Synthesis of RLC Circuits

INTRODUCTION

In the study of electrical networks, broadly there are two topics: 'Network Analysis' and 'Network Synthesis'. Any network consists of excitation, response and network function. In network analysis, network and excitation are given, whereas the response has to be determined. In network synthesis, excitation and response are given, and the network has to be determined. Thus, in network synthesis we are concerned with the realisation of a network for a given excitation-response characteristic. Also, there is one major difference between analysis and synthesis. In analysis, there is a unique solution to the problem. But in synthesis, the solution is not unique and many networks can be realised.

The first step in synthesis procedure is to determine whether the network function can be realised as a physical passive network. There are two main considerations; causality and stability. By *causality* we mean that a voltage cannot appear at any port before a current is applied or vice-versa. In other words, the response of the network must be zero for $t < 0$. For the network to be stable, the network function cannot have poles in the right half of the s-plane. Similarly, a network function cannot have multiple poles on the $j\omega$ axis.

HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when s is real.

(b) The roots of $P(s)$ have real parts which are zero or negative.

Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.

2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.

(a) $s^4 + 4s^3 + 3s + 2$

(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

Solution (a) In the given polynomial, the term s^2 is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

Test whether the polynomial $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ *is Hurwitz.*

Solution Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 3s)\, s^4 + 5s^2 + 4 \;(s$$

$$\underline{s^4 + 3s^2 \quad}$$

$$2s^2 + 4\Big)\, s^3 + 3s \left(\frac{1}{2}s\right.$$

$$\underline{s^3 + 2s}$$

$$s)\, 2s^2 + 4 \;(2s$$

$$\underline{2s^2 \quad}$$

$$4\Big)\, s \left(\frac{1}{4}s\right.$$

$$\underline{s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Test whether the polynomial $P(s) = s^3 + 4s^2 + 5s + 2$ is Hurwitz.

Solution Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)$ $n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$4s^2 + 2 \,\Big)\, s^3 + 5s \,\Big(\, \frac{1}{4}s$$

$$\underline{s^3 + \frac{2}{4}s}$$

$$\frac{9}{2}s \,\Big)\, 4s^2 + 2 \,\Big(\, \frac{8}{9}s$$

$$\underline{4s^2}$$

$$2 \,\Big)\, \frac{9}{2}s \,\Big(\, \frac{9}{4}s$$

$$\underline{\frac{9}{2}s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

Test whether the polynomial $P(s)=s^4+7s^3+6s^2+21s+8$ *is Hurwitz.*

Solution Even part of $P(s)=m(s)=s^4+6s^2+8$

Odd part of $P(s)=n(s)=7s^3+21s$

$$Q(s)=\frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$
\begin{array}{rl}
7s^3+21s\,\Big) & s^4+6s^2+8\Big(\frac{1}{7}s \\
 & \underline{s^4+3s^2} \\
 & 3s^2+8\,\Big)\,7s^3+21s\Big(\frac{7}{3}s \\
 & \qquad\qquad \underline{7s^3+\frac{56}{3}s} \\
 & \qquad\qquad \frac{7}{3}s\,\Big)\,3s^2+8\Big(\frac{9}{7}s \\
 & \qquad\qquad\qquad \underline{3s^2} \\
 & \qquad\qquad\qquad 8\,\Big)\,\frac{7}{3}s\Big(\frac{7}{24}s \\
 & \qquad\qquad\qquad\quad \underline{\frac{7}{3}s} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

Since all the quotient terms are positive, the polynomial $P(s)$ is Hurwitz.

Test whether the polynomial P(s) is Hurwitz.

$$P(s) = s^5 + s^3 + s$$

Solution Since the given polynomial contains odd functions only, it is not possible to perform continued fraction expansion.

$$P'(s) = \frac{d}{ds}P(s) = 5s^4 + 3s^2 + 1$$

$$Q(s) = \frac{P(s)}{P'(s)}$$

By continued fraction expansion,

$$
\begin{array}{l}
5s^4+3s^2+1\,\big)\,s^5+s^3+s\,\Big(\frac{1}{5}s \\
\qquad s^5+\frac{3}{5}s^3+\frac{1}{5}s \\
\hline
\qquad \frac{2}{5}s^3+\frac{4}{5}s\,\big)\,5s^4+3s^2+1\,\Big(\frac{25}{2}s \\
\qquad\qquad 5s^4+10s^2 \\
\hline
\qquad\qquad -7s^2+1\,\big)\,\frac{2}{5}s^3+\frac{4}{5}s\,\Big(-\frac{2}{35}s \\
\qquad\qquad\qquad \frac{2}{5}s^3-\frac{2}{35}s \\
\hline
\qquad\qquad\qquad \frac{26}{35}s\,\big)\,-7s^2+1\,\Big(-\frac{245}{26}s \\
\qquad\qquad\qquad\qquad -7s^2 \\
\hline
\qquad\qquad\qquad\qquad 1\,\big)\,\frac{26}{35}s\,\Big(\frac{26}{35}s \\
\qquad\qquad\qquad\qquad\qquad \frac{26}{35}s \\
\hline
\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

Since the third and fourth quotient terms are negative, $P(s)$ is not Hurwitz.

There is another method to test a Hurwitz polynomial. In this method, we construct the Routh–Hurwitz array for the required polynomial.

Let $$P(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0$$

The Routh–Hurwitz array is given by,

$$
\begin{array}{l|llll}
s^n & a_n & a_{n-2} & a_{n-4} & \ldots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \ldots \\
s^{n-2} & b_n & b_{n-1} & b_{n-2} & \ldots \\
s^{n-3} & c_n & c_{n-1} & & \ldots \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
. & . & & & \\
s^1 & . & & & \\
s^0 & . & & &
\end{array}
$$

The coefficients of s^n and s^{n-1} rows are directly written from the given equation.

where
$$b_n = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$$

$$b_{n-1} = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$$

$$b_{n-2} = \frac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$$

$$c_n = \frac{b_n a_{n-3} - a_{n-1}b_{n-1}}{b_n}$$

$$c_{n-1} = \frac{b_n a_{n-5} - a_{n-1}b_{n-2}}{b_n}$$

Hence, for polynomial $P(s)$ to be Hurwitz, there should not be any sign change in the first column of the array.

Test whether $P(s) = s^4 + 7s^3 + 6s^2 + 21s + 8$ *is Hurwitz.*

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 8 \\ s^3 & 7 & 21 & \\ s^2 & 3 & 8 & \\ s^1 & \frac{7}{3} & 0 & \\ s^0 & 8 & & \end{array}$$

Since all the elements in the first column are positive, the polynomial $P(s)$ is Hurwitz.

Determine whether $P(s) = s^4 + s^3 + 2s^2 + 3s + 2$ *is Hurwitz.*

Solution The Routh array is given by,

$$
\begin{array}{c|ccc}
s^4 & 1 & 2 & 2 \\
s^3 & 1 & 3 & \\
s^2 & -1 & 2 & \\
s^1 & 5 & 0 & \\
s^0 & 2 & &
\end{array}
$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Test whether the polynomial $P(s) = s^8 + 5s^6 + 2s^4 + 3s^2 + 1$ is Hurwitz.

Solution The given polynomial contains even functions only.

$$P'(s) = 8s^7 + 30s^5 + 8s^3 + 6s$$

The Routh array is given by,

$$\begin{array}{c|ccccc}
s^8 & 1 & 5 & 2 & 3 & 1 \\
s^7 & 8 & 30 & 8 & 6 & 0 \\
s^6 & 1.25 & 1 & 2.25 & 1 & \\
s^5 & 23.6 & -6.4 & -0.4 & 0 & \\
s^4 & 1.33 & 2.27 & 1 & & \\
s^3 & -46.6 & -18.14 & 0 & & \\
s^2 & 1.75 & 1 & & & \\
s^1 & 8.49 & & & & \\
s^0 & 1 & & & &
\end{array}$$

Since there is a sign change in the first column of the array, the polynomial is not Hurwitz.

Test whether $P(s) = s^5 + 12s^4 + 45s^3 + 60s^2 + 44s + 48$ is Hurwitz.

Solution The Routh array is given by,

$$
\begin{array}{c|ccc}
s^5 & 1 & 45 & 44 \\
s^4 & 12 & 60 & 48 \\
s^3 & 40 & 40 & \\
s^2 & 48 & 48 & \\
s^1 & 0 & 0 & \\
s^0 & & &
\end{array}
$$

Notes: When all the elements in any one row is zero, the following steps are followed:

(i) Write an auxiliary equation with the help of the coefficients of the row just above the row of zeros.

(ii) Differentiate the auxiliary equation and replace its coefficient in the row of zeros.

(iii) Proceed for the Routh test.

Auxiliary equation,

$$A(s) = 48s^2 + 48$$
$$A'(s) = 96s$$

$$\begin{array}{c|ccc} s^5 & 1 & 45 & 44 \\ s^4 & 12 & 60 & 48 \\ s^3 & 40 & 40 & \\ s^2 & 48 & 48 & \\ s^1 & 96 & 0 & \\ s^0 & 48 & & \end{array}$$

Since there is no sign change in the first column of the array, the polynomial $P(s)$ is Hurwitz.

Check whether $P(s) = 2s^6 + s^5 + 13s^4 + 6s^3 + 56s^2 + 25s + 25$ *is Hurwitz.*

Solution The Routh array is given by,

$$
\begin{array}{c|cccc}
s^6 & 2 & 13 & 56 & 25 \\
s^5 & 1 & 6 & 25 & \\
s^4 & 1 & 6 & 25 & \\
s^3 & 0 & 0 & 0 & \\
s^2 & & & & \\
s^1 & & & & \\
s^0 & & & &
\end{array}
$$

$$A(s) = s^4 + 6s^2 + 25$$
$$A'(s) = 4s^3 + 12s$$

Now, the Routh array will be given by,

$$\begin{array}{c|cccc}
s^6 & 2 & 13 & 56 & 25 \\
s^5 & 1 & 6 & 25 & \\
s^4 & 1 & 6 & 25 & \\
s^3 & 4 & 12 & & \\
s^2 & 3 & 25 & & \\
s^1 & -21.3 & & & \\
s^0 & 25 & & &
\end{array}$$

Since there is a sign change in the first column of the array, the polynomial $P(s)$ is not Hurwitz.

Determine the range of values of 'a' so that $P(s) = s^4 + s^3 + as^2 + 2s + 3$ is Hurwitz.

Solution The Routh array is given by,

$$\begin{array}{c|ccc} s^4 & 1 & a & 3 \\ s^3 & 1 & 2 & \\ s^2 & a-2 & 3 & \\ s^1 & \dfrac{2a-7}{a-2} & & \\ s^0 & 3 & & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive, i.e.,

$$a - 2 > 0$$
$$a > 2$$

and

$$\frac{2a-7}{a-2} > 0$$

$$a > \frac{7}{2}$$

Hence, $P(s)$ will be Hurwitz when $a > \dfrac{7}{2}$.

Determine the range of values of k so that the polynomial $P(s) = s^3 + 3s^2 + 2s + k$ is Hurwitz.

Solution The Routh array is given by,

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 3 & k \\ s^1 & \dfrac{6-k}{3} & 0 \\ s^0 & k & \end{array}$$

For the polynomial to be Hurwitz, all the terms in the first column of the array should be positive,

i.e.,
$$\frac{6-k}{3} > 0$$
$$6-k > 0$$

i.e., $k < 6$ and $k > 0$

Hence, $P(s)$ will be Hurwitz for $0 < k < 6$.

POSITIVE REAL FUNCTIONS

A function $F(s)$ is positive real if the following conditions are satisfied:

(a) $F(s)$ is real for real s.

(b) The real part of $F(s)$ is greater than or equal to zero when the real part of s is greater than or equal to zero, i.e.,

$$\operatorname{Re} F(s) \geq 0 \qquad \text{for } \operatorname{Re}(s) \geq 0$$

Properties of Positive Real Functions

1. If $F(s)$ is positive real then $\dfrac{1}{F(s)}$ is also positive real.
2. The sum of two positive real functions is positive real.
3. The poles and zeros of a positive real function cannot have positive real parts, i.e., they cannot be in the right half of the s plane.
4. Only simple poles with real positive residues can exist on the $j\omega$-axis.
5. The poles and zeros of a positive real function are real or occur in conjugate pairs.
6. The highest powers of the numerator and denominator polynomials may differ at most by unity. This condition prevents the possibility of multiple poles and zeros at $s = \infty$.
7. The lowest powers of the denominator and numerator polynomials may differ by at most unity. Hence, a positive real function has neither multiple poles nor zeros at the origin.

Necessary and Sufficient Conditions for Positive Real Functions

The necessary and sufficient conditions for a function with real coefficients $F(s)$ to be positive real are the following:

1. $F(s)$ must have no poles and zeros in the right half of the s-plane.
2. The poles of $F(s)$ on the $j\omega$-axis must be simple and the residues evaluated at these poles must be real and positive.
3. $\operatorname{Re} F(j\omega) \geq 0$ for all ω.

Testing of the Above Conditions Condition (1) requires that we test the numerator and denominator of $F(s)$ for roots in the right half of the s-plane, i.e., we must determine whether the numerator and denominator of $F(s)$ are Hurwitz. This is done through a continued fraction expansion of the odd to even or even to odd parts of the numerator and denominator.

Condition (2) is tested by making a partial-fraction expansion of $F(s)$ and checking whether the residues of the poles on the $j\omega$-axis are positive and real. Thus, if $F(s)$ has a pair of poles at $s = \pm j\omega_0$, a partial-fraction expansion gives terms of the form

$$\frac{K_1}{s-j\omega_0}+\frac{K_1^*}{s+j\omega_0}$$

Since residues of complex conjugate poles are themselves conjugate, $K_1 = K_1^*$ and should be positive and real.

Condition (3) requires that Re $F(j\omega)$ must be positive and real for all ω.

Now, to compute Re $F(j\omega)$ from $F(s)$, the numerator and denominator polynomials are separated into even and odd parts.

$$F(s)=\frac{m_1(s)+n_1(s)}{m_2(s)+n_2(s)}=\frac{m_1+n_1}{m_2+n_2}$$

Multiplying $N(s)$ and $D(s)$ by $m_2 - n_2$,

$$F(s)=\frac{m_1+n_1}{m_2+n_2}\frac{m_2-n_2}{m_2-n_2}=\frac{m_1m_2-n_1n_2}{m_2^2-n_2^2}+\frac{m_2n_1-m_1n_2}{m_2^2-n_2^2}$$

But the product of two even functions or odd functions is itself an even function, while the product of an even and odd function is odd.

$$\text{Ev } F(s) = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}$$

$$\text{Od } F(s) = \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2}$$

Now, substituting $s = j\omega$ in the even polynomial gives the real part of $F(s)$ and substituting $s = j\omega$ into the odd polynomial gives imaginary part of $F(s)$.

$$\text{Ev } F(s)\Big|_{s=j\omega} = \text{Re } F(j\omega)$$

$$\text{Od } F(s)\Big|_{s=j\omega} = j\,\text{Im } F(j\omega)$$

We have to test Re $F(j\omega) \geq 0$ for all ω.
The denominator of Re $F(j\omega)$ is always a positive quantity because

$$m_2^2 - n_2^2\Big|_{s=j\omega} \geq 0$$

Hence, the condition that Ev $F(j\omega)$ should be positive requires

$$m_1 m_2 - n_1 n_2\Big|_{s=j\omega} = A(\omega^2)$$

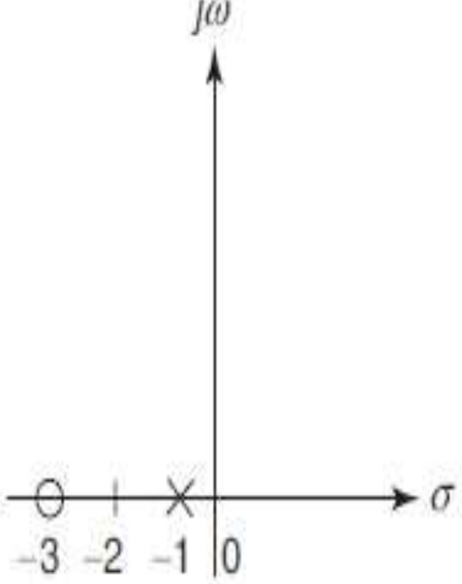
should be positive and real for all $\omega \geq 0$.

Test whether $F(s)=\dfrac{s+3}{s+1}$ *is a positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s+3}{s+1}$

The function $F(s)$ has pole at $s=-1$ and zero at $s=-3$ as shown in Fig. 10.1.



Thus, pole and zero are in the left half of the s-plane.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = 3$

Odd part of $N(s) = n_1 = s$

Even part of $D(s) = m_2 = 1$

Odd part of $D(s) = n_2 = s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (3)(1) - (s)(s) \big|_{s=j\omega} = 3 - s^2 \big|_{s=j\omega} = 3 + \omega^2$$

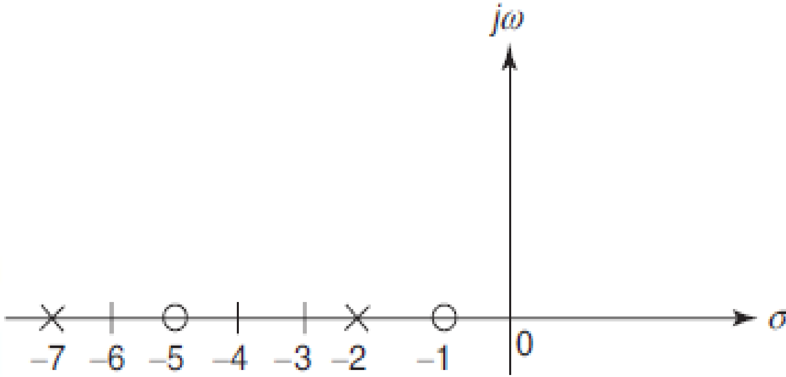
$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s)=\dfrac{s^2+6s+5}{s^2+9s+14}$ *is positive real function.*

Solution

(a) $F(s)=\dfrac{N(s)}{D(s)}=\dfrac{s^2+6s+5}{s^2+9s+14}=\dfrac{(s+5)(s+1)}{(s+7)(s+2)}$



The function $F(s)$ has poles at $s=-7$ and $s=-2$ and zeros at $s=-5$ an $s=-1$ as shown in Fig. 10.2.

Thus, all the poles and zeros are in the left half of the s plane.

(b) Since there is no pole on the $j\omega$ axis, the residue test is not carried out.

(c) Even part of $N(s) = m_1 = s^2 + 5$

Odd part of $N(s) = n_1 = 6s$

Even part of $D(s) = m_2 = s^2 + 14$

Odd part of $D(s) = n_2 = 9s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (s^2+5)\,(s^2+14) - (6s)(9s)\big|_{s=j\omega} = s^4 - 35s^2 + 70\ \big|_{s=j\omega} = \omega^4 + 35\omega^2 + 70$$

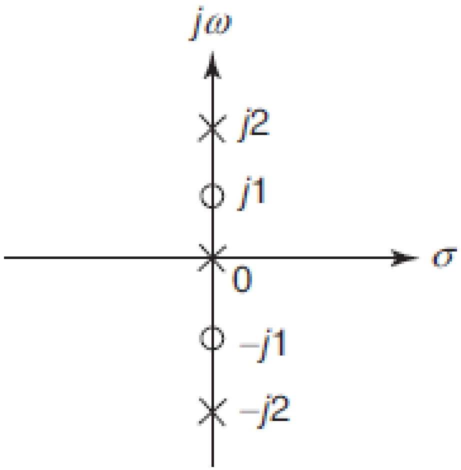
$A(\omega^2)$ is positive for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s) = \dfrac{s^2+1}{s^3+4s}$ *is positive real function.*

Solution

(a) $F(s) = \dfrac{N(s)}{D(s)} = \dfrac{s^2+1}{s^3+4s} = \dfrac{(s+j1)(s-j1)}{s(s+j2)(s-j2)}$



The function $F(s)$ has poles at $s = 0$, $s = -j2$ and $s = j2$ and zeros at $s = -j1$ and $s = j1$ as shown in Fig. 10.4.

Thus, all the poles and zeros are on the $j\omega$ axis.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s)=\frac{s^2+1}{s^3+4s}=\frac{s^2+1}{s(s^2+4)}$$

By partial-fraction expansion,

$$F(s)=\frac{K_1}{s}+\frac{K_2}{s+j2}+\frac{K_2^*}{s-j2}$$

The constants K_1, K_2 and K_2* are called residues.

$$K_1=s\,F(s)\big|_{s=0}=\left.\frac{s^2+1}{s^2+4}\right|_{s=0}=\frac{1}{4}$$

$$K_2=(s+j2)F(s)\big|_{s=-j2}=\left.\frac{s^2+1}{s(s-j2)}\right|_{s=-j2}=\frac{-4+1}{(-j2)(-j2-j2)}=\frac{3}{8}$$

$$K_2^*=K_2=\frac{3}{8}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = s^2 + 1$

Odd part of $N(s) = n_1 = 0$

Even part of $D(s) = m_2 = 0$

Odd part of $D(s) = n_2 = s^3 + 4s$

$$A\left(\omega^2\right) = m_1 m_2 - n_1 n_2 \,|_{s=j\omega} = (s^2 + 1)(0) - (0)(s^3 + 4s)\,|_{s=j\omega} = 0$$

$A(\omega^2)$ is zero for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s)=\dfrac{2s^3+2s^2+3s+2}{s^2+1}$ *is positive real function.*

Solution

(a) $$F(s)=\frac{N(s)}{D(s)}=\frac{2s^3+2s^2+3s+2}{s^2+1}=\frac{2s^3+2s^2+3s+2}{(s+j1)(s-j1)}$$

Since numerator polynomial cannot be easily factorized, we will prove whether $N(s)$ is Hurwitz.

Even part of $N(s)=m(s)=2s^2+2$

Odd part of $N(s)=n(s)=2s^3+3s$

By continued fraction expansion,

$$
\begin{array}{l}
2s^2+2 \,\big)\, 2s^3+3s \,\big(\, s \\
\qquad\qquad\;\; \underline{2s^3+2s} \\
\qquad\qquad\qquad\quad s \,\big)\, 2s^2+2 \,\big(\, 2s \\
\qquad\qquad\qquad\qquad\;\; \underline{2s^2} \\
\qquad\qquad\qquad\qquad\qquad 2 \,\big)\, s \,\big(\, \frac{1}{2}s \\
\qquad\qquad\qquad\qquad\qquad\quad\;\; \underline{s} \\
\qquad\qquad\qquad\qquad\qquad\quad\;\; 0
\end{array}
$$

Since all the quotient terms are positive, $N(s)$ is Hurwitz. This indicates that zeros are in the left half of the s plane.

The function $F(s)$ has poles at $s=-j1$ and $s=j1$.

Thus, all the poles and zeros are in the left half of the s plane.

(b) The poles on the $j\omega$ axis are simple. Hence, residue test is carried out.

$$F(s) = \frac{2s^3 + 2s^2 + 3s + 2}{s^2 + 1}$$

As the degree of the numerator is greater than that of the denominator, division is first carried out before partial-fraction expansion.

$$\begin{array}{r|l} s^2+1 & 2s^3 + 2s^2 + 3s + 2 \;\big(\; 2s+2 \\ & \underline{2s^3 \qquad\quad + 2s} \\ & \qquad 2s^2 + s + 2 \\ & \qquad \underline{2s^2 \qquad + 2} \\ & \qquad\qquad s \end{array}$$

$$F(s) = 2s + 2 + \frac{s}{s^2 + 1}$$

By partial-fraction expansion,

$$F(s) = 2s + 2 + \frac{K_1}{s + j1} + \frac{K_1^*}{s - j1}$$

$$K_1 = (s + j1)F(s)\,|_{s=-j1} = \frac{-j1}{-j1 - j1} = \frac{1}{2}$$

$$K_1^* = K_1 = \frac{1}{2}$$

Thus, residues are real and positive.

(c) Even part of $N(s) = m_1 = 2s^2 + 2$

Odd part of $N(s) = n_1 = 2s^3 + 3s$

Even part of $D(s) = m_2 = s^2 + 1$

Odd part of $D(s) = n_2 = 0$

$$A(\omega^2) = m_1 m_2 - n_1 n_2\,|_{s=j\omega} = (2s^2 + 2)(s^2 + 1) - (2s^3 + 3s)(0)\,|_{s=j\omega} = 2s^4 + 4s^2 + 2\,\,|_{s=j\omega} = 2(\omega^4 - 2\omega^2 + 1)$$

$$= 2(\omega^2 - 1)^2$$

$A(\omega^2) \geq 0$ for all $\omega \geq 0$.

Since all the three conditions are satisfied, the function is positive real.

Test whether $F(s)=\dfrac{s^2+s+6}{s^2+s+1}$ *is a positive real function.*

Solution

(a) $$F(s)=\frac{N(s)}{D(s)}=\frac{s^2+s+6}{s^2+s+1}=\frac{\left(s+\frac{1}{2}+j\frac{\sqrt{23}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{23}}{2}\right)}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}$$

The function $F(s)$ has zeros at $s=-\dfrac{1}{2}\pm j\dfrac{\sqrt{23}}{2}$ and poles at $s=-\dfrac{1}{2}\pm j\dfrac{\sqrt{3}}{2}$.

(b) There is no pole on the $j\omega$ axis. Hence, the residue test is not carried out.

(c) Even part of $N(s)=m_1=s^2+6$

Odd part of $N(s)=n_1=s$

Even part of $D(s)=m_2=s^2+1$

Odd part of $D(s)=n_2=s$

$$A(\omega^2) = m_1 m_2 - n_1 n_2 \big|_{s=j\omega} = (s^2+6)(s^2+1) - (s)(s) \big|_{s=j\omega} = s^4 + 6s^2 + 6 \big|_{s=j\omega} = \omega^4 - 6\omega^2 + 6$$

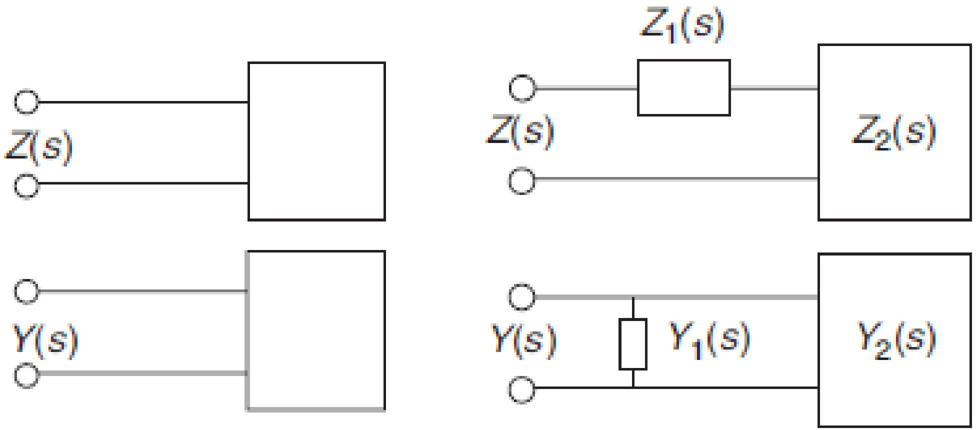
For $\omega = 2$, $A(\omega^2) = 16 - 24 + 6 = -2$

This condition is not satisfied.

Hence, the function $F(s)$ is not positve real.

ELEMENTARY SYNTHESIS CONCEPTS

We know that impedances and admittances of passive networks are positive real functions. Hence, addition of impedances of the two passive networks gives a function which is also a positive real function. Thus, $Z(s) = Z_1(s) + Z_2(s)$ is a positive real function, if $Z_1(s)$ and $Z_2(s)$ are positive real functions. Similarly, $Y(s) = Y_1(s) + Y_2(s)$ is a positive real function, if $Y_1(s)$ and $Y_2(s)$ are positive real functions. There is a special



terminology for synthesis procedure. We have,

$$Z(s) = Z_1(s) + Z_2(s)$$

$$Z_2(s) = Z(s) - Z_1(s)$$

Here, $Z_1(s)$ is said to have been removed from $Z(s)$ in forming the new function $Z_2(s)$ as shown in Fig. 10.5. If the removed network is associated with the pole or zero of the original network impedance then that pole or zero is also said to have been removed.

There are four important removal operations.

Removal of a Pole at Infinity

Consider an impedance function $Z(s)$ having a pole at infinity which means that the numerator polynomial is one degree greater than the degree of the denominator polynomial.

$$Z(s) = \frac{a_{n+1}s^{n+1} + a_n s^n + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Hs + \frac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0}$$

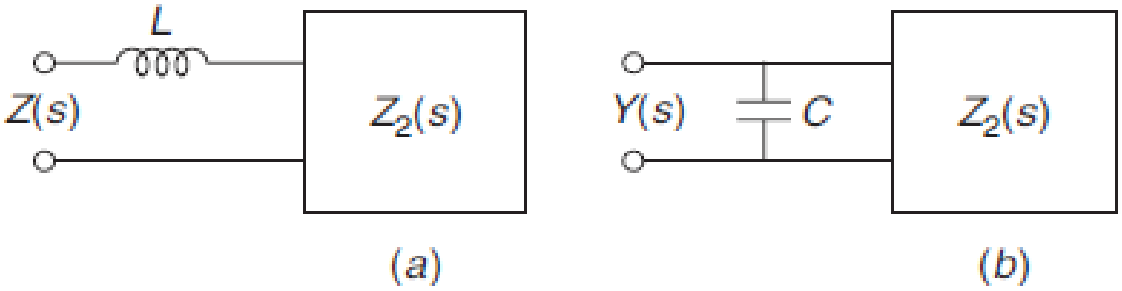
where $\quad H = \dfrac{a_{n+1}}{b_n}$

Let $\quad Z_1(s) = Hs$

and $\quad Z_2(s) = \dfrac{c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_1 s + b_0} = Z(s) - Hs$

$Z_1(s) = Hs$ represents impedance of an inductor of value H. Hence, the removal of a pole at infinity corresponds to the removal of an inductor from the network of Fig. 10.6(a).

If the given function is an admittance function $Y(s)$, then $Y_1(s) = Hs$ represents the admittance of a capacitor $Y_C(s) = Cs$. The network for $Y_1(s)$ is a capacitor of value $C = H$ as shown in Fig. 10.6(b).



Network interpretation of the removal of a pole at infinity

Removal of a Pole at Origin

If $Z(s)$ has a pole at the origin then it may be written as

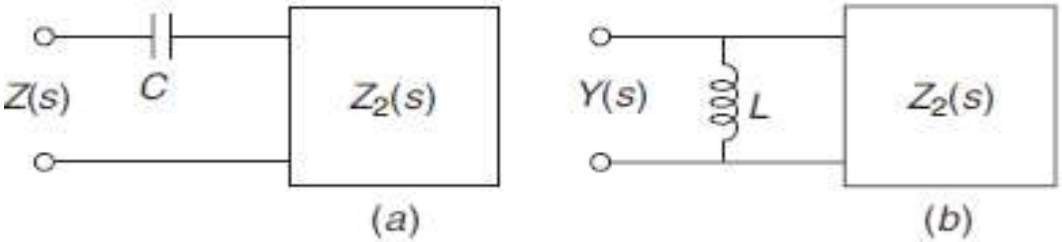
$$Z(s) = \frac{a_0 + a_1 s + \ldots + a_{n-1} s^{n-1} + a_n s^n}{b_1 s + b_2 s^2 + \ldots + b_m s^m} = \frac{K_0}{s} + \frac{d_1 + d_2 s + \ldots + d_n \mathbf{s}^{n-1}}{b_1 + b_2 s + \ldots + b_m s^{m-1}} = Z_1(s) + Z_2(s)$$

where $$K_0 = \frac{a_0}{b_1}$$

$Z_1(s) = \dfrac{K_0}{s}$ represents the impedance of a capacitor of value $\dfrac{1}{K_0}$.

If the given function is an admittance function $Y(s)$ then removal of $Y_1(s) = \dfrac{K_0}{s}$ corresponds to an inductor

of value $\dfrac{1}{K_0}$.



Network interpretation of the removal of a pole at origin

Thus, removal of a pole from the impedance function $Z(s)$ at the origin corresponds to the removal of a capacitor, and from admittance function $Y(s)$ corresponds to removal of an inductor as shown in Fig. 10.7.

Removal of Conjugate Imaginary Poles

If $Z(s)$ contains poles on the imaginary axis, i.e., at $s = \pm j\omega_1$ then $Z(s)$ will have factors $(s + j\omega_1)(s - j\omega_1) = s^2 + \omega_1{}^2$ in the denominator polynomial

$$Z(s) = \frac{p(s)}{(s^2 + \omega_1^2)\, q_1(s)}$$

By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s + j\omega_1} + \frac{K_1^*}{s - j\omega_1} + Z_2(s)$$

For a positive real function, $j\omega$ axis poles must themselves be conjugate and must have equal, positive and real residues.

$$K_1 = K_1^*$$

Hence,

$$Z(s) = \frac{2K_1 s}{s^2 + \omega_1^2} + Z_2(s)$$

Thus,

$$Z_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{\dfrac{s}{2K_1} + \dfrac{\omega_1^2}{2K_1 s}} = \frac{1}{Y_a + Y_b}$$

where $Y_a = \dfrac{s}{2K_1}$ is the admittance of a capacitor of value $C = \dfrac{1}{2K_1}$

and $Y_b = \dfrac{\omega_1^2}{2K_1 s}$ is the admittance of an inductor of value $L = \dfrac{2K_1}{\omega_1^2}$

If the given function is an admittance function $Y(s)$ then

$$Y_1(s) = \frac{2K_1 s}{s^2 + \omega_1^2} = \frac{1}{Z_a + Z_b} = \frac{1}{\dfrac{s}{2K_1} + \dfrac{\omega_1^2}{2K_1 s}}$$

where $Z_a = \dfrac{s}{2K_1}$ is the impedance of an inductor of value $L = \dfrac{1}{2K_1}$ and $Z_b = \dfrac{\omega_1^2}{2K_1 s}$ is the impedance of a capacitor of value $C = \dfrac{2K_1}{\omega_1^2}$.

Thus, removal of conjugate imaginary poles from impedance function $Z(s)$ corresponds to the removal of the parallel combination of $L - C$ and from admittance function $Y(s)$ corresponds to removal of series combination of $L - C$ as shown in Fig. 10.8.

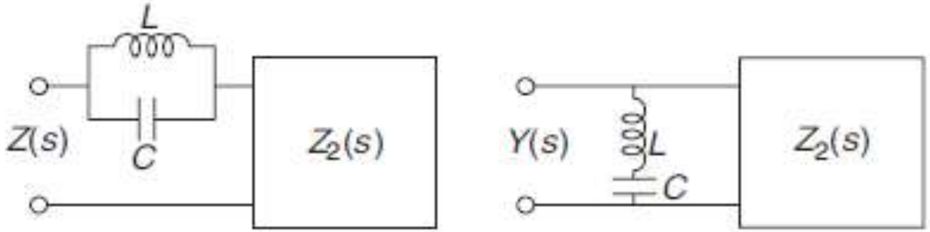


Fig. 10.8 *Network interpretation of the removal of conjugate imaginary poles*

Removal of a Constant

If a real number R_1 is subtracted from $Z(s)$ such that

$$Z_2(s) = Z(s) - R_1$$

$$Z(s) = R_1 + Z_2(s)$$

then R_1 represents a resistor.

If the given function is an admittance function $Y(s)$, then removal of $Y_1(s) = R_1$ represents a conductance of value R_1.

Thus, removal of a constant from impedance function $Z(s)$ corresponds to the removal of a resistance, and from admittance function $Y(s)$ corresponds to removal of a conductance.

Synthesize the impedance function $Z(s)=\dfrac{s^3+4s}{s^2+2}$.

Solution

By long division of $Z(s)$,

$$s^2+2\overline{\big)\,s^3+4s\,}\big(\,s$$
$$\underline{s^3+2s}$$
$$2s$$

$$Z(s)=s+\frac{2s}{s^2+2}=Z_1(s)+Z_2(s)$$

$Z_1(s)=s$ represents impedance of an inductor of value 1 H.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s^2+2}{2s}=\frac{s^2}{2s}+\frac{2}{2s}=\frac{1}{2}s+\frac{1}{s}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\dfrac{1}{2}s$ represents the admittance of a capacitor of value $\dfrac{1}{2}$F.

$Y_4(s)=\dfrac{1}{s}$ represents the admittance of an inductor of value 1 H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.9.

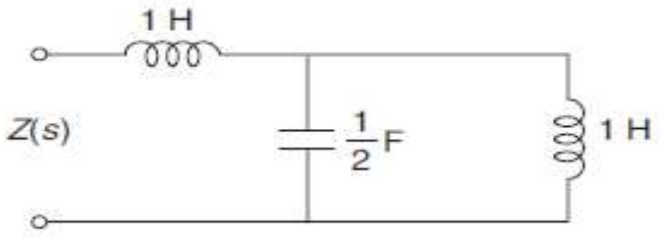


Fig. 10.9

Realise the network having impedance function

$$Z(s) = \frac{s^2 + 2s + 10}{s(s+5)}$$

Solution

By long division of $Z(s)$,

$$s^2 + 5s \overline{\big) \; s^2 + 2s + 10 \; \big(\frac{2}{s}}$$

$$\frac{\phantom{s^2 +{}} 2s + 10}{s^2}$$

$$Z(s) = \frac{2}{s} + \frac{s^2}{s^2 + 5s} = \frac{2}{s} + \frac{s}{s+5} = Z_1(s) + \mathbf{Z}_2(s)$$

$Z_1(s) = \dfrac{2}{s}$ represents the impedance of capacitor of value $\dfrac{1}{2}$F.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s+5}{s}=1+\frac{5}{s}=Y_3(s)+Y_4(s)$$

$Y_3(s) = 1$ represents the admittance of a resistor of value 1 Ω.

$Y_4(s)=\dfrac{5}{s}$ represents the admittance of an inductor of value $\dfrac{1}{5}$ H.

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.10.

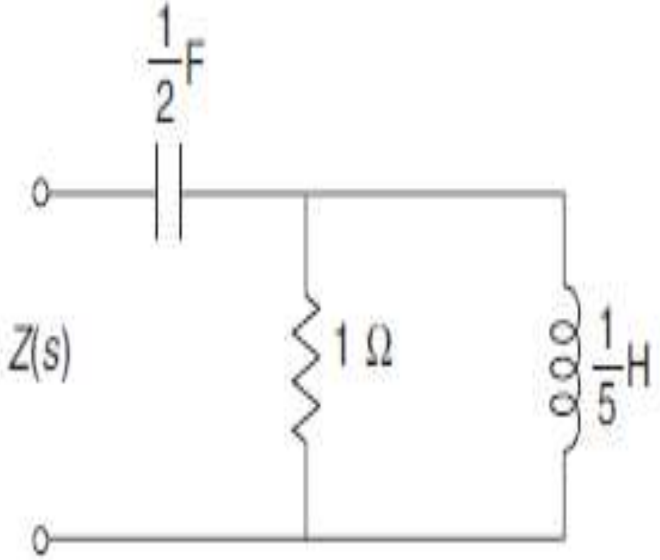


Fig. 10.10

Realise the network having impedance function $Z(s)=\dfrac{6s^3+5s^2+6s+4}{2s^3+2s}$.

Solution By long division of $Z(s)$,

$$\begin{array}{r|l} 2s^3+2s & 6s^3+5s^2+6s+4 \;(\,3 \\ & \underline{6s^3 \qquad\quad +6s} \\ & \qquad 5s^2 \qquad\quad +4 \end{array}$$

$$Z(s)=3+\frac{5s^2+4}{2s^3+2s}=Z_1(s)+Z_2(s)$$

$Z_1(s)=3$ represents the impedance of a resistor of value 3 Ω.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{2s^3+2s}{5s^2+4}$$

By long division of $Y_2(s)$,

$$\begin{array}{r|l} 5s^2+4 & 2s^3+2s \quad \left(\frac{2}{5}s\right. \\ & \underline{2s^3+\frac{8}{5}s} \\ & \qquad \frac{2}{5}s \end{array}$$

$$Y_2(s) = \frac{2}{5}s + \frac{\frac{2}{5}s}{5s^2+4} = Y_3(s) + Y_4(\boldsymbol{s})$$

$Y_3(s) = \frac{2}{5}s$ represents the admittance of a capacitor of value $\frac{2}{5}$F.

$$Z_4(s) = \frac{1}{Y_4(s)} = \frac{5s^2+4}{\frac{2}{5}s} = \frac{25s^2+20}{2s} = \frac{25}{2}s + \frac{10}{s} = Z_5(s) + Z_6(s)$$

$$Z_5(s) = \frac{25}{2}s \text{ represents the impedance of an inductor of value } \frac{25}{2}\text{H}.$$

$$Z_6(s) = \frac{10}{s} \text{ represents the impedance of a capacitor of value } \frac{1}{10}\text{F}.$$

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.11.

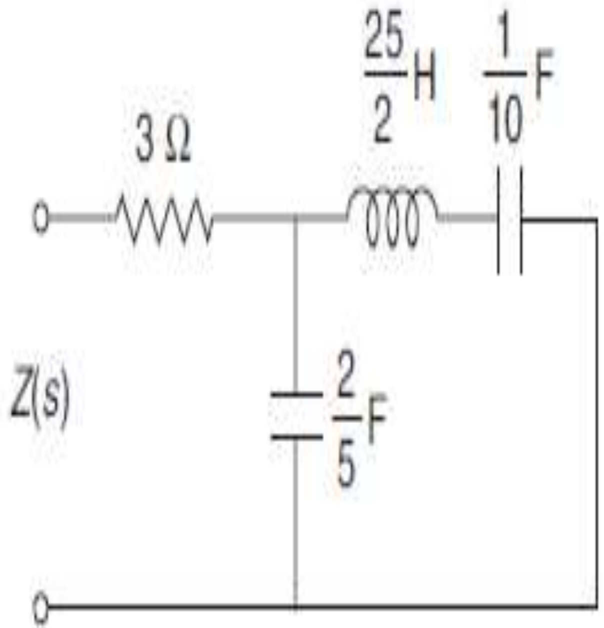


Fig. 10.11

Realise the network having impedance function

$$Z(s)=\frac{s^4+10s^2+7}{s^3+2s}$$

Solution By long division of $Z(s)$,

$$\begin{array}{r|l}
s^3+2s \;) & s^4+10s^2+7\;(\;s \\
 & \underline{s^4+2s^2} \\
 & \quad 8s^2+7
\end{array}$$

$$Z(s)=s+\frac{8s^2+7}{s^3+2s}=Z_1(s)+Z_2(s)$$

$Z_1(s)=s$ represents the impedance of an inductor of value 1 H.

$$Y_2(s)=\frac{1}{Z_2(s)}=\frac{s^3+2s}{8s^2+7}$$

By long division of $Y_2(s)$,

$$\begin{array}{rl} 8s^2+7 \big) & s^3+2s \left(\dfrac{1}{8}s \right. \\ & \underline{s^3+\dfrac{7}{8}s} \\ & \quad \dfrac{9}{8}s \end{array}$$

$$Y_2(s)=\frac{1}{8}s+\frac{\frac{9}{8}s}{8s^2+7}=Y_3(s)+Y_4(s)$$

$Y_3(s)=\dfrac{1}{8}s$ represents the admittance of a capacitor of value $\dfrac{1}{8}$F.

$$Z_4(s)=\frac{1}{Y_4(s)}=\frac{8s^2+7}{\frac{9}{8}s}=\frac{64}{9}s+\frac{56}{9s}=Z_5(s)+Z_6(s)$$

$Z_5(s) = \dfrac{64}{9}s$ represents the impedance of an inductor of value $\dfrac{64}{9}$ H.

$Z_6(s) = \dfrac{56}{9s}$ represents the impedance of a capacitor of value $\dfrac{9}{56}$ F.

The impedances are connected in the series branches, whereas the admittances are connected in the parallel branches. The network is shown in Fig. 10.12.

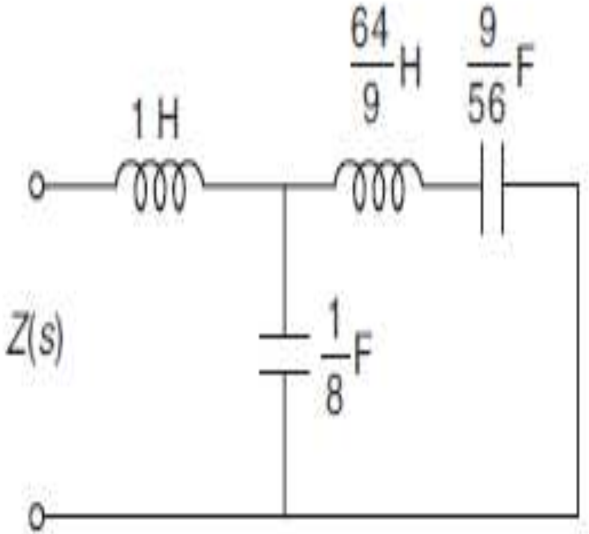


Fig. 10.12

REALISATION OF *LC* FUNCTIONS

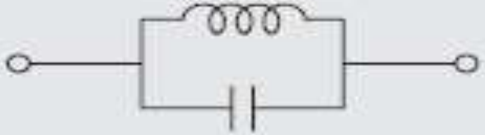
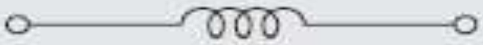
LC driving point immitance functions have the following properties.

1. It is the ratio of odd to even or even to odd polynomials.
2. The poles and zeros are simple and lie on the $j\omega$-axis.
3. The poles and zeros interlace on the $j\omega$-axis.
4. There must be either a zero or a pole at the origin and infinity.
5. The difference between any two successive powers of numerator and denominator polynomials is at most two. There cannot be any missing terms.
6. The highest powers of numerator and denominator polynomials must differ by unity; the lowest powers also differ by unity.

There are a number of methods of realising an *LC* function. But we will study only four basic forms—Foster I, Foster II, Cauer I and Cauer II forms. The Foster forms are obtained by partial-fraction expansion of $F(s)$, and the Cauer forms are obtained by continued fraction expansion of $F(s)$.

Foster Realisation

Table 10.1 *Realisation of Foster-I form of LC network*

Impedance function	Element
$\frac{K_0}{s} = \frac{1}{C_0 s}$	 $C_0 = \frac{1}{K_0}$
$\frac{2K_i s}{s^2 + \omega_i^2} = \frac{\left(\frac{1}{C_i}\right)s}{s^2 + \frac{1}{L_i C_i}}$	$L_i = \frac{2K_i}{\omega_i^2}$  $C_i = \frac{1}{2K_i}$
$K_\infty s = Ls$	 $L_\infty = K_\infty$

The network corresponding to Foster I form is shown in Fig. 10.15.

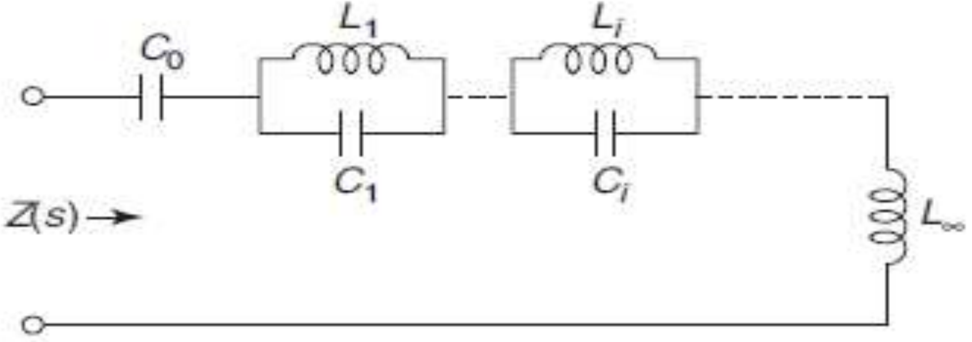
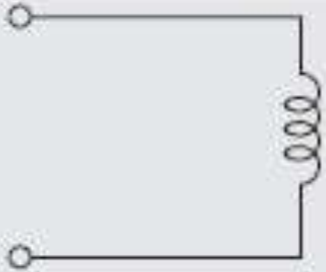
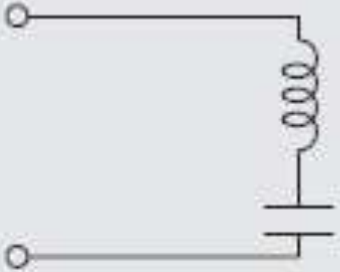
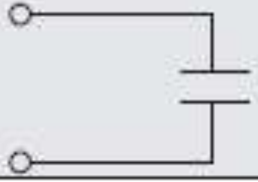


Fig. 10.15 *Foster-I form of LC network*

If $Z(s)$ has no pole at the origin then capacitor C_0 is not present in the network. Similarly, if there is no pole at ∞, inductor L_∞ is not present in the network.

Table 10.2 *Realisation of Foster-II form of LC network*

Admittance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{L_0 s}$	 $L_0 = \dfrac{1}{K_0}$
$\dfrac{2K_i s}{s^2 + \omega_i^2} = \dfrac{\left(\dfrac{1}{L_i}\right)s}{s^2 + \dfrac{1}{L_i C_i}}$	 $L_i = \dfrac{1}{2K_i}$, $C_i = \dfrac{2K_i}{\omega_i^2}$
$K_\infty s = Cs$	 $C_\infty = K_\infty$

The network corresponding to the Foster II form is shown in Fig. 10.16.

If $Y(s)$ has no pole at the origin then inductor L_0 is not present. Similarly, if there is no pole at infinity, capacitor C_∞ is not present.

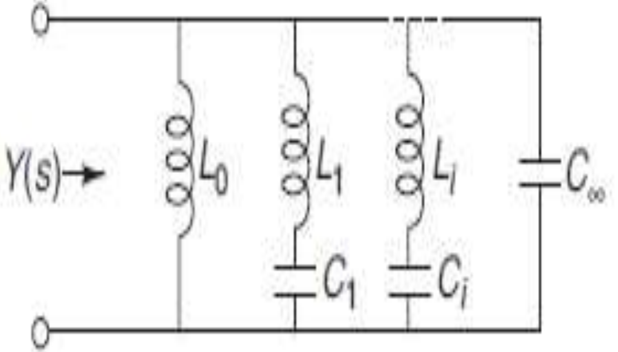


Fig. 10.16 *Foster-II form of LC network*

Cauer Realisation or Ladder Realisation

Cauer I Form Since the numerator and denominator polynomials of an LC function always differ in degrees by unity, there is always a zero or a pole at $s = \infty$. The Cauer I Form is obtained by successive removal of a pole or a zero at infinity from the function.

Consider an impedance function $Z(s)$ having a pole at infinity.

By removing the pole at infinity, we get

$$Z_2(s) = Z(s) - L_1 s$$

Now, $Z_2(s)$ has a zero at $s = \infty$. If we invert $Z_2(s)$, $Y_2(s)$ will have a pole at $s = \infty$.

By removing this pole,

$$Y_3(s) = Y_2(s) - C_2 s$$

Now $Y_3(s)$ has a zero at $s = \infty$, which we can invert and remove. This process continues until the remainder is zero. Each time we remove a pole, we remove an inductor or a capacitor depending on whether the function is an impedance or an admittance. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = L_1 s + \cfrac{1}{C_2 s + \cfrac{1}{L_3 s + \cfrac{1}{C_4 s + \dots}}}$$

Thus, the final structure is a ladder network whose series arms are inductors and shunt arms are capacitors. The Cauer I network is shown in Fig. 10.17.

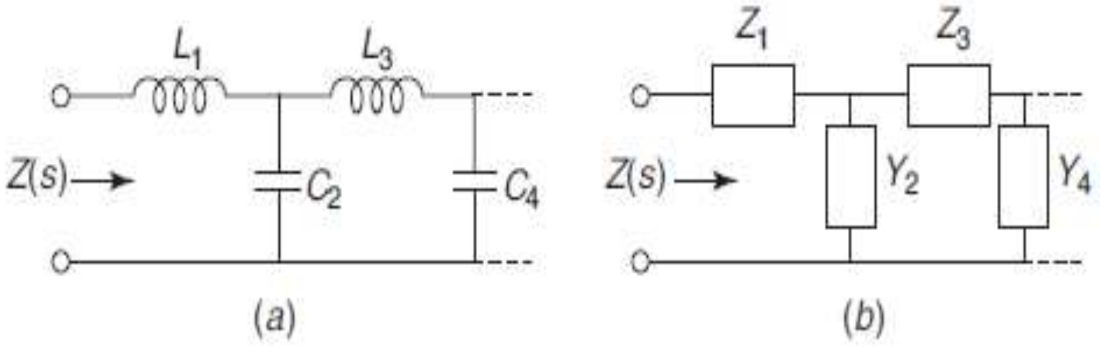


Fig. 10.17 *Cauer I form of LC network*

If the impedance function has zero at infinity, i.e., if degree of numerator is less than that of its denominator by unity, the function is first inverted and continued fraction expansion proceeds as usual. In this case, the first element is a capacitor as shown in Fig. 10.18.

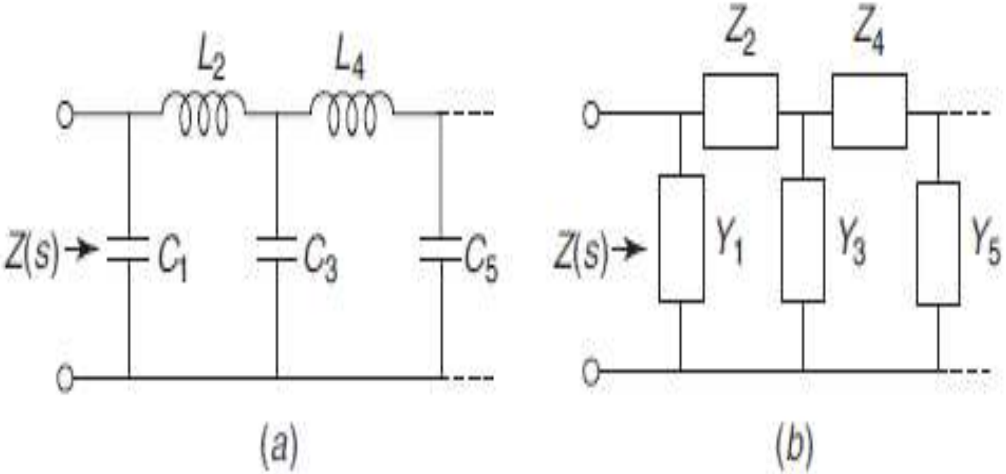


Fig. 10.18 *Cauer-I form of LC network*

Cauer II Form Since the lowest degrees of numerator and denominator polynomials of *LC* function must differ by unity, there is always a zero or a pole at $s = 0$. The Cauer II form is obtained by successive removal of a pole or a zero at $s = 0$ from the function.

In this method, continued fraction expansion of $Z(s)$ is carried out in terms of poles at the origin by removal of the pole at the origin, inverting the resultant function to create a pole at the origin which is removed and this process is continued until the remainder is zero. To do this, we arrange both numerator and denominator polynomials in ascending order and divide the lowest power of the denominator into the lowest power of the numerator. Then we invert the remainder and divide again. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{L_2 s} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{L_4 s} + \dots}}}$$

Thus, the final structure is a ladder network whose first element is a series capacitor and second element is a shunt inductor as shown in Fig. 10.19.

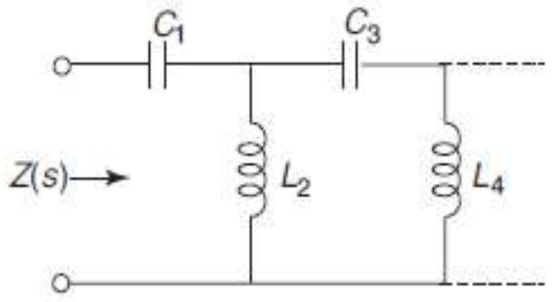


Fig. 10.19 *Cauer II form of LC network*

If the impedance function has a zero at the origin then the first element is a shunt inductor and the second element is a series capacitor as shown in Fig. 10.20.

Thus, the LC function $F(s)$ can be realised in four different forms. All these forms have the same number of elements and the number is equal to the number of poles and zeros of $F(s)$ including any at infinity.

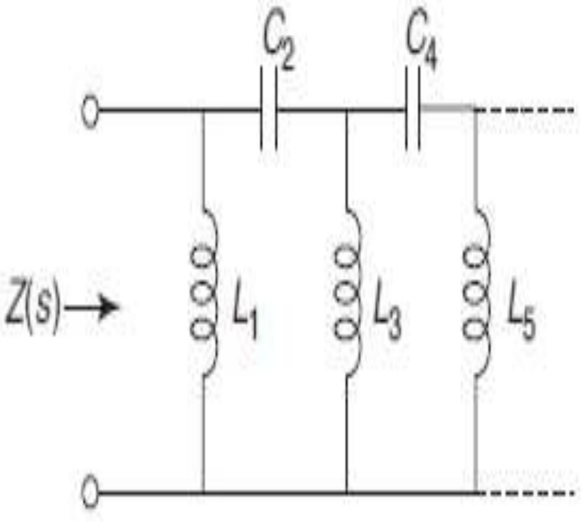
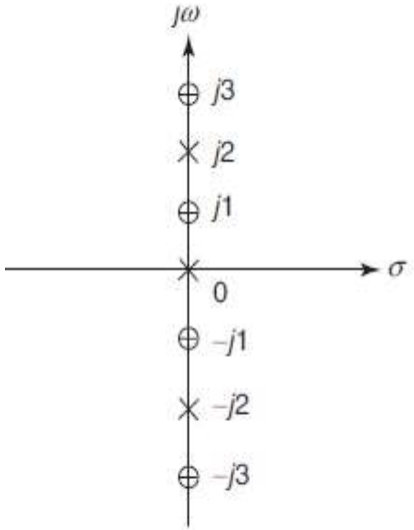


Fig. 10.20 *Cauer-II form of LC network*

Realise the Foster and Cauer forms of the following impedance function

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = \pm j2$ and zeros at $s = \pm j1$ and $s = \pm j3$ as shown in Fig. 10.23.



From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the $j\omega$ axis. Poles and zeros are interlaced. Hence, the given function is an LC function.

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. But degree of numerator is greater than degree of denominator. Hence, division is first carried out.

$$Z(s) = \frac{4(s^2+1)(s^2+9)}{s(s^2+4)} = \frac{4s^4+40s^2+36}{s^3+4s}$$

$$\begin{array}{r} s^3+4s\,)\,4s^4+40s^2+36\,(4s \\ \underline{4s^4+16s^2} \\ 24s^2+36 \end{array}$$

$$Z(s) = 4s + \frac{24s^2+36}{s^3+4s} = 4s + \frac{24s^2+36}{s(s^2+4)}$$

By partial-fraction expansion,

$$Z(s) = 4s + \frac{K_0}{s} + \frac{K_1}{s+j2} + \frac{K_1^*}{s-j2} = 4s + \frac{K_0}{s} + \frac{2K_1 s}{s^2+4}$$

where $$K_0 = sZ(s)\big|_{s=0} = \frac{4(1)(9)}{4} = 9$$

$$K_1 = \frac{(s^2+4)Z(s)}{2s}\bigg|_{s^2=-4} = \frac{4(-4+1)(-4+9)}{2(-4)} = \frac{15}{2}$$

$$Z(s) = 4s + \frac{9}{s} + \frac{15s}{s^2+4}$$

The first term represents the impedance of an inductor of 4 H. The second term represents the impedance of a capacitor of $\frac{1}{9}$ F. The third term represents the impedance of a parallel LC network.

For a parallel LC network,

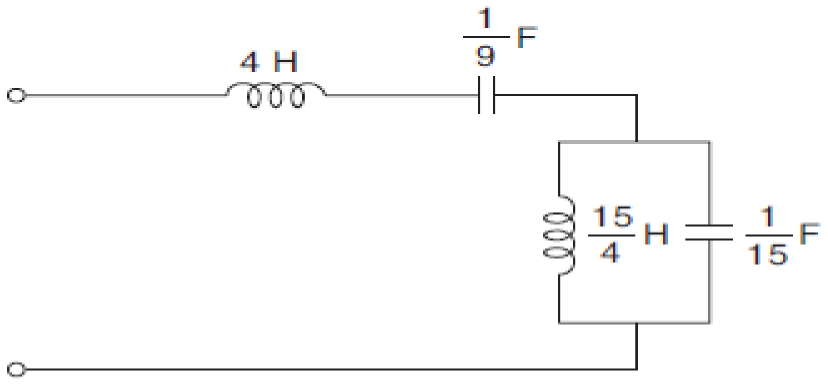
$$Z_{LC}(s) = \frac{\left(\frac{1}{C}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$C = \frac{1}{15}\text{F}$$

$$L = \frac{15}{4}\,\text{H}$$

The network is shown in Fig. 10.24.



Foster II Form The Foster II form is obtained by partial fraction expansion of the admittance function $Y(s)$.

$$Y(s) = \frac{s(s^2+4)}{4(s^2+1)(s^2+9)}$$

By partial-fraction expansion,

$$Y(s) = \frac{K_1}{s+j1} + \frac{K_1^*}{s-j1} + \frac{K_2}{s+j3} + \frac{K_2^*}{s-j3} = \frac{2K_1 s}{s^2+1} + \frac{2K_2 s}{s^2+9}$$

where
$$K_1 = \frac{(s^2+1)}{2s} Y(s)\bigg|_{s^2=-1} = \frac{(-1+4)}{8(-1+9)} = \frac{3}{64}$$

$$K_2 = \frac{(s^2+9)}{2s} Y(s)\bigg|_{s^2=-9} = \frac{(-9+4)}{8(-9+1)} = \frac{5}{64}$$

$$Y(s) = \frac{\left(\frac{3}{32}\right)s}{s^2+1} + \frac{\left(\frac{5}{32}\right)s}{s^2+9}$$

These two terms represent admittance of a series LC network. For a series LC network,

$$Y_{LC}(s) = \frac{\left(\frac{1}{L}\right)s}{s^2 + \frac{1}{LC}}$$

By direct comparison,

$$L_1 = \frac{32}{3}\text{H} \qquad C_1 = \frac{3}{32}\text{F}$$

$$L_2 = \frac{32}{5}\text{H} \qquad C_2 = \frac{5}{288}\text{F}$$

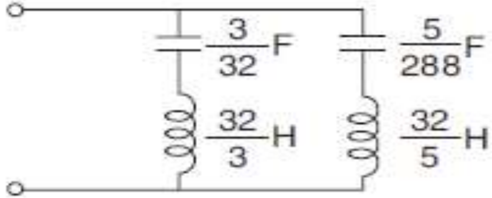


Fig. 10.25

The network is shown in Fig. 10.25.

Cauer I Form The Cauer I form is obtained from continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{4s^4 + 40s^2 + 36}{s^3 + 4s}$$

Since the degree of the numerator is greater than the degree of the denominator by one, it indicates the presence of a pole at infinity.

By continued fraction expansion,

$$s^3+4s\,\Big)\,4s^4+40s^2+36\,\Big(\,4s \leftarrow Z$$

$$\underline{4s^4+16s^2}$$

$$24s^2+36\,\Big)\,s^3+4s\,\Big(\,\frac{1}{24}s \leftarrow Y$$

$$\underline{s^3+\frac{3}{2}s}$$

$$\frac{5}{2}s\,\Big)\,24s^2+36\,\Big(\,\frac{48}{5}s \leftarrow Z$$

$$\underline{24s^2}$$

$$36\,\Big)\,\frac{5}{2}s\,\Big(\,\frac{5}{72}s \leftarrow Y$$

$$\underline{\frac{5}{2}s}$$

$$0$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.26.

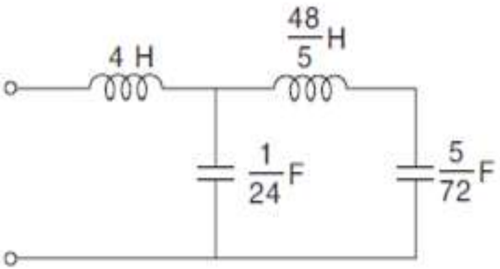


Fig. 10.26

Cauer II Form The Cauer II form is obtained from continued fraction expansion about pole at origin.

$$Z(s)=\frac{4(s^2+1)(s^2+9)}{s(s^2+4)}=\frac{4s^4+40s^2+36}{s^3+4s}$$

The function $Z(s)$ has a pole at origin. Arranging the numerator and denominator polynomials in ascending order of s,

$$Z(s)=\frac{36+40s^2+4s^4}{4s+s^3}$$

By continued fraction expansion,

$$4s+s^3\Big)\,36+40s^2+4s^4\Big(\frac{9}{s}\leftarrow Z$$

$$\underline{36+\;9s^2\qquad\qquad}$$

$$31s^2+4s^4\Big)\,4s+s^3\Big(\frac{4}{31s}\leftarrow Y$$

$$\underline{4s+\frac{16}{31}s^3}$$

$$\frac{15}{31}s^3\Big)\,31s^2+4s^4\Big(\frac{961}{15s}\leftarrow Z$$

$$\underline{31s^2\qquad\qquad}$$

$$4s^4\,\Big)\,\frac{15}{31}s^3\,\Big(\,\frac{15}{124s} \leftarrow Y$$

$$\frac{\frac{15}{31}s^3}{0}$$

The impedances are connected in the series branches whereas the admittances are connected in the parallel branches in a Cauer or ladder realisation. The network is shown in Fig. 10.27.

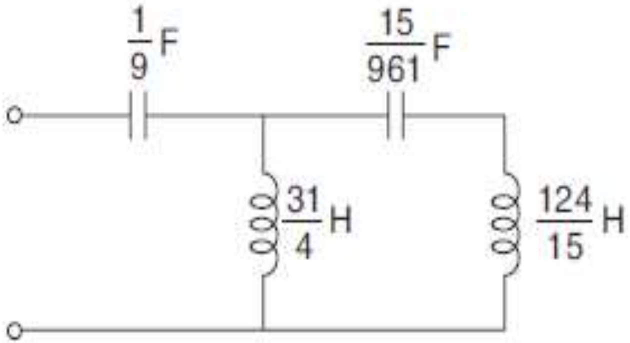


Fig. 10.27

REALISATION OF *RC* FUNCTIONS

RC driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency nearest to the origin is a pole.
4. The highest critical frequency farthest to the origin is a zero.
5. Residues evaluated at the poles of $Z_{RC}(s)$ are real and positive.
6. The slope $\frac{d}{d\sigma}Z_{RC}$ is negative.
7. $Z_{RC}(\infty) < Z_{RC}(0)$.

RC functions can also be realised in four different ways. The impedance function of RC networks is given by,

$$Z(s) = \frac{H(s+\sigma_1)(s+\sigma_3)\ldots}{s(s+\sigma_2)\ldots}$$

Table 10.3 *Realisation of Foster-I form of RC network*

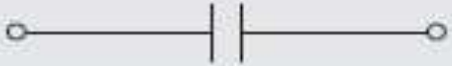
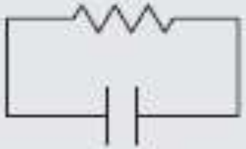
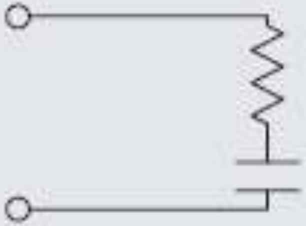
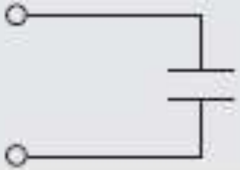
Impedance function	Element
$\dfrac{K_0}{s} = \dfrac{1}{C_0 s}$	 $C_0 = \dfrac{1}{K_0}$
$\dfrac{K_i}{s+\sigma_i} = \dfrac{(R_i)\left(\dfrac{1}{C_i s}\right)}{R_i + \dfrac{1}{C_i s}}$	$Ri = \dfrac{K_i}{\sigma_i}$  $C_i = \dfrac{1}{K_i}$
$K_\infty = R_\infty$	 $R_\infty = K_\infty$

Table 10.4 *Realisation of Foster II form of RC network*

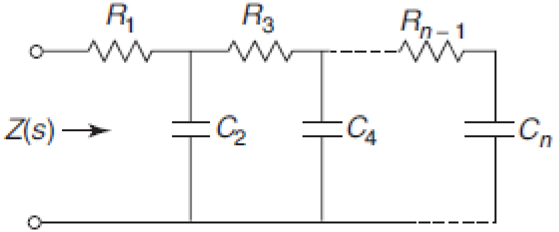
Admittance function	Element
$K_0 = \dfrac{1}{R_0}$	 $R_o = \dfrac{1}{K_o}$
$\dfrac{K_i}{s+\omega_i} = \dfrac{\left(\dfrac{1}{R_i}\right)s}{s+\dfrac{1}{R_iC_i}}$	 $R_i = \dfrac{1}{K_i}$ $C_i = \dfrac{K_i}{\sigma_i}$
$K_\infty s = C_\infty s$	 $C_\infty = K_\infty$

Cauer Realisation

Cauer I Form The Cauer I form is obtained by removal of the pole from the impedance function $Z(s)$ at $s = \infty$. This is the same as a continued fraction expansion of the impedance function about infinity. The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = R_1 + \cfrac{1}{C_2 s + \cfrac{1}{R_3 + \cfrac{1}{C_4 s + \ldots}}}$$

The network is shown in Fig. 10.40.



In the network shown in Fig. 10.40, if $Z(s)$ has a zero at $s = \infty$, the first element is the capacitor C_1. If $Z(s)$ is a constant at $s = \infty$, the first element is R_1. If $Z(s)$ has a pole at $s = 0$, the last element is C_n. If $Z(s)$ is a constant at $s = 0$, the last element is R_n.

Cauer II Form The Cauer II form is obtained by removal of the pole from the impedance function at the origin. This is the same as a continued fraction expansion of an impedance function about the origin.

If the given impedance function has a pole at the origin, it is removed as a capacitor C_1. The reciprocal of the remainder function has a minimum value at $s = 0$ which is removed as a constant of resistor R_2. If the original impedance has no pole at the origin, then the first capacitor is absent and the process is repeated with the removal of the constant corresponding to the resistor R_2.

The impedance $Z(s)$ can be written as a continued fraction expansion.

$$Z(s) = \frac{1}{C_1 s} + \cfrac{1}{\cfrac{1}{R_2} + \cfrac{1}{\cfrac{1}{C_3 s} + \cfrac{1}{\cfrac{1}{R_4} + \dots}}}$$

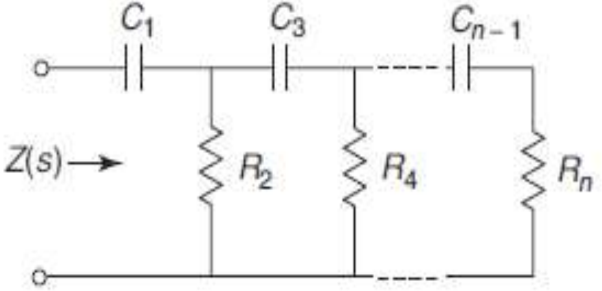


Fig. 10.41 *Caver-II form of RC network*

The network is shown in Fig. 10.41.

In the network shown in Fig. 10.41, if $Z(s)$ has a pole at $s = 0$, the first element is C_1. If $Z(s)$ is a constant at $s = 0$, the first element is R_2. If $Z(s)$ has a zero at $s = \infty$, the last element is C_n. If $Z(s)$ is constant at $s = \infty$, the last element is R_n.

Realise the Foster and Cauer forms of the impedance function

$$Z(s) = \frac{(s+1)(s+3)}{s(s+2)}$$

Solution The function $Z(s)$ has poles at $s = 0$ and $s = -2$ and zeros at $s = -1$ and $s = -3$ as shown in Fig. 10.44.

From the pole-zero diagram, it is clear that poles and zeros are simple and lie on the negative real axis. The poles and zeros are interlaced and the lowest critical frequency nearest to the origin is a pole. Hence, the function $Z(s)$ is an RC function.

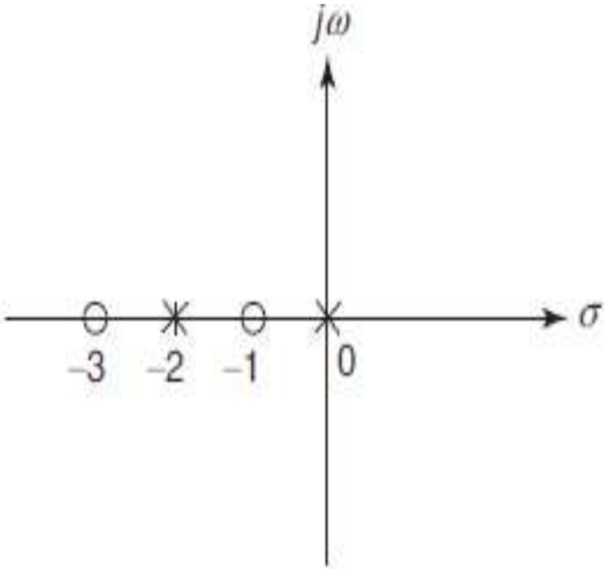


Fig. 10.44

Foster I Form The Foster I form is obtained by partial fraction expansion of impedance function $Z(s)$. Since the degree of the numerator is greater than the degree of the denominator, division is first carried out.

$$Z(s)=\frac{s^2+4s+3}{s^2+2s}$$

$$\begin{array}{r}
s^2+2s\,\big)\,s^2+4s+3\,\big(\,1 \\
\underline{s^2+2s\qquad} \\
2s+3\qquad
\end{array}$$

$$Z(s)=1+\frac{2s+3}{s^2+2s}=1+\frac{2s+3}{s(s+2)}$$

By partial-fraction expansion,

$$Z(s)=1+\frac{K_1}{s}+\frac{K_2}{s+2}$$

where

$$K_1=sZ(s)\big|_{s=0}=\frac{(1)(3)}{2}=\frac{3}{2}$$

$$K_2=(s+2)Z(s)\big|_{s=-2}=\frac{(-2+1)(-2+3)}{-2}=\frac{1}{2}$$

$$Z(s)=1+\frac{\frac{3}{2}}{s}+\frac{\frac{1}{2}}{s+2}$$

The first term represents the impedance of a resistor of 1 Ω. The second term represents the impedance of a capacitor of $\frac{2}{3}$ F. The third term represents the impedance of parallel RC circuit for which

$$Z_{RC}(s) = \frac{\frac{1}{C_i}}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R = \frac{1}{4}\,\Omega$$
$$C = 2\text{ F}$$

The network is shown in Fig. 10.45.

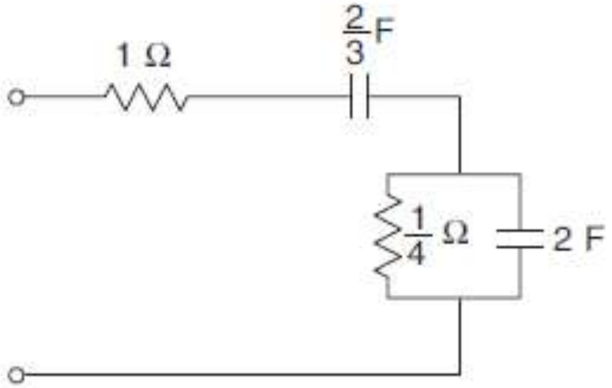


Fig. 10.45

Foster II Form The Foster II form is obtained by the partial-fraction expansion of admittance function $\frac{Y(s)}{s}$.

$$Y(s) = \frac{1}{Z(s)} = \frac{s(s+2)}{(s+1)(s+3)}$$

$$\frac{Y(s)}{s} = \frac{s+2}{(s+1)(s+3)}$$

By partial-fraction expansion,

$$\frac{Y(s)}{s} = \frac{K_1}{s+1} + \frac{K_2}{s+3}$$

where

$$K_1 = (s+1)\left.\frac{Y(s)}{s}\right|_{s=-1} = \frac{(-1+2)}{(-1+3)} = \frac{1}{2}$$

$$K_2 = (s+3)\left.\frac{Y(s)}{s}\right|_{s=-3} = \frac{(-3+2)}{(-3+1)} = \frac{1}{2}$$

$$\frac{Y(s)}{s} = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3}$$

$$Y(s) = \frac{\frac{1}{2}s}{s+1} + \frac{\frac{1}{2}s}{s+3}$$

These two terms represent the admittance of a series RC circuit. For a series RC circuit.

$$Y_{RC}(s) = \frac{\left(\frac{1}{R_i}\right)s}{s + \frac{1}{R_i C_i}}$$

By direct comparison,

$$R_1 = 2\,\Omega, \quad C_1 = \frac{1}{2}\,\text{F}$$

$$R_2 = 2\,\Omega, \quad C_2 = \frac{1}{6}\,\text{F}$$

The network is shown in Fig. 10.46.

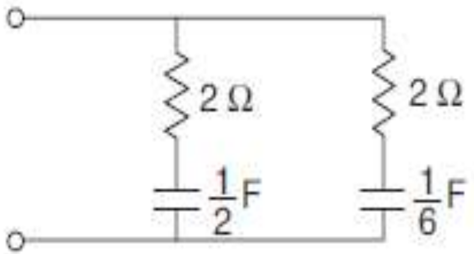


Fig. 10.46

Cauer I Form The Cauer I form is obtained by continued fraction expansion about the pole at infinity.

$$Z(s) = \frac{s^2 + 4s + 3}{s^2 + 2s}$$

By continued fraction expansion,

$$s^2 + 2s \,\big)\, s^2 + 4s + 3 \,\big(\, 1 \leftarrow Z$$

$$\underline{s^2 + 2s}$$

$$2s + 3 \,\big)\, s^2 + 2s \,\Big(\, \frac{1}{2}s \leftarrow Y$$

$$\underline{s^2 + \frac{3}{2}s}$$

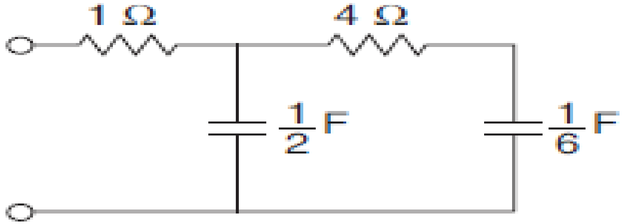
$$\frac{1}{2}s \,\Big)\, 2s + 3 \,\big(\, 4 \leftarrow Z$$

$$\underline{2s}$$

$$3 \,\Big)\, \frac{1}{2}s \,\Big(\, \frac{1}{6}s \leftarrow Y$$

$$\underline{\frac{1}{2}s}$$

$$0$$



Cauer II Form The Cauer II form is obtained from continued fraction expansion about the pole at the origin. Arranging the numerator and denominator polynomials of $Z(s)$ in ascending order of s,

$$Z(s)=\frac{3+4s+s^2}{2s+s^2}$$

By continued fraction expansion,

$$2s+s^2\,\big)\,3+4s+s^2\,\big(\,\frac{3}{2s}\leftarrow Z$$

$$\underline{3+\frac{3}{2}s}$$

$$\frac{5}{2}s+s^2\,\big)\,2s+s^2\,\big(\,\frac{4}{5}\leftarrow Y$$

$$\underline{2s+\frac{4}{5}s^2}$$

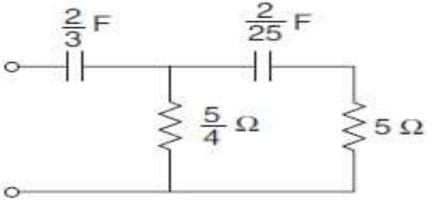
$$\frac{1}{5}s^2\,\big)\,\frac{5}{2}s+s^2\,\big(\,\frac{25}{2s}\leftarrow Z$$

$$\underline{\frac{5}{2}s}$$

$$s^2\,\big)\,\frac{1}{5}s^2\,\big(\,\frac{1}{5}\leftarrow Y$$

$$\underline{\frac{1}{5}s^2}$$

$$0$$



The impedances are connected in the series branches whereas admittances are connected in the parallel branches. The network is shown

REALISATION OF *RL* FUNCTIONS

RL driving point immittance functions have following properties:

1. The poles and zeros are simple and are located on the negative real axis of the s plane.
2. The poles and zeros are interlaced.
3. The lowest critical frequency is a zero which may be at $s = 0$.
4. The highest critical frequency is a pole which may be at infinity.
5. Residues evaluated at the poles of $Z_{RL}(s)$ are real and negative while that of $\frac{Z_{RL}(s)}{s}$ are real and positive.
6. The slope $\frac{d}{d\sigma} Z_{RL}$ is positive.
7. $Z_{RL}(0) < Z_{RL}(\infty)$.

Realise following RL impedance function in Foster-I and Foster-II form.

$$Z(s) = \frac{2(s+1)(s+3)}{(s+2)(s+6)}$$

Solution

Foster I Form The Foster I form is obtained by partial-fraction expansion of the impedance function $Z(s)$. By partial-fraction expansion,

$$Z(s) = \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where

$$K_1 = (s+2)Z(s)\Big|_{s=-2} = \frac{2(-2+1)(-2+3)}{(-2+6)} = -\frac{1}{2}$$

$$K_2 = (s+6)Z(s)\Big|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6+2)} = -\frac{15}{2}$$

Since residues of $Z(s)$ are negative, partial fraction expansion of $\frac{Z(s)}{s}$ is carried out.

$$\frac{Z(s)}{s} = \frac{2(s+1)(s+3)}{s(s+2)(s+6)}$$

By partial fraction expansion,

$$\frac{Z(s)}{s} = \frac{K_0}{s} + \frac{K_1}{s+2} + \frac{K_2}{s+6}$$

where

$$K_0 = s\frac{Z(s)}{s}\bigg|_{s=0} = \frac{2(1)(3)}{(2)(6)} = \frac{1}{2}$$

$$K_1 = (s+2)\frac{Z(s)}{s}\bigg|_{s=-2} = \frac{2(-2+1)(-2+\mathbf{3})}{(2)(-2+6)} = \frac{1}{4}$$

$$K_2 = (s+6)\frac{Z(s)}{s}\bigg|_{s=-6} = \frac{2(-6+1)(-6+3)}{(-6)(-6+2)} = \frac{5}{\mathbf{4}}$$

$$\frac{Z(s)}{s} = \frac{\frac{1}{2}}{s} + \frac{\frac{1}{4}}{s+2} + \frac{\frac{5}{4}}{s+6}$$

$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4}s}{s+2} + \frac{\frac{5}{4}s}{s+6}$$

The first term represents the impedance of the resistor of $\frac{1}{2}\,\Omega$. The other two terms represent the impedance of the parallel RL circuit for which

$$Z_{RL}(s) = \frac{R_i s}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{1}{4}\,\Omega, \quad L_1 = \frac{1}{8}\,\text{H}$$

$$R_2 = \frac{5}{4}\,\Omega, \quad L_2 = \frac{5}{24}\,\text{H}$$

The network is shown in Fig. 10.62.

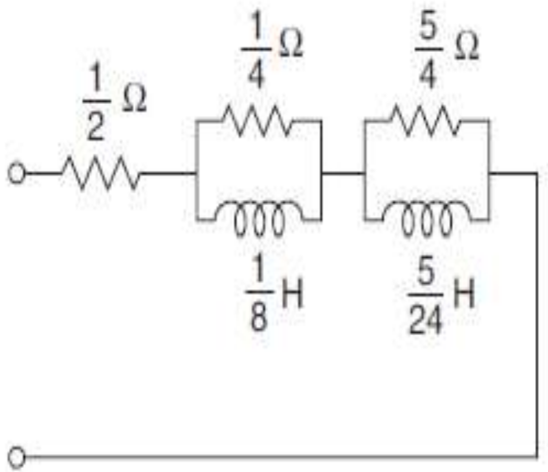


Fig. 10.62

Foster II Form The Foster II form is obtained by partial fraction expansion of $Y(s)$. Since the degree of the numerator is equal to the degree of the denominator, division is first carried out.

$$Y(s) = \frac{(s+2)(s+6)}{2(s+1)(s+3)} = \frac{s^2+8s+12}{2s^2+8s+6}$$

$$\begin{array}{r|l} 2s^2+8s+6 & s^2+8s+12 \left(\frac{1}{2} \right. \\ & \underline{s^2+4s+3} \\ & \quad\;\; 4s+9 \end{array}$$

$$Y(s) = \frac{1}{2} + \frac{4s+9}{2s^2+8s+6} = \frac{1}{2} + \frac{4s+9}{2(s+1)(s+3)}$$

By partial-fraction expansion,

$$Y_1(s) = \frac{4s+9}{2(s+1)(s+3)} = \frac{K_0}{s+1} + \frac{K_1}{s+3}$$

where

$$K_0 = (s+1)Y_1(s)\Big|_{s=-1} = \frac{(-4+9)}{2(-1+3)} = \frac{5}{4}$$

$$K_1 = (s+3)Y_1(s)\Big|_{s=-3} = \frac{(-12+9)}{2(-3+1)} = \frac{3}{4}$$

$$Y(s) = \frac{1}{2} + \frac{\frac{5}{4}}{s+1} + \frac{\frac{3}{4}}{s+3}$$

The first term represents the admittance of a resistor of 2 Ω. The other two terms represent the admittance of a series *RL* circuit. For a series *RL* circuit,

$$Y_{RL}(s) = \frac{\frac{1}{L_i}}{s + \frac{R_i}{L_i}}$$

By direct comparison,

$$R_1 = \frac{4}{5}\,\Omega, \quad L_1 = \frac{4}{5}\,\text{H}$$

$$R_2 = 4\,\Omega, \quad L_2 = \frac{4}{3}\,\text{H}$$

The network is shown in Fig. 10.63.

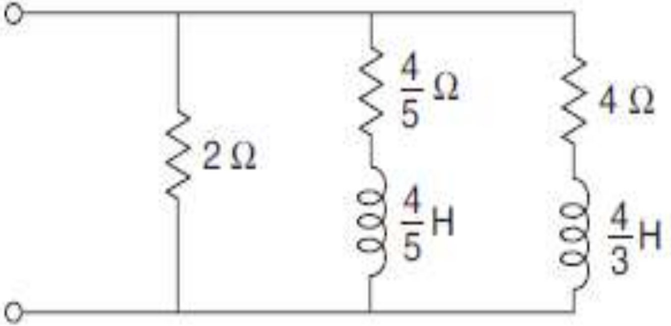


Fig. 10.63