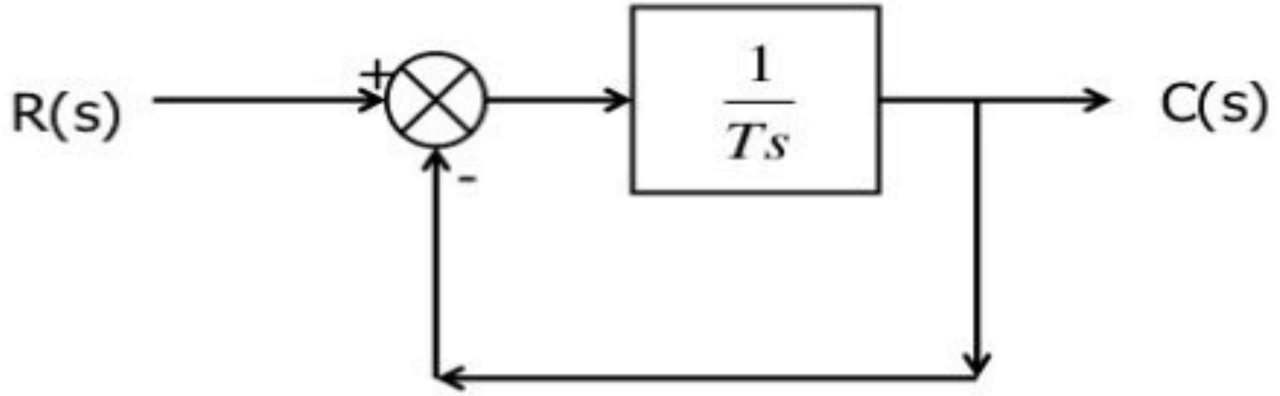
Analysis Of First Order System For Step Input

Consider a first order system as shown;



Here $G(s) = \dfrac{1}{Ts}$ and $H(s) = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{1}{Ts}}{1+\frac{1}{Ts}} = \frac{1}{1+Ts}$$

For step input;

$$r(t) = u(t) \qquad t>0$$
$$= 0 \qquad t<0$$

Taking Laplace transform;

$$R(\mathrm{s}) = L\{\mathrm{Ru(t)}\} = \frac{1}{s}$$

but

$$\frac{C(\mathrm{s})}{R(\mathrm{s})} = \frac{1}{1+Ts}$$

$$\therefore C(\mathrm{s}) = \frac{1}{1+Ts} \times R(\mathrm{s})$$

$$\therefore C(s) = \frac{1}{1+Ts} \times \frac{1}{s}$$

Using partial fraction;

$$\therefore C(s) = \frac{A}{s} + \frac{B}{s + \frac{1}{T}}$$

Solving;

$$\therefore A = s.C(s) \,|_{s=0} = 1$$

$$\therefore B = (s + \frac{1}{T})C(s) \,|_{s=-\frac{1}{T}} = -1$$

$$\therefore C(\mathrm{s}) = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

Taking Inverse Laplace transform;

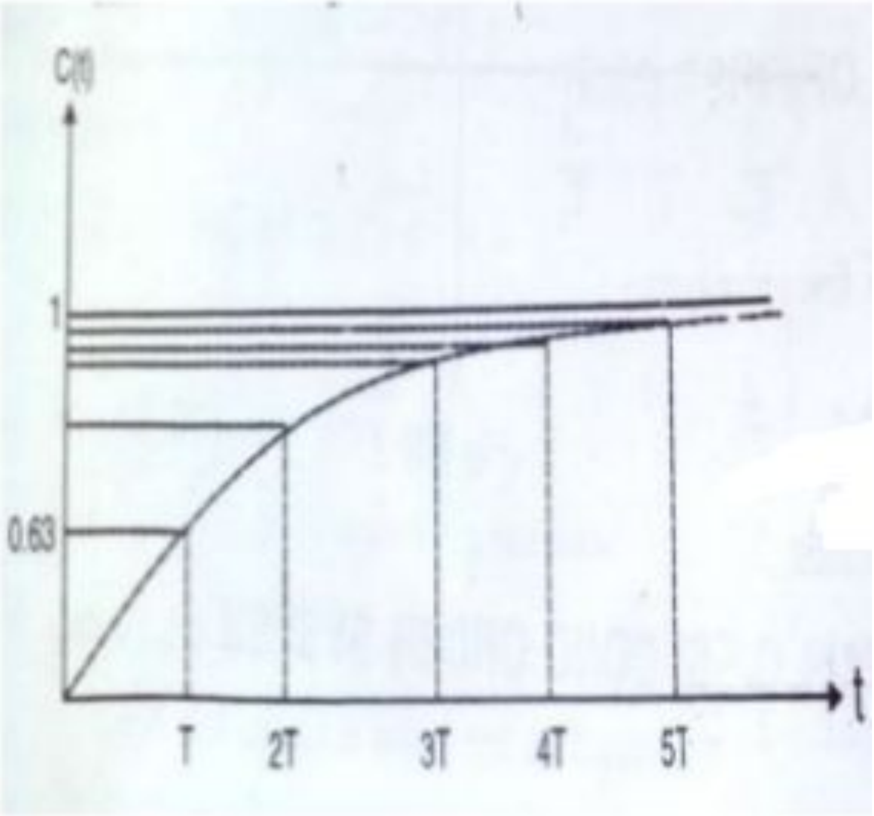
$$\therefore c(\mathrm{t}) = L^{-1}\{C(\mathrm{s})\} = L^{-1}\{\frac{1}{s}\} - L^{-1}\{\frac{1}{s + \frac{1}{T}}\}$$

$$\boxed{\therefore c(\mathrm{t}) = 1 - e^{-\frac{1}{T}t}}$$

Analysis Of First Order System For Step Input

Plot c(t) vs t;

Sr. No.	t	C(t)
1	T	0.632
2	2T	0.86
3	3T	0.95
4	4T	0.982
5	5T	0.993
6	∞	1



TIME CONSTANT (T)

- ✓ The value of $c(t)=1$ only at $t=\infty$.
- ✓ Practically the value of $c(t)$ is within 5% of final value at $t=3T$ and within 2% at $t=4T$.
- ✓ In practice $t=3T$ or $4T$ may be taken as steady state.
- ✓ How quickly the value reaches steady state is a function of the time constant of the system.
- ✓ Hence smaller T indicates quicker response.

Analysis Of Second Order Control System

- Analysis for Step Input
- Definition of damping
- Effect of Damping

Definition of damping

- **Damping**

Every system has a tendency to oppose the oscillatory behavior of the system which is known as **"Damping"**.

- **Damping factor (ξ)**

The damping in any system is measured by a factor or ratio which is known as damping ratio.

It is denoted by ξ (Zeta)

Analysis of second order system

when zeta is maximum ; it produces maximum opposition to the oscillatory behavior of system.

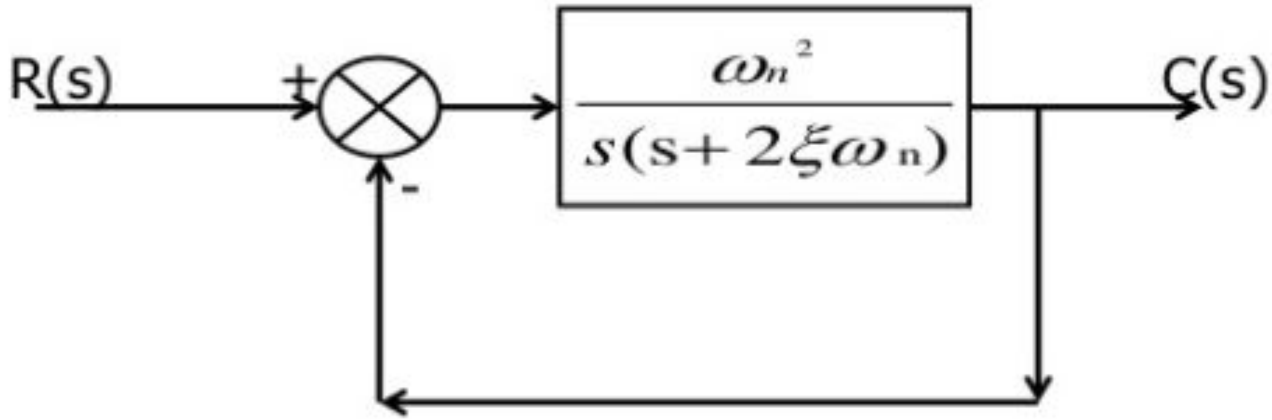
Natural frequency of oscillation :

When zeta is zero ; that means there is no opposition to the oscillatory behavior of a system then the system will oscillate naturally.

Thus when zeta is zero the system oscillates with max frequency.

Analysis Of Second Order Control System for step Input

Consider a second order system as shown;



Here $G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)}$ and $H(s) = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{\frac{\omega_n^2}{s(s+2\xi\omega_n)}}{1+\frac{\omega_n^2}{s(s+2\xi\omega_n)}} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

This is the standard form of the closed loop transfer function

These poles of transfer function are given by;

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$$

$$\therefore s = \frac{-2\xi\omega_n \pm \sqrt{(2\xi\omega_n)^2 - 4(\omega_n)^2}}{2}$$

$$= -\xi\omega_n \pm \sqrt{\xi^2\omega_n^2 - \omega_n^2}$$

$$= -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

Analysis Of Second Order Control System for step Input

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Now,\ r(t) = 1\ or\ R(s) = \frac{1}{s}$$

$$\therefore C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Analysis Of Second Order Control System for step Input

$$\frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{A}{s} + \frac{(BS+C)}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad \ldots\ldots(1)$$

$$\frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{A(s^2 + 2\zeta\omega_n s + \omega_n^2) + (Bs+c)S}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

After solving equation, we get
A=1, B=-1, C=$-2\zeta\omega_n$

Analysis Of Second Order Control System for step Input

$$\therefore C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2 - \zeta^2\omega_n^2}$$

$$= \frac{1}{s} \cdot \frac{\omega_n^2}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

From equation 1..

$$C(S) = \frac{1}{s} - \frac{s + 2s\zeta\omega_n}{(s + \zeta\omega_n)^2 + \omega_n^2(1 - \zeta^2)}$$

Analysis Of Second Order Control System for step Input

$$C(S) = \frac{1}{s} - \frac{s + 2\ \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2\,(1-\zeta^2)}$$

$$Putting,\ \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$\begin{aligned}
&= \frac{1}{s} - \frac{s + 2\ \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_n^2\,(1-\zeta^2)} \\
&= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} \\
&= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{\omega_d} \cdot \frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}
\end{aligned}$$

Analysis Of Second Order Control System for step Input

Taking inverse Laplace transform of above equation, we get,

$$\mathcal{L}^{-1}[C(s)] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\zeta\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2}\right] - \mathcal{L}^{-1}\left[\frac{\zeta\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\zeta\omega_n)^2+\omega_d^2}\right]$$

$$\therefore c(t) = 1 - e^{-\zeta\omega_n t}\cdot\cos\omega_d t - \frac{\zeta\omega_n}{\omega_d}\cdot e^{-\zeta\omega_n t}\cdot\sin\omega_d t$$

$$\because \mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1,\ \mathcal{L}^{-1}\left[\frac{s+\alpha}{(s+\alpha)^2+\omega^2}\right] = e^{-\alpha t}\cos\omega t,$$

$$\mathcal{L}^{-1}\left[\frac{\omega}{(s+\alpha)^2+\omega^2}\right] = e^{-\alpha t}\sin\omega t$$

Analysis Of Second Order Control System for step Input

The above expression of output c(t) can be rewritten as

$$c(t) = 1 - e^{-\zeta\omega_n t}\left(\cos\omega_d t + \frac{\zeta}{\sqrt{1-\zeta^2}}\cdot\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos\omega_d t + \zeta\cdot\sin\omega_d t\right)$$

$$\left[Say,\ \zeta = \cos\phi,\ hence,\ \sqrt{1-\zeta^2} = \sin\phi\right]$$

$$\therefore c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\phi\cos\omega_d t + \cos\phi\sin\omega_d t\right)$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \phi\right)$$

The error of the signal of the response is given by e(t) = r (t) – c(t), and hence.

$$e(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t + \phi\right)$$

Analysis Of Second Order Control System for step Input

From the above expression it is clear that the error of the signal is of oscillation type with exponentially decaying magnitude when ζwd and the time constant of exponential decay is $1/\zeta\omega_n$. Where, ω_d, is referred as damped frequency of the oscillation, and ω_n is natural frequency of the oscillation. The term ζ affects that damping a lot and hence this term is called damping ratio.

The poles are;

(i) Real and Unequal if $\sqrt{\xi^2 - 1} > 0$ (Over Damped)

i.e. $\xi > 1$ They lie on real axis and distinct

(ii) Real and equal if $\sqrt{\xi^2 - 1} = 0$ (Critically Damped)

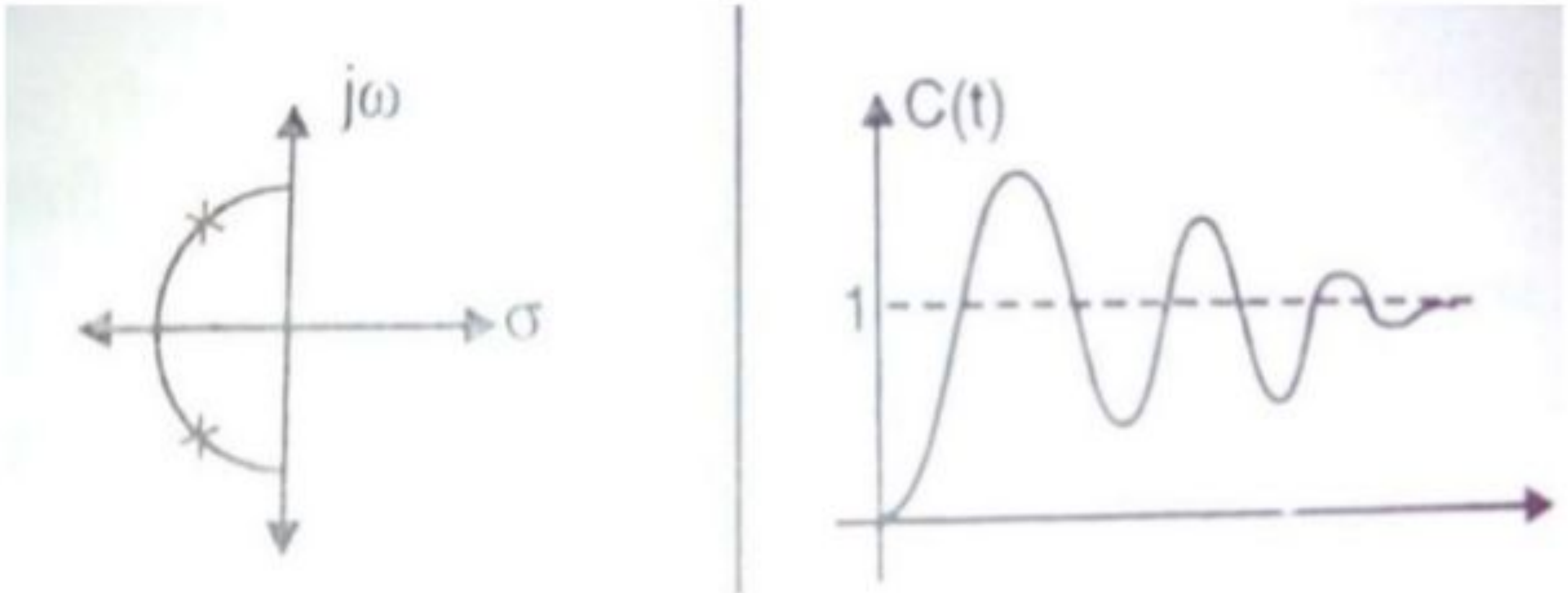
i.e. $\xi = 1$ They are repeated on real axis

(iii) Complex if $\sqrt{\xi^2 - 1} < 0$ Under Damped

i.e. $\xi < 1$ Poles are in second and third quadrant

Relation between ξ and pole locations

(i) $0 < \xi < 1$ Under damped

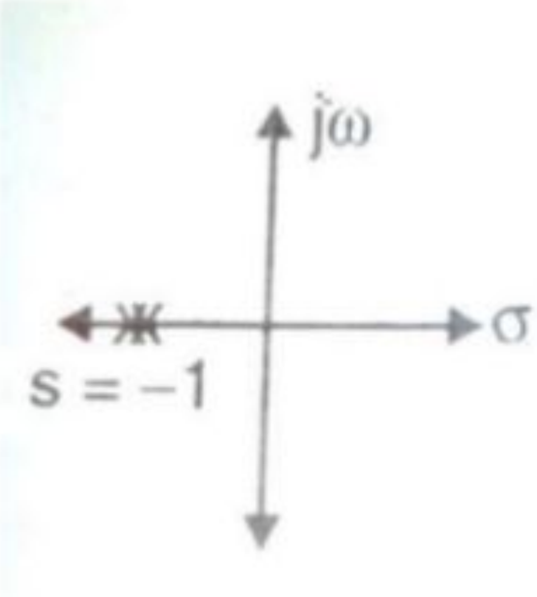


Pole Location

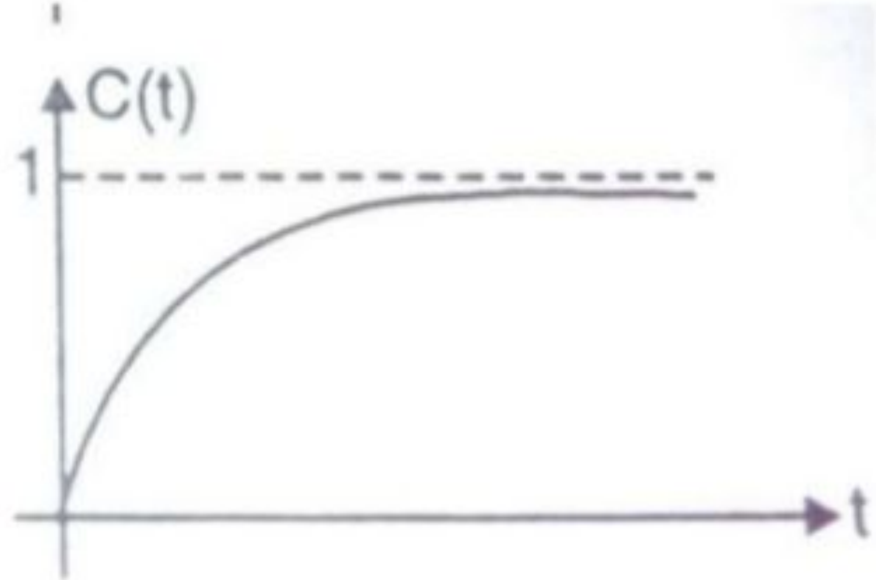
Step Response c(t)

Relation between ξ and pole locations

(ii) $\xi = 1$ Critically damped



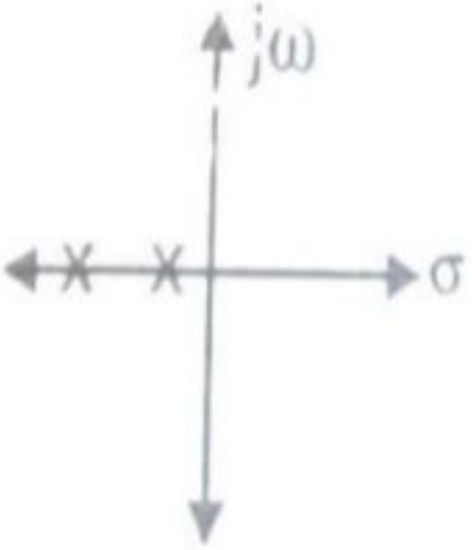
Pole Location



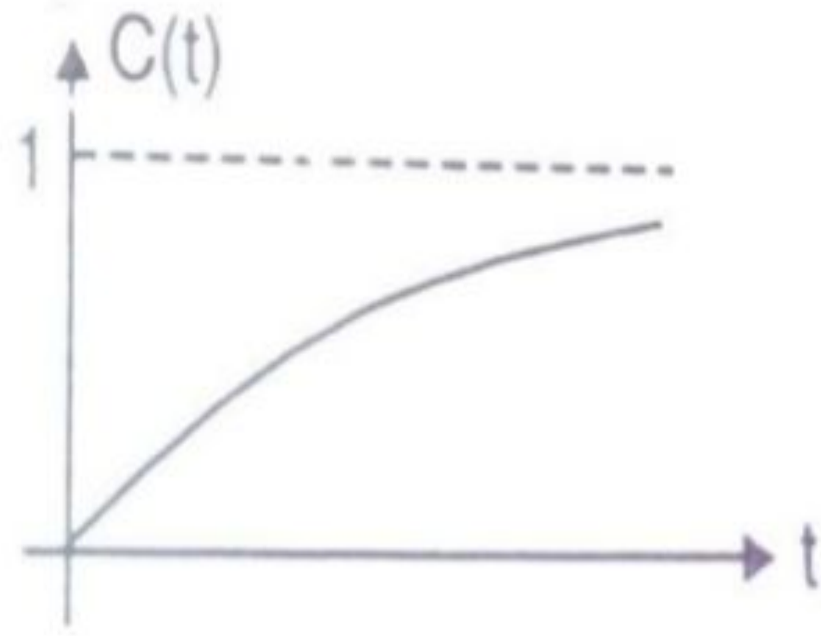
Step Response c(t)

Relation between ξ and pole locations

(iii) $\xi > 1$ over damped



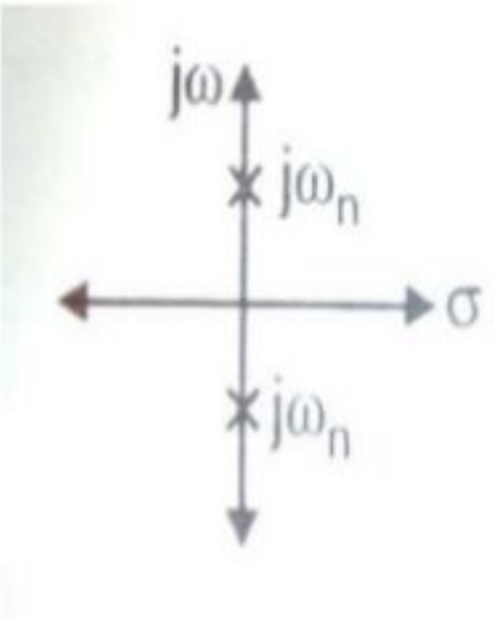
Pole Location



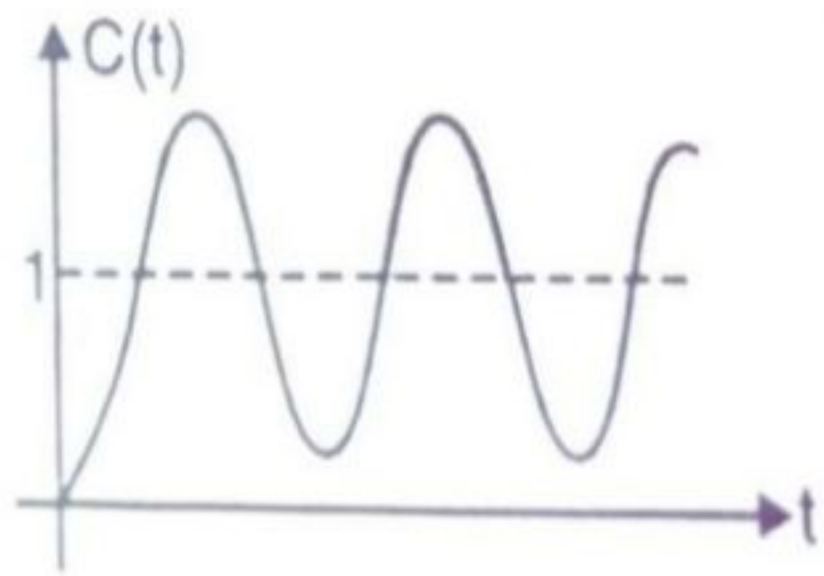
Step Response c(t)

Relation between ξ and pole locations

(iv) $\xi = 0$ Undamped



Pole Location



Step Response c(t)