Module No. 3

Fourier Analysis of Continuous and Discrete time Signals and Systems

Part - I

Fourier Analysis of Continuous Signals and Systems

Fourier Transform

1 Development of Fourier Transform From Fourier Series

The exponential form of Fourier series representation of a periodic signal is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Omega_0 t} \qquad .....(4.29)$$

$$\text{where, } c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jn\Omega_0 t}\, dt \qquad .....(4.30)$$

In the Fourier representation using equation (4.29), the c_n for various values of n are the spectral components of the signal x(t), located at intervals of fundamental frequency Ω_0. Therefore the frequency spectrum is discrete in nature.

The Fourier representation of a signal using equation (4.29) is applicable for periodic signals. For Fourier representation of non-periodic signals, let us consider that the fundamental period tends to infinity. When the fundamental period tends to infinity, the fundamental frequency Ω_0 tends to zero or becomes very small. Since fundamental frequency Ω_0 is very small, the spectral components will lie very close to each other and so the frequency spectrum becomes continuous.

In order to obtain the Fourier representation of a non-periodic signal let us consider that the fundamental frequency Ω_0 is very small.

Let, $\Omega_0 \rightarrow \Delta\Omega$

On replacing Ω_0 by $\Delta\Omega$ in equation (4.29) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\Delta\Omega t}$$

On substituting for c_n in the above equation from equation (4.30) (by taking τ as dummy variable for integration) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{1}{T} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t} \qquad .....(4.31)$$

We know that, $\Omega_0 = 2\pi F_0 = \dfrac{2\pi}{T}$; $\qquad \therefore \dfrac{1}{T} = \dfrac{\Omega_0}{2\pi}$

Since $\Omega_0 \rightarrow \Delta\Omega$, $\quad \dfrac{1}{T} = \dfrac{\Delta\Omega}{2\pi}$ $\qquad .....(4.32)$

On substituting for $\dfrac{1}{T}$ from equation (4.32) in equation (4.31) we get,

$$x(t) = \sum_{n=-\infty}^{+\infty} \left[\frac{\Delta\Omega}{2\pi} \int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t} = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t}\, \Delta\Omega$$

For non-periodic signals, the fundamental period T tends to infinity. On letting limit T tends to infinity in the above equation we get,

$$x(t) = \underset{T\to\infty}{\mathrm{Lt}} \ \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} \left[\int_{-T/2}^{T/2} x(\tau)\, e^{-jn\Delta\Omega\tau}\, d\tau \right] e^{jn\Delta\Omega t}\, \Delta\Omega$$

When $T \to \infty$; $\quad \sum \to \int$; $\quad \Delta\Omega \to \Omega$

$$\therefore x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x(\tau)\, e^{-jn\Omega\tau}\, d\tau \right] e^{jn\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{jn\Omega t}\, d\Omega \qquad .....(4.33)$$

$\boxed{\text{Since } \tau \text{ is a dummy variable, Let } \tau = t.}$

$$\text{where, } X(j\Omega) = \int_{-\infty}^{+\infty} x(\tau)\, e^{-jn\Omega\tau}\, d\tau = \int_{-\infty}^{+\infty} x(t)\, e^{-jn\Omega t}\, dt \qquad .....(4.34)$$

The equation (4.34) is Fourier transform of x(t) and equation (4.33) is inverse Fourier transform of x(t).

Since the equation (4.34) extracts the frequency components of the signal, transformation using equation (4.34) is also called ***analysis*** of the signal x(t). Since the equation (4.33) combines the frequency components of the signal, the inverse transformation using equation (4.33) is also called ***synthesis*** of the signal x(t).

Fourier Transform

Definition of Fourier Transform

Let, $x(t)$ = Continuous time signal

$X(j\Omega)$ = Fourier transform of $x(t)$

The Fourier transform of continuous time signal, $x(t)$ is defined as,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

Also, $X(j\Omega)$ is denoted as $\mathcal{F}\{x(t)\}$ where "$\mathcal{F}$" is the symbol used to denote the Fourier transform operation.

$$\boxed{\therefore\ \mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt} \qquad .....(4.35)$$

Note : *Sometimes the Fourier transform is expressed as a function of cyclic frequency F, rather than radian frequency Ω. The Fourier transform as a function of cyclic frequency F, is defined as,*

$$X(jF) = \int_{-\infty}^{+\infty} x(t)\, e^{-j2\pi Ft}\, dt$$

Condition for Existence of Fourier Transform

The Fourier transform of x(t) exists if it satisfies the following Dirichlet condition.

1. The x(t) be absolutely integrable.

   i.e., $\int_{-\infty}^{+\infty} x(t)\, dt < \infty$

2. The x(t) should have a finite number of maxima and minima within any finite interval.
3. The x(t) can have a finite number of discontinuities within any interval.

Definition of Inverse Fourier Transform

The ***inverse Fourier transform*** of X(jΩ) is defined as,

$$\boxed{x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega} \qquad .....(4.36)$$

The signals x(t) and X(jΩ) are called ***Fourier transform pair*** and can be expressed as shown below,

$$\boxed{x(t) \quad \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\Longleftrightarrow}} \quad X(j\Omega)}$$

Frequency Spectrum Using Fourier Transform

The $X(j\Omega)$ is a complex function of Ω. Hence it can be expressed as a sum of real part and imaginary part as shown below.

$$\therefore\ X(j\Omega) = X_r(j\Omega) + jX_i(j\Omega)$$

where, $X_r(j\Omega)$ = Real part of $X(j\Omega)$

$X_i(j\Omega)$ = Imaginary part of $X(j\Omega)$

The magnitude of $X(j\Omega)$ is called ***Magnitude spectrum.***

$$\boxed{\therefore\ \text{Magnitude spectrum, } |X(j\Omega)| = \sqrt{X_r^2(j\Omega) + X_i^2(j\Omega)}} \qquad .....(4.37)$$

(or)

$$\text{Magnitude spectrum, } |X(j\Omega)| = \sqrt{X(j\Omega)\ X^*(j\Omega)} \qquad .....(4.38)$$

where, $X^*(j\Omega)$ = Conjugate of $X(j\Omega)$

The phase of $X(j\Omega)$ is called ***Phase spectrum***.

$$\boxed{\therefore\ \text{Phase spectrum, } \angle X(j\Omega) = \tan^{-1}\frac{X_i(j\Omega)}{X_r(j\Omega)}} \qquad .....(4.39)$$

The magnitude spectrum will always have even symmetry and phase spectrum will have odd symmetry. The magnitude and phase spectrum together called ***frequency spectrum***.

Properties of Fourier Transform

1. Linearity

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

The linearity property of Fourier transform says that,

$$\mathcal{F}\{a_1\, x_1(t) + a_2\, x_2(t)\} = a_1\, X_1(j\Omega) + a_2\, X_2(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X_1(j\Omega) = \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt \quad \text{and} \quad X_2(j\Omega) = \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt \qquad .....(4.40)$$

Consider the linear combination $a_1\, x_1(t) + a_2\, x_2(t)$. On taking Fourier transform of this signal we get,

$$\mathcal{F}\{a_1\, x_1(t) + a_2\, x_2(t)\} = \int_{-\infty}^{+\infty} \left[a_1\, x_1(t) + a_2\, x_2(t)\right] e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} a_1\, x_1(t)\, e^{-j\Omega t}\, dt + \int_{-\infty}^{+\infty} a_2\, x_2(t)\, e^{-j\Omega t}\, dt$$

$$= a_1 \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt + a_2 \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt$$

Using equation (4.40)

$$= a_1\, X_1(j\Omega) + a_2\, X_2(j\Omega)$$

2. Time shifting

The time shifting property of Fourier transform says that,

$$\text{If } \mathcal{F}\{x(t)\} = X(j\Omega) \text{ then}$$

$$\mathcal{F}\{x(t-t_0)\} = e^{-j\Omega_0 t}\, X(j\Omega)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.41)$$

$$\therefore\ \mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{+\infty} x(t-t_0)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega(\tau + t_0)}\, d\tau$$

Let, $t - t_0 = \tau$
$\therefore t = \tau + t_0$
On differentiating
$dt = d\tau$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau} \times e^{-j\Omega t_0}\, d\tau = e^{-j\Omega t_0} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\tau}\, d\tau$$

Since τ is a dummy variable for ingetration we can change τ to t.

$$= e^{-j\Omega t_0} \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = e^{-j\Omega t_0}\, X(j\Omega)$$

Using equation (4.41)

3. Time scaling

The time scaling property of Fourier transform says that,

$$\text{If } \mathcal{F}\{x(t)\} = X(j\Omega) \text{ then}$$

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore\ \mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\Omega\left(\frac{\tau}{a}\right)} \frac{d\tau}{a}$$

$$\boxed{\text{Put, } at = \tau \ ; \ \therefore t = \frac{\tau}{a} \ ; \ dt = \frac{d\tau}{a}}$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau = \frac{1}{a} X\left(\frac{j\Omega}{a}\right)$$

The term $\int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau$ is similar to the form of Fourier transform except that Ω is replaced by $\left(\frac{\Omega}{a}\right)$.

$$\therefore \int_{-\infty}^{+\infty} x(\tau)\, e^{-j\left(\frac{\Omega}{a}\right)\tau}\, d\tau = X\left(\frac{j\Omega}{a}\right)$$

The above transform is applicable for positive values of "a".

If "a" happens to be negative then it can be proved that,

$$\mathcal{F}\{x(at)\} = -\frac{1}{a} X\left(\frac{j\Omega}{a}\right)$$

Hence in general,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right) \quad \text{for both positive and negative values of "a"}$$

4. Time reversal

The time reversal property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x(-t)\} = X(-j\Omega)$$

Proof :

From time scaling property we know that,

$$\mathcal{F}\{x(at)\} = \frac{1}{|a|} X\left(\frac{j\Omega}{a}\right)$$

Let, $a = -1$.

$$\therefore \mathcal{F}\{x(-t)\} = X(-j\Omega)$$

5. *Conjugation*

The conjugation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\mathcal{F}\{x^*(t)\} = X^*(-j\Omega)$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore\ \mathcal{F}\left\{x^*(t)\right\} = \int_{-\infty}^{+\infty} x^*(t)\, e^{-j\Omega t}\, dt$$

$$= \left[\int_{-\infty}^{+\infty} x(t)\, e^{j\Omega t}\, dt\right]^* = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t}\, dt\right]^*$$

$$= [X(-j\Omega)]^* = X^*(-j\Omega)$$

The term, $\int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t}\, dt$ is similar to the form of Fourier transform except that Ω is replaced by $-\Omega$.

$$\therefore\ \int_{-\infty}^{+\infty} x(t)\, e^{-j(-\Omega)t} dt = X(-j\Omega)$$

6. *Frequency shifting*

The frequency shifting property of Fourier transform says that,

$$\text{If } \mathcal{F}\{x(t)\} = X(j\Omega) \text{ then}$$

$$\mathcal{F}\{e^{j\Omega_0 t}\,x(t)\} = X(j(\Omega - \Omega_0))$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt$$

$$\therefore\ \mathcal{F}\{e^{j\Omega_0 t}\,x(t)\} = \int_{-\infty}^{+\infty} \left[e^{j\Omega_0 t}\,x(t)\right] e^{-j\Omega t}\,dt = \int_{-\infty}^{+\infty} x(t)\,e^{j\Omega_0 t}\,e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\,e^{-j(\Omega - \Omega_0)t}\,dt = X\big(j(\Omega - \Omega_0)\big)$$

The term $\int_{-\infty}^{+\infty} x(t)\,e^{-j(\Omega-\Omega_0)t}\,dt$ is similar to the form of Fourier transform except that Ω is replaced by $\Omega - \Omega_0$.

$$\therefore\ \int_{-\infty}^{+\infty} x(t)\,e^{-j(\Omega - \Omega_0)t}\,dt = X\big(j(\Omega - \Omega_0)\big)$$

7. Time differentiation

The differentiation property of Fourier transform says that,

$$\text{If } \mathcal{F}\{x(t)\} = X(j\Omega) \text{ then}$$

$$\mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = j\Omega\, X(j\Omega)$$

Proof :

Consider the definition of Fourier transform of x(t).

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.42)$$

$$\therefore \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = \int_{-\infty}^{+\infty} \left(\frac{d}{dt}x(t)\right) e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} e^{-j\Omega t}\left(\frac{d}{dt}x(t)\right) dt$$

$$\therefore \mathcal{F}\left\{\frac{d}{dt}x(t)\right\} = \left[e^{-j\Omega t}\, x(t)\right]_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (-j\Omega)\, e^{-j\Omega t}\, x(t)\, dt \qquad \boxed{\int uv = u\int v - \int\left[du\int v\right]}$$

$$= e^{-\infty}\, x(\infty) - e^{+\infty}\, x(-\infty) + j\Omega \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad \boxed{\begin{array}{c} x(-\infty) = 0 \\ e^{-\infty} = 0 \end{array}}$$

$$= j\Omega \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = j\Omega\, X(j\Omega) \qquad \boxed{\text{Using equation (4.42)}}$$

8. Time integration

The integration property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ and $X(0) = 0$ then

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = \frac{1}{j\Omega} X(j\Omega)$$

Proof :

Consider a continuous time signal x(t). Let X(jΩ) be Fourier transform of x(t). Since integration and differentiation are inverse operations, x(t) can be expressed as shown below.

$$\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = x(t)$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\left\{\frac{d}{dt}\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right]\right\} = \mathcal{F}\{x(t)\}$$

$$j\Omega\ \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = \mathcal{F}\{x(t)\}$$

Using time differentiation property of Fourier transform.

$$\therefore\ \mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)\, d\tau\right\} = \frac{1}{j\Omega} X(j\Omega)$$

$\boxed{\mathcal{F}\{x(t)\} = X(j\Omega)}$

9. Frequency differentiation

The frequency differentiation property of Fourier transform says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$, then

$$\mathcal{F}\{t\,x(t)\} = j\frac{d}{d\Omega}X(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

On differentiating the above equation with respect to Ω we get,

$$\frac{d}{d\Omega}X(j\Omega) = \frac{d}{d\Omega}\left(\int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt\right)$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{d\Omega}e^{-j\Omega t}\right) dt$$

Interchanging the order of integration and differentiation

$$\therefore\ \frac{d}{d\Omega}X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\left(-jt\,e^{-j\Omega t}\right)dt = \frac{1}{j}\int_{-\infty}^{+\infty}\left(t\,x(t)\right)e^{-j\Omega t}\,dt$$

$$-j = -j \times \frac{j}{j} = \frac{1}{j}$$

$$= \frac{1}{j}\mathcal{F}\left\{t\,x(t)\right\}$$

Using definition of Fourier transform.

$$\therefore\ \mathcal{F}\left\{t\,x(t)\right\} = j\frac{d}{d\Omega}X(j\Omega)$$

10. *Convolution theorem*

The convolution theorem of Fourier transform says that, Fourier transform of convolution of two signals is given by the product of the Fourier transform of the individual signals.

i.e., if $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$ and $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$ then,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\Omega)\, X_2(j\Omega) \qquad .....(4.43)$$

The equation (4.43) is also known as convolution property of Fourier transform.

With reference to chapter-2, section -2.9 we get,

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \qquad .....(4.44)$$

where τ is a dummy variable used for integration.

Proof :

Let $x_1(t)$ and $x_2(t)$ be two time domain signals. Now, by definition of Fourier transform,

$$X_1(j\Omega) = \mathcal{F}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt \quad .....(4.45)$$

$$X_2(j\Omega) = \mathcal{F}\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt \quad .....(4.46)$$

Using definition of Fourier transform we can write,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{\infty} [x_1(t) * x_2(t]\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{+\infty} x_1(\tau)\; x_2(t-\tau)\; d\tau\right] e^{-j\Omega t}\, dt \quad .....(4.47)$$

Let, $e^{-j\Omega t} = e^{j\Omega\tau} \times e^{-j\Omega\tau} \times e^{-j\Omega t} = e^{-j\Omega\tau} \times e^{-j\Omega(t-\tau)} = e^{-j\Omega\tau} \times e^{-j\Omega M}$ .....(4.48)

where, $M = t - \tau$ and so, $dM = dt$ .....(4.49)

Using equations (4.48) and (4.49), the equation (4.47) can be written as,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1(\tau)\; x_2(M)\; e^{-j\Omega\tau}\, e^{-j\Omega M}\, d\tau\, dM$$

$$= \int_{-\infty}^{+\infty} x_1(\tau)\, e^{-j\Omega\tau}\, d\tau \times \int_{-\infty}^{+\infty} x_2(M)\, e^{-j\Omega M}\, dM \quad .....(4.50)$$

In equation (4.50), τ and M are dummy variables used for integration, and so they can be changed to t.

Therefore equation (4.50) can be written as,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-j\Omega t}\, dt \times \int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, dt$$

$$= X_1(j\Omega)\, X_2(j\Omega)$$

Using equations (4.45) and (4.46)

11. Frequency convolution

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$.

The frequency convolution property of Fourier transform says that,

$$\mathcal{F}\{x_1(t)\, x_2(t)\} = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega - \lambda))\, d\lambda$$

Proof :

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$\therefore\ \mathcal{F}\{x_1(t)\, x_2(t)\} = \int_{t=-\infty}^{t=+\infty} x_1(t)\, x_2(t)\, e^{-j\Omega t}\, dt$$

By the definition of inverse Fourier transform we get,

$$x_1(t) = \mathcal{F}^{-1}\{X_1(j\Omega)\} = \frac{1}{2\pi} \int_{\Omega=-\infty}^{\Omega=+\infty} X_1(j\Omega)\, e^{j\Omega t}\, d\Omega = \frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, e^{j\lambda t}\, d\lambda$$

On substituting for $X_1(t)$ from equation (4.52) in equation (4.51) we get,

$$\mathcal{F}\{x_1(t)\,x_2(t)\} = \int_{t=-\infty}^{t=+\infty} \left[\frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, e^{j\lambda t}\, d\lambda\right] x_2(t)\, e^{-j\Omega t}\, dt$$

Here Ω is the vairable used for ingetration. Let us change Ω to λ.

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\left[\int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j\Omega t}\, e^{j\lambda t}\, dt\right] d\lambda$$

Interchanging the order of integration.

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\left[\int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j(\Omega-\lambda)t}\, dt\right] d\lambda$$

$$= \frac{1}{2\pi}\int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega-\lambda))\, d\lambda$$

The term, $\int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j(\Omega-\lambda)t}\, dt$ is similar to the form of Fourier transform except that Ω is replaced by $\Omega-\lambda$.

$$\therefore \int_{t=-\infty}^{t=+\infty} x_2(t)\, e^{-j(\Omega-\lambda)t} dt = X_2(j(\Omega-\lambda))$$

12. Parseval's relation

The Parseval's relation says that,

If $\mathcal{F}\{x(t)\} = X(j\Omega)$ then

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(j\Omega)|^2 \, d\Omega$$

Proof :

Let $x(t)$ be a continuous time signal and $x^*(t)$ be conjugate of $x(t)$.

Now, $|x(t)|^2 = x(t)\, x^*(t)$

On integrating the above equation with respect to t we get,

$$\int_{t=-\infty}^{t=+\infty} |x(t)|^2 \, dt = \int_{t=-\infty}^{t=+\infty} x(t)\, x^*(t)\, dt \qquad .....(4.53)$$

By definition of inverse Fourier transform, we can write,

$$x(t) = \mathcal{F}^{-1}\{X(j\Omega)\} = \frac{1}{2\pi} \int_{\Omega=-\infty}^{\Omega=+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

On taking conjugate of the above equation we get,

$$x^*(t) = \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, e^{-j\Omega t}\, d\Omega \qquad .....(4.54)$$

Using equation (4.54) the equation (4.53) can be written as,

$$\int_{t=-\infty}^{t=+\infty} |x(t)|^2\, dt = \int_{t=-\infty}^{t=+\infty} x(t)\left[\frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, e^{-j\Omega t}\, d\Omega\right] dt$$

Interchanging the order of integration.

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\left[\int_{t=-\infty}^{t=+\infty} x(t)\, e^{-j\Omega t}\, dt\right] d\Omega$$

Using definition of Fourier transform.

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} X^*(j\Omega)\, X(j\Omega)\, d\Omega$$

$X(j\Omega)\, X^*(j\Omega) = |X(j\Omega)|^2$

$$= \frac{1}{2\pi}\int_{\Omega=-\infty}^{\Omega=+\infty} |X(j\Omega)|^2\, d\Omega$$

Note : *The term $|X(j\Omega)|^2$ represents the distribution of energy as function of Ω and so it is called* ***energy density spectrum*** *or* ***energy spectral density*** *of the signal x(t).*

13. Duality

If $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$ and $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

and if $x_2(t) \equiv X_1(j\Omega)$, i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions

then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$, i.e., $X_2(j\Omega)$ are $2\pi x_2(-j\Omega)$ are similar functions

Alternatively duality property is expressed as shown below.

If $\quad x_2(t) \Leftrightarrow X_1(j\Omega)$

then $X_2(j\Omega) \Leftrightarrow 2\pi x_1(-j\Omega)$

Proof :

Let, $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$ and $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

Let, $x_2(t)$ and $X_1(j\Omega)$ are similar in form.

$$\therefore x_2(t) = X_1(j\Omega)\Big|_{j\Omega=t} \qquad .....(4.55)$$

By definition of inverse Fourier transform,

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\therefore x_1(-t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X_1(j\Omega)\, e^{-j\Omega t}\, d\Omega \qquad \boxed{\text{Replacing t by }-\text{t}}$$

$$\therefore x_1(-t)\Big|_{t=j\Omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(X_1(j\Omega)\Big|_{j\Omega=t}\right) e^{-j\Omega t}\, d\Omega \qquad \boxed{\text{interchanging } j\Omega \text{ and t}}$$

$$\therefore x_1(-j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_2(t)\, e^{-j\Omega t}\, d\Omega \qquad \boxed{\text{Using equation (4.55)}}$$

$$\therefore x_1(-t)\Big|_{t=j\Omega} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(X_1(j\Omega)\Big|_{j\Omega=t}\right)e^{-j\Omega t}\,d\Omega$$

interchanging $j\Omega$ and t

$$\therefore x_1(-j\Omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} x_2(t)\,e^{-j\Omega t}\,d\Omega$$

Using equation (4.55)

$$\therefore \int_{-\infty}^{+\infty} x_2(t)\,e^{-j\Omega t}\,d\Omega = 2\pi\,x_1(-j\Omega)$$

$$\therefore X_2(j\Omega) = 2\pi\,x_1(-j\Omega)$$

Using definition of Fourier transform

Note : For even function $x_1(-j\Omega) = x_1(j\Omega)$.

$$\therefore X_2(j\Omega) = 2\pi\,x_1(j\Omega)$$

14. Area under a time domain signal

$$\text{Area under } x(t) = \int_{-\infty}^{+\infty} x(t)\,dt$$

If $x(t)$ and $X(j\Omega)$ are Fourier transform pair,

then, $$\int_{-\infty}^{+\infty} x(t)\,dt = X(0)$$

where, $$X(0) = \underset{j\Omega \to 0}{\text{Lt}}\; X(j\Omega)$$

Proof :

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\,dt$$

$$\therefore X(0) = \underset{j\Omega\to 0}{\text{Lt}}\; X(j\Omega) = \underset{j\Omega\to 0}{\text{Lt}} \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{0}\,dt = \int_{-\infty}^{+\infty} x(t)\,dt$$

$$\therefore \int_{-\infty}^{+\infty} x(t)\,dt = X(0)$$

15. Area under a frequency domain signal

$$\text{Area under } X(j\Omega) = \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega$$

If $x(t)$ and $X(j\Omega)$ are Fourier transform pair,

$$\text{then,} \quad \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi\; x(0)$$

$$\text{where,}\; x(0) = \underset{t \to 0}{\text{Lt}}\; x(t)$$

Proof :

By definition of inverse Fourier transform,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$\therefore x(0) = \underset{t\to 0}{\text{Lt}}\; x(t) = \underset{t\to 0}{\text{Lt}}\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{0}\, d\Omega \;=\; \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega$$

$$\therefore \quad \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega \;=\; 2\pi\; x(0)$$

Table 4.3 : Summary of Properties of Fourier Transform

Let, $\mathcal{F}\{x(t)\} = X(j\Omega)$; $\mathcal{F}\{x_1(t)\} = X_1(j\Omega)$; $\mathcal{F}\{x_2(t)\} = X_2(j\Omega)$

Property	Time domain signal	Frequency domain signal
Linearity	$a_1 x_1(t) + a_2 x_2(t)$	$a_1 X_1(j\Omega) + a_2 X_2(j\Omega)$
Time shifting	$x(t - t_0)$	$e^{-j\Omega t_0} X(j\Omega)$
Time scaling	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{j\Omega}{a}\right)$
Time reversal	$x(-t)$	$X(-j\Omega)$
Conjugation	$x^*(t)$	$X^*(-j\Omega)$
Frequency shifting	$e^{j\Omega_0 t} x(t)$	$X(j(\Omega - \Omega_0))$
Time differentiation	$\frac{d}{dt} x(t)$	$j\Omega X(j\Omega)$
Time integration	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\frac{X(j\Omega)}{j\Omega} = \pi X(0)\, \delta(\Omega)$
Frequency differentiation	$t\ x(t)$	$j \frac{d}{d\Omega} X(j\Omega)$
Time convolution	$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t - \tau) d\tau$	$X_1(j\Omega)\ X_2(j\Omega)$

Frequency convolution (or Multiplication)	$x_1(t)\, x_2(t)$	$\frac{1}{2\pi} \int_{\lambda=-\infty}^{\lambda=+\infty} X_1(j\lambda)\, X_2(j(\Omega-\lambda))\, d\lambda$
Symmetry of real signals	x(t) is real	$X(j\Omega) = X^*(j\Omega)$ $\lvert X(j\Omega)\rvert = \lvert X(-j\Omega)\rvert;\ \angle X(j\Omega) = -\angle X(-j\Omega)$ $Re\{X(j\Omega)\} = Re\{X(-j\Omega)\}$ $Im\{X(j\Omega)\} = -Im\{X(-j\Omega)\}$
Real and even	x(t) is real and even	$X(j\Omega)$ are real and even
Real and odd	x(t) is real and odd	$X(j\Omega)$ are imaginary and odd
Duality	If $x_2(t) \equiv X_1(j\Omega)$ [i.e., $x_2(t)$ and $X_1(j\Omega)$ are similar functions] then $X_2(j\Omega) \equiv 2\pi x_1(-j\Omega)$ [i.e., $X_2(j\Omega)$ and $2\pi x_1(-j\Omega)$ are similar functions]	
Area under a frequency domain signal	$\int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi\, x(0)$	
Area under a time domain signal	$\int_{-\infty}^{+\infty} x(t)\, dt = X(0)$	
Parseval's relation	Energy in time domain is, $E = \int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt$	Energy in frequency domain is, $E = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2 d\Omega$
	$\int_{-\infty}^{+\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \lvert X(j\Omega)\rvert^2\, d\Omega$	

Fourier Transform of Some Important Signals

Fourier Transform of Unit Impulse Signal

The impulse signal is defined as,

$$x(t) = \delta(t) = \infty \quad ; \quad t = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$
$$= 0 \quad ; \quad t \neq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} \delta(t)\, e^{-j\Omega t}\, dt$$
$$= 1 \times e^{-j\Omega t}\Big|_{t=0} = 1 \times e^{0} = 1$$

$\boxed{\delta(t) \text{ exists only for } t = 0}$

$$\therefore \boxed{\mathcal{F}\{x(t)\} = 1}$$

The plot of impulse signal and its magnitude spectrum are shown in fig 4.18 and fig 4.19 respectively.

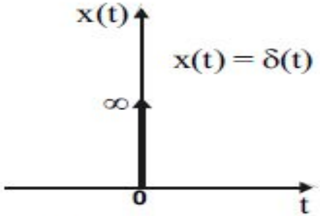


Fig 4.18 : *Impulse signal.*

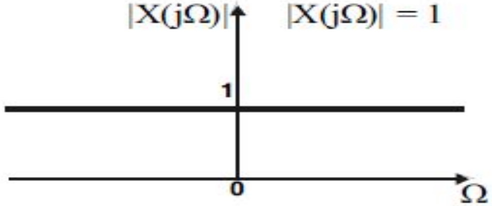


Fig 4.19 : *Magnitude spectrum of impulse signal.*

Fourier Transform of Single Sided Exponential Signal

The single sided exponential signal is defined as,

$$x(t) = A\,e^{-at} \quad ; \quad \text{for } t \geq 0$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{0}^{+\infty} A\,e^{-at}\,e^{-j\Omega t}\,dt$$

$$= \int_{0}^{+\infty} A\,e^{-(a+j\Omega)t}\,dt = \left[\frac{A\,e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_0^{+\infty}$$

$$= \left[\frac{A\,e^{-\infty}}{-(a+j\Omega)} - \frac{A\,e^{0}}{-(a+j\Omega)}\right] = \frac{A}{a+j\Omega} \qquad \boxed{e^{-\infty} = 0}$$

$$\therefore \boxed{\mathcal{F}\{A\,e^{-at}\,u(t)\} = \frac{A}{a+j\Omega}} \qquad .....(4.56)$$

The plot of exponential signal and its magnitude spectrum are shown in fig 4.20 and fig 4.21 respectively.

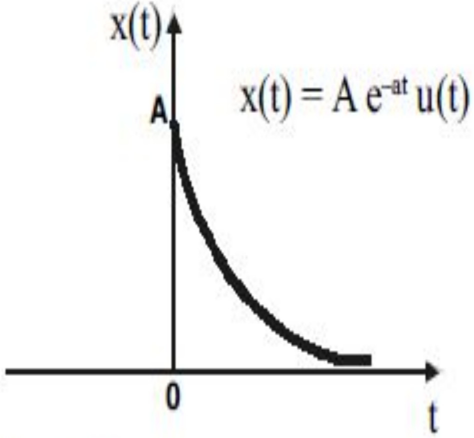


Fig 4.20: *Single sided exponential signal.*

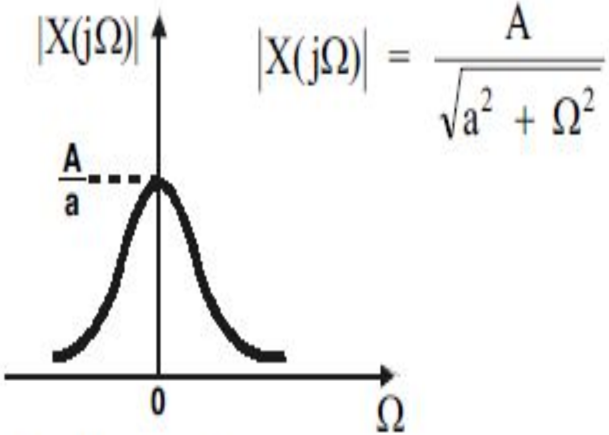


Fig 4.21 : *Magnitude spectrum of single sided exponential signal.*

Fourier Transform of Double Sided Exponential Signal

The double sided exponential signal is defined as,

$$x(t) = A\,e^{-a|t|} \;;\text{ for all } t$$

$$\therefore\; x(t) = A\,e^{+at} \;;\text{ for } t = -\infty \text{ to } 0$$

$$= A\,e^{-at} \;;\text{ for } t = 0 \text{ to } +\infty$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\Omega t}\,dt = \int_{-\infty}^{0} A\,e^{at}\,e^{-j\Omega t}\,dt + \int_{0}^{+\infty} A\,e^{-at}\,e^{-j\Omega t}\,dt$$

$$= \int_{-\infty}^{0} A\,e^{(a-j\Omega)t}\,dt + \int_{0}^{\infty} A\,e^{-(a+j\Omega)t}\,dt = \left[\frac{A\,e^{(a-j\Omega)t}}{a-j\Omega}\right]_{-\infty}^{0} + \left[\frac{A\,e^{-(a+j\Omega)t}}{-(a+j\Omega)}\right]_{0}^{\infty}$$

$$= \frac{A\,e^{0}}{a - j\Omega} - \frac{A\,e^{-\infty}}{a - j\Omega} + \frac{A\,e^{-\infty}}{-(a + j\Omega)} - \frac{A\,e^{0}}{-(a + j\Omega)} = \frac{A}{a - j\Omega} + \frac{A}{a + j\Omega}$$

$\boxed{e^{-\infty} = 0}$

$$= \frac{A(a + j\Omega) + A(a - j\Omega)}{(a - j\Omega)\,(a + j\Omega)} = \frac{2aA}{a^2 + \Omega^2}$$

$\boxed{(a+b)(a-b) = a^2 - b^2 \;\middle|\; j^2 = -1}$

$$\therefore \boxed{\mathcal{F}\left\{A\,e^{-a|t|}\right\} = \frac{2aA}{a^2 + \Omega^2}} \qquad .....(4.57)$$

The plot of double sided exponential signal and its magnitude spectrum are shown in fig 4.22 and fig 4.23 respectively.

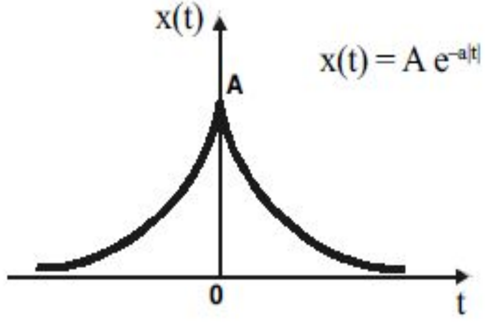


***Fig 4.22 :** Double sided exponential signal.*

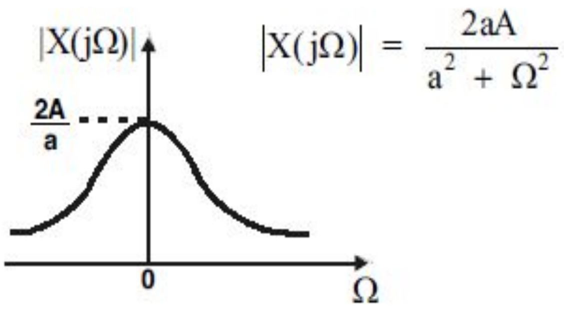


***Fig 4.23 :** Magnitude spectrum of double sided exponential signal.*

Fourier Transform of a Constant

Let, $x(t) = A$, where A is a constant.

If definition of Fourier transform is directly applied, the constant will not satisfy the condition,

$$\int_{-\infty}^{+\infty} |x(t)|\, dt < \infty$$

Hence the constant can be viewed as a double sided exponential with limit "a" tends to 0 as shown below.

Let $x_1(t)$ = Double sided exponential signal.

The double sided exponential signal is defined as,

$$x_1(t) = A\, e^{-a|t|}$$

$$\text{i.e,} \quad x_1(t) = A\, e^{at} \;;\quad \text{for } t = -\infty \text{ to } 0$$
$$= A\, e^{-at} \;;\quad \text{for } t = 0 \text{ to } +\infty$$

$$\therefore\ x(t) = A \underset{a \to 0}{\mathrm{Lt}}\ x_1(t)$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\underset{a \to 0}{\mathrm{Lt}}\ x_1(t)\right\}$$

$$\mathcal{F}\{x(t)\} = \underset{a \to 0}{\mathrm{Lt}}\ \mathcal{F}\{x_1(t)\}$$

$$X(j\Omega) = \underset{a \to 0}{\mathrm{Lt}}\ [X_1(j\Omega)]$$

$$\boxed{\mathcal{F}\{x(t)\} = X(j\Omega)} \quad \boxed{\mathcal{F}\{x_1(t)\} = X_1(j\Omega)}$$

$$= \underset{a \to 0}{\mathrm{Lt}}\ \frac{2aA}{\Omega^2 + a^2}$$

Using equation (4.57)

The above equation is 0 for all values of Ω except at $\Omega = 0$.

At $\Omega = 0$, the above equation represents an impulse of magnitude "k".

$$\therefore\ X(j\Omega) = k\,\delta(\Omega) \quad ; \quad \Omega = 0$$
$$= 0 \quad ; \quad \Omega \neq 0$$

The magnitude "k" can be evaluated as shown below.

$$k = \int_{-\infty}^{+\infty} \frac{2aA}{\Omega^2 + a^2}\, d\Omega = 2aA \int_{-\infty}^{+\infty} \frac{1}{\Omega^2 + a^2}\, d\Omega$$

$$\boxed{\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}}$$

$$= 2aA \left[\frac{1}{a} \tan^{-1}\left(\frac{\Omega}{a}\right)\right]_{-\infty}^{+\infty} = 2aA \left[\frac{1}{a} \tan^{-1}(+\infty) - \frac{1}{a} \tan^{-1}(-\infty)\right]$$

$$= 2aA \left[\frac{1}{a}\, \frac{\pi}{2} - \frac{1}{a}\left(-\frac{\pi}{2}\right)\right] = 2aA\left(\frac{\pi}{a}\right) = 2\pi A$$

$$\therefore\ \boxed{\mathcal{F}\{A\} = 2\pi A\,\delta(\Omega)} \qquad .....(4.58)$$

The plot of constant and its magnitude spectrum are shown in fig 4.24 and fig 4.25 respectively.

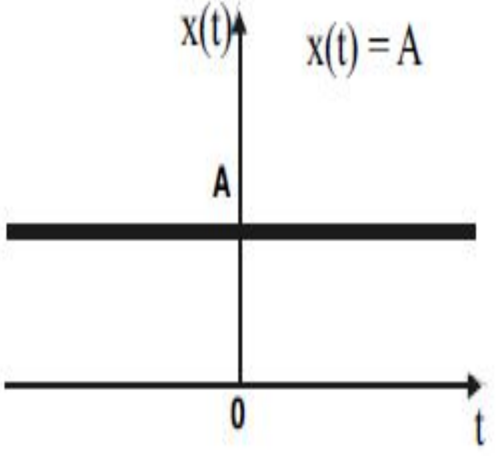


Fig 4.24 : *Constant.*

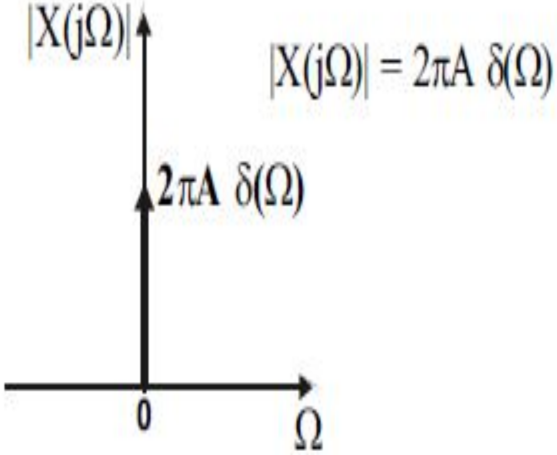


Fig 4.25 : *Magnitude spectrum of constant.*

Fourier Transform of Signum Function

The signum function is defined as,

$$x(t) = \mathrm{sgn}(t) = \begin{cases} 1 & ; \quad t > 0 \\ -1 & ; \quad t < 0 \end{cases}$$

The signum function can be expressed as a sum of two one sided exponential signal and taking limit "a" tends to 0 as shown below.

$$\therefore \ \mathrm{sgn}(t) = \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

$$\therefore \quad x(t) = \mathrm{sgn}(t) = \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at} u(t) - e^{at} u(-t) \right]$$

By definition of Fourier transform,

$$X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{+\infty} \underset{a \to 0}{\mathrm{Lt}} \left[e^{-at} u(t) - e^{at} u(-t) \right] e^{-j\Omega t}\, dt$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\int_{0}^{+\infty} e^{-at}\, e^{-j\Omega t}\, dt - \int_{-\infty}^{0} e^{at}\, e^{-j\Omega t}\, dt \right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\int_{0}^{+\infty} e^{-(a+j\Omega)t}\, dt - \int_{-\infty}^{0} e^{+(a-j\Omega)t}\, dt \right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right]_0^{\infty} - \left[\frac{e^{(a-j\Omega)t}}{(a-j\Omega)} \right]_{-\infty}^{0} \right] \qquad \boxed{e^0 = 1\ ;\ e^{-\infty} = 0}$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\frac{e^{-\infty}}{-(a+j\Omega)} - \frac{e^{0}}{-(a+j\Omega)} - \frac{e^{0}}{a-j\Omega} + \frac{e^{-\infty}}{a-j\Omega} \right]$$

$$= \underset{a \to 0}{\mathrm{Lt}} \left[\frac{1}{a+j\Omega} - \frac{1}{a-j\Omega} \right] = \frac{1}{j\Omega} + \frac{1}{j\Omega} = \frac{2}{j\Omega}$$

$$\therefore \boxed{\mathcal{F}\{\mathrm{sgn}(t)\} = \frac{2}{j\Omega}} \qquad .....(4.59)$$

The plot of signum function and its magnitude spectrum are shown in fig 4.26 and fig 4.27 respectively.

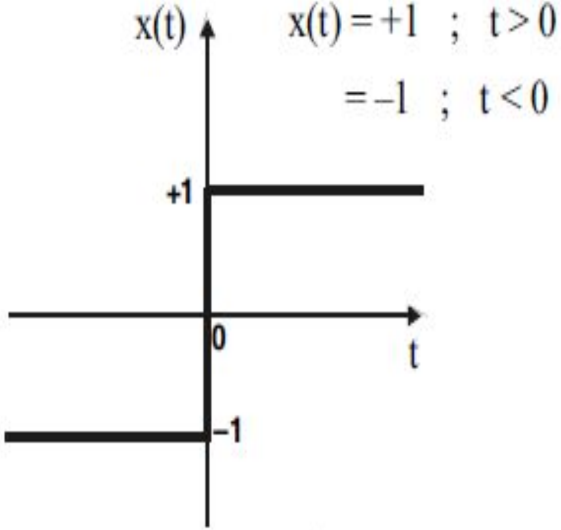


Fig 4.26 : *Signum function.*

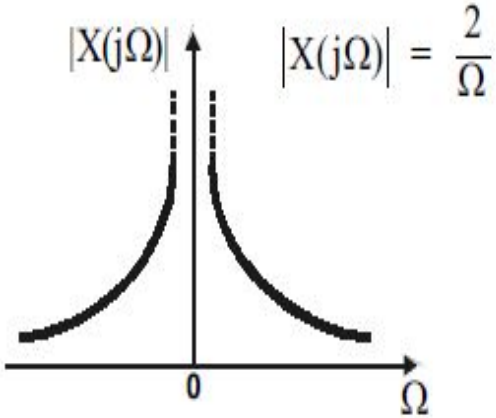


Fig 4.27 : *Magnitude spectrum of signum function.*

Fourier Transform of Unit Step Signal

The unit step signal is defined as,

$$u(t) = 1 \quad ; \quad t \geq 0$$
$$\quad\;\; = 0 \quad ; \quad t < 0$$

If can be proved that, $\text{sgn}(t) = 2u(t) - 1 \quad \Rightarrow \quad u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$

$$\therefore \; x(t) = u(t) = \frac{1}{2}\left[1 + \text{sgn}(t)\right]$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{1}{2}\left[1 + \text{sgn}(t)\right]\right\}$$

$$\therefore \; X(j\Omega) = \mathcal{F}\left\{\frac{1}{2}\right\} + \mathcal{F}\left\{\frac{1}{2}\,\text{sgn}(t)\right\} = \frac{1}{2}\,\mathcal{F}\{1\} + \frac{1}{2}\,\mathcal{F}\{\text{sgn}(t)\}$$

$$= \frac{1}{2}\left[2\pi\,\delta(\Omega)\right] + \frac{1}{2}\left[\frac{2}{j\Omega}\right] = \pi\,\delta(\Omega) + \frac{1}{j\Omega}$$

Using equations (4.58) and (4.59)

$$\therefore \; \boxed{\mathcal{F}\{u(t)\} = \pi\,\delta(\Omega) + \frac{1}{j\Omega}} \qquad .....(4.60)$$

The plot of unit step signal and its magnitude spectrum are shown in fig 4.28 and fig 4.29 respectively.

$$x(t) = u(t) = 1 \quad ; \quad t \geq 0$$
$$= 0 \quad ; \quad t < 0$$

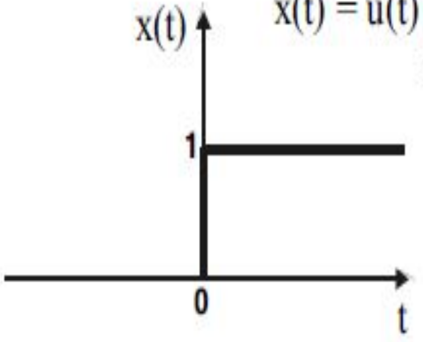


Fig 4.28 : *Unit step signal.*

$$|X(j\Omega)| = \pi\delta(\Omega) + \frac{1}{\Omega}$$

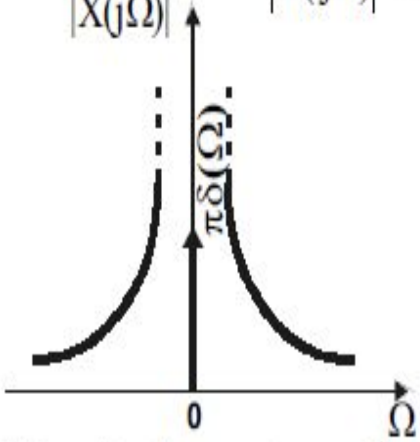


Fig 4.29 : *Magnitude spectrum of unit step signal.*

Fourier Transform of Complex Exponential Signal

The complex exponential signal is defined as,

$$x(t) = A\,e^{j\Omega_0 t}$$
$$= e^{j\Omega_0 t} A$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{e^{j\Omega_0 t} A\right\}$$
$$= \mathcal{F}\{A\}\Big|_{\Omega = \Omega - \Omega_0}$$
$$= 2\pi\,\delta(\Omega)\Big|_{\Omega = \Omega - \Omega_0}$$
$$= 2\pi\,\delta(\Omega - \Omega_0)$$

$$\boxed{\begin{array}{c}\underline{\text{Frequency shifting property.}}\\ \text{If } \mathcal{F}\{x(t)\} = X(j\Omega) \text{ then,}\\ \mathcal{F}\left\{e^{j\Omega_0 t}\,x(t)\right\} = X(j(\Omega - \Omega_0))\end{array}}$$

$\boxed{\text{Using frequency shifting property}}$

$\boxed{\text{Using equation (4.46)}}$

$$\therefore \boxed{\mathcal{F}\left\{A\,e^{j\Omega_0 t}\right\} = 2\pi A\,\delta(\Omega - \Omega_0)} \qquad .....(4.61)$$

Similarly, $$\boxed{\mathcal{F}\left\{e^{-j\Omega_0 t} A\right\} = 2\pi A\,\delta(\Omega + \Omega_0)} \qquad .....(4.62)$$

The signal $Ae^{-j\Omega_0 t}$ can be represented by a rotating vector of magnitude, "A", in clockwise direction in a complex plane with an angular speed of $\Omega_0 t$ as shown in fig.4.30 . The magnitude spectrum of $Ae^{-j\Omega_0 t}$ is shown in fig. 4.31.

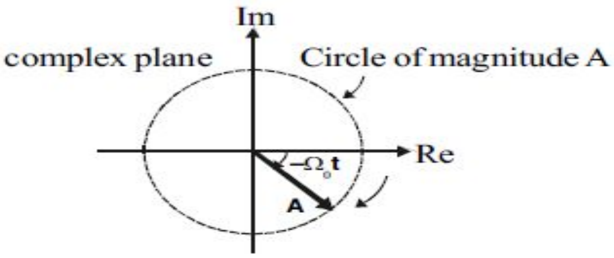


Fig 4.30 : Complex exponential signal.

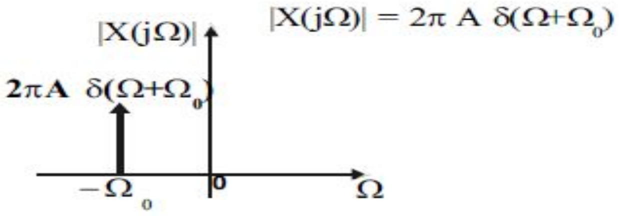


Fig 4.31 : Magnitude spectrum of $Ae^{-j\Omega_0 t}$.

The signal $Ae^{j\Omega_0 t}$ can be represented by a rotating vector of magnitude "A", in anticlockwise direction in a complex plane with an angular speed of $\Omega_0 t$ as shown in fig 4.32. The magnitude spectrum of $Ae^{+j\Omega_0 t}$ is shown in fig 4.33.

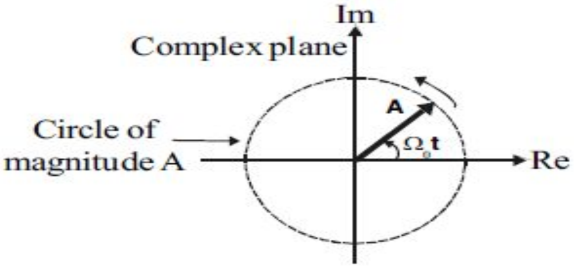


Fig 4.32 : Complex exponential signal.

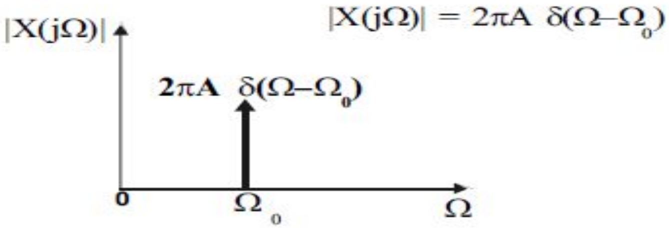


Fig 4.33: Magnitude spectrum of $Ae^{+j\Omega_0 t}$.

Fourier Transform of Sinusoidal Signal

The sinusoidal signal is defined as,

$$x(t) = A\sin\Omega_0 t = \frac{A}{2j}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{A}{2j}\left(e^{j\Omega_0 t} - e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2j}\left[\mathcal{F}\{e^{j\Omega_0 t}\} - \mathcal{F}\{e^{-j\Omega_0 t}\}\right]$$

Using equations (4.61) and (4.62).

$$= \frac{A}{2j}\left[2\pi\,\delta(\Omega-\Omega_0) - 2\pi\,\delta(\Omega+\Omega_0)\right] = \frac{A\pi}{j}\left[\delta(\Omega-\Omega_0) - \delta(\Omega+\Omega_0)\right]$$

$$\therefore \boxed{\mathcal{F}\{A\sin\Omega_0 t\} = \frac{A\pi}{j}\left[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right]} \quad .....(4.63)$$

The plot of sinusoidal signal and its spectrum are shown in fig 4.34 and fig 4.35.

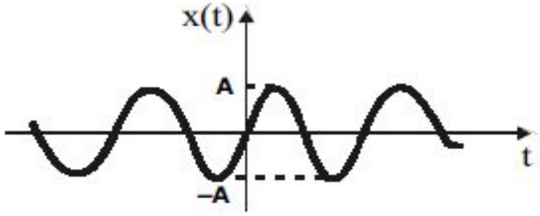


Fig 4.34 : Sinusoidal signal.

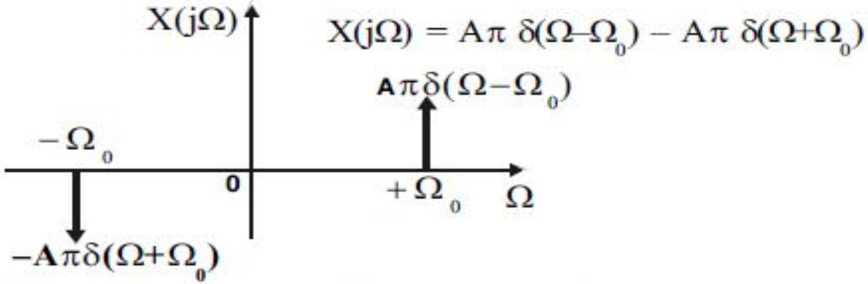


Fig 4.35 : Spectrum of sinusoidal signal.

Fourier Transform of Cosinusoidal Signal

The cosinusoidal signal is defined as,

$$x(t) = A\cos\Omega_0 t = \frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

On taking Fourier transform we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\frac{A}{2}\left(e^{j\Omega_0 t} + e^{-j\Omega_0 t}\right)\right\} = \frac{A}{2}\left[\mathcal{F}\{e^{j\Omega_0 t}\} + \mathcal{F}\{e^{-j\Omega_0 t}\}\right]$$

Using equations (4.61) and (4.62).

$$= \frac{A}{2}\left[2\pi\,\delta(\Omega - \Omega_0) + 2\pi\,\delta(\Omega + \Omega_0)\right] = A\pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]$$

$$\therefore \boxed{\mathcal{F}\{A\cos\Omega_0 t\} = A\pi\left[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)\right]} \qquad .....(4.64)$$

The plot of cosinusoidal signal and its magnitude spectrum are shown in fig 4.36 and fig 4.37.

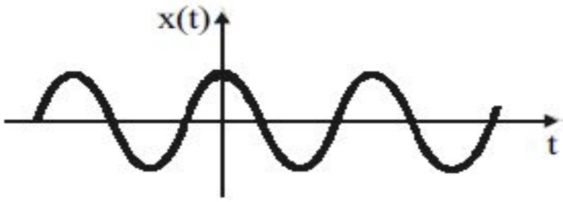


Fig 4.36 : Cosinusoidal signal.

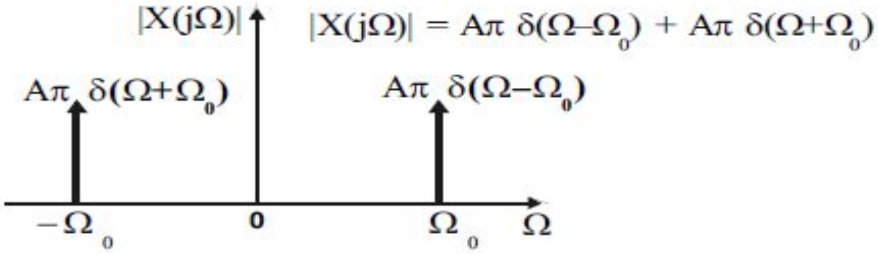
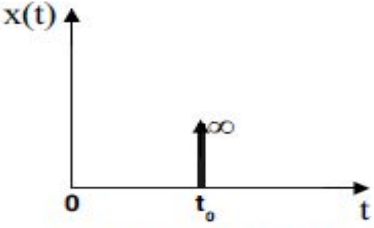


Fig 4.37 : Magnitude spectrum of cosinusoidal signal.

Table 4.4 : Fourier Transform of Standard Signals and their Magnitude Spectrum

<table>
<tr><th>x(t)</th><th>X(jΩ) and Magnitude Spectrum</th></tr>
<tr><td>

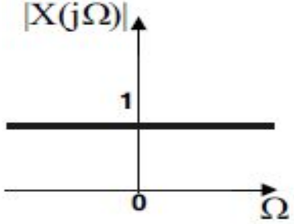
$x(t) = \delta(t - t_0)$



Shifted impulse signal

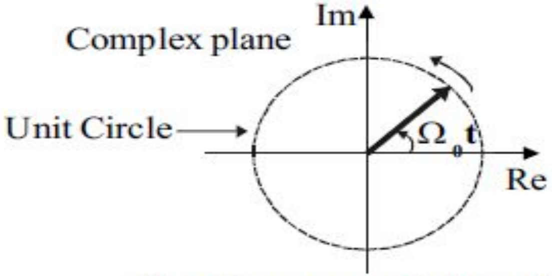
</td><td>

$X(j\Omega) = e^{-j\Omega t_0}$



</td></tr>
<tr><td>

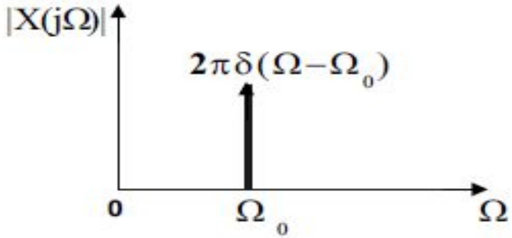
$x(t) = e^{j\Omega_0 t}$



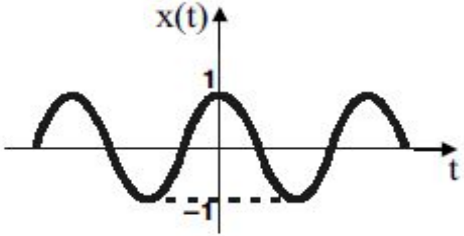
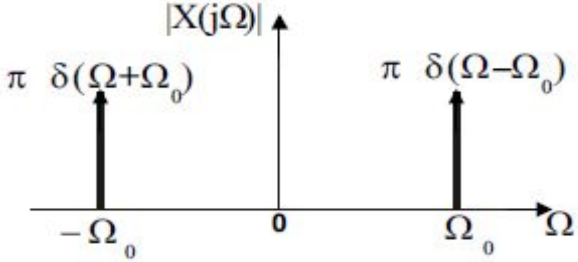
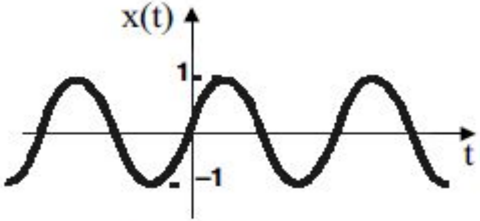
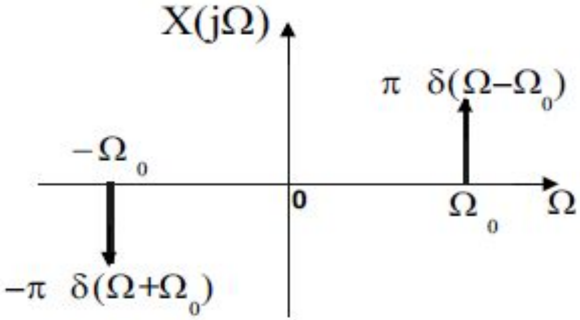
Complex exponential signal

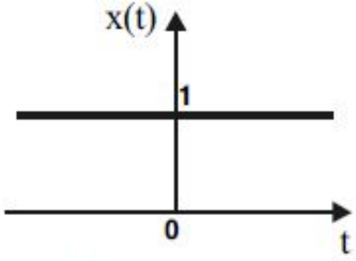
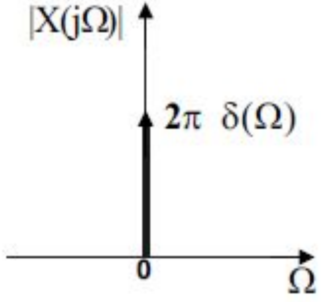
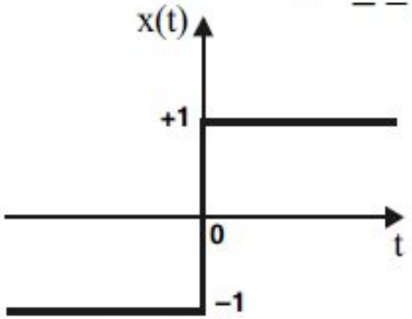
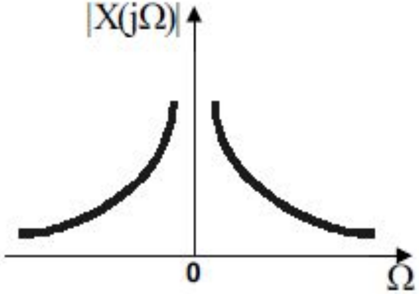
</td><td>

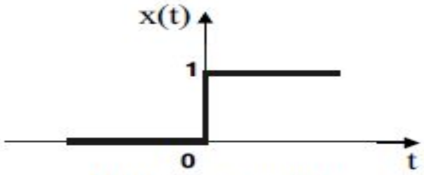
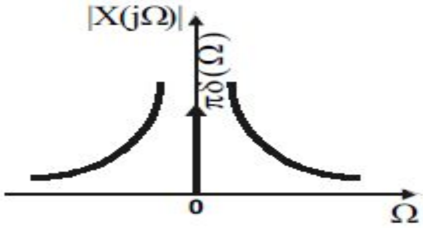
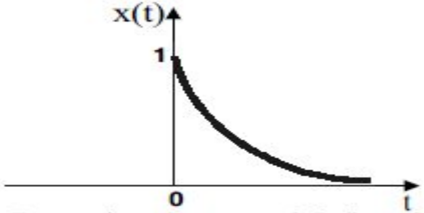
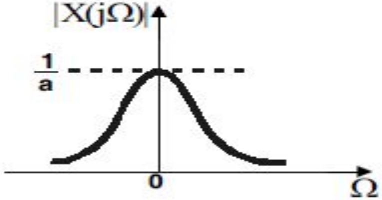
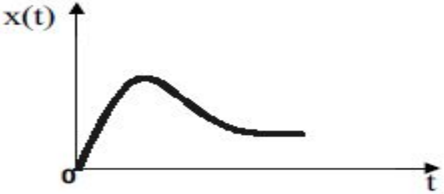
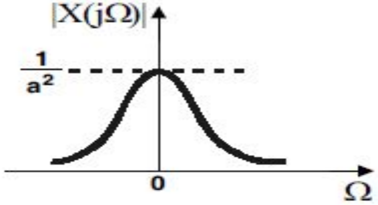
$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0)$

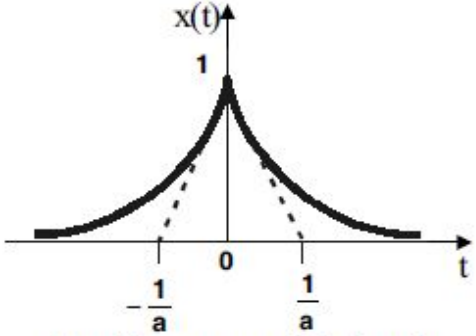
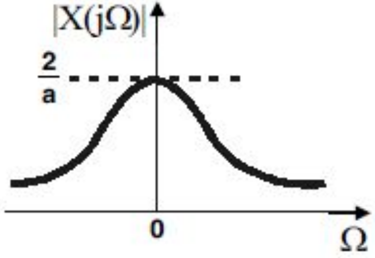
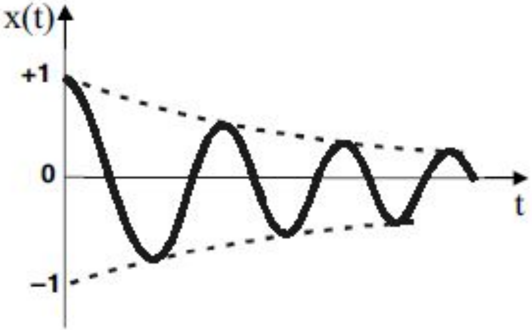
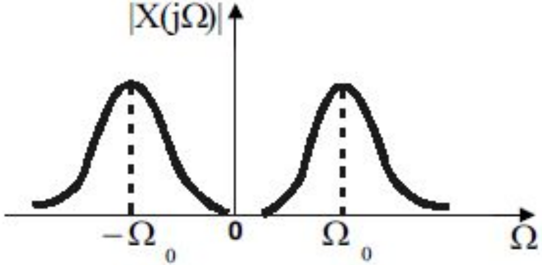


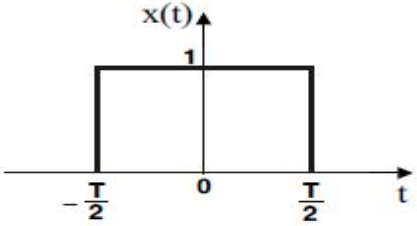
</td></tr>
</table>

$x(t) = \cos\Omega_0 t$  Cosinusoidal signal	$X(j\Omega) = \pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$ 
$x(t) = \sin\Omega_0 t$  Sinusoidal signal	$X(j\Omega) = \frac{\pi}{j}[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$ 

x(t)	**X(jΩ) and Magnitude Spectrum**
$x(t) = 1$  Constant	$X(j\Omega) = 2\pi\delta(\Omega)$ 
$x(t) = \mathrm{sgn}(t) = \dfrac{t}{\|t\|} = 1 \quad ; t > 0$ $= -1 \quad ; t < 0$  Signum signal	$X(j\Omega) = \dfrac{2}{j\Omega}$ 

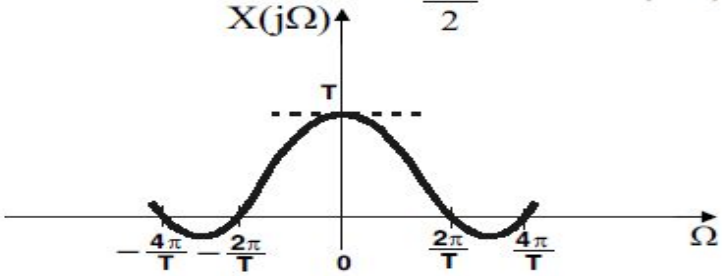
$$x(t) = u(t) = 1 \;;\; t \geq 0$$ $$= 0 \;;\; t < 0$$  Unit step signal	$$X(j\Omega) = \pi\delta(\Omega) + \frac{1}{j\Omega}$$ 
$$x(t) = e^{-at}\,u(t)$$  Decaying exponential signal	$$X(j\Omega) = \frac{1}{a + j\Omega}$$ 
$$x(t) = t\,e^{-at}\,u(t)$$  Product of ramp and decaying exponential signal	$$X(j\Omega) = \frac{1}{(a + j\Omega)^2}$$ 

x(t)	**X(jΩ) and Magnitude Spectrum**
$x(t) = e^{-a\|t\|}$  Double exponential signal	$X(j\Omega) = \frac{2a}{a^2 + \Omega^2}$ 
 Exponentially decaying cosinusoidal signal	$X(j\Omega) = \frac{a + j\Omega}{(a + j\Omega)^2 + \Omega_0^2}$ 

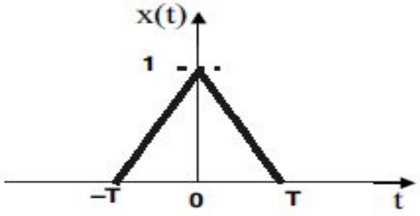


Rectangular pulse

$$X(j\Omega) = T\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}} = T\,\mathrm{sinc}\left(\frac{\Omega T}{2}\right)$$

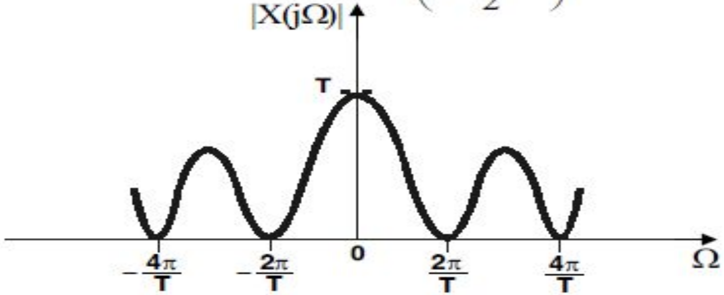


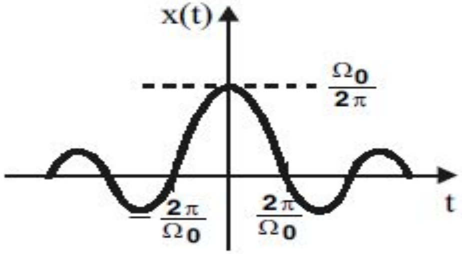
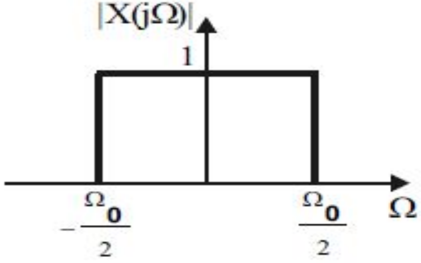
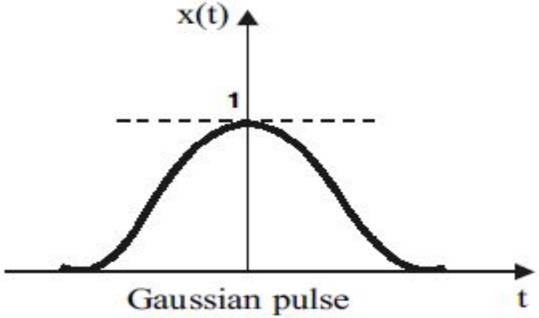
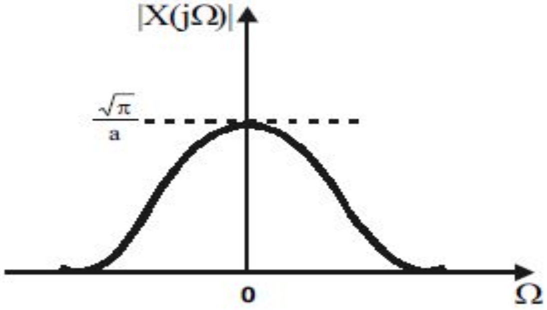
$$x(t) = 1 + \tfrac{t}{T} \;;\; t = -T \;\text{ to }\; 0$$
$$= 1 - \tfrac{t}{T} \;;\; t = \; 0 \;\text{ to }\; 0$$



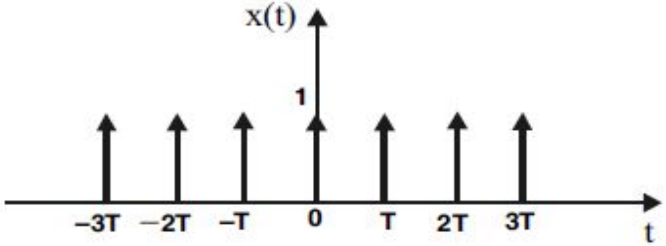
Triangular pulse

$$X(j\Omega) = T\left(\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}}\right)^2$$



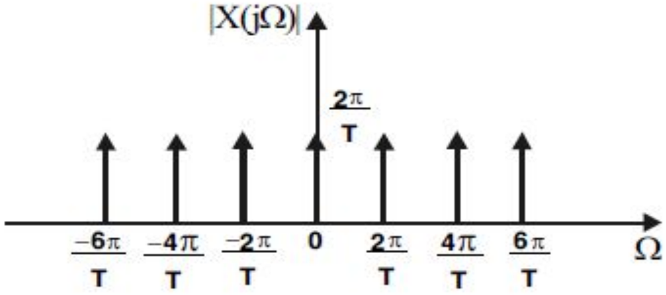
x(t)	X(jΩ) and Magnitude Spectrum
$x(t) = \frac{\Omega_0}{2\pi}\operatorname{sinc}\left(\frac{\Omega_0}{2}t\right) = \frac{1}{\pi}\frac{\sin\left(\frac{\Omega_0}{2}t\right)}{t}$  Sinc pulse	$X(j\Omega) = \left[u\left(\Omega + \frac{\Omega_0}{2}\right) - u\left(\Omega - \frac{\Omega_0}{2}\right)\right]$  $\boxed{\Omega_0 = \frac{2\pi}{T}}$
$x(t) = e^{-a^2t^2}$  Gaussian pulse	$X(j\Omega) = \frac{\sqrt{\pi}}{a}e^{-\left(\frac{\Omega}{2a}\right)^2}$ 

$$x(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$



Impulse train

$$X(j\Omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{+\infty} \delta\left(\Omega - \frac{2\pi n}{T}\right)$$



From table 4.4 the following observations are made.

1. The Fourier transform of a Gaussian pulse will be another Gaussian pulse.
2. The Fourier transform of an impulse train will be another impulse train.
3. The Fourier transform of a rectangular pulse will be a sinc pulse and viceversa.
4. The Fourier transform of a triangular pulse will be a squared sinc pulse.
5. The Fourier transform of a constant will be an impulse and vice-versa.

Standard Fourier Tranform Pairs

$x(t)$	$X(j\Omega)$
$\delta(t)$	1
$\delta(t-t_0)$	$e^{-j\Omega t_0}$
A where, A is constant	$2\pi A\delta(\Omega)$
$u(t)$	$\pi\delta(\Omega) + \frac{1}{j\Omega}$
$\mathrm{sgn}(t)$	$\frac{2}{j\Omega}$
$t\,u(t)$	$\frac{1}{(j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, \ldots..$	$\frac{1}{(j\Omega)^n}$
$t^n u(t)$ where, $n = 1, 2, 3, \ldots..$	$\frac{n!}{(j\Omega)^{n+1}}$

$e^{-at}u(t)$	$\dfrac{1}{j\Omega + a}$
$t\,e^{-at}u(t)$	$\dfrac{1}{(j\Omega + a)^2}$
$Ae^{-a\lvert t\rvert}$	$\dfrac{2Aa}{a^2 + \Omega^2}$
$Ae^{j\Omega_0 t}$	$2\pi A\,\delta(\Omega - \Omega_0)$
$\sin\Omega_0 t$	$\dfrac{\pi}{j}\left[\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)\right]$
$\cos\Omega_0 t$	$\pi[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$

Fourier Transform of a Periodic Signal

Let, $x(t)$ = Continuous time periodic signal

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \text{Fourier transform of } x(t)$$

The exponential form of Fourier series representation of $x(t)$ is given by,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n \; e^{jn\Omega_0 t} \qquad \boxed{\text{From equation (4.9)}}$$

On taking Fourier transform of the above equation we get,

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n \, e^{jn\Omega_0 t}\right\} = \sum_{n=-\infty}^{+\infty} c_n \; \mathcal{F}\left\{e^{jn\Omega_0 t}\right\}$$

$$= \sum_{n=-\infty}^{+\infty} c_n \, 2\pi \, \delta(\Omega - n\Omega_0) = 2\pi \sum_{n=-\infty}^{+\infty} c_n \; 2\pi \, \delta(\Omega - n\Omega_0) \qquad \boxed{\text{Using equation (4.61)}}$$

$$\begin{aligned} = \; ..... &+ 2\pi c_{-3} \, \delta(\Omega + 3\Omega_0) + 2\pi c_{-2} \, \delta(\Omega + 2\Omega_0) + 2\pi c_{-1} \, \delta(\Omega + \Omega_0) \\ &+ 2\pi c_0 \, \delta(\Omega) + 2\pi c_1 \, \delta(\Omega - \Omega_0) + 2\pi c_2 \, \delta(\Omega - 2\Omega_0) \\ &+ 2\pi \, c_3 \, \delta(\Omega - 3\Omega_0) + ... \end{aligned} \qquad .....(4.65)$$

The magnitude of each term of equation (4.65) represents an impulse, located at its harmonic frequency in the magnitude spectrum. Hence we can say that the Fourier transform of a periodic continuous time signal consists of impulses located at the harmonic frequencies of the signal. The magnitude of each impulse is 2π times the magnitude of Fourier coefficient, i.e., the magnitude of n^{th} impulse is $2\pi\ |c_n|$.

Analysis of LTI Continuous Time System Using Fourier Transform

Transfer Function of LTI Continuous Time System in Frequency Domain

The ratio of Fourier transform of output and the Fourier transform of input is called ***transfer function*** of LTI continuous time system in frequency domain.

Let, $x(t)$ = Input to the continuous time system

$y(t)$ = Output of the continuous time system

$\therefore$ $X(j\Omega)$ = Fourier transform of $x(t)$

$Y(j\Omega)$ = Fourier transform of $y(t)$

$$\text{Now, Transfer function} = \frac{Y(j\Omega)}{X(j\Omega)} \qquad \text{.....(4.66)}$$

The transfer function of LTI continuous time system in frequency domain can be obtained from the differential equation governing the input-output relation of an LTI continuous time system, (refer chapter-2, equation (2.13)),

$$\frac{d^N}{dt^N}y(t) + a_1\frac{d^{N-1}}{dt^{N-1}}y(t) + a_2\frac{d^{N-2}}{dt^{N-2}}y(t) + \ldots\ldots + a_{N-1}\frac{d}{dt}y(t) + a_N\,y(t) = b_0\frac{d^M}{dt^M}x(t)$$
$$+ b_1\frac{d^{M-1}}{dt^{M-1}}x(t) + b_2\frac{d^{M-2}}{dt^{M-2}}x(t) + \ldots\ldots + b_{M-1}\frac{d}{dt}x(t) + b_M\,x(t)$$

On taking Fourier transform of the above equation and rearranging the resultant equation as a ratio of $Y(j\Omega)$ and $X(j\Omega)$, the transfer function of LTI continuous time system in frequency domain is obtained.

Impulse Response and Transfer Function

Consider an LTI continuous time system, $\mathcal{H}$ shown in fig 4.38. Let x(t) and y(t) be the input and output of the system respectively.

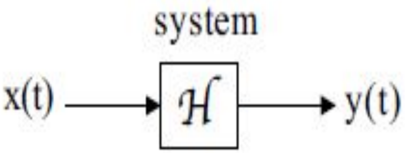


Fig 4.38.

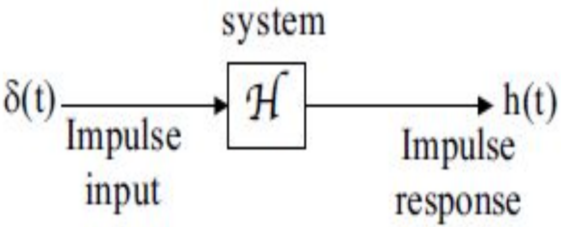


Fig 4.39.

For a continuous time system $\mathcal{H}$, if the input is impulse signal δ(t) as shown in fig 4.39, then the output is called ***impulse response***, which is denoted by h(t).

The importance of impulse response is that the response for any input to LTI system is given by convolution of input and impulse response.

Symbolically, the convolution operation is denoted as,

$$y(t) = x(t) * h(t) \qquad .....(4.67)$$

where, " $*$ " is the symbol for convolution.

Mathematically, the convolution operation is defined as,

$$y(t) = x(t) * h(t) = \int_{-\infty}^{+\infty} x(\tau)\, h(t - \tau)\, d\tau$$

where, τ is the dummy variable for integration.

Let, $H(j\Omega)$ = Fourier transform of $h(t)$

$X(j\Omega)$ = Fourier transform of $x(t)$

$Y(j\Omega)$ = Fourier transform of $y(t)$

Now, by convolution property of Fourier transform we get,

$$\mathcal{F}\{x(t) * h(t)\} = X(j\Omega)\, H(j\Omega) \qquad .....(4.68)$$

Using equation (4.67), the equation (4.68) can be written as,

$$\mathcal{F}\{y(t)\} = X(j\Omega)\, H(j\Omega)$$

$$\therefore\ Y(j\Omega) = X(j\Omega)\, H(j\Omega)$$

$$\therefore\ H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)} \qquad .....(4.69)$$

From equations (4.66) and (4.69) we can say that the ***transfer function in frequency domain*** is given by Fourier transform of impulse response.

$$\boxed{\therefore\ H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}} \qquad .....(4.70)$$

Response of LTI Continuous Time System Using Fourier Transform

Consider the transfer function of LTI continuous time system, $H(j\Omega)$.

$$H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}$$

Now, response in frequency domain, $Y(j\Omega) = H(j\Omega)\,X(j\Omega)$

The response function $Y(j\Omega)$ will be a rational function of $j\Omega$, and so $Y(j\Omega)$ can be expressed as a ratio of two factorized polynomial in $j\Omega$ as shown below.

$$Y(j\Omega) = \frac{(j\Omega + z_1)\,(j\Omega + z_2)\,(j\Omega + z_3)\,.....}{(j\Omega + p_1)\,(j\Omega + p_2)\,(j\Omega + p_3)\,.....} \qquad .....(4.71)$$

By partial fraction expansion technique the equation (4.71) can be expressed as shown below.

$$Y(j\Omega) = \frac{k_1}{j\Omega + p_1} + \frac{k_2}{j\Omega + p_2} + \frac{k_3}{j\Omega + p_3} + \,..... \qquad .....(4.72)$$

where, k_1, k_2, k_3, ... are residues.

Now the time domain response y(t) can be obtained by taking inverse Fourier transform of equation (4.72). The inverse Fourier transform of each term in equation (4.72) can be obtained by comparing the terms with standard Fourier transform pair listed in table 4.5.

From table - 4.5, we get, $\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \dfrac{1}{a + j\Omega}$ .....(4.73)

Using equation (4.73), the inverse Fourier transform of equation (4.72) can be obtained as shown below.

$$y(t) = k_1\, e^{-p_1 t}\, u(t) + k_2\, e^{-p_2 t}\, u(t) + k_3\, e^{-p_3 t}\, u(t) + \ldots.. \quad .....(4.74)$$

Since the transfer function is defined with zero initial conditions, the response obtained by using equation (4.74) is the time domain steady state (or forced) response of the LTI continuous time system.

Note: *Only steady state or forced response alone can be computed via frequency domain*

Frequency Response of LTI Continuous Time System

The output y(t) of an LTI continuous time system is given by convolution of h(t) and x(t).

$$\therefore y(t) = x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{+\infty} h(\tau)\; x(t - \tau)\, d\tau \qquad .....(4.75)$$

Consider a special class of input (sinusoidal input), $\boxed{Ae^{j\Omega t} = A(\cos\Omega t + j\sin\Omega t)}$

$$x(t) = A\, e^{j\Omega t} \qquad .....(4.76)$$

where, A = Amplitude ; Ω = Angular frequency in rad/sec

$$\therefore x(t - \tau) = Ae^{j\Omega(t - \tau)} \qquad .....(4.77)$$

On substituting for $x(t - \tau)$ from equation (4.77) in equation (4.75) we get,

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\; Ae^{j\Omega(t - \tau)}\, d\tau$$

$$\therefore y(t) = \int_{-\infty}^{+\infty} h(\tau)\, Ae^{j\Omega t}\, e^{-j\Omega\tau} d\tau$$

$$= Ae^{j\Omega t} \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega\tau} d\tau \qquad .....(4.78)$$

By the definition of Fourier transform,

$\boxed{\text{Replace t by } \tau.}$

$$H(j\Omega) = \mathcal{F}\{h(t)\} = \int_{-\infty}^{+\infty} h(t)\, e^{-j\Omega t} dt = \int_{-\infty}^{+\infty} h(\tau)\, e^{-j\Omega\tau} d\tau \qquad .....(4.79)$$

Using equations (4.76) and (4.79), the equation (4.78) can be written as,

$$y(t) = x(t)\ H(j\Omega) \qquad .....(4.80)$$

From equation (4.80) we can say that if a complex sinusoidal signal is given as input signal to an LTI continuous time system, then the output is also a sinusoidal signal of the same frequency modified by $H(j\Omega)$. Hence $H(j\Omega)$ is called the ***frequency response*** of the continuous time system.

Since the $H(j\Omega)$ is a complex function of Ω, the multiplication of $H(j\Omega)$ with input produces a change in the amplitude and phase of the input signal, and the modified input signal is the output signal. Therefore, an LTI system is characterized in the frequency domain by its frequency response.

The function $H(j\Omega)$ is a complex quantity and so it can be expressed as magnitude function and phase function.

$$\therefore H(j\Omega) = |H(j\Omega)| \angle H(j\Omega)$$

where, $|H(j\Omega)|$ = Magnitude function

$\angle H(j\Omega)$ = Phase function

The sketch of magnitude function and phase function with respect to Ω will give the frequency response graphically.

Let, $H(j\Omega) = H_r(j\Omega) + jH_i(j\Omega)$

where, $H_r(j\Omega)$ = Real part of $H(j\Omega)$

$H_i(j\Omega)$ = Imaginary part of $H(j\Omega)$

The ***magnitude function*** is defined as,

$$|H(j\Omega)|^2 = H(j\Omega)\, H^*(j\Omega) = [H_r(j\Omega) + jH_i(j\Omega)]\, [H_r(j\Omega) - jH_i(j\Omega)]$$

where, $H^*(j\Omega)$ is complex conjugate of $H(j\Omega)$

$$\therefore |H(j\Omega)|^2 = H_r^2(j\Omega) + H_i^2(j\Omega) \quad \Rightarrow \quad |H(j\Omega)| = \sqrt{H_r^2(j\Omega) + H_i^2(j\Omega)}$$

The ***phase function*** is defined as,

$$\angle H(j\Omega) = \text{Arg}[H(j\Omega)] = \tan^{-1}\left[\frac{H_i(j\Omega)}{H_r(j\Omega)}\right]$$

From equation (4.70) we can say that the frequency response $H(j\Omega)$ of an LTI continuous time system is same as transfer function in frequency domain and so, the frequency response is also given by the ratio of Fourier transform of output to Fourier transform of input.

$$\boxed{\therefore \text{ Frequency response, } H(j\Omega) = \frac{Y(j\Omega)}{X(j\Omega)}} \qquad .....(4.81)$$

Advantages of frequency response analysis

1. The practical testing of systems can be easily carried with available sinusoidal signal generators and precise measurement equipments .
2. The transfer function of complicated systems can be determined experimentally by frequency response tests.
3. The design and parameter adjustment is carried out more easily in frequency domain.
4. In frequency domain, the effects of noise disturbance and parameter variations are relatively easy to visualize and incorporate corrective measures.
5. The frequency response analysis and designs can be extended to certain nonlinear systems.

Relation Between Fourier and Laplace Transform

Let x(t) be a continuous time signal, defined for all t.

The definition of Laplace tranform of x(t) is,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

On substituting $s = \sigma + j\Omega$ in the above definition of Laplace tranform we get,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-(\sigma + j\Omega)t}\, dt \qquad .....(4.82)$$

The definition of Fourier transform of x(t) is,

$$\mathcal{F}\{x(t)\} = X(j\Omega) = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt \qquad .....(4.83)$$

On comparing equations (4.82) and (4.83) we can say that, the Fourier transform of a continuous time signal, is obtained by letting $\sigma = 0$ (i.e., $s = j\Omega$) in the Laplace transform. Summary of this relation is presented in table 4.6, for causal signals.

$$\boxed{\therefore\ X(j\Omega) = X(s)\big|_{s = j\Omega}}$$

Since $s = \sigma + j\Omega$, we can say that, the Laplace transform is a generalized transform and Fourier transform is a particular transform when $s = j\Omega$. Since $s = j\Omega$ represents the points on an imaginary axis in the s-plane, we can say that, the Fourier transform is an evaluation of the Laplace transform along the imaginary axis in the s-plane.

Since Fourier transform is evaluation of Laplace transform along imaginary axis, the ROC of X(s) should include the imaginary axis. For all causal signals, the imaginary axis is included in ROC. Therefore for all causal signals the Fourier transform exist.

Summary of Laplace and Fourier Tranform for Causal Signals

$x(t)$ for $t = 0$ to ∞	$X(s)$	$X(j\Omega)$ $\left[X(j\Omega) = X(s)\big\|_{s = j\Omega}\right]$
$\delta(t)$	1	1
$u(t)$	$\frac{1}{s}$	$\frac{1}{j\Omega}$
$t\,u(t)$	$\frac{1}{s^2}$	$\frac{1}{(j\Omega)^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{s^n}$	$\frac{1}{(j\Omega)^n}$
$t^n u(t)$ where, $n = 1, 2, 3, ...$	$\frac{n!}{s^{n+1}}$	$\frac{n!}{(j\Omega)^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\frac{1}{j\Omega + a}$

$t\,e^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\frac{1}{(j\Omega+a)^2}$
$\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2+\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2+\Omega_0^2}=\frac{\Omega_0}{\Omega_0^2-\Omega^2}$
$\cos\Omega_0 t\;u(t)$	$\frac{s}{s^2+\Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2+\Omega_0^2}=\frac{j\Omega}{\Omega_0^2-\Omega^2}$
$\sinh\Omega_0 t\,u(t)$	$\frac{\Omega_0}{s^2-\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega)^2-\Omega_0^2}=\frac{-\Omega_0}{\Omega^2+\Omega_0^2}$
$\cosh\Omega_0 t\,u(t))$	$\frac{s}{s^2-\Omega_0^2}$	$\frac{j\Omega}{(j\Omega)^2-\Omega_0^2}=\frac{-j\Omega}{\Omega^2+\Omega_0^2}$
$e^{-at}\sin\Omega_0 t\,u(t)$	$\frac{\Omega_0}{(s+a)^2+\Omega_0^2}$	$\frac{\Omega_0}{(j\Omega+a)^2+\Omega_0^2}$
$e^{-at}\cos\Omega_0 t\,u(t)$	$\frac{s+a}{(s+a)^2+\Omega_0^2}$	$\frac{j\Omega+a}{(j\Omega+a)^2+\Omega_0^2}$

Example 4.13

Determine the Fourier transform of following continuous time domain signals.

a) $x(t) = 1 - t^2$; for $|t| < 1$

$= 0$; for $|t| > 1$

b) $x(t) = e^{-at} \cos \Omega_0 t \, u(t)$

Solution

a) Given that, $x(t) = 1 - t^2$; for $|t| < 1$

$\therefore x(t) = 1 - t^2$; for $t = -1$ to $+1$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-1}^{+1} (1 - t^2)\, e^{-j\Omega t}\, dt = \int_{-1}^{+1} e^{-j\Omega t}\, dt - \int_{-1}^{+1} t^2\, e^{-j\Omega t}\, dt$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[t^2\, \frac{e^{-j\Omega t}}{-j\Omega} - \int 2t\, \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{-1}^{1}$$

$$\boxed{\int uv = u\int v - \int \left[du \int v\right]}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{j\Omega}\int t\, e^{-j\Omega t}\, dt\right]_{-1}^{1}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{j\Omega}\left(t\, \frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right)\right]_{-1}^{1}$$

$$\boxed{\int uv = u\int v - \int \left[du \int v\right]}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2}{(j\Omega)^2}\left(-\, t\, e^{-j\Omega t} + \int e^{-j\Omega t}\, dt\right)\right]_{-1}^{1}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} - \frac{2}{\Omega^2}\left(-\, t\, e^{-j\Omega t} + \frac{e^{-j\Omega t}}{-j\Omega}\right)\right]_{-1}^{1}$$

$$= \left[-\frac{e^{-j\Omega t}}{j\Omega}\right]_{-1}^{1} - \left[-\frac{t^2 e^{-j\Omega t}}{j\Omega} + \frac{2t\, e^{-j\Omega t}}{\Omega^2} + \frac{2e^{-j\Omega t}}{j\Omega^3}\right]_{-1}^{1}$$

$$= -\frac{e^{-j\Omega}}{j\Omega} + \frac{e^{j\Omega}}{j\Omega} - \left[-\frac{e^{-j\Omega}}{j\Omega} + \frac{2\, e^{-j\Omega}}{\Omega^2} + \frac{2\, e^{-j\Omega}}{j\Omega^3} + \frac{e^{j\Omega}}{j\Omega} + \frac{2\, e^{j\Omega}}{\Omega^2} - \frac{2e^{j\Omega}}{j\Omega^3}\right]$$

$$= -\frac{e^{-j\Omega}}{j\Omega} + \frac{e^{j\Omega}}{j\Omega} + \frac{e^{-j\Omega}}{j\Omega} - \frac{2\, e^{-j\Omega}}{\Omega^2} - \frac{2\, e^{-j\Omega}}{j\Omega^3} - \frac{e^{j\Omega}}{j\Omega} - \frac{2\, e^{j\Omega}}{\Omega^2} + \frac{2\, e^{j\Omega}}{j\Omega^3}$$

$$= -\frac{2}{\Omega^2}\left(e^{j\Omega} + e^{-j\Omega}\right) + \frac{2}{j\Omega^3}\left(e^{j\Omega} - e^{-j\Omega}\right)$$

$\sin\theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$	$\cos\theta = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

$$= -\frac{2}{\Omega^2}\, 2\cos\Omega + \frac{2}{j\Omega^3}\, 2j\sin\Omega$$

$$= -\frac{4\cos\Omega}{\Omega^2} + \frac{4\sin\Omega}{\Omega^3}$$

$$= \frac{4}{\Omega^2}\left(\frac{\sin\Omega}{\Omega} - \cos\Omega\right)$$

(b) Given that, $\mathbf{x(t) = e^{-at} \cos\Omega_0 t \, u(t)}$

Since $u(t) = 1$, for $t \geq 0$, we can write,

$$x(t) = e^{-at} \cos \Omega_0 t \quad ; \quad \text{for} \quad t \geq 0$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_0^{\infty} e^{-at} \cos\Omega_0 t\, e^{-j\Omega t}\, dt = \int_0^{\infty} e^{-at} \left(\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} \right) e^{-j\Omega t}\, dt$$

$$= \frac{1}{2} \int_0^{\infty} e^{-at}\, e^{j\Omega_0 t}\, e^{-j\Omega t}\, dt + \frac{1}{2} \int_0^{\infty} e^{-at}\, e^{-j\Omega_0 t}\, e^{-j\Omega t}\, dt$$

$$= \frac{1}{2} \int_0^{\infty} e^{-(a - j\Omega_0 + j\Omega)t}\, dt + \frac{1}{2} \int_0^{\infty} e^{-(a + j\Omega_0 + j\Omega)t}\, dt$$

$$= \frac{1}{2} \left[\frac{e^{-(a - j\Omega_0 + j\Omega)t}}{-(a - j\Omega_0 + j\Omega)} \right]_0^{\infty} + \frac{1}{2} \left[\frac{e^{-(a + j\Omega_0 + j\Omega)t}}{-(a + j\Omega_0 + j\Omega)} \right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(a - j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a - j\Omega_0 + j\Omega)}\right] + \frac{1}{2}\left[\frac{e^{-\infty}}{-(a + j\Omega_0 + j\Omega)} - \frac{e^{0}}{-(a + j\Omega_0 + j\Omega)}\right]$$

$$= \frac{1}{2}\left[0 + \frac{1}{a - j\Omega_0 + j\Omega}\right] + \frac{1}{2}\left[0 + \frac{1}{a + j\Omega_0 + j\Omega}\right] \qquad \boxed{e^{-\infty} = 0\,;\ e^{0} = 1}$$

$$= \frac{1}{2}\left[\frac{1}{(a + j\Omega) - j\Omega_0} + \frac{1}{(a + j\Omega) + j\Omega_0}\right] \qquad \boxed{(a+b)(a-b) = a^2 - b^2 \ \Big|\ j^2 = -1}$$

$$= \frac{1}{2}\left[\frac{(a + j\Omega) + j\Omega_0 + (a + j\Omega) - j\Omega_0}{(a + j\Omega)^2 + \Omega_0^2}\right]$$

$$= \frac{1}{2}\,\frac{2\,(a + j\Omega)}{(a + j\Omega)^2 + \Omega_0^2} = \frac{a + j\Omega}{(a + j\Omega)^2 + \Omega_0^2}$$

Example 4.14

Determine the Fourier transform of the rectangular pulse shown in fig 4.14.1.

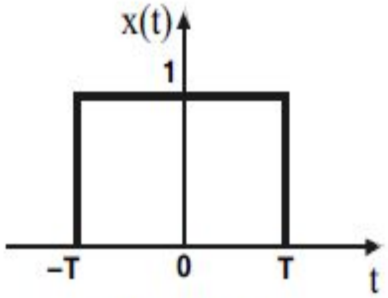


Fig 4.14.1.

Solution

The mathematical equation of the rectangular pulse is,

$$x(t) = 1 \quad ; \quad \text{for } t = -T \text{ to } +T$$

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{+T} 1 \times e^{-j\Omega t}\, dt = \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{+T}$$

$$= \frac{e^{-j\Omega T}}{-j\Omega} - \frac{e^{j\Omega T}}{-j\Omega} = \frac{1}{j\Omega}\left(e^{j\Omega T} - e^{-j\Omega T}\right) = \frac{1}{j\Omega}\, 2j\, \sin\Omega T$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$= 2\,\frac{\sin\Omega T}{\Omega} = 2T\,\frac{\sin\Omega T}{\Omega T}$$

$$\boxed{\frac{\sin\theta}{\theta} = \operatorname{sinc}\theta}$$

$$= 2T\, \operatorname{sinc}\Omega T$$

Example 4.15

Determine the Fourier transform of the triangular pulse shown in fig 4.15.1.

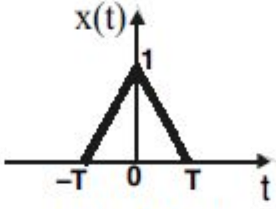


Fig 4.15.1.

Solution

The mathematical equation of triangular pulse is,

$$x(t) = 1 + \frac{t}{T} \quad ; \quad \text{for } t = -T \text{ to } 0$$

$$= 1 - \frac{t}{T} \quad ; \quad \text{for } t = 0 \text{ to } T$$

(Please refer example 4.11 for the mathematical equation of triangular pulse).

By definition of Fourier transform,

$$\mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-T}^{0} \left(1 + \frac{t}{T}\right) e^{-j\Omega t}\, dt + \int_{0}^{T} \left(1 - \frac{t}{T}\right) e^{-j\Omega t}\, dt$$

$$= \int_{-T}^{0} e^{-j\Omega t}\, dt + \frac{1}{T}\int_{-T}^{0} t\, e^{-j\Omega t}\, dt + \int_{0}^{T} e^{-j\Omega t}\, dt - \frac{1}{T}\int_{0}^{T} t\, e^{-j\Omega t}\, dt$$

$$\boxed{\int uv = u\int v - \int \left[du \int v\right]}$$

$$= \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} + \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{-T}^{0} + \left[\frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T} - \frac{1}{T}\left[t\,\frac{e^{-j\Omega t}}{-j\Omega} - \int 1 \times \frac{e^{-j\Omega t}}{-j\Omega}\, dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\,e^{-j\Omega t} - \int e^{-j\Omega t}\,dt\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\,e^{-j\Omega t} - \int e^{-j\Omega t}\,dt\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{-T}^{0} - \frac{1}{j\Omega T}\left[t\,e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{-T}^{0} - \frac{1}{j\Omega}\left[e^{-j\Omega t}\right]_{0}^{T} + \frac{1}{j\Omega T}\left[t\,e^{-j\Omega t} - \frac{e^{-j\Omega t}}{-j\Omega}\right]_{0}^{T}$$

$$= -\frac{1}{j\Omega}\left[e^{0} - e^{j\Omega T}\right] - \frac{1}{j\Omega T}\left[0 - \frac{e^{0}}{-j\Omega} + T\,e^{j\Omega T} + \frac{e^{j\Omega T}}{-j\Omega}\right] - \frac{1}{j\Omega}\left[e^{-j\Omega T} - e^{0}\right]$$

$$+ \frac{1}{j\Omega T}\left[T\,e^{-j\Omega T} - \frac{e^{-j\Omega T}}{-j\Omega} - 0 + \frac{e^{0}}{-j\Omega}\right]$$

$$= -\frac{1}{j\Omega} + \frac{e^{j\Omega T}}{j\Omega} - 0 + \frac{1}{T\Omega^2} - \frac{e^{j\Omega T}}{j\Omega} - \frac{e^{j\Omega T}}{T\Omega^2} - \frac{e^{-j\Omega T}}{j\Omega} + \frac{1}{j\Omega} + \frac{e^{-j\Omega T}}{j\Omega} - \frac{e^{-j\Omega T}}{T\Omega^2} - 0 + \frac{1}{T\Omega^2}$$

$$= \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\left(e^{j\Omega T} + e^{-j\Omega T}\right) = \frac{2}{T\Omega^2} - \frac{1}{T\Omega^2}\,2\cos\Omega T \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right)$$

Alternatively the above result can be expressed as shown below.

$$\mathcal{F}\{x(t)\} = \frac{2}{T\Omega^2}\left(1 - \cos\Omega T\right) = \frac{2}{T\Omega^2}\left(1 - \cos 2\left(\frac{\Omega T}{2}\right)\right)$$

$$= \frac{2}{T\Omega^2}\left(2\sin^2\frac{\Omega T}{2}\right) = T\,\frac{4}{T^2\Omega^2}\sin^2\frac{\Omega T}{2} = T\,\frac{\sin^2\left(\frac{\Omega T}{2}\right)}{\left(\frac{\Omega T}{2}\right)^2} \qquad \boxed{\sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$= T\left(\frac{\sin\frac{\Omega T}{2}}{\frac{\Omega T}{2}}\right)^2 = T\left(\operatorname{sinc}\frac{\Omega T}{2}\right)^2 \qquad \boxed{\frac{\sin\theta}{\theta} = \operatorname{sinc}\theta}$$

Example 4.16

Determine the inverse Fourier transform of the following functions, using partial fraction expansion technique.

a) $X(j\Omega) = \dfrac{3(j\Omega) + 14}{(j\Omega)^2 + 7(j\Omega) + 12}$ b) $X(j\Omega) = \dfrac{j\Omega + 7}{(j\Omega + 3)^2}$

Solution

a) Given that, $\mathbf{X(j\Omega)} = \dfrac{\mathbf{3(j\Omega) + 14}}{\mathbf{(j\Omega)^2 + 7(j\Omega) + 12}} = \dfrac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)}$

By partial fraction expansion technique we can write,

$$X(j\Omega) = \frac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)} = \frac{k_1}{j\Omega + 3} + \frac{k_2}{j\Omega + 4}$$

$$k_1 = \frac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)} \times (j\Omega + 3)\Bigg|_{j\Omega = -3} = \frac{3(-3) + 14}{-3 + 4} = 5$$

$$k_2 = \frac{3(j\Omega) + 14}{(j\Omega + 3)(j\Omega + 4)} \times (j\Omega + 4)\Bigg|_{j\Omega = -4} = \frac{3(-4) + 14}{-4 + 3} = -2$$

$$\therefore X(j\Omega) = \frac{5}{j\Omega + 3} - \frac{2}{j\Omega + 4}$$

We know that, $\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \dfrac{1}{j\Omega + a}$

Using equation (2), the inverse Fourier transform of equation (1) is,

$$x(t) = 5\,e^{-3t}\,u(t) - 2\,e^{-4t}\,u(t)$$

b) Given that, $\mathbf{X(j\Omega) = \dfrac{j\Omega + 7}{(j\Omega + 3)^2}}$

By partial fraction expansion technique $X(j\Omega)$ can be written as,

$$\therefore\ X(j\Omega) = \frac{k_1}{(j\Omega + 3)^2} + \frac{k_2}{j\Omega + 3}$$

$$k_1 = \frac{j\Omega + 7}{(j\Omega + 3)^2} \times (j\Omega + 3)^2 \Bigg|_{j\Omega = -3} = -3 + 7 = 4$$

$$k_2 = \frac{d}{d(j\Omega)}\left[\frac{j\Omega + 7}{(j\Omega + 3)^2} \times (j\Omega + 3)^2\right]\Bigg|_{j\Omega = -3} = \frac{d}{d(j\Omega)}[j\Omega + 7]\Bigg|_{j\Omega = -3} = 1$$

$$\therefore\ X(j\Omega) = \frac{4}{(j\Omega + 3)^2} + \frac{1}{j\Omega + 3} \qquad .....(3)$$

We know that, $\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \dfrac{1}{j\Omega + a}$ .....(4)

$$\mathcal{F}\left\{t\,e^{-at}\,u(t)\right\} = \frac{1}{(j\Omega + a)^2} \qquad .....(5)$$

Using equations (4) and (5), the inverse Fourier transform of equation (3) is,

$$x(t) = 4t\,e^{-3t}\,u(t) + e^{-3t}\,u(t) = (4t + 1)\,e^{-3t}\,u(t)$$

Example 4.17

Determine the convolution of $x_1(t) = e^{-2t}\,u(t)$ and $x_2(t) = e^{-6t}\,u(t)$, using Fourier transform.

Solution

Let, $X_1(j\Omega)$ = Fourier transform of $x_1(t)$

$X_2(j\Omega)$ = Fourier transform of $x_2(t)$

By convolution property of Fourier transform,

$$\mathcal{F}\{x_1(t) * x_2(t)\} = X_1(j\Omega)\,X_2(j\Omega)$$

Let,
$$\begin{aligned} X(j\Omega) &= X_1(j\Omega)\,X_2(j\Omega) \\ &= \mathcal{F}\{e^{-2t}\,u(t)\} \times \mathcal{F}\{e^{-6t}\,u(t)\} \\ &= \frac{1}{j\Omega + 2} \times \frac{1}{j\Omega + 6} \end{aligned}$$

By partial fraction expansion technique $X(j\Omega)$ can be expressed as,

$$X(j\Omega) = \frac{1}{(j\Omega + 2)\,(j\Omega + 6)} = \frac{k_1}{j\Omega + 2} + \frac{k_2}{j\Omega + 6}$$

$$k_1 = \frac{1}{(j\Omega + 2)\,(j\Omega + 6)} \times (j\Omega + 2)\Bigg|_{j\Omega = -2} = \frac{1}{-2 + 6} = \frac{1}{4} = 0.25$$

$$k_2 = \frac{1}{(j\Omega + 2)(j\Omega + 6)} \times (j\Omega + 6)\Bigg|_{j\Omega = -6} = \frac{1}{-6 + 2} = -\frac{1}{4} = -0.25$$

$$\therefore\ X(j\Omega) = \frac{0.25}{j\Omega + 2} - \frac{0.25}{j\Omega + 6}$$

On taking inverse Fourier transform of the above equation we get,

$$x(t) = 0.25\, e^{-2t}\, u(t) - 0.25\, e^{-6t}\, u(t)$$

$$= 0.25(e^{-2t} - e^{-6t})\, u(t)$$

$$\boxed{\mathcal{F}\left\{e^{-at}\, u(t)\right\} = \frac{1}{j\Omega + a}}$$

Example 4.18

The impulse response of an LTI system is $h(t) = 2\,e^{-3t}u(t)$.

Find the response of the system for the input $x(t) = 2e^{-5t}u(t)$, using Fourier transform.

Solution

Given that, $x(t) = 2\,e^{-5t}u(t)$.

$$\therefore\ X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\left\{2\,e^{-5t}\,u(t)\right\} = \frac{2}{j\Omega + 5} \qquad .....(1)$$

$$\boxed{\mathcal{F}\left\{e^{-at}\,u(t)\right\} = \frac{1}{j\Omega + a}}$$

Given that, $h(t) = 2\,e^{-3t}u(t)$.

$$\therefore\ H(j\Omega) = \mathcal{F}\{h(t)\} = \mathcal{F}\left\{2\,e^{-3t}\,u(t)\right\} = \frac{2}{j\Omega + 3} \qquad .....(2)$$

For LTI system, the response, $y(t) = = x(t) * h(t)$ .....(3)

On taking Fourier transform of equation (3) we get,

$$\mathcal{F}\{y(t)\} = \mathcal{F}\{x(t) * h(t)\}$$

Let, $\mathcal{F}\{y(t)\} = Y(j\Omega)$.

$$\therefore Y(j\Omega) = \mathcal{F}\{x(t) * h(t)\}$$

Using convolution property of Fourier transform.

$$= X(j\Omega)\, H(j\Omega)$$

$$= \frac{2}{j\Omega + 5} \times \frac{2}{j\Omega + 3} = \frac{4}{(j\Omega + 5)(j\Omega + 3)}$$

Using equations (1) and (2)

By partial fraction expansion technique, the above equation can be written as,

$$Y(j\Omega) = \frac{4}{(j\Omega + 5)(j\Omega + 3)} = \frac{k_1}{j\Omega + 5} + \frac{k_2}{j\Omega + 3}$$

$$k_1 = \frac{4}{(j\Omega + 5)(j\Omega + 3)} \times (j\Omega + 5)\Bigg|_{j\Omega = -5} = \frac{4}{-5 + 3} = -2$$

$$k_2 = \frac{4}{(j\Omega + 5)(j\Omega + 3)} \times (j\Omega + 3)\Bigg|_{j\Omega = -3} = \frac{4}{-3 + 5} = 2$$

$$\therefore Y(j\Omega) = -\frac{2}{j\Omega + 5} + \frac{2}{j\Omega + 3}$$

On taking inverse Fourier transform of $Y(j\Omega)$ we get $y(t)$.

$$y(t) = \mathcal{F}^{-1}\{Y(j\Omega)\} = \mathcal{F}^{-1}\left\{-\frac{2}{j\Omega + 5} + \frac{2}{j\Omega + 3}\right\}$$

$$= -2\, e^{-5t}\, u(t) + 2\, e^{-3t}\, u(t) = 2\left(e^{-3t} - e^{-5t}\right) u(t)$$

Example 4.20

Determine the Fourier transform of the periodic impulse function shown in fig 4.20.1.

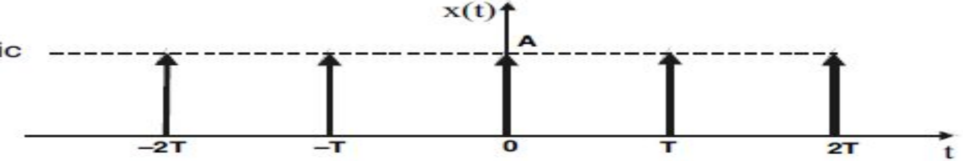


Fig 4.20.1.

Solution

The mathematical equation for one period of the periodic impulse function is,

$$x(t) = A\,\delta(t) \quad ; \quad \text{for } t = -\frac{T}{2} \text{ to } +\frac{T}{2}$$

The Fourier coefficient c_n is given by,

$$c_n = \frac{1}{T}\int_{-T/2}^{+T/2} x(t)\, e^{-jn\Omega_0 t}\, dt = \frac{1}{T}\int_{-T/2}^{+T/2} A\,\delta(t)\, e^{-jn\Omega_0 t}\, dt = \frac{A}{T}\, e^{-jn\Omega_0 t}\Big|_{t=0} = \frac{A}{T} \quad \boxed{\Omega_0 = \frac{2\pi}{T}} \quad \ldots(1)$$

The Exponential Fourier series representation of the periodic impulse train is,

$$x(t) = \sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t}$$

On taking Fourier transform of the above equation we get,

$$\mathcal{F}\{x(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{+\infty} c_n\, e^{jn\Omega_0 t}\right\}$$

$$\therefore X(j\Omega) = \sum_{n=-\infty}^{+\infty} c_n\, \mathcal{F}\{e^{jn\Omega_0 t}\} \quad \boxed{\mathcal{F}\{x(t)\} = X(j\Omega)}$$

$$= \sum_{n=-\infty}^{+\infty} c_n\, 2\pi\,\delta(\Omega - n\Omega_0) \quad \boxed{\mathcal{F}\{e^{jn\Omega_0 t}\} = 2\pi\,\delta(\Omega - n\Omega_0)}$$

$$= \sum_{n=-\infty}^{+\infty} \frac{A}{T}\, 2\pi\,\delta(\Omega - n\Omega_0) = \sum_{n=-\infty}^{+\infty} A\Omega_0\, \delta(\Omega - n\Omega_0)$$

On substituting for c_n from equation (1)

The magnitude spectrum of $X(j\Omega)$ is shown in fig 1, which is also a periodic impulse function of Ω.

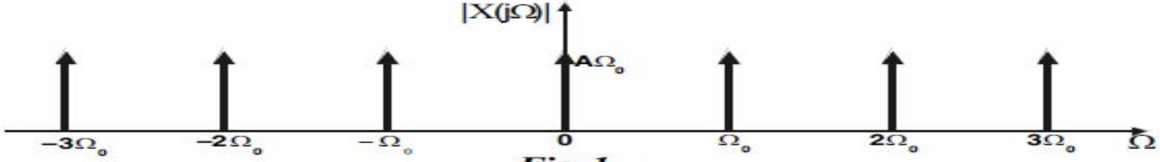


Fig 1.

Example 4.21

Find the Fourier transform and sketch the magnitude and phase spectrum for the signal, $x(t) = e^{-at}\,u(t)$.

Solution

Given that, $x(t) = e^{-at}\,u(t)$.

The Fourier transform of x(t) is, (refer section 4.11 for Fourier transform of $e^{-at}\,u(t)$).

$$X(j\Omega) = \mathcal{F}\{x(t)\} = \mathcal{F}\{e^{-at}\,u(t)\} = \frac{1}{a + j\Omega}$$

$$= \frac{1}{a + j\Omega} \times \frac{a - j\Omega}{a - j\Omega} = \frac{a - j\Omega}{(a + j\Omega)\,(a - j\Omega)}$$

$$= \frac{a - j\Omega}{a^2 + \Omega^2} = \frac{a}{a^2 + \Omega^2} - j\frac{\Omega}{a^2 + \Omega^2}$$

The $X(j\Omega)$ is calculated for a = 0.5 and a = 1.0 and tabluted in table 1and table 2 respectively. Using the values listed in table 1and table 2, the magnitude and phase spectrum are sketched as shown in fig 1 and fig 2 respectively.

Table 1 : Frequency Spectrum for a = 0.5

Ω	$X(j\Omega)$	$\lvert X(j\Omega)\rvert$	$\angle X(j\Omega)$ in rad
−8	$0.0080 + j\,0.12 = 0.12\ \angle 1.506 = 0.12\ \angle 0.48\pi$	0.12	0.48π
−6	$0.014 + j\,0.167 = 0.166\ \angle 1.4866 = 0.166\ \angle 0.473\pi$	0.166	0.473π
−4	$0.03 + j\,0.246 = 0.248\ \angle 1.45 = 0.248\ \angle 0.46\pi$	0.248	0.46π
−3	$0.054 + j\,0.324 = 0.328\ \angle 1.40 = 0.328\ \angle 0.45\pi$	0.328	0.45π
−2	$0.118 + j\,0.47 = 0.485\ \angle 1.325 = 0.485\ \angle 0.422\pi$	0.485	$0.422\,\pi$
−1	$0.4 + j\,0.8 = 0.89\ \angle 1.11 = 0.89\ \angle 0.353\pi$	0.89	0.353π
0	$2 + j\,0 = 2\ \angle 0 = 2\ \angle 0$	2.0	0
1	$0.4 - j\,0.8 = 0.89\ \angle{-1.11} = 0.89\ \angle{-0.353\pi}$	0.89	-0.353π

Ω	$X(j\Omega)$	$\lvert X(j\Omega)\rvert$	$\angle X(j\Omega)$ in rad
2	$0.118 - j\,0.47 \;= 0.485\ \angle{-1.325} \;= 0.485\,\angle{-0.422\pi}$	0.485	-0.422π
3	$0.054 - j\,0.324 \;= 0.328\ \angle{-1.40} \;= 0.282\,\angle{-0.45\,\pi}$	0.282	-0.45π
4	$0.03 - j\,0.246 \;= 0.248\ \angle{-1.45} \;= 0.248\ \angle{-0.46\,\pi}$	0.248	-0.46π
6	$0.014 - j\,0.167 \;= 0.166\ \angle{-1.4866} \;= 0.078\ \angle{-0.473\pi}$	0.166	-0.473π
8	$0.0080 - j\,0.12 \;= 0.12\ \angle{-1.506} \;= 0.12\ \angle{-0.48\,\pi}$	0.12	$-0.48\,\pi$

Table 2 : Frequency Spectrum for a = 1

Ω	$X(j\Omega)$	$\lvert X(j\Omega)\rvert$	$\angle X(j\Omega)$ in rad
-8	$0.015 + j0.123 = 0.124\angle 1.45 = 0.124\angle 0.461\pi$	0.124	0.461π
-6	$0.03 + j0.162 = 0.165\angle 1.39 = 0.165\angle 0.442\pi$	0.165	0.442π
-4	$0.059 + j0.24 = 0.25\angle 1.33 = 0.25\angle 0.423\pi$	0.25	0.423π
-3	$0.1 + j0.3 = 0.316\angle 1.25 = 0.316\angle 0.398\pi$	0.316	0.398π
-2	$0.2 + j0.4 = 0.45\angle 1.11 = 0.45\angle 0.353\pi$	0.45	0.353π
-1	$0.5 + j0.5 = 0.707\angle 0.785 = 0.707\angle 0.25\pi$	0.707	0.25π
0	$1 + j0 = 1\angle 0 = 1\angle 0$	1.0	0

Ω	$X(j\Omega)$	$\lvert X(j\Omega)\rvert$	$\angle X(j\Omega)$ in rad
1	$0.5 - j0.5 = 0.707\angle{-0.785} = 0.707\angle{-0.25\pi}$	0.707	-0.25π
2	$0.2 - j0.4 = 0.45\angle{-1.11} = 0.45\angle{-0.353\pi}$	0.45	-0.353π
3	$0.1 - j0.3 = 0.316\angle{-1.25} = 0.316\angle{-0.398\pi}$	0.316	-0.398π
4	$0.059 - j0.24 = 0.25\angle{-1.33} = 0.25\angle{-0.423\pi}$	0.25	-0.423π
6	$0.03 - j0.162 = 0.165\angle{-1.39} = 0.165\angle{-0.442\pi}$	0.165	-0.442π
8	$0.015 - j0.123 = 0.124\angle{-1.45} = 0.124\angle{-0.461\pi}$	0.124	-0.461π

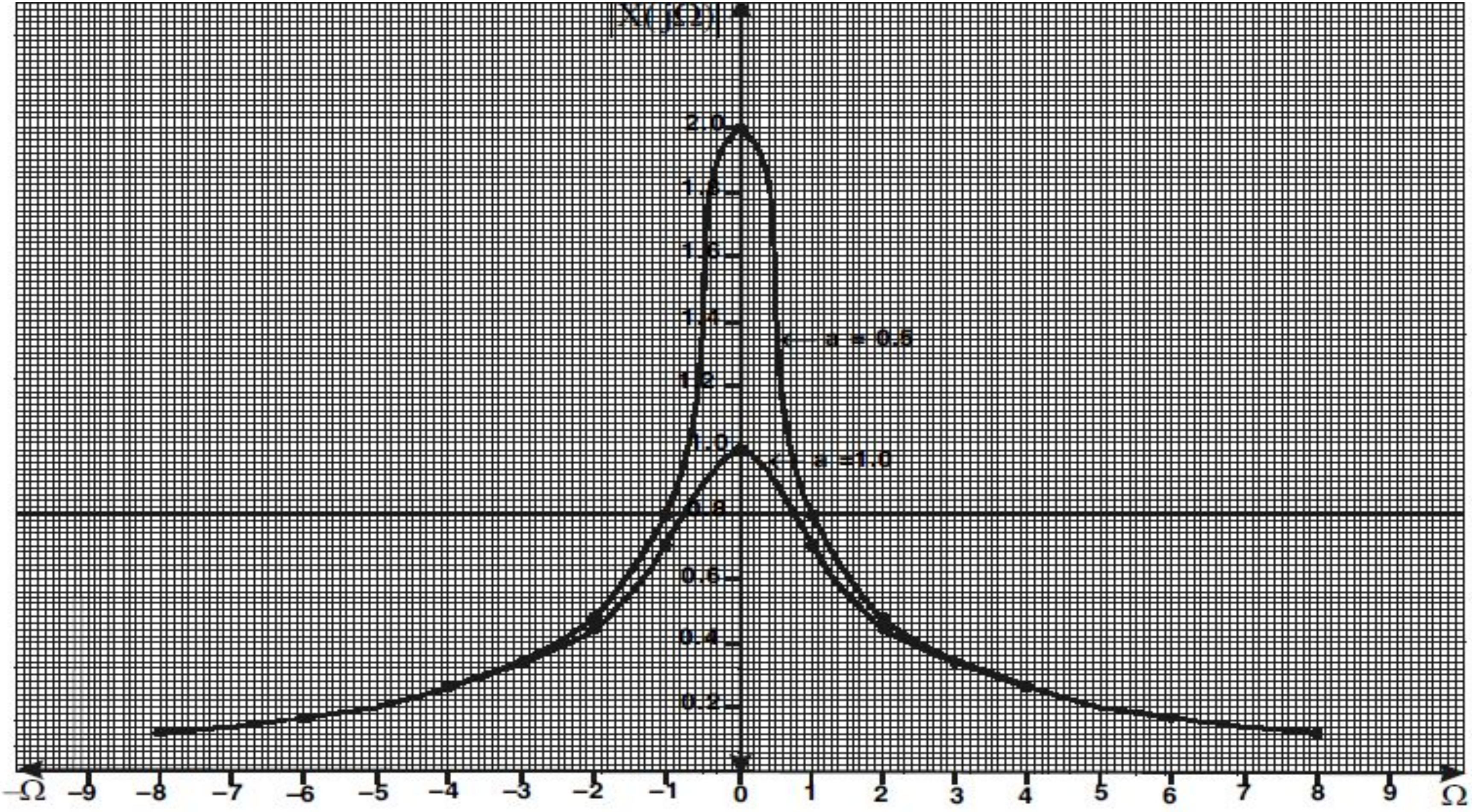


***Fig 1 :** Magnitude spectrum of $X(j\Omega)$.*

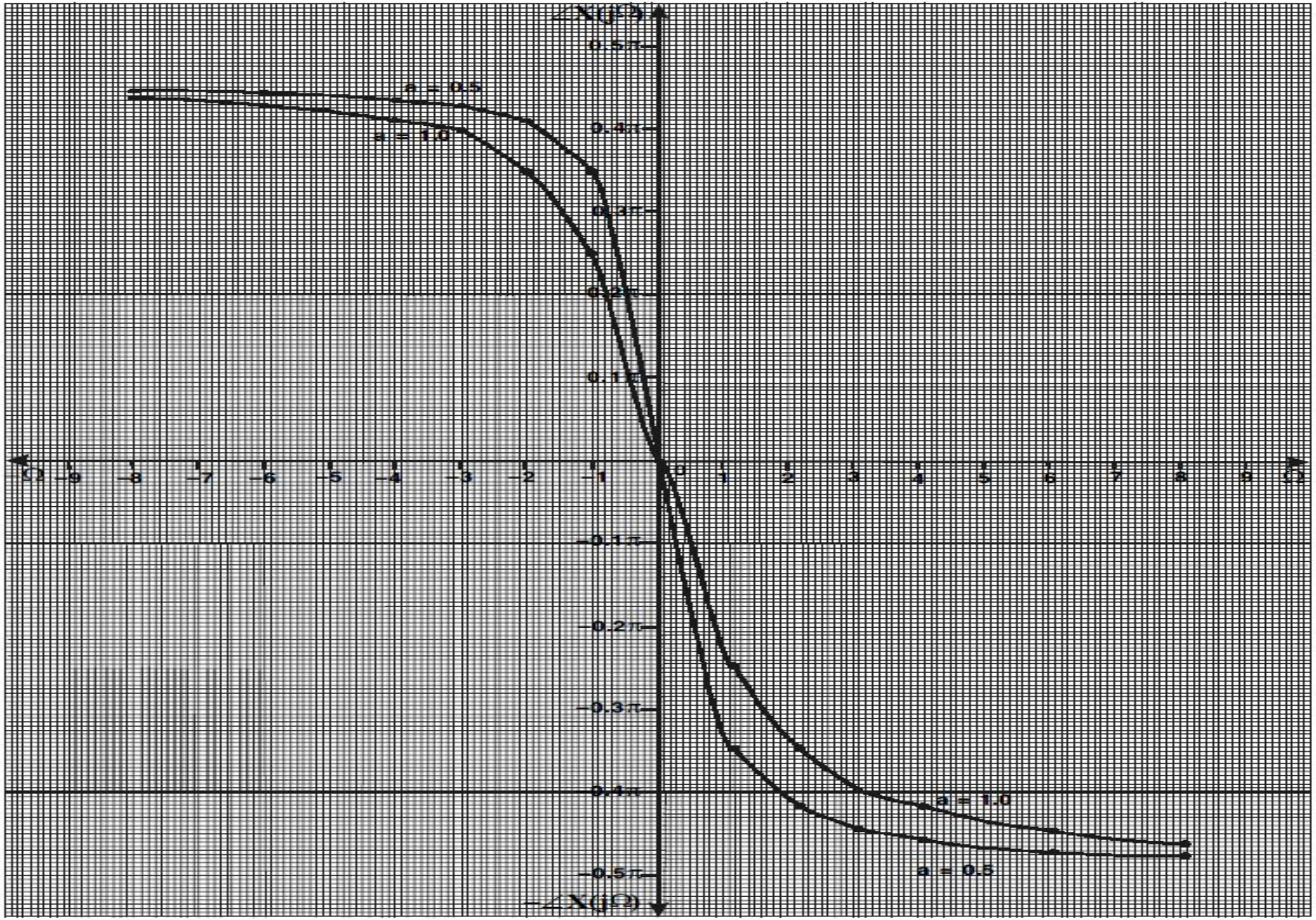


Fig 2: *Phase spectrum of X(jΩ).*

Find the Fourier transform of the signal $e^{-3|t|}u(t)$.

<u>Solution</u>

$$x(t) = e^{-3|t|} = e^{-3t} \text{ for } t > 0$$

$$= e^{3t} \text{ for } t < 0$$

The Fourier transform of x(t) is,

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt = \int_{-\infty}^{0} e^{3t}\, e^{-j\Omega t}\, dt + \int_{0}^{\infty} e^{-3t}\, e^{-j\Omega t}\, dt = \int_{-\infty}^{0} e^{(3-j\Omega)t}\, dt + \int_{0}^{\infty} e^{-(3+j\Omega)t}\, dt$$

$$= \left[\frac{e^{(3-j\Omega)t}}{3-j\Omega}\right]_{-\infty}^{0} + \left[\frac{e^{-(3+j\Omega)t}}{-(3+j\Omega)}\right]_{0}^{\infty} = \left[\frac{e^{0}}{3-j\Omega} - \frac{e^{-\infty}}{3-j\Omega}\right] + \left[-\frac{e^{-\infty}}{3+j\Omega} + \frac{e^{0}}{3+j\Omega}\right]$$

$$\boxed{e^{-\infty} = 0\,;\ \ e^{0} = 1}$$

$$= \frac{1}{3-j\Omega} + \frac{1}{3+j\Omega} = \frac{3+j\Omega+3-j\Omega}{3^2+\Omega^2} = \frac{6}{3^2+\Omega^2}$$

$$\boxed{(a+b)(a-b) = a^2-b^2} \ \boxed{j^2 = -1}$$

For the signal shown in fig Q4.12. Find a) X(j0) b) $\int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega$.

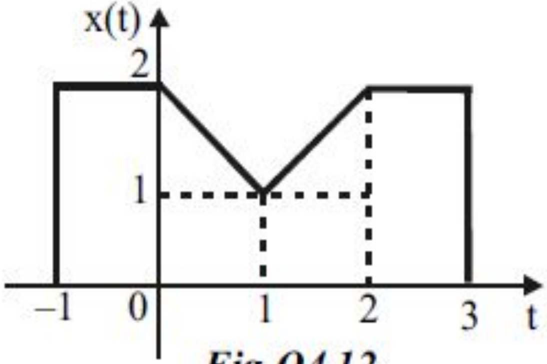


Fig Q4.12.

Solution

a) $X(j0) = \int_{-\infty}^{+\infty} x(t)\, dt$ = Area of the signal

$$= \text{Area of rectangle} - \text{Area of triangle} = 4 \times 2 - \frac{1}{2} \times 2 \times 1 = 8 - 1 = 7$$

b) By definition of inverse Fourier transform,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{j\Omega t}\, d\Omega$$

On letting $t = 0$ in the above equation we get,

$$x(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\Omega)\, e^{0}\, d\Omega$$

$\boxed{e^0 = 1}$

$\boxed{\text{from fig Q4.12 },\ x(0) = 2}$

$$\therefore \int_{-\infty}^{+\infty} X(j\Omega)\, d\Omega = 2\pi \times x(0) = 2\pi \times 2 = 4\pi$$

Find the inverse Fourier transform of, $X(j\Omega) = \dfrac{1}{(4 + j\Omega)^2}$***. Using convolution property.***

<u>Solution</u>

Let, $X(j\Omega) = \dfrac{1}{(4+j\Omega)^2} = \dfrac{1}{4+j\Omega} \times \dfrac{1}{4+j\Omega} = X_1(j\Omega) \times X_2(j\Omega)$

where, $X_1(j\Omega) = \dfrac{1}{4+j\Omega}$ and $X_2(j\Omega) = \dfrac{1}{4+j\Omega}$

$$\therefore\ x_1(t) = \mathcal{F}^{-1}\{X_1(j\Omega)\} = \mathcal{F}^{-1}\left\{\frac{1}{4+j\Omega}\right\} = e^{-4t}\,u(t)$$

$$x_2(t) = e^{-4t}\,u(t)$$

By time convolution property,

$$x(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\,x_1(t-\tau)\,d\tau$$

$$= \int_0^t e^{-4\tau}\,e^{-4(t-\tau)}\,d\tau = \int_0^t e^{-4t}\,e^{-4\tau}e^{4\tau}\,d\tau \quad = e^{-4t}\int_0^t e^{-4\tau+4\tau}\,d\tau$$

$$= e^{-4t}\int_0^t d\tau \ = e^{-4t}[T]_0^t = e^{-4t}[t-0] \ = te^{-4t}\ ;\ \ t \ge 0 \quad = te^{-4t}\,u(t)$$

Using frequency shifting property