* Expectations: $(E(x))$

* For discrete variable:

let X be the discrete variable and $x_1, x_2, \cdots, x_n, \cdots$ are values of X with the probabilities $P(x_1), P(x_2), \cdots, P(x_n), \cdots$ respectively then the expectation of X is denoted by '$E(X)$' and is defined as

$$E(X) = P(x_1)\, x_1 + P(x_2)\, x_2 + \cdots + P(x_n)\, x_n + \cdots$$

i.e. $$E(X) = \sum_i P(x_i) \cdot x_i$$

In short, $$\boxed{E(X) = \sum_i P_i\, x_i}$$ where, $\sum_i P_i = 1$

* For contineous Variable;

let X be the contineous random variable with probability density function $f(x)$ then the Expectations of X is denoted by $E(X)$ and is

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx}$$ where, $\int_{-\infty}^{\infty} f(x)\, dx = 1$

Note that ① Expectation of constant is constant

② If X and Y are any two variables

then $E(X+Y) = E(X) + E(Y)$

and $E(X-Y) = E(X) - E(Y)$

③ If X and Y are independent variables

then $E(XY) = E(X) \cdot E(Y)$

Example 1: Three urns contains respectively 3 green and 2 white balls, 5 green and 6 white balls, 2 green and 4 white balls. One ball is drawn from each urn. find the expected number of white ball drawn.

Solution:

let X be the number of white ball drawn from urn and If white ball is drawn then $X=1$ and if green ball is drawn the $X=0$

Note that $E(X) = \sum_i p_i x_i$

* for first urn : $E(X_1) = p_1 x_1 + p_2 x_2$

$$= \frac{2}{5}(1) + \frac{3}{5}(0)$$

$$= \frac{2}{5}$$

* for second urn : $E(X_2) = \frac{6}{11}(1) + \frac{5}{11}(0)$

$$= \frac{6}{11}$$

* for third urn : $E(X_3) = \frac{4}{6}(1) + \frac{2}{6}(0)$

$$= \frac{2}{3}$$

$\therefore$ The Required Expectation $= E(X_1) + E(X_2) + E(X_3)$

$$= \frac{2}{5} + \frac{6}{11} + \frac{2}{3}$$

$$= \frac{266}{165}$$

$$E(X) = \underline{\underline{1.61}}$$

<u>Ex ②</u>. There are 10 counters in a bag, 6 of which are worth 5 rupees each while the remaining 4 are equal but unknown value. If the expectation of drawing a single counter at random is 4 rupees, find the unknown value.

<u>Solution:</u>

let k be the value of remaining 4 counters

Now from the given data

$p(x_1) = p(\text{of counter worth of ₹ } 5) = \frac{6}{10}$

$p(x_2) = p(\text{of Counter of unknown value}) = \frac{4}{10}$

$\therefore \quad E(X) = \sum_i p_i x_i$

$\Rightarrow \quad 4 = \frac{6}{10}(5) + \frac{4}{10}(k)$

$\therefore \quad 40 = 30 + 4k$

$\Rightarrow \quad 4k = 10$

$\Rightarrow \quad \boxed{k = 2.5 \text{ ₹}}$

Example 3. The daily consumption of electric power (in millon kwh) is a random variable X with probability density function

$$f(x) = \begin{cases} kx e^{-\frac{x}{3}} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

find the value of k, the expectation of x and the probability that on a given day the electric consumption is more than expected value.

Solution:

Note that the total probability is 1

$$\therefore P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\Rightarrow \int_0^{\infty} kx e^{-\frac{x}{3}}\, dx = 1$$

$$\Rightarrow k \int_0^{\infty} x e^{-\frac{x}{3}}\, dx = 1$$

$$\Rightarrow k \left[x \left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}} \right) - (1) \left(\frac{e^{-\frac{x}{3}}}{(-\frac{1}{3})(-\frac{1}{3})} \right) \right]_0^{\infty} = 1$$

$$\left(\because \int uv\, dx = uv_1 - u' v_2 + u'' v_3 - \dots \right)$$

$$\Rightarrow \quad k\left[-3x\, e^{-\frac{x}{3}} - 9\, e^{-\frac{x}{3}}\right]_0^{\infty} = 1$$

$$\Rightarrow \quad k\left[(0-0)-(0-9)\right] = 1$$

$$\Rightarrow \quad 9k = 1$$

$$\Rightarrow \quad \boxed{k = \frac{1}{9}}$$

Now, to find Expectation of X

since, $f(x) = \begin{cases} \frac{1}{9} x e^{-\frac{x}{3}} & , \text{for } x > 0 \\ 0 & , \text{for } x \leq 0 \end{cases}$

Note that $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_0^{\infty} x \cdot \frac{1}{9} x e^{-\frac{x}{3}}$$

$$= \frac{1}{9} \int_0^{\infty} x^2 e^{-\frac{x}{3}}\, dx$$

But $\int uv\, dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \cdots$

$$\therefore \quad E(X) = \frac{1}{9}\left[(x^2)\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right) - (2x)\left(\frac{e^{-\frac{x}{3}}}{(-\frac{1}{3})(-\frac{1}{3})}\right) + 2\left(\frac{e^{-\frac{x}{3}}}{(-\frac{1}{3})(-\frac{1}{3})(-\frac{1}{3})}\right)\right]_0^{\infty}$$

$$= \frac{1}{9}\left[-3x^2 e^{-\frac{x}{3}} - 18x e^{-\frac{x}{3}} - 54 e^{-\frac{x}{3}}\right]_0^{\infty}$$

$$= \frac{1}{9}\left[(0-0-0) - (0-0-54)\right]$$

$$= \frac{1}{9}(54)$$

$$\Rightarrow \quad \underline{E(X) = 6}$$

To find the probability that on a given day the electric consumption is more than expected value.

$$\therefore \quad P(X>6) = \int_6^\infty f(x)\,dx = \frac{1}{9}\int_6^\infty x\,e^{-\frac{x}{3}}\,dx.$$

$$= \frac{1}{9}\left[(x)\left(\frac{e^{-\frac{x}{3}}}{-\frac{1}{3}}\right) - (1)\left(\frac{e^{-\frac{x}{3}}}{\frac{1}{9}}\right)\right]_6^\infty$$

$$= \frac{1}{9}\left[(0-0)-(-18e^{-2}-9e^{-2})\right]$$

$$= 3e^{-2} = \underline{\underline{0.406}}.$$

Ex. 4. find the expection of number of failures preceeding the first success is an infinite series of independent trials with constant probabilities p and q of success and failure respectively.

solution: from the given data we can make the probability distribution table

X	:	0	1	2	3	...
$P(X=x)$	:	p	qp	q^2p	q^3p	...

Since, we may get sucess in the first trial where the number of failures $X=0$ and the probability is p; we may get success in the second trials when number of failures $X=1$ and the probability is qp and so on.

$$\therefore \quad E(X) = \sum_i p_i x_i = p_1x_1 + p_2x_2 + \cdots$$

$$= p(0) + qp(1) + q^2p(2) + \cdots$$

$$= qp(1+2q+3q^2+\cdots) = qp\left[\frac{1}{(1-q)^2}\right] = \frac{qp}{p^2}$$

$$= \underline{\underline{\frac{q}{p}}}$$