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Signals and Systems
Tutorial No: 4

Convolve $x(n)=\left(\frac{1}{3}\right)^n u(n)$ with $h(n)=\left(\frac{1}{2}\right)^n u(n)$ using convolution formula.

Solution,

$x(n)=\left(\frac{1}{3}\right)^n u(n)$, $h(n)=\left(\frac{1}{2}\right)^n u(n)$

Using convolution sum formula,

$$y(n)=\sum_{k=-\infty}^{\infty} x(k)\cdot h(n-k)$$

$x(k)=\left(\frac{1}{3}\right)^n\cdot u(k)$, $h(n-k)=\left(\frac{1}{2}\right)^{n-k}\cdot u(n-k)$

$$y(n)=\sum_{k=-\infty}^{\infty}\left(\frac{1}{3}\right)^k\cdot u(k)\cdot\left(\frac{1}{2}\right)^n 2^k u(n-k)$$

$$=\left(\frac{1}{2}\right)^n\sum_{k=-\infty}^{\infty}\left(\frac{2}{3}\right)^k\cdot u(k)\cdot u(n-k)$$

$$=\left(\frac{1}{2}\right)^n\sum_{k=0}^{n}\left(\frac{2}{3}\right)^k$$

$$y(n)=\left(\frac{1}{2}\right)^n\left[\frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}\right] \qquad \left(\because \sum_{k=0}^{n-1} c^N=\frac{1-c^N}{1-c}\right)$$

$$y(n)=\left(\frac{1}{2}\right)^n\cdot 3\left[1-\left(\frac{2}{3}\right)^{n+1}\right]$$

Q2) Perform convolution of $x_1(t)$ and $x_2(t)$ using convolution theorem and sketch resultant waveform. where:

$x_1(t) = u(t) - u(t-1)$

$x_2(t) = u(t) - u(t-2)$.

→ Solution,

$x_1(t) = u(t) - u(t-1)$

Using Laplace Theorem,

$$X_1(s) = L\{x_1(t)\} = L\{u(t) - u(t-1)\}$$

$$X_1(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

$x_2(t) = u(t) - u(t-2)$

$$X_2(s) = L\{x_2(t)\} = L\{u(t)\} - L\{u(t-2)\}$$

$$= \frac{1}{s} - \frac{e^{-2s}}{s}$$

Convolution theorem of Laplace Theorem,

$$L\{x_1(t) * x_2(t)\} = X_1(s) \cdot X_2(s)$$

$$= \left(\frac{1}{s} - \frac{e^{-s}}{s}\right) \cdot \left(\frac{1}{s} - \frac{e^{-2s}}{s}\right)$$

$$= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2}$$

$$x_3(t) = x_1(t) * x_2(t) = L^{-1}\left\{\frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} + \frac{e^{-3s}}{s^2}\right\}$$

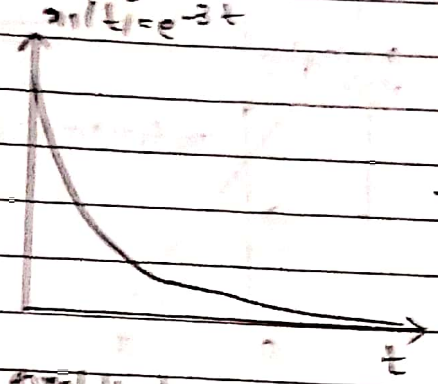
Using $\left(L\{(t-a) \cdot u(t-a)\} = \frac{e^{-as}}{s^2}\right)$

$$x_3(t) = t u(t) - (t-1) \cdot u(t-1) - (t-2) u(t-2) + (t-3) \cdot u(t$$

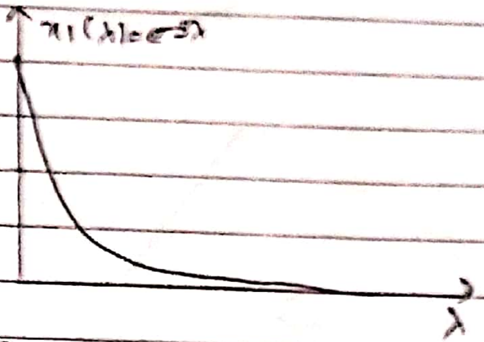
ii) Graphical method-

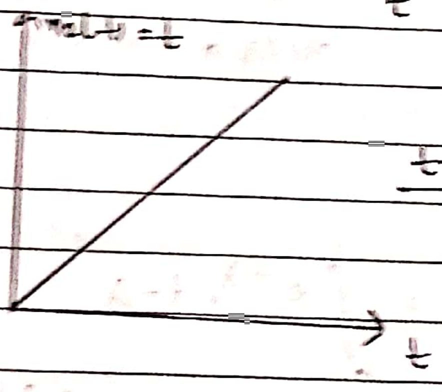
$$x_3(t) = \int_{\lambda=-\infty}^{\infty} x_1(\lambda)\cdot x_2(t-\lambda)\,d\lambda$$

① Change of index-

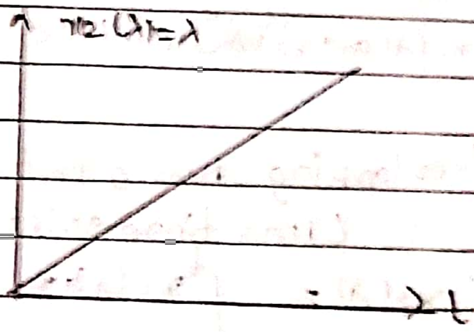


$\xrightarrow{t \to \lambda}$

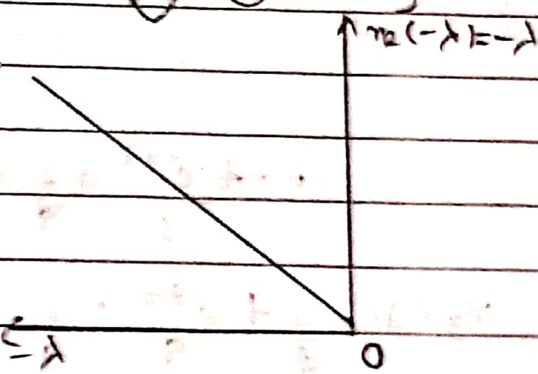




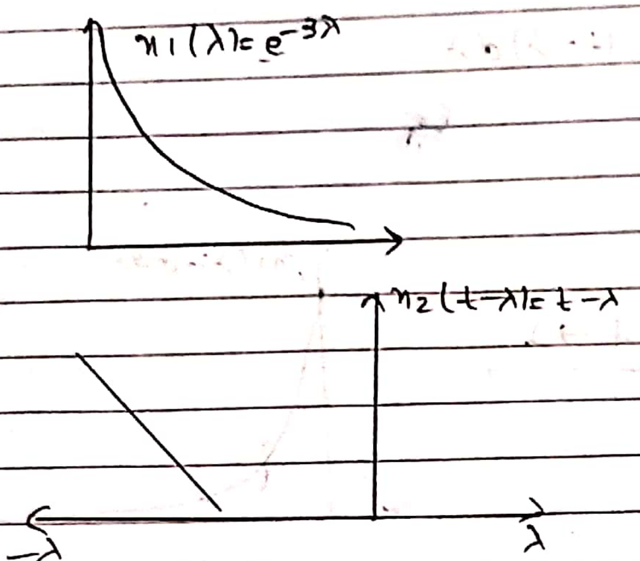
$\xrightarrow{t \to \lambda}$



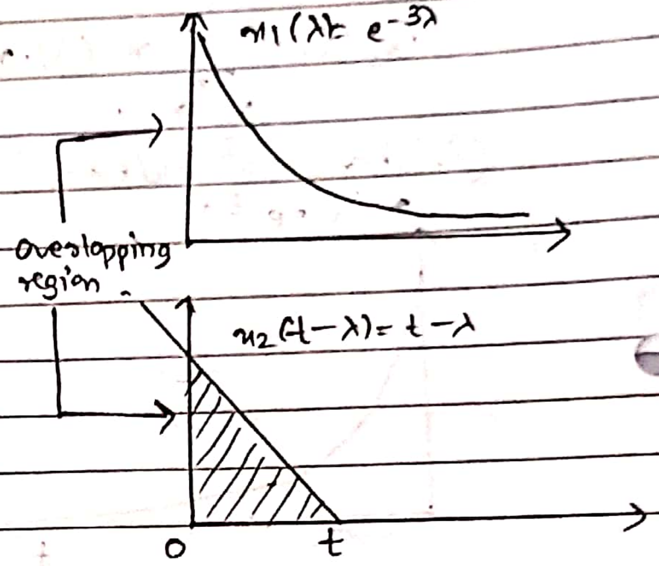
② Folding



③ Shifting $x_2(-\lambda)$ by t units



$x_1(\lambda)$ and $x_2(t-\lambda)$ when $t<0$



$x_1(\lambda)$ and $x_2(t-\lambda)$, when $t>0$

Overlapping from 0 to t,

When time shift $t>0$

$$\therefore x_3(\lambda) = \int_0^t x_1(\lambda)\cdot x_2(t-\lambda) = \int_0^t e^{-3\lambda}\cdot t-\lambda$$

$$= \int_0^t t\cdot e^{-3\lambda} - \int_0^t \lambda\cdot e^{-3\lambda} = \left[\frac{t e^{-3\lambda}}{3}\right]_0^t - \left[\frac{\lambda\cdot e^{-3\lambda}}{-3} - \frac{e^{-3\lambda}}{9}\right]_0^t$$

$$= -\frac{t\cdot e^{-3t}}{3} + \frac{t}{9} + \frac{te^{-3t}}{3} + \frac{e^{-2t}}{9} - \frac{1}{9}$$

$$x_3(\lambda) = \frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9};\ t\geq 0$$

$$x_3(\lambda) = \frac{1}{9}\left[e^{-3t}+3t-1\right]\cdot u(t) \text{ for all } t.$$

When $t = 0$ to 1;

$u(t) = 1,\ u(t-1) = 0,\ u(t-2) = 0,\ u(t-3) = 0$

$\therefore x_3(t) = t \times 1 - 0 - 0 + 0$

$x_3(t) = t$

When $1 \leq t \leq 2$;

$u(t) = 1,\ u(t-1) = 1,\ u(t-2) = 0,\ u(t-3) = 0$

$x_3(t) = 1$

When $2 \leq t \leq 3$

$u(t) = 1,\ u(t-1) = 1,\ u(t-2) = 1,\ u(t-3) = 0,$

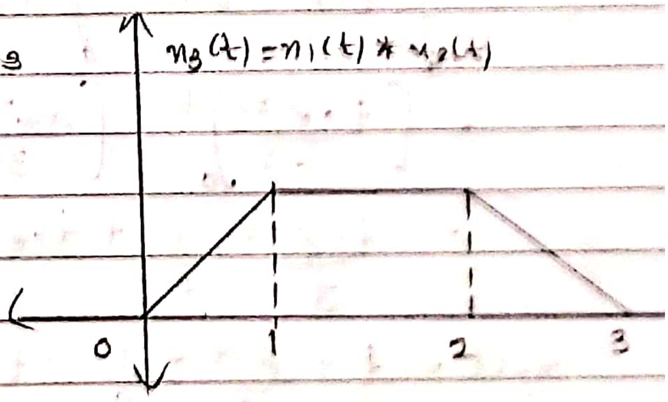
$x_3(t) = 3 - t$

When $t > 3$,

$u(t) = 1,\ u(t-1) = 1,\ u(t-2) = 1,\ u(t-3) = 1$

$\therefore x_3(t) = t - (t-1) - (t-2) + t - 3 = 0$

$\therefore$ The resultant waveform is



Q.3) Perform convolution of $x_1(t) = e^{-3t}u(t)$ and $x_2(t) = t \cdot u(t)$ using mathematical method and also by graphical method.

→ ① Mathematical method -

$x_1(t) = e^{-3t}u(t) \Rightarrow x_1(t) = e^{-3t} \; ; t \geq 0$

$x_2(t) = t \cdot u(t) \Rightarrow x_2(t) = t \; ; t \geq 0$

$x_3(t) = x_1(t) * x_2(t)$

$= \int_{\lambda = \lambda_1}^{\lambda_2} x_1(\lambda) \cdot x_2(t-\lambda) \, d\lambda$

As $x_1(t)$ and $x_2(t)$ are causal; limits of integration are 0 to t

$= \int_0^t e^{-3\lambda} \cdot (t-\lambda) \, d\lambda$

$= \int_0^t t \cdot e^{-3\lambda} \, d\lambda - \int_0^t \lambda \cdot e^{-3\lambda} \, d\lambda$

$= \left[\frac{t \cdot e^{-3\lambda}}{-3} \right]_0^t - \left[\frac{\lambda e^{-3\lambda}}{3} - \frac{e^{-3\lambda}}{9} \right]_0^t$

$= -\frac{t \cdot e^{-3t}}{3} + \frac{t}{3} + \frac{t \cdot e^{-3t}}{3} - \frac{e^{-3t}}{9} - \frac{1}{9}$

$x_3(t) = \frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9} \; ; t \geq 0$

$x_3(t) = \frac{1}{3}\left(e^{-3t} + 3t - 1\right) \cdot u(t)$

Write notes on relation of ESD, PSD with auto correlation

i) Relation of ESD to auto correlation :

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)\, dt = x(\tau) * x(-\tau)$$

$$F[R_{xx}(\tau)] = X(\omega)\cdot X(-\omega) = X(\omega)\cdot X^{*}(\omega)$$
$$= |X(\omega)|^2$$
$$= \psi_x(\omega)$$

$R_{xx}(\tau) \to \psi_x(\omega) \to R_{xx}(\tau)$ and ESD make Fourier transform pair.

ii) Relation of PSD to auto correlation:

$$R_{xx}(\tau) = \lim_{T\to\infty} \int_{-T/2}^{T/2} x(t)\cdot x(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} x_T(t)\cdot x(t-\tau)\, dt$$

$$R_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T} \left[x_T(\tau) * x_T(-\tau) \right]$$

$$F[R_{xx}(\tau)] = \lim_{T\to\infty} \frac{1}{T}\cdot X_T(\omega)\cdot X_T(-\omega) = \lim_{T\to\infty} \frac{1}{T} X_T(\omega)\cdot X_T^{*}(\omega)$$

$$= \lim_{T\to\infty} \frac{1}{T}\cdot |X_T(\omega)|^2$$

$$= G_x(\omega).$$

$R_{xx}(\tau) \leftrightarrow G_x(\omega) \to R_{xx}(\tau)$ and PSD $G_x(\omega)$ make a Fourier transform pair

Compute the convolution $y(n) = x(n) * h(n)$ using tabulation method, where $x(n) = \{1, 1, 0, 1, 1\}$ and $h(n) = \{1, -2, 3, 4\}$
$\uparrow \qquad\qquad\qquad \uparrow$

$n_1 + n_2 = -2 + -3 = -5$

Number of terms $= N_1 + N_2 - 1 = 5 + 4 - 1 = 8$

Last term $= -5 + 8 - 1 = 2$

-5	-4	-3	-2	-1	0	1	2	3	4	5
			1	1	0	1	1			
		1	2	-3	4					
					4	-3	-2	1		
4	-3	-2	1							
	4	-3	-2	1						
		4	-3	-2	1					
			4	-3	-2	1				
				4	-3	-2	1			
					4	-3	-2	1		
						4	-3	-2	1	
							4	-3	-2	1

$$y(n) = \sum_{m=-\infty}^{\infty} x(m)\, h(n-m)$$

$y(-5) = x(m)\, h(-5-m)$
$= 1$

$y(-4) = -1$

$y(-3) = -5$

$y(-2) = 2$

$y(-1) = 3$

$y(0) = -5$

$y(1) = 1$

$y(2) = 4$

$\rightarrow \therefore y(n) = \{1, -1, -5, 2, 3, -5, 1, 4\}$
$\uparrow$

FOR EDUCATIONAL USE

Determine crosscorrelation of sequence $x(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$

and $y(n) = \{1, \underset{\uparrow}{3}, 1\}$

$r_{xy}(m)$ be cross correlation of $x(n)$ and $y(n)$

$$r_{xy} = \sum_{m=-\infty}^{\infty} x(n) \cdot y(n-m)$$

$x(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$, $y(n) = \{1, \underset{\uparrow}{3}, 1\}$

4 samples starts from $n=1$

3 samples starts from $n=-1$

$\therefore$ Total samples in $y(n) = 4+3-1 = 6$

$\therefore$ $y(n)$ starts at $n=-1$

ends at $n=4$

$y(-n) = \{1, \underset{\uparrow}{3}, 1\}$

	1	3	1
1	1	3	1
1	1	3	1
2	2	6	2
2	2	6	2

$r_{xy} = \{1, \underset{\uparrow}{4}, 6, 9, 8, 2\}$

1) Consider two LTI system connected in series. Their impulse responses are $h_1(n)$ and $h_2(n)$ respectively. Find output of systems if $x(n)$ is input applied to

$x(n) = \{\underset{\uparrow}{1}, 2\}$, $h_1(n) = \{1, 0, \underset{\uparrow}{-1}\}$, $h_2(n) = \{\underset{\uparrow}{2}, 1, -1\}$

As $h_1(n)$ and $h_2(n)$ connected in series, their equivalent impulse responses.

$$h(n) = h_1(n) * h_2(n)$$

By Matrix method,

$$\begin{array}{c|ccc} & 1 & 0 & \overset{\downarrow}{-1} \\ \hline \rightarrow 2 & 2 & 0 & -2 \\ 1 & 1 & 0 & -1 \\ -1 & -1 & 0 & 1 \end{array}$$

$\therefore h(n) = \{2, 1, \underset{\uparrow}{-3}, -1, 1\}$

Final output of system $y(n) = x(n) * h(n)$

$$\begin{array}{c|ccccc} & 2 & 1 & -3 & -1 & 1 \\ \hline 1 & 2 & 1 & -3 & -1 & 1 \\ 2 & 4 & 2 & -6 & -2 & 2 \end{array}$$

$y(n) = \{2, 5, \underset{\uparrow}{-1}, -7, -1, 2\}$