7 Frequency Domain Analysis of RLC Circuits

7.1 INTRODUCTION

Time-domain analysis is the conventional method of analysing a network. For a simple network with first-order differential equation of network variable, this method is very useful. But as the order of network variable equation increases, this method of analysis becomes very tedious. For such applications, frequency domain analysis using Laplace transform is very convenient. Time-domain analysis, also known as *classical method,* is difficult to apply to a differential equation with excitation functions which contain derivatives. Laplace transform methods prove to be superior. The Laplace transform method has the following advantages:

(1) Solution of differential equations is a systematic procedure.
(2) Initial conditions are automatically incorporated.
(3) It gives the complete solution, i.e., both complementary and particular solution in one step.

Laplace transform is the most widely used integral transform. It is a powerful mathematical technique which enables us to solve linear differential equations by using algebraic methods. It can also be used to solve systems of simultaneous differential equations, partial differential equations and integral equations. It is applicable to continuous functions, piecewise continuous functions, periodic functions, step functions and impulse functions. It has many important applications in mathematics, physics, optics, electrical engineering, control engineering, signal processing and probability theory.

7.2 LAPLACE TRANSFORMATION

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

where s is the complex frequency variable.

$$s = \sigma + j\omega$$

The function $f(t)$ must satisfy the following condition to possess a Laplace transform,

$$\int_0^\infty |f(t)|e^{-\sigma t}\,\mathrm{d}t < \infty$$

where σ is real and positive.
The inverse Laplace transform $L^{-1}\{F(s)\}$ is

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st}\, ds$$

7.3 LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

1. *Constant Function k*

The Laplace transform of a constant function is

$$L\{k\} = \int_0^\infty k e^{-st} dt = k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{s}$$

2. *Function t^n*

The Laplace transform of $f(t)$ is

$$L\{t^n\} = \int_0^\infty t^n e^{-st} dt$$

Putting $st = x,\; dt = \dfrac{dx}{s}$

$$L\{t^n\} = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty x^n e^{-x} dx = \frac{\sqrt{n+1}}{s^{n+1}},\; s > 0,\, n+1 > 0$$

If n is a positive integer, $\sqrt{n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

3. *Unit-Step Function*

The unit-step function (Fig 7.1) is defined by the equation,

$$\begin{aligned} u(t) &= 1 \qquad & t > 0 \\ &= 0 \qquad & t < 0 \end{aligned}$$

The Laplace transform of unit step function is

$$L\{u(t)\} = \int_0^\infty 1 \cdot e^{-st}\, dt = \left[-\frac{e^{-st}}{s}\right]_0^\infty = \frac{1}{s}$$

Fig. 7.1 *Unit-step function*

4. *Delayed or Shifted Unit-Step Function*

The delayed or shifted unit-step function (Fig 7.2) is defined by the equation

$$\begin{aligned} u\,(t - a) &= 1 \qquad & t > a \\ &= 0 \qquad & t < a \end{aligned}$$

The Laplace transform of $u\,(t - a)$ is

$$L\{u(t-a)\} = \int_a^\infty 1 \cdot e^{-st}\, dt = \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$$

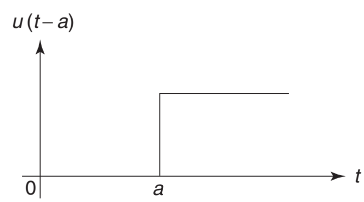

Fig. 7.2 *Shifted unit-step function*

5. Unit-Ramp Function

The unit-ramp function (Fig 7.3) is defined by the equation

$$r(t) = t \qquad t > 0$$
$$= 0 \qquad t < 0$$

The Laplace transform of the unit-ramp function is

$$L\{r(t)\} = \int_0^\infty t\, e^{-st} = \frac{1}{s^2}$$

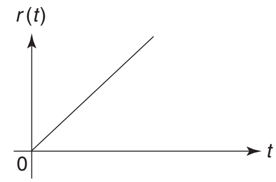

Fig. 7.3 *Unit-ramp function*

6. Delayed Unit-Ramp Function

The delayed unit-ramp function (Fig 7.4) is defined by the equation

$$r\,(t-a) = t \qquad t > a$$
$$= 0 \qquad t < a$$

The Laplace transform of $r\,(t-a)$ is

$$L\{r(t-a)\} = \int_a^\infty t\, e^{-st}\, dt = \frac{e^{-as}}{s^2}$$

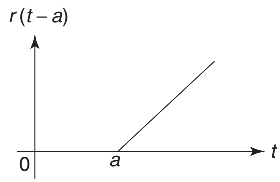

Fig. 7.4 *Delayed unit-ramp function*

7. Unit-Impulse Function

The unit-impulse function (Fig 7.5) is defined by the equation

$$\delta(t) = 0 \qquad t \neq 0$$

and $$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad t = 0$$

The Laplace transform of the unit-impulse function is

$$L\{\delta(t)\} = \int_0^\infty \delta(t)\, e^{-st}\, dt = 1$$

Fig. 7.5 *Unit-impulse function*

8. Exponential Function (e^{at})

The Laplace transform of the exponential function (Fig 7.6) is

$$L\left\{e^{at}\right\} = \int_0^\infty e^{at}\, e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$

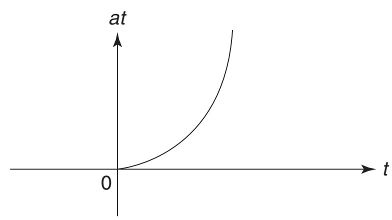

Fig. 7.6 *Exponential function*

9. Sine Function

We know that $\sin\omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$.

The Laplace transform of the sine function is

$$L\{\sin\omega t\} = L\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = \frac{1}{2j}\left[L\{e^{j\omega t}\} - L\{e^{-j\omega t}\}\right] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

10. Cosine Function

We know that $\cos\omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$.

The Laplace transform of the cosine function is

$$L\{\cos\omega t\} = L\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2}\left[L\{e^{j\omega t}\} + L\{e^{-j\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

11. *Hyperbolic sine function*

We know that $\sinh \omega t = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic sine function is

$$L\{\sinh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} - e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} - L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{\omega}{s^2-\omega^2}$$

12. *Hyperbolic cosine function*

We know that $\cosh \omega t = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic cosine function is

$$L\{\cosh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} + e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} + L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s-\omega}\right] = \frac{s}{s^2-\omega^2}$$

13. *Exponentially Damped Function*

Laplace transform of an exponentially damped function $e^{-at} f(t)$ is

$$L\left\{e^{-at} f(t)\, dt\right\} = \int_{0^-}^{\infty} f(t) e^{-at}\, e^{-st}\, dt = \int_{0^-}^{\infty} f(t) e^{-(s+a)t}\, dt = F(s+a)$$

Thus, the transform of the function $e^{-at} f(t)$ is obtained by putting $(s+a)$ in place of s in the transform of $f(t)$.

$$L\left\{e^{-at} \sin \omega t\right\} = \frac{\omega}{(s+a)^2+\omega^2} \qquad L\left\{e^{-at} \sinh \omega t\right\} = \frac{\omega}{(s+a)^2-\omega^2}$$

$$L\left\{e^{-at} \cos \omega t\right\} = \frac{s+a}{(s+a)^2+\omega^2} \qquad L\left\{e^{-at} \cosh \omega t\right\} = \frac{s+a}{(s+a)^2-\omega^2}$$

7.4 PROPERTIES OF LAPLACE TRANSFORM

7.4.1 Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$

where a and b are constants.

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^{\infty} \{af_1(t) + bf_2(t)\} e^{-st} dt = a\int_0^{\infty} f_1(t) e^{-st} dt + b\int_0^{\infty} f_2(t) e^{-st}\, dt = aF_1(s) + bF_2(s)$$

7.4.2 Time Scaling

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{f(at)\} = \int_0^{\infty} f(at) e^{-st}\, dt$$

Putting $at = x,\ dt = \dfrac{dx}{a}$

$$L\{(f(at)\} = \int_0^\infty f(x)\, e^{-s\left(\frac{x}{a}\right)} \frac{dx}{a} = \frac{1}{a}\int_0^\infty f(x)\, e^{-\left(\frac{s}{a}\right)x}\, dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

7.4.3 Frequency-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)t\, e^{-st}\ d$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-at} f(t) e^{-st}\ dt = \int_0^\infty f(t) e^{-(s+a)t}\ dt = F(s+a)$$

7.4.4 Time-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{f(t-a)\} = e^{-as}F(s)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t) e^{-st}\ dt$$

$$L\{f(t-a)\} = \int_0^\infty f(t-a)\, e^{-st}\ dt$$

Putting $t - a = x, \quad dt = dx$

When $t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\} = \int_0^\infty f(x) e^{-s(a+x)}\ dx = e^{-as}\int_0^\infty f(x) e^{-sx}\ dx = e^{-as}\int_0^\infty f(t)\, e^{-st}\ dt = e^{-as}F(s)$$

7.4.5 Multiplication by *t* (Frequency-Differentiation Theorem)

If $L\{f(t)\} = F(s)$ then $L\{t\, f(t)\} = -\dfrac{d}{ds}F(s)$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t) e^{-st}\ dt$$

Differentiating both the sides w.r.t s using DUIS,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t) e^{-st} dt = \int_0^\infty \frac{\partial}{\partial s} f(t) e^{-st}\ dt$$

$$= \int_0^\infty (-t)\ f(t) e^{-st}\ dt = \int_0^\infty \{-t\, f(t)\} e^{-st} dt = -L\{t\ f(t)\}$$

$$L\{t\ f(t)\} = (-1)\frac{d}{ds}F(s)$$

7.4.6 Division by t (Frequency-Integration Theorem)

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\, ds$

Proof

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)\, e^{-st}\ dt$$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^{\infty} F(s)\, ds = \int_s^{\infty}\int_0^{\infty} f(t)e^{-st} dt\ ds$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^{\infty} F(s) ds = \int_0^{\infty}\left[\int_s^{\infty} f(t)e^{-st} ds\right] dt = \int_0^{\infty}\left[\frac{1}{-t} f(t)e^{-st}\right]_s^{\infty} dt = \int_0^{\infty} \frac{f(t)}{t} e^{-st} dt$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s) ds$$

7.4.7 Time-Differentiation Theorem: Laplace Transform of Derivatives

If $L\{f(t)\} = F(s)$ then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

In general,

$$L\{f''(t)\} = s''F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{(n-1)}(0)$$

Proof

$$L\{f'(t)\} = \int_0^{\infty} f'(t)\, e^{-st}\ dt$$

Integrating by parts,

$$L\{f'(t)\} = \left[f(t)e^{-st}\right]_0^{\infty} - \int_0^{\infty} (-s) f(t)e^{-st} dt = -f(0) + s\int_0^{\infty} f(t)e^{-st} dt = -f(0) + sL\{f(t)\}$$

Similarly,

$$L\{f''\ t\} = -f' + sL\{f'\ t\} = -f'(0) + s\left[-f(0) + sL\{f(t)\}\right] = -f'(0) - sf(0) + s^2 L\{f(t)\}$$

In general, $$L\left\{f^n(t)\right\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)$$

7.4.8 Time-Integration Theorem: Laplace Transform of Integral

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t) dt\right\} = \dfrac{F(s)}{s}$

Proof

$$L\left\{\int_0^t f(t)dt\right\} = \int_0^\infty \left\{\int_0^t f(t)\,dt\right\} e^{-st}\,dt$$

Integrating by parts,

$$L\left\{\int_0^t f(t)dt\right\} = \left[\int_0^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty \left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{d}{dt}\int_0^t f(t)dt\right)\right]dt = \int_0^\infty \frac{1}{s} f(t)e^{-st}dt = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

7.4.9 Initial Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\int_0^\infty f'(t)e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to\infty}[f'(t)e^{-st}]dt + f(0) = 0 + f(0) = f(0) = \lim_{t\to 0} f(t)$$

7.4.10 Final Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim_{t\to\infty} f(t) = \lim_{s\to\infty} sF(s)$

Proof We know that

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\,e^{-st}dt + f(0)$$

$$\lim_{s\to\infty} sF(s) = \lim_{s\to 0}\int_0^\infty f'(t)\,e^{-st}dt + f(0) = \int_0^\infty \lim_{s\to 0}\left[f'(t)e^{-st}\right]dt + f(0) = \int_0^\infty f'(t)dt + f(0)$$

$$= \left|f(t)\right|_0^\infty + f(0) = \lim_{t\to\infty} f(t) - f(0) + f(0) = \lim_{t\to\infty} f(t)$$

7.5 INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Partial fraction expansion

7.5.1 Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

Example 7.1 *Find the inverse Laplace transform of* $\dfrac{s^2-3s+4}{s^3}$.

Solution

$$F(s)=\frac{s^2-3s+4}{s^3}=\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}$$

$$L^{-1}\{F(s)\}=1-3t+2t^2$$

Example 7.2 *Find the inverse Laplace transform of* $\dfrac{3s+4}{s^2+9}$.

Solution

$$F(s)=\frac{3s+4}{s^2+9}=\frac{3s}{s^2+9}+\frac{4}{s^2+9}$$

$$L^{-1}\{F(s)\}=3\cos 3t+\frac{4}{3}\sin 3t$$

Example 7.3 *Find the inverse Laplace transform of* $\dfrac{4s+15}{16s^2-25}$.

Solution

$$F(s)=\frac{4s+15}{16s^2-25}=\frac{4s+15}{16\left(s^2-\frac{25}{16}\right)}=\frac{1}{4}\frac{s}{s^2-\frac{25}{16}}+\frac{15}{16}\frac{1}{s^2-\frac{25}{16}}$$

$$L^{-1}\{F(s)\}=\frac{1}{4}\cosh\frac{5}{4}t+\frac{3}{4}\sinh\frac{5}{4}t$$

Example 7.4 *Find the inverse Laplace transform of* $\dfrac{2s+2}{s^2+2s+10}$.

Solution

$$F(s)=\frac{2s+2}{s^2+2s+10}=\frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\}=2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}=2e^{-t}\cos 3t$$

Example 7.5 *Find the inverse Laplace transform of* $\dfrac{3s+7}{s^2-2s-3}$.

Solution

$$F(s)=\frac{3s+7}{s^2-2s-3}=\frac{3(s-1)+10}{(s-1)^2-4}=3\frac{(s-1)}{(s-1)^2-4}+10\frac{1}{(s-1)^2-4}$$

$$L^{-1}\{F(s)\}=3e^{t}L^{-1}\left\{\frac{s}{s^2-4}\right\}+10e^{t}L^{-1}\left\{\frac{1}{s^2-4}\right\}=3e^{t}\cosh 2t+5e^{t}\sinh 2t$$

7.5.2 Partial Fraction Expansion

Any function $F(s)$ can be written as $\dfrac{P(s)}{Q(s)}$ where $P(s)$ and $Q(s)$ are polynomials in s. For performing partial fraction expansion, the degree of $P(s)$ must be less than the degree of $Q(s)$. If not, $P(s)$ must be divided by $Q(s)$, so that the degree of $P(s)$ becomes less than that of $Q(s)$. Assuming that the degree of $P(s)$ is less than that $Q(s)$, four possible cases arise depending upon the factors of $Q(s)$.

Case I Factors are linear and distinct,

$$F(s) = \frac{P(s)}{(s+a)(s+b)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B}{s+b}$$

Case II Factors are linear and repeated,

$$F(s) = \frac{P(s)}{(s+a)(s+b)^n}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s+a} + \frac{B_1}{s+b} + \frac{B_2}{(s+b)^2} + \ldots + \frac{B_n}{(s+b)^n}$$

Case III Factors are quadratic and distinct,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)}$$

By partial-fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{Cs+D}{s^2+cs+d}$$

Case IV Factors are quadratic are repeated,

$$F(s) = \frac{P(s)}{(s^2+as+b)(s^2+cs+d)^n}$$

By partial-fraction expansion,

$$F(s) = \frac{As+B}{s^2+as+b} + \frac{C_1s+D_1}{s^2+cs+d} + \frac{C_2s+D_2}{(s^2+cs+d)^2} + \ldots + \frac{C_ns+D_n}{(s^2+cs+d)^n}$$

Example 7.6 *Find the inverse Laplace transform of* $\frac{s+2}{s(s+1)(s+3)}$.

Solution

$$F(s) = \frac{s+2}{s(s+1)(s+3)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3}$$

$$A = sF(s)\Big|_{s=0} = \frac{s+2}{(s+1)(s+3)}\Bigg|_{s=0} = \frac{2}{3}$$

$$B = (s+1)F(s)\big|_{s=-1} = \left.\frac{s+2}{s(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$C = (s+3)F(s)\big|_{s=-3} = \left.\frac{s+2}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Example 7.7 *Find the inverse Laplace transform of* $\dfrac{s+2}{s^2(s+3)}$.

Solution

$$F(s) = \frac{s+2}{s^2(s+3)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$\begin{aligned} s+2 &= As(s+3) + B(s+3) + Cs^2 \\ &= As^2 + 3As + Bs + 3B + Cs^2 \\ &= (A+C)s^2 + (3A+B)s + 3B \end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned} A + C &= 0 \\ 3A + B &= 1 \\ 3B &= 2 \end{aligned}$$

Solving these equations,

$$A = \frac{1}{9},\ B = \frac{2}{3},\ C = -\frac{1}{9}$$

$$F(s) = \frac{1}{9}\cdot\frac{1}{s} + \frac{2}{3}\cdot\frac{1}{s^2} - \frac{1}{9}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9}L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{9} + \frac{2}{3}t - \frac{1}{9}e^{-3t}$$

Example 7.8 *Find the inverse Laplace transform of* $\dfrac{s^2 - 15s - 11}{(s+1)(s-2)^2}$.

Solution

$$F(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^2}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{B}{s-2}+\frac{C}{(s-2)^2}$$

$$\begin{aligned}5s^2-15s-11 &= A(s-2)^2+B(s+1)(s-2)+C(s+1)\\ &= A(s^2-4s+4)+B(s^2-s-2)+C(s+1)\\ &= As^2-4As+4A+Bs^2-Bs-2B+Cs+c\\ &= (A+B)s^2-(4A+B-C)s+(4A-2B+C)\end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned}A+B&=5\\ 4A+B-C&=15\\ 4A-2B+C&=-11\end{aligned}$$

Solving these equations,

$$\begin{aligned}A&=1\\ B&=4\\ C&=-7\end{aligned}$$

$$F(s)=\frac{1}{s+1}+\frac{4}{s-2}-\frac{7}{(s-2)^2}$$

$$L^{-1}\{F(s)\}=L^{-1}\left\{\frac{1}{s+1}\right\}+4L^{-1}\left\{\frac{1}{s-2}\right\}-7L^{-1}\left\{\frac{1}{(s-2)^2}\right\}=e^{-t}+4e^{2t}-7te^{2t}$$

Example 7.9 *Find the inverse Laplace transform of* $\dfrac{3s+1}{(s+1)(s^2+2)}$.

Solution

$$F(s)=\frac{3s+1}{(s+1)(s^2+2)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s+1}+\frac{Bs+C}{s^2+2}$$

$$\begin{aligned}3s+1 &= A(s^2+2)+(Bs+C)(s+1)\\ &= As^2+2A+Bs^2+Bs+Cs+C\\ &= (A+B)s^2+(B+C)s+(2A+C)\end{aligned}$$

Comparing coefficients of s^2, s^1 and s^0,

$$\begin{aligned}A+B&=0\\ B+C&=3\\ 2A+C&=1\end{aligned}$$

Solving these equations,

$$A=-\frac{2}{3},\ B=\frac{2}{3},\ C=\frac{7}{3}$$

$$F(s) = -\frac{2}{3}\cdot\frac{1}{s+1} + \frac{2}{3}\cdot\frac{s}{s^2+2} + \frac{7}{3}\cdot\frac{1}{s^2+2}$$

$$L^{-1}\{F(s)\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\} + \frac{2}{3}L^{-1}\left\{\frac{s}{s^2+2}\right\} + \frac{7}{3}L^{-1}\left\{\frac{1}{s^2+2}\right\} = -\frac{2}{3}e^{-t} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

Example 7.10 *Find the inverse Laplace transform of* $\dfrac{s}{(s^2+1)(s^2+4)}$.

Solution

$$F(s) = \frac{s}{(s^2+1)(s^2+4)} = \frac{s}{3}\left[\frac{s^2+4-s^2-1}{(s^2+1)(s^2+4)}\right] = \frac{1}{3}\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right]$$

$$L^{-1}\{F(s)\} = \frac{1}{3}\left[L^{-1}\left\{\frac{s}{s^2+1}\right\} - L^{-1}\left\{\frac{s}{s^2+4}\right\}\right] = \frac{1}{3}[\cos t - \cos 2t]$$

7.6 FREQUENCY DOMAIN REPRESENTAION OF RLC CIRCUITS

Voltage–current relationships of network elements can also be represented in the frequency domain.

1. Resistor For the resistor, the v–i relationship in time domain is

$$v(t) = R\,i(t)$$

The corresponding frequency–domain relation are given as

$$V(s) = RI(s)$$

The transformed network is shown in Fig 7.7.

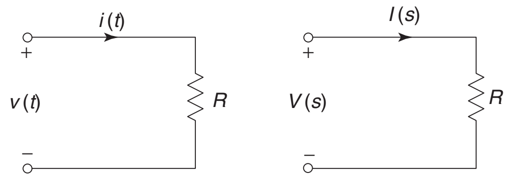

Fig. 7.7 *Resistor*

2. Inductor For the inductor, the v–i relationships in time domain are

$$v(t) = L\frac{di}{dt}$$

$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

The corresponding frequency-domain relation are given as

$$V(s) = Ls\,I(s) - L\,i(0)$$

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

The transformed network is shown in Fig 7.8.

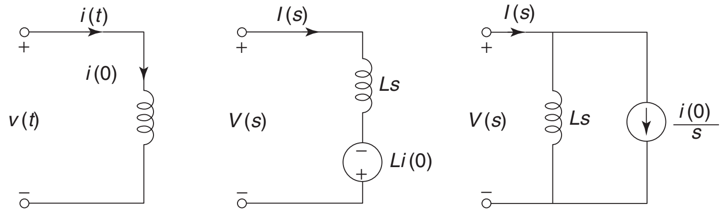

Fig. 7.8 *Inductor*

3. Capacitor For capacitor, the v–i relationships in time domain are

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

$$i(t) = C\frac{dv}{dt}$$

The corresponding frequency–domain relations are given as

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

$$I(s) = CsV(s) - Cv(0)$$

The transformed network is shown in Fig 7.9.

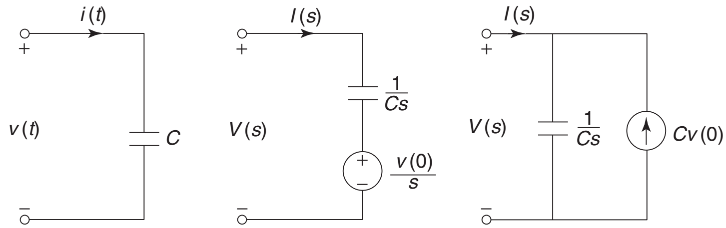

Fig. 7.9 *Capacitor*

7.7 RESISTOR–INDUCTOR CIRCUIT

Consider a series RL circuit as shown in Fig. 7.10. The switch is closed at time $t = 0$.

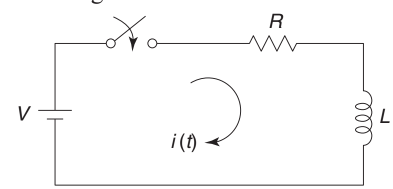

Fig. 7.10 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 7.11.

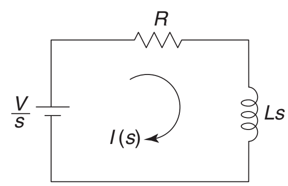

Fig. 7.11 *Transformed network*

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \frac{R}{L}}$$

$$A = sI(s)\big|_{s=0} = s \times \left.\frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)}\right|_{s=0} = \frac{V}{R}$$

$$B = \left(s+\frac{R}{L}\right)I(s)\Big|_{s=-\frac{R}{L}} = \left(s+\frac{R}{L}\right)\times\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\Bigg|_{s=-\frac{R}{L}} = -\frac{V}{R}$$

$$I(s) = \frac{\frac{V}{R}}{s} + \frac{\left(-\frac{V}{R}\right)}{s+\frac{R}{L}}$$

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R} - \frac{V}{R}e^{-\frac{R}{L}t}$$

$$= \frac{V}{R}\left[1-e^{-\frac{R}{L}t}\right] \qquad \text{for } t>0$$

Example 7.11 *In the network of Fig. 7.12, the switch is moved from the position 1 to 2 at t = 0, steady-state condition having been established in the position 1. Determine i (t) for t > 0.*

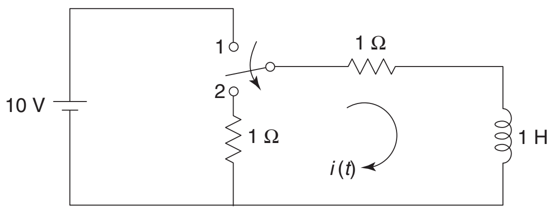

Fig. 7.12

Solution At $t = 0^-$, the network is shown in Fig 7.13. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{10}{1} = 10 \text{ A}$$

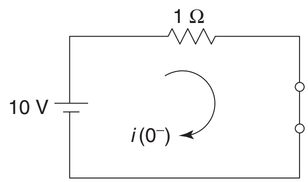

Fig. 7.13

Since the current through the inductor cannot change instantaneously,

$$i\,(0^+) = 10 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.14.

Applying KVL to the mesh for $t > 0$,

$$-I(s) - I(s) - sI(s) + 10 = 0$$

$$I(s)(s+2) = 10$$

$$I(s) = \frac{10}{s+2}$$

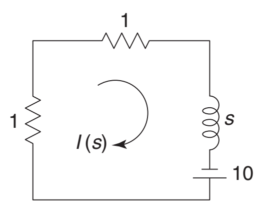

Fig. 7.14

Taking inverse Laplace transform,

$$i\,(\text{t}) = 10e^{-2t} \quad \text{for } t > 0$$

Example 7.12 *The network of Fig. 7.15 was initially in the steady state with the switch in the position a. At $t = 0$, the switch goes from a to b. Find an expression for voltage $v(t)$ for $t > 0$.*

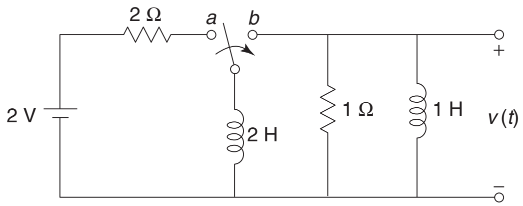

Fig. 7.15

Solution At $t = 0^-$, the network is shown in Fig 7.16. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor of 2H acts as a short circuit.

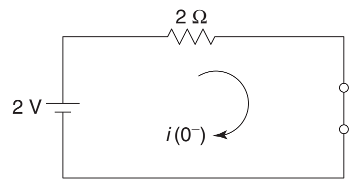

Fig. 7.16

$$i(0^-) = \frac{2}{2} = 1 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.17.

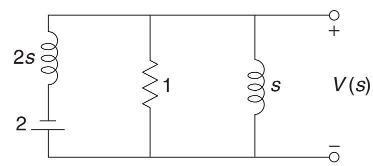

Fig. 7.17

Applying KCL at the node for $t > 0$,

$$\frac{V(s)+2}{2s} + \frac{V(s)}{1} + \frac{V(s)}{s} = 0$$

$$V(s)\left(1 + \frac{3}{2s}\right) = -\frac{1}{s}$$

$$V(s) = \frac{-\frac{1}{s}}{\frac{2s+3}{2s}} = -\frac{2}{2s+3} = -\frac{1}{s+1.5}$$

Taking the inverse Laplace transform,

$$v(t) = -e^{-1.5t} \qquad \text{for } t > 0$$

Example 7.13 *In the network of Fig. 7.18, the switch is opened at $t = 0$. Find $i(t)$.*

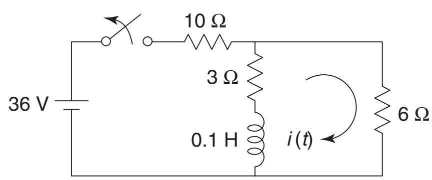

Fig. 7.18

Solution At $t = 0^-$, the network is shown in Fig. 7.19. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the inductor acts as a short circuit.

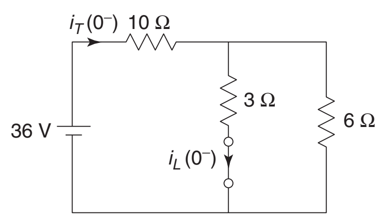

Fig. 7.19

$$i_T(0^-) = \frac{36}{10 + (3 \| 6)} = \frac{36}{10+2} = 3 \text{ A}$$

$$i_L(0^-) = 3 \times \frac{6}{6+3} = 2 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2\text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.20

Applying KVL to the mesh for $t > 0$,

$$-0.2 - 0.1s\,I(s) - 3I(s) - 6I(s) = 0$$

$$0.1sI(s) + 9I(s) = -0.2$$

$$I(s) = \frac{-0.2}{0.1s+9} = \frac{-2}{s+90}$$

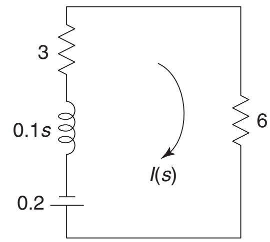

Fig. 7.20

Taking inverse Laplace transform,

$$i(t) = -2e^{-90t} \qquad \text{for } t > 0$$

Example 7.14 *The network shown in Fig. 7.21 has acquired steady-state with the switch closed for $t < 0$. At $t = 0$, the switch is opened. Obtain $i(t)$ for $t > 0$.*

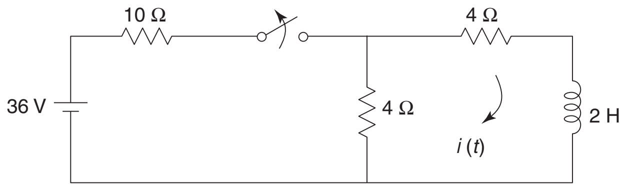

Fig. 7.21

Solution At $t = 0^-$, the network is shown in Fig 7.22. At $t = 0^-$, the switch is closed and the network has acquired steady-state. Hence, the inductor acts as a short circuit.

$$i_T(0^-) = \frac{36}{10+(4\,||\,4)} = \frac{36}{10+2} = 3\text{ A}$$

$$i(0^-) = 3 \times \frac{4}{4+4} = 1.5\text{ A}$$

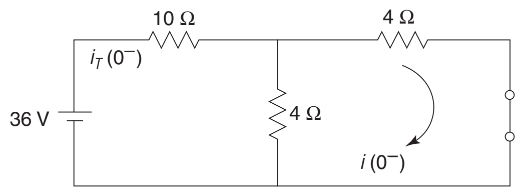

Fig. 7.22

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 1.5\text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.23.

Applying KVL to the mesh for $t > 0$,

$$-4I(s) - 4I(s) - 2sI(s) + 3 = 0$$

$$8I(s) + 2sI(s) = 3$$

$$I(s) = \frac{3}{2s+8} = \frac{1.5}{s+4}$$

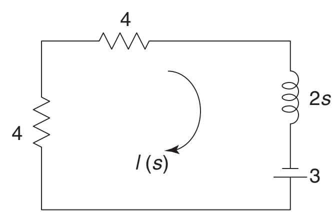

Fig. 7.23

Taking the inverse Laplace transform,

$$i(t) = 1.5e^{-4t} \qquad \text{for } t > 0$$

Example 7.15 *In the network shown in Fig. 7.24, the switch is closed at $t = 0$, the steady-state being reached before $t = 0$. Determine current through inductor of 3 H.*

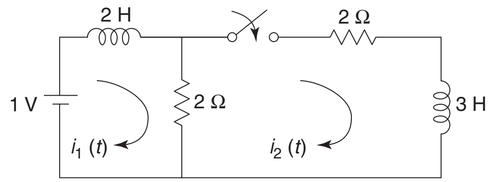

Fig. 7.24

Solution At $t = 0^-$, the network is shown in Fig. 7.25. At $t = 0^-$, steady-state condition is reached. Hence, the inductor of 2 H acts as a short circuit.

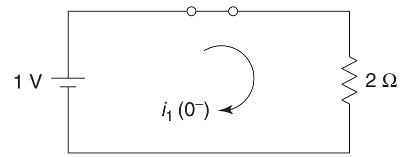

Fig. 7.25

$$i_1(0^-) = \frac{1}{2} \text{ A}$$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_1(0^+) = \frac{1}{2} \text{ A}$$

$$i_2(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 17.26.

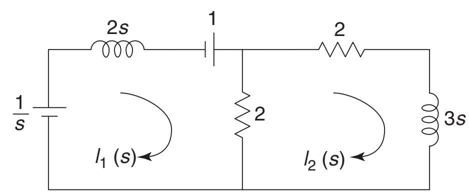

Fig. 7.26

Applying KVL to Mesh 1,

$$\frac{1}{s} - 2s\,I_1(s) + 1 - 2[I_1(s) - I_2(s)] = 0$$

$$(2+2s)\,I_1(s) - 2I_2(s) = 1 + \frac{1}{s}$$

Applying KVL to Mesh 2,

$$-2\,[I_2(s) - I_1(s)] - 2I_2(s) - 3s\,I_2\,(s) = 0$$

$$-2I_1(s) + (4+3s)\,I_2(s) = 0$$

By Cramer's rule,

$$I_2(s) = \frac{\begin{vmatrix} 2+2s & 1+\frac{1}{s} \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 2+2s & -2 \\ -2 & 4+3s \end{vmatrix}} = \frac{\frac{2}{s}(s+1)}{(2+2s)(4+3s)-4} = \frac{s+1}{s(3s^2+7s+2)} = \frac{s+1}{3s\left(s+\frac{1}{3}\right)(s+2)} = \frac{\frac{1}{3}(s+1)}{s(s+2)\left(s+\frac{1}{3}\right)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+\frac{1}{3}}$$

$$A = s\,I_2(s)\big|_{s=0} = \left.\frac{\frac{1}{3}(s+1)}{(s+2)\left(s+\frac{1}{3}\right)}\right|_{s=0} = \frac{1}{2}$$

$$B = (s+2)I_2(s)\big|_{S=-2} = \left.\frac{\frac{1}{3}(s+1)}{s\left(s+\frac{1}{3}\right)}\right|_{s=-2} = -\frac{1}{10}$$

$$C = \left(s+\frac{1}{3}\right) I_2(s)\Big|_{s=-\frac{1}{3}} = \left.\frac{\frac{1}{3}(s+1)}{s(s+2)}\right|_{s=-\frac{1}{3}} = -\frac{2}{5}$$

$$I_2(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{10}\frac{1}{s+2} - \frac{2}{5}\frac{1}{s+\frac{1}{3}}$$

Taking inverse Laplace transform

$$i_2(t) = \frac{1}{2} - \frac{1}{10}e^{-2t} - \frac{2}{5}e^{-\frac{1}{3}t} \quad \text{for } t > 0$$

Example 7.16 *In the network of Fig. 7.27, the switch is closed at t = 0 with the network previously unenergised. Determine currents $i_1(t)$.*

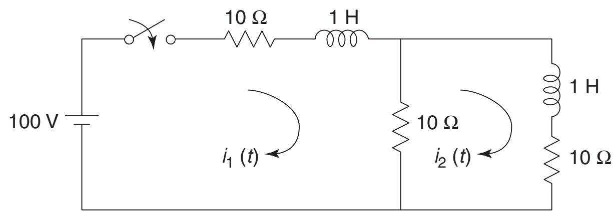

Fig. 7.27

Solution For $t > 0$, the transformed network is shown in Fig. 7.28.

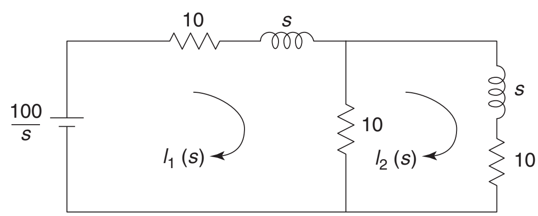

Fig. 7.28

Applying KVL to Mesh 1,

$$\frac{100}{s} - 10I_1(s) - sI_1(s) - 10\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+20)I_1(s) - 10I_2(s) = \frac{100}{s}$$

Applying KVL to Mesh 2,

$$-10\left[I_2(s) - I_1(s)\right] - s\, I_2(s) - 10\, I_2(s) = 0$$

$$-10\, I_1(s) + (s+20)I_2(s) = 0$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{100}{s}(s+20)}{(s+20)^2 - 100} = \frac{100(s+20)}{s(s^2+40s+300)} = \frac{100(s+20)}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_1(s)=\frac{A}{s}+\frac{B}{s+10}+\frac{C}{s+30}$$

$$A=s\,I_1(s)\big|_{s=0}=\left.\frac{100(s+20)}{(s+10)(s+30)}\right|_{s=0}=\frac{20}{3}$$

$$B=(s+10)I_1(s)\big|_{s=-10}=\left.\frac{100(s+20)}{s(s+30)}\right|_{s=-10}=-5$$

$$C=(s+30)I_1(s)\big|_{s=-30}=\left.\frac{100(s+20)}{s(s+10)}\right|_{s=-30}=-\frac{5}{3}$$

$$I_1(s)=\frac{20}{3}\frac{1}{s}-\frac{5}{s+10}-\frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_1(t)=\frac{20}{3}-5e^{-10t}-\frac{5}{3}e^{-30t}$$

Similarly,

$$I_2(s)=\frac{\begin{vmatrix} s+20 & \frac{100}{s} \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}}=\frac{\frac{1000}{s}}{(s+20)^2-100}=\frac{1000}{s(s^2+40s+300)}=\frac{1000}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_2(s)=\frac{A}{s}+\frac{B}{s+10}+\frac{C}{s+30}$$

$$A=sI_2(s)\big|_{s=0}=\left.\frac{1000}{(s+10)(s+30)}\right|_{s=0}=\frac{10}{3}$$

$$B=(s+10)I_2(s)\big|_{s=-10}=\left.\frac{1000}{s(s+30)}\right|_{s=-10}=-5$$

$$C=(s+30)I_2(s)\big|_{s=-30}=\left.\frac{1000}{s(s+10)}\right|_{s=-30}=\frac{5}{3}$$

$$I_2(s)=\frac{10}{3}\frac{1}{s}-\frac{5}{s+10}+\frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_2(t)=\frac{10}{3}-5e^{-10t}+\frac{5}{3}e^{-30t} \qquad \text{for } t>0$$

7.8 RESISTOR–CAPACITOR CIRCUIT

Consider a series *RC* circuit as shown in Fig. 7.29. The switch is closed at time $t = 0$.

For $t > 0$, the transformed network is shown in Fig. 7.30.

Applying KVL to the mesh,

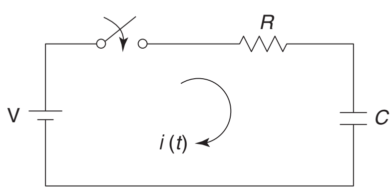

Fig. 7.29 *RC circuit*

$$\frac{V}{s} - RI(s) - \frac{1}{Cs}I(s) = 0$$

$$\left(R + \frac{1}{Cs}\right)I(s) = \frac{V}{s}$$

$$I(s) = \frac{\frac{V}{s}}{R + \frac{1}{Cs}} = \frac{\frac{V}{s}}{\frac{RCs+1}{Cs}} = \frac{\frac{V}{R}}{s + \frac{1}{RC}}$$

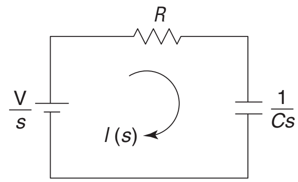

Fig. 7.30 *Transformed network*

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R}e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

Example 7.17 *In the network of Fig. 7.31, the switch is moved from a to b at t = 0. Determine i (t) and v_c (t).*

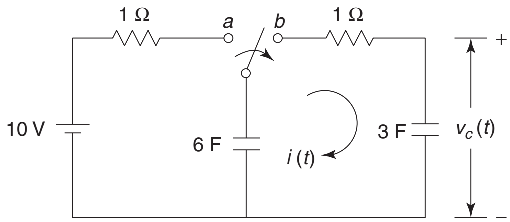

Fig. 7.31

Solution At $t = 0^-$, the network is shown in Fig. 7.32. At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor of 6 F acts as an open circuit.

$$v_{6\,F}(0^-) = 10 \text{ V}$$
$$i(0^-) = 0$$
$$v_{3\,F}(0^-) = 0$$

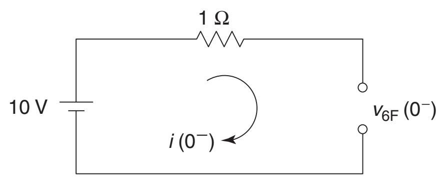

Fig. 7.32

Since voltage across the capacitor cannot change instantaneously,

$$v_{6\,F}(0^+) = 10 \text{ V}$$
$$v_{3\,F}(0^+) = 0$$

For $t > 0$, the transformed network is shown in 7.33.

Applying KVL to the mesh for $t > 0$,

$$\frac{10}{s} - \frac{1}{6s}I(s) - I(s) - \frac{1}{3s}I(s) = 0$$

$$\frac{1}{6s}I(s) + I(s) + \frac{1}{3s}I(s) = \frac{10}{s}$$

$$I(s) = \frac{10}{s\left(1 + \frac{1}{6s} + \frac{1}{3s}\right)} = \frac{60}{6s+3} = \frac{10}{s+0.5}$$

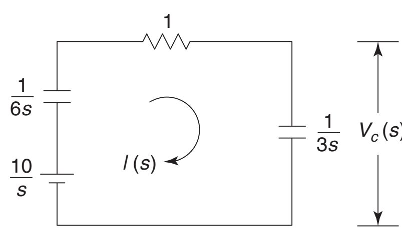

Fig. 7.33

Taking the inverse Laplace transform,

$$i(t) = 10e^{-0.5t} \qquad \text{for } t > 0$$

Voltage across the 3 F capacitor is given by

$$V_c(s) = \frac{1}{3s} I(s) = \frac{10}{3s(s+0.5)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.5}$$

$$A = sV_c(s)\Big|_{s=0} = \frac{10}{3(s+0.5)}\Bigg|_{s=0} = \frac{20}{3}$$

$$B = (s+0.5)V_c(s)\Big|_{s=-0.5} = \frac{10}{3s}\Bigg|_{s=-0.5} = -\frac{20}{3}$$

$$V_c(s) = \frac{20}{3}\frac{1}{s} - \frac{20}{3}\frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v_c(t) = \frac{20}{3} - \frac{20}{3}e^{-0.5t}$$

$$= \frac{20}{3}(1 - e^{-0.5t}) \qquad \text{for } t > 0$$

Example 7.18 *The switch in the network shown in Fig. 7.34 is closed at t = 0. Determine the voltage cross the capacitor.*

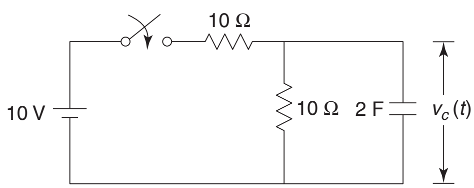

Fig. 7.34

Solution At $t = 0^-$, the capacitor is uncharged.

$$v_c(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.35.

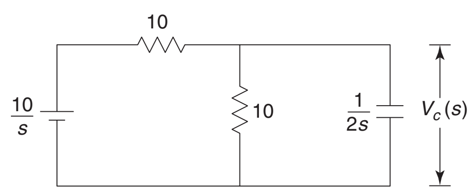

Fig. 7.35

Applying KCL at the node for $t > 0$,

$$\frac{V_c(s) - \frac{10}{s}}{10} + \frac{V_c(s)}{10} + \frac{V_c(s)}{\frac{1}{2s}} = 0$$

$$2sV_c(s) + 0.2V_c(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(2s+0.2)} = \frac{0.5}{s(s+0.1)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.1}$$

$$A = sV_c(s)\Big|_{s=0} = \left.\frac{0.5}{s+0.1}\right|_{s=0} = \frac{0.5}{0.1} = 5$$

$$B = (s+0.1)V_c(s)\Big|_{s=-0.1} = \left.\frac{0.5}{s}\right|_{s=-0.1} = -\frac{0.5}{0.1} = -5$$

$$V_c(s) = \frac{5}{s} - \frac{5}{s+0.1}$$

Taking inverse Laplace transform,

$$v_c(t) = 5 - 5e^{-0.1t} \qquad \text{for } t > 0$$

Example 7.19 *In the network of Fig. 7.36, the switch is closed for a long time and at t = 0, the switch is opened. Determine the current through the capacitor.*

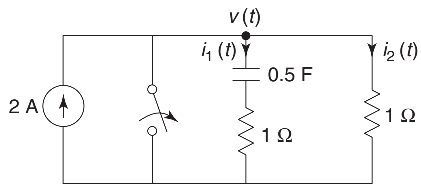

Fig. 7.36

Solution At $t = 0^-$, the network is shown in Fig. 7.37. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v_c(0^-) = 0$$

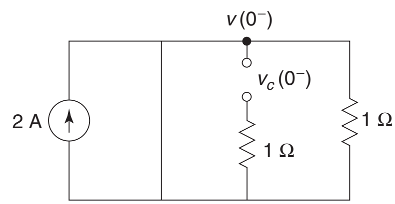

Fig. 7.37

Since voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.38.

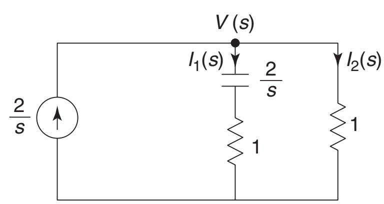

Fig. 7.38

Applying KVL to two parallel branches,

$$\frac{2}{s}I_1(s) + I_1(s) = I_2(s)$$

Applying KCL at the node for $t > 0$,

$$\frac{2}{s} = I_1(s) + I_2(s)$$

$$\frac{2}{s}I_1(s)+I_1(s)=\frac{2}{s}-I_1(s)$$

$$\frac{2}{s}I_1(s)+2I_1(s)=\frac{2}{s}$$

$$I_1(s)=\frac{\frac{2}{s}}{\frac{2}{s}+2}=\frac{1}{s+1}$$

Taking the inverse Laplace transform,

$$i_1(t)=e^{-t} \qquad \text{for } t>0$$

Example 7.20 *In the network of Fig. 7.39, the switch is moved from a to b, at t = 0. Find v(t).*

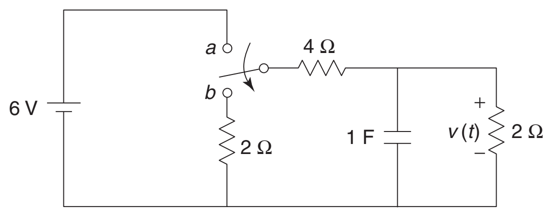

Fig. 7.39

Solution At $t = 0^-$, the network is shown in Fig 7.40. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v(0^-)=6\times\frac{2}{4+2}=2\text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+)=2\text{ V}$$

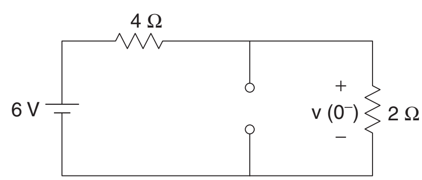

Fig. 7.40

For $t > 0$, the transformed network is shown in Fig. 7.41.

Applying KCL at the node for $t > 0$,

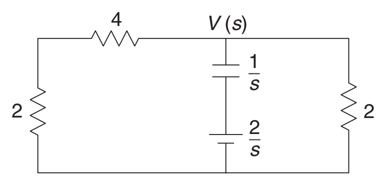

Fig. 7.41

$$\frac{V(s)}{6}+\frac{V(s)-\frac{2}{s}}{\frac{1}{s}}+\frac{V(s)}{2}=0$$

$$V(s)\left(\frac{2}{3}+s\right)=2$$

$$V(s)=\frac{2}{s+\frac{2}{3}}$$

Taking the inverse Laplace transform,

$$v(t)=2e^{-\frac{2}{3}t} \qquad \text{for } t>0$$

Example 7.21 *The network shown in Fig. 7.42 has acquired steady-state at $t < 0$ with the switch open. The switch is closed at $t = 0$. Determine $v(t)$.*

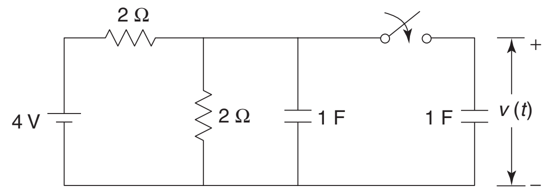

Fig. 7.42

Solution At $t = 0^-$, the network is shown in Fig 7.43. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor of 1 F acts as an open circuit.

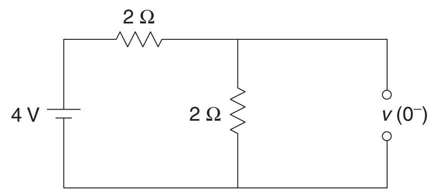

Fig. 7.43

$$v(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 7.44.

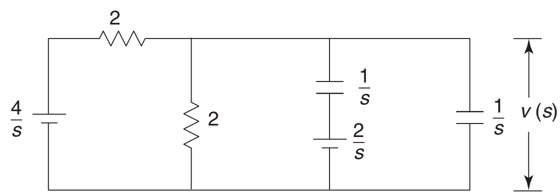

Fig. 7.44

Applying KCL at the node for $t > 0$,

$$\frac{V(s) - \frac{4}{s}}{2} + \frac{V(s)}{2} + \frac{V(s) - \frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{\frac{1}{s}} = 0$$

$$2sV(s) + V(s) = \frac{2}{s} + 2$$

$$V(s) = \frac{\frac{2}{s} + 2}{2s+1} = \frac{2s+2}{s(2s+1)} = \frac{2}{s} - \frac{2}{2s+1} = \frac{2}{s} - \frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v(t) = 2 - e^{-0.5t} \qquad \text{for } t > 0$$

7.9 RESISTOR–INDUCTOR–CAPACITOR CIRCUIT

Consider a series *RLC* circuit shown in Fig. 7.45. The switch is closed at time $t = 0$.

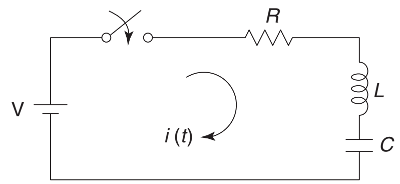

Fig. 7.45 *RLC circuit*

For $t > 0$, the transformed network is shown in Fig. 7.46.

Applying KVL to the mesh,

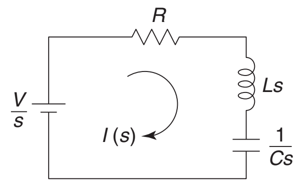

Fig. 7.46 *Transformed network*

$$\frac{V}{s} - RI(s) - Ls\,I(s) - \frac{1}{Cs}I(s) = 0$$

$$\left(R + Ls + \frac{1}{Cs}\right)I(s) = \frac{V}{s}$$

$$\left(\frac{LCs^2 + RCs + 1}{Cs}\right)I(s) = \frac{V}{s}$$

$$I(s) = \frac{\dfrac{V}{s}}{\dfrac{LCs^2 + RCs + 1}{Cs}} = \frac{\dfrac{V}{L}}{s^2 + \dfrac{R}{L}s + \dfrac{1}{LC}} = \frac{\dfrac{V}{L}}{(s - s_1)(s - s_2)}$$

where s_1 and s_2 are the roots of the equation $s^2 + \left(\frac{R}{L}\right)s + \left(\frac{1}{LC}\right) = 0$.

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \beta$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha - \beta$$

where
$$\alpha = \frac{R}{2L}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

and
$$\beta = \sqrt{\alpha^2 - \omega_0^2}$$

By partial-fraction expansion, of $I(s)$,

$$I(s)=\frac{A}{s-s_1}+\frac{B}{s-s_2}$$

$$A=(s-s_1)\,I(s)\big|_{s=s_1}=\frac{\frac{V}{L}}{s_1-s_2}$$

$$B=(s-s_2)\,I(s)\big|_{s=s_2}=\frac{\frac{V}{L}}{s_2-s_1}=-\frac{\frac{V}{L}}{s_1-s_2}$$

$$I(s)=\frac{V}{L(s_1-s_2)}\left[\frac{1}{s-s_1}-\frac{1}{s-s_2}\right]$$

Taking the inverse Laplace transform,

$$i(t)=\frac{V}{L(s_1-s_2)}\left[e^{s_1t}-e^{s_2t}\right]=k_1\,e^{s_1t}+k_2\,e^{s_2t}$$

where k_1 and k_2 are constants to be determined and s_1 and s_2 are the roots of the equation. Now depending upon the values of s_1 and s_2, we have 3 cases of the response.

Case I When the roots are real and unequal, it gives an overdamped response.

$$\frac{R}{2L}>\frac{1}{\sqrt{LC}}$$

$$\alpha>\omega_0$$

In this case, the solution is given by

$$i(t)=e^{-\alpha t}\left(k_1e^{\beta t}+k_2e^{-\beta t}\right)$$

or $$i(t)=k_1\,e^{s_1t}+k_2\,e^{s_2t}\qquad \text{for } t>0$$

Case II When the roots are real and equal, it gives a critically damped response.

$$\frac{R}{2L}=\frac{1}{\sqrt{LC}}$$

$$\alpha=\omega_0$$

In this case, the solution is given by

$$i(t)=e^{-\alpha t}\,(k_1+k_2\,t)\qquad \text{for } t>0$$

Case III When the roots are complex conjugate, it gives an underdamped response.

$$\frac{R}{2L}<\frac{1}{\sqrt{LC}}$$

$$\alpha<\omega_0$$

In this case, the solution is given by

$$i(t)=k_1\,e^{s_1t}+k_2\,e^{s_2t}$$

where $$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

Let $$\sqrt{\alpha^2-\omega_0^2}=\sqrt{-1}\sqrt{\omega_0^2-\alpha^2}=j\omega_d$$

where $$j=\sqrt{-1}$$

and $$\omega_d=\sqrt{\omega_0^2-\alpha^2}$$

Hence

$$i(t) = e^{-\alpha t}\left(k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t}\right)$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\left\{\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2}\right\} + j(k_1 - k_2)\left\{\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}\right\}\right]$$

$$= e^{-\alpha t}\left[(k_1 + k_2)\cos\omega_d t + j(k_1 - k_2)\sin\omega_d t\right] \qquad \text{for } t > 0$$

Example 7.22 *The switch in Fig. 7.47 is opened at time $t = 0$. Determine the voltage $v(t)$ for $t > 0$.*

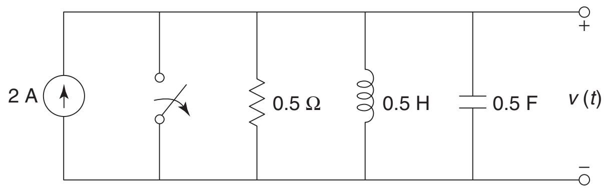

Fig. 7.47

Solution At $t = 0^-$, the network is shown in Fig. 7.48. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit.

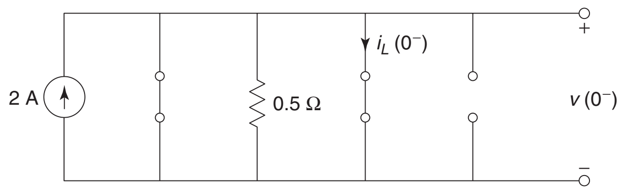

Fig. 7.48

$$i_L(0^-) = 0$$
$$v(0^-) = 0$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$i_L(0^+) = 0$$
$$v(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.49.

Applying KCL at the node for $t > 0$,

$$\frac{V(s)}{0.5} + \frac{V(s)}{0.5s} + \frac{V(s)}{\frac{1}{0.5s}} = \frac{2}{s}$$

$$2V(s) + \frac{2}{s}V(s) + 0.5sV(s) = \frac{2}{s}$$

$$V(s) = \frac{\frac{2}{s}}{\frac{2}{s} + 0.5s + 2} = \frac{4}{s^2 + 4s + 4} = \frac{4}{(s+2)^2}$$

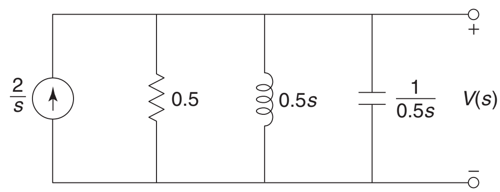

Fig. 7.49

Taking inverse Laplace transform,

$$v(t) = 4t\, e^{-2t} \qquad \text{for } t > 0$$

Example 7.23 *In the network of Fig. 7.50, the switch is closed and steady-state is attained. At $t = 0$, switch is opened. Determine the current through the inductor.*

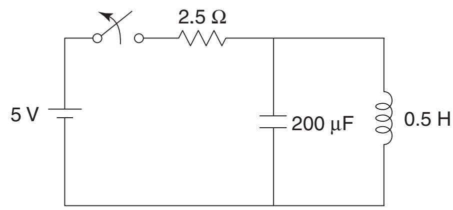

Fig. 7.50

Solution At $t = 0^-$, the network is shown in Fig. 7.51. At $t = 0^-$, the switch is closed and steady-state condition is attained. Hence, the inductor acts as a short circuit and the capacitor acts as an open circuit. Current through inductor is same as the current through the resistor.

$$i_L(0^-) = \frac{5}{2.5} = 2 \text{ A}$$

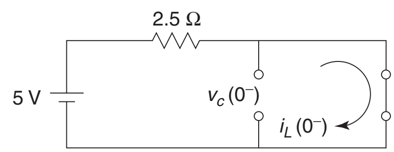

Fig. 7.51

Voltage across the capacitor is zero as it is connected in parallel with a short.

$$v_c(0^-) = 0$$

Since voltage across the capacitor and current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2 \text{ A}$$

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.52. Applying KVL to the mesh for $t > 0$,

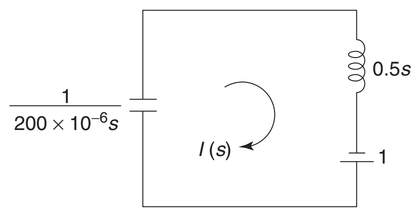

Fig. 7.52

$$-\frac{1}{200 \times 10^{-6} s} I(s) - 0.5s\, I(s) + 1 = 0$$

$$0.5s\, I(s) - 1 + 5000 \frac{I(s)}{s} = 0$$

$$I(s) = \frac{1}{0.5s + \frac{5000}{s}} = \frac{2s}{s^2 + 10000}$$

Taking inverse Laplace transform,

$$i(t) = 2 \cos 100t \qquad \text{for } t > 0$$

Example 7.24 *In the network shown in Fig. 7.53, the switch is opened at t = 0. Steady-state condition is achieved before t = 0. Find i(t).*

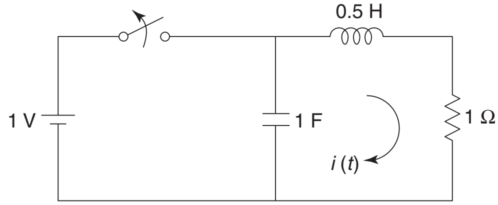

Fig. 7.53

Solution At $t = 0^-$, the network is shown in Fig 7.54. At $t = 0^-$, the switch is closed and steady-state condition is achieved. Hence, the capacitor acts as an open circuit and the inductor acts as a short circuit.

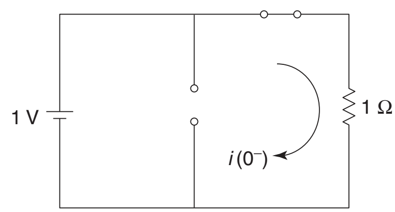

Fig. 7.54

$$v_c(0^-) = 1 \text{ V}$$
$$i(0^-) = 1 \text{ A}$$

Since current through the inductor and voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 1 \text{ V}$$
$$i(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.55.

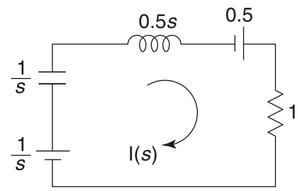

Fig. 7.55

Applying KVL to the mesh for $t > 0$,

$$\frac{1}{s} - \frac{1}{s} I(s) - 0.5s\, I(s) + 0.5 - I(s) = 0$$

$$0.5 + \frac{1}{s} = \frac{1}{s} I(s) + 0.5\, s\, I(s) + I(s)$$

$$I(s)\left[1 + \frac{1}{s} + 0.5s\right] = 0.5 + \frac{1}{s}$$

$$I(s) = \frac{s+2}{s^2+2s+2} = \frac{(s+1)+1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + \frac{1}{(s+1)^2+1}$$

Taking the inverse Laplace transform,

$$i(t) = e^{-t}\cos t + e^{-t}\sin t \qquad \text{for } t > 0$$

Example 7.25 *In the network shown in Fig. 7.56, the switch is closed at t = 0. Find the currents $i_1(t)$ and $i_2(t)$ when initial current through the inductor is zero and initial voltage on the capacitor is 4 V.*

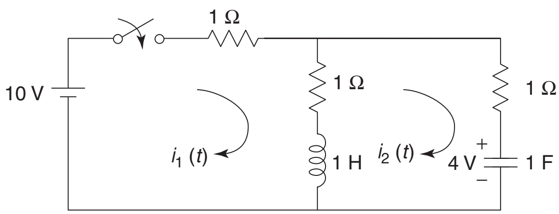

Fig. 7.56

Solution For $t > 0$, the transformed network is shown in Fig. 7.57.

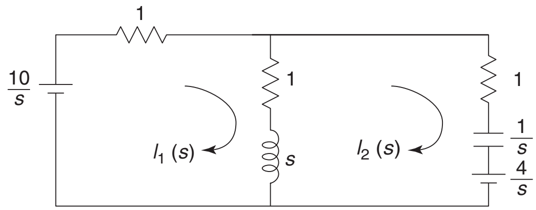

Fig. 7.57

Applying KVL to Mesh 1,

$$\frac{10}{s} - I_1(s) - (1+s)\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+2)I_1(s) - (s+1)I_2(s) = \frac{10}{s}$$

Applying KVL to Mesh 2,

$$-(s+1)\left[I_2(s) - I_1(s)\right] - I_2(s) - \frac{1}{s}I_2(s) - \frac{4}{s} = 0$$

$$-(s+1)I_1(s) + \left(s+2+\frac{1}{s}\right)I_2(s) = -\frac{4}{s}$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{10}{s} & -(s+1) \\ -\frac{4}{s} & s+2+\frac{1}{s} \end{vmatrix}}{\begin{vmatrix} s+2 & -(s+1) \\ -(s+1) & s+2+\frac{1}{s} \end{vmatrix}} = \frac{\left(\frac{10}{s}\right)\left(\frac{s^2+2s+1}{s}\right) - (s+1)\left(\frac{4}{s}\right)}{(s+2)\left(\frac{s^2+2s+1}{s}\right) - (s+1)^2} = \frac{\frac{10}{s^2}(s+1)^2 - (s+1)\frac{4}{s}}{(s+2)\frac{(s+1)^2}{s} - (s+1)^2}$$

$$= \frac{\frac{10}{s^2}(s+1) - \frac{4}{s}}{(s+2)\frac{(s+1)}{s} - (s+1)} = \frac{3s+5}{s(s+1)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+1}$$

$$A = sI_1(s)\big|_{s=0} = \left.\frac{3s+5}{s+1}\right|_{s=0} = 5$$

$$B = (s+1)I_1(s)\big|_{s=-1} = \left.\frac{3s+5}{s}\right|_{s=-1} = -2$$

$$I_1(s) = \frac{5}{s} - \frac{2}{s+1}$$

Taking inverse Laplace transform,

$$i_1(t) = 5 - 2e^{-t} \qquad \text{for } t > 0$$

Similarly,

$$I_2(s) = \frac{\begin{vmatrix} s+2 & \frac{10}{s} \\ -(s+1) & -\frac{4}{s} \end{vmatrix}}{\begin{vmatrix} s+2 & -(s+1) \\ -(s+1) & s+2+\frac{1}{s} \end{vmatrix}} = \frac{3s+1}{(s+1)^2} = \frac{3s+3-2}{(s+1)^2} = \frac{3(s+1)-2}{(s+1)^2} = \frac{3}{s+1} - \frac{2}{(s+1)^2}$$

Taking inverse Laplace transform,

$$i_2(t) = 3e^{-t} - 2te^{-t} \qquad \text{for } t > 0$$

7.10 RESPONSE OF RL CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RL* circuit shown in Fig. 7.58. When the switch is closed at $t = 0$, $i(0^-) = i(0^+) = 0$.

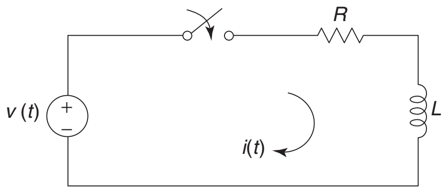

Fig. 7.58 *RL circuit*

For $t > 0$, the transformed network is shown in Fig. 7.59.
Applying KVL to the mesh,

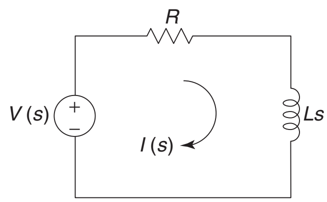

Fig. 7.59 *Transformed network*

$$V(s) - R\,I(s) - Ls\,I(s) = 0$$

$$I(s) = \frac{V(s)}{R+Ls} = \frac{1}{L}\frac{V(s)}{s+\frac{R}{L}}$$

(a) When the unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{1}{L}\frac{\frac{1}{s}}{s+\frac{R}{L}}$$

$$= \frac{1}{L}\frac{1}{s\left(s+\frac{R}{L}\right)}$$

By partial-fraction expansion,

$$I(s)=\frac{1}{L}\left(\frac{A}{s}+\frac{B}{s+\frac{R}{L}}\right)$$

$$A=s\,I(s)\big|_{s=0}=\left.\frac{1}{s+\frac{R}{L}}\right|_{s=0}=\frac{L}{R}$$

$$B=\left(s+\frac{R}{L}\right)I(s)\Big|_{s=-\frac{R}{L}}=\left.\frac{1}{s}\right|_{s=-\frac{R}{L}}=-\frac{L}{R}$$

$$I(s)=\frac{1}{L}\left(\frac{L}{R}\frac{1}{s}-\frac{L}{R}\frac{1}{s+\frac{R}{L}}\right)$$

$$=\frac{1}{R}\left(\frac{1}{s}-\frac{1}{s+\frac{R}{L}}\right)$$

Taking inverse Laplace transform,

$$i(t)=\frac{1}{R}[1-e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t>0$$

(b) When unit ramp signal is applied,

$$v(t)=r(t)=t \qquad \text{for } t>0$$

Taking Laplace transform,

$$V(s)=\frac{1}{s^2}$$

$$I(s)=\frac{1}{L}\frac{1}{s^2\left(s+\frac{R}{L}\right)}$$

By partial-faction expansion,

$$\frac{1}{L}\frac{1}{s^2\left(s+\frac{R}{L}\right)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+\frac{R}{L}}$$

$$\frac{1}{L}=As\left(s+\frac{R}{L}\right)+B\left(s+\frac{R}{L}\right)+Cs^2$$

Putting $s=0$,

$$B=\frac{1}{R}$$

Putting $s=-\frac{R}{L}$,

$$C=\frac{L}{R^2}$$

Comparing coefficients of s^2,

$$A + C = 0$$

$$A = -C = -\frac{L}{R^2}$$

$$I(s) = -\frac{L}{R^2}\frac{1}{s} + \frac{1}{R}\frac{1}{s^2} + \frac{L}{R^2}\frac{1}{s+\frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = -\frac{L}{R^2} + \frac{1}{R}t + \frac{L}{R^2}e^{-\left(\frac{R}{L}\right)t}$$

$$= \frac{1}{R}t - \frac{L}{R^2}[1 - e^{-\left(\frac{R}{L}\right)t}] \qquad \text{for } t > 0$$

(c) When unit impulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{1}{L}\frac{1}{s+\frac{R}{L}}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{L}e^{-\left(\frac{R}{L}\right)t} \qquad \text{for } t > 0$$

Example 7.26 *At $t = 0$, unit pulse voltage of unit width is applied to a series RL circuit as shown in Fig. 7.60. Obtain an expression for $i(t)$.*

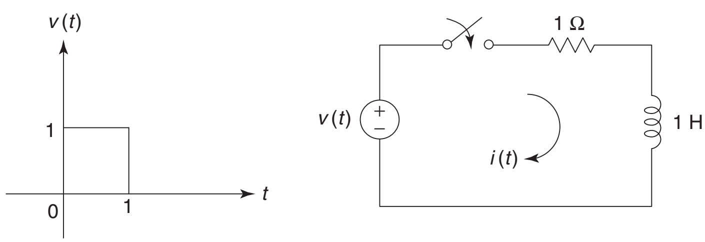

Fig. 7.60

Solution

$$v(t) = u(t) - u(t-1)$$

$$V(s) = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1 - e^{-s}}{s}$$

For $t > 0$, the transformed network is shown in Fig. 7.61.
Applying KVL to the mesh,

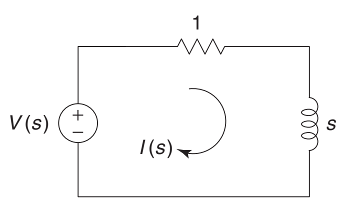

Fig. 7.61

$$V(s)-I(s)-sI(s)=0$$

$$I(s)=\frac{V(s)}{s+1}$$

$$=\frac{1-e^{-s}}{s(s+1)}$$

$$=\frac{1}{s(s+1)}-\frac{e^{-s}}{s(s+1)}$$

$$=\frac{1}{s}-\frac{1}{s+1}-\frac{e^{-s}}{s}+\frac{e^{-s}}{s+1}$$

Taking inverse Laplace transform,

$$i(t)=u(t)-e^{-t}u(t)-u(t-1)+e^{-(t-1)}u(t-1)$$

$$=(1-e^{-t})u(t)-[1-e^{-(t-1)}]u(t-1) \qquad \text{for } t>0$$

Example 7.27 *For the network shown in Fig. 7.62, determine the current i(t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

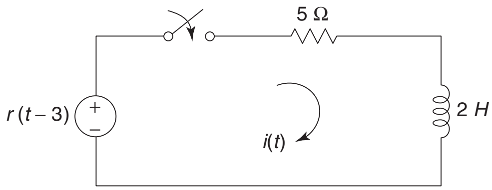

Fig. 7.62

Solution For $t>0$, the transformed network is shown in Fig. 7.63.

Applying KVL to the mesh for $t>0$,

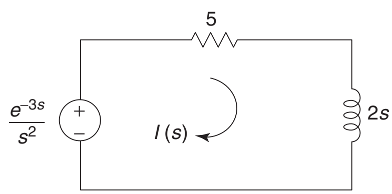

Fig. 7.63

$$\frac{e^{-3s}}{s^2}-5\,I(s)-2s\,I(s)=0$$

$$5\,I(s)+2s\,I(s)=\frac{e^{-3s}}{s^2}$$

$$I(s)=\frac{e^{-3s}}{s^2(2s+5)}=\frac{0.5\,e^{-3s}}{s^2(s+2.5)}$$

By partial-fraction expansion,

$$\frac{0.5}{s^2(s+2.5)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s+2.5}$$

$$0.5=As(s+2.5)+B(s+2.5)+Cs^2$$

$$=As^2+2.5As+Bs+2.5B+Cs^2$$

$$=(A+C)s^2+(2.5A+B)s+2.5\,B$$

Comparing coefficients of s^2, s and s^0,

$$A+C=0$$

$$2.5A+B=0$$

$$2.5B=0.5$$

Solving these equations,

$$A = -0.08$$
$$B = 0.2$$
$$C = 0.08$$

$$I(s) = e^{-3s}\left(-\frac{0.08}{s}+\frac{0.2}{s^2}+\frac{0.08}{s+2.5}\right)$$
$$= -0.08\frac{e^{-3s}}{s}+0.2\frac{e^{-3s}}{s^2}+0.08\frac{e^{-3s}}{s+2.5}$$

Taking inverse Laplace transform,

$$i(t) = -0.08u(t-3)+0.2r(t-3)+0.08e^{-2.5(t-3)}u(t-3)$$

Example 7.28 *Determine the expression for $v_L(t)$ in the network shown in Fig. 7.64. Find $v_L(t)$ when (i) $v_s(t) = \delta(t)$, and (ii) $v_s(t) = e^{-t}u(t)$.*

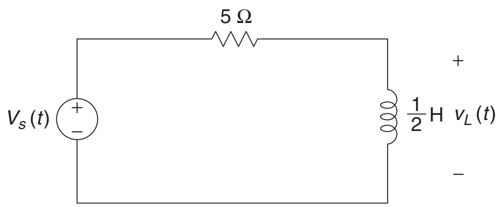

Fig. 7.64

Solution For $t > 0$, the transformed network is shown in Fig. 7.65.
By voltage-division rule,

$$V_L(s) = V_s(s)\times\frac{\frac{s}{2}}{\frac{s}{2}+5} = \frac{s}{s+10}V_s(s)$$

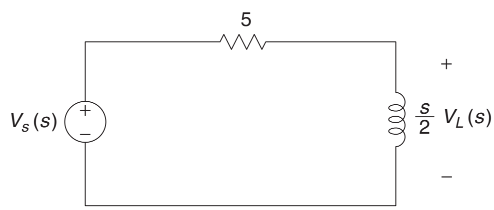

Fig. 7.65

(a) For impulse input,

$$V_s(s) = 1$$
$$V_L(s) = \frac{s}{s+10} = \frac{s+10-10}{s+10} = 1-\frac{10}{s+10}$$

Taking inverse Laplace transform,

$$V_L(t) = \delta(t)-10e^{-10t}u(t) \qquad \text{for } t>0$$

(b) For $v_s(t) = e^{-t}u(t)$,

$$V_s(s) = \frac{1}{s+1}$$
$$V_L(s) = \frac{s}{(s+10)(s+1)}$$

By partial-fraction expansion,

$$V_L(s) = \frac{A}{s+10} + \frac{B}{s+1}$$

$$A = (s+10)V_L(s)\big|_{s=-10} = \left.\frac{s}{s+1}\right|_{s=-10} = \frac{10}{9}$$

$$B = (s+1)V_L(s)\big|_{s=-1} = \left.\frac{s}{s+10}\right|_{s=-1} = -\frac{1}{9}$$

$$V_L(s) = \frac{10}{9}\frac{1}{s+10} - \frac{1}{9}\frac{1}{s+1}$$

Taking inverse Laplace transform,

$$v_L(t) = \frac{10}{9}e^{-10t}u(t) - \frac{1}{9}e^{-t}u(t)$$

$$= \left(\frac{10}{9}e^{-10t} - \frac{1}{9}e^{-t}\right)u(t) \qquad \text{for } t > 0$$

Example 7.29 *For the network shown in Fig. 7.66, determine the current i (t) when the switch is closed at t = 0. Assume that initial current in the inductor is zero.*

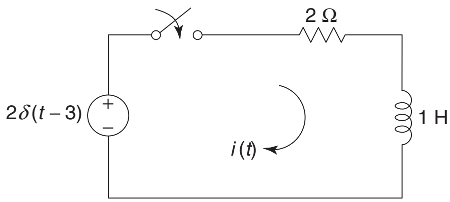

Fig. 7.66

Solution For $t > 0$, the transformed network is shown in Fig. 7.67.

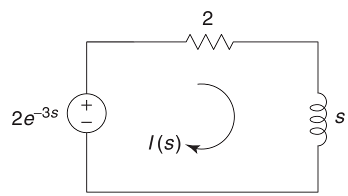

Fig. 7.67

Applying KVL to the mesh for $t > 0$,

$$2e^{-3s} - 2I(s) - sI(s) = 0$$

$$2I(s) + sI(s) = 2e^{-3s}$$

$$I(s) = \frac{2e^{-3s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = 2e^{-2(t-3)}u(t-3) \qquad \text{for } t > 0$$

Example 7.30 *Determine the current i(t) in the network shown in Fig. 7.68, when the switch is closed at t = 0.*

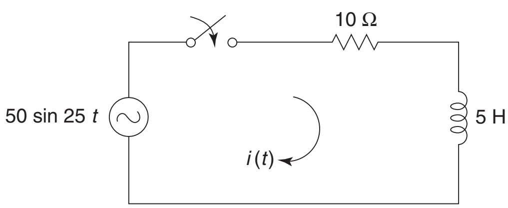

Fig. 7.68

Solution For $t > 0$, the transformed network is shown in Fig. 7.69.

Applying KVL to the mesh for $t > 0$,

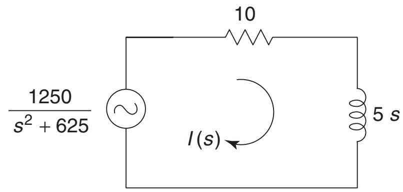

Fig. 7.69

$$\frac{1250}{s^2+625} - 10I(s) - 5sI(s) = 0$$

$$I(s) = \frac{250}{(s^2+625)(s+2)}$$

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+625} + \frac{C}{s+2}$$

$$250 = (As+B)(s+2) + C(s^2+625)$$

$$= (A+C)s^2 + (2A+B)s + (2B+625C)$$

Comparing coefficients,

$$A + C = 0$$

$$2A + B = 0$$

$$2B + 625C = 250$$

Solving the equations,

$$A = -0.397$$

$$B = 0.795$$

$$C = 0.397$$

$$I(s) = \frac{-0.397s + 0.795}{s^2+625} + \frac{0.397}{s+2} = -\frac{0.397s}{s^2+625} + \frac{0.795}{s^2+625} + \frac{0.397}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = -0.397\cos 25t + 0.032\sin 25t + 0.397e^{-2t} \qquad \text{for } t > 0$$

Example 7.31 *Find impulse response of the current i(t) in the network shown in Fig. 7.70.*

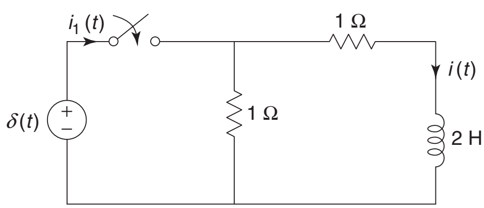

Fig. 7.70

Solution The transformed network is shown in Fig. 7.71.

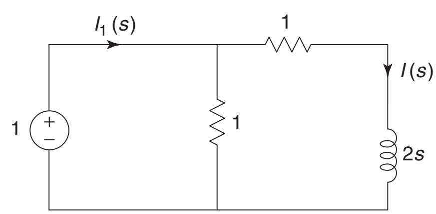

Fig. 7.71

$$Z(s) = \frac{1(2s+1)}{2s+1+1} = \frac{2s+1}{2s+2}$$

$$I_1(s) = \frac{V(s)}{Z(s)} = \frac{1}{\frac{2s+1}{2s+2}} = \frac{2s+2}{2s+1}$$

By current-division rule,

$$I(s) = I_1(s)\times\frac{1}{2s+2} = \frac{1}{2s+2}\times\frac{2s+2}{2s+1} = \frac{1}{2s+1} = \frac{1}{2}\frac{1}{s+0.5}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{2}e^{-0.5t}u(t) \qquad \text{for } t>0$$

Example 7.32 *The network shown in Fig. 7.72 is at rest for $t<0$. If the voltage $v(t) = u(t)\cos t + A\delta(t)$ is applied to the network, determine the value of A so that there is no transient term in the current response i(t).*

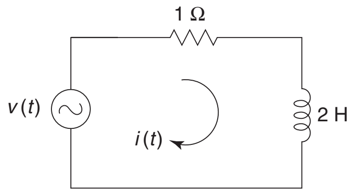

Fig. 7.72

$$v(t) = u(t)\cos t + A\delta(t)$$

$$V(s) = \frac{s}{s^2+1} + A$$

Solution For $t>0$, the transformed network is shown in Fig. 7.73.

Applying KVL to the mesh for $t>0$,

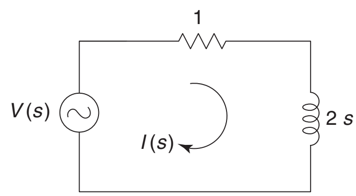

Fig. 7.73

$$V(s) = 2sI(s) + I(s) = \frac{s}{s^2+1} + A$$

$$I(s) = \frac{s + A(s^2+1)}{2\left(s+\frac{1}{2}\right)(s^2+1)} = \frac{K_1}{s+\frac{1}{2}} + \frac{K_2 s + K_3}{s^2+1}$$

The transient part of the response is given by the first term. Hence, for the transient term to vanish, $K_1 = 0$.

$$K_1 = \left(s+\frac{1}{2}\right)I(s)\Big|_{s=-\frac{1}{2}} = \frac{\frac{-1}{2} + A\left(\frac{5}{4}\right)}{2\left(\frac{5}{4}\right)}$$

When $K_1 = 0$

$$\frac{5}{4}A = \frac{1}{2}$$

$$A = \frac{2}{5} = 0.4$$

7.11 RESPONSE OF RC CIRCUIT TO VARIOUS FUNCTIONS

Consider a series *RC* circuit as shown in Fig. 7.74.

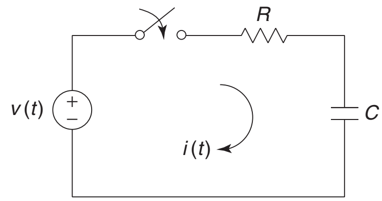

Fig. 7.74 *RC circuit*

For $t > 0$, the transformed network is shown in Fig. 7.75.

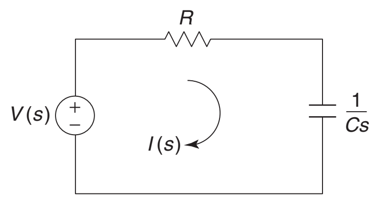

Fig. 7.75 *Transformed network*

Applying KVL to the mesh,

$$V(s) - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$I(s) = \frac{V(s)}{\frac{1}{Cs} + R} = \frac{sV(s)}{R\left(s + \frac{1}{RC}\right)}$$

(a) When unit step signal is applied,

$$v(t) = u(t)$$

Taking Laplace transform,

$$V(s) = \frac{1}{s}$$

$$I(s) = \frac{s \times \frac{1}{s}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R\left(s + \frac{1}{RC}\right)}$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R} e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(b) When unit ramp signal is applied,

$$v(t) = r(t) = t$$

Taking Laplace transform,

$$V(s) = \frac{1}{s^2}$$

$$I(s) = \frac{s \times \frac{1}{s^2}}{R\left(s + \frac{1}{RC}\right)} = \frac{\frac{1}{R}}{s\left(s + \frac{1}{RC}\right)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{RC}}$$

$$A = s\,I(s)\big|_{s=0} = \left.\frac{\frac{1}{R}}{s + \frac{1}{RC}}\right|_{s=0} = C$$

$$B = \left(s + \frac{1}{RC}\right) I(s)\Big|_{s=-\frac{1}{RC}} = \left.\frac{\frac{1}{R}}{s}\right|_{s=-\frac{1}{RC}} = -C$$

$$I(s) = \frac{C}{s} - \frac{C}{s + \frac{1}{RC}}$$

Taking inverse Laplace transform,

$$i(t) = C - Ce^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

(c) When unit inpulse signal is applied,

$$v(t) = \delta(t)$$

Taking Laplace transform,

$$V(s) = 1$$

$$I(s) = \frac{s}{R\left(s + \frac{1}{RC}\right)} = \frac{s + \frac{1}{RC} - \frac{1}{RC}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R}\left(1 - \frac{\frac{1}{RC}}{s + \frac{1}{RC}}\right)$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[\delta(t) - \frac{1}{RC}e^{-\frac{1}{RC}t}\right] \qquad \text{for } t > 0$$

Example 7.33 *A rectangular voltage pulse of unit height and T-seconds duration is applied to a series RC network at t = 0. Obtain the expression for the current i(t). Assume the capacitor to be initially uncharged.*

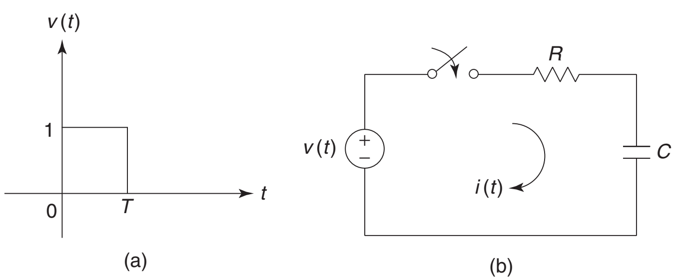

Fig. 7.76

Solution

$$v(t) = u(t) - u(t - T)$$

$$V(s) = \frac{1}{s} - \frac{e^{-sT}}{s} = \frac{1-e^{-sT}}{s}$$

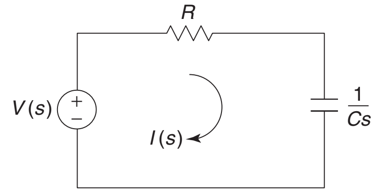

Fig. 7.77

For $t > 0$, the transformed network is shown in Fig. 7.77.
Applying KVL to the mesh for $t > 0$,

$$V(s) - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$I(s) = \frac{V(s)}{R + \frac{1}{Cs}} = \frac{\frac{1}{R}s}{s + \frac{1}{RC}} V(s) = \frac{1-e^{-sT}}{R\left(s + \frac{1}{RC}\right)} = \frac{1}{R}\left[\frac{1}{s + \frac{1}{RC}} - \frac{e^{-sT}}{s + \frac{1}{RC}}\right]$$

Taking inverse Laplace transform,

$$i(t) = \frac{1}{R}\left[e^{-\left(\frac{1}{RC}\right)t} u(t) - e^{-\left(\frac{1}{RC}\right)(t-T)} u(t-T)\right] \qquad \text{for } t > 0$$

Example 7.34 *For the network shown in Fig. 7.78, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

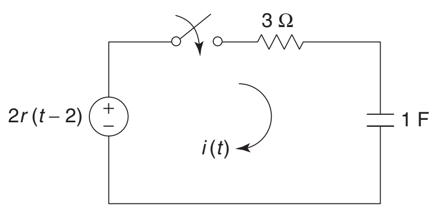

Fig. 7.78

Solution For $t > 0$, the transformed network is shown in Fig. 7.79.
Applying KVL to the mesh for $t > 0$,

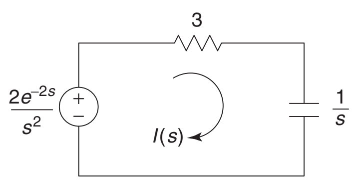

Fig. 7.79

$$\frac{2e^{-2s}}{s^2} - 3\, I(s) - \frac{1}{s} I(s) = 0$$

$$\left(3 + \frac{1}{s}\right) I(s) = \frac{2e^{-2s}}{s^2}$$

$$I(s) = \frac{2e^{-2s}}{s^2\left(3 + \frac{1}{s}\right)} = \frac{0.67e^{-2s}}{s(s+0.33)}$$

By partial-fraction expansion,

$$\frac{0.67}{s(s+0.33)} = \frac{A}{s} + \frac{B}{s+0.33}$$

$$A = \left.\frac{0.67}{s+0.33}\right|_{s=0} = 2$$

$$B = \left.\frac{0.67}{s}\right|_{s=-0.33} = -2$$

$$I(s) = e^{-2s}\left(\frac{2}{s} - \frac{2}{s+0.33}\right) = 2\frac{e^{-2s}}{s} - 2\frac{e^{-2s}}{s+0.33}$$

Taking inverse Laplace transform,

$$i(t) = 2u(t-2) - 2e^{-0.33(t-2)}u(t-2) \qquad \text{for } t > 0$$

Example 7.35 *For the network shown in Fig. 7.80, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

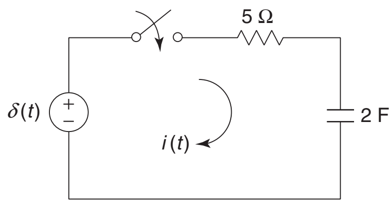

Fig. 7.80

Solution For $t > 0$, the transformed network is shown in Fig. 7.81.

Applying KVL to the mesh for $t > 0$,

$$1 - 5\,I(s) - \frac{1}{2s}I(s) = 0$$

$$\left(5 + \frac{1}{2s}\right)I(s) = 1$$

$$I(s) = \frac{1}{5+\frac{1}{2s}}$$

$$= \frac{2s}{10s+1}$$

$$= \frac{0.2s}{s+0.1}$$

$$= \frac{0.2(s+0.1-0.1)}{s+0.1}$$

$$= 0.2\left(1 - \frac{0.1}{s+0.1}\right)$$

$$= 0.2 - \frac{0.02}{s+0.1}$$

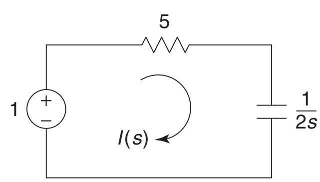

Fig. 7.81

Taking inverse Laplace transform,

$$i(t) = 0.2\delta(t) - 0.02\,e^{-0.1t}u(t)$$

Example 7.36 *For the network shown in Fig. 7.82, find the response $v_0(t)$.*

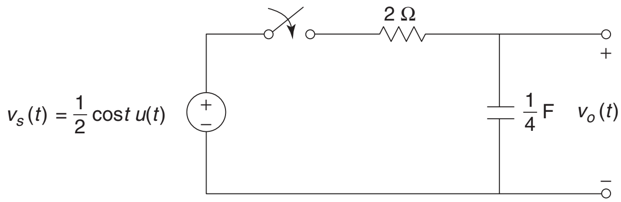

Fig. 7.82

Solution For $t > 0$, the transformed network is shown in Fig. 7.83.

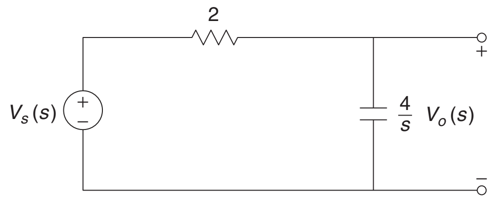

Fig. 7.83

$$V_s(s) = \frac{1}{2}\frac{s}{s^2+1}$$

By voltage-division rule,

$$V_o(s) = V_s(s) \times \frac{\frac{4}{s}}{2+\frac{4}{s}} = \frac{2V_s(s)}{s+2} = \frac{s}{(s^2+1)(s+2)}$$

By partial-fraction expansion,

$$V_o(s) = \frac{As+B}{s^2+1} + \frac{C}{s+2}$$

$$s = (As+B)(s+2) + c(s^2+1)$$

$$s = (A+C)s^2 + (2A+B)s + (2B+C)$$

Comparing coefficient of s^2, s and s^0,

$$A + C = 0$$

$$2A + B = 1$$

$$2B + C = 0$$

Solving the equations,

$$A = 0.4, \quad B = 0.2, \quad C = -0.4$$

$$V_o(s) = \frac{0.4s+0.2}{s^2+1} - \frac{0.4}{s+2} = \frac{0.4s}{s^2+1} + \frac{0.2}{s^2+1} - \frac{0.4}{s+2}$$

Taking the inverse Laplace transform,

$$i(t) = 0.4\cos t + 0.2\sin t - 0.4e^{-2t} \qquad \text{for } t > 0$$

Example 7.37 *Find the impulse response of the voltage across the capacitor in the network shown in Fig. 7.84. Also determine response $v_c(t)$ for step input.*

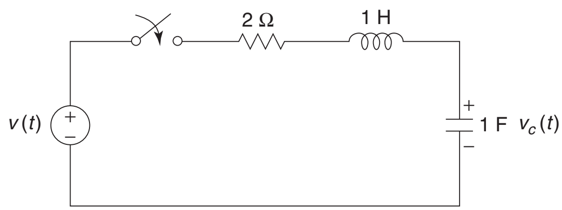

Fig. 7.84

Solution For $t > 0$, the transformed network is shown in Fig. 7.85.

By voltage-division rule,

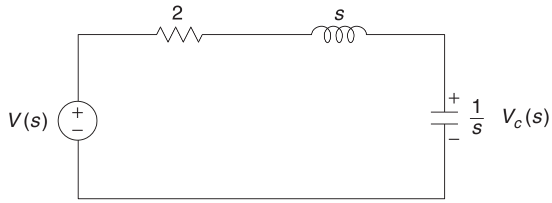

Fig. 7.85

$$V_c(s) = V(s) \times \frac{\frac{1}{s}}{2 + s + \frac{1}{s}}$$

$$= \frac{V(s)}{s^2 + 2s + 1} = \frac{V(s)}{(s+1)^2}$$

(a) For impulse input,

$$V(s) = 1$$

$$V_c(s) = \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = te^{-t}u(t) \qquad \text{for } t > 0$$

(b) For step input,

$$V(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(s+1)^2}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$$1 = A(s+1)^2 + Bs(s+1) + Cs$$

$$= A(s^2 + 2s + 1) + B(s^2 + s) + Cs$$

$$= (A + B)s^2 + (2A + B + C)s + A$$

Comparing coefficient of s^2, s^1 and s^0,

$$A = 1$$

$$A + B = 0$$

$$B = -A = -1$$

$$2A + B + C = 0$$

$$C = -2A - B = -2 + 1 = -1$$

$$V_c(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Taking inverse Laplace transform,

$$v_c(t) = u(t) - e^{-t}u(t) - te^{-t}u(t)$$

$$= (1 - e^{-t} - te^{-t})ut \qquad \text{for } t > 0$$

Example 7.38 *For the network shown in Fig. 7.86, determine the current i(t) when the switch is closed at t = 0 with zero initial conditions.*

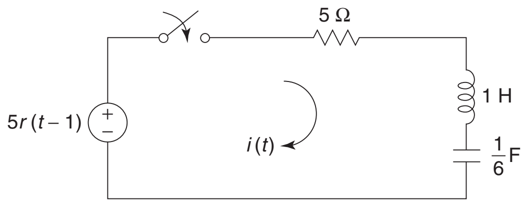

Fig. 7.86

Solution For $t > 0$, the transformed network is shown in Fig. 7.87.

Applying KVL to the mesh for $t > 0$,

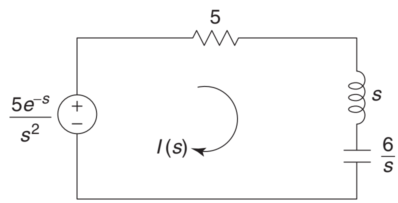

Fig. 7.87

$$\frac{5e^{-s}}{s^2} - 5I(s) - sI(s) - \frac{6}{s}I(s) = 0$$

$$5I(s) + sI(s) + \frac{6}{s}I(s) = \frac{5e^{-s}}{s^2}$$

$$I(s) = \frac{5e^{-s}}{s(s^2+5s+6)} = \frac{5e^{-s}}{s(s+3)(s+2)}$$

By partial-fraction expansion,

$$\frac{1}{s(s+3)(s+2)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s+2}$$

$$A = \left.\frac{1}{(s+3)(s+2)}\right|_{s=0} = \frac{1}{6}$$

$$B = \left.\frac{1}{s(s+2)}\right|_{s=-3} = \frac{1}{3}$$

$$C = \left.\frac{1}{s(s+3)}\right|_{s=-2} = -\frac{1}{2}$$

$$I(s) = 5e^{-s}\left[\frac{1}{6s} + \frac{1}{3(s+3)} - \frac{1}{2(s+2)}\right] = \frac{5}{6}\frac{e^{-s}}{s} + \frac{5}{3}\frac{e^{-s}}{s+3} - \frac{5}{2}\frac{e^{-s}}{s+2}$$

Taking inverse Laplace transform,

$$i(t) = \frac{5}{6}u(t-1) + \frac{5}{3}e^{-3(t-1)}u(t-1) - \frac{5}{2}e^{-2(t-1)}u(t-1) \qquad \text{for } t > 0$$

Example 7.39 *For the network shown in Fig. 7.88, the switch is closed at t = 0. Determine the current i(t) assuming zero initial conditions.*

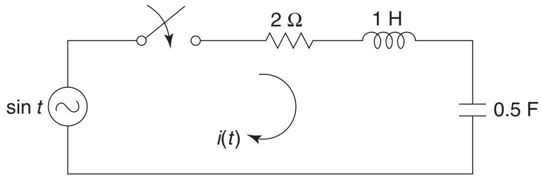

Fig. 7.88

Solution For $t > 0$, the transformed network is shown in Fig. 7.89.

Applying KVL to the mesh for $t > 0$,

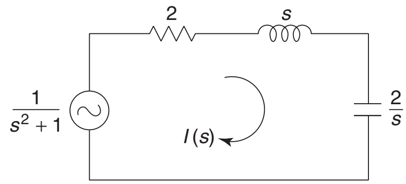

Fig. 7.89

$$\frac{1}{s^2+1} - 2I(s) - s\,I(s) - \frac{2}{s}I(s) = 0$$

$$\left(2 + s + \frac{2}{s}\right)I(s) = \frac{1}{s^2+1}$$

$$I(s) = \frac{s}{(s^2+1)(s^2+2s+2)}$$

By partial-fraction expansion,

$$I(s) = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2}$$

$$s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1)$$

$$= As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Cs + Ds^2 + D$$

$$= (A+C)s^3 + (2A+B+D)s^2 + (2A+2B+C)s + (2B+D)$$

Comparing coefficients of s^3, s^2, s^1 and s^0,

$$A + C = 0$$

$$2A + B + D = 0$$

$$2A + 2B + C = 1$$

$$2B + D = 0$$

Solving these equations,

$$A = 0.2,\ B = 0.4,\ C = -0.2,\ D = -0.8$$

$$I(s) = \frac{0.2s+0.4}{s^2+1} - \frac{0.2s+0.8}{s^2+2s+2}$$

$$= \frac{0.2s}{s^2+1} + \frac{0.4}{s^2+1} - \frac{0.2s+0.2+0.6}{(s+1)^2+(1)^2}$$

$$= \frac{0.2s}{s^2+1} + \frac{0.4}{s^2+1} - \frac{0.2(s+1)}{(s+1)^2+1} - \frac{0.6}{(s+1)^2+1}$$

Taking inverse Laplace transform,

$$i(t) = 0.2\cos t + 0.4\sin t - 0.2\,e^{-t}\cos t - 0.6\,e^{-t}\sin t$$

$$= 0.2\cos t + 0.4\sin t - e^{-t}(0.2\cos t + 0.6\sin t) \qquad \text{for } t > 0$$

Example 7.40 *For the network shown in Fig. 7.90, the switch is closed at $t = 0$. Determine the current $i(t)$ assuming zero initial conditions in the network elements.*

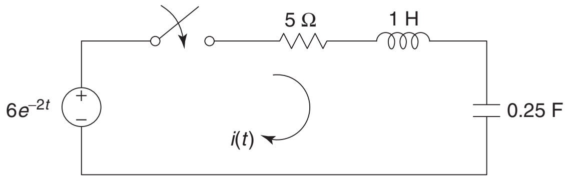

Fig. 7.90

Solution For $t > 0$, the transformed network is shown in Fig. 7.91.

Applying KVL to the mesh for $t > 0$,

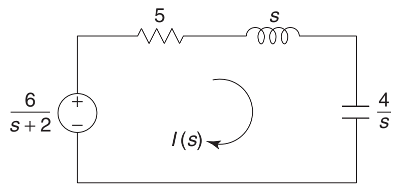

Fig. 7.91

$$\frac{6}{s+2} - 5I(s) - s\,I(s) - \frac{4}{s}I(s) = 0$$

$$\left(5 + s + \frac{4}{s}\right)I(s) = \frac{6}{s+2}$$

$$I(s) = \frac{6s}{(s+2)(s^2+5s+4)}$$

$$= \frac{6s}{(s+2)(s+1)(s+4)}$$

By partial-fraction expansion,

$$I(s) = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$A = (s+2)I(s)\big|_{s=-2} = \left.\frac{6s}{(s+1)(s+4)}\right|_{s=-2} = 6$$

$$B = (s+1)I(s)\big|_{s=-1} = \left.\frac{6s}{(s+2)(s+4)}\right|_{s=-1} = -2$$

$$C = (s+4)I(s)\big|_{s=-4} = \left.\frac{6s}{(s+2)(s+1)}\right|_{s=-4} = -4$$

$$I(s) = \frac{6}{s+2} - \frac{2}{s+1} - \frac{4}{s+4}$$

Taking inverse Laplace transform,

$$i(t) = 6e^{-2t}u(t) - 2e^{-t}u(t) - 4\,e^{-4t}u(t) \qquad \text{for } t > 0$$

Example 7.41 *The network shown has zero initial conditions. A voltage $v_i(t) = \delta(t)$ applied to two terminal network produces voltage $v_o(t) = [e^{-2t} + e^{-3t}]\,u(t)$. What should be $v_i(t)$ to give $v_o(t) = t\,e^{-2t}\,u(t)$?*

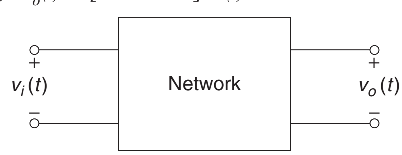

Fig. 7.92

Solution For $v_i(t) = \delta(t)$,

$$V_i(s) = 1$$

$$v_o(t) = [e^{-2t} + e^{-3t}]u(t)$$

$$V_o(s) = \frac{1}{s+2} + \frac{1}{s+3}$$

System function $H(s) = \dfrac{V_o(s)}{V_i(s)}$

$$= \frac{1}{s+2} + \frac{1}{s+3} = \frac{2s+5}{(s+2)(s+3)} \qquad \text{...(i)}$$

For $v_o(t) = te^{-2t}u(t)$,

$$V_o(s) = \frac{1}{(s+2)^2}$$

From Eq. (i),

$$V_i(s) = \frac{V_o(s)}{H(s)} = \frac{1}{(s+2)^2} \times \frac{(s+2)(s+3)}{2s+5} = \frac{(s+3)}{2(s+2.5)(s+2)}$$

By partial-fraction expansion,

$$V_i(s) = \frac{A}{s+2} + \frac{B}{s+2.5}$$

$$A = 1$$

$$B = -0.5$$

$$V_i(s) = \frac{1}{s+2} - \frac{0.5}{s+2.5}$$

Taking inverse Laplace transform,

$$v_i(t) = e^{-2t} - 0.5e^{-2.5t} \qquad \text{for } t > 0$$

Example 7.42 *A unit impulse applied to two terminal black box produces a voltage $v_o(t) = 2e^{-t} - e^{-3t}$. Determine the terminal voltage when a current pulse of 1 A height and a duration of 2 seconds is applied at the terminal.*

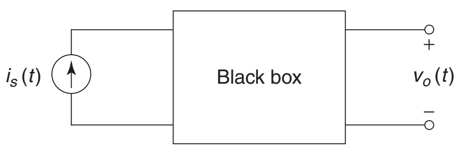

Fig. 7.93

Solution

$$v_o(t) = 2e^{-t} - e^{-3t}$$

$$V_o(s) = \frac{2}{s+1} - \frac{1}{s+3}$$

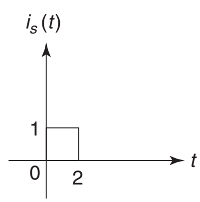

Fig. 7.94

When $i_s(t) = \delta(t)$,

$$I_s(s) = 1$$

$$V_o(s) = Z(s)\, I_s(s) \qquad \ldots\text{(i)}$$

$$Z(s) = \frac{V_o(s)}{I_s(s)} = \frac{2}{s+1} - \frac{1}{s+3}$$

When $i_s(t)$ is a pulse of 1 A height and a duration of 2 seconds then,

$$i_s(t) = u(t) - u(t-2)$$

$$I_s(s) = \frac{1}{s} - \frac{e^{-2s}}{s}$$

From Eq. (i),

$$V_o(s) = \left[\frac{2}{s+1} - \frac{1}{s+3}\right]\left[\frac{1}{s} - \frac{e^{-2s}}{s}\right]$$

$$= \frac{2}{s(s+1)} - \frac{1}{s(s+3)} - \frac{2e^{-2s}}{s(s+1)} + \frac{e^{-2s}}{s(s+3)}$$

$$= 2\left[\frac{1}{s} - \frac{1}{s+1}\right] - \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right] - 2e^{-2s}\left[\frac{1}{s} - \frac{1}{s+1}\right] + \frac{e^{-2s}}{3}\left[\frac{1}{s} - \frac{1}{s+3}\right]$$

Taking the inverse Laplace transform,

$$v(t) = 2[u(t) - e^{-t}u(t)] - \frac{1}{3}[u(t) - e^{-3t}u(t)] - 2[u(t-2) - e^{-(t-2)}u(t-2)] + \frac{1}{3}[u(t-2) - e^{-3(t-2)}u(t-2)]$$

for $t > 0$

Exercises

7.1 For the network shown in Fig 7.95, the switch is closed at $t = 0$. Find the current i1(t) for $t > 0$.

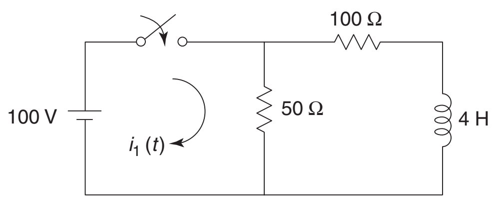

Fig. 7.95

$[i_1(t) = 3 - e^{-25\,t}]$

7.2 Determine the current $i(t)$ in the network of Fig. 7.96, when the switch is closed at $t = 0$. The inductor is initially unenergized.

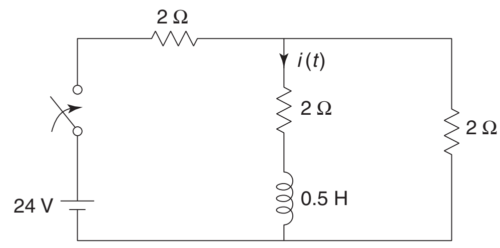

Fig. 7.96

$[i(t) = 4(1 - e^{-6t})]$

7.3 In the network of Fig. 7.97, after the switch has been in the open position for a long time, it is closed at $t = 0$. Find the voltage across the capacitor.

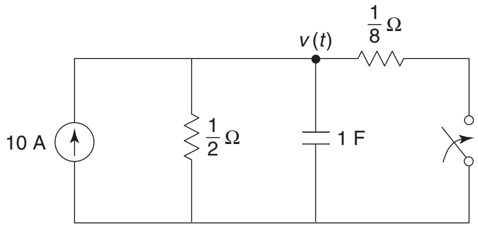

Fig. 7.97

$[v(t) = 1 + 4\,e^{-10t}]$

7.4 The circuit of Fig. 7.98, has been in the condition shown for a long time. At $t = 0$, switch is closed. Find $v(t)$ for $t > 0$.

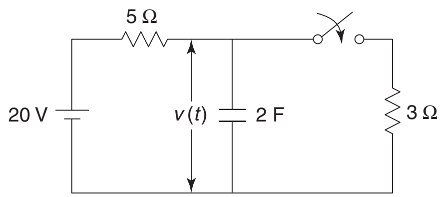

Fig. 7.98

$[v(t) = 7.5 + 12.5\,e^{-(4/15)t}]$

7.5 Figure 7.99 shows a circuit which is in the steady-state with the switch open. At $t = 0$, the switch is closed. Determine the current i (t). Find its value at $t = 0.114\ \mu$ seconds.

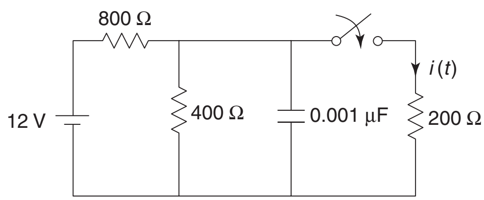

Fig. 7.99

$[i(t) = 0.00857 + 0.01143\,e^{-8.75 \times 10^6 t}, 0.013\text{ A}]$

7.6 Find $i(t)$ for the network shown in Fig. 7.100.

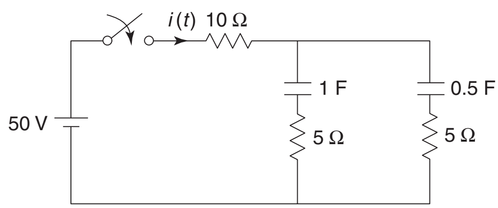

Fig. 7.100

$[i(t) = 0.125\,e^{-0.308t} + 3.875\,e^{-0.052t}]$

7.7 Determine $v(t)$ in the network of Fig. 7.101 where $i_L(0^-) = 15$ A and $v_c(0^-) = 5$ V.

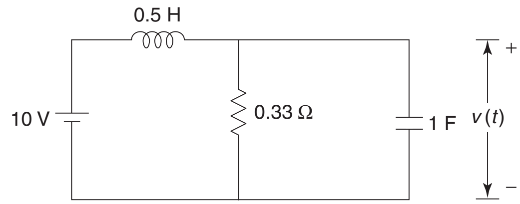

Fig. 7.101

$[v(t) = 10 - 10e^{-t} + 5e^{-2t}]$

7.8 The network shown in Fig. 7.102 has acquired steady state with the switch at position 1 for $t < 0$. At $t = 0$, the switch is thrown to the position 2. Find $v(t)$ for $t > 0$.

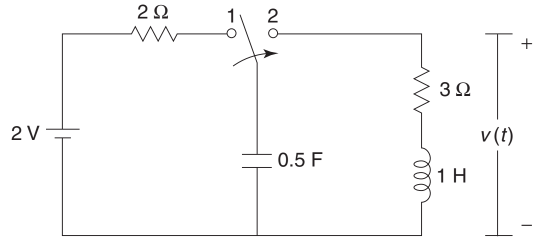

Fig. 7.102

$[v(t) = 4e^{-t} - 2\,e^{-2t}]$

7.9 In the network shown in Fig. 7.103, the switch is closed at $t = 0$. Find current $i_1(t)$ for $t > 0$.

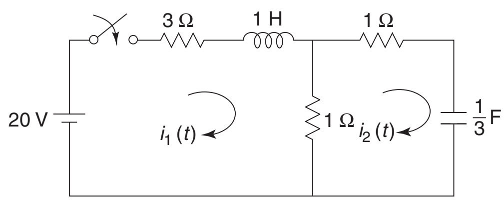

Fig. 7.103

$[i_1(t) = 5 + 5e^{-2t} - 10e^{-3t}]$

7.10 In the network shown in Fig. 7.104, the switch is closed at $t = 0$. Find the current through the 30 Ω resistor.

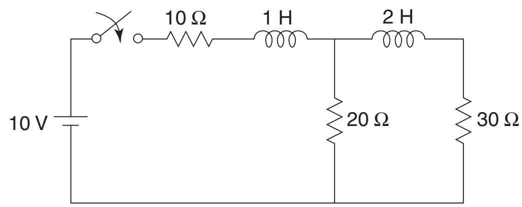

Fig. 7.104

$[i(t) = 0.1818 - 0.265\,e^{-13.14t} + 0.083\,e^{-41.86t}]$

7.11 The network shown in Fig. 7.105 is in steady state with s_1 closed and s_2 open. At $t = 0$, s_1 is opened and s_2 is closed. Find the current through the capacitor.

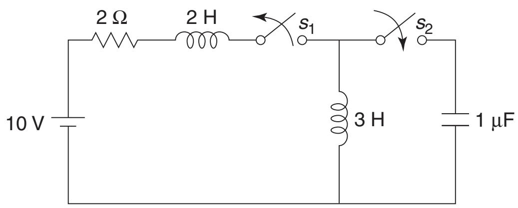

Fig. 7.105

$[i(t) = 5 \cos (0.577 \times 10^3\, t)]$

7.12 In the network shown in Fig. 7.106, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

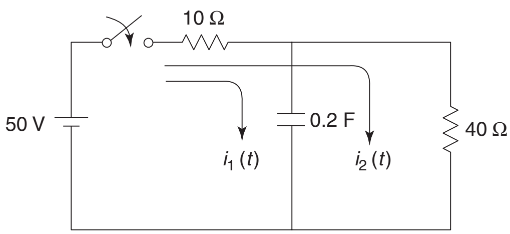

Fig. 7.106

$[i_1(t) = 5\, e^{-0.625t},\ i_2(t) = 1 - e^{-0.625t}\,]$

7.13 For the network shown in Fig. 7.107, find currents $i_1(t)$ and $i_2(t)$ for $t > 0$.

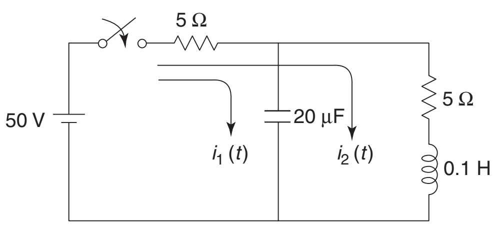

Fig. 7.107

$$\begin{bmatrix} i_1(t) = 0.101e^{-100.5t} + 10.05e^{-9949.5t} \\ i_2(t) = 5 - 5.05e^{-100.5t} + 0.05e^{-9949.5t} \end{bmatrix}$$

7.14 In the network shown in Fig. 7.108, the switch is opened at $t = 0$, the steady state having been established previously. Find $i(t)$ for $t > 0$.

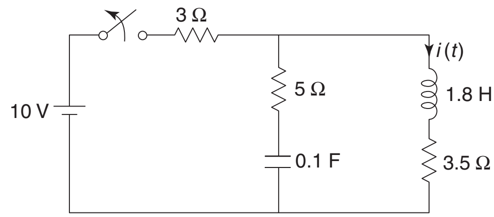

Fig. 7.108

$[i(t) = 1.5124e^{-2.22t} + 3.049e^{-2.5t}]$

7.15 Find the current $i(t)$ in the network of Fig. 7.109, if the switch is closed at $t = 0$. Assume initial conditions to be zero.

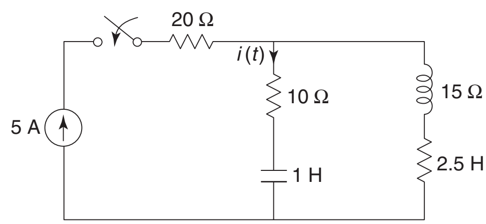

Fig. 7.109

$[i(t) = 3 + 0.57e^{-7.14\,t}]$

7.16 In the network shown in Fig. 7.110, find the voltage $v(t)$ for $t > 0$.

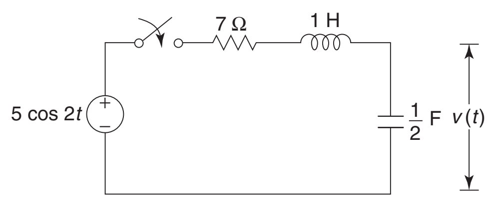

Fig. 7.110

$$\left[v(t) = -\frac{6}{5}e^{-t} + \frac{9}{10}e^{-6t} + \frac{3}{10}\cos 2t + \frac{21}{10}\sin 2t \right]$$

7.17 For the network shown in Fig. 7.111, determine $v(t)$ when the input is

(i) an impulse function $[e^{-t}\, u(t)]$

(ii) $i(t) = 4e^{-t}\, u(t)$ $[4t\, e^{-t}\, u(t)]$

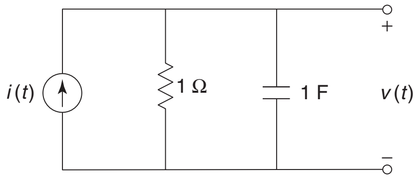

Fig. 7.111

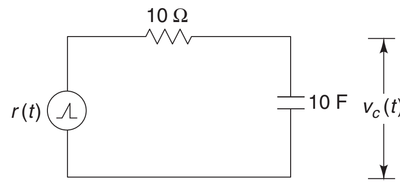

Fig. 7.112

7.18 For a unit-ramp input shown in Fig. 7.112, find the response $v_c(t)$ for $t > 0$.

$[v_c(t) = -100\ u(t) + 100e^{-0.01t}\ u(t) + tu(t)]$

Objective-Type Questions

7.1 If the Laplace transform of the voltage across a capacitor of value $\frac{1}{2}$ F is

$$V_c(s) = \frac{1}{s^2 + 1}$$

the value of the current through the capacitor at $t = 0^+$ is

(a) 0 (b) 2 A

(c) $\frac{1}{2}$ A (d) 1 A

7.2 The response of an initially relaxed linear constant parameter network to a unit impulse applied at $t = 0$ is $4\,e^{-2t}\,u(t)$. The response of this network to a unit-step function will be

(a) $2[1 - e^{-2t}]\ u(t)$
(b) $4[e^{-t} - e^{-2t}]\ u(t)$
(c) $\sin 2t$
(d) $(1 - 4\,e^{-4t})\ u(t)$

7.3 The Laplace transform of a unit-ramp function starting at $t = a$ is

(a) $\dfrac{1}{(s+a)^2}$ (b) $\dfrac{e^{-as}}{(s+a)^2}$

(c) $\dfrac{e^{-as}}{s^2}$ (d) $\dfrac{a}{s^2}$

7.4 The Laplace transform of $e^{at} \cos \alpha t$ is equal to

(a) $\dfrac{s-\alpha}{(s-\alpha)^2 + \alpha^2}$

(b) $\dfrac{s+\alpha}{(s-\alpha)^2 + \alpha^2}$

(c) $\dfrac{1}{(s-\alpha)^2}$

(d) none of the above

7.5 The circuit shown in Fig. 7.113 has initial current $i(0^-) = 1$ A through the inductor and an initial voltage $v_c(0^-) = -1$ V across the capacitor. For input $v(t) = u(t)$, the Laplace transform of the current $i(t)$ for $t \geq 0$ is

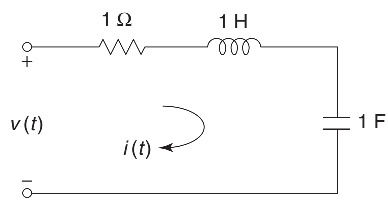

Fig. 7.113

(a) $\dfrac{s}{s^2+s+1}$ (b) $\dfrac{s+2}{s^2+s+1}$

(c) $\dfrac{s-2}{s^2+s+1}$ (d) $\dfrac{s-2}{s^2+s+1}$

7.6 A square pulse of 3 volts amplitude is applied to an *RC* circuit shown in Fig. 7.114. The capacitor is initially uncharged. The output voltage v_0 at time $t = 2$ seconds is

(a) 3 V (b) –3 V
(c) 4 V (d) –4 V

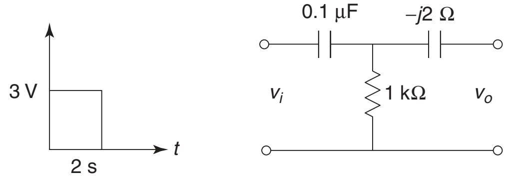

Fig. 7.114

7.7 A 2 mH inductor with some initial current can be represented as shown in Fig. 7.115. The value of the initial current is

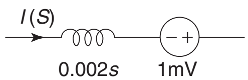

Fig. 7.115

(a) 0.5 A (b) 2 A
(c) 1 A (d) 0

7.8 A current impulse 5 $\delta(t)$ is forced through a capacitor C. The voltage $v_c(t)$ across the capacitor is given by

(a) $5\ t$ (b) $5\ u(t) - C$

(c) $\frac{5}{C}t$ (d) $\frac{5u(t)}{C}$

7.9 In the circuit shown in Fig. 7.116, it is desired to have a constant direct current $i(t)$ through the ideal inductor L. The nature of the voltage source $v(t)$ must be

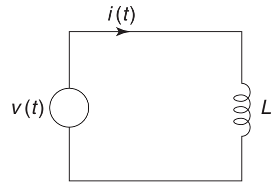

Fig. 7.116

(a) a constant voltage
(b) a linearly increasing voltage
(c) an ideal impulse
(d) as exponential increasing voltage

7.10 When a unit-impulse voltage is applied to an inductor of 1 H, the energy supplied by the source is

(a) ∞ (b) 1 J

(c) $\frac{1}{2}$ J (d) 0

Answers to Objective-Type Questions

7.1 (c)	7.2 (a)	7.3 (c)	7.4 (a)	7.5 (b)	7.6 (b)
7.7 (a)	7.8 (d)	7.9 (c)	7.10 (c)		