Spectral density

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10.22 SPECTRAL DENSITIES

As a matter of fact, spectral densities are used to represent random process in frequency-domain. In this article, we shall discuss frequency-domain description of random processes. Frequency-domain methods are very powerful tools for analysis of various types of signals. We have already discussed some powerful tools such as Fourier series, Fourier transform, Laplace transform for the analysis of deterministic signals. In this article, let us extend these tools to the stationary random processes.

4.2 Spectral Density Functions :

- In this section, we will categorize signals and systems by concentrating on the "energy" or "power" of a signal. In order to do so let us get introduced to a new term called "spectral density". **The spectral density of a signal, is used for defining the distribution of energy or power per unit bandwidth as a function of frequency.**
- The spectral density of energy signals is called as "Energy Spectral Density (ESD)" while that of the power signals is called as "Power Spectral Density (PSD)". Let us discuss about them now.

5.8 ENERGY SPECTRAL DENSITY (ESD)

Parseval's theorem relates the total signal energy in a signal $x(t)$ to its Fourier transform through

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} |X(\omega)|^2 df$$

Parseval's theorem states that the total energy E_x may be determined either by computing the energy per unit time ($|x(t)|^2$) and integrating over all time or by computing the energy per unit frequency $|X(\omega)|^2$ and integrating over all frequencies. For this reason, $|X(\omega)|^2$ represents energy per unit bandwidth and is often referred to as the *energy spectral density* (per unit bandwidth in hertz) or *energy density spectrum* of the signal $x(t)$ and is denoted by $\Psi_x(\omega)$. Hence,

$$\Psi_x(\omega) = |X(\omega)|^2 \tag{5.90}$$

The units of ESD depend on the units of the underlying signal $x(t)$. For example, if the signal unit is volts (V), its Fourier transform has units of V/Hz and its ESD has units of $(\mathrm{V/Hz})^2$ or $(V\,\mathrm{s})^2$.

5.8.2 Relation of ESD to Autocorrelation

The autocorrelation function $R_{xx}(\tau)$ of an energy signal is defined as [Eq. (2.44)]:

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)\, dt$$

$$R_{x}x(\tau) = x(\tau) * x(-\tau)$$

Taking the Fourier transform of the above equation gives

$$\mathcal{F}[R_{xx}(\tau)] = X(\omega)X(-\omega)$$

$$= X(\omega)X^{*}(\omega)$$

$$\mathcal{F}[R_{xx}(\tau)] = |X(\omega)|^2$$

$$\mathcal{F}[R_{xx}(\tau)] = \Psi_x(\omega)$$

$$R_{xx}(\tau) \longleftrightarrow \Psi_x(\omega) \tag{5.92}$$

Thus, the autocorrelation function $R_{xx}(\tau)$ and ESD makes a Fourier transform pair.

Energy Signals

- A signal can be categorized into energy signal or power signal:
- An **energy signal** has a finite energy, $0 < E < \infty$.
- In other words, energy signals have values only in the limited time duration.
- For example, a signal having only one square pulse is energy signal.
- A signal that decays exponentially has finite energy, so, it is also an energy signal.
- The power of an energy signal is 0, because of dividing finite energy by infinite time (or length).

$$E(t) = \lim_{L \to \infty} \int_{-L}^{L} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E[n] = \lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

Power Signals

- A **Power signal** has finite power , 0 < P < ∞.
- A **power signal** is not limited in time.
- It always exists from beginning to end and it never ends.
- For example, sine wave in infinite length is power signal.
- The power of an energy signal is 0, because of dividing finite energy by infinite time (or length).

$$P(t) = \lim_{L \to \infty} \frac{1}{2L} \int_{-L}^{L} |x(t)|^2 \, dt = \lim_{L \to \infty} \frac{E(t)}{2L}$$

$$P[n] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

5.9 POWER SPECTRAL DENSITY (PSD)

PSD has the same relation to power signals as ESD has to energy signals. The expression of PSD may be obtained by assuming the power signal as a limiting case of an energy signal.

Consider a power signal of infinite duration. Since the total energy of a power signal cannot be found, let us first find the ESD of a truncated version $x_T(t)$ of the infinite duration signal $x(t)$. The signal $x_T(t)$ may be expressed as

$$x_T(t) = \begin{cases} x(t), & -\frac{T}{2} < t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases} \tag{5.93}$$

$$x_T(t) = \text{rect}\left(\frac{t}{T}\right) x(t)$$

The truncated signal $x_T(t)$ is of finite duration therefore it is an energy signal. Now, if $x_T(t) \longleftrightarrow X_T(\omega)$, then using the Parseval's theorem, we have

$$E_{x_T} = \int_{-\infty}^{\infty} |x_T(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X_T(\omega)|^2 d\omega$$

$$\int_{-\infty}^{\infty} |x_T(t)|^2 \, dt = \int_{-\infty}^{\infty} |X_T(\omega)|^2 df$$

Substituting the value of $x_T(t)$ from Eq. (5.93) into the above equation yields

$$\int_{-T/2}^{T/2} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X_T(\omega)|^2 df$$

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 \, dt = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(\omega)|^2 df$$

The LHS of the above equation represents the average power P_x of the signal $x(t)$. Therefore,

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} |X_T(\omega)|^2 df$$

$$= \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} df$$

$$P_x = \int_{-\infty}^{\infty} G_x(\omega) df \tag{5.94}$$

where

$$G_x(\omega) = \lim_{T\to\infty} \frac{|X_T(\omega)|^2}{T} \tag{5.95}$$

is the PSD (average power per unit bandwidth). The units of PSD depend on the units of the underlying signal $x(t)$. If the signal unit is amperes (A), the units of PSD are A^2/Hz. If the signal unit is volts (V), the units of PSD are V^2/Hz.

5.9.2 Relation of PSD to Autocorrelation

The autocorrelation function $R_{xx}(\tau)$ of a power signal is defined as [Eq. (2.45)]:

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)\,dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} x_{\mathrm{T}}(t)x_{\mathrm{T}}(t-\tau)\,dt$$

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}[x_{\mathrm{T}}(\tau) * x_{\mathrm{T}}(-\tau)]$$

Taking the Fourier transform of the above equation, we get

$$\mathcal{F}[R_{xx}(\tau)] = \lim_{T\to\infty} \frac{1}{T} X_{\mathrm{T}}(\omega)X_{\mathrm{T}}(-\omega)$$

$$= \lim_{T\to\infty} \frac{1}{T} X_{\mathrm{T}}(\omega)X_{\mathrm{T}}^*(\omega)$$

$$\mathcal{F}[R_{xx}(\tau)] = \lim_{T\to\infty} \frac{1}{T} |X_{\mathrm{T}}(\omega)|^2$$

$$\mathcal{F}[R_{xx}(\tau)] = G_x(\omega)$$

$$R_{xx}(\tau) \longleftrightarrow G_x(\omega) \tag{5.97}$$

Thus, the autocorrelation function $R_{xx}(\tau)$ and PSD $G_x(\omega)$ makes a Fourier transform pair.

10.22.1. Power Spectral Density (psd)*

Power spectral density $S_X(\omega)$ of a wide-sense stationary (WSS) random process X(t) may be defined as

$$S_X(\omega) = \text{Fourier transform } \{R_X(\tau)\}$$

$$= \int_{-\infty}^{\infty} R_X(\tau)\, e^{-j\omega\tau} d\tau \qquad ...(10.137)$$

where $R_X(\tau)$ = Autocorrelation function of random process X(t).

> **DO YOU KNOW?**
>
> The zero-frequency value of the power spectral density (psd) of a stationary process equals the total area under the graph of the autocorrelation function.

Thus, we can say that the Fourier transform of autocorrelation function $R_X(\tau)$ of a random process X(t) is called power spectral density (psd) of random process. Equation (10.137) was intially developed by Albert Einstein. Later it was developed by N. Weiner and Khenchine. Hence, this equation is known as Einstein-Weiner Khenchine (EWK) equation. Power spectral density (psd) is used to measure the energy of the signal in the frequency band $[\omega, \omega + d\omega]$.

10.22.2. Energy Spectral Density (ESD)

The energy spectral density (psd) $\psi_X(\omega)$ may be defined as a measure of density of the energy contained in random process X(t) in Joules per Hertz. It may be noted that since the amplitude spectrum of a real-valued random process X(t) is an even function of ω, the energy spectral density of such a signal is symmetrical about the vertical axis passing through the origin.

Thus, total energy of the random process X(t) is defined as

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi_X(\omega)\, d\omega \qquad \ldots(10.145)$$