* Taylor's and Laurent's Series :-

→ power series in complex number :-

the power series in powers of $(z-a)$ is of the form $\sum_{n=1}^{\infty} C_n (z-a)^n$ —— (1)

where, z is complex number and C_n's are constants.

Note that : The power series (1) is convergent for $|z-a| < R$, for some real Number R.

therefore the number 'R' is called as Radius of convergence.

* To find the Radius of convergence :-

consider the power series $\sum_{n=1}^{\infty} C_n (z-a)^n$

(1) $$R = \lim_{n \to \infty} \left| \frac{C_n}{C_{n+1}} \right|$$

(2) $$R = \lim_{n \to \infty} (C_n)^{-\frac{1}{n}}$$

Example Find the radius of convergence of following

(i) $\sum_{n=0}^{\infty} \frac{z^n}{3^n+1}$ (ii) $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} \cdot z^n$

solution:

(1) Given power series is $\sum_{n=0}^{\infty} \frac{1}{3^n+1} z^n$

compare with $\sum_{n=0}^{\infty} C_n z^n$, we get

$$C_n = \frac{1}{3^n+1}$$

$$\therefore R \doteq \lim_{n\to\infty} \left| \frac{C_n}{C_{n+1}} \right| = \lim_{n\to\infty} \left| \frac{1}{3^n+1} \times \frac{3^{n+1}+1}{1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3^{n+1}+1}{3^n+1} \right|$$

$$= \lim_{n\to\infty} \left| \frac{3^n\left(3+\frac{1}{3^n}\right)}{3^n\left(1+\frac{1}{3^n}\right)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{\left(3+\frac{1}{3^n}\right)}{\left(1+\frac{1}{3^n}\right)} \right|$$

$$= \frac{3+0}{1+0}$$

$$= 3.$$

$\therefore$ Radius of convergence : $\underline{\underline{R = 3.}}$

(2) Given power series is $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2} \cdot z^n$

compare with $\sum_{n=1}^{\infty} C_n z^n$, we get

$$C_n = \left(1+\frac{1}{n}\right)^{n^2}$$

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$$\therefore \quad R = \lim_{n\to\infty} (c_n)^{-\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{-\frac{1}{n}}$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{-n}$$

$$= e^{-1} = \frac{1}{e}$$

$\therefore$ Radius of convergence : $\underline{\underline{R = \frac{1}{e}}}$

Homework.

Example : find the Radius of convergence of following

① $\displaystyle\sum_{n=0}^{1} \frac{n+1}{(n+2)(n+3)} z^n$. ② $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n^p}$

* Taylor's series Expansion:

let C be the circle with centre at z_0 and $f(z)$ be analytic everywhere inside C then Taylor's series expansion of $f(z)$ is

$$f(z) = f(z_0) + (z-z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \cdots \quad \text{—(1)}$$

<u>Note that:</u> (1) Above series of $f(z)$ is convergent at every point inside C

(2) If we put $z = z_0 + h$ then equation (1) becomes

$$f(z_0+h) = f(z_0) + h f'(z_0) + \frac{h^2}{2!} f''(z_0) + \cdots$$

(3) If we put $z_0 = 0$ then equation (1) becomes

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \cdots$$

is known as Maclaurin's series.

* Important power series of the function:-

→ $(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \cdots$, where $|z| < 1$

→ $(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \cdots$, where $|z| < 1$

→ $(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \cdots$, where $|z| < 1$

→ $(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \cdots$, where $|z| < 1$

* Examples on Taylor's series:

Example ① obtain Taylor's expansion of

$$f(z) = \frac{z+2}{(z-1)(z-4)} \quad \text{at} \quad z=2$$

Solution: given function is $f(z) = \dfrac{z+2}{(z-1)(z-4)}$

clearly, degree of Numerators is less than degree of Denomenators

Now, consider the partial fraction

$$\frac{z+2}{(z-1)(z-4)} = \frac{A}{z-1} + \frac{B}{z-4}$$

$$\Rightarrow \quad \frac{z+2}{(z-1)(z-4)} = \frac{A(z-4)+B(z-1)}{(z-1)(z-4)}$$

$$\Rightarrow \quad A(z-4)+B(z-1) = z+2$$

if $z=1$ then $A(1-4)+0 = 1+2 \Rightarrow A=-1$

if $z=4$ then $A(0)+B(4-1) = 4+2 \Rightarrow B=2$

$$\therefore \quad \frac{z+2}{(z-1)(z-4)} = -\frac{1}{z-1} + \frac{2}{z-4}$$

we have to find series expansion in the power of $(z-2)$

$$\frac{z+2}{(z-1)(z-4)} = -\frac{1}{(z-2)+1} + \frac{2}{(z-2)-2}$$

$$= -\frac{1}{[1+(z-2)]} + \frac{2}{-2\left[1-\left(\frac{z-2}{2}\right)\right]}$$

$$= -\frac{1}{1+(z-2)} - \frac{1}{1-\left(\frac{z-2}{2}\right)}$$

$$= -[1+(z-2)]^{-1} - \left[1-\left(\frac{z-2}{2}\right)\right]^{-1}$$

we know, $[1+z]^{-1} = 1-z+z^2-z^3+\cdots$ and $[1-z]^{-1} = 1+z+z^2+z^3+\cdots$

$$\Rightarrow \frac{z+2}{(z-1)(z-4)} = -\left[1-(z-2)+(z-2)^2-(z-2)^3+\cdots\right]$$

$$-\left[1+\left(\frac{z-2}{2}\right)+\left(\frac{z-2}{2}\right)^2+\left(\frac{z-2}{2}\right)^3+\cdots\right]$$

$$= -\sum_{n=0}^{\infty}(-1)^n(z-2)^n - \sum_{n=0}^{\infty}\left(\frac{z-2}{2}\right)^n$$

$$= -\sum_{n=0}^{\infty}\left[(-1)^n+\frac{1}{2^n}\right](z-2)^n$$

i.e. $$f(z) = -\sum\left[(-1)^n+\frac{1}{2^n}\right](z-2)^n, \quad |z-2|<1$$

Example ②. obtain Taylor's expansion of $f(z) = \frac{1-z}{z^2}$ in the powers of $(z-1)$

solution: Given: $f(z) = \frac{1-z}{z^2}$

$$\Rightarrow f(z) = -\frac{(z-1)}{(z-1+1)^2}$$

$$= -\frac{(z-1)}{[1+(z-1)]^2}$$

$$= -(z-1)\left[1+(z-1)\right]^{-2}$$

$$= -(z-1)\left[1-2(z-1)+3(z-1)^2-4(z-1)^3+\cdots\right]$$

$$\left(\because [1+z]^{-2} = 1-2z+3z^2-4z^3+\cdots\right)$$

$$= \left[-(z-1)+2(z-1)^2-3(z-1)^3+4(z-1)^4-\cdots\right]$$

$$= \sum_{n=1}^{\infty}(-1)^n\, n\,(z-1)^n$$

i.e. $$f(z) = \sum_{n=1}^{\infty}(-1)^n\cdot n\,(z-1)^n$$