$$

■ Example 1.71

Find the energy of the following DT signal

$$x[n] = \left(\frac{1}{2}\right)^n \qquad n \geq 0$$

$$= 3^n \qquad n < 0$$

(*Anna University, April, 2005*)

Solution:

$$E = \left[\sum_{-\infty}^{-1} (3)^{2n} + \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{2n}\right]$$

$$= \left[\sum_{-\infty}^{-1} (9)^{n} + \sum_{0}^{\infty} \left(\frac{1}{4}\right)^{n}\right]$$

$$= \left[\sum_{1}^{\infty} (9)^{-n} + \frac{1}{\left(1-\frac{1}{4}\right)}\right]$$

$$= \left[\sum_{1}^{\infty} \left(\frac{1}{9}\right)^{n} + \frac{4}{3}\right]$$

$$= \left[\frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \cdots\right] + \frac{4}{3}$$

$$= \frac{1}{9}\left[1 + \frac{1}{9} + \frac{1}{9^2} + \frac{1}{9^3} + \cdots\right] + \frac{4}{3}$$

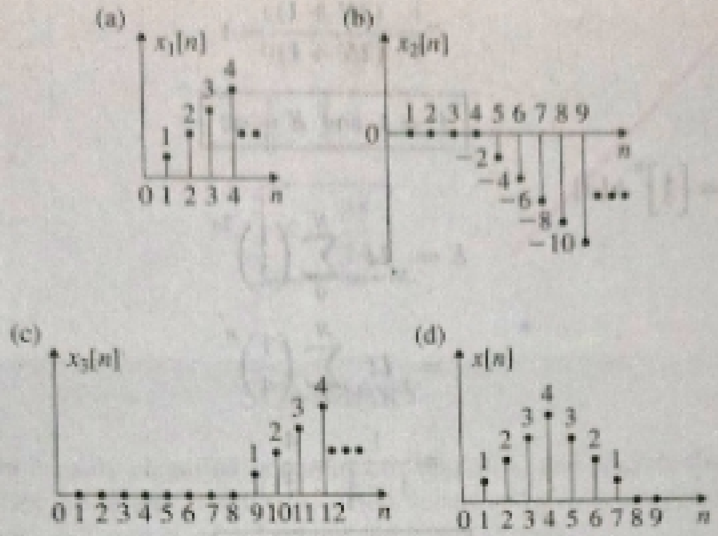

Figure 1.76 DT energy signal of Example 1.69.

Solution:

$$x[n] = \text{ramp}[n] - 2\,\text{ramp}[n-4] + \text{ramp}[n-8]$$
$$= x_1[n] + x_2[n] + x_3[n]$$

$x_1[n]$, $x_2[n]$ and $x_3[n]$ are shown in Figures 1.76(a), (b) and (c) respectively. Figure 1.76(d) represents $x[n]$. From Figure 1.76(d), the energy of the signal $x[n]$ is obtained as

$$E = 1^2 + 2^2 + 3^2 + 4^2 + 3^2 + 2^2 + 1^2$$

$$\boxed{E = 44}$$

■ Example 1.70

Determine the value of power and energy of each of the following signals:

(a) $x[n] = e^{j(\frac{\pi n}{2}+\frac{\pi}{8})}$

(b) $x[n] = \left(\frac{1}{2}\right)^n u[n]$

(Anna University, April, 2008)

Solution:

(a) $x[n] = e^{j(\frac{\pi n}{2}+\frac{\pi}{8})}$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{-N}^{N} |e^{j(\frac{\pi n}{2}+\frac{\pi}{8})}|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{-N}^{N} 1$$

(b) $x[n] = a^n u[n]$ where $|a| = 1$

$$E = \underset{N\to\infty}{Lt} \sum_{0}^{N} 1^n = \underset{N\to\infty}{Lt} (N+1)$$

$$\boxed{E = \infty}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{0}^{N} (1)^n$$

$$P == \underset{N\to\infty}{Lt} \frac{(N+1)}{(2N+1)}$$

$$= \underset{N\to\infty}{Lt} \frac{N(1+\frac{1}{N})}{N(2+\frac{1}{N})}$$

$$\boxed{P = \frac{1}{2}}$$

(c) $x[n] = a^n u[n]$ where $a > 1$

$$E = \underset{N\to\infty}{Lt} \sum_{0}^{N.} a^n$$

$$= 1 + a + a^2 + \cdots + a^N$$

$$\boxed{E = \infty}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} a^n$$

$$= \underset{N\to\infty}{Lt} \frac{1}{(N+1)} \frac{(1-a^{N+1})}{(1-a)}$$

$$\boxed{P = \infty}$$

The signal is neither energy nor power signal.

■ Example 1.69

Find the energy of the following signal:

$$x[n] = \text{ramp}[n] - 2\,\text{ramp}[n-4] + \text{ramp}[n-8]$$

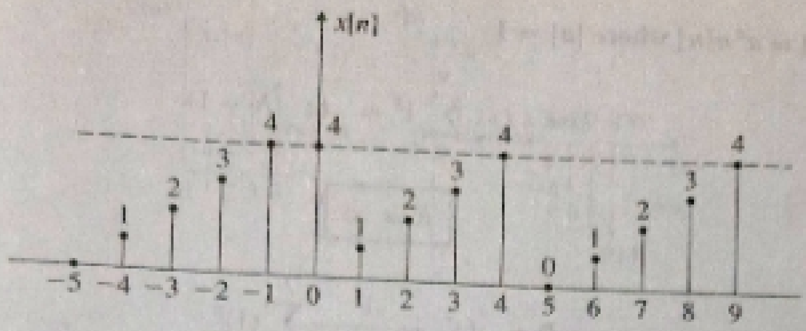

Figure 1.75 $x[n]$ of Example 1.67.

$$\boxed{P = 6}$$

Average energy per period is

$$E = \sum_{n=0}^{4} n^2$$
$$= [0 + 1 + 4 + 9 + 16]$$
$$\boxed{E = 30}$$

■ Example 1.68

Find the energy and power of the following signal:

$$x[n] = a^n u[n]$$

for the following cases:

(a) $|a| < 1$

(b) $|a| = 1$

(c) $|a| > 1$

Solution:

(a) $x[n] = a^n u[n]$ where $|a| < 1$ and $n \geq 0$

$$E = \sum_{n=0}^{\infty} (a^n)^2$$
$$= 1 + a^2 + a^4 + \ldots$$
$$\boxed{E = \frac{1}{1 - |a|^2}}$$
$$P = 0$$

Solution:

$$E = \underset{N\to\infty}{Lt} \sum_{n=0}^{N} \left(\frac{1}{3}\right)^{2n}$$

$$= \underset{N\to\infty}{Lt} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^{n}$$

$$= 1 + \frac{1}{9} + \left(\frac{1}{9}\right)^2 + \ldots$$

$$= \frac{1}{1-\frac{1}{9}}$$

$$\boxed{\begin{array}{c} E = \dfrac{9}{8} \\ P = 0 \end{array}}$$

■ Example 1.66

Find the energy of the following sequence shown in below:

$$x[n] = n \qquad 0 \le n \le 4$$

Solution:

$$x[n] = n$$

$$= \{0, 1, 2, 3, 4\}$$

$$E = \sum_{n=0}^{4} n^2$$

$$= 0 + 1 + 4 + 9 + 16$$

$$\boxed{E = 30}$$

■ Example 1.67

Determine the average power and the energy per period of the sequence shown in Figure 1.75.

Solution: The fundamental period N of the signal is 5. Hence, the average power per period is

$$P = \frac{1}{5}\sum_{n=0}^{4} n^2$$

$$= \frac{1}{5}[0 + 1 + 4 + 9 + 16]$$

But $|e^{j(n\pi+\theta)}| = 1$ and $\sum_{-N}^{N} 1 = (2N+1)$

$$P = \underset{N\to\infty}{Lt}\, 4\frac{(2N+1)}{(2N+1)} = 4$$

$$\boxed{\begin{array}{c} P = 4 \\ E = \infty \end{array}}$$

(f) $x[n] = \cos\frac{\pi}{2}n$

$$P = \frac{1}{(2N+1)}\sum_{-N}^{N}\cos^2\frac{\pi}{2}n$$

Since $\sum_{-N}^{N}\cos\pi n = 0$,

$$P = \underset{N\to\infty}{Lt}\frac{1}{(2N+1)}\sum_{-N}^{N}\frac{(1+\cos\pi n)}{2}$$

$$= \frac{1}{2}\underset{N\to\infty}{Lt}\frac{(2N+1)}{(2N+1)}$$

$$= \frac{1}{2}$$

$$\boxed{\begin{array}{c} P = \frac{1}{2} \\ E = \infty \end{array}}$$

■ Example 1.65

Determine the energy of the signal shown in Figure 1.74 whose

$$x[n] = \left(\frac{1}{3}\right)^n u[n]$$

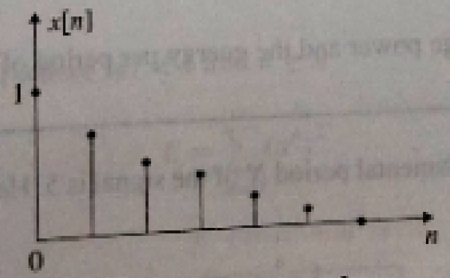

Figure 1.74 $x[n] = \left(\frac{1}{3}\right)^2 u[n]$.

(Anna University, December, 2007)

(c) $x[n] = \text{ramp}\, n;\ n \geq 0$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} |x[n]|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} n^2$$

But $\sum_{n=0}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$

$$P = \underset{N\to\infty}{Lt} \frac{N(N+1)(2N+1)}{(2N+1)6}$$

$$\boxed{P = \infty}$$

$$E = \underset{N\to\infty}{Lt} \sum_{n=0}^{N} n^2$$

$$= \underset{N\to\infty}{Lt} \frac{N(N+1)(2N+1)}{6} = \infty$$

$$\boxed{E = \infty}$$

The signal $x[n] = n$ is neither power signal nor energy signal.

(d) $x[n] = A$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=-\infty}^{\infty} A^2$$

$$= \underset{N\to\infty}{Lt} \frac{A^2}{(2N+1)}(2N+1) \qquad \left[\sum_{n=-\infty}^{\infty} 1 = (2N+1)\right]$$

$$\boxed{\begin{array}{c} P = A^2 \\ E = \infty \end{array}}$$

(e) $x[n] = 2e^{j(\pi n+\theta)}$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{-N}^{N} |2e^{j(n\pi+\theta)}|^2$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} 4 \sum_{-N}^{N} |e^{j(n\pi+\theta)}|^2$$

■ Example 1.64

Determine whether the following signals are energy signals or power signals:

(a) $x[n] = A\delta[n]$

(b) $x[n] = u[n]$

(c) $x[n] = \text{ramp}\, n$

(d) $x[n] = A$

(e) $x[n] = 2e^{j(\pi n+\theta)}$

(f) $x[n] = \cos\frac{\pi}{2}n$

Solution:

(a) $x[n] = A\delta[n]$

$$x[n] = A\delta[n]$$
$$= A \qquad n = 0$$
$$= 0 \qquad n \neq 0$$

$$\text{Energy } E = \sum_{n=0}^{0}(A)^2$$

$$\boxed{E = A^2}$$

For unit impulse, $A = 1$ and $E = 1$.

(b) $x[n] = u[n];\ n \geq 0$

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=0}^{N} 1$$

But $\sum_{n=0}^{N} 1 = (N+1)$

$$P = \underset{N\to\infty}{Lt} \frac{(N+1)}{(2N+1)}$$

$$= \underset{N\to\infty}{Lt} \frac{N(1+\frac{1}{N})}{N(2+\frac{1}{N})} = \frac{1}{2}$$

$$\boxed{\begin{array}{c} P = \frac{1}{2} \\ E = \infty \end{array}}$$

$$x_0[n] = \{-1.5, .5, -1, -.5, 0, .5, 1, -.5, 1.5\}$$
$$\uparrow$$

Even and odd components of $x[n]$ are represented in Figures 1.73(a) and (b) respectively.

1.9.3 Energy and Power of DT Signals

For a discrete time signal $x[n]$, the total energy is defined as

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{1.57}$$

The average power is defined as

$$P = \underset{N\to\infty}{Lt} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2 \tag{1.58}$$

From the definitions of energy and power, the following inferences are derived:

1. $x[n]$ is an energy sequence iff $0 < E < \infty$. For finite energy signal, the average power $P = 0$.
2. $x[n]$ is a power sequence iff $0 < P < \infty$. For a sequence with average power P being finite, the total energy $E = \infty$.
3. Periodic signal is a power signal and *vice versa* is not true. Here the energy of the signal per period is finite.
4. Signals which do not satisfy the definitions of total energy and average power, neither termed as power signal nor energy signal. The following summation formulae are very often used while evaluating the average power and total energy of DT sequence.

1.

$$\sum_{n=0}^{N-1} a^n = \frac{(1-a^n)}{(1-a)} \qquad a \neq 1 \tag{1.59}$$
$$= N \qquad a = 1$$

2.

$$\sum_{n=0}^{\infty} a^n = \frac{1}{(1-a)} \qquad a < 1 \tag{1.60}$$

3.

$$\sum_{n=m}^{\infty} a^n = \frac{a^m}{(1-a)} \qquad a < 1 \tag{1.61}$$

4.

$$\sum_{n=0}^{\infty} na^n = \frac{a}{(1-a)^2} \qquad a < 1 \tag{1.62}$$

Solution:

1. $x[n]$ is represented in Figure 1.72(a).
2. $x[-n]$ is obtained by folding $x[n]$ which is represented in Figure 1.72(b).
3. $-x[n]$ is obtained by inverting $x[-n]$ of Figure 1.72(b). This is represented in Figure 1.72(c).
4. $x_e[n] = \frac{1}{2}[x[n] + x[-n]]$. Figures 1.72(a) and (b) sample wise are added and their amplitudes are divided by the factor 2. This gives $x_e[n]$ and is represented in Figure 1.72(d).
5. $x_0[n] = \frac{1}{2}[x[n] - x[-n]]$. Figures 1.72(a) and (c) sample wise are added and their amplitudes are divided by a factor 2 to get $x_0[n]$. This is represented in Figure 1.72(e).

■ Example 1.63

Find the even and odd components of the following DT signal and sketch the same.

$$x[n] = \{-2, 1, 2, -1, 3\}$$

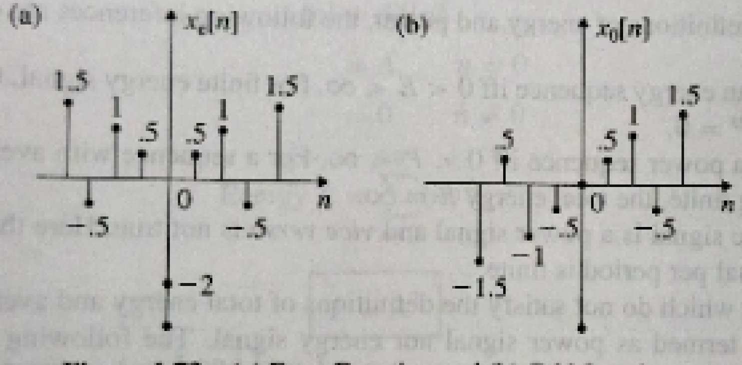

Figure 1.73 (a) Even Function and (b) Odd function.

(Anna University, December, 2007)

Solution:

$$x[n] = \{-2, 1, 2, -1, 3\}$$

$$x[-n] = \{3, -1, 2, 1, \underset{\uparrow}{-2}\}$$

$$x_e[n] = \frac{1}{2}\{x[n] + x[-n]\}$$

$$= \frac{1}{2}[\{\underset{\uparrow}{-2}, 1, 2, -1, 3\} + \{3, -1, 2, 1, \underset{\uparrow}{-2}\}]$$

$$= \{1.5, -.5, 1, .5, \underset{\uparrow}{-2}, .5, 1, -.5, 1.5\}$$

$$x_0[n] = \frac{1}{2}[x[n] - x[-n]]$$

$$= \frac{1}{2}[\{\underset{\uparrow}{-2}, 1, 2, -1, 3\} - \{3, -1, 2, 1, \underset{\uparrow}{-2}\}]$$

$$x_e[n] = \frac{1}{2}\,[x[n] + x[-n]]$$

$$= \frac{1}{2}\,[\{-2, 1, \underset{\uparrow}{3}, -5, 4\} + \{4, -5, \underset{\uparrow}{3}, 1, -2\}]$$

$$= \frac{1}{2}\,[(-2+4), (1-5), \underset{\uparrow}{(3+3)}, (-5+1), (4-2)]$$

$$\boxed{x_e[n] = \{1, -2, \underset{\uparrow}{3}, -2, 1\}}$$

$$x_0[n] = \frac{1}{2}\,[x[n] - x[-n]]$$

$$= \frac{1}{2}\,[\{-2, 1, \underset{\uparrow}{3}, -5, 4\} + \{-4, 5, \underset{\uparrow}{-3}, -1, 2\}]$$

$$= \frac{1}{2}\,[(-2-4), (1+5), \underset{\uparrow}{(3-3)}, (-5-1), (4+2)]$$

$$\boxed{x_0[n] = \{-3, 3, \underset{\uparrow}{0}, -3, 3\}}$$

Odd and even components by graphical method.

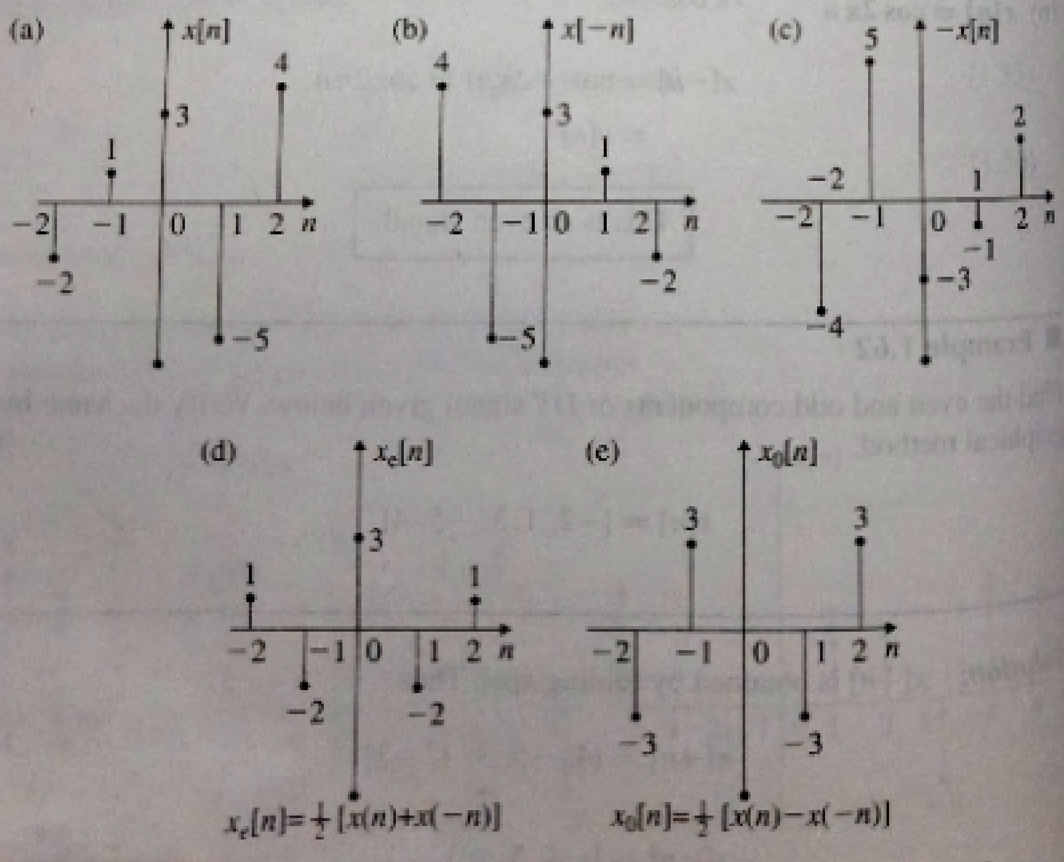

Figure 1.72 Graphical determination of even and odd function from $x[n]$.

- The product of two even signals or of two odd signals is an even signal.
- The product of an odd and an even signal is an odd signal.
- At $n = 0$, the odd signal is zero.

The even and odd signals are represented in Figures 1.71(a) and (b) respectively.

■ Example 1.61

Determine whether the following functions are odd or even:

(a) $x[n] = \sin 2\pi n$

(b) $x[n] = \cos 2\pi n$

Solution:

(a) $\mathbf{x[n] = \sin 2\pi n}$

$$x[-n] = \sin(-2\pi n) = -\sin 2\pi n$$
$$= -x[n]$$

$$\boxed{\text{This is an odd signal.}}$$

(b) $\mathbf{x[n] = \cos 2\pi n}$

$$x[-n] = \cos(-2\pi n) = \cos 2\pi n$$
$$= x[n]$$

$$\boxed{\text{This is an even signal.}}$$

■ Example 1.62

Find the even and odd components of DT signal given below. Verify the same by graphical method.

$$x[n] = \{-2, 1, \underset{\uparrow}{3}, -5, 4\}$$

Solution: $x[-n]$ is obtained by folding $x[n]$. Thus

$$x[-n] = \{4, -5, \underset{\uparrow}{3}, 1, -2\}$$

$$-x[-n] = \{-4, 5, \underset{\uparrow}{-3}, -1, 2\}$$

$$N_2 = 8 \quad \text{for } m_2 = 3$$

$$\frac{N_1}{N_2} = \frac{3}{8}$$

$$8N_1 = 3N_2 = N = 24$$

$$\boxed{N = 24}$$

The signal is periodic with fundamental period $N = 24$.

1.9.2 Odd and Even DT Signals

Similar to continuous time signals, DT signals are also classified as odd and even signals. The relationships are analogous to CT signals.

A discrete time signal $x[n]$ is said to be an even signal if

$$x[-n] = x[n] \tag{1.52}$$

A discrete time signal $x[n]$ is said to be an odd signal if

$$x[-n] = -x[n] \tag{1.53}$$

The signal $x[n]$ can be expressed as the sum of odd and even signals as

$$x[n] = x_e[n] + x_0[n] \tag{1.54}$$

The even and odd components of $x[n]$ can be expressed as

$$x_e[n] = \frac{1}{2}\left[x[n] + x[-n]\right] \tag{1.55}$$

$$x_0[n] = \frac{1}{2}\left[x[n] - x[-n]\right] \tag{1.56}$$

It is to be noted that

- An even function has an odd part which is zero.
- An odd function has an even part which is zero.

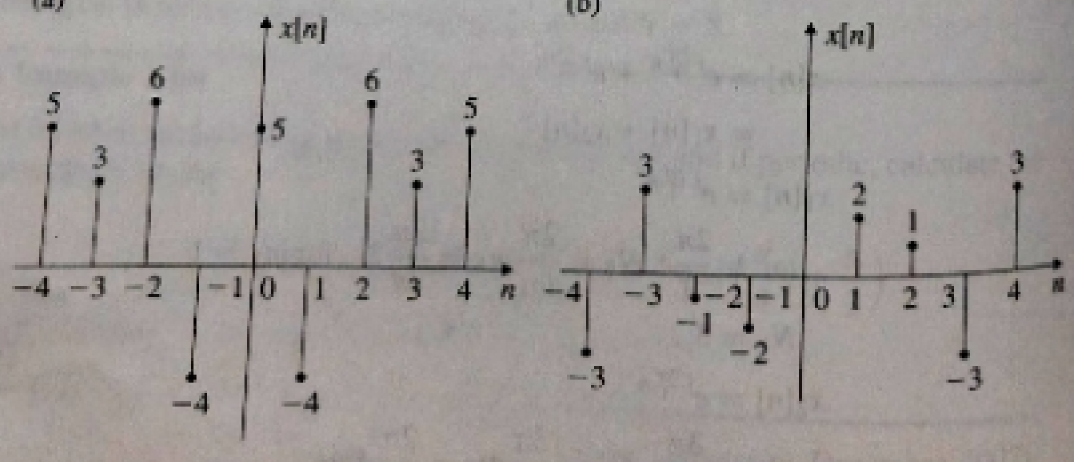

Figure 1.71 (a) Even Function and (b) Odd function.

Solution:

(a)

$$x[n] = \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{8}n\right) + 3\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$$
$$= x_1[n] + x_2[n] + x_3[n]$$
$$x_1[n] = \cos\frac{\pi}{2}n$$
$$\omega_1 = \frac{\pi}{2}; \quad N_1 = \frac{2\pi}{\omega_1} = \frac{2\pi\, 2}{\pi} \quad \text{for } m_1 = 1$$
$$N_1 = 4$$
$$x_2[n] = \sin\left(\frac{\pi}{8}n\right)$$
$$\omega_2 = \frac{\pi}{8}; \quad N_2 = \frac{2\pi}{\omega_2}m_2 = \frac{2\pi\, 8}{\pi} \quad \text{for } m_2 = 1$$
$$N_2 = 16$$
$$x_3[n] = 3\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$$
$$\omega_3 = \frac{\pi}{4}; \quad N_3 = \frac{2\pi}{\omega_3}m_3 = \frac{2\pi\, 4}{\pi} \quad \text{for } m_3 = 1$$
$$N_3 = 8$$

$\omega_1 = \frac{\pi}{2}$

$N_1 = \frac{2\pi}{\omega_1}$

$\frac{2\pi}{\pi/2} m$

3

$\pi/4$

To find the LCM of N_1, N_2 and N_3.

$$\begin{array}{r|rrr} 4 & 4, & 8, & 16 \\ \hline 2 & 1, & 2, & 4 \\ \hline & 1, & 1, & 2 \end{array}$$

$$\begin{array}{r|rrr} 2 & 4 & 8 & 16 \\ \hline 2 & 2 & 4 & 8 \\ 2 & 1 & 2 & 4 \\ 2 & 1 & 1 & 2 \\ & 1 & 1 & 1 \end{array}$$

$$\text{LCM} = 4 \times 2 \times 2 = 16$$

$$\boxed{N = 16}$$

The signal is periodic.

$2 \times 2 \times 2 \times 2 = 16$

(b)

$$x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$$
$$= x_1[n] + x_2[n]$$
$$x_1[n] = e^{j\frac{2\pi}{3}n}$$
$$\omega_1 = \frac{2\pi}{3}; \; N_1 = \frac{2\pi}{\omega_1}m_1 = \frac{2\pi}{2\pi}3 \quad \text{for } m_1 = 1$$
$$N_1 = 3$$
$$x_2[n] = e^{j\frac{3\pi}{4}n}$$
$$\omega_2 = \frac{3\pi}{4}; \; N_2 = \frac{2\pi}{\omega_2}m_2 = \frac{2\pi}{3\pi}4m_2$$

■ Example 1.59

Determine whether the following signal is periodic. If periodic find its fundamental period.

$$x[n] = \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)$$

(Anna University, December, 2006)

Solution:

$$x[n] = \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{n\pi}{4}\right)$$

$$= x_1[n]x_2[n]$$

$$x_1[n] = \cos\frac{n\pi}{2}$$

$$\omega_1 = \frac{\pi}{2}$$

$$N_1 = \frac{2\pi}{\omega_1}m_1 = \frac{2\pi}{\pi}2m_1 = 4 \quad \text{for } m_1 = 1$$

$$x_2[n] = \cos\frac{n\pi}{4}$$

$$\omega_2 = \frac{\pi}{4}$$

$$N_2 = \frac{2\pi}{\omega_2}m_2 = \frac{2\pi}{\pi}4m_2 = 8 \quad \text{for } m_2 = 1$$

$$\frac{N_1}{N_2} = \frac{4}{8} = \frac{1}{2} \quad \text{or}$$

$$2N_1 = N_2 = N$$

$$\boxed{N = 8}$$

The signal is periodic and the fundamental period $N = 8$.

■ Example 1.60

Test whether the following signals are periodic or not and if periodic, calculate the fundamental period.

(a) $x[n] = \cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{8}n\right) + 3\cos\left(\frac{\pi}{4}n + \frac{\pi}{3}\right)$

(b) $x[n] = e^{j\frac{2\pi}{3}n} + e^{j\frac{3\pi}{4}n}$

(Anna University, December, 2007)

(c) $x[n] = \sin^2 \frac{\pi}{4} n$

$$x[n] = \sin^2 \frac{\pi}{4} n$$
$$= \frac{1}{2} - \frac{1}{2} \cos \frac{2\pi}{4} n$$
$$= x_1[n] + x_2[n]$$
$$x_1[n] = \frac{1}{2} = \frac{1}{2}(1)^n \text{ is periodic with } N_1 = 1$$
$$x_2[n] = -\frac{1}{2} \cos \frac{\pi}{2} n$$
$$\omega_0 = \frac{\pi}{2}$$
$$N_2 = \frac{2\pi}{\omega_0} m = 4m = 4 \qquad \text{for } m = 1$$
$$\frac{N_1}{N_2} = \frac{1}{4}$$
$$\text{or} \quad 4N_1 = N_2 = N$$
$$\boxed{N = 4}$$

■ Example 1.58

Find the periodicity of the DT signal

$$x[n] = \sin \frac{2\pi}{3} n + \cos \frac{\pi}{2} n$$

(Anna University, December, 2007)

Solution:

$$x[n] = \sin \frac{2\pi}{3} n + \cos \frac{\pi}{2} n$$
$$= x_1[n] + x_2[n]$$
$$x_1[n] = \sin \frac{2}{3} \pi n$$
$$\omega_1 = \frac{2}{3} \pi$$
$$N_1 = \frac{2\pi}{\omega_1} m_1 = \frac{2\pi}{2\pi} 3m_1 = 3 \quad \text{for } m_1 = 1$$
$$x_2[n] = \cos \frac{\pi}{2} n$$
$$\omega_2 = \frac{\pi}{2}$$
$$N_2 = \frac{2\pi}{\omega_2} m_2 = \frac{2\pi}{\pi} 2m_2 = 4 \quad \text{for } m_2 = 1$$

For $x[n]$ to be periodic with period N,

$$x[n+N] = x_1[n+N] + x_2[n+N]$$

$$x[n] = x[n+N]$$

$$x_1[n+mN_1] + x_2[n+kN_2] = x_1[n+N] + x_2[n+N]$$

The above equation is satisfied if

$$mN_1 = kN_2 = N$$

m and k which are integers are chosen to satisfy the above equation. It implies that N is the LCM of N_1 and N_2.

On similar line it can be proved that if $x_1[n]$ and $x_2[n]$ are periodic signals with fundamental period N_1 and N_2 respectively, then $x[n] = x_1[n]x_2[n]$ is periodic if

$$mN_1 = kN_2 = N$$

■ Example 1.57

Find whether the following signals are periodic. If periodic, determine the fundamental period

(a) $x[n] = e^{j\pi n}$

(b) $x[n] = \cos\left[\frac{n}{8} - \pi\right]$

(c) $x[n] = \sin^2 \frac{\pi}{4} n$

Solution:

(a) $\boldsymbol{x[n] = e^{j\pi n}}$

$$\omega_0 = \pi$$

$$N = \frac{2\pi}{\omega_0} m$$

$$\boxed{N = \frac{2\pi}{\pi} = 2} \quad \text{if } m = 1$$

$x[n]$ is periodic with fundamental period 2.

(b) $\boldsymbol{x[n] = \cos\left[\frac{n}{8} - \pi\right]}$

$$\omega_0 = \frac{1}{8}$$

$$N = \frac{2\pi}{\omega_0} m = 16\pi m$$

For any integer value of m, N is not integer. Hence, $x[n]$ is not periodic.

$$\boxed{x[n] \text{ is not periodic}}$$

$$= e^{j\omega_0 n} \quad \text{if } e^{j\omega_0 N} = 1$$

$$\omega_0 N = m2\pi \quad \text{where } m \text{ is any integer.}$$

$$\boxed{N = m\frac{2\pi}{\omega_0}}$$

or

$$\frac{\omega_0}{2\pi} = \frac{m}{N} = \text{rational number.}$$

Thus, $e^{j\omega_0 n}$ is periodic if $\frac{m}{N}$ is rational. For $m = 1$, $N = N_0$. The corresponding frequency $F_0 = \frac{1}{N_0}$ is the fundamental frequency. F_0 is expressed in cycles and not Hz. Similarly ω_0 is expressed in radians and not in radians per second.

■ Example 1.55

Consider the following DT Signal.

$$x[n] = \sin(\omega_0 n + \phi)$$

Under what condition, the above signal is periodic.

Solution:

$$x[n] = \sin(\omega_0 n + \phi)$$

$$x[n+N] = \sin(\omega_0(n+N) + \phi)$$

$$= \sin(\omega_0 n + \omega_0 N + \phi)$$

$$= \sin(\omega_0 n + \phi) \quad \text{if } \omega_0 N = 2\pi m$$

$$= x[n]$$

$$\boxed{\frac{\omega_0}{2\pi} = \frac{m}{N} = \text{rational}}$$

■ Example 1.56

If $x_1[n]$ and $x_2[n]$ are periodic then show that the sum of the composite signal $x[n] = x_1[n] + x_2[n]$ is also periodic with the least common multiple (LCM) of the fundamental period of individual signal.

Solution: Let N_1 and N_2 be the fundamental periods of $x_1[n]$ and $x_2[n]$ respectively. Since both $x_1[n]$ and $x_2[n]$ are periodic,

$$x_1[n] = x_1[n + mN_1]$$

$$x_2[n] = x_2[n + kN_2]$$

$$x[n] = x_1[n] + x_2[n]$$

$$= x_1[n + mN_1] + x_2[n + kN_2]$$

1.9 Classification of Discrete Time Signals

Like continuous time signals, discrete time signals are also classified as

1. Periodic and non-periodic signals.
2. Odd and even signals.
3. Power and energy signals.

They are discussed below with suitable examples.

1.9.1 Periodic and Non-Periodic DT Signals

A discrete time signal (sequence) $x[n]$ is said to be periodic with period N which is a positive integer if

$$x[n+N] = x[n] \quad \text{for all } n \tag{1.50}$$

Consider the DT sequence shown in Figure 1.70. The signal gets repeated for every N. For Figure 1.70, the following equation is written:

$$x[n+mN] = x[n] \quad \text{for all } n \tag{1.51}$$

where m is any integer. The smallest positive integer N in Equation (1.51) is called the fundamental period N_0. Any sequence which is not periodic is said to be non-periodic or aperiodic.

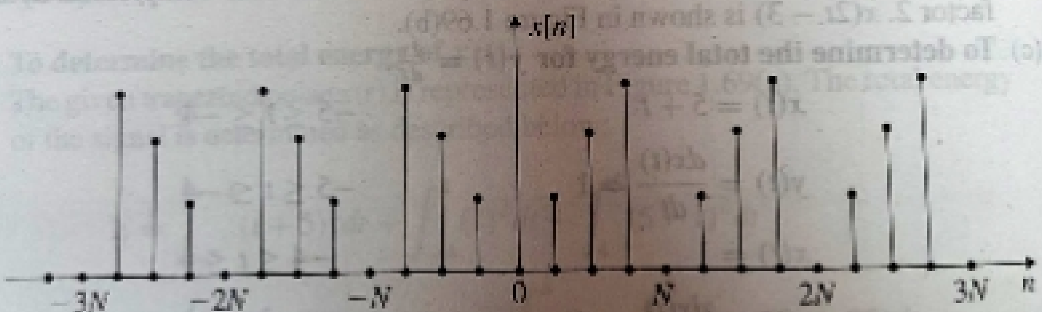

Figure 1.70 Periodic sequence.

■ **Example 1.54**

Show that complex exponential sequence $x[n] = e^{j\omega_0 n}$ is periodic and find the fundamental period.

Solution:

$$x[n] = e^{j\omega_0 n}$$

$$x[n+N] = e^{j\omega_0(n+N)}$$

$$= e^{j\omega_0 n} e^{j\omega_0 N}$$

Solution:

(1) $x[n] = (-1)^n$ is tabulated for $-2 \le n \le 2$.

n	-2	-1	0	1	2
$x[n]$	1	-1	1	-1	1

The samples corresponding to the above table are sketched and shown as $x[n]$ in Figure 1.51(a).

(2) Consider the step sequence $u[n+2]$ for $-2 \le n \le 2$. The samples are shown in Figure 1.51(b).

(3) Consider unit step sequence $u[n+3]$ for $-3 \le n \le 3$. This is represented in Figure 1.51(c). From Figure 1.51(c), $-2u[3n+3]$ is obtained by amplitude inversion and multiplication and time scaling (compression). This is represented in Figure 1.51(d) for $-2 \le n \le 2$.

(4) Consider the step sequence $2[u[n] - u[n-1]]$ for $n \ge 0$. This is represented in Figure 1.51(e). This is nothing but the sample of strength 2 at $n = 0$.

(5) Now, by adding the samples in Figure 1.51(b), Figures 1.51(d) and (e), it can be easily verified that

$$\boxed{x[n] = u[n+2] - 2u[3n+3] + 2[u[n] - u[n-1]]} \quad -2 \le n \le 2$$

■ Example 1.21

Given

$$x[n] = \{1, 2, \underset{\uparrow}{3}, -4, 6\}$$

Plot the signal $x[-n-1]$.

(Anna University, May, 2007)

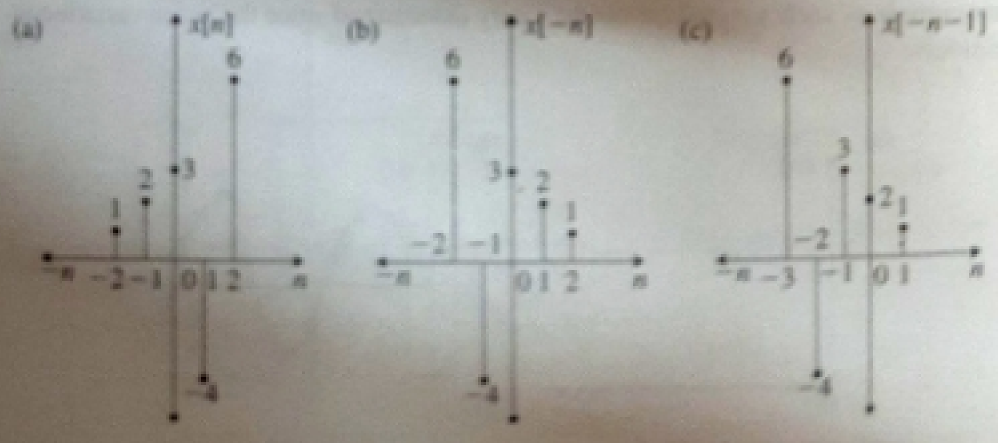

Figure 1.52 DT sequences of Example 1.21.

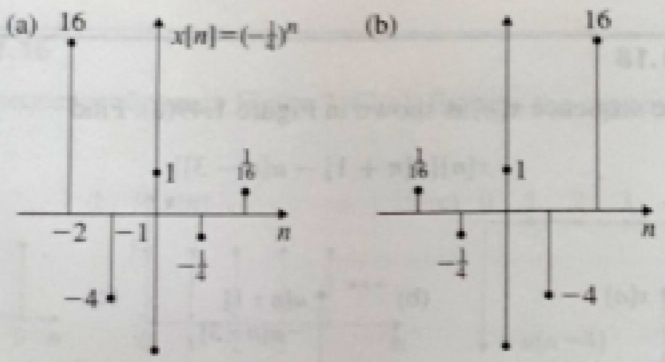

Figure 1.50 DT sequences of Example 1.19.

The samples of $x[n]$ are plotted and shown in Figure 1.50. $x[n] = (-\frac{1}{4})^n$ is represented in Figure 1.50(a) and $x[n] = (-4)^n$ is represented in Figure 1.50(b).

■ Example 1.20

Express

$$x[n] = (-1)^n \qquad -2 \leq n \leq 2$$

as a sum of scaled and shifted step function.

(Anna University, May, 2007)

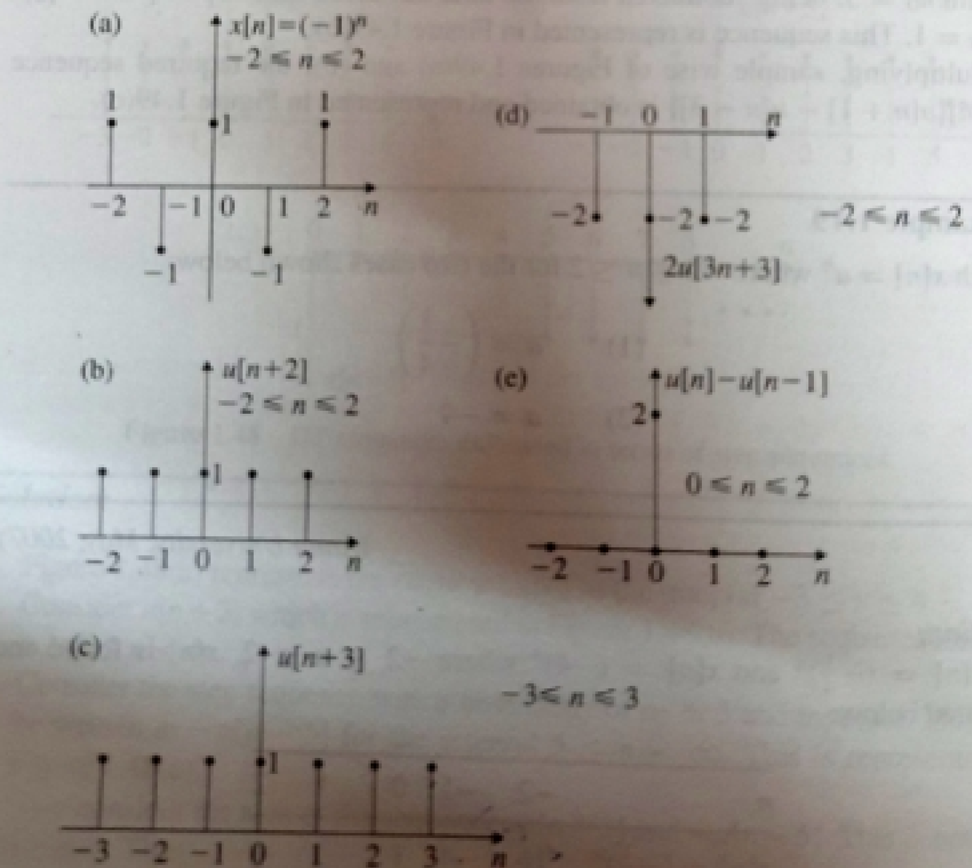

Figure 1.51 DT sequences of Example 1.20.

Example 1.18

A discrete time sequence $x[n]$ is shown in Figure 1.49(a). Find

$$x[n]\{u[n+1] - u[n-3]\}$$

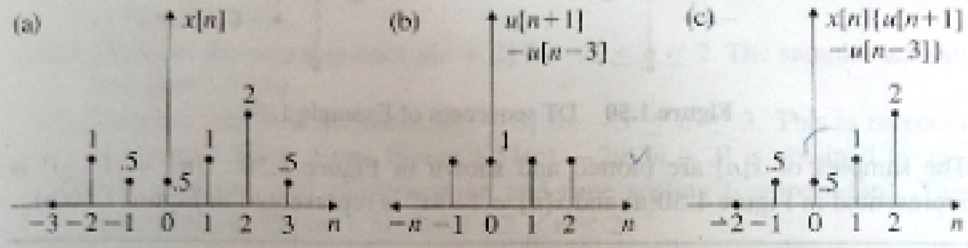

Figure 1.49 Multiplication of DT sequences.

Solution:

1. $x[n]$ sequence is represented in Figure 1.49(a).
2. $\{u[n+1] - u[n-3]\}$ sequence is nothing but the time delayed unit step sequence with $n_0 = 3$, being subtracted from the time advanced unit step sequence with $n_0 = 1$. This sequence is represented in Figure 1.49(b).
3. Multiplying, sample wise of Figures 1.49(a) and (b), the required sequence $x[n]\{u[n+1] - u[n-3]\}$ is obtained and represented in Figure 1.49(c).

Example 1.19

Sketch $x[n] = a^n$ where $-2 \le n \le 2$ for the two cases shown below:

$$(1) \quad a = \left(-\frac{1}{4}\right)$$

$$(2) \quad a = -4$$

(*Anna University, May, 2007*)

Solution:

For $x[n] = (-\frac{1}{4})^n$ and $x[n] = (-4)^n$ where $-2 \le n \le 2$, $x[n]$ is found and tabulated below:

n	-2	-1	0	1	2
$x[n] = (-\frac{1}{4})^n$	16	-4	1	$-\frac{1}{4}$	$\frac{1}{16}$
$x[n] = (-4)^n$	$\frac{1}{16}$	$-\frac{1}{4}$	1	-4	16

■ **Example 1.16**

Consider the sequence shown in Figure 1.47(a). Express the sequence in terms of step function.

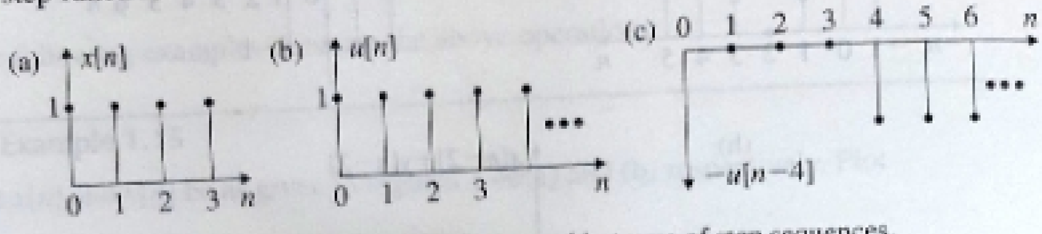

Figure 1.47 Sequences expressed in terms of step sequences.

Solution: The unit step sequence $u[n]$ is shown in Figure 1.47(b). The unit negative step sequence with a time delay of $n_0 = 4$ is shown in Figure 1.47(b). It is evident form Figure 1.47 that $\{u[n] - u[n-4]\}$ gives the required $x[n]$ sequence which is represented in Figure 1.47(a). Thus, $x[n] = \{u[n] - u[n-4]\}$.

■ **Example 1.17**

Consider the sequence shown in Figure 1.48(a). Express the sequence in terms of step function.

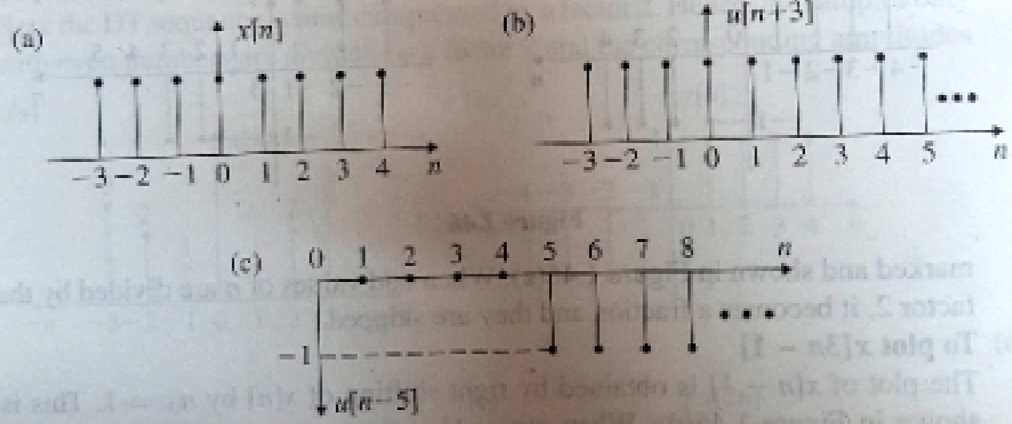

Figure 1.48 DT sequences expressed in terms of step sequences.

Solution:

1. Figure 1.48(a) represents the sequence $x[n]$ in the interval $-3 \leq n \leq 4$.
2. Consider $u[n+3]$ which is represented in Figure 1.48(b). The sequence interval is $-3 \leq n < \infty$.
3. Consider the step sequence with a time delay of $n_0 = 5$ and inverted. This can be written as $-u[n-5]$ for the interval $5 \leq n < \infty$. This is represented in Figure 1.48(c).
4. Now consider the sum of the sequences $u[n+3]$ and $-u[n-5]$. This is nothing but $x[n]$. Thus

$$\boxed{x[n] = u[n+3] - u[n-5]}$$

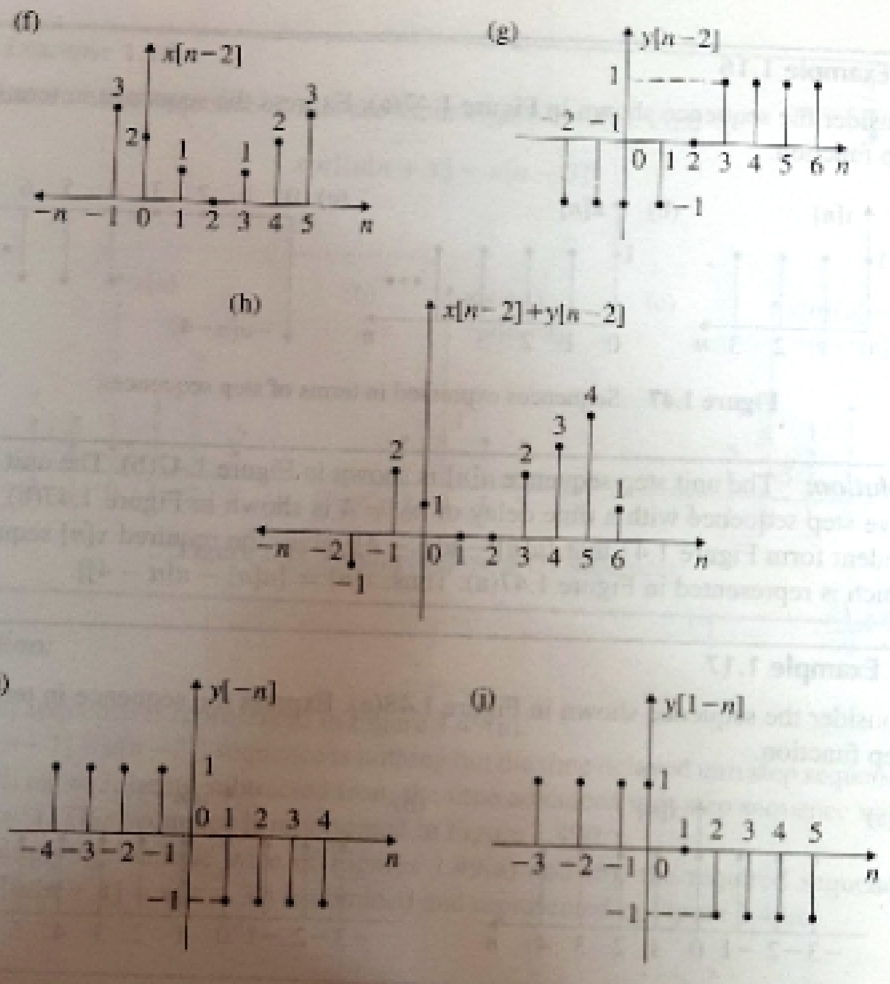

Figure 1.46

marked and shown in Figure 1.46(c). When odd values of n are divided by the factor 2, it becomes a fraction and they are skipped.

(b) To plot $x[3n-1]$

The plot of $x[n-1]$ is obtained by right shifting of $x[n]$ by $n_0 = 1$. This is shown in Figure 1.46(d). When $x[n-1]$ is time compressed by a factor 3, $x[3n-1]$ is obtained. Only integers which are divisible by 3 in the sequence $x[n-1]$ are to be taken to plot $x[3n-1]$. Thus samples for $n = 0$ and $n = 3$, will be plotted as shown in Figure 1.46(e).

(c) To plot $x[n-2] + y[n-2]$

The sequence $x[n-2]$ is obtained by right shifting of $x[n]$ by 2 and is shown in Figure 1.46(f). Similarly, the sequence $y[n-2]$ is obtained by right shifting of $y[n]$ by 2 and is shown in Figure 1.46(g). The sequence $x[n-2] + y[n-2]$ is obtained by summing up the sequences in Figures 1.46(f) and (g) for all n and is shown in Figure 1.46(h).

(d) To plot $y[1-n]$

The sequence $y[-n]$ is obtained by folding $y[n]$ and is shown in Figure 1.46(i). $y[-n]$ is right shifted by 1 sample to get the sequence $y[1-n]$. This is shown in Figure 1.46(j).

3. Using time shifting, plot $Ax[-n+n_0]$ where $n_0 > 0$. The time shift is to be right of $x[-n]$ by n_0 samples.
4. Using time scaling, plot $Ax[-\frac{n}{a}+n_0]$ where a is in integer. In the above case, keeping amplitude constant, time is expanded by a.

The following examples illustrate the above operations:

■ Example 1.15

Let $x[n]$ and $y[n]$ be as given in Figures 1.46(a) and (b) respectively. Plot

(a) $x[2n]$

(b) $x[3n-1]$

(c) $x[n-2]+y[n-2]$

(d) $y[1-n]$

(*Anna University, December, 2006*)

Solution:

(a) **To plot $x[2n]$**

Here the DT sequence is time compressed by a factor 2. Hence, the samples only with even numbers are divided by a factor 2 and the corresponding amplitudes

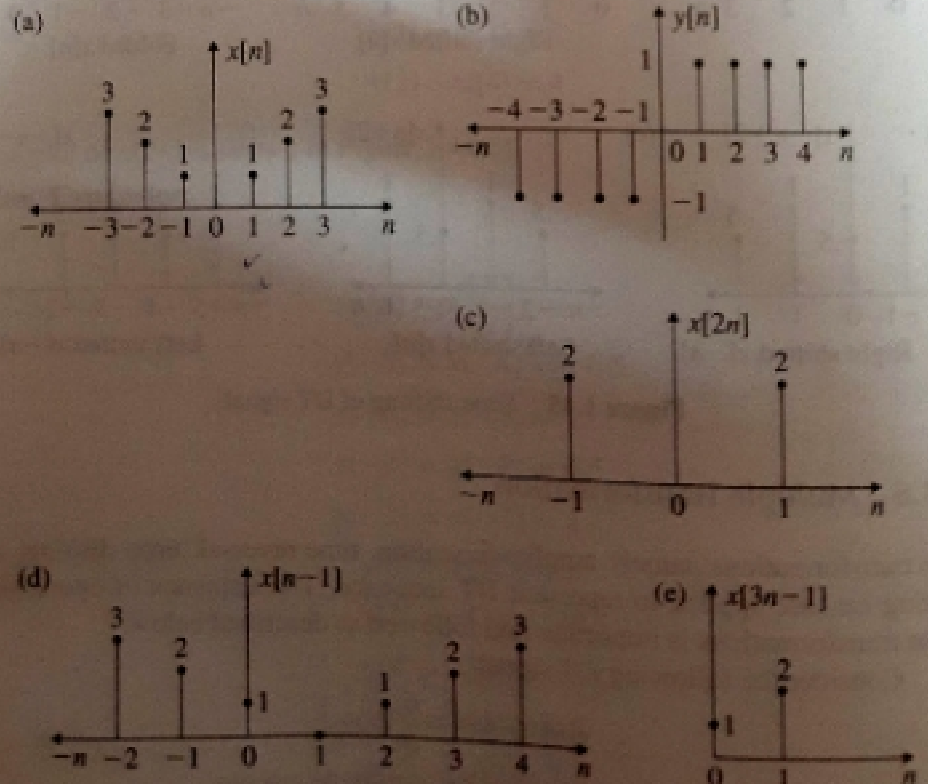

Figure 1.46 Two discrete sequences.

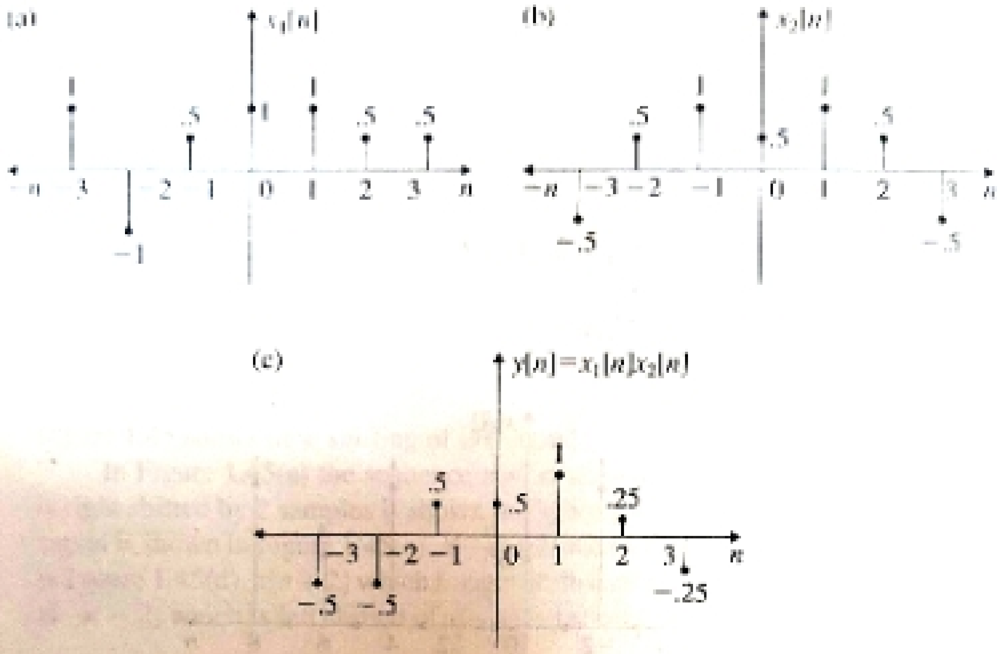

Figure 1.42 Multiplications of two DT signals.

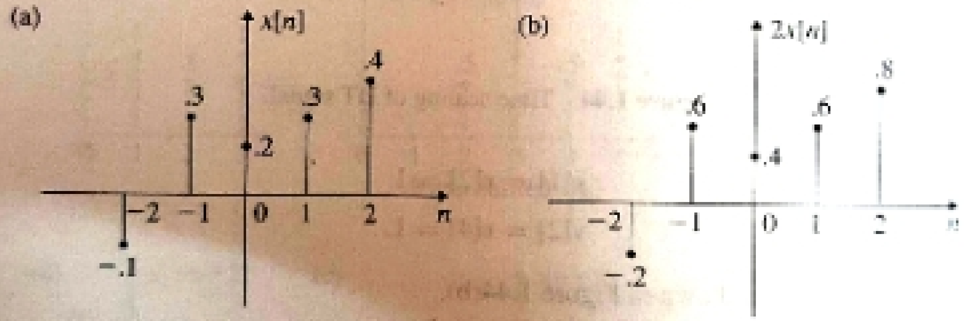

Figure 1.43 Amplitude scaling of DT signals.

1.7.4 Time Scaling of DT Signal

The time compression or expansion of a DT signal in time is known as time scaling. Consider the signal $x[n]$ shown in Figure 1.44(a). The time compressed signal $x[2n]$ and time expanded signal $x[\frac{n}{2}]$ are shown in Figures 1.44(b) and (c) respectively. One should note that while doing compression and expansion of DT signal, **only for integer value of n the samples exist. For non-integer value of n, the samples do not exist.**

Time Compression

Let

$$y[n] = x[2n]$$
$$y[-2] = x[-4] = -0.5$$
$$y[-1] = x[-2] = 0.5$$
$$y[0] = x[0] = 0.5$$

1.7 Basic Operations on Discrete Time Signals

The basic operations that are applied to continuous time signals are also applicable to discrete time signals. The time t in CT signal is replaced by n in DT signals. The basic operations as applied to DT signals are explained below.

1.7.1 Addition of Discrete Time Sequence

Addition of discrete time sequence is done by adding the signals at every instant of time. Consider the signals $x_1[n]$ and $x_2[n]$ shown in Figures 1.41(a) and (b) respectively. The addition of these signals at every n is done and represented as $x[n] = x_1[n] + x_2[n]$. This is shown in Figure 1.41(c).

1.7.2 Multiplication of DT Signals

The multiplication of two DT signals $x_1[n]$ and $x_2[n]$ is obtained by multiplying the signal values at each instant of time n. Consider the signal $x_1[n]$ and $x_2[n]$ represented in Figures 1.42(a) and (b). At each instant of time n, the samples of $x_1[n]$ and $x_2[n]$ are multiplied and represented as shown in Figure 1.42(c).

1.7.3 Amplitude Scaling of DT Signal

Let $x[n]$ be a discrete time signal. The signal $Ax[n]$ is represented by multiplying the amplitude of the sequence by A at each instant of time n. Consider the signal $x[n]$ shown in Figure 1.43(a). The signal $2x[n]$ is represented and shown in Figure 1.43(b).

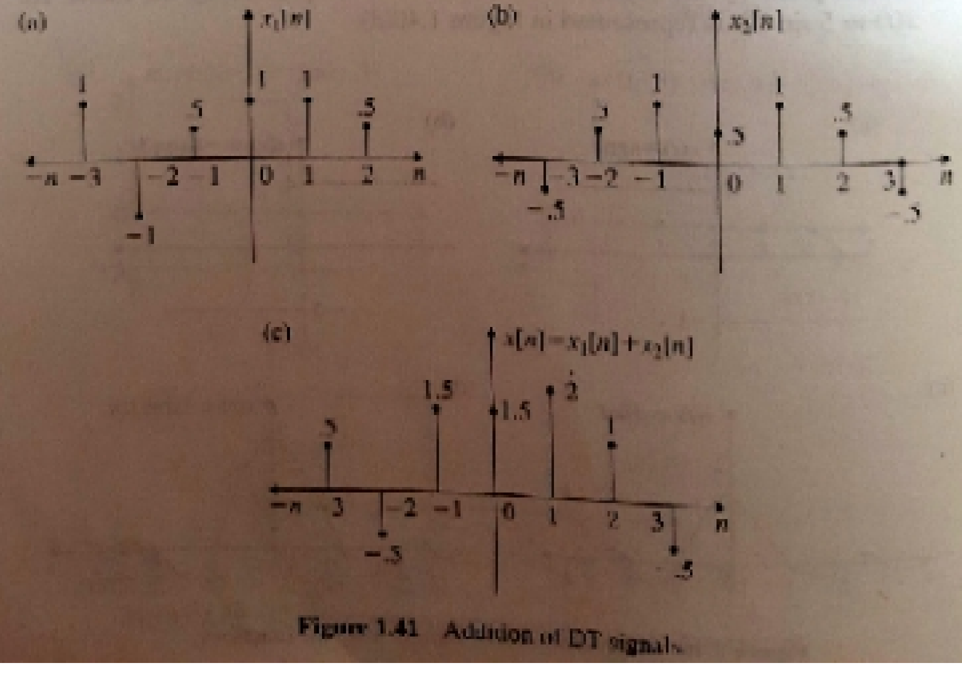

Figure 1.41 Addition of DT signals.