Chapter 3
Frequency Domain Analysis of Continuous Time System using Laplace Transform

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Need of Laplace Transform

The Laplace transform is used to transform a time signal to complex frequency domain. (The complex frequency domain is also known as Laplace domain or s-domain). This transformation was first proposed by Laplace (in the year 1780) and later adopted for various engineering applications for solving differential equations. Hence this transformation is called ***Laplace transform.***

In signals and systems the Laplace transform is used to transform a time domain system to s-domain. In time domain the equations governing a system will be in the form of differential equations. While transforming the system to s-domain, the differential equations are transformed to simple algebraic equations and so the analysis of systems will be much easier in s-domain.

Complex Frequency

The ***complex frequency*** is defined as,

Complex frequency, $s = \sigma + j\Omega$

where, σ = ***Neper frequency*** in neper per second

Ω = ***Radian (or Real) frequency*** in radian per second

The complex frequency is involved in the time domain signal of the form Ke^{st}. The signal Ke^{st} can be thought of as an universal signal which represents all types of signals and takes a particular form for various choices of σ and Ω as shown below.

Let, $x(t) = A\,e^{st} = A\,e^{(\sigma + j\Omega)t}$

Case i : ***When*** $\sigma = 0,\ \Omega = \Omega_0$

On substituting $\sigma = 0$ in equation (3.1) we get,

$$\begin{aligned} x(t) &= A\,e^{j\Omega_0 t} \\ &= A\,(\cos\Omega_0 t + j\,\sin\Omega_0 t) \\ &= A\cos\Omega_0 t + j\,A\sin\Omega_0 t \end{aligned} \qquad .....(3.2)$$

The real part of equation (3.2) represents a cosinusoidal signal and the imaginary part represents a sinusoidal signal.

i.e., $Re[x(t)] = A\cos\Omega_0 t$ ..... Cosinusoidal signal

$Im[x(t)] = A\sin\Omega_0 t$ ..... Sinusoidal signal

> *Note : Re - stands for 'real part of'*
> *Im - stands for 'imaginary part of'*

***Case ii :* When $\Omega = 0$**

On substituting $\Omega = 0$ in equation (3.1) we get,

$$x(t) = A\, e^{\sigma t} \qquad \text{....(3.3)}$$

In equation (3.3) if σ is positive then the signal will be an exponentially increasing signal.

In equation (3.3) if σ is negative then the signal will be an exponentially decreasing signal.

i.e., $x(t) = A\, e^{\sigma t}$ ..... Exponentially increasing signal

$x(t) = A\, e^{-\sigma t}$ ..... Exponentially decreasing signal

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***Case iii :* When $\sigma = 0$ and $\Omega = 0$**

On substituting $\sigma = 0$ and $\Omega = 0$ in equation (3.1) we get,

$$x(t) = A\, e^{0} = A \qquad \text{.....(3.4)}$$

The equation (3.4) represents a step signal.

Case iv : ***When $\sigma \neq 0$, $\Omega \neq 0$ and $\Omega = \Omega_0$.***

When both σ and Ω are non-zero and when $\Omega = \Omega_0$, the equation (3.1) can be expressed as shown below.

$$x(t) = Ae^{(\sigma + j\Omega_0)t} = Ae^{\sigma t}\, Ae^{j\Omega_0 t}$$

$$= Ae^{\sigma t}(\cos\Omega_0 t + j\sin\Omega_0 t)$$

$$= Ae^{\sigma t}\cos\Omega_0 t + j\,Ae^{\sigma t}\sin\Omega_0 t) \qquad .....(3.5)$$

The real part of equation (3.5) represents an exponentially increasing/decreasing cosinusoidal signal.

The imaginary part of equation (3.5) represents an exponentially increasing/decreasing sinusoidal signal.

i.e., $\mathrm{Re}[x(t)] = Ae^{\sigma t}\cos\Omega_0 t$

→ $Ae^{\sigma t}\cos\Omega_0 t$ ..... Exponentially increasing cosinusoidal signal

→ $Ae^{-\sigma t}\cos\Omega_0 t$ ..... Exponentially decreasing cosinusoidal signal

$\mathrm{Im}[x(t)] = Ae^{\sigma t}\sin\Omega_0 t$

→ $Ae^{\sigma t}\sin\Omega_0 t$ ..... Exponentially increasing sinusoidal signal

→ $Ae^{-\sigma t}\sin\Omega_0 t$ ..... Exponentially decreasing sinusoidal signal

Complex Frequency Plane or s-Plane

The complex frequency is defined as,

$$\text{Complex frequency, } s = \sigma + j\Omega$$

$$\text{where, } \sigma = \text{Real part of } s$$

$$\Omega = \text{Imaginary part of } s$$

The σ and Ω can take values from $-\infty$ to $+\infty$. A two dimensional complex plane with values of σ on horizontal axis and values Ω on vertical axis as shown in fig 3.1 is called ***complex frequency plane*** or ***s-plane.***

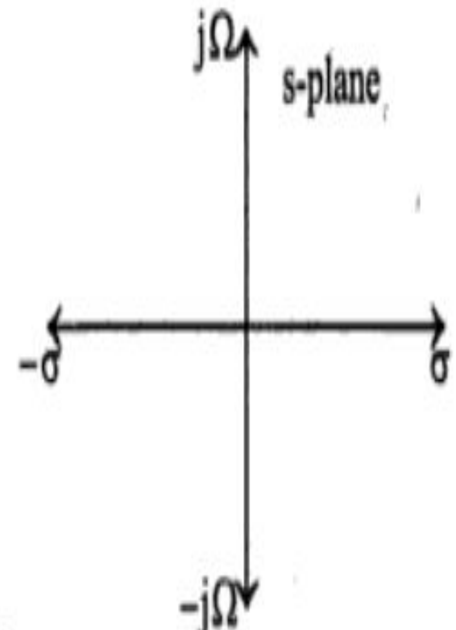

Fig 3.1: Complex frequency plane or s-plane.

Definition of Laplace Transform

In order to transform a time domain signal x(t) to s-domain, multiply the signal by e^{-st} and then integrate from $-\infty$ to ∞. The transformed signal is represented as X(s) and the transformation is denoted by the script letter $\mathcal{L}$.

Symbolically the ***Laplace transform*** of x(t) is denoted as,

$$X(s) = \mathcal{L}\{x(t)\}$$

Let x(t) be a continuous time signal defined for all values of t. Let X(s) be Laplace transform of x(t). Now the ***Laplace transform*** of x(t) is defined as,

$$\boxed{\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt} \qquad .....(3.6)$$

If x(t) is defined for $t \geq 0$, (i.e., if x(t) is causal) then,

$$\boxed{\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{+\infty} x(t)\, e^{-st}\, dt} \qquad .....(3.7)$$

The definition of Laplace transform as given by equation (3.6) is called ***Two sided Laplace transform*** or ***Bilateral Laplace Transform*** and the definition of Laplace transform as given by equation (3.7) is called ***One sided Laplace transform*** or ***Unilateral Laplace transform***.

Definition of Inverse Laplace Transform

The s-domain signal X(s) can be transformed to time domain signal x(t) by using inverse Laplace transform.

The ***Inverse Laplace transform*** of X(s) is defined as,

$$\boxed{\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(s)\, e^{st}\, ds}$$

The signal x(t) and X(s) are called ***Laplace transform pair*** and can be expressed as,

$$\boxed{x(t) \overset{\mathcal{L}}{\underset{\mathcal{L}^{-1}}{\rightleftarrows}} X(s)}$$

Existence of Laplace Transform

The computation of Laplace transform involves integral of x(t) from $t = -\infty$ to $+\infty$. Therefore Laplace transform of a signal exists if the integral, $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges (i.e., finite). The integral will converge if the signal x(t) is sectionally continuous in every finite interval of t and if it is of exponential order as t approaches infinity.

A causal signal x(t) is said to be ***exponential order*** if a real, positive constant σ (where σ is real part of s) exists such that the function, $e^{-\sigma t}|x(t)|$ approaches zero as t approaches infinity.

i.e., if, $\underset{t \to \infty}{\mathrm{Lt}}\; e^{-\sigma t}\,|x(t)| = 0$, then x(t) is of exponential order.

For a causal signal, if $\underset{t \to \infty}{\mathrm{Lt}}\; e^{-\sigma t}\,|x(t)| = 0$ for $\sigma > \sigma_c$, and if $\underset{t \to \infty}{\mathrm{Lt}}\; e^{-\sigma t}|x(t)| = \infty$ for $\sigma < \sigma_c$, then σ_c is called ***abscissa of convergence***, (where σ_c is a point on real axis in s-plane).

The integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges only if the real part of s is greater than the abscissa of convergence σ_c. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges is called ***Region Of Convergence (ROC)***. Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence.

3.2 Region of Convergence

The Laplace transform of a signal is given by $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges is called ***Region Of Convergence (ROC)***. The ROC for the following three types of signals are discussed here.

Case i : Right sided (causal) signal

Case ii : Left sided (anticausal) signal

Case iii : Two sided signal.

Case i : ***Right sided (causal) signal***

Let, $x(t) = e^{-at}u(t)$, where $a > 0$

$= e^{-at}$ for $t \geq 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} e^{-at}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{+\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{+\infty} e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \frac{e^{-(\sigma + j\Omega + a)\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)} = -\frac{e^{-(\sigma + a)\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} \qquad \boxed{\begin{array}{c}\text{Put,}\\ s = \sigma + j\Omega\end{array}}$$

$$\therefore \mathcal{L}\{x(t)\} = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

$$\text{where, } k = \sigma + a = \sigma - (-a)$$

When $\sigma > -a$, $k = \sigma - (-a) =$ Positive, $\therefore e^{-k\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $k = \sigma - (-a) =$ Negative, $\therefore e^{-k\infty} = e^{+\infty} = \infty$

Hence we can say that, X(s) converges, when $\sigma > -a$, and does not converge for $\sigma < -a$.

$\therefore$ Abscissa of convergence, $\sigma_c = -a$.

When $\sigma > -a$, the X(s) is given by,

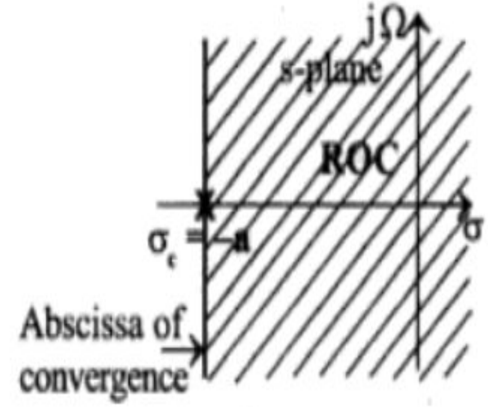

Fig 3.2 : ROC of $x(t) = e^{-at}u(t)$.

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = -\frac{0 \times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence, $\sigma_c = -a$, as shown in fig 3.2.

Case ii : ***Left sided (anticausal) signal***

Let, $x(t) = e^{-bt}u(-t) = e^{-bt}$ for $t \le 0$, where $b > 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} e^{-bt}\, u(-t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{-bt}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s+b)t}\, dt = \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0} = \frac{e^{0}}{-(s+b)} - \frac{e^{(\sigma + j\Omega + b)\infty}}{-(s+b)}$$

$$= -\frac{1}{s+b} + \frac{e^{(\sigma+b)\times\infty} e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{e^{k\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

Put, $s = \sigma + j\Omega$

where, $k = \sigma + b = \sigma - (-b)$

When $\sigma > -b$, $k = \sigma - (-b) =$ Positive, $\therefore e^{k\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $k = \sigma - (-b) =$ Negative, $\therefore e^{k\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when $\sigma < -b$, and does not converge for $\sigma > -b$.

$\therefore$ Abscissa of convergence, $\sigma_c = -b$.

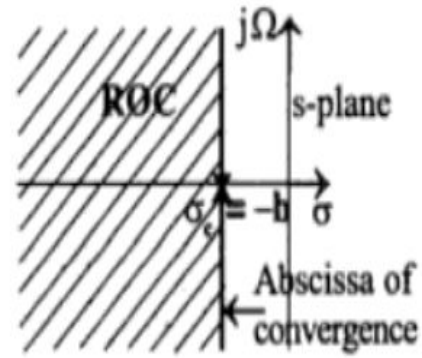

Fig 3.3 : ROC of $x(t) = e^{-bt}u(-t)$.

When $\sigma > -b$, $k = \sigma - (-b) = \text{Positive}$, $\therefore e^{k\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $k = \sigma - (-b) = \text{Negative}$, $\therefore e^{k\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when $\sigma < -b$, and does not converge for $\sigma > -b$.

$\therefore$ Abscissa of convergence, $\sigma_c = -b$.

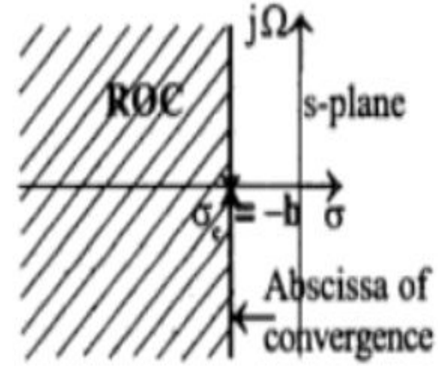

Fig 3.3 : ROC of $x(t) = e^{-bt}u(-t)$.

When $\sigma < -b$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{1}{s+b} + \frac{e^{k\times\infty}\, e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{0 \times e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b}$$

Therefore for an anticausal signal the ROC includes all points on the s-plane to the left of abscissa of convergence, $\sigma_c = -b$, as shown in fig 3.3.

Case iii: ***Two sided signal***

Let, $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$, where $a > 0,\ b > 0,$ and $a > b$ (i.e., $-a < -b$)

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} \left[e^{-at}\, u(t) + e^{-bt}\, u(-t)\right] e^{-st}\, dt$$

$$= \int_{0}^{+\infty} e^{-at}\, e^{-st}\, dt + \int_{-\infty}^{0} e^{-bt}\, e^{-st}\, dt = \int_{0}^{+\infty} e^{-(s+a)t}\, dt + \int_{-\infty}^{0} e^{-(s+b)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} + \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0} = \left[\frac{e^{-(\sigma+j\Omega+a)t}}{-(s+a)}\right]_0^{\infty} + \left[\frac{e^{-(\sigma+j\Omega+b)t}}{-(s+b)}\right]_{-\infty}^{0}$$

$$= \frac{e^{-(\sigma+j\Omega+a)\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)} + \frac{e^{0}}{-(s+b)} - \frac{e^{(\sigma+j\Omega+b)\infty}}{-(s+b)} \qquad \boxed{\text{Put, } s = \sigma + j\Omega}$$

$$= -\frac{e^{-p\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

where, $p = \sigma + a = \sigma - (-a)$ and $q = \sigma + b = \sigma - (-b)$

When $\sigma > -a$, $p = \sigma - (-a) = \text{Positive}$, $\therefore e^{-p\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $p = \sigma - (-a) = \text{Negative}$, $\therefore e^{-p\infty} = e^{+\infty} = \infty$

When $\sigma > -b$, $q = \sigma - (-b) = \text{Positive}$, $\therefore e^{q\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $q = \sigma - (-b) = \text{Negative}$, $\therefore e^{q\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when σ lies between $-a$ and $-b$ (i.e., $-a < \sigma < -$
does not converge for $\sigma < -a$ and $\sigma > -b$.

$\therefore$ Abscissa of convergences, $\sigma_{c1} = -a$ and $\sigma_{c2} = -b$.

When $-a < \sigma < -b$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-p\times\infty}\,e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\,e^{j\Omega\times\infty}}{s+b}$$

$$= -\frac{0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{0\times e^{j\Omega\times\infty}}{s+b}$$

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-p\times\infty}\,e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\,e^{j\Omega\times\infty}}{s+b}$$

$$= -\frac{0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{0\times e^{j\Omega\times\infty}}{s+b}$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

Therefore for a two sided signal the ROC includes all points on the s-plane in the region in between two abscissa of convergences, $\sigma_{c1} = -a$ and $\sigma_{c2} = -b$, as shown in fig 3.4.

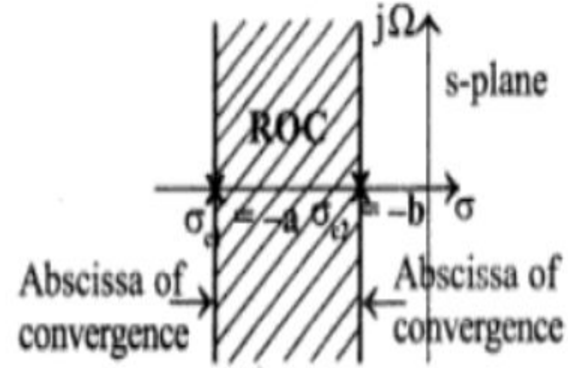

Fig 3.4 : ROC of $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$.

Example 3.1

Determine the Laplace transform of the following continuous time signals and their ROC.

a) $x(t) = A\,u(t)$ b) $x(t) = t\,u(t)$ c) $x(t) = e^{-3t}\,u(t)$ d) $x(t) = e^{-3t}\,u(-t)$ e) $x(t) = e^{-4|t|}$

Solution

a) Given that, $x(t) = A\,u(t) = A\ ;\ t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{0}^{\infty} A\,e^{-st}\,dt = A\int_{0}^{\infty} e^{-st}\,dt$$

$$\boxed{\begin{array}{c}\text{Put,}\\ s = \sigma + j\Omega\end{array}}$$

$$= A\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma+j\Omega)t}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma+j\Omega)\infty}}{-s} - \frac{e^0}{-s}\right] = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$
When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, $X(s)$ converges when $\sigma > 0$.

When $\sigma > 0$, the $X(s)$ is given by,

$$X(s) = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = A\left[\frac{0\times e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = \frac{A}{s}$$

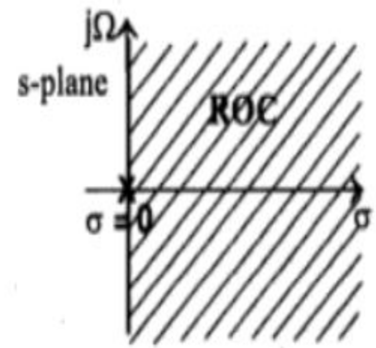

Fig 1 : ROC of x(t) = A u(t).

$\therefore\ \mathcal{L}\{A\,u(t)\} = \dfrac{A}{s}$; with ROC as all points is s-plane to the right of line passing through $\sigma = 0$.

(or ROC is right half of s-plane).

b) Given that, $x(t) = t\,u(t) = t\ ;\ t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$\boxed{\int u\,v = u \int v - \int \left[du \int v\right]}$$

$$= \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1 \times \frac{e^{-st}}{-s}\, dt = \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty} = \left[t\,\frac{e^{-(\sigma + j\Omega)t}}{-s}\right]_0^{\infty} - \left[\frac{e^{-(\sigma + j\Omega)t}}{s^2}\right]_0^{\infty}$$

$$\boxed{\begin{array}{c}\text{Put,}\\ s = \sigma + j\Omega\end{array}}$$

$$= \left[\infty \times \frac{e^{-(\sigma + j\Omega)\infty}}{-s} - 0 \times \frac{e^{0}}{-s} - \frac{e^{-(\sigma + j\Omega)\infty}}{s^2} + \frac{e^{0}}{s^2}\right]$$

$$= \left[\infty \times \frac{e^{-\sigma \times \infty}\, e^{-j\Omega \times \infty}}{-s} - 0 - \frac{e^{-\sigma \times \infty}\, e^{-j\Omega \times \infty}}{s^2} + \frac{1}{s^2}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$

When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, $X(s)$ converges when $\sigma > 0$.

When $\sigma > 0$, the $X(s)$ is given by,

$$X(s) = \left[\infty \times \frac{e^{-\sigma \times \infty}\, e^{-j\Omega \times \infty}}{-s} - \frac{e^{-\sigma \times \infty}\, e^{-j\Omega \times \infty}}{s^2} + \frac{1}{s^2}\right]$$

$$= \left[\infty \times \frac{0 \times e^{-j\Omega \times \infty}}{-s} - \frac{0 \times e^{-j\Omega \times \infty}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

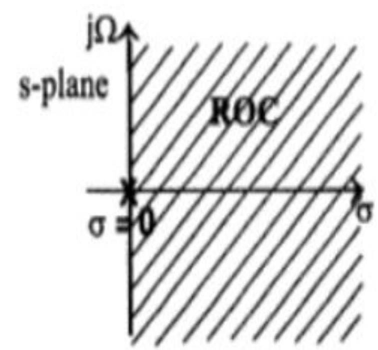

Fig 2 : ROC of x(t) = tu(t).

$\therefore \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}$; with ROC as all points is s-plane to the right of line passing through $\sigma = 0$.

(or ROC is right half of s-plane).

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c) Given that, $x(t) = e^{-3t}\,u(t) = e^{-3t}$; $t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-3t}\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+3)t}\, dt$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_0^{\infty} = \frac{e^{-(s+3)\infty}}{-(s+3)} - \frac{e^{0}}{-(s+3)} = -\frac{e^{-(\sigma+j\Omega+3)\infty}}{s+3} + \frac{1}{s+3}$$

Put, $s = \sigma + j\Omega$

$$= -\frac{e^{-(\sigma+3)\times\infty}\, e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3}$$

where, $k = \sigma + 3 = \sigma - (-3)$

When, $\sigma > -3$, $k = \sigma - (-3)$ = Positive. $\therefore e^{-k\infty} = e^{-\infty} = 0$

When, $\sigma < -3$, $k = \sigma - (-3)$ = Negative. $\therefore e^{-k\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > -3$.

When $\sigma > -3$, the X(s) is given by,

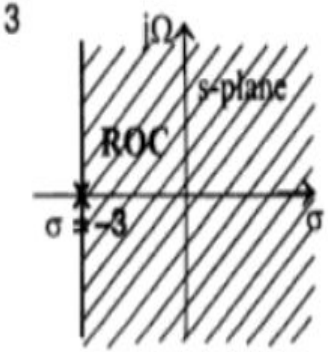


Fig 3 : ROC of $x(t) = e^{-3t}u(t)$.

$$X(s) = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{0\times e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = \frac{1}{s+3}$$

$\therefore \mathcal{L}\{e^{-3t}\,u(t)\} = \frac{1}{s+3}$; with ROC as all points in s-plane to the right of line passing through $\sigma = -3$.

b) Given that, $x(t) = \cos\Omega_0 t\, u(t) = \cos\Omega_0 t \quad ; \quad t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty \cos\Omega_0 t\, e^{-st}\, dt = \int_0^\infty \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\, e^{-st}\, dt \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s - j\Omega_0)t} + e^{-(s + j\Omega_0)t}\right) dt = \frac{1}{2}\left[\frac{e^{-(s - j\Omega_0)t}}{-(s - j\Omega_0)} + \frac{e^{-(s + j\Omega_0)t}}{-(s + j\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s - j\Omega_0)} + \frac{e^{-\infty}}{-(s + j\Omega_0)} - \frac{e^{0}}{-(s - j\Omega_0)} - \frac{e^{0}}{-(s + j\Omega_0)}\right]$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s - j\Omega_0} + \frac{1}{s + j\Omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s + j\Omega_0 + s - j\Omega_0}{(s - j\Omega_0)(s + j\Omega_0)}\right] = \frac{1}{2}\left[\frac{2s}{s^2 + \Omega_0^2}\right] = \frac{s}{s^2 + \Omega_0^2} \qquad \boxed{(a+b)(a-b) = a^2 - b^2} \quad \boxed{j^2 = -1}$$

$$\therefore \mathcal{L}\{\cos\Omega_0 t\, u(t)\} = \frac{s}{s^2 + \Omega_0^2}$$

c) Given that, $x(t) = \cosh\Omega_0 t\, u(t) = \cosh\Omega_0 t \quad ; \quad t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty \cosh\Omega_0 t\, e^{-st}\, dt = \int_0^\infty \frac{e^{\Omega_0 t} + e^{-\Omega_0 t}}{2}\, e^{-st}\, dt$$

$$\boxed{\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}}$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-\Omega_0)t} + e^{-(s+\Omega_0)t}\right) dt = \frac{1}{2}\left[\frac{e^{-(s-\Omega_0)t}}{-(s-\Omega_0)} + \frac{e^{-(s+\Omega_0)t}}{-(s+\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-\Omega_0)} + \frac{e^{-\infty}}{-(s+\Omega_0)} - \frac{e^{0}}{-(s-\Omega_0)} - \frac{e^{0}}{-(s+\Omega_0)}\right]$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s-\Omega_0} + \frac{1}{s+\Omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+\Omega_0+s-\Omega_0}{(s-\Omega_0)(s+\Omega_0)}\right] = \frac{1}{2}\left[\frac{2s}{s^2-\Omega_0^2}\right] = \frac{s}{s^2-\Omega_0^2}$$

$$\boxed{(a+b)(a-b) = a^2 - b^2}$$

$$\therefore \mathcal{L}\{\cosh\Omega_0 t\, u(t)\} = \frac{s}{s^2-\Omega_0^2}$$

Example 3.3

Determine the Laplace transform of the signals shown below.

a)

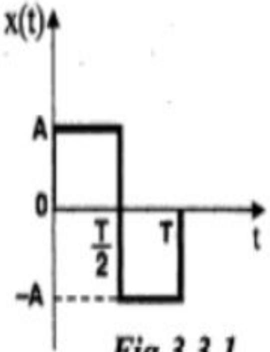


Fig 3.3.1.

b)

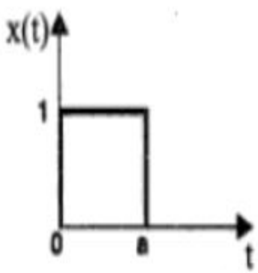


Fig 3.3.2.

c)

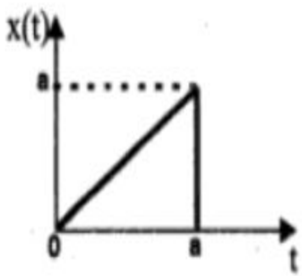


Fig 3.3.3.

d)

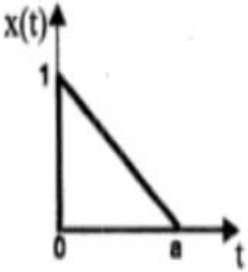
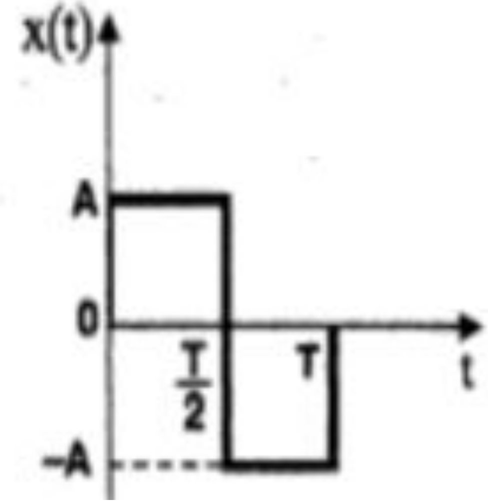


Fig 3.3.4.

Solution

a)



The mathematical equation of the signal shown in fig 3.3.1 is,

$$x(t) = A \quad ;\text{for } 0 < t < T/2$$
$$= -A \quad ;\text{for } T/2 < t < T$$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{T} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{T/2} A\, e^{-st}\, dt + \int_{T/2}^{T} (-A)\, e^{-st}\, dt = \left[\frac{A\, e^{-st}}{-s}\right]_{0}^{T/2} + \left[\frac{-A\, e^{-st}}{-s}\right]_{T/2}^{T}$$

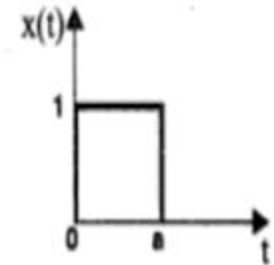
$$= \left[\frac{A\, e^{\frac{-sT}{2}}}{-s} - \frac{A\, e^{0}}{-s}\right] + \left[\frac{A\, e^{-sT}}{s} - \frac{A\, e^{\frac{-sT}{2}}}{s}\right]$$

$$= -\frac{A\, e^{\frac{-sT}{2}}}{s} + \frac{A}{s} + \frac{A\, e^{-sT}}{s} - \frac{A\, e^{\frac{-sT}{2}}}{s}$$

$$= \frac{A}{s}\left[1 + e^{-sT} - 2\, e^{\frac{-sT}{2}}\right] = \frac{A}{s}\left[1 - e^{\frac{-sT}{2}}\right]^{2}$$

$$\boxed{(a-b)^2 = a^2 + b^2 - 2ab}$$

b)



The mathematical equation of the signal shown in fig 3.3.2 is,

$$x(t) = 1 \quad ;\text{for } 0 \le t \le a$$
$$= 0 \quad ;\text{for } t > a$$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{a} 1 \times e^{-st}\, dt = \int_{0}^{a} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^a$$

$$= \frac{e^{-as}}{-s} - \frac{e^{0}}{-s} = -\frac{e^{-as}}{s} + \frac{1}{s} = \frac{1}{s}\left(1 - e^{-as}\right)$$

c)

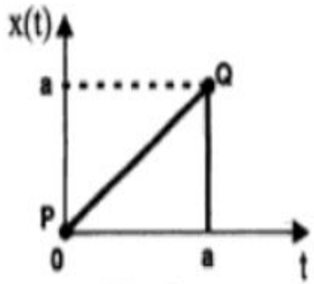


Fig 1.

To Find Mathematical Equation for x(t)

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point - P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point - Q = $[t_2, x(t_2)] = [a, a]$

On substituting the coordinates of points - P and Q in equation - (1) we get,

$$\frac{x(t)-0}{0-a}=\frac{t-0}{0-a} \Rightarrow \frac{x(t)}{-a}=\frac{t}{-a} \Rightarrow x(t)=t$$

$$\therefore x(t) = t \quad ;\text{for } t=0 \text{ to } a$$

$$= 0 \quad ;\text{for } t>a$$

To Evaluate Laplace transform of x(t)

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{0}^{a} t\, e^{-st}\, dt$$

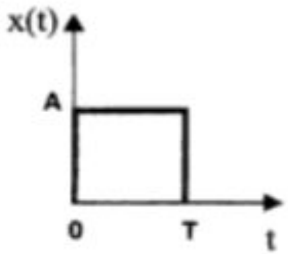
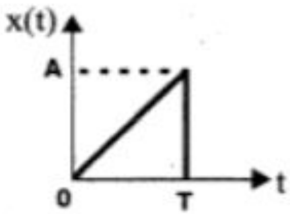
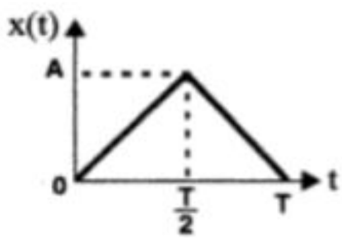
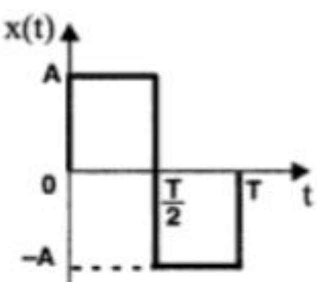
$$= \left[t \times \frac{e^{-st}}{-s} - \int 1 \times \frac{e^{-st}}{-s}\, dt\right]_0^a = \left[-\frac{t\,e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^a$$

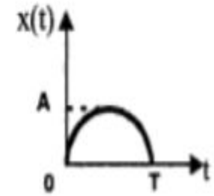
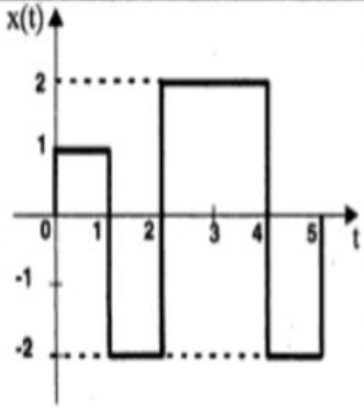
$$= \left[-\frac{a\,e^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2}\right] = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{a\,e^{-as}}{s}$$

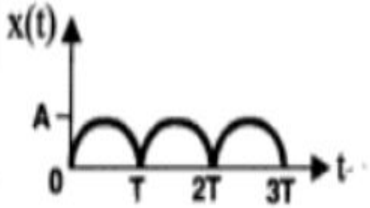
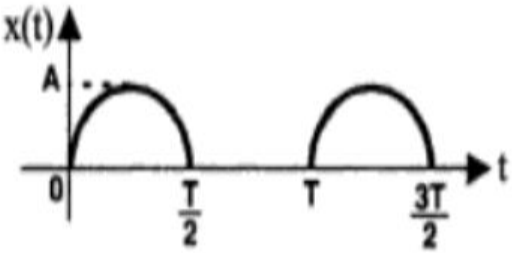
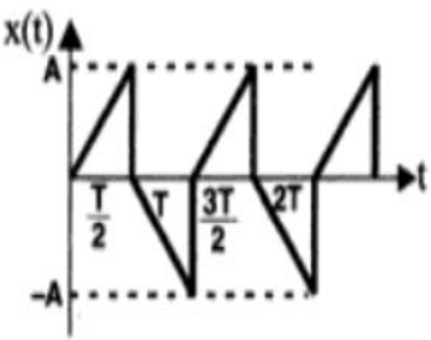
$$= \frac{1}{s^2}\left[1 - e^{-as}(1+as)\right]$$

$\int uv = u\int v - \int\left[du \int v\right]$	
$u = t$	$v = e^{-st}$

Table 3.1 : Laplace Transform of Some Standard Signals

Waveform	x(t)	$X(s) = \mathcal{L}\{x(t)\}$
	$x(t) = A \;;\; 0 < t < T$ $= 0 \;;\; t > T$	$X(s) = \frac{A}{s}(1 - e^{-sT})$
	$x(t) = \frac{At}{T} \;;\; 0 < t < T$ $= 0 \;;\; t > T$	$X(s) = \frac{A}{Ts^2}\left[1 - e^{-sT}(1 + sT)\right]$
	$x(t) = \frac{2At}{T} \;;\; 0 < t < \frac{T}{2}$ $= 2A - \frac{2At}{T} \;;\; \frac{T}{2} < t < T$	$X(s) = \frac{2A}{Ts^2}\left(1 - e^{\frac{-sT}{2}}\right)^2$
	$x(t) = A \;;\; 0 < t < \frac{T}{2}$ $= -A \;;\; \frac{T}{2} < t < T$	$X(s) = \frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$

	$x(t) = A \sin t \;;\; 0 < t < T$ $\quad = 0 \;;\; t > T$	$X(s) = \dfrac{A}{s^2+1}\left(e^{-sT}+1\right)$
	$x(t) = 1 \;;\; 0 < t < 1$ $\quad = -2 \;;\; 1 < t < 2$ $\quad = 2 \;;\; 2 < t < 4$ $\quad = -2 \;;\; 4 < t < 5$ $\quad = 0 \;;\; t > 5$	$X(s) = \dfrac{1}{s}\left(1 - 3e^{-s} + 4e^{-2s} - 4e^{-4s} + 2e^{-5s}\right)$

Waveform	x(t)	$X(s) = \mathcal{L}\{x(t)\}$
	$x(t) = A \sin t \,;\, 0 < t < T$ and $x(t + nT) = x(t)$	$X(s) = \dfrac{A}{s^2 + 1}\left(\dfrac{1 + e^{-sT}}{1 - e^{-sT}}\right)$
	$x(t) = A \sin t \,;\, 0 < t < \frac{T}{2}$ $= 0 \,;\, \frac{T}{2} < t < T$ and $x(t + nT) = x(t)$	$X(s) = \dfrac{A}{\left(s^2 + 1\right)\left(1 - e^{-\frac{sT}{2}}\right)}$
	$x(t) = \dfrac{2At}{T} \,;\, 0 < t < \frac{T}{2}$ $= A - \dfrac{2At}{T} \,;\, \frac{T}{2} < t < T$ and $x(t + nT) = x(t)$	$X(s) = \dfrac{2A\left[1 - \left(1 + \frac{Ts}{2}\right)e^{-\frac{sT}{2}}\right]}{Ts^2\left(1 + e^{-\frac{sT}{2}}\right)}$

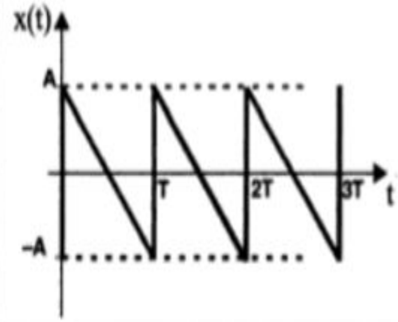
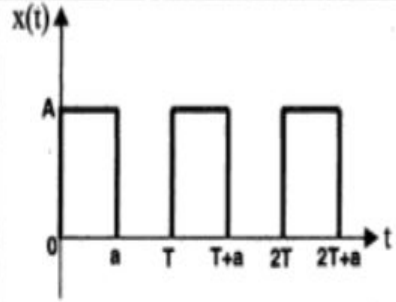
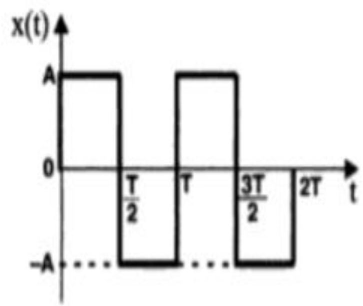
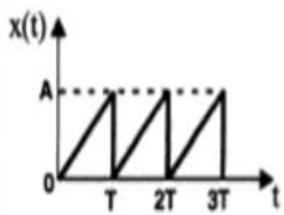
	$x(t) = A - \frac{2At}{T}; 0 < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{2A}{Ts}\left(\frac{T}{2}\,\frac{1 + e^{-sT}}{1 - e^{-sT}} - \frac{1}{s}\right)$
	$x(t) = A \;;\; 0 < t < a$ $= 0 \;;\; a < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{A}{s}\,\frac{1 - e^{-as}}{1 - e^{-sT}}$
	$x(t) = A \;;\; 0 < t < \frac{T}{2}$ $= -A \;;\; \frac{T}{2} < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{A}{s}\left(\frac{1 - e^{-\frac{sT}{2}}}{1 + e^{-\frac{sT}{2}}}\right)$
	$x(t) = \frac{At}{T} \;;\; 0 < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{A}{Ts^2}\left[\frac{1 - e^{-sT}(1 + sT)}{1 - e^{-sT}}\right]$

Table 3.2 : Some Standard Laplace Transform Pairs

Note : σ = Real part of s

x(t)	X(s)	ROC
$\delta(t)$	1	Entire s-plane
$u(t)$	$\frac{1}{s}$	$\sigma > 0$
$t\,u(t)$	$\frac{1}{s^2}$	$\sigma > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, n = 1, 2, 3, ...	$\frac{1}{s^n}$	$\sigma > 0$
$e^{-at}\,u(t)$	$\frac{1}{s+a}$	$\sigma > -a$
$-e^{-at}\,u(-t)$	$\frac{1}{s+a}$	$\sigma < -a$
$t^n\,u(t)$ where, n = 1, 2, 3	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$t\,e^{-at}\,u(t)$	$\frac{1}{(s+a)^2}$	$\sigma > -a$

$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{(s+a)^n}$	$\sigma > -a$
$t^n e^{-at} u(t)$ where, $n = 1, 2, 3, ...$	$\frac{n!}{(s+a)^{n+1}}$	$\sigma > -a$
$\sin\Omega_0 t\, u(t)$	$\frac{\Omega_0}{s^2 + \Omega_0^2}$	$\sigma > 0$
$\cos\Omega_0 t\, u(t)$	$\frac{s}{s^2 + \Omega_0^2}$	$\sigma > 0$
$\sinh\Omega_0 t\, u(t)$	$\frac{\Omega_0}{s^2 - \Omega_0^2}$	$\sigma > \Omega_0$
$\cosh\Omega_0 t\, u(t)$	$\frac{s}{s^2 - \Omega_0^2}$	$\sigma > \Omega_0$
$e^{-at} \sin\Omega_0 t\, u(t)$	$\frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$	$\sigma > -a$
$e^{-at} \cos\Omega_0 t\, u(t)$	$\frac{s+a}{(s+a)^2 + \Omega_0^2}$	$\sigma > -a$

3.3 Properties and Theorems of Laplace Transform

The properties and theorems of Laplace transform are listed in table 3.3. The proof of properties and theorems are presented in this section.

1. Amplitude Scaling

In amplitude scaling, if the amplitude (or magnitude) of a time domain signal is multiplied by a constant A, then its Laplace transform is also multiplied by the same constant.

i.e., if $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{A\, x(t)\} = A\, X(s)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.8)$$

$$\mathcal{L}\{A\, x(t)\} = \int_{-\infty}^{+\infty} A\, x(t)\, e^{-st}\, dt$$

$$= A \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$= A\, X(s)$$

Using equation (3.8)

2. *Linearity*

The linearity property states that, Laplace transform of weighted sum of the two or more signals is equal to similar weighted sum of Laplace transforms of the individual signals.

i.e., if $\mathcal{L}\{x_1(t)\} = X_1(s)$ and $\mathcal{L}\{x_2(t)\} = X_2(s)$, then

$$\mathcal{L}\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(s) + a_2 X_2(s)$$

Proof :

By definition of Laplace transform,

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt \qquad .....(3.9)$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt \qquad .....(3.10)$$

$$\mathcal{L}\{a_1 x_1(t) + a_2 x_2(t)\} = \int_{-\infty}^{+\infty} [a_1 x_1(t) + a_2 x_2(t)]\, e^{-st}\, dt$$

$$= a_1 \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt + a_2 \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt$$

$$= a_1 X_1(s) + a_2 X_2(s)$$

Using equations (3.9) and (3.10)

3. Time Differentiation

The time differentiation property states that if a causal signal x(t) is piecewise continuous, and Laplace transform of x(t) is X(s) then, Laplace transform of $\frac{d}{dt}x(t)$ is given by sX(s) − x(0).

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0) \quad ; \text{ where, } x(0) \text{ is value of } x(t) \text{ at } t = 0.$$

Proof :

By definition of Laplace transform, the Laplace transform of a causal signal is given by,

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t)\, e^{-st}\, dt \qquad .....(3.11)$$

$$\therefore \mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt = \int_0^{\infty} e^{-st}\, \frac{dx(t)}{dt}\, dt$$

$\int u\,v = u\int v - \int\left[du \int v\right]$	
$u = e^{-st}$	$v = \frac{dx(t)}{dt}$

$$= \left[e^{-st}\, x(t)\right]_0^{\infty} - \int_0^{\infty} -s\, e^{-st}\, x(t)\, dt$$

$$= e^{-\infty}\, x(\infty) - e^{0}\, x(0) + s\int_0^{\infty} x(t)\, e^{-st}\, dt$$

$\boxed{e^{-\infty} = 0 \text{ and } e^{0} = 1}$

$$= s\int_0^{\infty} x(t)\, e^{-st}\, dt - x(0) = s\,X(s) - x(0)$$

Using equation (3.11)

4. Time Integration

The time integration property states that, if a causal signal x(t) is continuous and Laplace transform of x(t) is X(s), then the Laplace transform of $\int x(t)\,dt$ is given by, $\frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\big|_{t=0}}{s}$

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\int x(t)\,dt\right\} = \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$$

Proof :

By definition of Laplace transform, the Laplace transform of a causal signal is given by,

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)\,e^{-st}\,dt \qquad .....(3.12)$$

$$\therefore \mathcal{L}\left\{\int x(t)\,dt\right\} = \int_0^\infty \left[\int x(t)\,dt\right] e^{-st}\,dt$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = \int x(t)\,dt$	$v = e^{-st}$

$$= \left[\left[\int x(t)\,dt\right]\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty x(t)\,\frac{e^{-st}}{-s}\,dt$$

$$= \left[\int x(t)\,dt\right]\Big|_{t=\infty}\frac{e^{-\infty}}{-s} - \left[\int x(t)\,dt\right]\Big|_{t=0}\frac{e^{0}}{-s} + \frac{1}{s}\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= \frac{1}{s}\left[\int x(t)\,dt\right]\Big|_{t=0} + \frac{1}{s}\int_0^\infty x(t)\,e^{-st}\,dt \qquad \boxed{e^{-\infty} = 0 \text{ and } e^{0} = 1}$$

$$= \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s} \qquad \boxed{\text{Using equation (3.12)}}$$

5. Frequency shifting

The frequency shifting property of Laplace transform says that,

If, $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{e^{\pm at}\, x(t)\} = X(s \mp a) \qquad \left[\text{i.e., } \mathcal{L}\{e^{at}\, x(t)\} = X(s-a) \text{ and } \mathcal{L}\{e^{-at}\, x(t)\} = X(s+a)\right]$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.13)$$

$$\therefore \mathcal{L}\{e^{\pm at}\, x(t)\} = \int_{-\infty}^{+\infty} e^{\pm at}\, x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-(s \mp a)t}\, dt$$

$$= X(s \mp a)$$

The term $\int_{-\infty}^{+\infty} x(t)\, e^{-(s \mp a)t}\, dt$ is similar to the form of definition of Laplace transform (equation(3.13)) except that s is replaced by $(s \mp a)$.

$$\therefore \int_{-\infty}^{+\infty} x(t)\, e^{-(s \mp a)t} dt = X(s \mp a)$$

6. Time shifting

The time shifting property of Laplace transform says that,

If, $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x(t \pm a)\} = e^{\pm as}\, X(s) \qquad \left[\text{i.e., } \mathcal{L}\{x(t+a)\} = e^{as}\, X(s) \text{ and } \mathcal{L}\{x(t-a)\} = e^{-as}\, X(s)\right]$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.14)$$

Let, $t \pm a = \tau$
$\therefore t = \tau \mp a$
On differentiating
$dt = d\tau$

$$\therefore \mathcal{L}\{x(t \pm a)\} = \int_{-\infty}^{+\infty} x(t \pm a)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-s(\tau \mp a)}\, d\tau$$

Since τ is a dummy variable for integration we can change τ to t.

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-s\tau} \times e^{\pm as}\, d\tau = e^{\pm as} \int_{-\infty}^{+\infty} x(\tau)\, e^{-s\tau}\, d\tau$$

Using equation (3.14)

$$= e^{\pm as} \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = e^{\pm as}\, X(s)$$

7. Frequency Differentiation

The frequency differentiation property of Laplace transform says that,

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

On differentiating the above equation with respect to s we get,

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left(\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt\right)$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{ds}e^{-st}\right)dt = \int_{-\infty}^{+\infty} x(t)\left(-t\,e^{-st}\right)dt$$

Interchanging the order of integration and differentiation

$$= \int_{-\infty}^{+\infty} (-t\,x(t))\,e^{-st}\,dt = \mathcal{L}\{-t\,x(t)\} = -\mathcal{L}\{t\,x(t)\}$$

$$\therefore \mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s)$$

8. Frequency Integration

The frequency integration property of Laplace transform says that,

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\frac{1}{t}\,x(t)\right\} = \int_{s}^{\infty} X(s)\,ds$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

On integrating the above equation with respect to s between limits s to ∞ we get,

$$\int_{s}^{\infty} X(s)\,ds = \int_{s}^{\infty}\left[\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt\right]ds$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\int_{s}^{\infty} e^{-st}\,ds\right]dt$$

Interchanging the order of integrations.

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} dt = \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right]dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[0 + \frac{e^{-st}}{t}\right]dt = \int_{-\infty}^{+\infty}\left[\frac{1}{t}\,x(t)\right]e^{-st}\,dt = \mathcal{L}\left\{\frac{1}{t}\,x(t)\right\}$$

9. Time scaling

The time scaling property of Laplace transform says that,

If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|}\, X\left(\frac{s}{a}\right)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.15)$$

$$\therefore\ \mathcal{L}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-s\left(\frac{\tau}{a}\right)}\, \frac{d\tau}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau = \frac{1}{a}\, X\left(\frac{s}{a}\right)$$

Put, $at = \tau$

$\therefore\ t = \dfrac{\tau}{a}$

On differentiating

$dt = \dfrac{d\tau}{a}$

The above transform is applicable for positive values of "a".

If "a" happens to be negative it can be proved that,

$$\mathcal{L}\{x(at)\} = -\frac{1}{a}\, X\left(\frac{s}{a}\right)$$

Hence in general,

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|}\, X\left(\frac{s}{a}\right) \text{ for both positive and negative values of "a"}$$

The term $\int_{-\infty}^{+\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau$ is similar to the form of definition of Laplace transform(equation(3.15)) except that s is replaced by $\left(\frac{s}{a}\right)$.

$$\therefore\ \int_{-\infty}^{+\infty} x(\tau) e^{-\left(\frac{s}{a}\right)\tau}\, d\tau = X\left(\frac{s}{a}\right)$$

10. Periodicity

The periodicity property of Laplace transform says that,

If $x(t) = x(t+nT)$, and $x_1(t)$ be one period of $x(t)$, and $\mathcal{L}\{x_1(t)\} = \int_0^T x_1(t)\, e^{-st}\, dt$, then

$$\mathcal{L}\{x(t + nT)\} = \frac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

Proof :

By definition of Laplace transform,

$$\mathcal{L}\{x(t + nT)\} = \int_0^\infty x(t + nT)\, e^{-st}\, dt$$

$$= \int_0^T x_1(t)\, e^{-st}\, dt + \int_T^{2T} x_1(t - T)\, e^{-s(t+T)}\, dt + \int_{2T}^{3T} x_1(t - 2T)\, e^{-s(t+2T)}\, dt + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots + \int_{pT}^{(p+1)T} x_1(t - pT)\, e^{-s(t+pT)}\, dt + \ldots\ldots\ldots$$

$$= \sum_{p=0}^{\infty} \int_{pT}^{(p+1)T} x_1(t - pT)\, e^{-s(t+pT)}\, dt$$

$$= \sum_{p=0}^{\infty} \int_{0}^{T} x_1(t)\, e^{-st}\, e^{-psT}\, dt$$

The periodic signal will be identical in every period and so, $x_1(t+pT) = x_1(t)$.

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\sum_{p=0}^{\infty} e^{-psT} \right) dt$$

Interchanging the order of integration and summation

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\sum_{p=0}^{\infty} e^{-sT} \right)^{p} dt$$

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\frac{1}{1 - e^{-sT}} \right) dt$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \frac{1}{1 - C}$

$$= \frac{1}{1 - e^{-sT}} \int_{0}^{T} x_1(t)\, e^{-st}\, dt$$

The term $\frac{1}{1 - e^{-sT}}$ is independent of **t**

11. Initial Value Theorem

The initial value theorem states that, if x(t) and its derivative are Laplace transformable then,

$$\underset{t \to 0}{\mathrm{Lt}}\ x(t) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$$

i.e., Initial value of signal, $x(0) = \underset{t \to 0}{\mathrm{Lt}}\ x(t) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$

Proof :

We know that, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\,X(s) - x(0)$

On taking limit $s \to \infty$ on both sides of the above equation we get,

$$\underset{s \to \infty}{\mathrm{Lt}}\ \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \underset{s \to \infty}{\mathrm{Lt}}\ [s\,X(s) - x(0)]$$

By definition of Laplace transform, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt$

$$\underset{s \to \infty}{\mathrm{Lt}} \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = \underset{s \to \infty}{\mathrm{Lt}}\ [s\,X(s) - x(0)]$$

Here $\frac{dx(t)}{dt}$ and $x(0)$ are not functions of s

$$\int_0^\infty \frac{dx(t)}{dt}\left(\underset{s \to \infty}{\mathrm{Lt}}\ e^{-st}\right) dt = \left(\underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)\right) - x(0)$$

$\boxed{\underset{s \to \infty}{\mathrm{Lt}}\ e^{-st} = 0}$

$$0 = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s) - x(0)$$

$$\therefore\ x(0) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$$

$\boxed{x(0) = \underset{t \to 0}{\mathrm{Lt}}\ x(t)}$

$$\therefore\ \underset{t \to 0}{\mathrm{Lt}}\ x(t) = \underset{s \to \infty}{\mathrm{Lt}}\ s\,X(s)$$

12. Final Value Theorem

The final value theorem states that if x(t) and its derivative are Laplace transformable then $\underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ s\,X(s)$

i.e., Final value of signal, $x(\infty) = \underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ s\,X(s)$

Proof :

We know that, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\,X(s) - x(0)$

On taking limit $s \to 0$ on both sides of the above equation we get,

$$\underset{s \to 0}{Lt}\ \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \underset{s \to 0}{Lt}\ [s\,X(s) - x(0)]$$

By definition of Laplace transform

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt$$

$$\underset{s \to 0}{Lt} \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = \underset{s \to 0}{Lt}\ [s\,X(s) - x(0)]$$

Here $\frac{dx(t)}{dt}$ and $x(0)$ are not functions of s

$$\int_0^\infty \frac{dx(t)}{dt}\left(\underset{s \to 0}{Lt}\ e^{-st}\right) dt = \left(\underset{s \to 0}{Lt}\ s\,X(s)\right) - x(0)$$

$$\boxed{\underset{s \to 0}{Lt}\ e^{-st} = 1}$$

$$\int_0^\infty \frac{dx(t)}{dt}\, dt = \underset{s \to 0}{Lt}\ s\,X(s) - x(0)$$

$$[x(t)]_0^\infty = \underset{s \to 0}{Lt}\ s\,X(s) - x(0)$$

$$x(\infty) - x(0) = \underset{s \to 0}{Lt}\ s\,X(s) - x(0)$$

$$\therefore x(\infty) = \underset{s \to 0}{Lt}\ s\,X(s)$$

$$\therefore \underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ s\,X(s)$$

$$\boxed{x(\infty) = \underset{t \to \infty}{Lt}\ x(t)}$$

13. Convolution Theorem

The convolution theorem of Laplace transform says that, Laplace transform of convolution of two signals is given by the product of the Laplace transform of the individual signals.

i.e., if $\mathcal{L}\{x_1(t)\} = X_1(s)$ and $\mathcal{L}\{x_2(t)\} = X_2(s)$ then,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s) \qquad \text{.....(3.16)}$$

The equation (3.16) is also known as convolution property of Laplace transform.

With reference to chapter-2, section 2.9 we get,

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda \qquad \text{....(3.17)}$$

where, λ is a dummy variable used for integration.

Proof :

Let $x_1(t)$ and $x_2(t)$ be two time domain signals.

By definition of Laplace transform,

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt \qquad \text{.....(3.18)}$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt \qquad \text{.....(3.19)}$$

Let $x_3(t)$ be the signal obtained by convolution of $x_1(t)$ and $x_2(t)$. Now from equation (3.17) we get,

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(t-\lambda)\, d\lambda \qquad \text{.....(3.20)}$$

Let, $\mathcal{L}\{x_3(t)\} = X_3(s)$. Now by definition of Laplace transform we can write,

$$X_3(s) = \mathcal{L}\{x_3(t)\} = \int_{-\infty}^{+\infty} x_3(t)\, e^{-st}\, dt \qquad \text{..... (3.21)}$$

On substituing for $x_3(t)$ from equation (3.20) in equation (3.21) we get,

$$X_3(s) = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(t-\lambda)\; d\lambda\right] e^{-st}\, dt \qquad \text{..... (3.22)}$$

Let, $e^{-st} = e^{s\lambda} \times e^{-s\lambda} \times e^{-st} = e^{-s\lambda} \times e^{-s(t-\lambda)} = e^{-s\lambda} \times e^{-sM}$ ..... (3.23)

where, $M = t - \lambda$ and so, $dM = dt$ .....(3.24)

Using equations (3.23) and (3.24), the equation (3.22) can be written as,

$$X_3(s) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(M)\, e^{-s\lambda}\; e^{-sM}\; d\lambda\; dM$$

$$= \int_{-\infty}^{+\infty} x_1(\lambda)\, e^{-s\lambda}\; d\lambda \times \int_{-\infty}^{+\infty} x_2(M)\, e^{-sM}\; dM \qquad \text{.....(3.25)}$$

In equation (3.25), λ and M are dummy variables used for integration, and so they can be changed to t.

Therefore equation (3.25) can be written as,

$$X_3(s) = \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt \times \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt$$

$$= X_1(s)\, X_2(s)$$

Using equations (3.18) and (3.19)

$$\therefore\; \mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s)$$

Table 3.3 : Properties of Laplace Transform

Note : $\mathcal{L}\{x(t)\} = X(s);\quad \mathcal{L}\{x_1(t)\} = X_1(s);\quad \mathcal{L}\{x_2(t)\} = X_2(s)$

Property	Time domain signal	s-domain signal
Amplitude scaling	$A\,x(t)$	$A\,X(s)$
Linearity	$a_1x_1(t) \pm a_2x_2(t)$	$a_1X_1(s) \pm a_2X_2(s)$
Time differentiation	$\frac{d}{dt}x(t)$	$s\,X(s) - x(0)$
	$\frac{d^n}{dt^n}x(t)$ where $n = 1, 2, 3$	$s^n X(s) - \sum_{K=1}^{n} s^{n-K} \left.\frac{d^{(K-1)}x(t)}{dt^{K-1}}\right\vert_{t=0}$
Time integration	$\int x(t)\,dt$	$\frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]_{t=0}}{s}$
	$\int \text{.....} \int x(t)\,(dt)^n$ where $n = 1, 2, 3$	$\frac{X(s)}{s^n} + \sum_{K=1}^{n} \frac{1}{s^{n-K+1}} \left[\int \text{....} \int x(t)\,(dt)^k\right]_{t=0}$
Frequency shifting	$e^{\pm at}\,x(t)$	$X(s \mp a)$
Time shifting	$x(t \pm \alpha)$	$e^{\pm \alpha s}\,X(s)$

Frequency differentiation	$t\,x(t)$	$-\dfrac{dX(s)}{ds}$
	$t^n\,x(t)$ where $n = 1, 2, 3 \ldots\ldots$	$(-1)^n \dfrac{d^n}{ds^n} X(s)$
Frequency integration	$\dfrac{1}{t}\,x(t)$	$\displaystyle\int_s^{\infty} X(s)\,ds$
Time scaling	$x(at)$	$\dfrac{1}{\lvert a\rvert} X\left(\dfrac{s}{a}\right)$
Periodicity	$x(t + nT)$	$\dfrac{1}{1 - e^{-sT}} \displaystyle\int_0^{T} x_1(t)\,e^{-st}\,dt$ where, $x_1(t)$ is one period of $x(t)$.
Initial value theorem	$\underset{t \to 0}{\text{Lt}}\; x(t) = x(0)$	$\underset{s \to \infty}{\text{Lt}}\; s\,X(s)$
Final value theorem	$\underset{t \to \infty}{\text{Lt}}\; x(t) = x(\infty)$	$\underset{s \to 0}{\text{Lt}}\; s\,X(s)$
Convolution theorem	$x_1(t) * x_2(t) = \displaystyle\int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(t-\lambda)\,d\lambda$	$X_1(s)\,X_2(s)$

Example 3.5

Determine Laplace transform of periodic square wave shown in fig 3.5.1.

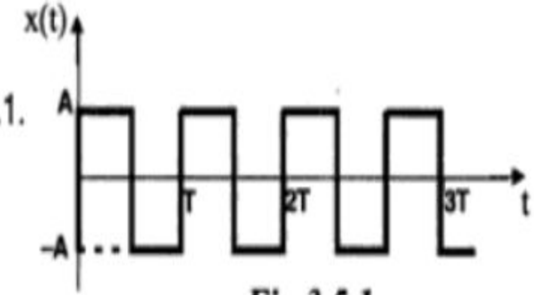


Fig 3.5.1.

Solution

The given waveform satisfy the condition, $x(t+nT) = x(t)$, and so it is periodic.

Let $x_1(t)$ be one period of $x(t)$. The equation for one period of the periodic waveform of fig 3.5.1 is,

$$x_1(t) = A \quad ; \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= -A \quad ; \text{for } t = \frac{T}{2} \text{ to } T$$

From periodicity property of Laplace transform,

If $X(s) = \mathcal{L}\{x(t)\}$, and if $x(t) = x(t + nT)$ then, $X(s) = \dfrac{1}{1 - e^{-sT}} \displaystyle\int_0^T x_1(t)\, e^{-st}\, dt$, where $x_1(t)$ is one period of $x(t)$.

$$\therefore \mathcal{L}\{x(t)\} = X(s) = \frac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

Using the result of example 3.3(a), the above equation can be written as,

From example 3.3(a) we get,

$$\int_0^T x_1(t)\, e^{-st}\, dt = \frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$$

$$X(s) = \frac{1}{1 - e^{-sT}} \left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

If $X(s) = \mathcal{L}\{x(t)\}$, and if $x(t) = x(t + nT)$ then, $X(s) = \dfrac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$, where $x_1(t)$ is one period of $x(t)$.

$$\therefore\ \mathcal{L}\{x(t)\} = X(s) = \frac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

Using the result of example 3.3(a), the above equation can be written as,

From example 3.3(a) we get,

$$\int_0^T x_1(t)\, e^{-st}\, dt = \frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$$

$$X(s) = \frac{1}{1 - e^{-sT}}\left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

$$= \frac{1}{\left(1 + e^{-\frac{sT}{2}}\right)\left(1 - e^{-\frac{sT}{2}}\right)}\left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

$$\boxed{a^2 - b^2 = (a + b)\,(a - b)}$$

$$= \frac{A}{s}\left(\frac{1 - e^{-\frac{sT}{2}}}{1 + e^{-\frac{sT}{2}}}\right)$$

Ex. 4.3.1 : Calculate laplace transform of

$x(t) = e^{-2t} u(t) - e^{2t} u(-t)$

and plot the ROC.

Soln. :

$$\text{Let } x(t) = x_1(t) + x_2(t)$$

The given equation is,

$$x(t) = e^{-2t} u(t) - e^{2t} u(-t)$$

Comparing Equations (1) and (2) we get,

$$x_1(t) = e^{-2t} u(t) \text{ and } x_2(t) = -e^{-2t} u(-t)$$

Then using linearity property we can write,

$$X(s) = X_1(s) + X_2(s)$$

Now recall standard laplace transform pair,

$$e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad \text{ROC } \sigma > -a$$

$$\therefore \quad e^{-2t} u(t) \xleftrightarrow{LT} \frac{1}{s+2} \qquad \text{ROC } \sigma > -2$$

Similarly we have standard laplace transform pair,

$$-e^{-at} u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad \text{ROC } \sigma < -a$$

$$\therefore \quad -e^{+2t}u(-t) \xleftrightarrow{LT} \frac{1}{s-2} \qquad \text{ROC } \sigma < 2$$

e : Here the function is $-e^{2t}u(-t)$ while the standard equation is for $-e^{-at}u(-t)$. Thus in th case $a = -2$. That means the given equation can be written as $-e^{-(-2)t}u(-t)$.

Putting Equations (4) and (5) in Equation (3) we get,

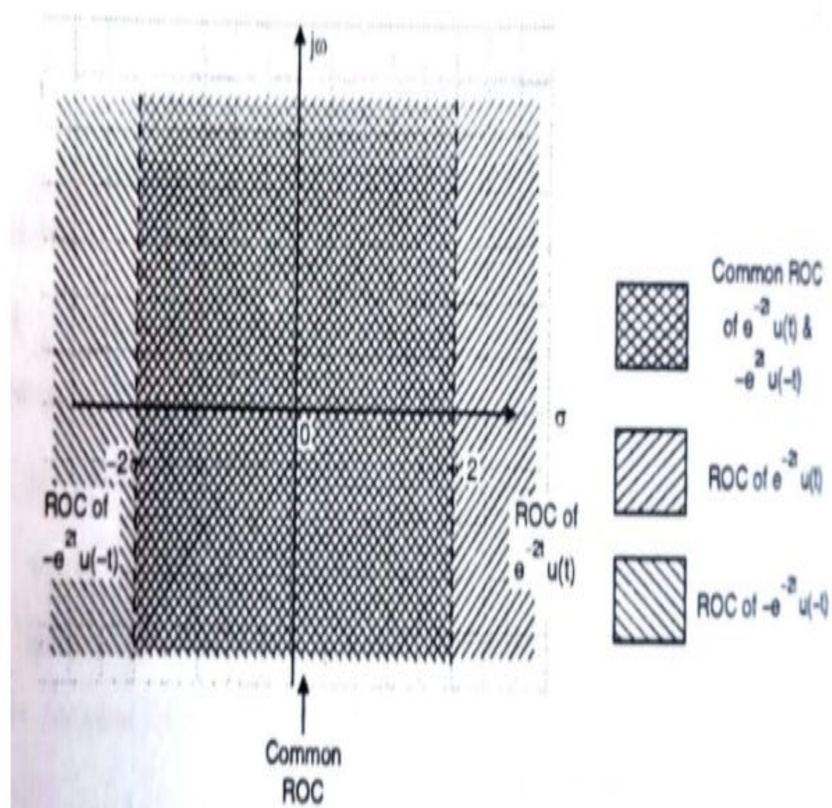
$$X(s) = \frac{1}{s+2} + \frac{1}{s-2}$$

C : ROC is intersection of two ROCs. That means,

$$\text{ROC}: \quad \sigma > -2 \quad \text{and} \quad \sigma < 2$$

$$\therefore -2 < \sigma < 2$$

$$\boxed{\therefore X(s) = \frac{1}{s+2} + \frac{1}{s-2} \quad \text{ROC}: -2 < \sigma < 2}$$



3.2 : Find the L.T of $x(t) = 5e^{4t} + 6t^3 - 3\sin 5t + 2\cos 2t$ using linearity property.

:

$$x(t) = 5e^{4t} + 6t^3 - 3\sin 5t + 2\cos 2t$$

Taking LT on both sides and using **linearity property.**

$$L\{x(t)\} = 5L\{e^{4t}\} + 6L\{t^3\} - 3L\{\sin 5t\} + 2L\{\cos 2t\}$$

We know,

$$e^{at} \xleftrightarrow{LT} \frac{1}{s-a}$$

Remember one important laplace transform pair

$$t^n \xleftrightarrow{LT} \frac{n!}{s^{n+1}}$$

Similarly we know,

$$\sin at \xleftrightarrow{LT} \frac{a}{a^2+s^2}$$

$$\cos at \xleftrightarrow{LT} \frac{s}{a^2+s^2}$$

$$\therefore L\{x(t)\} = 5\left(\frac{1}{s-4}\right) + 6\left[\frac{3!}{s^{(3+1)}}\right] - 3\left[\frac{5!}{5^2+s^2}\right] + 2\left[\frac{s}{2^2+s^2}\right]$$

$$\therefore x(s) = \left(\frac{5}{s-4}\right) + \left(\frac{36}{s^4}\right) - \left(\frac{15}{s^2+25}\right) + \left(\frac{2s}{s^2+4}\right)$$

3.3 : Find the laplace transform of

$$x(t) = e^{-a|t|}$$

.: Here 'a' is some constant. First we will decide the nature of x (t).

Let $a = \frac{1}{2}$. We will plot the given function x (t) for some values of t.

For $t = 0 \Rightarrow x(0) = e^{-\frac{1}{2}|0|} = e^0 = 1$

For $t = 1 \Rightarrow x(1) = e^{-\frac{1}{2}|1|} = e^{-1/2} = 0.606$

For $t = 2 \Rightarrow x(2) = e^{-\frac{1}{2}|2|} = e^{-1} = 0.37$

For $t = 3 \Rightarrow x(3) = e^{-\frac{1}{2}|3|} = 0.223$

Now we will calculate x (t) for some negative values of 't'

For $t = -1 \Rightarrow x(-1) = e^{-\frac{1}{2}|-1|} = e^{-1/2} = 0.606$

For $t = -2 \Rightarrow x(-2) = e^{-\frac{1}{2}|-2|} = e^{-1} = 0.37$

For $t = -3 \Rightarrow x(-3) = e^{-\frac{1}{2}|-3|} = e^{-3/2} = 0.223$

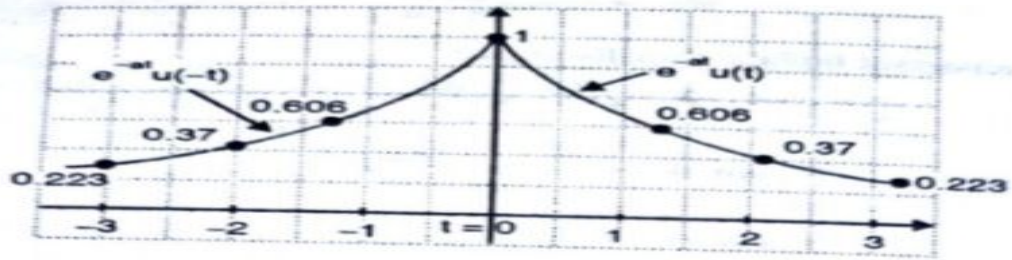


Fig. P. 4.3.3

Thus the given function can be expressed as,

$$x(t) = e^{-at}u(t) + e^{at}u(-t) \quad \text{...(1)}$$

$$\text{Let } x(t) = x_1(t) + x_2(t) \quad \text{...(2)}$$

Using linearity property we can write,

$$X(s) = X_1(s) + X_2(s) \quad \text{...(3)}$$

Comparing Equations (1) and (2) we get,

$$x_1(t) = e^{-at}u(t) \quad \text{and} \quad x_2(t) = e^{at}u(-t)$$

$$\text{Consider } x_1(t) = e^{-at}u(t)$$

We have standard laplace transform pair

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad \text{ROC} : \sigma > -a$$

$$\therefore X_1(s) = \frac{1}{s+a} \qquad \text{ROC} : \sigma > -a$$

$$\text{Now consider } X_2(s) = e^{at}u(-t)$$

We have standard laplace transform pair

$$-e^{-at}u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad \text{ROC} : \sigma < -a$$

$$\therefore e^{-at}u(-t) \xleftrightarrow{LT} -\frac{1}{s+a} \qquad \text{ROC} : \sigma < -a$$

$$\therefore\ X_2(s) = -\frac{1}{s-a} \qquad \text{ROC} : \sigma < +a$$

ting these values in Equation (3) we get,

$$X(s) = \frac{1}{s+a} - \frac{1}{s-a} = \frac{-2a}{s^2-a^2}$$

r $X_1(s)$ ROC is $\sigma > -a$ and for $X_2(s)$ ROC is $\sigma < +a$. Thus combined ROC is

$$\sigma > -a \text{ and } \sigma < a$$

$$\therefore \text{ROC is } -a < \sigma < +a$$

: Find Laplace transform of,

$$x(t) = (t-3)^2$$

$$\text{Here } x(t) = (t-3)^2 \quad \text{for} \quad t > 3$$

aking L.T on both sides.

$$X(s) = L\{(t-3)^2\}$$

$$\text{We know } t^n \xleftrightarrow{LT} \frac{n!}{s^{n+1}}$$

$$\therefore \quad t^2 \xleftrightarrow{LT} \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

Now using time shifting property.

$$L\{x(t-t_0)\} = e^{-st_0} \cdot X(s)$$

$$\text{Here } t_0 = 3$$

$$\therefore \quad X(s) = e^{-3s} \times L\{t^2\}$$

$$\therefore \quad X(s) = e^{-3s} \times \frac{2}{s^3}$$

$$\therefore \quad X(s) = \frac{2e^{-3s}}{s^3}$$

Example 3.7

Determine the Laplace transform of the following signals using properties of Laplace transform.

a) $x(t) = (t^2 - 2t)\, u(t-1)$ b) Unit ramp signal starting at $t = a$. c)

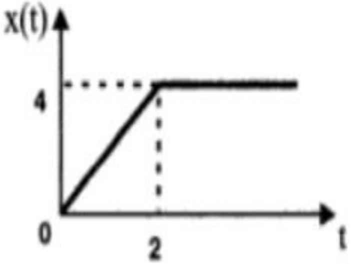


Fig 3.7.1.

Solution

a) Given that, $x(t) = (t^2 - 2t)\, u(t-1) = t^2\, u(t-1) - 2t\, u(t-1)$

From table 3.2 we get,

$$\mathcal{L}\{t^2\, u(t)\} = \frac{2}{s^3} \quad .....(1)$$

$$\mathcal{L}\{t\, u(t)\} = \frac{1}{s^2} \quad ; \quad \therefore \mathcal{L}\{2t\, u(t)\} = \frac{2}{s^2} \quad .....(2)$$

From time delay property of Laplace transform,

If $\mathcal{L}\{x(t)\, u(t)\} = X(s)$, then $\mathcal{L}\{x(t)\, u(t-a)\} = e^{-as}\, X(s)$

$$\therefore X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\left\{\left(t^2 - 2t\right) u(t-1)\right\} = \mathcal{L}\left\{t^2\, u(t-1)\right\} - \mathcal{L}\{2t\, u(t-1)\}$$

$$= e^{-s}\, \mathcal{L}\left\{t^2\, u(t)\right\} - e^{-s}\, \mathcal{L}\{2t\, u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= e^{-s}\, \frac{2}{s^3} - e^{-s}\, \frac{2}{s^2} = 2e^{-s}\left(\frac{1-s}{s^3}\right) = \frac{2e^{-s}(1-s)}{s^3} \qquad \boxed{\text{Using equations (1) and (2)}}$$

b) Given that, Unit ramp signal starting at $t = a$.

The unit ramp starting at $t = a$, is unit ramp delayed by "a" units of time. The unit ramp waveform and the ramp waveform starting at $t = a$ are shown in fig 1 and fig 2 respectively. The equation of unit ramp and delayed ramp are given below.

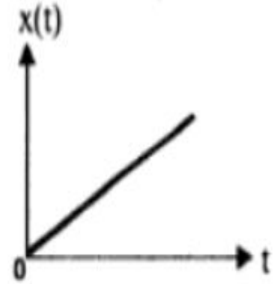


Fig 1 : Ramp.

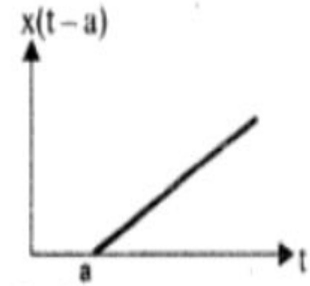


Fig 2 : Ramp starting at $t = a$.

Unit ramp, $x(t) = t\,u(t)$

Delayed unit ramp, $x(t-a) = (t-a)\,u(t-a)$

From table 3.2 we get,

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2} \qquad .....(3)$$

From time delay property of Laplace transform,

If $\mathcal{L}\{x(t)\,u(t)\} = X(s)$, then $\mathcal{L}\{x(t)\,u(t-a)\} = e^{-as}\,X(s)$

Therefore Laplace transform of delayed ramp signal is,

$$\mathcal{L}\{x(t-a)\} = e^{-as}\,\mathcal{L}\{x(t)\}$$

$$= e^{-as}\,\mathcal{L}\{t\,u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= e^{-as}\,\frac{1}{s^2} = \frac{e^{-as}}{s^2} \qquad \boxed{\text{Using equation (3)}}$$

c)

The given signal can be decomposed into two signals as shown in fig 4 and fig 5.

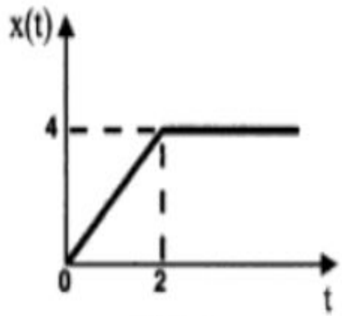


Fig 3.

$\Rightarrow$

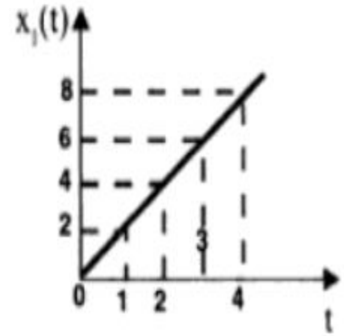


Fig 4.

+

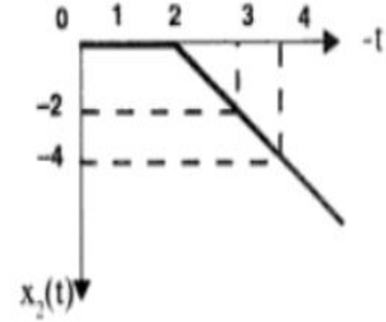


Fig 5.

The mathematical equations of signals, $x_1(t)$ and $x_2(t)$ are given below.

$$x_1(t) = 2t\,u(t)$$

$$x_2(t) = -2(t-2)\,u(t-2)$$

$$\therefore \quad x(t) = x_1(t) + x_2(t) = 2t\,u(t) - 2(t-2)\,u(t-2)$$

From table 3.2 we get,

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2} \; ; \quad \therefore \mathcal{L}\{2t\,u(t)\} = \frac{2}{s^2} \qquad .....(4)$$

$$\therefore X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{x_1(t) + x_2(t)\} = \mathcal{L}\{2t\,u(t) - 2(t-2)\,u(t-2)\}$$

$$= \mathcal{L}\{2t\,u(t)\} - \mathcal{L}\{2(t-2)\,u(t-2)\}$$

$$= \mathcal{L}\{2t\,u(t)\} - e^{-2s}\,\mathcal{L}\{2t\,u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= \frac{2}{s^2} - e^{-2s}\frac{2}{s^2} = \frac{2\left(1 - e^{-2s}\right)}{s^2} \qquad \boxed{\text{Using equation (4)}}$$

Example 3.8

Let, $X(s) = \mathcal{L}\{x(t)\}$. Determine the initial value, $x(0)$ and the final value, $x(\infty)$, for the following signals using initial value and final value theorems.

a) $X(s) = \dfrac{1}{s(s-1)}$ b) $X(s) = \dfrac{s+1}{s^2+2s+2}$ c) $X(s) = \dfrac{7s+6}{s(3s+5)}$

d) $X(s) = \dfrac{s^2+1}{s^2+6s+5}$ e) $X(s) = \dfrac{s+5}{s^2(s+9)}$

Solution

a) Given that, $X(s) = \dfrac{1}{s(s-1)}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{\mathrm{Lt}}\ s\,X(s) = \underset{s\to\infty}{\mathrm{Lt}}\ s\,\frac{1}{s(s-1)} = \underset{s\to\infty}{\mathrm{Lt}}\ \frac{1}{(s-1)} = \underset{s\to\infty}{\mathrm{Lt}}\ \frac{1}{s\left(1-\frac{1}{s}\right)}$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ \frac{1}{s}\,\frac{1}{\left(1-\frac{1}{s}\right)} = \frac{1}{\infty}\,\frac{1}{\left(1-\frac{1}{\infty}\right)} = 0 \times \frac{1}{1-0} = 0$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{\mathrm{Lt}}\ sX(s) = \underset{s\to 0}{\mathrm{Lt}}\ s\,\frac{1}{s(s-1)} = \underset{s\to 0}{\mathrm{Lt}}\ \frac{1}{(s-1)} = \frac{1}{0-1} = -1$$

b) Given that, $X(s) = \dfrac{s+1}{s^2+2s+2}$

Initial value, $x(0) = \lim_{s \to \infty} sX(s) = \lim_{s \to \infty} s\dfrac{s+1}{s^2+2s+2} = \lim_{s \to \infty} s\dfrac{s\left(1+\dfrac{1}{s}\right)}{s^2\left(1+\dfrac{2}{s}+\dfrac{2}{s^2}\right)}$

$$= \lim_{s \to \infty} \frac{1+\dfrac{1}{s}}{1+\dfrac{2}{s}+\dfrac{2}{s^2}} = \frac{1+\dfrac{1}{\infty}}{1+\dfrac{2}{\infty}+\dfrac{2}{\infty}} = \frac{1+0}{1+0+0} = 1$$

Final value, $x(\infty) = \lim_{s \to 0} sX(s) = \lim_{s \to 0} s\,\dfrac{s+1}{s^2+2s+2}$

$$= 0 \times \frac{0+1}{0+0+2} = 0$$

Example 3.9

Perform convolution of $x_1(t)$ and $x_2(t)$ using convolution theorem of Laplace transform.

a) $x_1(t) = u(t+5),\ x_2(t) = \delta(t-7)$

b) $x_1(t) = u(t-2),\ x_2(t) = \delta(t+6)$

c) $x_1(t) = u(t+1),\ x_2(t) = r(t-2)$; where $r(t) = t\,u(t)$

Solution

$\boxed{\begin{array}{l} \mathcal{L}\{u(t)\} = \dfrac{1}{s} \text{ and } \mathcal{L}\{\delta(t)\} = 1 \\ \text{if } \mathcal{L}\{x(t)\} = X(s) \text{ then } \mathcal{L}\{x(t \pm a)\} = e^{\pm as}X(s) \end{array}}$

a) Given that, $x_1(t) = u(t+5)$, $x_2(t) = \delta(t-7)$

$$x_1(t) = u(t+5)$$

$$\therefore X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t+5)\} = e^{5s}\,\mathcal{L}\{u(t)\} = e^{5s} \times \frac{1}{s} = \frac{e^{5s}}{s} \quad .....(1)$$

$$x_2(t) = \delta(t-7)$$

$$\therefore X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{\delta(t-7)\} = e^{-7s}\,\mathcal{L}\{\delta(t)\} = e^{-7s} \times 1 = e^{-7s} \quad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{e^{5s}}{s} \times e^{-7s} = \frac{e^{5s-7s}}{s} = \frac{e^{-2s}}{s}$$

$\boxed{\text{Using equations (1) and (2)}}$

$$\therefore x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Bigg|_{t=t-2} = u(t)\big|_{t=t-2} = u(t-2)$$

b) Given that, $x_1(t) = u(t-2)$, $x_2(t) = \delta(t+6)$

$$\boxed{\begin{array}{l} \mathcal{L}\{u(t)\} = \frac{1}{s} \text{ and } \mathcal{L}\{\delta(t)\} = 1 \\ \text{if } \mathcal{L}\{x(t)\} = X(s) \text{ then } \mathcal{L}\{x(t \pm a)\} = e^{\pm as} X(s) \end{array}}$$

$$x_1(t) = u(t-2)$$

$$\therefore X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t-2)\} = e^{-2s}\,\mathcal{L}\{u(t)\} = e^{-2s} \times \frac{1}{s} = \frac{e^{-2s}}{s} \qquad .....(1)$$

$$x_2(t) = \delta(t+6)$$

$$\therefore X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{\delta(t+6)\} = e^{6s}\,\mathcal{L}\{\delta(t)\} = e^{6s} \times 1 = e^{6s} \qquad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{e^{-2s}}{s} \times e^{6s} = \frac{e^{-2s+6s}}{s} = \frac{e^{4s}}{s} \qquad \boxed{\text{Using equations (1) and (2)}}$$

$$\therefore x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{e^{4s}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Bigg|_{t=t+4} = u(t)\big|_{t=t+4} = u(t+4)$$

3.4 Poles and Zeros of Rational Function of s

Let X(s) be Laplace transform of x(t). When X(s) is expressed as a ratio of two polynomials in s, then the s-domain signal X(s) is called a ***rational function*** of s.

The zeros and poles are two critical complex frequencies at which a rational function of s takes two extreme values, such as zero and infinity respectively.

Let X(s) is expressed as a ratio of two polynomials in s as shown in equation (3.26).

$$X(s) = \frac{P(s)}{Q(s)}$$

$$= \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + \ldots\ldots + b_{M-1} s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + \ldots\ldots + a_{N-1} s + a_N} \qquad \ldots\ldots(3.26)$$

where, P(s) = Numerator polynomial of X(s)

Q(s) = Denominator polynomial of X(s)

In equation (3.26) let us scale the coefficients of numerator polynomial by b_0 and the coefficients of denominator polynomial by a_0, and the equation (3.26) can be expressed in factorized form as shown in equation (3.27).

$$X(S) = \frac{b_0\left(s^M + \frac{b_1}{b_0}s^{M-1} + \frac{b_2}{b_0}s^{M-2} + ... + \frac{b_{M-1}}{b_0}s + \frac{b_M}{b_0}\right)}{a_0\left(s^N + \frac{a_1}{a_0}s^{N-1} + \frac{a_2}{a_0}s^{N-2} + ... + \frac{a_{N-1}}{a_0}s + \frac{a_N}{a_0}\right)}$$

$$= G\,\frac{(s - z_1)(s - z_2)\,.....\,(s - z_M)}{(s - p_1)(s - p_2)\,.....\,(s - p_N)} \qquad .....(3.27)$$

where, $G = \frac{b_0}{a_0}$ = Scaling factor

$z_1, z_2, ..., z_M$ = Roots of numerator polynomial, P(s)

$p_1, p_2, ..., p_N$ = Roots of denominator polynomial, Q(s)

In equation (3.27) if the value of s is equal to any one of the root of numerator polynomial then the signal X(s) will become zero.

Therefore the roots of numerator polynomial $z_1, z_2, ..., z_M$ are called ***zeros*** of X(s). Since s is complex frequency, the ***zeros*** can be defined as values of complex frequencies at which the signal X(s) becomes zero.

In equation (3.27) if the value of s is equal to any one of the root of numerator polynomial then the signal X(s) will become zero.

Therefore the roots of numerator polynomial z_1, z_2, ..., z_M are called ***zeros*** of X(s). Since s is complex frequency, the ***zeros*** can be defined as values of complex frequencies at which the signal X(s) becomes zero.

In equation (3.27), if the value of s is equal to any one of the roots of the denominator polynomial then the signal X(s) will become infinite. Therefore the roots of denominator polynomial p_1, p_2, ..., p_N are called ***poles*** of X(s). Since **s** is complex frequency, the ***poles*** can be defined as values of complex frequencies at which the signal X(s) become infinite. *Since the signal X(s) attains infinte value at poles, the ROC of X(s) does not include poles.*

In a realizable system, *the number of zeros will be less than or equal to number of poles.* Also for every zero, we can associate one pole. (When number of finite zeros are less than poles, the missing zeros are assumed to exist at infinity).

Let z_i be the zero associated with the pole p_i. If we evaluate $|X(s)|$ for various values of s, then $|X(s)|$ will be zero for $s = z_i$ and infnite for $s = p_i$. Hence the plot of $|X(s)|$ in a three dimensional plane will look like a pole (or pillar like structure) and so the point $s = p_i$ is called a pole.

3.4.3 Properties of ROC

The various concepts of ROC that has been discussed in section 3.2 and 3.4.2 are summarized as properties of ROC and given below.

Property-1 : The ROC of X(s) consists of strips parallel to the $j\Omega$ - axis in the s-plane.

Property-2 : If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s- plane.

Property-3 : If x(t) is right sided, and if the line passing through $Re(s) = \sigma_0$ is in ROC, then all values of s for which $Re(s) > \sigma_0$ will also be in ROC.

Property-4 : If x(t) is left sided, and if the line passing through $Re(s) = \sigma_0$ is in ROC, then all values of s for which $Re(s) < \sigma_0$ will also be in ROC.

Property-5 : If x(t) is two sided, and if the line passing through $Re(s) = \sigma_0$ is in ROC, then the ROC will consists of a strip in the s-plane that includes the line passing through $Re(s) = \sigma_0$.

Property-6 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), then its ROC is bounded by poles or extends to infinity.

Property-7 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), then ROC does not include any poles of X(s).

Property-8 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), and if x(t) is right sided, then ROC is the region in s-plane to the right of the rightmost pole.

Property-9 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), and if x(t) is left sided, then ROC is the region in s-plane to the left of the leftmost pole.

3.5 Inverse Laplace Transform

The ***Inverse Laplace transform*** of X(s) is defined as,

$$\boxed{\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(s)\, e^{st}\, ds}$$

Performing inverse Laplace transform by using the above fundamental definition is tedious. But the inverse Laplace transform by partial fraction expansion method will be much easier. In this method the s-domain signal is expressed as a sum of first order and second order sections. Then the inverse Laplace transform is obtained by comparing each section with standard transform pair, (listed in table 3.2).

In the following section the inverse Laplace transform by partial fraction exapansion method is explained with example.

3.5.1 Inverse Laplace Transform by Partial Fraction Expansion Method

Let Laplace transform of $x(t)$ be $X(s)$. The s-domain signal $X(s)$ will be a ratio of two polynomials in s (i.e., rational function of s). The roots of the denominator polynomial are called poles. The roots of numerator polynomials are called zeros. (For definition of poles and zeros please refer section 3.4). In signals and systems, three different types of s-domain signals are encountered. They are,

Case i : Signals with separate poles.

Case ii : Signals with multiple poles.

Case iii : Signals with complex conjugate poles.

Case - i : ***When s-Domain Signal X(s) has Distinct Poles***

Let, $$X(s) = \frac{K}{s(s + p_1)(s + p_2)} \quad .....(3.29)$$

By partial fraction expansion technique, the equation (3.29) can be expressed as,

$$X(s) = \frac{K}{s(s + p_1)(s + p_2)} = \frac{K_1}{s} + \frac{K_2}{s + p_1} + \frac{K_3}{s + p_2} \quad .....(3.30)$$

The residues K_1, K_2 and K_3 are given by,

$$K_1 = X(s) \times s \,\big|_{s=0}$$
$$K_2 = X(s) \times (s + p_1) \,\big|_{s=-p_1}$$
$$K_3 = X(s) \times (s + p_2) \,\big|_{s=-p_2}$$

We Know that, $\mathcal{L}\{u(t)\} = \frac{1}{s}$, $\mathcal{L}\left\{e^{-at}\, u(t)\right\} = \frac{1}{s + a}$

By using the above standard Laplace transform pairs the inverse Laplace transform of equation (3.30) can be obtained as shown below.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s + p_1} + \frac{K_3}{s + p_2}\right\}$$

$$\therefore x(t) = K_1\, \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + K_2\, \mathcal{L}^{-1}\left\{\frac{1}{s + p_1}\right\} + K_3\, \mathcal{L}^{-1}\left\{\frac{1}{s + p_2}\right\}$$

$$= K_1\, u(t) + K_2\, e^{-p_1 t}\, u(t) + K_3\, e^{-p_2 t}\, u(t)$$

Determine the inverse Laplace transform of $X(s) = \dfrac{2}{s(s+1)(s+2)}$

Solution

Given that, $X(s) = \dfrac{2}{s(s+1)(s+2)}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{2}{s(s+1)(s+2)} = \frac{K_1}{s} + \frac{K_2}{s+1} + \frac{K_3}{s+2}$$

The residue K_1 is obtained by multiplying X(s) by s and letting s = 0.

$$K_1 = X(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)} \times s\Big|_{s=0} = \frac{2}{(s+1)(s+2)}\Big|_{s=0} = \frac{2}{1 \times 2} = 1$$

The residue K_2 is obtained by multiplying X(s) by (s + 1) and letting s = −1.

$$K_2 = X(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)(s+2)} \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+2)}\Big|_{s=-1} = \frac{2}{-1(-1+2)} = -2$$

The residue K_3 is obtained by multiplying X(s) by (s + 2) and letting s = −2.

$$K_3 = X(s) \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)} \times (s+2)\Big|_{s=-2} = \frac{2}{s(s+1)}\Big|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

$$\therefore X(s) = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

$$\text{Now, } x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}\right\}$$

$$= u(t) - 2e^{-t}u(t) + e^{-2t}u(t)$$

$$= (1 - 2e^{-t} + e^{-2t})\,u(t) = (1 - e^{-t})^2\,u(t)$$

$\boxed{(x-y)^2 = x^2 - 2xy + y^2}$

Case - ii : When s-Domain Signal X(s) has Multiple Poles

Let, $$X(s) = \frac{K}{s(s+p_1)(s+p_2)^2} \quad .....(3.31)$$

By partial fraction expansion technique, the equation (3.31) can be expressed as,

$$X(s) = \frac{K}{s(s+p_1)(s+p_2)^2} = \frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{(s+p_2)^2} + \frac{K_4}{(s+p_2)} \quad .....(3.32)$$

The residues K_1, K_2, K_3, and K_4 are given by,

$$K_1 = X(s) \times s \Big|_{s=0} \quad ; \quad K_2 = X(s) \times (s+p_1) \Big|_{s=-p_1}$$

$$K_3 = X(s) \times (s+p_2)^2 \Big|_{s=-p_2} \quad ; \quad K_4 = \frac{d}{ds}\left[X(s) \times (s+p_2)^2\right]\Big|_{s=-p_2}$$

We know that, $\mathcal{L}\{u(t)\} = \frac{1}{s}$, $\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}$, $\mathcal{L}\left\{t\,e^{-at}\,u(t)\right\} = \frac{1}{(s+a)^2}$

By using the above standard Laplace transform pairs the inverse Laplace transform of equation (3.31) can be obtained as shown below.

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{K_1}{s} + \frac{K_2}{s+p_1} + \frac{K_3}{(s+p_2)^2} + \frac{K_4}{s+p_2}\right\}$$

$$\therefore\ x(t) = K_1\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + K_2\,\mathcal{L}^{-1}\left\{\frac{1}{s+p_1}\right\} + K_3\,\mathcal{L}^{-1}\left\{\frac{1}{(s+p_2)^2}\right\} + K_4\,\mathcal{L}^{-1}\left\{\frac{1}{s+p_2}\right\}$$

$$= K_1\,u(t) + K_2\,e^{-p_1 t}\,u(t) + K_3\,t\,e^{-p_2 t}\,u(t) + K_4\,e^{-p_2 t}\,u(t)$$

Determine the inverse Laplace transform of $X(s) = \dfrac{2}{s(s+1)(s+2)^2}$

Solution

Given that, $X(s) = \dfrac{2}{s(s+1)(s+2)^2}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{K}{s(s+1)(s+2)^2} = \frac{K_1}{s} + \frac{K_2}{(s+1)} + \frac{K_3}{(s+2)^2} + \frac{K_4}{(s+2)}$$

The residue K_1 is obtained by multiplying X(s) by s and letting $s = 0$.

$$K_1 = X(s) \times s\Big|_{s=0} = \frac{2}{s(s+1)(s+2)^2} \times s\Bigg|_{s=0} = \frac{2}{(s+1)(s+2)^2}\Bigg|_{s=0} = \frac{2}{1 \times 2^2} = 0.5$$

The residue K_2 is obtained by multiplying X(s) by $(s+1)$ and letting $s = -1$.

$$K_2 = X(s) \times (s+1)\Big|_{s=-1} = \frac{2}{s(s+1)\ (s+2)^2} \times (s+1)\Bigg|_{s=-1} = \frac{2}{-1(-1+2)^2} = -2$$

The residue K_3 is obtained by multiplying X(s) by $(s+2)^2$ and letting $s = -2$.

$$K_3 = X(s) \times (s+2)^2\Big|_{s=-2} = \frac{2}{s(s+1)(s+2)^2} \times (s+2)^2\Bigg|_{s=-2} = \frac{2}{-2(-2+1)} = 1$$

The residue K_4 is obtained by differentiating the product $X(s) \times (s+2)^2$ with respect to s and then letting $s = -2$.

$$K_4 = \frac{d}{ds}\left[X(s) \times (s+2)^2\right]\Bigg|_{s=-2} = \frac{d}{ds}\left[\frac{2}{s(s+1)\,(s+2)^2} \times (s+2)^2\right]_{s=-2}$$

$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2}$	
$u = 2$	$v = s(s+1)$ $= s^2 + s$

$$= \frac{d}{ds}\left[\frac{2}{s(s+1)}\right]\Bigg|_{s=-2} = \frac{-2(2s+1)}{s^2(s+1)^2}\Bigg|_{s=-2} = \frac{-2(2(-2)+1)}{(-2)^2\,(-2+1)^2} = 1.5$$

$$\therefore\ X(s) = \frac{2}{s(s+1)\,(s+2)^2} = \frac{0.5}{s} - \frac{2}{s+1} + \frac{1}{(s+2)^2} + \frac{1.5}{s+2}$$

$$\text{Now, } x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{0.5}{s} - \frac{2}{s+1} + \frac{1}{(s+2)^2} + \frac{1.5}{s+2}\right\}$$

$$= 0.5\,u(t) - 2\,e^{-t}\,u(t) + t\,e^{-2t}\,u(t) + 1.5\,e^{-2t}\,u(t)$$

$$= \left(0.5 - 2\,e^{-t} + t\,e^{-2t} + 1.5\,e^{-2t}\right)u(t)$$

Case iii : ***When s-Domain Signal X(s) has Complex Conjugate Poles***

Let, $$X(s) = \frac{K}{(s + p_1)(s^2 + bs + c)} \qquad .....(3.33)$$

By partial fraction expansion technique, the equation (3.33) can be expressed as,

$$X(s) = \frac{K}{(s + p_1)(s^2 + bs + c)} = \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{s^2 + bs + c} \qquad ..... (3.34)$$

The residue K_1 is given by, $K_1 = X(s) \times (s + p_1) \big|_{s=-p_1}$

The residues K_2 and K_3 are solved by cross multiplying the equation (3.34) and then equating the coefficients of like power of s.

Finally express X(s) as shown below,

Arranging, $s^2 + bs$, in the form of $(x + y)^2$

$(x + y)^2 = x^2 + 2xy + y^2$

$$X(s) = \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{s^2 + bs + c}$$

$$= \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{s^2 + 2 \times \frac{b}{2} s + \left(\frac{b}{2}\right)^2 + c - \left(\frac{b}{2}\right)^2}$$

$$= \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{\left(s + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)}$$

Put, $\frac{b}{2} = a$ and $c - \frac{b^2}{4} = \Omega_0^2$

$$= \frac{K_1}{s + p_1} + \frac{K_2 s + K_3}{(s + a)^2 + \Omega_0^2}$$

$$X(s) = \frac{K_1}{s+p_1} + K_2 \frac{s + \frac{K_3}{K_2}}{(s+a)^2 + \Omega_0^2} = \frac{K_1}{s+p_1} + K_2 \frac{s + a + \frac{K_3}{K_2} - a}{(s+a)^2 + \Omega_0^2}$$

$$= \frac{K_1}{s+p_1} + K_2 \frac{s + a + K_4}{(s+a)^2 + \Omega_0^2} \qquad \boxed{\text{Put, } \frac{K_3}{K_2} - a = K_4}$$

$$= \frac{K_1}{s+p_1} + K_2 \frac{s+a}{(s+a)^2 + \Omega_0^2} + \frac{K_2 K_4}{\Omega_0} \frac{\Omega_0}{(s+a)^2 + \Omega_0^2} \qquad \boxed{\text{Put, } \frac{K_2 K_4}{\Omega_0} = K_5}$$

$$\therefore\ X(s) = \frac{K_1}{s+p_1} + K_2 \frac{s+a}{(s+a)^2 + \Omega_0^2} + K_5 \frac{\Omega_0}{(s+a)^2 + \Omega_0^2} \qquad \text{.....(3.35)}$$

We know that, $\mathcal{L}\left\{e^{-at} u(t)\right\} = \frac{1}{s+a}$; $\quad \mathcal{L}\left\{e^{-at} \cos\Omega_0 t\, u(t)\right\} = \frac{s+a}{(s+a)^2 + \Omega_0^2}$;

$$\mathcal{L}\left\{e^{-at} \sin\Omega_0 t\, u(t)\right\} = \frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$$

Example 3.14

Determine the inverse Laplace transform of $X(s) = \dfrac{1}{(s+2)(s^2+s+1)}$

Solution

Given that, $\mathbf{X(s)} = \dfrac{1}{(s+2)(s^2+s+1)}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{K_1}{s+2} + \frac{K_2 s + K_3}{s^2+s+1}$$

The residue K_1 is obtained by multiplying X(s) by (s + 2) and letting s = −2.

$$\therefore K_1 = X(s) \times (s+2)\Big|_{s=-2} = \frac{1}{(s+2)(s^2+s+1)} \times (s+2)\Bigg|_{s=-2} = \frac{1}{(-2)^2 - 2 + 1} = \frac{1}{3}$$

To solve K_2 and K_3, cross multiply the following equation and substitute the value of K_1. Then equate the coefficients of like power of s.

$$\frac{1}{(s+2)(s^2+s+1)} = \frac{K_1}{s+2} + \frac{K_2 s + K_3}{s^2+s+1}$$

$$1 = K_1\left(s^2 + s + 1\right) + (K_2 s + K_3)(s+2)$$

$$1 = \frac{1}{3}\left(s^2 + s + 1\right) + K_2 s^2 + 2K_2 s + K_3 s + 2K_3$$

To solve K_2 and K_3, cross multiply the following equation and substitute the value of K_1. Then equate the coefficients of like power of s.

$$\frac{1}{(s+2)(s^2+s+1)} = \frac{K_1}{s+2} + \frac{K_2 s + K_3}{s^2+s+1}$$

$$1 = K_1 (s^2 + s + 1) + (K_2 s + K_3)(s + 2)$$

$$1 = \frac{1}{3}(s^2 + s + 1) + K_2 s^2 + 2K_2 s + K_3 s + 2K_3$$

$$1 = \frac{s^2}{3} + \frac{s}{3} + \frac{1}{3} + K_2 s^2 + 2K_2 s + K_3 s + 2K_3$$

$$1 = \left(\frac{1}{3} + K_2\right) s^2 + \left(\frac{1}{3} + 2K_2 + K_3\right) s + \frac{1}{3} + 2K_3$$

On equating the coefficients of s^2 terms,

$$0 = \frac{1}{3} + K_2 \quad \Rightarrow \quad K_2 = -\frac{1}{3}$$

On equating the coefficients of s terms,

$$0 = \frac{1}{3} + 2K_2 + K_3 \quad \Rightarrow \quad K_3 = -\frac{1}{3} - 2K_2 = -\frac{1}{3} - 2 \times \left(-\frac{1}{3}\right) = \frac{1}{3}$$

$$\therefore X(s) = \frac{1}{(s+2)(s^2+s+1)} = \frac{\frac{1}{3}}{s+2} + \frac{\frac{-1}{3}s + \frac{1}{3}}{s^2+s+1} = \frac{\frac{1}{3}}{s+2} + \frac{-\frac{1}{3}(s-1)}{s^2+s+1}$$

Arranging, s^2+s, in the form of $(x+y)^2$

$(x+y)^2 = x^2+2xy+x^2$

$$= \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{s-1}{(s^2 + (2 \times 0.5s) + 0.5^2) + (1 - 0.5^2)} = \frac{1}{3}\frac{1}{s+2} - \frac{1}{3}\frac{s-1}{(s+0.5)^2 + 0.75}$$

$$= \frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{s+0.5-1-0.5}{(s+0.5)^2+\left(\sqrt{0.75}\right)^2} = \frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{s+0.5-1.5}{(s+0.5)^2+0.866^2}$$

$$= \frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{s+0.5}{(s+0.5)^2+0.866^2} + \frac{1}{3}\times\frac{1.5}{0.866}\,\frac{0.866}{(s+0.5)^2+0.866^2}$$

$$\therefore\ x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\,\frac{1}{s+2} - \frac{1}{3}\,\frac{s+0.5}{(s+0.5)^2+0.866^2} + 0.577\,\frac{0.866}{(s+0.5)^2+0.866^2}\right\}$$

$$= \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} - \frac{1}{3}\,\mathcal{L}^{-1}\left\{\frac{s+0.5}{(s+0.5)^2+(0.866)^2}\right\} + 0.577\,\mathcal{L}^{-1}\left\{\frac{0.866}{(s+0.5)^2+0.866^2}\right\}$$

$$= \frac{1}{3}\,e^{-2t}\,u(t) - \frac{1}{3}\,e^{-0.5t}\cos 0.866t\,u(t) + 0.577\,e^{-0.5t}\sin 0.866t\,u(t)$$

3.5.2 Inverse Laplace Transform Using Convolution Theorem

The convolution theorem of equation (3.16) is useful to evaluate the inverse Laplace transform of complicated s-domain signals.

Let x(t) be inverse Laplace transform of X(s). Let, the s-domain signal X(s) be expressed as a product of two s-domain signals $X_1(s)$ and $X_2(s)$. Let, $x_1(t)$ and $x_2(t)$ be inverse Laplace transform of $X_1(s)$ and $X_2(s)$ respectively.

Now, the inverse Laplace transform of $X_1(s)$ and $X_2(s)$ are computed separately to get $x_1(t)$ and $x_2(t)$, and then the inverse Laplace transform of X(s) is obtained by convolution of $x_1(t)$ and $x_2(t)$, as shown below.

Let, $X(s) = X_1(s)\,X_2(s), \quad \mathcal{L}^{-1}\{X(s)\} = x(t), \quad \mathcal{L}^{-1}\{X_1(s)\} = x_1(t), \quad \mathcal{L}^{-1}\{X_2(s)\} = x_2(t).$

First determine inverse Laplace transform of $X_1(s)$ and $X_2(s)$, to get $x_1(t)$ and $x_2(t)$.

Now, by convolution property,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

Here, $X(s) = X_1(s)\,X_2(s)$.

$$\therefore X(s) = \mathcal{L}\{x_1(t) * x_2(t)\}$$

On taking inverse Laplace transform of the above equation we get,

$$x(t) = \mathcal{L}^{-1}\{X(s)\} = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda \qquad \boxed{\text{Using equation (3.17)}}$$

Example 3.15

Find the inverse Laplace transform of the s-domain signal, $X(s) = \dfrac{4}{s^2(s^2+16)}$ using convolution theorem.

Solution

Let, $X(s) = \dfrac{4}{s^2(s^2+16)} = \dfrac{4}{s^2(s^2+4^2)} = \dfrac{4}{s^2+4^2} \times \dfrac{1}{s^2} = X_1(s)\,X_2(s)$

where, $X_1(s) = \dfrac{4}{s^2+4^2}$ and $X_2(s) = \dfrac{1}{s^2}$

$$x_1(t) = \mathcal{L}^{-1}\{X_1(s)\} = \mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} = \sin 4t\, u(t)$$

$$x_2(t) = \mathcal{L}^{-1}\{X_2(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t\, u(t)$$

$$\therefore x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\{X_1(s)\,X_2(s)\}$$

By convolution theorem, the inverse Laplace transform of $X_1(s)\,X_2(s)$ is given by convolution of $x_1(t)$ and $x_2(t)$.

$$\therefore x(t) = \mathcal{L}^{-1}\{X_1(s)\,X_2(s)\} = x_1(t) * x_2(t)$$

Since $x_1(t)$ and $x_2(t)$ are causal, limits of integration are changed to 0 to t.

$$= \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda = \int_0^t \sin 4\lambda\,(t-\lambda)\, d\lambda = \int_0^t (t-\lambda)\sin 4\lambda\, d\lambda$$

$\int u\,v = u\int v - \int\left[du \int v\right]$; $u = t - \lambda$, $v = \sin 4\lambda$

$$= \left[(t-\lambda)\left(\frac{-\cos 4\lambda}{4}\right) - \int (-1)\left(\frac{-\cos 4\lambda}{4}\right) d\lambda\right]_0^t$$

$$= \left[-(t-\lambda)\left(\frac{\cos 4\lambda}{4}\right) - \frac{\sin 4\lambda}{16}\right]_0^t$$

$$= \left[-(t-t)\left(\frac{\cos 4t}{4}\right) - \frac{\sin 4t}{16} + (t-0)\frac{\cos 0}{4} + \frac{\sin 0}{16}\right]$$

$\cos 0 = 1, \quad \sin 0 = 0$

$$= 0 - \frac{\sin 4t}{16} + \frac{t}{4} + 0 = \frac{1}{4}\left(t - \frac{\sin 4t}{4}\right);\ t \geq 0 = \frac{1}{4}\left(t - \frac{\sin 4t}{4}\right)u(t)$$

Example 3.16

Find the inverse Laplace transform of the following s-domain signals.

a) $X(s) = \dfrac{3s^2 + 8s + 23}{(s+3)\,(s^2 + 2s + 10)}$ b) $X(s) = \dfrac{8s^2 + 11s}{(s+2)\,(s+1)^3}$

Solution

a) **Given that,** $X(s) = \dfrac{3s^2 + 8s + 23}{(s+3)\,(s^2 + 2s + 10)}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{3s^2 + 8s + 23}{(s+3)\,(s^2 + 2s + 10)} = \frac{K_1}{s+3} + \frac{K_2\,s + K_3}{s^2 + 2s + 10}$$

The residue K_1 is obtained by multiplying X(s) by (s + 3) and letting s = –3.

$$\therefore K_1 = X(s) \times (s+3)\Big|_{s=-3} = \frac{3s^2 + 8s + 23}{(s+3)\,(s^2 + 2s + 10)} \times (s+3)\Bigg|_{s=-3}$$

$$= \frac{3 \times (-3)^2 + 8 \times (-3) + 23}{(-3)^2 + 2 \times (-3) + 10} = \frac{27 - 24 + 23}{9 - 6 + 10} = \frac{26}{13} = 2$$

To solve K_2 and K_3 cross multiply the following equation and substitute the value of K_1. Then equate the coefficients of like power of s.

$$\frac{3s^2+8s+23}{(s+3)(s^2+2s+10)} = \frac{K_1}{s+3} + \frac{K_2 s + K_3}{s^2+2s+10}$$

$$3s^2+8s+23 = K_1\left(s^2+2s+10\right) + (K_2 s + K_3)(s+3)$$

$$3s^2+8s+23 = K_1 s^2 + 2K_1 s + 10K_1 + K_2 s^2 + 3K_2 s + K_3 s + 3K_3$$

$$3s^2+8s+23 = (K_1+K_2)s^2 + (2K_1+3K_2+K_3)s + 10K_1 + 3K_3$$

On equating the coefficients of s^2 terms, we get,

$$3 = K_1 + K_2 \qquad \Rightarrow \qquad K_2 = 3 - K_1 = 3 - 2 = 1$$

On equating the coefficients of s terms, we get,

$$8 = 2K_1 + 3K_2 + K_3 \qquad \Rightarrow \qquad K_3 = 8 - 2K_1 - 3K_2 = 8 - 2\times 2 - 3\times 1 = 1$$

$$\therefore X(s) = \frac{3s^2+8s+23}{(s+3)(s^2+2s+10)} = \frac{2}{s+3} + \frac{s+1}{s^2+2s+10}$$

$$= \frac{2}{s+3} + \frac{s+1}{s^2+2s+1+9} = \frac{2}{s+3} + \frac{s+1}{(s+1)^2+3^2}$$

$$\therefore x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{2}{s+3} + \frac{s+1}{(s+1)^2+3^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{2}{s+3}\right\} + \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+3^2}\right\} = 2e^{-3t}\,u(t) + e^{-t}\cos 3t\; u(t)$$

Example 3.17

Find the inverse Laplace transform of $X(s) = \frac{4}{(s+2)(s+4)}$ if the ROC is,

i) $-2 > \text{Re}\{s\} > -4$ ii) $\text{Re}\{s\} < -4$ iii) $\text{Re}\{s\} > -2$

Solution

Given that, $X(s) = \frac{4}{(s+2)(s+4)}$

By partial fraction expansion technique, X(s) can be expressed as,

$$X(s) = \frac{4}{(s+2)(s+4)} = \frac{K_1}{s+2} + \frac{K_2}{s+4}$$

The residue K_1 is obtained by multiplying X(s) by (s + 2) and letting s = –2.

$$\therefore K_1 = X(s) \times (s+2)\Big|_{s=-2} = \frac{4}{(s+2)(s+4)} \times (s+2)\Bigg|_{s=-2} = \frac{4}{-2+4} = 2$$

The residue K_2 is obtained by multiplying X(s) by (s + 4) and letting s = –4.

$$\therefore K_2 = X(s) \times (s+4)\Big|_{s=-4} = \frac{4}{(s+2)(s+4)} \times (s+4)\Bigg|_{s=-4} = \frac{4}{-4+2} = -2$$

$$\therefore X(s) = \frac{4}{(s+2)(s+4)} = \frac{2}{s+2} - \frac{2}{s+4}$$

Case i :

Given that ROC lies between lines passing through $s=-2$ to $s=-4$. Hence $x(t)$ will be two sided signal.

The term corresponding to the pole, $p=-2$ will be anticausal signal and the term corresponding to the pole, $p=-4$ will be causal signal.

$$\therefore\ x(t) = -2\,e^{-2t}\,u(-t)\ -2\,e^{-4t}\,u(t)$$

Case ii :

Given that ROC is left of the line passing through $s=-4$. Hence $x(t)$ will be anticausal signal.

$$\therefore\ x(t) = -2\,e^{-2t}\,u(-t)\ +2\,e^{-4t}\,u(-t) = 2\left[e^{-4t}-e^{-2t}\right]u(-t)$$

Case iii :

Given that ROC is right of the line passing through $s=-2$. Hence $x(t)$ will be causal signal.

$$\therefore\ x(t) = 2\,e^{-t}\,u(t)\ -2\,e^{-4t}\,u(t) = 2\left[e^{-t}-e^{-4t}\right]u(t)$$

3.6 Analysis of LTI Continuous Time System Using Laplace Transform

3.6.1 Transfer Function of LTI Continuous Time System

In general, the input-output relation of a LTI (Linear Time Invariant) continuous time system is represented by the constant coefficient differential equation shown below, (equation (3.36)).

$$\frac{d^N}{dt^N}y(t) + a_1 \frac{d^{N-1}}{dt^{N-1}}y(t) + a_2 \frac{d^{N-2}}{dt^{N-2}}y(t) + ... + a_{N-1}\frac{d}{dt}y(t) + a_N\, y(t)$$

$$= b_0 \frac{d^M}{dt^M}x(t) + b_1 \frac{d^{M-1}}{dt^{M-1}}x(t) + b_2 \frac{d^{M-2}}{dt^{M-2}}x(t) + ... + b_{M-1}\frac{d}{dt}x(t) + b_M\, x(t) \quad .....(3.36)$$

where, N = Order of the system and $M \leq N$.

On taking Laplace transform of equation (3.36) with zero initial conditions we get,

$$s^N\, Y(s) + a_1\, s^{N-1}\, Y(s) + a_2\, s^{N-2}\, Y(s) + ... + a_{N-1}\, s\, Y(s) + a_N\, Y(s)$$

$$= b_0\, s^M\, X(s) + b_1\, s^{M-1}\, X(s) + b_2\, s^{M-2}\, X(s) + ... + b_{M-1}\, s\, X(s) + b_M\, X(s)$$

$$Y(s)\left(s^N + a_1\, s^{N-1} + a_2\, s^{N-2} + ... + a_{N-1}\, s + a_N\right)$$

$$= X(s)\left(b_0\, s^M + b_1\, s^{M-1} + b_2\, s^{M-2} + ... + b_{M-1}\, s + b_M\right)$$

$$\therefore \quad \frac{Y(s)}{X(s)} = \frac{b_0\, s^M + b_1\, s^{M-1} + b_2\, s^{M-2} + ... + b_{M-1}\, s + b_M}{s^N + a_1\, s^{N-1} + a_2\, s^{N-2} + ... + a_{N-1}\, s + a_N} \quad .....(3.37)$$

The ***transfer function*** of a continuous time system is defined as the ratio of Laplace transform of output and Laplace transform of input. Hence the equation (3.37) is the transfer function of an LTI continuous time system.

The equation (3.37) is a rational function of s (i.e., ratio of two polynomials in s). The numerator and denominator polynomials of equation (3.36) can be expressed in the factorized form as shown in equation (3.38).

$$\frac{Y(s)}{X(s)} = G\,\frac{(s-z_1)(s-z_2)(s-z_3)\,.....\,(s-z_M)}{(s-p_1)(s-p_2)(s-p_3)\,.....\,(s-p_N)} \qquad .....(3.38)$$

where, $z_1, z_2, z_3, ... z_M$ are roots of numerator polynomial

(or zeros of continuous time system)

$p_1, p_2, p_3, ... p_N$ are roots of denominator polynomial

(or poles of continuous time system)

3.6.2 Impulse Response and Transfer Function

Let, $x(t)$ = Input of a LTI continuous time system

$y(t)$ = Output / Response of the LTI continuous time system for the input $x(t)$

$h(t)$ = Impulse response (i.e., response for impulse input)

Now, the response $y(t)$ of the continuous time system is given by convolution of input and impulse response. (Refer chapter - 2, section 2.9.1, equation (2.23)).

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\lambda)\, h(t-\lambda)\, d\lambda \qquad \text{....(3.39)}$$

On taking Laplace transform of equation(3.39) we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

If $\mathcal{L}\{x(t)\} = X(s)$
and $\mathcal{L}\{h(t)\} = H(s)$
then by convolution property
$\mathcal{L}\{x(t) * h(t)\} = X(s)\, H(s)$

Using convolution property of Laplace transform the above equation can be written as,

$$Y(s) = X(s)\, H(s)$$

$$\therefore\ H(s) = \frac{Y(s)}{X(s)} \qquad \text{.....(3.40)}$$

$$\therefore\ H(s) = \frac{Y(s)}{X(s)} = G\,\frac{(s-z_1)\,(s-z_2)\,(s-z_3)\ .....\ (s-z_M)}{(s-p_1)\,(s-p_2)\,(s-p_3)\ .....\ (s-p_N)} \qquad \boxed{\text{Using equation (3.38)}}$$

From equation (3.40) we can conclude that *the transfer function of LTI continuous time system is also given by Laplace transform of the impulse response.*

Alternatively we can say that *the inverse Laplace transform of transfer function is the impulse response of the system.*

$$\boxed{\therefore \text{ Impulse response, } \quad h(t) = \mathcal{L}^{-1}\{H(s)\} = \mathcal{L}^{-1}\left\{\frac{Y(s)}{X(s)}\right\}} \qquad \boxed{\text{Using equation (3.40)}}$$

3.6.3 Response of LTI Continuous Time System Using Laplace Transform

In general, the input-output relation of an LTI (Linear Time Invariant) continuous time system is represented by the constant coefficient differential equation shown below, (equation (3.41)).

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + ... + a_{N-1}\frac{d}{dt}y(t) + a_N\, y(t)$$

$$= b_0\frac{d^M}{dt^M}x(t) + b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + ... + b_{M-1}\frac{d}{dt}x(t) + b_M\, x(t) \quad .....(3.41)$$

The solution of the above differential equation (equation (3.41)) is the ***(total) response*** y(t) of LTI system, which consists of two parts.

In signals and systems the two parts of the solution y(t) are called zero-input response $y_{zi}(t)$ and zero-state response $y_{zs}(t)$.

$$\therefore \text{ Response, } \; y(t) = y_{zi}(t) + y_{zs}(t)$$

Total Response

The ***total response*** y(t) is the response of the system due to input signal and initial output (or intial stored energy). The total response is obtained from the differential equation governing the system (equation(3.41)) for specific input signal x(t) for $t \geq 0$ and with non-zero initial conditions.

On taking Laplace transform of equation (3.41) with non-zero initial conditions for both input and output, and then substituting for X(s) we can form an equation for Y(s).

The total response y(t) is given by inverse Laplace transform of Y(s).

Alternatively the total response y(t) is given by sum of zero-input response $y_{zi}(t)$ and zero-state response $y_{zs}(t)$.

$$\therefore \text{ Total response, } \; y(t) = y_{zi}(t) + y_{zs}(t)$$

3.6.4 Convolution and Deconvolution Using Laplace Transform

Convolution

The ***convolution*** operation is performed to find the response y(t) of LTI continuous time system from the input x(t) and impulse response h(t).

If $\mathcal{L}\{x(t)\} = X(s)$
and $\mathcal{L}\{h(t)\} = H(s)$
then by convolution property
$\mathcal{L}\{x(t) * h(t)\} = X(s)\,H(s)$

$\therefore$ Response, $y(t) = x(t) * h(t)$

On taking Laplace transform of the above equation we get,

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{x(t) * h(t)\}$$

$$\therefore\ Y(s) = X(s)\,H(s) \qquad .....(3.42)$$

$\therefore$ Response, $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$= \mathcal{L}^{-1}\{X(s)\,H(s)\}$$

Procedure :
1. Take Laplace transform of x(t) to get X(s).
2. Take Laplace transform of h(t) to get H(s).
3. Compute the product of X(s)H(s). Let, X(s)H(s) = Y(s).
4. Take inverse Laplace transform of Y(s) to get y(t).

Deconvolution

The ***deconvolution*** operation is performed to extract the input x(t) of an LTI continuous time system from the response y(t) of the system.

From equation (3.42) get,

$$X(s) = \frac{Y(s)}{H(s)}$$

On taking inverse Laplace transform of the above equation we get,

$$\text{Input, } x(t) = \mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{Y(s)}{H(s)}\right\}$$

Procedure : 1. Take Laplace transform of y(t) to get Y(s).

2. Take Laplace transform of h(t) to get H(s).
3. Divide Y(s) by H(s) to get X(s) (i.e., X(s) = Y(s) / H(s)).
4. Take inverse Laplace transform of X(s) to get x(t).

3.6.5 Stability in s-Domain

ROC of a Stable LTI System

Let H(s) be Laplace transform of h(t). Now by definition of Laplace transform we get,

$$H(s) = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} h(t)\, e^{-(\sigma+j\Omega)t}\, dt$$

$\boxed{\text{Put, } s = \sigma + j\Omega}$

Let us evaluate H(s) for $\sigma = 0$.

$$\therefore H(s) = \int_{-\infty}^{+\infty} h(t) e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} e^{-j\Omega t}\, h(t)\, dt = \left[e^{-j\Omega t}\left(\int h(t)\, dt\right) - \int -j\Omega e^{-j\Omega t}\left(\int h(t)\, dt\right) dt\right]_{-\infty}^{+\infty} \quad .....(3.43)$$

For a stable LTI system, $\int h(t)\, dt$ is constant.

$\boxed{\int uv = u\int v - \int\left[du \int v\right]}$

Therefore, in equation (3.43), put, $\int h(t)\, dt = A$, where A is constant.

$$\therefore H(s) = \left[e^{-j\Omega t} A + j\Omega \int e^{-j\Omega t} A\, dt\right]_{-\infty}^{+\infty} = A\left[e^{-j\Omega t} + j\Omega \int e^{-j\Omega t}\, dt\right]_{-\infty}^{+\infty} \quad .....(3.44)$$

The evaluation of equation (3.44), is evaluation of Laplace transform for $s = j\Omega$, (i.e., evaluation of H(s) along imaginary axis) and so we can say that H(s) exists if the ROC includes the imaginary axis. Hence *for a stable LTI continuous time system the ROC should include the imaginary axis of s-plane.*

Location of Poles for Stability of Causal Systems

Let h(t) be impulse response of an LTI causal system. Now if h(t) satisfies the condition,

$$\boxed{\int_0^{\infty} h(t)\, dt < \infty} \qquad \text{....(3.45)}$$

then the LTI continuous time causal system is stable.

Let, $h(t) = e^{at}\, u(t)$

$$\text{Now, } \int_0^{\infty} h(t)\, dt = \int_0^{\infty} e^{at}\, u(t)\, dt = \int_0^{\infty} e^{at}\, dt = \left[\frac{e^{at}}{a}\right]_0^{\infty} = \frac{e^{a\times\infty}}{a} - \frac{e^{a\times 0}}{a} = \frac{e^{a\times\infty}}{a} - \frac{1}{a}$$

Let a be negative, and let $k = -a$, so that k is always positive.

$$\text{Now, } \int_0^\infty h(t)\,dt = \frac{e^{a\times\infty}}{a} - \frac{1}{a} = \frac{e^{-k\infty}}{-k} + \frac{1}{k} = \frac{e^{-\infty}}{-k} + \frac{1}{k} \qquad \boxed{e^{-\infty} = 0}$$

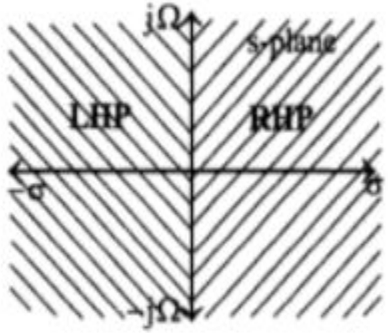
$$= \frac{0}{-k} + \frac{1}{k} = \frac{1}{k} = \text{Constant, and so system is stable.} \qquad \text{....(3.46)}$$

Let a be positive.

$$\text{Now, } \int_0^\infty h(t)\,dt = \frac{e^{a\times\infty}}{a} - \frac{1}{a} \qquad \boxed{e^{\infty} = \infty}$$

$$= \frac{e^{a\infty}}{a} - \frac{1}{a} = \frac{\infty}{a} - \frac{1}{a} = \infty\text{, and so system is unstable.} \qquad \text{.....(3.47)}$$

From the above discussion, the stability condition of equation (3.45) can be transformed as a condition on location of poles of transfer function of the LTI continuous time causal system in s-plane.



Consider the s-plane shown in fig 3.10. The area to the right of vertical axis is called right half plane (RHP) and the area to the left of vertical axis is called left half plane (LHP).

The transfer function of a continuous time system is given by Laplace transform of its impulse response.

Let, $h(t) = e^{at}\, u(t)$

$$\therefore \text{Transfer function, } H(s) = \mathcal{L}\{h(t)\} = \mathcal{L}\{e^{at}\, u(t)\} = \frac{1}{s - a}$$

Here, the transfer function H(s) has pole at s = a.

If, $a < 0$, (i.e., if a is negative), then the pole will lie on the left half of s-plane, and from equation (3.46) we can say that the causal system is stable.

If, $a > 0$, (i.e., if a is positive), then the pole will lie on the right half of s-plane. and from equation (3.47) we can say that the causal system is unstable.

Therefore we can say that, *for a stable LTI continuous time causal system the poles should lie on the left half of s-plane.*

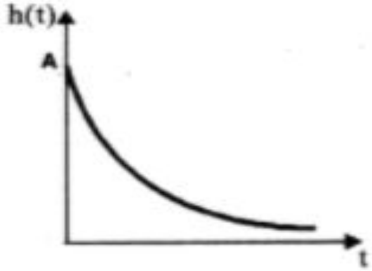
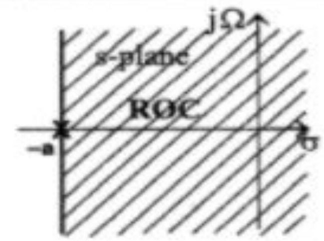
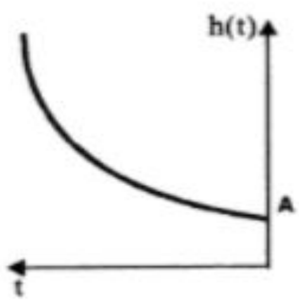
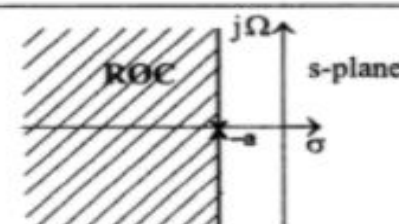
General Condition for Stability in s-Plane

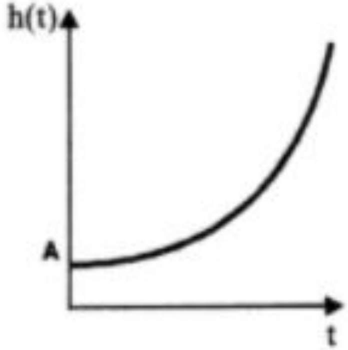
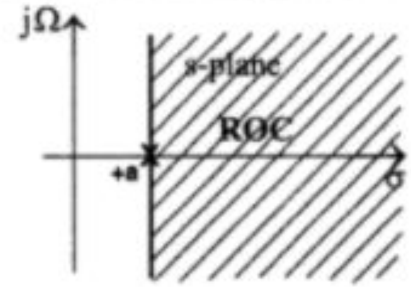
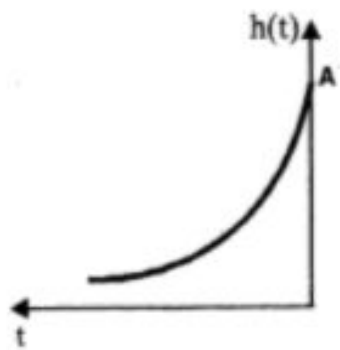
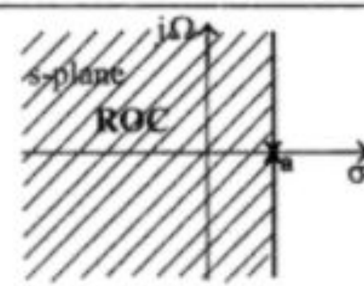
On combining the condition for location of poles and the ROC we can say that,

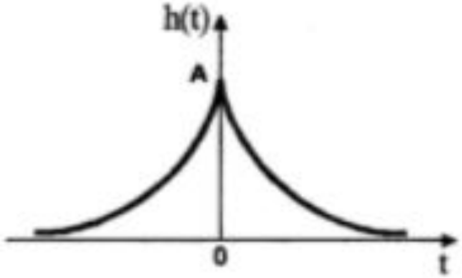
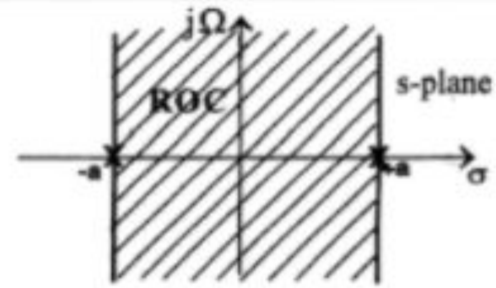
1. *For a stable LTI continuous time causal system, the poles should lie on the left half of s-plane and the imaginary axis should be included in the ROC.*
2. *For a stable LTI continuous time noncausal system, the imaginary axis should be included in the ROC.*

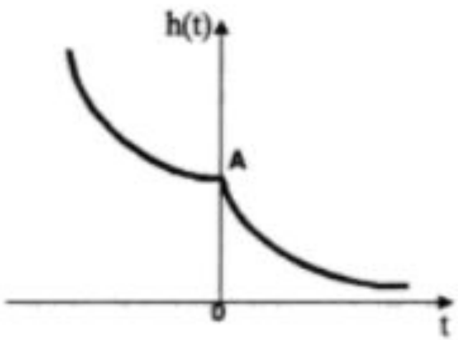
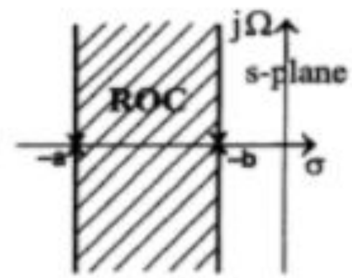
The various types of impulse response of LTI continuous time system and their transfer functions and the locations of poles

Table-3.4 : Impulse Response and Location of Poles of Transfer Function in s-Plane

Impulse response **h(t)**	**Transfer function** **$H(s) = \mathcal{L}\{h(t)\}$**	**Location of poles in** **s-plane and ROC**
$h(t) = A\,e^{-at}\,u(t)$; $a > 0$ 	$H(s) = \dfrac{A}{s+a}$ Pole at $s = -a$. ROC is $\sigma > -a$, where σ is real part of s.	 The pole at $s = -a$, lies on left half of s-plane. ROC includes imaginary axis. Causal system. Since pole lies on LHP and the imaginary axis is included in ROC, the system is stable.
$h(t) = A\,e^{-at}\,u(-t)$; $a > 0$ 	$H(s) = -\dfrac{A}{s+a}$ Pole at $s = -a$. ROC is $\sigma < -a$, where σ is real part of s.	 The pole at $s = -a$, lies on left half of s-plane. The ROC does not include imaginary axis. Noncausal system. Since imaginary axis is not included in ROC, the system is unstable.

$h(t) = A\, e^{at}\, u(t)\ ;\ a > 0$ 	$H(s) = \dfrac{A}{s-a}$ Pole at $s = +a$. ROC is $\sigma > +a$, where σ is real part of s.	 The pole at $s = +a$, lies on right half of s-plane. ROC does not include imaginary axis. Causal system. Since pole lies on RHP and imaginary axis is not included in ROC, the system is unstable.
$h(t) = A\, e^{at}\, u(-t)\ ;\ a > 0$ 	$H(s) = -\dfrac{A}{s-a}$ Pole at $s = +a$. ROC is $\sigma < +a$, where σ is real part of s.	 The pole at $s = +a$, lies on right half of s-plane. The ROC includes imaginary axis. Noncausal system. Since imaginary axis is included in ROC, the system is stable.

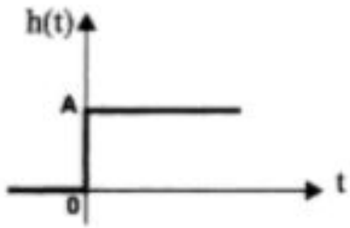
<table>
<tr><th>Impulse response
h(t)</th><th>Transfer function
$H(s) = \mathcal{L}\{h(t)\}$</th><th>Location of poles in
s-plane and ROC</th></tr>
<tr><td>$h(t) = A\,e^{-a|t|}\;;\;\; a > 0$

</td><td>$$H(s) = \frac{A}{s+a} - \frac{A}{s-a}$$
Poles at $s = -a, +a$.
ROC is $-a < \sigma < +a$,
where σ is real part of s.</td><td>

The pole at $s = +a$, lies on RHP and pole at $s = -a$, lies on LHP. ROC includes imaginary axis. Noncausal system. Since the imaginary axis is included in ROC, the system is stable.</td></tr>
<tr><td>$h(t) = A\,e^{-at}u(t) + A\,e^{-bt}u(-t)$
where $a > 0, b > 0, a > b$

</td><td>$$H(s) = \frac{A}{s+a} - \frac{A}{s+b}$$
Poles at $s = -a, -b$.
ROC is $-a < \sigma < -b$,
where σ is real part of s.</td><td>

The poles at $s = -a, -b$, lie on LHP. The ROC does not include imaginary axis. Noncausal system. Since the imaginary axis is not included in ROC, the system is unstable.</td></tr>
</table>

<table>
<tr>
<td>

$h(t) = A\,u(t)$

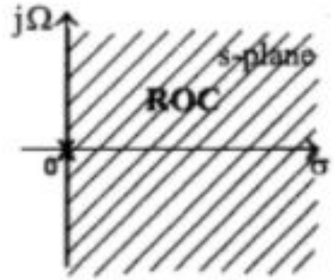


</td>
<td>

$$H(s) = \frac{A}{s}$$

Pole at s = 0.
ROC is $\sigma > 0$,
where σ is real part of s.

</td>
<td>



The pole at s = 0 lies on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.

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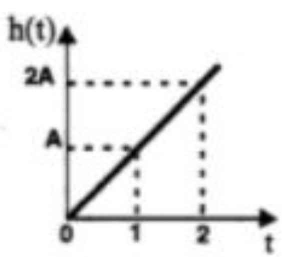
<tr>
<td>

$h(t) = A\,t\,u(t)$

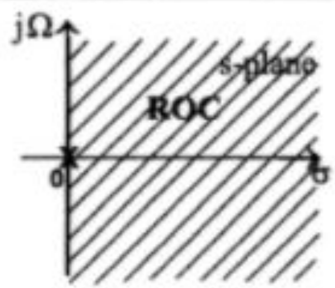


</td>
<td>

$$H(s) = \frac{A}{s^2}$$

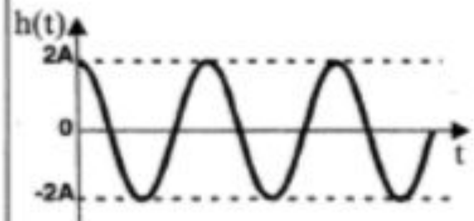
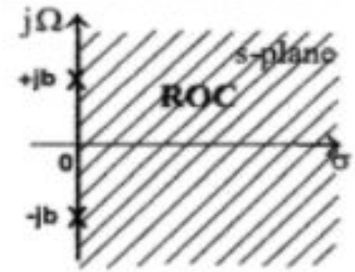
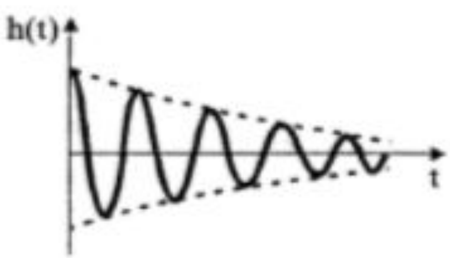
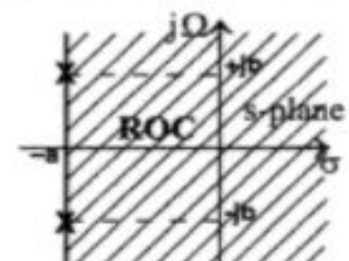
Double pole at s = 0.
ROC is $\sigma > 0$,
where σ is real part of s.

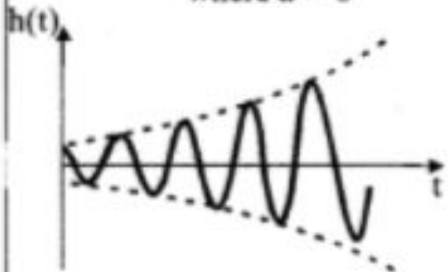
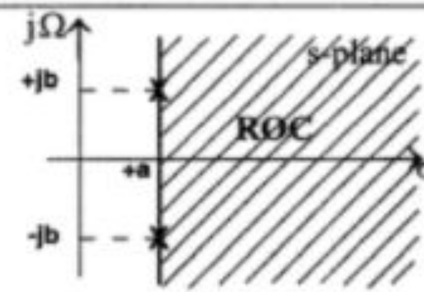
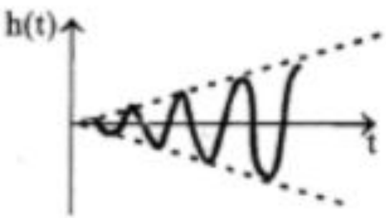
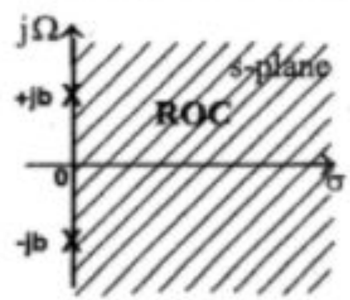
</td>
<td>



The poles at s = 0 lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.

</td>
</tr>
</table>

Impulse respor h(t)	**Transfer function H(s) = $\mathcal{L}\{h(t)\}$**	**Location of poles in s-plane and ROC**
$h(t) = 2A \cos bt\, u(t)$ 	$H(s) = \dfrac{A}{s + jb} + \dfrac{A}{s - jb}$ Poles at $s = -jb, +jb$. ROC is $\sigma > 0$, where σ is real part of s.	The poles at $s = -jb, +jb$, lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.
$h(t) = 2A\, e^{-at} \cos bt\, u(t)$, where $a > 0$ 	$H(s) = \dfrac{A}{s + a + jb} + \dfrac{A}{s + a - jb}$ Poles at $s = -a - jb, -a + jb$. ROC is $\sigma > -a$, where σ is real part of s.	The poles at $s = -a-jb, -a+jb$, lie on left half of s-plane. The ROC includes the imaginary axis. Causal system. Since poles lie on LHP and the imaginary axis is included in ROC, the system is stable.

$h(t) = 2A\, e^{at} \cos bt\, u(t)$, where $a > 0$ 	$H(s) = \dfrac{A}{s - a + jb} + \dfrac{A}{s - a - jb}$ Poles at $s = a - jb,\ a + jb$. ROC is $\sigma > a$, where σ is real part of s.	The poles at $s = a - jb,\ a + jb$, lie on right half of s-plane. The ROC does not include imaginary axis. Causal system. Since poles lie on RHP and the imaginary axis is not included in ROC, the system is unstable.
$h(t) = 2A\, t \cos bt\, u(t)$ 	$H(s) = \dfrac{A}{(s + jb)^2} + \dfrac{A}{(s - jb)^2}$ Double poles at $s = -jb,\ +jb$. ROC is $\sigma > 0$, where σ is real part of s.	The poles at $s = -jb,\ +jb$, lie on imaginary axis. The ROC does not include the imaginary axis. Causal system. Since the imaginary axis is not included in ROC, the system is unstable.

ANALYSIS AND CHARACTERIZATION OF LTI SYSTEMS USING THE LAPLACE TRANSFORM

If $x(t)$ and $y(t)$ are the input and output of an LTI system with impulse response $h(t)$ [Fig. 9.18], then

$$y(t) = x(t) * h(t)$$

Application of the time convolution property to the above equation yields

$$\mathcal{L}[y(t)] = \mathcal{L}[x(t)] * \mathcal{L}[h(t)]$$

$$Y(s) = X(s)H(s)$$

or

$$H(s) = \frac{Y(s)}{X(s)}$$

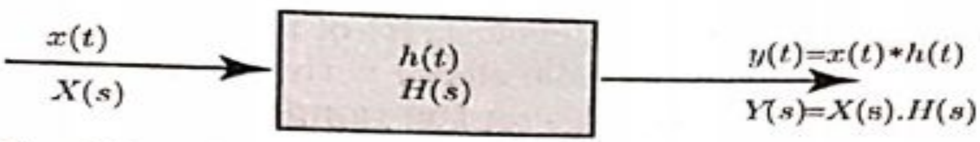


Fig. 9.18 LTI system

$H(s)$ is commonly referred to as the *system function* or *transfer function*. The transfer function $H(s)$ of an LTI system is equal to the ratio of the Laplace transform of the output signal to the Laplace transform of the input signal when all initial conditions are zero. For $s = j\omega$, $H(s)$ is the frequency response of the LTI system.

The Transfer Function and Differential-equation System Description

The transfer function may be related directly to the differential-equation description of an LTI system by using the bilateral Laplace transform. The relationship between the input and output of an Nth-order LTI system is described by the differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Taking the Laplace transform of the above equation, we obtain

$$\sum_{k=0}^{N} a_k s^k Y(s) = \sum_{k=0}^{M} b_k s^k X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

$H(s)$ is a ratio of polynomials in s and is thus termed a rational transfer function.

Impulse Response and Step Response

The impulse response $h(t)$ is defined as the output of an LTI system due to a unit impulse signal input $\delta(t)$ applied at time $t = 0$. Also,

$$h(t) \longleftrightarrow H(s)$$

where $H(s)$ is the system function or transfer function.

The step response $s(t)$ is defined as the output of an LTI system due to a unit step input signal, i.e., $x(t) = u(t)$. The step response of an LTI system with impulse response $h(t)$ is given by

$$\begin{aligned} s(t) &= h(t) * x(t) \\ &= h(t) * u(t) \end{aligned}$$

$$= \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\, d\tau$$

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

Therefore,

$$\mathcal{L}[s(t)] = \mathcal{L}\left[\int_{-\infty}^{t} h(\tau)\, d\tau\right]$$

$$S(s) = \frac{H(s)}{s}$$

Conversely,

$$h(t) = \frac{ds(t)}{dt}$$

$$\mathcal{L}[h(t)] = \mathcal{L}\left[\frac{ds(t)}{dt}\right]$$

$$H(s) = s\, S(s)$$

Example Find the transfer function and the impulse response of a causal LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - 2x(t)$$

Solution

Given that

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} - 2x(t)$$

Taking the Laplace transform of the above equation, we obtain

$$s^2Y(s) + 2sY(s) + Y(s) = s^2X(s) - 2X(s)$$

$$Y(s)(s^2 + 2s + 1) = X(s)(s - 2)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s-2}{s^2 + 2s + 1}$$

$$H(s) = \frac{s-2}{(s+1)^2} = \frac{s-3+1}{(s+1)^2} = \frac{s+1}{(s+1)^2} - \frac{3}{(s+1)^2}$$

$$H(s) = \frac{1}{s+1} - \frac{3}{(s+1)^2}$$

$$e^{-t}u(t) \longleftrightarrow \frac{1}{s+1}$$

$$3t\, e^{-t}u(t) \longleftrightarrow \frac{3}{(s+1)^2}$$

Therefore, the impulse response

$$h(t) = e^{-t}u(t) - 3t\, e^{-t}u(t)$$

Example The unit step response of an LTI system is $s(t) = 2\,e^{-t}u(t)$. Determine its system function and the impulse response.

Solution

Given that the unit step response

$$s(t) = 2\,e^{-t}u(t)$$

$$\mathcal{L}[s(t)] = \mathcal{L}[2\,e^{-t}u(t)]$$

$$S(s) = \frac{2}{s+1}$$

The relationship between $\mathcal{L}[h(t)] = H(s)$ and $\mathcal{L}[s(t)] = S(s)$ is given by

$$H(s) = s\,S(s) = \frac{2s}{s+1} = \frac{2s+2-2}{s+1} = \frac{2(s+1)}{s+1} - \frac{2}{s+1}$$

$$H(s) = 2 - \frac{2}{s+1}$$

The inverse transform of this equation yields

$$h(t) = 2\delta(t) - 2\,e^{-t}u(t)$$

Example An LTI system has a unit step response given by $s(t) = (1 - e^{-t} - t\,e^{-t})u(t)$. For a certain input $x(t)$, the output is observed to be equal to $y(t) = (2 - 3\,e^{t} + e^{-3t})u(t)$. What is $x(t)$?

Solution
Given that

$$y(t) = (2 - 3\,e^{-t} + e^{-3t})u(t)$$
$$Y(s) = \frac{2}{s} - \frac{3}{s+1} + \frac{1}{s+3}$$
$$Y(s) = \frac{2(s^2 + 4s + 3) - 3s(s+3) + s(s+1)}{s(s+1)(s+3)} = \frac{6}{s(s+1)(s+3)}$$

Consider the given unit step response

$$s(t) = (1 - e^{-t} - t\,e^{-t})u(t)$$
$$s(t) = u(t) - e^{-t}u(t) - t\,e^{-t}u(t)$$

The Laplace transform of this equation yields

$$S(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

We know that [Eq. (9.53)]

$$H(s) = s\,S(s) = s\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$
$$H(s) = 1 - \frac{s}{s+1} - \frac{s}{(s+1)^2}$$
$$= \frac{(s+1)^2 - s(s+1) - s}{(s+1)^2}$$
$$= \frac{s^2 + 2s + 1 - s^2 - s - s}{(s+1)^2}$$
$$H(s) = \frac{1}{(s+1)^2} = \frac{Y(s)}{X(s)}$$

Therefore,

$$X(s) = \frac{Y(s)}{H(s)} = \frac{6/(s(s+1)(s+3))}{1/((s+1)^2)}$$

$$= \frac{6(s+1)}{s(s+3)}$$

$$X(s) = \frac{2}{s} + \frac{4}{s+3}$$

The inverse transform of this equation yields

$$x(t) = 2u(t) + 4\,e^{-3t}u(t)$$

Causality

For a causal LTI system, the impulse response $h(t) = 0$ for $t < 0$ and thus is right-sided. We know that if a signal is right-sided and of infinite duration, then the ROC is the region in the s-plane to the right of the rightmost pole. Consequently, *the ROC associated with the system function for a causal system is a right-half plane.* However, the converse of this statement is not necessarily true. If $H(s)$ is rational, then we can determine whether the system is causal simply by checking to see if its ROC is a right-half plane, that is, *for a system with a rational system function, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole.*

In an exactly analogous manner, we can deal with the concept of anticausality. For an anticausal LTI system, the impulse response $h(t) = 0$ for $t > 0$ and is thus left-sided. We know that if a signal is left-sided and of infinite duration, then the ROC is the region in the s-plane to the left of the leftmost pole. Consequently, *the ROC associated with the system function for an anticausal system is a left-half plane.* However, the converse of this statement is not necessarily true.

Example For the following system functions, check whether the corresponding LTI system is causal, anticausal, or noncausal.

(a) $H_1(s) = \dfrac{1}{s^2+5s+6} \quad \Re\{s\} > -2$

(b) $H_2(s) = \dfrac{1}{s^2+5s+6} \quad \Re\{s\} < -3$

(c) $H_3(s) = \dfrac{1}{s^2+5s+6} \quad -2 < \Re\{s\} < -3$

(d) $H_4(s) = \dfrac{e^{2s}}{s+1} \quad \Re\{s\} > -1$

Solution

(a) Consider the given system function

$$H_1(s) = \frac{1}{s^2+5s+6} = \frac{1}{(s+2)(s+3)} \quad \Re\{s\} > -2$$

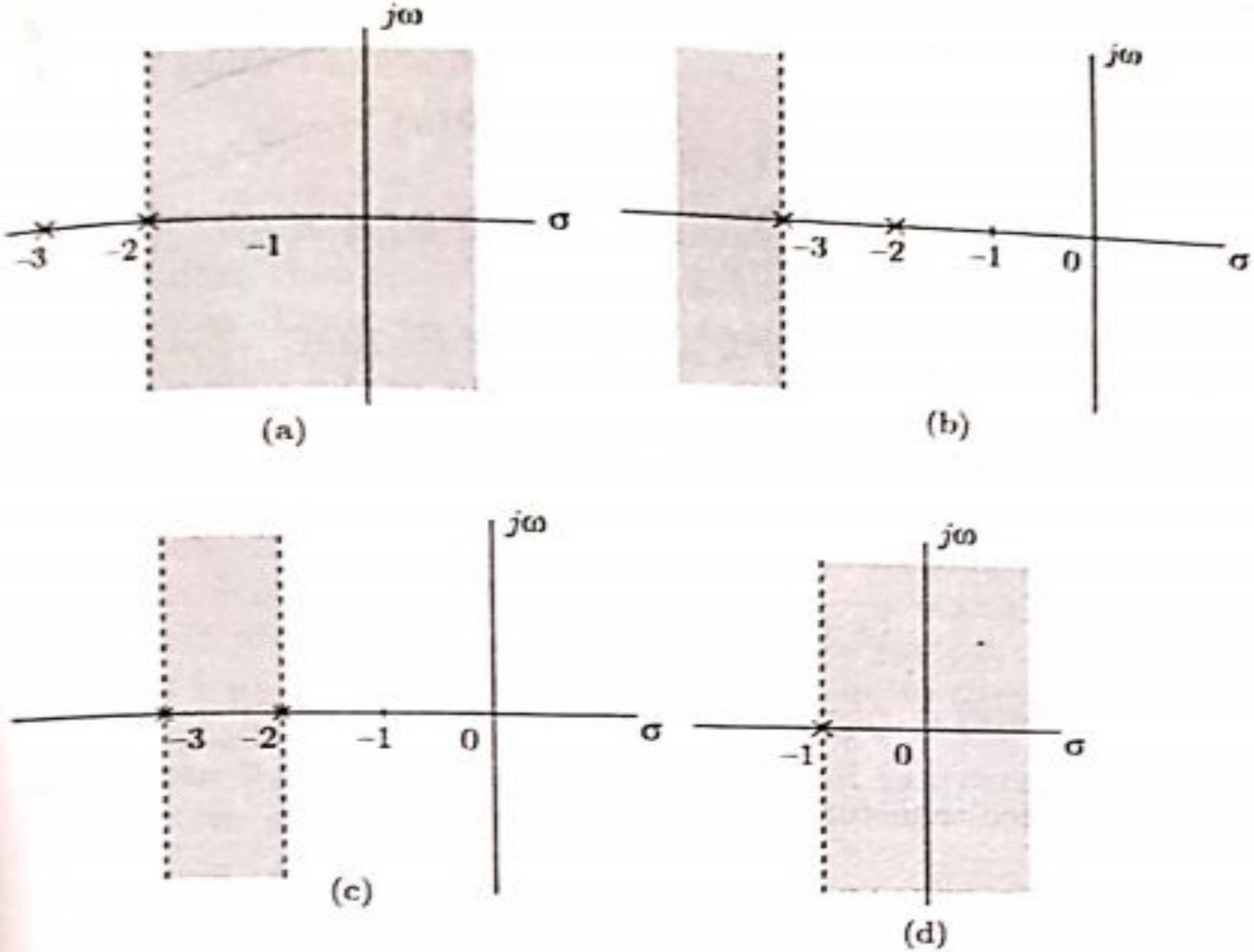
Since the given system function $H_1(s)$ is rational and its ROC [shown in Fig. 9.19(a)] is to the right of the rightmost pole, the corresponding LTI system is causal. Also it can be verified by its impulse response. Given that

$$H_1(s) = \frac{1}{(s+2)(s+3)}$$
$$= \frac{1}{s+2} - \frac{1}{s+3}$$

Taking the inverse Laplace transform, we obtain

$$h_1(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

Since $h_1(t) = 0$ for $t < 0$, this system is causal.



(a) ROC for $H_1(s)$, (b) ROC for $H_2(s)$, (c) ROC for $H_3(s)$, and (d) ROC for $H_4(s)$

(b) Consider the given system function

$$H_2(s) = \frac{1}{s^2 + 5s + 6} \quad \Re\{s\} < -3$$

$$H_2(s) = \frac{1}{(s+2)(s+3)} \quad \Re\{s\} < -3$$

Since the given system function $H_2(s)$ is rational and its ROC [shown in Fig. 9.19(b)] is to the left of the leftmost pole, the corresponding LTI system is anticausal. Also it can be verified by its impulse response. Given that

$$H_2(s) = \frac{1}{(s+2)(s+3)}$$

$$= \frac{1}{s+2} - \frac{1}{s+3}$$

Taking the inverse Laplace transform, we obtain

$$h_2(t) = -e^{-2t}u(-t) + e^{-3t}u(-t)$$

Since $h_2(t) = 0$ for $t > 0$, this system is anticausal.

(c) Consider the given system function

$$H_3(s) = \frac{1}{s^2 + 5s + 6} \qquad -2 < \Re\{s\} < -3$$

$$H_3(s) = \frac{1}{(s+2)(s+3)} \qquad -2 < \Re\{s\} < -3$$

Since the given system function $H_3(s)$ is rational and its ROC [shown in Fig. 9.19(c)] is neither to the left of the leftmost pole nor to the right of the rightmost pole, but it is a strip, the corresponding LTI system is noncausal. Also it can be verified by its impulse response. Given that

$$H_3(s) = \frac{1}{(s+2)(s+3)}$$

$$H_3(s) = \frac{1}{s+2} - \frac{1}{s+3} \qquad -2 < \Re\{s\} < -3$$

Taking the inverse Laplace transform, we obtain

$$h_3(t) = e^{-2t}u(t) + e^{-3t}u(-t)$$

Since $h_3(t)$ is two-sided, this system is noncausal.

(d) Consider the given system function

$$H_4(s) = \frac{e^{2s}}{s+1} \qquad \Re\{s\} > -1$$

(d) Consider the given system function

$$H_4(s) = \frac{e^{2s}}{s+1} \quad \Re\{s\} > -1$$

Since the given system function $H_4(s)$ is not rational, we cannot determine whether the system is causal by checking its ROC. For this system, the ROC [shown in Fig. 9.19(a)] is to the right of the rightmost pole; therefore, the impulse response must be right-sided. We know that

$$e^{-at}u(t) \longleftrightarrow \frac{1}{s+a} \quad \Re\{s\} > -a$$

$$e^{-t}u(t) \longleftrightarrow \frac{1}{s+1} \quad \Re\{s\} > -1$$

$$e^{-(t+2)}u(t+2) \longleftrightarrow \frac{e^{2s}}{s+1} \quad \Re\{s\} > -1$$

Therefore,

$$h_4(t) = e^{-(t+2)}u(t+2)$$

Since $h_4(t)$ is two-sided, this system is noncausal.

Stability

An LTI system is stable if the impulse response $h(t)$ is *absolutely integrable*, that is,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

This implies that the Fourier transform exists, and thus the ROC includes the $j\omega$-axis in the s-plane. Thus, *an LTI system is stable if and only if the ROC of its system function $H(s)$ includes the $j\omega$-axis.*

Stability of a Causal LTI System

Consider a causal LTI system with a rational system function $H(s)$. Since the system is causal, the ROC is to the right of the rightmost pole. Consequently, for this system to be stable (i.e., for the ROC to include the $j\omega$-axis), the rightmost pole of $H(s)$ must be to the left of the $j\omega$-axis. That is, *a causal system with rational system function $H(s)$ is stable if and only if all the poles of $H(s)$ lie in the left-half of the s-plane.*

Example For the following system functions, check whether the corresponding LTI system is causal and stable.

(a) $H_1(s) = \dfrac{1}{s^2 - s - 6} \quad \Re\{s\} > 3$

(b) $H_2(s) = \dfrac{1}{s^2 - s - 6} \quad \Re\{s\} < -2$

(c) $H_3(s) = \dfrac{1}{s^2 - s - 6} \quad -2 < \Re\{s\} < 3$

Solution

(a) Consider the given system function

$$H_1(s) = \frac{1}{s^2 - s - 6} = \frac{1}{(s-3)(s+2)} \quad \Re\{s\} > 3$$

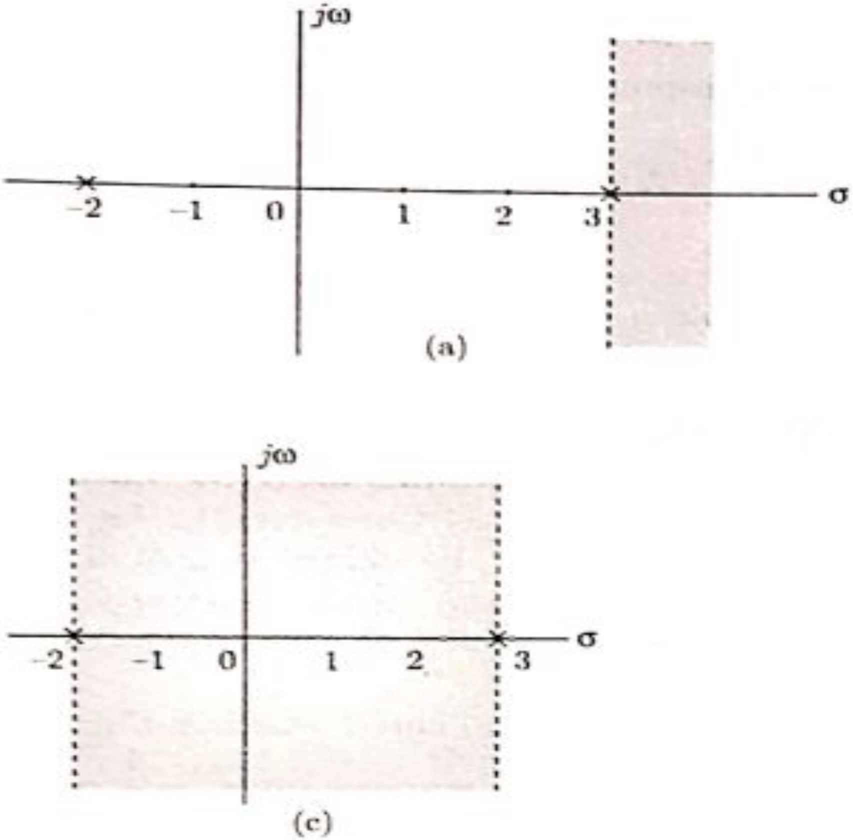
Since the given system function $H_1(s)$ is rational and its ROC [shown in Fig. 9.20(a)] is to the right of the rightmost pole, the corresponding LTI system is causal. Also, note that the ROC does not include the $j\omega$-axis, and, consequently, the corresponding system is unstable. This can be verified by its impulse response. Given that

$$\begin{aligned} H_1(s) &= \frac{1}{(s-3)(s+2)} \\ &= \frac{1/5}{s-3} - \frac{1/5}{s+2} \end{aligned}$$

Taking the inverse Laplace transform, we obtain

$$h_1(t) = \frac{1}{5}[e^{3t}u(t) - e^{-2t}u(t)]$$

Note that $h_1(t) = 0$ for $t < 0$, and $\int_{-\infty}^{\infty} h_1(t)\,dt = \infty$, i.e., the impulse response is not absolutely integrable, so this system is causal and unstable.



ROC for (a) causal and unstable system $H_1(s)$, (b) anticausal and unstable system $H_2(s)$ and (c) noncausal and stable system $H_3(s)$.

(b) Consider the given system function

$$H_2(s) = \frac{1}{s^2 - s - 6} \quad \Re\{s\} < -2$$

$$H_2(s) = \frac{1}{(s-3)(s+2)} \quad \Re\{s\} < -2$$

Since the given system function $H_2(s)$ is rational and its ROC [shown in Fig. 9.20(b)] is to the left of the leftmost pole, the corresponding LTI system is anticausal. Also, note that the ROC does not include the $j\omega$-axis, and, consequently, the corresponding system is unstable. This can be verified by its impulse response. Given that

$$H_2(s) = \frac{1}{(s-3)(s+2)} \quad \Re\{s\} < -2$$

$$= \frac{1/5}{s-3} - \frac{1/5}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$h_2(t) = \frac{1}{5}[-e^{3t}u(-t) + e^{-2t}u(-t)]$$

Note that $h_2(t) = 0$ for $t > 0$, and $\int_{-\infty}^{\infty} h_2(t)\,dt = \infty$, i.e., the impulse response is not absolutely integrable, so this system is anticausal and unstable.

(c) Consider the given system function

$$H_3(s) = \frac{1}{s^2 - s - 6} \quad -2 < \Re\{s\} < 3$$

$$H_3(s) = \frac{1}{(s-3)(s+2)} \quad -2 < \Re\{s\} < 3$$

Since the given system function $H_3(s)$ is rational and its ROC [shown in Fig. 9.20(c)] is a strip, the corresponding LTI system is noncausal. Also, note that the ROC includes the $j\omega$-axis, and, consequently, the corresponding system is stable. This can be verified by its impulse response. Given that

$$H_3(s) = \frac{1}{(s-3)(s+2)} \quad -2 < \Re\{s\} < 3$$

$$= \frac{1/5}{s-3} - \frac{1/5}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$h_3(t) = \frac{1}{5}[-e^{3t}u(-t) - e^{-2t}u(t)]$$

Note that $h_3(t)$ is two-sided, and $\int_{-\infty}^{\infty} h_3(t)\,dt < \infty$, i.e., the impulse response is absolutely integrable, so this system is noncausal and stable.

Example For the following system functions, check whether the corresponding LTI system is causal and stable.

$$H(s) = \frac{1}{s^2 + 5s + 6} \quad \Re\{s\} > -2$$

Solution

Consider the given system function

$$H(s) = \frac{1}{s^2 + 5s + 6} = \frac{1}{(s+2)(s+3)} \quad \Re\{s\} > -2$$

Since the given system function $H(s)$ is rational and its ROC [shown in Fig. 9.21] is to the right of the rightmost pole, the corresponding LTI system is causal. Also, note that the ROC includes the $j\omega$-axis, and, consequently, the corresponding system is stable. This can be verified by its impulse response. Given that

$$H(s) = \frac{1}{(s+2)(s+3)} \quad \Re\{s\} > -2$$

$$= \frac{1}{s+2} - \frac{1}{s+3}$$

Taking the inverse Laplace transform, we obtain

$$h(t) = e^{-2t}u(t) - e^{-3t}u(t)$$

Note that $h(t) = 0$ for $t < 0$, and $\int_{-\infty}^{\infty} h(t)\,dt < \infty$, i.e., the impulse response is absolutely integrable, so this system is causal and stable.

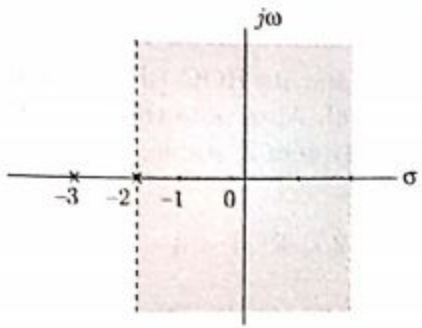


Fig. ROC for Example 9.34

Example Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system impulse response.

(a) Determine $H(s)$ as a ratio of two polynomials in s. Sketch the pole-zero pattern of $H(s)$.

(b) Determine $h(t)$ for each of the following cases:

1. The system is stable.
2. The system is causal.
3. The system is neither stable nor causal.

Solution

(a) Consider the given differential equation

$$\frac{d^2y(t)}{dt^2} - \frac{dy(t)}{dt} - 2y(t) = x(t)$$

Taking the Laplace transform of the above equation, we obtain

$$s^2Y(s) - sY(s) - 2Y(s) = X(s)$$

$$Y(s)[s^2 - s - 2] = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 - s - 2}$$

$$H(s) = \frac{1}{(s-2)(s+1)}$$

The system has poles at -1 and 2. The pole-zero pattern is shown in Fig. 9.24(a).
(b) Using partial fraction expansion

$$H(s) = \frac{1/3}{s-2} - \frac{1/3}{s+1}$$

1. For this system to be stable, the ROC must include the $j\omega$-axis. To include the $j\omega$-axis, its ROC must be in the region $-1 < \Re\{s\} < 2$ [shown in Fig. 9.24(b)].

The first term has a pole at $s = 2$. Here the ROC is to the left of this pole, so this pole corresponds to the anti-causal (left-sided) signal. Therefore,

$$-\frac{1}{3}\, e^{2t} u(-t) \longleftrightarrow \frac{1/3}{s-2}$$

The pole of the second term is at -1. The ROC lies to the right of this pole, so this pole corresponds to a causal (right-sided) signal. Therefore,

$$\frac{1}{3}\, e^{-t} u(t) \longleftrightarrow \frac{1/3}{s+1}$$

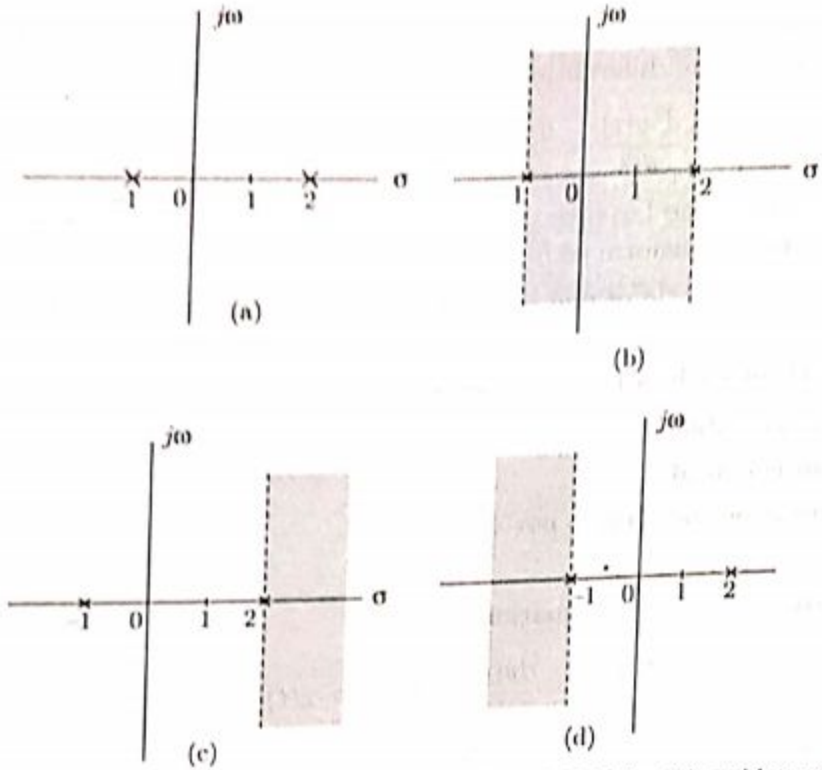


Fig. (a) Pole-zero pattern of $H(s)$ and ROC for (b) stable system, (c) causal system, and (d) neither stable nor causal system of Example 9.37

and hence we obtain

$$h(t) = -\frac{1}{3}\, e^{2t}u(-t) - \frac{1}{3}\, e^{-t}u(t)$$

2. For this system to be causal, the ROC lies to the right of the rightmost pole, i.e., $\Re\{s\} > 2$ [shown in Fig. 9.24(c)].

Since the ROC, $\Re\{s\} > 2$, is to the right of the rightmost pole, all the poles correspond to causal (right-sided) signals. Therefore,

$$\frac{1}{3}\, e^{2t}u(t) \longleftrightarrow \frac{1/3}{s-2}$$

$$\frac{1}{3}\, e^{-t}u(t) \longleftrightarrow \frac{1/3}{s+1}$$

and hence we obtain

$$h(t) = \frac{1}{3}\, e^{2t}u(t) - \frac{1}{3}\, e^{-t}u(t)$$

leftmost pole, all the poles correspond to anti-causal (left-sided) signals. Therefore,

$$-\frac{1}{3}\, e^{2t}u(-t) \longleftrightarrow \frac{1/3}{s-2}$$

$$-\frac{1}{3}\, e^{-t}u(-t) \longleftrightarrow \frac{1/3}{s+1}$$

and hence we obtain

$$h(t) = -\frac{1}{3}\, e^{2t}u(-t) + \frac{1}{3}\, e^{-t}u(-t)$$

Example Find the response $y(t)$ of a noncausal system with the transfer function

$$H(s) = \frac{-1}{s-1} \quad \Re\{s\} < 1$$

to the input $x(t) = e^{-2t}u(t)$.

Solution

Given that

$$x(t) = e^{-2t}u(t)$$

$$\mathcal{L}[x(t)] = X(s) = \frac{1}{s+2} \quad \Re\{s\} > -2$$

We know that [Eq. (9.50)]

$$Y(s) = X(s)H(s) = \frac{-1}{(s-1)(s+2)}$$

The ROC of $Y(s)$ is in the region $-2 < \Re\{s\} < 1$. Using partial fraction expansion, we obtain

$$Y(s) = \frac{-1/3}{s-1} + \frac{1/3}{s+2} \quad -2 < \Re\{s\} < 1$$

Taking the inverse Laplace transform, we obtain

$$y(t) = \frac{1}{3}[e^{t}u(-t) + e^{-2t}u(t)]$$