* Discrete random variable :

let X be a random variable. If X takes finite or Countably infinite values $x_0, x_1, \cdots$ then X is called discrete random variable.

for ex. 1) $X = 0, 1, 2, 3, 4, \cdots$

2) $X = 0, 2, 4, 6, \cdots$

* Contineous random variable.

let X be a random variable. If X takes uncountably infinite values in given interval. then X is called contineous random variable.

for ex. i) $X = [0, 1]$

* probability mass function (p.m.f.) :
(probability density function)

let X be a discrete random variable. and $x_1, x_2, \cdots x_n, \cdots$ be the possible values of X and $p(x_1), p(x_2), \cdots, p(x_n), \cdots$ be the curresponding probabilities then the function p is called probability mass function if

1) $p(x_i) \geq 0$ for all i

ii) $\sum_i p(x_i) = 1$

* Comulative distribution function : (c.d.f.)

let X be discrete random variable and $x_1, x_2, \cdots, x_n, \cdots$ be the possible values of X and $P(x_1), P(x_2), \cdots, P(x_n), \cdots$ be the corresponding probability such that $P(x_i) \geq 0$ for all i and $\sum_i P(x_i) = 1$

we define a function F as

$$F(x_i) = P(X \leq x_i) \quad , i = 1, 2, 3, \cdots$$

i.e. $$\boxed{F(x_i) = P(x_1) + P(x_2) + \cdots + P(x_i)}$$

then function F is called cumulative distribution function

Note that :

$$F(x_1) = P(x_1)$$
$$F(x_2) = P(x_1) + P(x_2)$$
$$F(x_3) = P(x_1) + P(x_2) + P(x_3)$$
$$\vdots$$

and so on

Ex. 1 A pair of fair dice is rolled once. let X be the random variable whose value for any outcomes is the sum of two number on dice

i> find the probability function of X and construct the probability table

ii> find the probability that X is an odd number

iii> find the probability that X lies between 3 and 9

solution: Given Experiment: A pair of fair dice is rolled once

Random variable X = sum of two number on dice

∴ sample space = S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6)
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6)
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)
(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6)
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

1> ∴ $n(S) = 36$

* probability distribution table :

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

ii) If X is odd number

i.e if $X = 3, 5, 7, 9, 11$

$\therefore P(X \text{ is odd}) = P(X=3) + P(X=5) + P(X=7) + P(X=9) + P(X=11)$

$$= \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36}$$

$$= \frac{18}{36}$$

$$= \frac{1}{2}$$

iii) If X lies between 3 and 9

i.e $X = 3, 4, 5, 6, 7, 8, 9$

$\therefore P(3 \leq X \leq 9) = P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8) + P(X=9)$

$$= \frac{2}{36} + \frac{3}{36} + \frac{4}{36} + \frac{5}{36} + \frac{6}{36} + \frac{5}{36} + \frac{4}{36}$$

$$= \frac{29}{36}$$

Ex. 2. A random variable X has following Probability function.

X	1	2	3	4	5	6	7
	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

find i) k ii) $P(X<5)$ iii) $P\left(\frac{X<5}{2<X\leq 6}\right)$

iv) $P\left(\frac{X=4}{3\leq X\leq 5}\right)$

Solution:

i) Note that $\sum P(x_i) = 1$

$$\Rightarrow \quad P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$$

$$\Rightarrow \quad k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2 = 1$$

$$\Rightarrow \quad 8k^2 + 7k - 1 = 0$$

$$\Rightarrow \quad (8k-1)(k+1) = 0$$

$$\Rightarrow \quad k = \frac{1}{8} \quad \text{or} \quad k = -1$$

But $k = -1$ is not possible $\left(\because P(x_i) \geq 0, \forall x_i\right)$

Hence, $\boxed{k = \frac{1}{8}}$

Now we put $k = \frac{1}{8}$ in given probability distribution table

X	1	2	3	4	5	6	7
$P(X=x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{64}$	$\frac{9}{64}$	$\frac{2}{64}$	$\frac{4}{64}$

ii) $P(X < 5) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$$= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{1}{64}$$

$$= \frac{49}{64}$$

iii) $P\left(\frac{X<5}{2<X\leq 6}\right) = P\left[(X<5)/(2<X\leq 6)\right]$

$$= \frac{P\left[(X<5) \cap (2<X\leq 6)\right]}{P(2<X\leq 6)} \qquad \left(\because P(A/B) = \frac{P(A\cap B)}{P(B)}\right)$$

$$= \frac{P[(X=1,2,3,4) \cap (X=3,4,5,6)]}{P(X=3,4,5,6)}$$

$$= \frac{P(X=3,4)}{P(X=3,4,5,6)}$$

$$= \frac{P(X=3)+P(X=4)}{P(X=3)+P(X=4)+P(X=5)+P(X=6)}$$

$$= \frac{\frac{3}{8}+\frac{1}{64}}{\frac{3}{8}+\frac{1}{64}+\frac{9}{64}+\frac{2}{64}}$$

$$= \frac{\frac{25}{64}}{\frac{36}{64}}$$

$$= \underline{\underline{\frac{25}{36}}}$$

iv) $P\left(\frac{X=4}{3 \leq X \leq 5}\right) = P\left[(X=4)/(3 \leq X \leq 5)\right]$

$$= \frac{P[(X=4) \cap (3 \leq X \leq 5)]}{P(3 \leq X \leq 5)} \quad \left(\because P(A/B) = \frac{P(A \cap B)}{P(B)}\right)$$

$$= \frac{P[(X=4) \cap (X=3,4,5)]}{P(X=3,4,5)}$$

$$= \frac{P(X=4)}{P(X=3,4,5)}$$

$$= \frac{\frac{1}{64}}{\frac{3}{8}+\frac{1}{64}+\frac{9}{64}}$$

$$= \frac{\frac{1}{64}}{\frac{34}{64}}$$

$$= \underline{\underline{\frac{1}{34}}}$$

* Probability density function of contineous Random variable :

let $f(x)$ be the contineous function defined on $[a, b]$ and X be the contineous random variable then probability density function of X is

$$\boxed{P(\alpha \leq X \leq \beta) = \int_{\alpha}^{\beta} f(x)\, dx}, \text{ where } a < \alpha < \beta < b$$

Ex 1. A contineous random variable X has following probability law $f(x) = kx^2$, $0 \leq x \leq 2$

Determine k and find the probability that

i> $0.2 \leq X \leq 0.5$

ii> $X \geq \frac{3}{4}$ given that $X \geq \frac{1}{2}$

solution! given: $f(x) = kx^2$, $x \in [0, 2]$

Note that the Total probability is 1

$$\Rightarrow \quad P(0 \leq X \leq 2) = 1$$

$$\Rightarrow \quad \int_0^2 f(x)\, dx = 1$$

$$\Rightarrow \quad \int_0^2 kx^2\, dx = 1$$

$$\Rightarrow \quad k\left[\frac{x^3}{3}\right]_0^2 = 1$$

$$\Rightarrow \quad k\left[\frac{(2)^3}{3} - \frac{(0)^3}{3}\right] = 1 \quad \Rightarrow \boxed{k = \frac{3}{8}}$$

i> $P(0.2 \leq X \leq 0.5) = \int_{0.2}^{0.5} f(x)\, dx$

$= \int_{0.2}^{0.5} k x^2\, dx = k \int_{0.2}^{0.5} x^2\, dx$

$= k \left[\frac{x^3}{3}\right]_{0.2}^{0.5}$

$= k \left[\frac{(0.5)^3}{3} - \frac{(0.2)^3}{3}\right]$

$= \frac{3}{8}\left[\frac{(0.5)^3}{3} - \frac{(0.2)^3}{3}\right] \quad \left(\because k = \frac{3}{8}\right)$

$= 0.0123$

ii> To find probability that $X \geq \frac{3}{4}$ given that $X \geq \frac{1}{2}$

let $A = (X \geq \frac{3}{4})$ and $B = (X \geq \frac{1}{2})$

$\therefore P(A) = P(X \geq \frac{3}{4}) = P(\frac{3}{4} \leq X \leq 2)$

$= \int_{3/4}^{2} f(x)\, dx = \int_{3/4}^{2} k x^2\, dx$

$= k \left[\frac{x^3}{3}\right]_{3/4}^{2} = \frac{3}{8}\left[\frac{(2)^3}{3} - \frac{(\frac{3}{4})^3}{3}\right]$

$= 0.947$

and $P(B) = P(X \geq \frac{1}{2}) = P(\frac{1}{2} \leq X \leq 2)$

$= \int_{\frac{1}{2}}^{2} f(x)\, dx = \int_{\frac{1}{2}}^{2} k x^2\, dx$

$= k \left[\frac{x^3}{3}\right]_{\frac{1}{2}}^{2} = \frac{3}{8}\left[\frac{(2)^3}{3} - \frac{(\frac{1}{2})^3}{3}\right]$

$= 0.984$

$\therefore \quad P(A \cap B) = P\left[\left(X \geq \frac{3}{4}\right) \cap \left(X \geq \frac{1}{2}\right)\right]$

$= P\left(X \geq \frac{3}{4}\right) = P(A) = 0.947$

Therefore, $P(A/B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{0.947}{0.984}$

$\Rightarrow \quad P(A/B) = 0.96$

Ex 2. let X be a contineous random variable with probability distribution

$$p(x) = \begin{cases} \frac{x}{6} + k & , \text{ if } 0 \leq x \leq 3 \\ 0 & , \text{ elsewhere} \end{cases}$$

Evaluate k and find $p(1 \leq X \leq 2)$

Solution: Note that the total probability is 1

$\therefore \quad P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1.$

$\Rightarrow \quad \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$

$\Rightarrow \quad 0 + \int_{0}^{3} \left(\frac{x}{6} + k\right) dx + 0 = 1$

$\Rightarrow \quad \left[\frac{x^2}{12} + kx\right]_0^3 = 1$

$\Rightarrow \quad \left[\left(\frac{(3)^2}{12} + k(3)\right) - \left(\frac{(0)^2}{12} + k(0)\right)\right] = 1$

$\Rightarrow \quad \frac{9}{12} + 3k = 1$

$\Rightarrow \quad 3k = 1 - \frac{9}{12} \quad \Rightarrow \quad 3k = \frac{3}{12}$

$\Rightarrow \quad \boxed{k = \frac{1}{12}}$

Now, $P(1 \leq X \leq 2) = \int_1^2 f(x)\,dx = \int_1^2 \left(\frac{x}{6} + k\right) dx$

$$= \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx$$

$$= \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \left[\left(\frac{(2)^2}{12} + \frac{(2)}{12}\right) - \left(\frac{(1)^2}{12} + \frac{(1)}{12}\right)\right]$$

$$= \frac{4}{12} + \frac{1}{6} - \frac{1}{12} - \frac{1}{12}$$

$$= \frac{1}{3}$$

Ex 3. Let X be a contineous random variable with p.d.f. $f(x) = kx(1-x)$, $0 \leq x \leq 1$

find k and

determine a number 'b' such that $P(X \leq b) = P(X \geq b)$

Solution: Note that total probability $= 1$

$\therefore \quad P(0 \leq X \leq 1) = 1$

$\Rightarrow \int_0^1 f(x)\,dx = 1 \quad \Rightarrow \int_0^1 kx(1-x)\,dx = 1$

$\Rightarrow k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \quad \Rightarrow k\left[\frac{(1)^2}{2} - \frac{(1)^3}{3}\right] = 1$

$\Rightarrow \boxed{k = 6}$

Note that total probability $= 1$

$\therefore \quad P(0 \leq X \leq b) = P(b \leq X \leq 1)$

$\Rightarrow \int_0^b f(x)\,dx = \int_b^1 f(x)\,dx$

$\Rightarrow \int_0^b 6x(1-x)\,dx = \int_b^1 6x(1-x)\,dx$

$$\Rightarrow \quad 6\int_0^b (x - x^2)\,dx = 6\int_b^1 (x-x^2)\,dx$$

$$\Rightarrow \quad 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_b^1$$

$$\Rightarrow \quad 6\left[\left(\frac{b^2}{2} - \frac{b^3}{3}\right) - \left(\frac{(0)^2}{2} - \frac{(0)^3}{3}\right)\right] = 6\left[\left(\frac{(1)^2}{2} - \frac{(1)^3}{3}\right) - \left(\frac{b^2}{2} - \frac{b^3}{3}\right)\right]$$

$$\Rightarrow \quad 3b^2 - 2b^3 = 1 - 3b^2 + 2b^3$$

$$\Rightarrow \quad 6b^2 - 4b^3 - 1 = 0$$

$$\Rightarrow \quad 4b^3 - 6b^2 + 1 = 0$$

$$\Rightarrow \quad \boxed{b = \frac{1}{2}}$$