* Cauchy Integral Formula :-

Statement : Let $f(z)$ be the analytic everywhere on and inside a simple closed contour C taken in the anticlockwise direction.

If z_0 is any point interior to C then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0}\, dz$$

that is

$$\boxed{\int_C \frac{f(z)}{z-z_0}\, dz = 2\pi i\, f(z_0)}$$

⇒ **Important Note:**

The General Cauchy Integral formula is

$$\boxed{\int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0)}$$

In perticular,

* $\displaystyle\int_C \frac{f(z)}{(z-z_0)^2}\, dz = \frac{2\pi i}{1!} f'(z_0)$, for $n=1$

* $\displaystyle\int_C \frac{f(z)}{(z-z_0)^3}\, dz = \frac{2\pi i}{2!} f''(z_0)$, for $n=2$

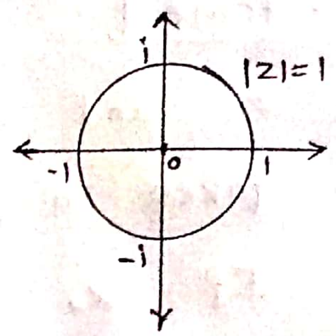
⋮

and so on.

* Examples on Cauchy Integral formula :

Example ① Evaluate $\int_C \frac{z^3-6}{3z-i}\,dz$, where C is $|z|=1$

Solution! Given : $C: \quad |z|=1$



let $f(z) = z^3 - 6$

Note that $\int_C \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\int_C \frac{z^3-6}{\left(z-\frac{i}{3}\right)}$ — ①

clearly, $f(z) = z^3-6$ is analytic everywhere on and inside C

only $z_0 = \frac{i}{3}$ lie inside C with order 1

$\therefore$ By cauchy integral formula

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\Rightarrow \int_C \frac{z^3-6}{z-\frac{i}{3}}\,dz = 2\pi i\left(\left(\frac{i}{3}\right)^3 - 6\right)$$

$$= 2\pi i\left(\frac{-i}{27} - 6\right)$$

$$= -2\pi i\left(\frac{i}{27} + 6\right)$$

$\therefore$ equation ① becomes

$$\int_C \frac{z^3-6}{3z-i}\,dz = \frac{1}{3}\int_C \frac{z^3-6}{z-\frac{i}{3}}\,dz$$

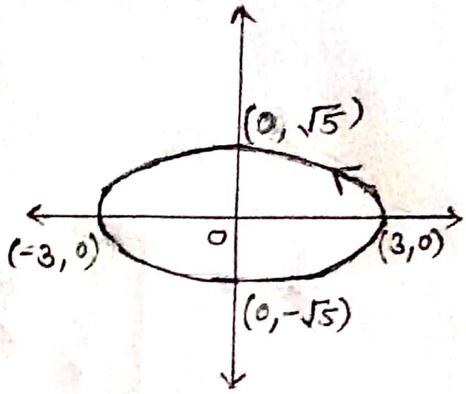
$$= \frac{1}{3}\left[-2\pi i\left(\frac{i}{27}+6\right)\right]$$

$$= \frac{-2\pi i}{3}\left(\frac{i}{27}+6\right)$$

i.e. $$\boxed{\int_C \frac{z^3-6}{3z-i}\,dz = -\frac{2\pi i}{3}\left(\frac{i}{27}+6\right)}$$

Example ② Evaluate $\int_C \frac{e^{3z}}{z-i}\,dz$, where C is the curve $|z-2|+|z+2|=6$

Solution: Given: $C: |z-2|+|z+2|=6$



Note that $|z-2|+|z+2|=6$

$\Rightarrow |(x+iy)-2|+|(x+iy)+2|=6$

$\Rightarrow |(x-2)+iy|+|(x+2)+iy|=6$

$\Rightarrow \sqrt{(x-2)^2+y^2}+\sqrt{(x+2)^2+y^2}=6$

if $y=0$ then $\sqrt{(x-2)^2}+\sqrt{(x+2)^2}=6$

$\Rightarrow x-2+x+2=6 \Rightarrow x=3$

if $x=0$ then $\sqrt{(-2)^2+y^2}+\sqrt{(2)^2+y^2}=6$

$\Rightarrow 2\sqrt{y^2+4}=6$

$\Rightarrow \sqrt{y^2+4}=3$

$\Rightarrow y^2+4=9$

$\Rightarrow y^2=5$

$\Rightarrow y=\pm\sqrt{5}$

$\therefore$ Intersection points of ellipse with x-axis are $(-3,0)$, $(3,0)$

and Intersection points of ellipse with y axis are $(0,\sqrt{5})$, $(0,-\sqrt{5})$

Now, let $f(z)=e^{3z}$

then clearly, $f(z)=e^{3z}$ is analytic everywhere on and inside C

only $z_0=i$ lie inside C with order 1

$\therefore$ By Cauchy integral formula,

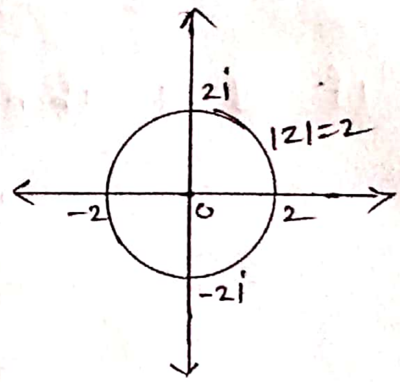
$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$= 2\pi i\, f(i)$$

$$= 2\pi i\, e^{3i}$$

$$\Rightarrow \boxed{\int_C \frac{e^{3z}}{z-i}\,dz = 2\pi i\, e^{3i}}$$

Example ③ Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z|=2$



Solution: $C: |z|=2$

Note that $\int_C \frac{3z^2+z}{z^2-1}\,dz = \int_C \frac{3z^2+z}{(z+1)(z-1)}$

here, Both $z=1$, $z=-1$ lie inside C

$\therefore$ we use partial fraction

Consider, $\frac{1}{(z+1)(z-1)} = \frac{A}{z+1} + \frac{B}{z-1}$

$$\Rightarrow \frac{1}{(z+1)(z-1)} = \frac{(z-1)A + (z+1)B}{(z+1)(z-1)}$$

$$\Rightarrow A(z-1) + B(z+1) = 1$$

Now, If $z=-1$ then $A(-1-1) + B(0) = 1 \Rightarrow A = -\frac{1}{2}$

If $z=1$ then $A(0) + B(1+1) = 1 \Rightarrow B = \frac{1}{2}$

$$\therefore \frac{1}{(z+1)(z-1)} = \frac{-1}{2(z+1)} + \frac{1}{2(z-1)}$$

i.e. $\dfrac{1}{(z+1)(z-1)} = \dfrac{1}{2(z-1)} - \dfrac{1}{2(z+1)}$

$\Rightarrow \dfrac{3z^2+z}{(z+1)(z-1)} = \dfrac{3z^2+z}{2(z-1)} - \dfrac{3z^2+z}{2(z+1)}$

$\Rightarrow \displaystyle\int_C \frac{3z^2+z}{(z+1)(z-1)}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{(z-1)}\,dz - \frac{1}{2}\int_C \frac{3z^2+z}{(z+1)}\,dz$

clearly, $f(z) = 3z^2+z$ is analytic everywhere on and inside C

$\therefore$ By cauchy integral formula,

$$\int_C \frac{3z^2+z}{(z+1)(z-1)}\,dz = \frac{1}{2}\,2\pi i\, f(1) - \frac{1}{2}\,2\pi i\, f(-1)$$

$$= \frac{1}{2}\,2\pi i\,\left(3(1)^2+1\right) - \frac{1}{2}\,2\pi i\,\left(3(-1)^2-1\right)$$

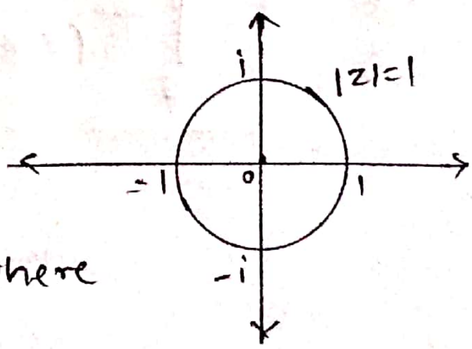
$$= \frac{1}{2}\,2\pi i\,(4) - \frac{1}{2}\,2\pi i\,(2)$$

$$= 4\pi i - 2\pi i$$

$$= 2\pi i$$

i.e. $\boxed{\displaystyle\int_C \frac{3z^2+z}{z^2-1}\,dz = 2\pi i}$

Example (4) Evaluate $\displaystyle\int_C \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3}\,dz$ where C is $|z|=1$



Solution: Given: C : $|z|=1$

let $f(z) = \sin^6 z$

clearly, $f(z)$ is analytic everywhere on and inside C

only $z_0 = \frac{\pi}{6}$ lie inside C with order 3

$\therefore$ By genral Cauchy integral formula for $n=3$ is

$$\int_C \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!}\,f''(z_0)$$

$$= \pi i\, f''\left(\frac{\pi}{6}\right) \quad \text{——} ①$$

Since, $f(z) = \sin^6 z$

$f'(z) = 6\sin^5 z \cdot \cos z$

$f''(z) = 6\left[5\sin^4 z \cdot \cos^2 z + \sin^5 z \cdot (-\sin z)\right]$

i.e. $f''(z) = 6\left[5\sin^4 z \cdot \cos^2 z - \sin^6 z\right]$

$\Rightarrow f''\left(\frac{\pi}{6}\right) = 6\left[5\cdot\sin^4\left(\frac{\pi}{6}\right)\cdot\cos^2\left(\frac{\pi}{6}\right) - \sin^6\left(\frac{\pi}{6}\right)\right]$

$= 6\left[5\cdot\left(\frac{1}{2}\right)^4\cdot\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^6\right]$

$= 6\left[5\cdot\frac{1}{16}\cdot\frac{3}{4} - \frac{1}{64}\right]$

$= 6\left[\frac{15}{64} - \frac{1}{64}\right]$

$= \frac{21}{16}$

$\therefore$ equation ① becomes,

$$\int_C \frac{f(z)}{(z-z_0)^3}\,dz = \pi i\,\frac{21}{16}$$

i.e. $$\boxed{\int_C \frac{\sin^6 z}{(z-z_0)^3}\,dz = \frac{21\pi i}{16}}$$

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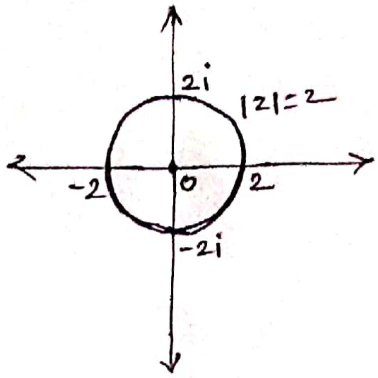
Example (5) Evaluate $\int_C \frac{dz}{z^2(z+4)}$, where C is the circle $|z| = 2$

Solution: Given: $C: |z| = 2$



Note that:

$$\int_C \frac{dz}{z^2(z+4)} = \int_C \frac{\left(\frac{1}{z+4}\right)}{z^2} dz$$

let $f(z) = \frac{1}{z+4}$

here, $z_0 = 0$, $z_1 = -4$ are singular points

clearly, $f(z)$ is analytic everywhere on and inside C

$\therefore$ only $z_0 = 0$ lie inside C

$\therefore$ By general cauchy integral formula, for $n=2$

$$\int_C \frac{f(z)}{(z-z_0)^2} dz = \frac{2\pi i}{1!} \cdot f'(z_0)$$

$$\Rightarrow \int_C \frac{\frac{1}{z+4}}{z^2} dz = 2\pi i \, f'(0)$$

$$= 2\pi i \left(-\frac{1}{(0+4)^2}\right) \quad \left(\because f'(z) = -\frac{1}{(z+4)^2}\right)$$

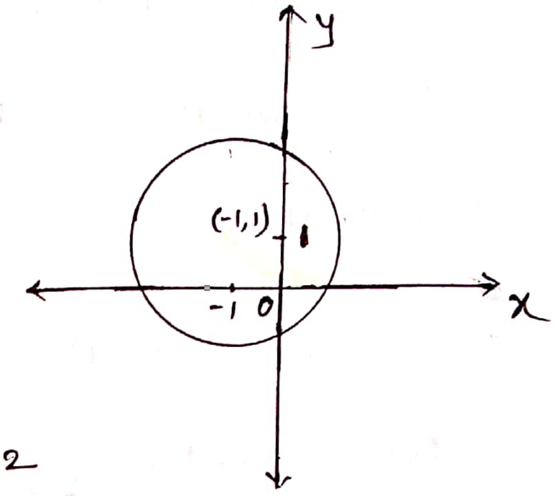
$$= 2\pi i \left(-\frac{1}{16}\right)$$

$$= -\frac{\pi i}{8}$$

i.e. $$\boxed{\int_C \frac{1}{z^2(z+4)} dz = -\frac{\pi i}{8}}$$

Example (6) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where C is the circle $|z+1-i| = 2$

Solution: Given: $C: |z+1-i| = 2$



Note that $|z+1-i| = 2$

$\Rightarrow |z-(-1+i)| = 2$

which is equation of circle with centre $(-1, 1)$ and radius 2

Now, $z^2 + 2z + 5 = 0$

$$\Rightarrow z = \frac{-2 \pm \sqrt{(2)^2 - 20}}{2} = \frac{-2 \pm 4i}{2}$$

$\Rightarrow z = -1-2i, -1+2i$

Clearly, the point $z = -1+2i = (-1, 2)$ lies inside C and the point $z = -1-2i = (-1,-2)$ lies outside C

Now,
$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{z+4}{[z-(-1-2i)][z-(-1+2i)]}\,dz$$

$$= \int_C \frac{\frac{z+4}{[z-(-1-2i)]}}{z-(-1+2i)}\,dz$$

$\therefore$ we take $f(z) = \frac{z+4}{z-(-1-2i)}$

$\Rightarrow$ $f(z)$ is analytic everywhere on and inside C and $z_0 = -1+2i$ lies inside C with order 1

$\therefore$ By Cauchy integral formula,

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

$$\Rightarrow \int_C \frac{\frac{z+4}{[z-(-1-2i)]}}{z-(-1+2i)}\,dz = 2\pi i\, f(-1+2i)$$

$$\Rightarrow \int_C \frac{z+4}{z^2+2z+5}\,dz = 2\pi i\left[\frac{(-1+2i)+4}{(-1+2i)-(-1-2i)}\right]$$

$$= 2\pi i\left[\frac{3+2i}{4i}\right]$$

$$= \frac{\pi}{2}(3+2i)$$

$$\therefore \boxed{\int_C \frac{z+4}{z^2+2z+5}\,dz = \frac{\pi}{2}(3+2i)}$$

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