* Orthonormal Basis :

— Inner product :

let $\bar{u} = (u_1, u_2, \cdots, u_n)$ and $\bar{v} = (v_1, v_2, \cdots, v_n)$ be the element of R^n then inner product of $\bar{u}$ and $\bar{v}$ is

$$\langle \bar{u}, \bar{v} \rangle = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

Note that : i> If $\langle \bar{u}, \bar{v} \rangle = 0$ then $\bar{u}$ and $\bar{v}$ are orthogonal to each other

ii> If $\langle \bar{u}, \bar{u} \rangle = 1$ then $\bar{u}$ is said to be unit vector $\left(\because \langle \bar{u}, \bar{u} \rangle = \| u \|^2 \right)$

* orthogonal set :

let $S = \{ \bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n \}$ be the set of vectors in R^n then S is said to be orthogonal set if $\langle \bar{v}_i, \bar{v}_j \rangle = 0$, for all $i \neq j$

(i.e. Any two disjoint vectors are orthogonal to each other)

* Orthonormal set :

let $S = \{ \bar{v}_1, \bar{v}_2, \cdots, \bar{v}_n \}$ be the set of vectors in R^n then S is said to be orthonormal set if

i> $\langle v_i, v_j \rangle = 0$, for all $i \neq j$ (orthogonal)

ii> $\langle v_i, v_i \rangle = 1$, for all i (unit vector)

Ex. 1 Check whether the following set of vectors in R^3 are orthogonal with respect to inner product

$$\left\{ \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\}$$

Solution: let $\bar{V}_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$

and $\bar{V}_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$, $\bar{V}_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Now,

$$\langle \bar{V}_1, \bar{V}_2 \rangle = \left\langle \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) \right\rangle$$

$$= \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)$$

$$= \frac{4}{9} - \frac{2}{9} - \frac{2}{9}$$

$$= 0$$

$$\langle \bar{V}_2, \bar{V}_3 \rangle = \left\langle \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\rangle$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right)$$

$$= \frac{2}{9} + \frac{2}{9} - \frac{4}{9}$$

$$= 0$$

$$\langle \bar{V}_1, \bar{V}_3 \rangle = \left\langle \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) \right\rangle$$

$$= \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) + \left(-\frac{2}{3}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)$$

$$= \frac{2}{9} - \frac{4}{9} + \frac{2}{9}$$

$$= 0$$

Hence, $\{\bar{V}_1, \bar{V}_2, \bar{V}_3\}$ is orthogonal set

Ex ② which of the following set are orthonormal with respect to the inner product P_2 defined by $\langle p, q \rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

where, $p = a_0 + a_1 x + a_2 x^2$, $q = b_0 + b_1 x + b_2 x^2$

(a) x , $\frac{1}{2} + x^2$, $1 - x - \frac{1}{2}x^2$

Solution: let $v_1 = x$, $v_2 = \frac{1}{2} + x^2$, $v_3 = 1 - x - \frac{1}{2}x^2$

$$\therefore \langle v_1, v_2 \rangle = \langle (0, 1, 0), (\tfrac{1}{2}, 0, 1) \rangle$$

$$= (0)(\tfrac{1}{2}) + (1)(0) + (0)(1)$$

$$= 0$$

$$\langle v_2, v_3 \rangle = \langle (\tfrac{1}{2}, 0, 1), (1, -1, -\tfrac{1}{2}) \rangle$$

$$= (\tfrac{1}{2})(1) + (0)(-1) + (1)(-\tfrac{1}{2})$$

$$= \tfrac{1}{2} + 0 - \tfrac{1}{2}$$

$$= 0$$

$$\langle v_1, v_3 \rangle = \langle (0, 1, 0), (1, -1, -\tfrac{1}{2}) \rangle$$

$$= (0)(1) + (1)(-1) + (0)(-\tfrac{1}{2})$$

$$= -1 \neq 0$$

$\therefore$ the set $\{x, \frac{1}{2} + x^2, 1 - x - \frac{1}{2}x^2\}$ is not orthogonal

$\Rightarrow$ $\{x, \frac{1}{2} + x^2, 1 - x - \frac{1}{2}x^2\}$ is NOT orthonormal set

Ex. 3 cheack whether the following vectors are orthonormal

$$u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \quad u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Solution:

$$\langle u_1, u_2\rangle = \left\langle \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} - \frac{1}{2}$$

$$= 0$$

$\therefore$ $\{u_1, u_2\}$ is orthogonal set

Now, $\langle u_1, u_1\rangle = \left\langle \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)\right\rangle$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

and $\langle u_2, u_2\rangle = \left\langle \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\rangle$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2} + \frac{1}{2}$$

$$= 1$$

Hence Both u_1 and u_2 are unit vectors

$\therefore$ $\{u_1, u_2\}$ is orthonormal set.

* Gram-Schmidt process :-

working Rule:

step 1 Given: $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$

let $\bar{v}_1 = \bar{u}_1$

step 2. $$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$$

step 3 $$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

Then $\{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ is orthogonal set.

and $\left\{\frac{\bar{v}_1}{\|v_1\|}, \frac{\bar{v}_2}{\|v_2\|}, \frac{\bar{v}_3}{\|v_3\|}\right\}$ is orthonormal set.

Example 1. Construct an orthonormal basis of R^2 by applying Gram-schmidt orthogonalisation to $S = \{(3,1),(4,2)\}$

Solution: let $u_1 = (3,1)$, $u_2 = (4,2)$

Step 1 $v_1 = u_1 = (3,1)$

Step 2 $$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 \quad \text{——} (1)$$

consider, $$\begin{aligned}\langle u_2, v_1\rangle &= \langle (4,2), (3,1)\rangle \\ &= (4)(3) + (2)(1) \\ &= 14\end{aligned}$$

$$\|v_1\|^2 = \|(3,1)\|^2 = (3)^2 + (1)^2 = 10$$

$$\therefore \; v_2 = (4,2) - \frac{14}{10}(3,1) \qquad \text{(using eq}^n \text{ 1)}$$

$$= (4,2) - \left(\frac{21}{5}, \frac{7}{5}\right)$$

$$= \left(4 - \frac{21}{5}, 2 - \frac{7}{5}\right)$$

$$= \left(-\frac{1}{5}, \frac{3}{5}\right)$$

$\therefore$ the set $\{v_1, v_2\} = \left\{ (3,1), \left(-\frac{1}{5}, \frac{3}{5}\right) \right\}$ is orthogonal set

Now, $\|v_1\| = \|(3,1)\| = \sqrt{(3)^2 + (1)^2} = \sqrt{10}$

$\|v_2\| = \left\|\left(-\frac{1}{5}, \frac{3}{5}\right)\right\| = \sqrt{\left(-\frac{1}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \sqrt{\frac{2}{5}}$

$$\therefore \; \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{10}}(3,1) = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

and $\dfrac{v_2}{\|v_2\|} = \dfrac{1}{\sqrt{\frac{2}{5}}}\left(-\frac{1}{5}, \frac{3}{5}\right) = \sqrt{\frac{5}{2}}\left(-\frac{1}{5}, \frac{3}{5}\right) = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$

therefore, The orthonormal Basis are

$$\left\{ \frac{v_1}{\|v_2\|}, \frac{v_2}{\|v_2\|} \right\} = \left\{ \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) \right\}$$

Example ②: Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into orthonormal bases where

$u_1 = (1,1,1)$, $u_2 = (0,1,1)$, $u_3 = (0,0,1)$

solution: step 1: $v_1 = u_1 = (1,1,1)$

step 2: $$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1 \quad \text{——— ①}$$

* $\langle u_2, v_1\rangle = \langle (0,1,1), (1,1,1)\rangle$

$= (0)(1) + (1)(1) + (1)(1)$

$= 2$

* $\|v_1\|^2 = (1)^2 + (1)^2 + (1)^2 = 3$

equation ① becomes

$$v_2 = (0,1,1) - \frac{2}{3}(1,1,1) = (0,1,1) + \left(-\frac{2}{3}, -\frac{2}{3}, -\frac{2}{3}\right)$$

$$= \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step3 $$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$$

* $\langle u_3, v_1\rangle = \langle (0,0,1), (1,1,1)\rangle = 0+0+1 = 1$

* $\langle u_3, v_2\rangle = \langle (0,0,1), \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)\rangle = 0+0+\frac{1}{3} = \frac{1}{3}$

* $\|v_1\|^2 = (1)^2 + (1)^2 + (1)^2 = 3$

* $\|v_2\|^2 = \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{6}{9} = \frac{2}{3}$

equation ② becomes

$$V_3 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) + \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right) - \frac{1}{2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0,0,1) + \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right) + \left(\frac{1}{3}, -\frac{1}{6}, -\frac{1}{6}\right)$$

$$= \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Therefore, $V_1 = (1,1,1)$, $V_2 = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $V_3 = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$

forms the orthonal basis for R^3

* $\|V_1\| = \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$

* $\|V_2\| = \sqrt{\left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2} = \sqrt{\frac{6}{9}} = \frac{\sqrt{6}}{3}$

* $\|V_3\| = \sqrt{(0)^2 + \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}$

Therefore, orthonormal basis for R^3 is

$$\left\{\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|}\right\} = \left\{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}.$$

Homework.

Ex ③ let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis where

$u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$