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Maths Tutorial 2

Calculus of Variation

Q.1) Find extremals of $\int_{x_1}^{x_2} \frac{y'^2}{x^2}\,dx$

→ Solution,

we have $F = \frac{y'^2}{x^2}$

Since F does not contain y explicitly, we shall use formula $\frac{\partial F}{\partial y'} = c$

Now, $\frac{\partial F}{\partial y'} = \frac{2y'}{x^2}$

But, as $\frac{\partial F}{\partial y'} = c$

$\therefore \frac{2y'}{x^2} = c$

$\therefore y' = \frac{c x^2}{2}$ $\therefore \frac{dy}{dx} = \frac{c}{2} x^2$

Integrating above equation,

$\int \frac{dy}{dx} = \int \frac{c}{2} x^2 dx$

$\therefore y = \frac{c}{6} x^3 + c'$

Taking the constants, suitably $y = c_1 x^3 + c_2$.

Show that isoperimetric problem $I[y(x)] = \int_{x_1}^{x_2} y'^2 dx$ subject to the condition

$$\int_{x_1}^{x_2} y\, dx = k$$ is a parabola

Solution,

we have to find $y = f(x)$ such that

$$\int_{x_1}^{x_2} F\, dx = \int_{x_1}^{x_2} y'^2\, dx \quad \text{——— (1)}$$

is minimum subject to condition,

$$\int_{x_1}^{x_2} G\, dx = \int_{x_1}^{x_2} y\, dx = k \quad \text{——— (2)}$$

To use Lagrange's equation, we multiply (2) by λ and add to (1)

$$\therefore H = F + \lambda G = \int_{x_1}^{x_2} (y'^2 + \lambda y)\, dx \quad \text{——— (3)}$$

Since the integrand is free from x, we use

$$\therefore F - y' \frac{\partial F}{\partial y'} = c \quad \text{——— (4)}$$

$$\text{——— (5)}$$

where $F = H = y'^2 + \lambda y$

Hence from (4) using (5), we get

$$y'^2 + \lambda y - y' \cdot 2y' = c \quad \therefore -y'^2 + \lambda y = c$$

$$\therefore y'^2 - \lambda y = -c = c_1$$

$$\therefore y' = \sqrt{c_1 + \lambda y}$$

$$\therefore \frac{dy}{\sqrt{c_1 + \lambda y}} = dx$$

Integrating, we get

$$\frac{2}{\lambda}\sqrt{c_1 + \lambda y} = x + c_2$$

$$\therefore \sqrt{c_1+\lambda y} = \frac{\lambda}{2}(x+c_2)$$

$$\therefore (c_1+\lambda y) = \left(\frac{\lambda}{2}\right)^2 (x+c_2)^2$$

$$\therefore \lambda y = \frac{\lambda^2 x^2}{4} + \frac{\lambda^2}{2} c_2 x + \frac{\lambda^2}{4} c_2^2 - c_1$$

$$\therefore y = \frac{\lambda x^2}{4} + \frac{c_2 \lambda}{2} x + \left[\frac{\lambda c_2^2}{4} - \frac{c_1}{\lambda}\right] = \frac{\lambda x^2}{4} + c' x + c''$$

This is a parabola.

Using Rayleigh-Ritz method, solve the boundary value problem $I = \int_0^1 \left(xy + \frac{1}{2} y'^2\right) dx$, $0 \le x \le 1$

Solution,

We have to extremise

$$I = \int_0^1 \left(xy + \frac{1}{2} y'^2\right) dx \quad \text{--- (1)}$$

where $F = xy + \frac{1}{2} y'^2$ --- (2)

Assume the trial solution,

$$\bar{y}(x) = c_0 + c_1 x + c_2 x^2 \quad \text{--- (3)}$$

By data $\bar{y}(0) = 0 \therefore c_0 = 0$ and $\bar{y}(1) = 0$

$$\therefore c_1 + c_2 = 0, \therefore c_2 = -c_1$$

$$\therefore \bar{y}(x) = c_1 x - c_1 x^2 = c_1(x - x^2) \quad \text{--- (4)}$$

$$\bar{y}'(x) = c_1(1-2x)$$

Putting these values in (1), we get,

$$I = \int_0^1 \left\{ x(c_1(x-x^2)) + \frac{1}{2} c_1^2 (1-2x)^2 \right\} dx$$

$$= \int_0^1 \left[c_1(x^2 - x^3) + \frac{1}{2} c_1^2 (1 - 4x + 4x^2)\right] dx$$

$$= c_1 \left[\int_0^1 x^2 dx - \int_0^1 x^3 dx\right] + \frac{1}{2} c_1^2 \left[\int_0^1 1 dx - 4\int_0^1 x dx + 4\int_0^1 x^2 dx\right]$$

$$= c_1 \left\{\left[\frac{x^3}{3}\right]_0^1 - \left[\frac{x^4}{4}\right]_0^1\right\} + \frac{1}{2} c_1^2 \left\{[x]_0^1 - 4\left[\frac{x^2}{2}\right]_0^1 + 4\left[\frac{x^3}{3}\right]_0^1\right\}$$

$$= c_1 \left(\frac{1}{3} - \frac{1}{4}\right) + \frac{1}{2} c_1^2 \left[1 - 2 + \frac{4}{3}\right]$$

$$I = c_1 \left(\frac{1}{12}\right) + \frac{1}{2} c_1^2 \left(\frac{1}{3}\right) = \frac{c_1}{12} + \frac{c_1^2}{6}$$

Its stationary values are given by, $\frac{dI}{dc_1} = 0$

$$\therefore \frac{1}{12} + \frac{2c_1}{6} = 0 \quad \therefore c_1 = -\frac{1}{4}$$

Putting this value in (4),

we get $\bar{y}(x) = -\frac{1}{4}(x - x^2) = \frac{1}{4} x(x-1)$ //.

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Maths Tutorial 3

Statistical Technique

Calculate Rank ~~Co~~ Correlation Coefficient for the following data.

X :	53	98	95	81	75	61	59	55
y :	47	25	32	37	30	40	39	45

Solution,

X	R_1	y	R_2	D R_1-R_2	D^2
53	1	47	8	-7	49
98	8	25	1	7	49
95	7	32	3	4	16
81	6	37	4	2	4
75	5	30	2	3	9
61	4	40	6	-2	4
59	3	39	5	-2	4
55	2	45	7	-5	25
					$\Sigma D^2 = 160$

$$R = 1 - \frac{6(\Sigma D^2)}{N^3 - N} = 1 - \frac{6 \times 160}{8^3 - 8} = 1 - \frac{960}{504}$$

$$= 1 - 1.9048$$

$$= -0.9048$$

$\therefore \boxed{R = -0.9048}$

FOR EDUCATIONAL USE

Ex 2) Find the equation of line of regression of y on x for the following data.

x :	5	6	7	8	9	10	11
y :	11	14	14	15	12	17	16

Sr.No	x	x^2	y	y^2	xy
1	5	25	11	121	55
2	6	36	14	196	84
3	7	49	14	196	98
4	8	64	15	225	120
5	9	81	12	144	108
6	10	100	17	289	170
7	11	121	16	256	176
N=7	$\Sigma x = 56$	$\Sigma x^2 = 476$	$\Sigma y = 99$	$\Sigma y^2 = 1427$	$\Sigma xy = 811$

Line of regression of y on x is

$$y = a + bx$$

The normal equations are

$\Sigma y = Na + b\Sigma x \quad \therefore 99 = 7a + 56b$ —— ①

$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \therefore 811 = 56a + 476b$ —— ②

Multiply ① by 56 and ② by 7, subtract

$\therefore 5544 = 392a + 3136b$

$\underline{5677 = 392a + 3332b}$

$-133 = \quad -196b$

$\therefore b = \dfrac{133}{196} = 0.6789$

$\therefore 99 = 7a + 56 \times \frac{1.33}{196}$ $\therefore 7a = 99 - 38$

$\therefore 7a = 61$ $\therefore \boxed{a = 8.7143}$

The equation of line of regression of y on x is

$$\boxed{y = 8.7143 + 0.6786x}$$

Fit a parabola to given data

X :	1	2	3	4	5
y :	10	12	8	10	14

Let equation of parabola be $y = a + bx + cx^2$

X	y	X^2	X^3	X^4	Xy	X^2y
1	10	1	1	1	10	1
2	12	4	8	16	24	48
3	8	9	27	81	24	72
4	10	16	64	256	40	160
5	14	25	125	625	70	350
$\Sigma X = 15$	$\Sigma y =$ 54	ΣX^2 55	$\Sigma X^3 =$ 225	$\Sigma X^4 =$ 979	ΣXy = 168	$\Sigma X^2 y$ = 631

Normal equations are

$$\Sigma y = Na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$

$$\Sigma X^2 y = a\Sigma X^2 + b\Sigma x^3 + c\Sigma X^4$$

$$54 = 5a + 15b + 55c \quad \text{—— (1)}$$
$$168 = 15a + 55b + 225c \quad \text{—— (2)}$$
$$631 = 55a + 225b + 979c \quad \text{—— (3)}$$

Solving the equations simultaneously,

$$a = 9.5, \quad b = 0.171, \quad c = 0.0714$$

Hence equation of parabola is

$$\boxed{y = 9.5 + 0.171x + 0.0714x^2}$$

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Maths Tutorial 4
Probability Distribution

1) A bag contains 7 red and 3 black balls and another bag contains 4 red and 6 black balls. One ball is transferred from the first to second bag and then ball is drawn from the second bag. If this ball happened to be red, find the probability that a black ball was transferred.

Solution

Let A_1 = transferred black ball from first bag to second bag

A_2 = transferred red ball from 1st to 2nd bag

B = Drawn ball is red

$P(A_1) = \frac{3}{10}$, $P(A_2) = \frac{7}{10}$

$P(B|A_1) = \frac{4C_1}{10} = \frac{4}{10}$ $P(B|A_2) = \frac{5C_1}{10} = \frac{5}{10} = \frac{1}{2}$

By Bayes theorem,

$$P(A|B) = \frac{P(A_1) \cdot P(B|A_1)}{P(A_1) P(B|A_1) + P(A_2) \cdot P(B|A_2)}$$

$$= \frac{3/10 \cdot 4/10}{\frac{3}{10} \cdot \frac{4}{10} + \frac{7}{10} \cdot \frac{1}{2}}$$

$$= \frac{12/100}{\frac{12}{100} + \frac{7}{20}}$$

$$= \frac{12/100}{47/100}$$

$$= \frac{12}{47}$$

) Find the value of k if the function,

$f(x) = kx^2(1-x^3)$, $0 \le x \le 1$

$= 0$, otherwise

is a probability density function; Also find $P(0 \le x \le \frac{1}{2})$ and mean and variance.

Solution,

$f(x) = kx^2(1-x^3)$, $0 \le x \le 1$

$= 0$, otherwise

$$\int_0^1 kx^2(1-x^3) = 1.$$

$$k\int_0^1 x^2 - \int_0^1 x^5 = 1$$

$$k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1$$

$$\frac{k}{3} - \frac{k}{6} = 1$$

$$\boxed{k=6}$$

$\therefore P(0 \le x \le \frac{1}{2})$

$$= \int_0^{1/2} 6x^2(1-x^3)$$

$$= 6\int_0^{1/2} x^2 - x^5$$

$$= 6\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^{1/2}$$

$$= 6 \times \left(\frac{1}{8\times 3} - \frac{1}{64\times 6}\right)$$

$$= \frac{1}{4} - \frac{1}{64}$$

$$= \frac{15}{64}$$

$$E(x) = \int x f(x) = \mu_1'$$

$$= 6\int_0^1 x \cdot x^2(1-x^3)\,dx$$

$$= 6\int_0^1 (x^3 - x^6)\,dx$$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{7}\right]$$

$$= \frac{18}{28} = \frac{9}{14}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = 6\int_0^1 x^2(x^2(1-x^3))\,dx$$

$$= 6\int_0^1 (x^4 - x^7)\,dx$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1$$

$$= 6\left[\frac{1}{5} - \frac{1}{8}\right]$$

$$= \frac{18}{40} = \frac{9}{20}$$

$$\text{Variance} = \mu_2' - \mu_1' = \frac{9}{20} - \frac{91}{196} = \frac{441-405}{980} = \frac{36}{980} = \frac{9}{245}.$$

3) The marks obtained by 1000 students in an examination are found to be normally distributed with 70 and standard deviation 5. Estimate the number of students whose marks will be
i) between 60 and 75
ii) More than 75.

→ Solution,

$Z = \dfrac{X - M}{\sigma}$

$M = 70$

$\sigma = 5$

$X = 60, \quad z = -2$

$X = 75, \quad z = 1$

$\therefore P(60 \le X \le 75) = P(0 \le z \le 2) + P(0 \le z \le 1)$

$= 0.4772 + 0.3413$

$= 0.8185$

Number of students $= N \times P$

$= 1000 \times 0.8185$

$= 818$ students

$P(X \geq 75) = P(Z \geq 1)$

$= 0.5 - 0.3413 = 0.1587$

$NP = 1000 \times 0.1587$

$= 159$ students

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Maths Tutorial 5
Linear Algebra (Vector space)

Show that a line passing through origin in R^2 is a subspace of R^2

Solution,

Now by definition,

$$R^2 = \{(x, y) \mid x, y \in R\}$$

Let $W = \{(x, y) \mid x = y \mid x, y \in R\}$

where $x = y$ is a line passing through origin.

Let (x_1, y_1) & (x_2, y_2) be two elements of W & $\alpha \in R$,

then $x_1 = y_1$ & $x_2 = y_2$

1) $\alpha \; (x_1, y_1)$

$= (\alpha x_1, \alpha y_1)$

as $x_1 = y_1$ then $\alpha x_1 = \alpha y_1$

2) $(x_1, y_1) + (x_2, y_2)$

$(x_1 + x_2, y_1 + y_2)$

To be a part of W,

$x_1 + x_2 = y_1 + y_2$

$x_1 - y_1 = y_2 - x_2$

$0 = 0$ (as (x_1, y_1) & (x_2, y_2) are part of W)

So here W is a subspace of R^2

2) Verify Cauchy Schwartz inequality for $u = (-4, 2, 1)$ and $v = (8, -4, -2)$, then Also find the angle between u a…

→ Solution,

$u = (-4, 2, 1)$ & $v = (8, -4, -2)$

Checking for Cauchy Schwartz inequality,

$$|u \cdot v| \leq \|u\| \cdot \|v\|$$

$$u \cdot v = (-4 \times 8) + (2 \times -4) + (1 \times -2)$$

$$= -32 + (-8) + (-2)$$

$$= -42$$

$$|u \cdot v| = 42$$

$$\|u\| = \sqrt{(-4)^2 + (2)^2 + (1)^2}$$

$$= \sqrt{21}$$

$$\|v\| = \sqrt{(8)^2 + (-4)^2 + (-2)^2}$$

$$= \sqrt{84}$$

$$\|u\| \cdot \|v\| = \sqrt{21} \times \sqrt{84}$$

$$= 42$$

$$\therefore |u \cdot v| \leq \|u\| \cdot \|v\|$$

$$42 \leq 42$$

It satisfies Cauchy Schwartz Property.

To find Angle between u & v

$$\theta = \cos^{-1}\left(\frac{\bar{u} \cdot \bar{v}}{\|\bar{u}\| \cdot \|\bar{v}\|}\right)$$

$$= \cos^{-1}\left(\frac{-42}{42}\right)$$

$$= \cos^{-1}(-1)$$

$$\boxed{\theta \approx 3.14 \text{ radians.}}$$

Let $\mathbb{R}^3$ be the Euclidean Inner product. Use Gram Schmidt process to transform for basis $\{u_1, u_2, u_3\}$ into orthonormal basis where

$u_1 = (1,1,1)$, $u_2 = (-1,1,0)$, $u_3 = (1,2,1)$

Solution,

Using Gram Schmidt process,

Step 1

$\bar{v}_1 = \bar{u}_1$

$\bar{v}_1 = (1,1,1)$

Step 2: $\bar{v}_2 = \bar{u}_2 - \dfrac{\langle \bar{u}_2 \cdot \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} \bar{v}_1$

$= (-1,1,0) - \dfrac{((-1)+(1)+0)}{3}(1,1,1)$

$\bar{v}_2 = (-1,1,0)$

Step 3: $\vec{v}_3 = u_3 - \dfrac{\langle u_3 \cdot v_1 \rangle}{(v_1)^2} * - \dfrac{\langle u_3\, v_2 \rangle}{\|v_2\|^2} v_2$

$v_3 = (1,2,1) - \dfrac{[(1)+(2)+(1)]}{3}(1,1,1) - \dfrac{[(-1)+(2)+(0)]}{2}$

$= (1,2,1) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right) - \left(\frac{-1}{2}, \frac{1}{2}, 0\right)$

$= \left(\frac{3}{2}, \frac{3}{2}, 1\right) - \left(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}\right)$

$= \left(\frac{1}{6}, \frac{1}{6}, \frac{-1}{3}\right)$

To find orthonormal sets we need to divide it by $\|v_1\|, \|v_2\|, \|v_3\|$

$\Rightarrow \left(\frac{1}{6\sqrt{3}}, \frac{1}{6\sqrt{2}}, -\frac{\sqrt{6}}{3}\right)$ //