Classification of Continuous Time Signals

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1. Deterministic and Nondeterministic signals
2. Periodic and Nonperiodic signals
3. Symmetric and Antisymmetric signals (Even and Odd signals)
4. Energy and Power signals
5. Causal and Noncausal signals

Deterministic and Nondeterministic Signals

The signal that can be completely specified by a mathematical equation is called a ***deterministic signal***. The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.

Examples of deterministic signals: $x_1(t) = At$

$$x_2(t) = X_m \sin \Omega_0 t$$

The signal whose characteristics are random in nature is called a ***nondeterministic signal.*** The noise signals from various sources like electronic amplifiers, oscillators, radio receivers, etc., are best examples of nondeterministic signals.

Periodic and Non Periodic Signal

A periodic signal will have a definite pattern that repeats again and again over a certain period of time. Therefore the signal which satisfies the condition,

$\boxed{x(t+T) = x(t)}$ is called a ***periodic signal.***

A signal which does not satisfy the condition, $x(t+T) = x(t)$ is called an ***aperiodic or nonperiodic signal***. In periodic signals, the term T is called the ***fundamental time period*** of the signal. Hence, inverse of T is called the ***fundamental frequency***, F_0 in cycles/sec or Hz, and $2\pi F_0 = \Omega_0$ is called the ***fundamental angular frequency*** in rad/sec.

The sinusoidal signals and complex exponential signals are always periodic with a periodicity of T,

Periodic and Non Periodic Signal

Proof :

a) Cosinusoidal signal

Let, $x(t) = A\cos\Omega_0 t$

$\therefore x(t+T) = A\cos\Omega_0(t+T) = A\cos(\Omega_0 t + \Omega_0 T)$

$= A\cos\left(\Omega_0 t + \frac{2\pi}{T}T\right)$

$= A\cos(\Omega_0 t + 2\pi) = A\cos\Omega_0 t = x(t)$

$\boxed{\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}}$

$\boxed{\cos(\theta + 2\pi) = \cos\theta}$

b) Sinusoidal signal

Let, $x(t) = A\sin\Omega_0 t$

$\therefore x(t+T) = A\sin\Omega_0(t+T) = A\sin(\Omega_0 t + \Omega_0 T)$

$= A\sin\left(\Omega_0 t + \frac{2\pi}{T}T\right)$

$= A\sin(\Omega_0 t + 2\pi) = A\sin\Omega_0 t = x(t)$

$\boxed{\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}}$

$\boxed{\sin(\theta + 2\pi) = \sin\theta}$

c) Complex exponential signal

Let, $x(t) = Ae^{j\Omega_0 t}$

$\therefore x(t+T) = Ae^{j\Omega_0(t+T)} = Ae^{j\Omega_0 t}\,e^{j\Omega_0 T} = Ae^{j\Omega_0 t}\,e^{j\frac{2\pi}{T}T} = Ae^{j\Omega_0 t}\,e^{j2\pi}$

$= Ae^{j\Omega_0 t}(\cos 2\pi + j\sin 2\pi) = Ae^{j\Omega_0 t}(1 + j\,0) = x(t)$

$\boxed{\cos 2\pi = 1,\ \sin 2\pi = 0}$

Periodic and Non Periodic Signal

When a continuous time signal is a mixture of two periodic signals with fundamental time periods T_1 and T_2, then the continuous time signal will be periodic, if the ratio of T_1 and T_2 (i.e., T_1/T_2) is a rational number. Now the periodicity of the continuous time signal will be the LCM (Least Common Multiple) of T_1 and T_2.

Note : 1. The ratio of two integers is called a rational number.

Example of rational number : $\frac{5}{2}, \frac{7}{9}, \frac{8}{11}$.

Example of non-rational number : $\frac{\sqrt{2}}{5}, \frac{7}{2\pi}, \frac{4}{\sqrt{7}}$.

2. When T_1/T_2 is a rational number, then F_{01}/F_{02} and Ω_{01}/Ω_{02} are also rational numbers.

Examples on Periodic and Non Periodic Signal

Verify whether the following continuous time signals are periodic. If periodic, find the fundamental period.

a) $x(t) = 2\cos\frac{t}{4}$ b) $x(t) = e^{\alpha t}$; $\alpha > 1$ c) $x(t) = e^{\frac{-j2\pi t}{7}}$ d) $x(t) = 3\cos\left(5t + \frac{\pi}{6}\right)$ e) $x(t) = \cos^2\left(2t - \frac{\pi}{4}\right)$

Solution

a) Given that, $x(t) = 2\cos\frac{t}{4}$

The given signal is a cosinusoidal signal, which is always periodic.

On comparing x(t) with the standard form "$A\cos 2\pi F_0 t$" we get,

$$2\pi F_0 = \frac{1}{4} \quad \Rightarrow \quad F_0 = \frac{1}{8\pi}$$

Period, $T = \frac{1}{F_0} = 8\pi$

$\therefore$ x(t) is periodic with period, $T = 8\pi$.

Examples on Periodic and Non Periodic Signal

d) Given that, $x(t) = 3\cos\left(5t + \frac{\pi}{6}\right)$

The given signal is a cosinusoidal signal, which is always periodic.

$$\therefore\ x(t+T) = 3\cos\left(5(t+T) + \frac{\pi}{6}\right) = 3\cos\left(5t + 5T + \frac{\pi}{6}\right) = 3\cos\left(\left(5t + \frac{\pi}{6}\right) + 5T\right)$$

Let $5T = 2\pi$, $\quad \therefore T = \frac{2\pi}{5}$

$$\therefore\ x(t+T) = 3\cos\left(\left(5t + \frac{\pi}{6}\right) + 5 \times \frac{2\pi}{5}\right) = 3\cos\left(\left(5t + \frac{\pi}{6}\right) + 2\pi\right)$$

$$= 3\cos\left(5t + \frac{\pi}{6}\right) = x(t)$$

For integer values of M, $\cos(\theta + 2\pi M) = \cos\theta$

Since $x(t+T) = x(t)$, the signal $x(t)$ is periodic with period, $T = \frac{2\pi}{5}$

Examples on Periodic and Non Periodic Signal

e) Given that, $x(t) = \cos^2\left(2t - \frac{\pi}{3}\right)$

$$x(t) = \cos^2\left(2t - \frac{\pi}{3}\right) = \frac{1 + \cos 2\left(2t - \frac{\pi}{3}\right)}{2} = \frac{1 + \cos\left(4t - \frac{2\pi}{3}\right)}{2}$$

$$\boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$\therefore\ x(t + T) = \frac{1 + \cos\left(4(t + T) - \frac{2\pi}{3}\right)}{2} = \frac{1 + \cos\left(4t + 4T - \frac{2\pi}{3}\right)}{2}$$

$$= \frac{1 + \cos\left(4t - \frac{2\pi}{3} + 4T\right)}{2}$$

Let $4T = 2\pi$, $\therefore\ T = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\therefore\ x(t + T) = \frac{1 + \cos\left(4t - \frac{2\pi}{3} + 4 \times \frac{\pi}{2}\right)}{2} = \frac{1 + \cos\left(\left(4t - \frac{2\pi}{3}\right) + 2\pi\right)}{2}$$

$$= \frac{1 + \cos\left(4t - \frac{2\pi}{3}\right)}{2} = \frac{1 + \cos 2\left(2t - \frac{\pi}{3}\right)}{2} = \cos^2\left(2t - \frac{\pi}{3}\right) = x(t)$$

Since $x(t + T) = x(t)$, the signal $x(t)$ is periodic with period, $T = \frac{\pi}{2}$

$$\boxed{\text{For integer values of } M,\ \cos(\theta + 2\pi M) = \cos\theta}$$

(c) **Given that, $x(t) = 5\cos 4\pi t + 3\sin 8\pi t$**

Let, $x_1(t) = 5\cos 4\pi t$

Let T_1 be the periodicity of $x_1(t)$. On comparing $x_1(t)$ with the standard form "$A\cos 2\pi F_{01}t$", we get,

$$F_{01} = 2\ ; \qquad \therefore \text{ Period, } T_1 = \frac{1}{F_{01}} = \frac{1}{2}$$

Let, $x_2(t) = 3\sin 8\pi t$

Let T_2 be the periodicity of $x_2(t)$. On comparing $x_2(t)$ with the standard form "$A\sin 2\pi F_{02}t$", we get,

$$F_{02} = 4\ ; \qquad \therefore \text{ Period, } T_2 = \frac{1}{F_{02}} = \frac{1}{4}$$

Now, $\dfrac{T_1}{T_2} = T_1 \times \dfrac{1}{T_2} = \dfrac{1}{2} \times \dfrac{4}{1} = 2$

Since $x_1(t)$ and $x_2(t)$ are periodic and the ratio of T_1 and T_2 is a rational number, the signal $x(t)$ is also periodic. Let T be the periodicity of $x(t)$. Now, the periodicity of $x(t)$ is the LCM (Least Common Multiple) of T_1 and T_2, which is calculated as shown below.

$$T_1 = \frac{1}{2} = \frac{1}{2} \times 4 = 2$$

$$T_2 = \frac{1}{4} = \frac{1}{4} \times 4 = 1$$

Note : *To find LCM, first convert T_1 and T_2 to integers by multiplying by a common number. Find LCM of integer values of T_1 and T_2. Then divide this LCM by the common number.*

Now LCM of 2 and 1 is 2.

$$\therefore \text{ Period, } T = 2 \div 4 = 2 \times \frac{1}{4} = \frac{1}{2}$$

Proof : $x(t + T) = 5\cos 4\pi(t + T) + 3\sin 8\pi(t + T)$

$$= 5\cos(4\pi t + 4\pi T) + 3\sin(8\pi t + 8\pi T)$$

$$= 5\cos\left(4\pi t + 4\pi \times \frac{1}{2}\right) + 3\sin\left(8\pi t + 8\pi \times \frac{1}{2}\right)$$

$$= 5\cos(4\pi t + 2\pi) + 3\sin(8\pi t + 2\pi)$$

$$= 5\cos 4\pi t + 3\sin 8\pi t = x(t)$$

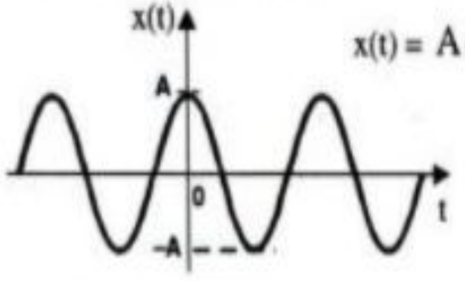
Put, $T = \dfrac{1}{2}$

For integer values of M,
$\cos(\theta + 2\pi M) = \cos\theta$
$\sin(\theta + 2\pi M) = \sin\theta$

Symmetric (Even)and Antisymmetric (Odd) Signals

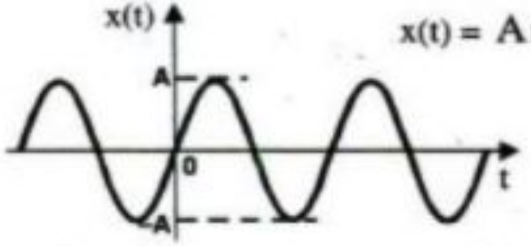
The signals may exhibit symmetry or antisymmetry with respect to $t = 0$.

When a signal exhibits symmetry with respect to $t = 0$ then it is called an ***even signal.*** Therefore, the even signal satisfies the condition, $\mathbf{x(-t) = x(t)}$.



Symmetric or Even signal.

When a signal exhibits antisymmetry with respect to $t = 0$, then it is called an ***odd signal***. Therefore, the odd signal satisfies the condition, $\mathbf{x(-t) = -x(t)}$.



Antisymmetric or Odd signal.

Symmetric (Even)and Antisymmetric (Odd) Signals

Since $\cos(-\theta) = \cos\theta$, the cosinusoidal signals are even signals and since $\sin(-\theta) = -\sin\theta$, the sinusoidal signals are odd signals.

A continuous time signal x(t) which is neither even nor odd can be expressed as a sum of even and odd signal.

Let, $x(t) = x_e(t) + x_o(t)$

where, $x_e(t)$ = Even part of x(t) and $x_o(t)$ = Odd part of x(t)

Now, it can be proved that,

$$\boxed{\begin{aligned} x_e(t) &= \frac{1}{2}[x(t) + x(-t)] \\ x_o(t) &= \frac{1}{2}[x(t) - x(-t)] \end{aligned}}$$

Calculation of even and odd part of signals

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Proof :

Let, $x(t) = x_e(t) + x_o(t)$ .....(2.1)

On replacing t by –t in equation (2.1) we get,

$$x(-t) = x_e(-t) + x_o(-t) \quad \text{.....(2.2)}$$

Since $x_e(t)$ is even, $x_e(-t) = x_e(t)$

Since $x_o(t)$ is odd, $x_o(-t) = -x_o(t)$

Hence the equation (2.2) can be written as,

$$x(-t) = x_e(t) - x_o(t) \quad \text{.....(2.3)}$$

On adding equations (2.1) & (2.3) we get,

$$x(t) + x(-t) = 2\,x_e(t)$$

$$\therefore x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

On subtracting equation (2.3) from equation (2.1) we get,

$$x(t) - x(-t) = 2\,x_o(t)$$

$$\therefore x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

Properties of signals with Symmetry

1. When a signal is even, then its odd part will be zero.
2. When a signal is odd, then its even part will be zero.
3. The product of two odd signals will be an even signal.
4. The product of two even signals will be an even signal.
5. The product of an even and odd signal will be an odd signal.

Examples on Even and Odd Signals

Determine the even and odd part of the following continuous time signals.

a) $x(t) = e^t$ b) $x(t) = 3 + 2t + 5t^2$ c) $x(t) = \sin 2t + \cos t + \sin t \cos 2t$

b) Given that, $x(t) = 3 + 2t + 5t^2$

$$\therefore x(-t) = 3 + 2(-t) + 5(-t)^2$$
$$= 3 - 2t + 5t^2$$

Even part, $x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[3 + 2t + 5t^2 + 3 - 2t + 5t^2]$

$$= \frac{1}{2}[6 + 10t^2] = 3 + 5t^2$$

Odd part, $x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[3 + 2t + 5t^2 - 3 + 2t - 5t^2]$

$$= \frac{1}{2}[4t] = 2t$$

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c) Given that, $x(t) = \sin 2t + \cos t + \sin t \cos 2t$

$$\therefore x(-t) = \sin 2(-t) + \cos(-t) + \sin(-t) \cos 2(-t)$$
$$= -\sin 2t + \cos t - \sin t \cos 2t$$

Even part, $x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[\sin 2t + \cos t + \sin t \cos 2t - \sin 2t + \cos t - \sin t \cos 2t]$

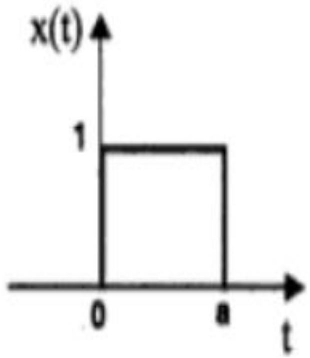
$$= \frac{1}{2}[2 \cos t] = \cos t$$

Odd part, $x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[\sin 2t + \cos t + \sin t \cos 2t + \sin 2t - \cos t + \sin t \cos 2t]$

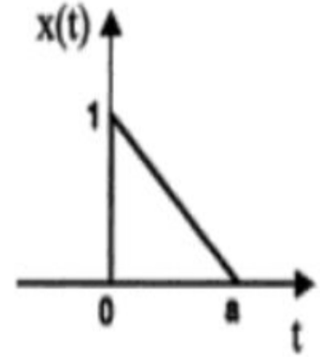
$$= \frac{1}{2}[2 \sin 2t + 2 \sin t \cos 2t] = \sin 2t + \sin t \cos 2t$$

Sketch the even and odd parts of the following signals.

a)



b)

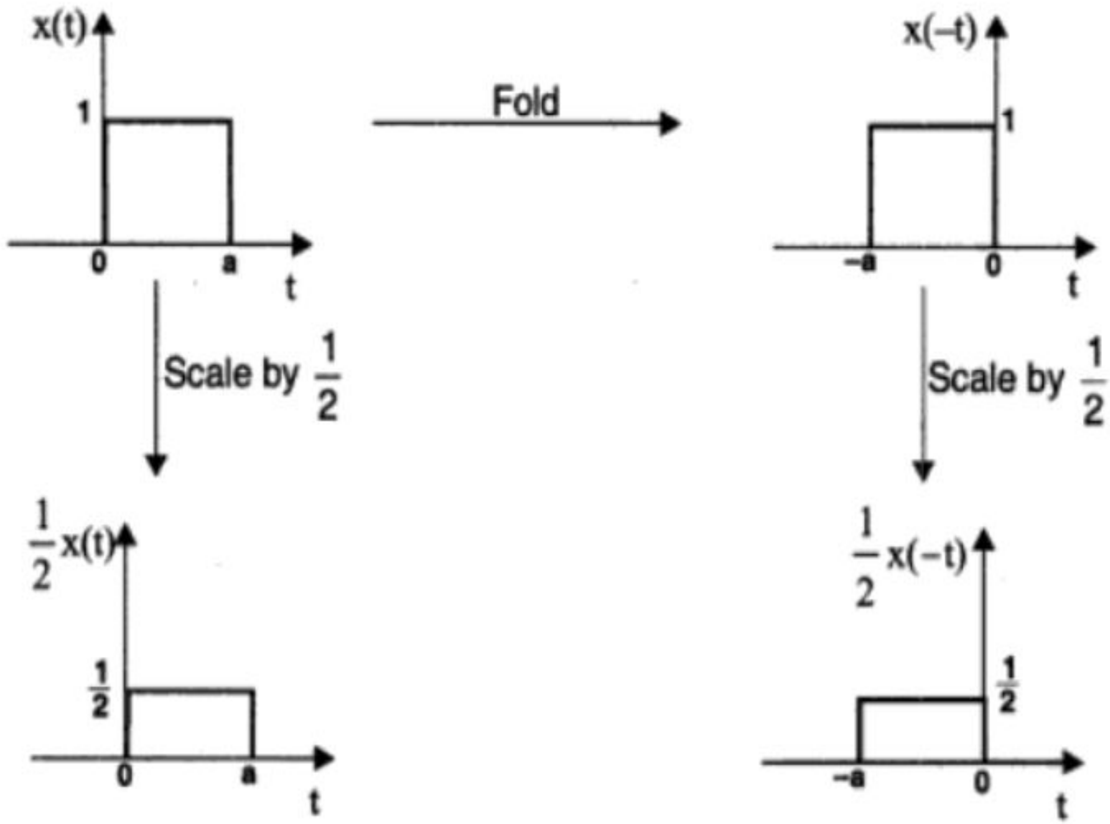


Solution

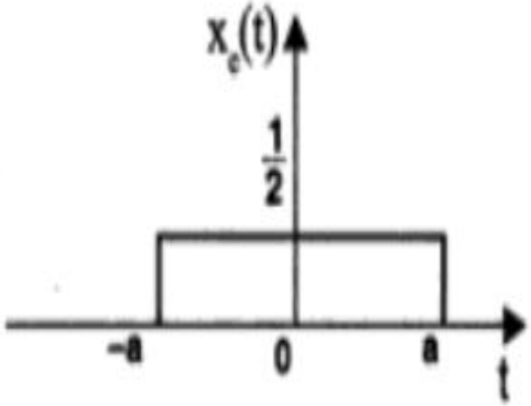
a) The even part of the signal is given by, $x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$ .....(1)

The odd part of the signal is given by, $x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$ .....(2)

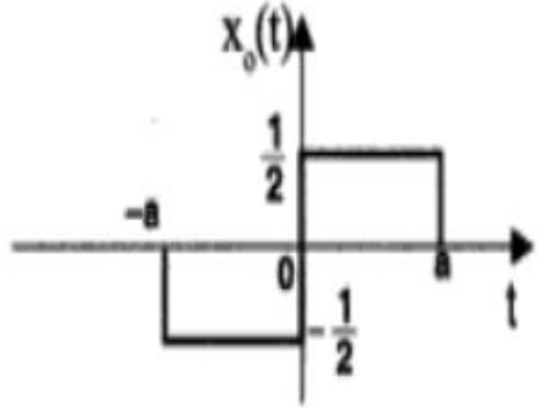
From equations (1) and (2), it is observed that the even and odd parts of the signal can be obtained from the folded and scaled versions of the signal. Hence the given signal is folded, scaled and then graphically added and subtracted to get the even and odd parts as shown below.



$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$$



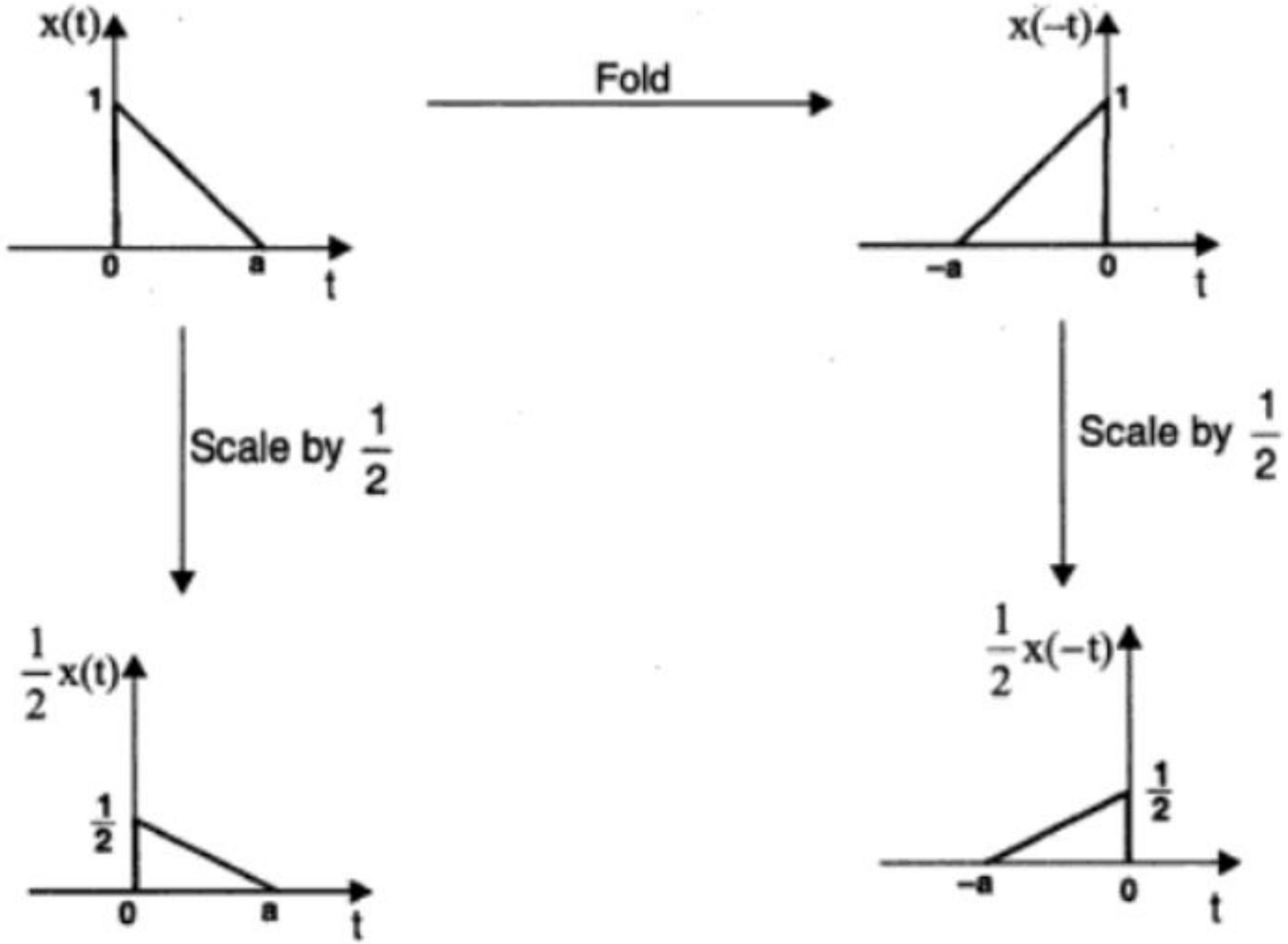
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$$



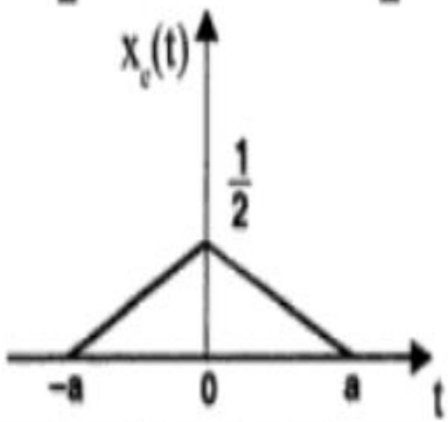
b) The even part of the signal is given by, $x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}x(t) + \frac{1}{2}x(-t)$ .....(1)

The odd part of the signal is given by, $x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}x(t) - \frac{1}{2}x(-t)$ .....(2)

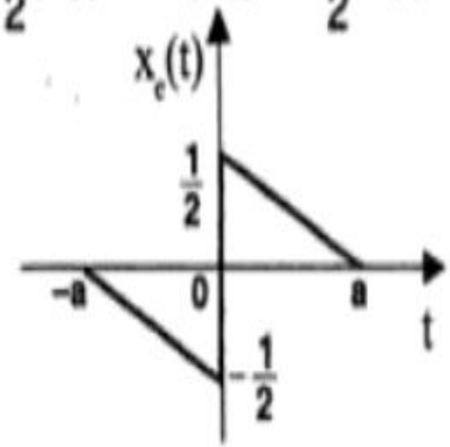
From equations (1) and (2), it is observed that the even and odd parts of the signal can be obtained from the folded and scaled versions of the signal. Hence the given signal is folded, scaled and then graphically added and subtracted to get the even and odd parts as shown below.


x(t)
1
0
a
t
Fold
x(−t)
1
−a
0
t
Scale by 1/2
Scale by 1/2
1/2 x(t)
1/2
0
a
t
1/2 x(−t)
1/2
−a
0
t

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}\,x(t) + \frac{1}{2}x(-t)$$



$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}\,x(t) - \frac{1}{2}x(-t)$$



Energy and Power signals

Energy signals

The signals which have finite energy are called ***energy signals***. The nonperiodic signals like exponential signals will have constant energy and so nonperiodic signals are energy signals.

The ***energy*** E of a continuous time signal x(t) is defined as,

$$\boxed{\text{Energy, } E = \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \text{ in joules}}$$

Power signals

The signals which have finite average power are called ***power signals***. The periodic signals like sinusoidal and complex exponential signals will have constant power and so periodic signals are power signals.

The average ***power*** of a continuous time signal x(t) is defined as,

$$\boxed{\text{Power, } P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \text{ in watts}}$$

Energy and Power signals

For periodic signals, the average power over one period will be same as average power over an infinite interval.

$$\boxed{\therefore \text{For periodic signals,}\quad \text{power, } P = \frac{1}{T}\int_0^T |x(t)|^2\, dt}$$

For energy signals, the energy will be finite (or constant) and average power will be zero. For power signals the average power is finite (or constant) and energy will be infinite.

i.e., For energy signal, E is constant (i.e., $0 < E < \infty$) and $P = 0$.

For power signal, P is constant (i.e., $0 < P < \infty$) and $E = \infty$.

For energy signals, the energy will be finite (or constant) and average power will be zero.

For power signals the average power is finite (or constant) and energy will be infinite.

Proof :

The energy of a signal x(t) is defined as ,

$$E = \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \qquad .....(2.4)$$

The power of a signal is defined as ,

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt = \underset{T \to \infty}{Lt} \frac{1}{2T} \underset{T \to \infty}{Lt} \int_{-T}^{T} |x(t)|^2 \, dt \qquad .....(2.5)$$

Using equation (2.4), the equation (2.5) can be written as,

$$P = \underset{T \to \infty}{Lt} \frac{1}{2T} \times E \qquad .....(2.6)$$

In equation (2.6), When E = constant,

$$P = E \times \underset{T \to \infty}{Lt} \frac{1}{2T}$$

$$= E \times \frac{1}{2 \times \infty} = E \times 0 = 0$$

From the above analysis, we can say that when a signal has finite energy the power will be zero. Also, from the above analysis we can say that the power is finite only when energy is infinite.

Examples on Energy and Power signals

Determine the power and energy for the following continuous time signals.

a) $x(t) = e^{-2t}\, u(t)$

b) $x(t) = e^{j\left(2t+\frac{\pi}{4}\right)}$

c) $x(t) = 3\cos 5\Omega_0 t$

Examples on Energy and Power signals

a. $x(t) = e^{-2t}\, u(t)$

Here, $x(t) = e^{-2t}\, u(t)$; for all t

$\therefore\ x(t) = e^{-2t}$; for $t \geq 0$

$$\therefore \int_{-T}^{T} |x(t)|^2\, dt = \int_{0}^{T} \left(\left|e^{-2t}\right|\right)^2 dt = \int_{0}^{T} \left(e^{-2t}\right)^2 dt = \int_{0}^{T} e^{-4t}\, dt = \left[\frac{e^{-4t}}{-4}\right]_0^T$$

$$= \left[\frac{e^{-4T}}{-4} - \frac{e^{0}}{-4}\right] = \left[\frac{1}{4} - \frac{e^{-4T}}{4}\right]$$

$$\text{Energy, } E = \underset{T \to \infty}{\text{Lt}} \int_{-T}^{T} |x(t)|^2\, dt = \underset{T \to \infty}{\text{Lt}} \left[\frac{1}{4} - \frac{e^{-4T}}{4}\right]$$

$$= \frac{1}{4} - \frac{e^{-\infty}}{4} = \frac{1}{4} - \frac{0}{4} = \frac{1}{4} \text{ joules}$$

$$\text{Power, } P = \underset{T \to \infty}{\text{Lt}} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt = \underset{T \to \infty}{\text{Lt}} \frac{1}{2T}\left[\frac{1}{4} - \frac{e^{-4T}}{4}\right]$$

$$= \frac{1}{\infty}\left[\frac{1}{4} - \frac{e^{-\infty}}{4}\right] = 0 \times \left[\frac{1}{4} - 0\right] = 0$$

Since energy is constant and power is zero, the given signal is an energy signal.

Examples on Energy and Power signals

c) Given that, $x(t) = 3\cos 5\Omega_0 t$

$$\therefore \int_{-T}^{T} |x(t)|^2\,dt = \int_{-T}^{T} \left(|3\cos 5\Omega_0 t|\right)^2 dt = \int_{-T}^{T} \left|(3\cos 5\Omega_0 t)^2\right| dt = \int_{-T}^{T} (3\cos 5\Omega_0 t)^2\,dt$$

$$= \int_{-T}^{T} 9\cos^2 5\Omega_0 t\,dt = 9\int_{-T}^{T}\left(\frac{1+\cos 10\Omega_0 t}{2}\right) dt \qquad \boxed{\cos^2\theta = \frac{1+\cos 2\theta}{2}}$$

$$= \frac{9}{2}\int_{-T}^{T} (1+\cos 10\Omega_0 t)\,dt = \frac{9}{2}\left[t + \frac{\sin 10\Omega_0 t}{10\Omega_0}\right]_{-T}^{T}$$

$$= \frac{9}{2}\left[T + \frac{\sin 10\Omega_0 T}{10\Omega_0} - \left(-T + \frac{\sin 10\Omega_0(-T)}{10\Omega_0}\right)\right]$$

$$= \frac{9}{2}\left[2T + 2\frac{\sin 10\Omega_0 T}{10\Omega_0}\right] = \frac{9}{2}\left[2T + 2\frac{\sin 10\frac{2\pi}{T}T}{10\frac{2\pi}{T}}\right] \qquad \boxed{\sin(-\theta) = -\sin\theta} \quad \boxed{\Omega_0 = 2\pi F_0 = \frac{2\pi}{T}}$$

$$= \frac{9}{2}\left[2T + \frac{T}{10\pi}\sin 20\pi\right] = \frac{9}{2}\left[2T + \frac{T}{10\pi}\times 0\right] = 9T \qquad \boxed{\begin{array}{l}\text{For integer M,}\\ \sin \pi M = 0\end{array}}$$

$$\text{Energy, } E = \underset{T\to\infty}{Lt}\int_{-T}^{T} |x(t)|^2\,dt = \underset{T\to\infty}{Lt}\; 9T = \infty$$

$$\text{Power, } P = \underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T} |x(t)|^2\,dt = \underset{T\to\infty}{Lt}\frac{1}{2T}\times 9T = \underset{T\to\infty}{Lt}\frac{9}{2} = \frac{9}{2} = 4.5 \text{ watts}$$

Since energy is infinite and power is constant, the given signal is a power signal.

Causal and Non causal Signals

A signal is said to be ***causal***, if it is defined for $t \geq 0$.

Therefore if $x(t)$ is causal, then $x(t) = 0$, for $t < 0$.

A signal is said to be ***noncausal***, if it is defined for either $t \leq 0$, or for both $t \leq 0$ and $t > 0$.

Therefore if $x(t)$ is noncausal, then $x(t) \neq 0$, for $t < 0$.

When a noncausal signal is defined only for $t \leq 0$, it is called ***anticausal signal***.

Examples of causal and noncausal signals

$$\left.\begin{array}{ll} \text{Step signal,} & x(t) = A \quad ; t \geq 0 \\ \text{Unit step signal,} & x(t) = u(t) = 1 \quad ; t \geq 0 \\ \text{Exponential signal,} & x(t) = A\, e^{bt}\, u(t) \\ \text{Complex exponential signal,} & x(t) = A\, e^{j\Omega_0 t}\, u(t) \end{array}\right\} \text{Causal signals}$$

$$\left.\begin{array}{ll} \text{Exponential signal,} & x(t) = A\, e^{bt} \quad ; \text{for all } t \\ \text{Complex exponential signal,} & x(t) = A\, e^{j\Omega_0 t} \quad ; \text{for all } t \end{array}\right\} \text{Noncausal signals}$$

Note : On multiplying a noncausal signal by u(t), it becomes causal.

2.6 Continuous Time System

A ***continuous time system*** (or Analog system) is a physical device that operates on a continuous time signal (or an analog signal) called input or excitation, according to some well defined rule, to produce another continuous time signal (or an analog signal) called output or response. We can say that the input signal x(t) is transformed by the system into a signal y(t), and the transformation can be expressed mathematically as shown in equation (2.11).The diagrammatic representation of continuous time system is shown in fig 2.36.

$$\text{Response,}\quad y(t) = \mathcal{H}\{x(t)\} \qquad \text{.....(2.11)}$$

where, $\mathcal{H}$ denotes the transformation (also called an operator).

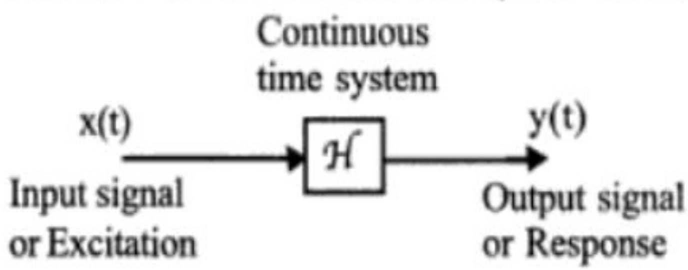


Fig 2.36 : Representation of continuous time system.

2.8 Classification of Continuous Time Systems

The continuous time systems are classified based on their characteristics. Some of the classifications of continuous time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. Feedback and nonfeedback systems

2.8.1 Static and Dynamic Systems

A continuous time system is called ***static*** or ***memoryless*** if its output at any instant of time t depends at most on the input signal at the same time but not on the past or future input. In any other case, the system is said to be ***dynamic*** or to have memory.

Example :

$$\left.\begin{array}{l} y(t) = a\,x(t) \\ \\ y(t) = t\,x(t) + 6\,x^3(t) \end{array}\right\}\ \text{Static systems}$$

$$\left.\begin{array}{l} y(t) = t\,x(t) + 3\,x(t^2) \\ \\ y(t) = x(t) + 3\,x(t-2) \end{array}\right\}\ \text{Dynamic systems}$$

2.8.2 Time Invariant and Time Variant Systems

A system is said to be ***time invariant*** if its input-output characteristics does not change with time.

Definition : A relaxed system $\mathcal{H}$ is ***time invariant*** or ***shift invariant*** if and only if

$$x(t) \xrightarrow{\mathcal{H}} y(t) \text{ implies that, } x(t-m) \xrightarrow{\mathcal{H}} y(t-m)$$

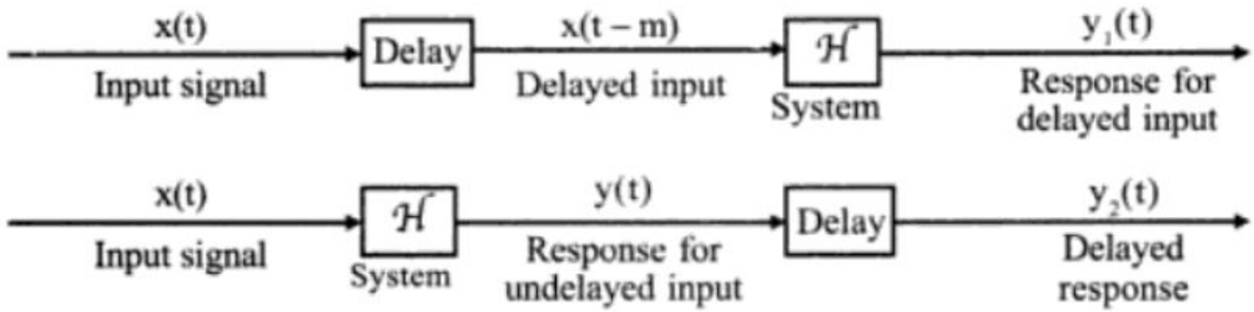
for every input signal x(t) and every time shift m.

i.e., in time invariant systems, if $\mathbf{y(t) = \mathcal{H}\{x(t)\}}$ then $\mathbf{y(t-m) = \mathcal{H}\{x(t-m)\}}$.

Alternative Definition for Time Invariance

A system $\mathcal{H}$ is ***time invariant*** if the response to a shifted (or delayed) version of the input is identical to a shifted (or delayed) version of the response based on the unshifted (or undelayed) input.

The diagrammatic explanation of the above definition of time invariance is shown in fig 2.41.



If $y_1(t) = y_2(t)$ then the system is time invariant

***Fig 2.41 :** Diagrammatic explanation of time invariance.*

Procedure to test for time invariance

1. Delay the input signal by m units of time and determine the response of the system for this delayed input signal. Let this response be $y_1(t)$.

2. Delay the response of the system for unshifted input by m units of time. Let this delayed response be $y_2(t)$.

3. Check whether $y_1(t) = y_2(t)$. If they are equal then the system is time invariant. Otherwise the system is time variant.

Example 2.11

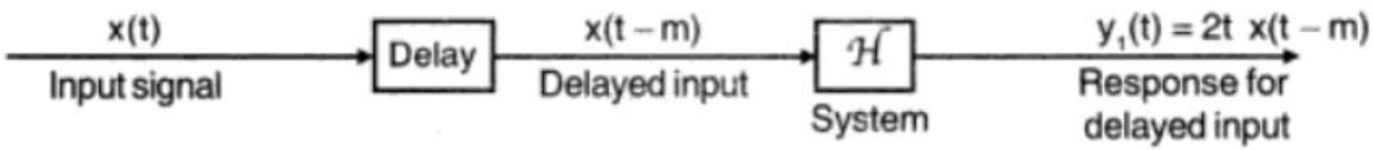
State whether the following systems are time invariant or not.

a) $y(t) = 2t\, x(t)$ b) $y(t) = x(t) \sin 20\pi t$ c) $y(t) = 3x(t^2)$ d) $y(t) = x(-t)$

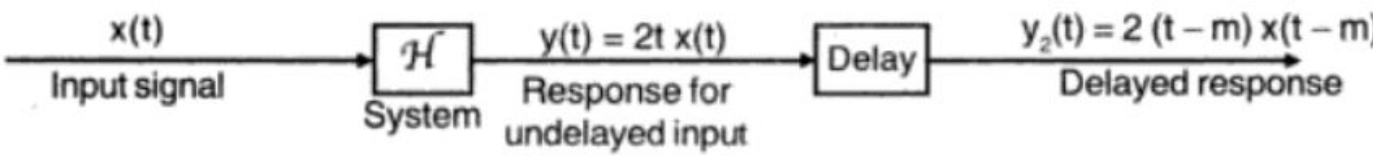
Solution

a) Given that, $y(t) = 2t\, x(t)$

<u>**Test 1** : Response for delayed input</u>



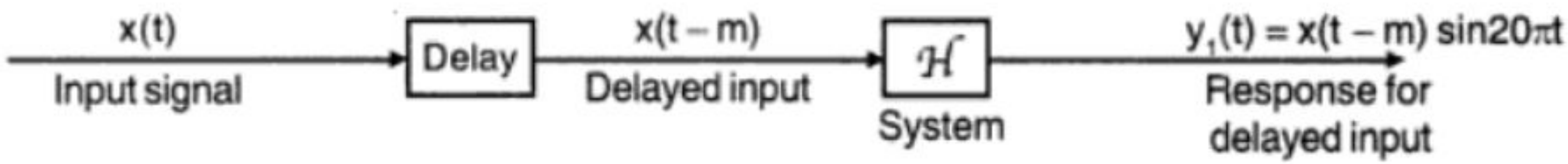
<u>**Test 2** : Delayed response</u>



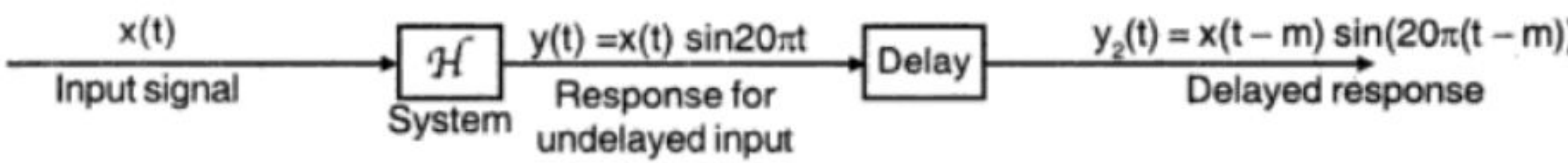
<u>Conclusion</u> : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

b) Given that, $y(t) = x(t) \sin 20\pi t$

Test 1 : Response for delayed input



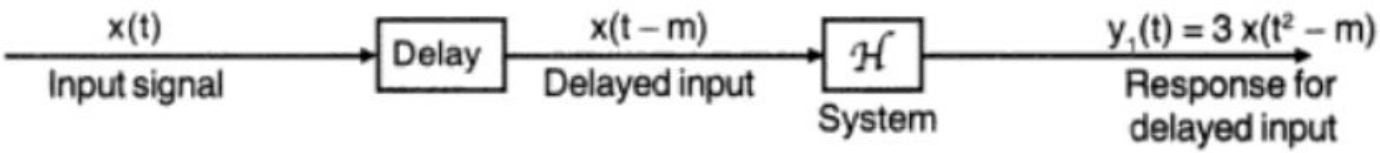
Test 2 : Delayed response



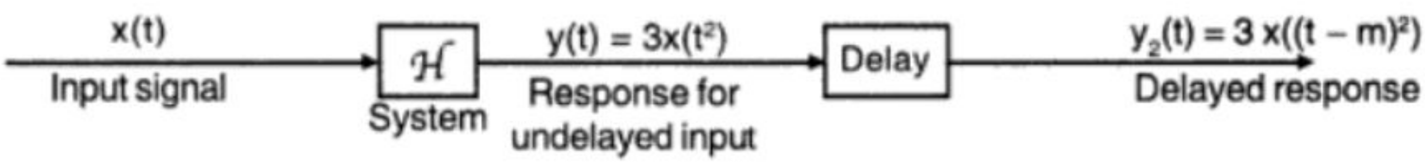
Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

c) **Given that, $y(t) = 3x(t^2)$**

Test 1 : Response for delayed input



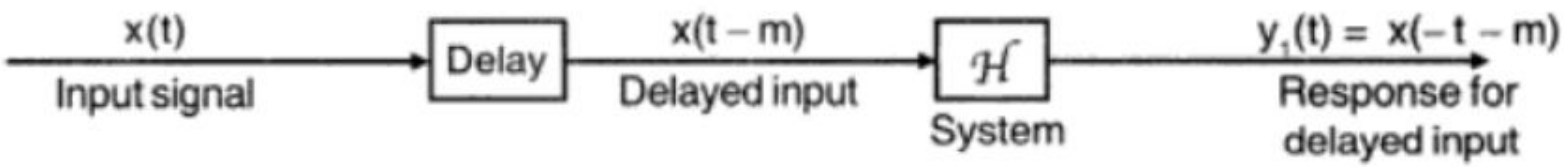
Test 2 : Delayed response



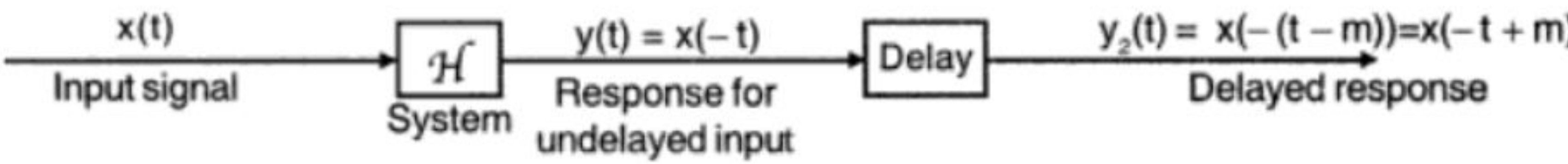
Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

d) Given that, $y(t) = x(-t)$

Test 1 : Response for delayed input



Test 2 : Delayed response



Conclusion : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

2.8.3 Linear and Nonlinear Systems

A ***linear system*** is the one that satisfies the superposition principle.

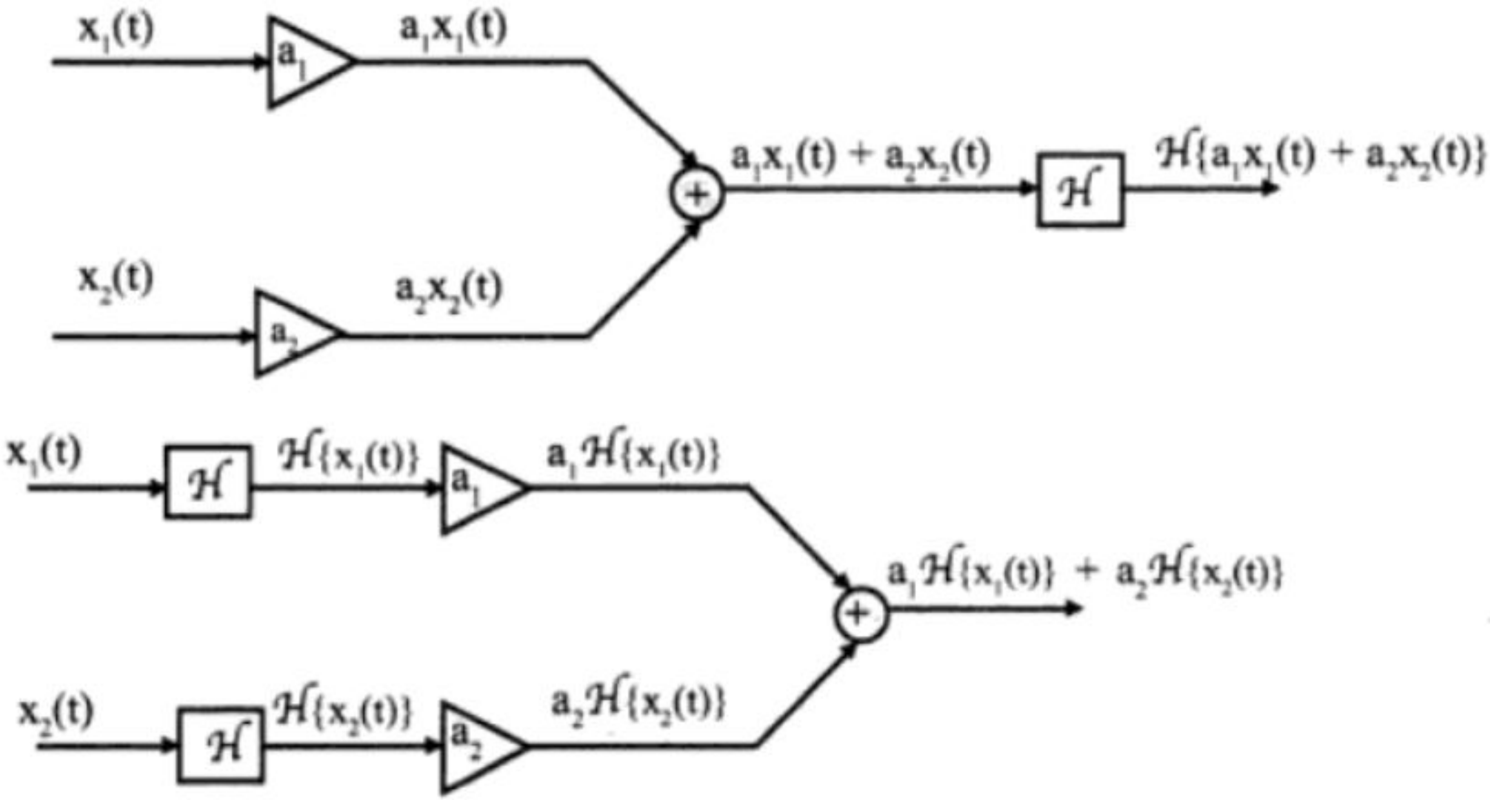
The ***principle of superposition*** requires that the response of a system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses to each of the individual input signals.

Definition : A relaxed system $\mathcal{H}$ is ***linear*** if

$$\mathcal{H}\{a_1\, x_1(t) + a_2\, x_2(t)\} = a_1\, \mathcal{H}\{x_1(t)\} + a_2\, \mathcal{H}\{x_2(t)\}$$

for any arbitrary input signal $x_1(t)$ and $x_2(t)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the superposition principle as given by the above definition, the system is ***nonlinear***. The diagrammatic explanation of linearity is shown in fig. 2.42.



The system, $\mathcal{H}$ is linear if and only if, $\mathcal{H}\{a_1\,x_1(t) + a_2\,x_2(t)\} = a_1\,\mathcal{H}\{x_1(t)\} + a_2\,\mathcal{H}\{x_2(t)\}$

***Fig 2.42 :** Diagrammatic explanation of linearity.*

Procedure to test for linearity

1. Let $x_1(t)$ and $x_2(t)$ be two inputs to the system $\mathcal{H}$, and $y_1(t)$ and $y_2(t)$ be the corresponding responses.

2. Consider a signal, $x_3(t) = a_1\, x_1(t) + a_2\, x_2(t)$ which is a weighed sum of $x_1(t)$ and $x_2(t)$.

3. Let $y_3(t)$ be the response for $x_3(t)$.

4. Check whether $y_3(t) = a_1\, y_1(t) + a_2\, y_2(t)$. If equal then the system is linear, otherwise it is nonlinear.

Example 2.13

Test the following systems for linearity.

a) $y(t) = t\,x(t)$, b) $y(t) = x(t^2)$, c) $y(t) = x^2(t)$, d) $y(t) = A\,x(t) + B$, e) $y(t) = e^{x(t)}$.

Solution

a) Given that, $y(t) = t\,x(t)$

Let $\mathcal{H}$ be the system operating on $x(t)$ to produce, $y(t) = \mathcal{H}\{x(t)\} = t\,x(t)$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\; y_1(t) = \mathcal{H}\{x_1(t)\} = t\,x_1(t) \qquad .....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = t\,x_2(t) \qquad .....(2)$$

Let $x_3(t) = a_1\,x_1(t) + a_2\,x_2(t)$.

A linear combination of inputs $x_1(t)$ and $x_2(t)$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore\; y_3(t) = \mathcal{H}\{\,x_3(t)\} = \mathcal{H}\{\,a_1\,x_1(t) + a_2\,x_2(t)\}$$

$$= t(a_1\,x_1(t) + a_2\,x_2(t)) = a_1\,t\,x_1(t) + a_2\,t\,x_2(t)$$

$$= a_1\,y_1(t) + a_2\,y_2(t))$$

Using equations (1) and (2)

Since, $y_3(t) = a_1\,y_1(t) + a_2\,y_2(t)$, the given system is linear.

b) Given that, $y(t) = x(t^2)$

Let $\mathcal{H}$ be the system operating on $x(t)$ to produce, $y(t) = \mathcal{H}\{x(t)\} = x(t^2)$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = x_1(t^2) \qquad \text{....(1)}$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = x_2(t^2) \qquad \text{.....(2)}$$

Let $x_3(t) = a_1\, x_1(t) + a_2\, x_2(t)$. $\boxed{\text{A linear combination of inputs } x_1(t) \text{ and } x_2(t)}$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore\ y_3(t) = \mathcal{H}\{x_3(t)\} = \mathcal{H}\{a_1\, x_1(t) + a_2\, x_2(t)\}$$

$$= (a_1\, x_1(t^2) + a_2\, x_2(t^2))$$

$$= a_1\, y_1(t) + a_2\, y_2(t)) \qquad \boxed{\text{Using equations (1) and (2)}}$$

Since, $y_3(t) = a_1\, y_1(t) + a_2\, y_2(t)$, the given system is linear.

c) Given that, $y(t) = x^2(t)$

Let $\mathcal{H}$ be the system operating on x(t) to produce, $y(t) = \mathcal{H}\{x(t)\} = x^2(t)$.

Consider two signals $x_1(t)$ and $x_2(t)$.

Let $y_1(t)$ and $y_2(t)$ be the response of the system $\mathcal{H}$ for inputs $x_1(t)$ and $x_2(t)$ respectively.

$$\therefore\ y_1(t) = \mathcal{H}\{x_1(t)\} = x_1^2(t) \qquad .....(1)$$

$$y_2(t) = \mathcal{H}\{x_2(t)\} = x_2^2(t) \qquad .....(2)$$

Let $x_3(t) = a_1 x_1(t) + a_2 x_2(t)$.

A linear combination of inputs $x_1(t)$ and $x_2(t)$

Let $y_3(t)$ be the response of the system $\mathcal{H}$ for input $x_3(t)$.

$$\therefore y_3(t) = \mathcal{H}\{x_3(t)\} = \mathcal{H}\{a_1 x_1(t) + a_2 x_2(t)\} = (a_1 x_1(t) + a_2 x_2(t))^2$$

$$= a_1^2\, x_1^2(t) + a_2^2\, x_2^2(t) + 2\, a_1\, a_2\, x_1(t)\, x_1(t)$$

$$= a_1^2\, y_1(t) + a_2^2\, y_2(t) + 2\, a_1\, a_2\, x_1(t)\, x_1(t)$$

Using equations (1) and (2)

Here, $y_3(t) \neq a_1 y_1(t) + a_2 y_2(t)$. Hence the given system is nonlinear.

2.8.4 Causal and Noncausal Systems

Definition : A system is said to be ***causal*** if the output of the system at any time t depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

If the system output at any time t depends on future inputs or outputs then the system is called a ***noncausal*** system.

The causality refers to a system that is realizable in real time. It can be shown that an LTI system is causal if and only if the impulse response is zero for $t < 0$, (i.e., $h(t) = 0$ for $t < 0$).

Example 2.15

Test the casuality of the following systems.

a) $y(t) = x(t) - x(t-1)$ b) $y(t) = x(t) + 2\,x(3-t)$ c) $y(t) = t\,x(t)$

d) $y(t) = x(t) + \int_0^t x(\lambda)\,d\lambda$ e) $y(t) = x(t) + \int_0^{3t} x(\lambda)\,d\lambda$ f) $y(t) = 2x(t) + \dfrac{dx(t)}{dt}$

Solution

a) Given that, $y(t) = x(t) - x(t-1)$

When $t = 0$, $y(0) = x(0) - x(-1)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past input $x(-1)$.

When $t = 1$, $y(1) = x(1) - x(0)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past input $x(0)$.

From the above analysis we can say that for any value of t, the system output depends on present and past inputs. Hence the system is causal.

b) Given that, $y(t) = x(t) + 2\,x(3-t)$

When $t=-1$, $y(-1) = x(-1) + 2\,x(4)$ $\Rightarrow$ The response at $t=-1$, i.e., $y(-1)$ depends on the present input $x(-1)$ and future input $x(4)$.

When $t=0$, $y(0) = x(0) + 2\,x(3)$ $\Rightarrow$ The response at $t=0$, i.e., $y(0)$ depends on the present input $x(0)$ and future input $x(3)$.

When $t=1$, $y(1) = x(1) + 2\,x(2)$ $\Rightarrow$ The response at $t=1$, i.e., $y(1)$ depends on the present input $x(1)$ and future input $x(2)$.

When $t=2$, $y(2) = x(2) + 2\,x(1)$ $\Rightarrow$ The response at $t=2$, i.e., $y(2)$ depends on the present input $x(2)$ and past input $x(1)$.

From the above analysis we can say that for $t<2$, the system output depends on present and future inputs. Hence the system is noncausal.

c) Given that, $y(t) = t\,x(t)$

When $t = 0$, $y(0) = 0 \times x(0)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $t = 1$, $y(1) = 1 \times x(1)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

When $t = 2$, $y(2) = 2 \times x(2)$ $\Rightarrow$ The response at $t = 2$, i.e., $y(2)$ depends on the present input $x(2)$.

From the above analysis we can say that the response for any value of t depends on the present input. Hence the system is causal.

d) Given that, $y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda$

$$y(t) = x(t) + \int_0^t x(\lambda)\, d\lambda = x(t) + \left[z(\lambda)\right]_0^t = x(t) + z(t) - z(0), \qquad \text{where, } z(\lambda) = \int x(\lambda)\, d\lambda$$

When $t = 0$, $y(0) = x(0) + z(0) - z(0)$ $\Rightarrow$ The response at $t = 0$, i.e., $y(0)$ depends on present input.

When $t = 1$, $y(1) = x(1) + z(1) - z(0)$ $\Rightarrow$ The response at $t = 1$, i.e., $y(1)$ depends on present and past input.

When $t = 2$, $y(2) = x(2) + z(1) - z(0)$ $\Rightarrow$ The response at $t = 2$, i.e., $y(2)$ depends on present and past input.

From the above analysis we can say that the response for any value of t depends on the present and past input. Hence the system is causal.

2.8.5 Stable and Unstable Systems

Definition : An arbitrary relaxed system is said to be ***BIBO stable*** (Bounded Input-Bounded Output stable) if and only if every bounded input produces a bounded output.

Let x(t) be the input of continuous time system and y(t) be the response or output for x(t).

The term ***bounded input*** refers to finite value of the input signal x(t) for any value of t. Hence if input x(t) is bounded then there exists a constant M_x such that $|x(t)| \le M_x$ and $M_x < \infty$, for all t.

Examples of bounded input signal are step signal, decaying exponential signal and impulse signal.
Examples of unbounded input signal are ramp signal and increasing exponential signal.

The term ***bounded output*** refers to finite and predictable output for any value of t. Hence if output y(t) is bounded then there exists a constant M_y such that $|y(t)| \le M_y$ and $M_y < \infty$, for all t.

In general, the test for stability of the system is performed by applying specific input. On applying a bounded input to a system if the output is bounded then the system is said to be BIBO stable.

Condition for Stability of an LTI System

For an LTI (Linear Time Invariant) system, the condition for BIBO stability can be transformed to a condition on impulse response, h(t). For BIBO stability of an LTI continuous time system, the integral of impulse response should be finite.

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\, dt < \infty \text{ , for stability of an LTI system.}$$

Proof :

The response of a system y(t) for any input x(t) is given by convolution of the input and impulse response.

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad \text{.....(2.18)}$$

On taking the absolute value on both sides of equation (2.18), we get,

$$|y(t)| = \left| \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau \right| = \int_{-\infty}^{+\infty} |h(\tau)\, x(t-\tau)|\, d\tau$$

$$= \int_{-\infty}^{+\infty} |h(\tau)|\, |x(t-\tau)|\, d\tau \qquad .....(2.19)$$

If the input x(t) is bounded then there exists a constant M_x, such that $|x(t-\tau)| \le M_x < \infty$. Hence equation(2.19) can be written as,

$$|y(t)| = M_x \int_{-\infty}^{+\infty} |h(\tau)|\, d\tau \qquad .....(2.20)$$

From equation (2.20) we can say that the output y(t) is bounded, If the impulse response satisfies the condition,

$$\int_{-\infty}^{+\infty} |h(\tau)|\, d\tau < \infty$$

Since τ is a dummy variable in the above condition we can replace τ by t.

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$$

Example 2.17

Test the stability of the following systems.

a) $y(t) = \cos(x(t))$ b) $y(t) = x(-t-2)$ c) $y(t) = t\,x(t)$

Solution

a) Given that, $y(t) = \cos(x(t))$

The given system is a nonlinear system, and so the test for stability should be performed for specific inputs.

The value of $\cos\theta$ lies between −1 to +1 for any value of θ. Therefore the output $y(t)$ is bounded for any value of input $x(t)$. Hence the given system is stable.

---

b) Given that, $y(t) = x(-t-2)$

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

The operations performed by the system on the input signal are folding and shifting. A bounded input signal will remain bounded even after folding and shifting. Therefore in the given system, the output will be bounded as long as input is bounded. Hence the given system is BIBO stable.

---

c) Given that, $y(t) = t\,x(t)$

The given system is a time variant system, and so the test for stability should be performed for specific inputs.

Case i : Let $x(t)$ tends to ∞ or constant, as **t** tends to infinity. In this case, $y(t) = t\,x(t)$ will be infinity as **t** tends to inifnity and so the system is unstable.

Case ii : Let $x(t)$ tends to 0, as **t** tends to infinity. In this case $y(t) = t\,x(t)$ will be zero as **t** tends to infinity and so the system is stable.

Example 2.18

Test the stability of the LTI systems, whose impulse responses are given below.

a) $h(t) = e^{-5|t|}$ b) $h(t) = e^{4t}\, u(t)$ c) $h(t) = e^{-4t}\, u(t)$

d) $h(t) = t\, e^{-3t}\, u(t)$ e) $h(t) = t \cos t\, u(t)$ f) $h(t) = e^{-t} \sin t\, u(t)$

Solution

a) Given that, $h(t) = e^{-5|t|}$

For stability, $\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} \left|e^{-5|t|}\right| dt = \int_{-\infty}^{+\infty} e^{-5|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{5t}\, dt + \int_{0}^{+\infty} e^{-5t}\, dt = \left[\frac{e^{5t}}{5}\right]_{-\infty}^{0} + \left[\frac{e^{-5t}}{-5}\right]_{0}^{\infty}$$

$$= \frac{e^{0}}{5} - \frac{e^{-\infty}}{5} + \frac{e^{-\infty}}{-5} - \frac{e^{0}}{-5} = \frac{1}{5} - 0 + 0 + \frac{1}{5} = \frac{2}{5}$$

Here, $\int_{-\infty}^{+\infty} |h(t)|\, dt = \frac{2}{5}$ = constant. Hence the system is stable.

b) Given that, $h(t) = e^{4t}\, u(t)$

For stability, $\int_{-\infty}^{+\infty} |h(t)|\, dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} \left|e^{4t}\, u(t)\right| dt = \int_{-\infty}^{+\infty} e^{4t}\, u(t)\, dt$$

$$= \int_{0}^{+\infty} e^{4t}\, dt = \left[\frac{e^{4t}}{4}\right]_0^{\infty} = \frac{e^{\infty}}{4} - \frac{e^{0}}{4} = \infty - \frac{1}{4} = \infty$$

Here, $\int_{-\infty}^{+\infty} |h(t)|\, dt = \infty$. Hence the system is unstable.

d) Given that, $h(t) = t\,e^{-3t}\,u(t)$

For stability, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

$$\therefore \int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left|t\,e^{-3t}\,u(t)\right| dt = \int_{0}^{+\infty} t\,e^{-3t}\,dt$$

$$= \left[t\,\frac{e^{-3t}}{-3} - \int 1 \times \frac{e^{-3t}}{-3}\,dt\right]_0^{\infty} = \left[-\frac{t\,e^{-3t}}{3} - \frac{e^{-3t}}{9}\right]_0^{\infty}$$

$$\boxed{\int u\,v = u\int v - \int\left[du\int v\right]}$$

$$= -\frac{\infty \times e^{-\infty}}{3} - \frac{e^{-\infty}}{9} + \frac{0 \times e^{0}}{3} + \frac{e^{0}}{9}$$

$$= -\frac{\infty \times 0}{3} - 0 + 0 + \frac{1}{9} = \frac{1}{9}$$

Since, $\displaystyle\int_{-\infty}^{+\infty} |h(t)|\,dt = \frac{1}{9}$ = constant, the system is stable.