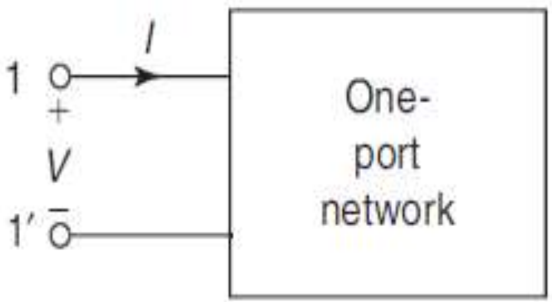
# Network Funcion

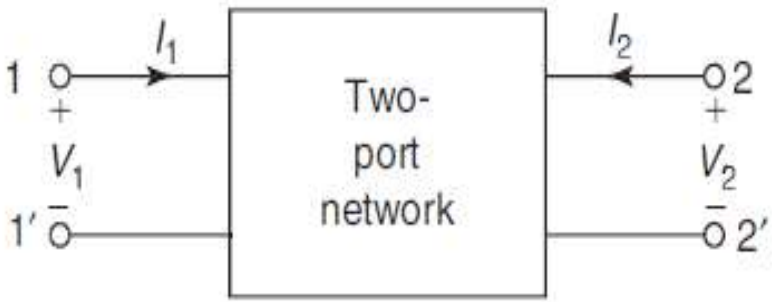
# INTRODUCTION

A network function gives the relation between currents or voltages at different parts of the network. It is broadly classified as *driving point* and *transfer function*. It is associated with terminals and ports.

Any network may be represented schematically by a rectangular box. Terminals are needed to connect any network to any other network or for taking some measurements. Two such associated terminals are called *terminal pair* or *port*. If there is only one pair of terminals in the network, it is called a one-port network. If there are two pairs of terminals, it is called a two-port network. The port to which energy source is connected is called the *input port*. The port to which load is connected is known as the *output port*. One such network having only one pair of terminals $(1 - 1')$ is shown in Fig. 8.1 (a) and is called *one-port network*. Figure 8.1 (b) shows a two-port network with two pairs of terminals. The terminals $1 - 1'$ together constitute a port. Similarly, the terminals $2 - 2'$ constitute another port.




(a)




(b)

A voltage and current are assigned to each of the two ports. $V_1$ and $I_1$ are assigned to the input port, whereas $V_2$ and $I_2$ are assigned to the output port. It is also assumed that currents $I_1$ and $I_2$ are entering into the network at the upper terminals 1 and 2 respectively.

# DRIVING-POINT FUNCTIONS

If excitation and response are measured at the same ports, the network function is known as the driving-point function. For a one-port network, only one voltage and current are specified and hence only one network function (and its reciprocal) can be defined.

1. ***Driving-point Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at the same port. It is denoted by $Z(s)$.

$$Z(s) = \frac{V(s)}{I(s)}$$

2. ***Driving-point Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at the same port. It is denoted by $Y(s)$.

$$Y(s)=\frac{I(s)}{V(s)}$$

For a two-port network, the driving-point impedance function and driving-point admittance function at port 1 are

$$Z_{11}(s)=\frac{V_1(s)}{I_1(s)}$$

$$Y_{11}(s)=\frac{I_1(s)}{V_1(s)}$$

Similarly, at port 2,

$$Z_{22}(s)=\frac{V_2(s)}{I_2(s)}$$

$$Y_{22}(s)=\frac{I_2(s)}{V_2(s)}$$

# TRANSFER FUNCTIONS

The transfer function is used to describe networks which have at least two ports. It relates a voltage or current at one port to the voltage or current at another port. These functions are also defined as the ratio of a response transform to an excitation transform. Thus, there are four possible forms of transfer functions.

1. ***Voltage Transfer Function*** It is defined as the ratio of the voltage transform at one port to the voltage transform at another port. It is denoted by $G(s)$.

$$G_{12}(s) = \frac{V_2(s)}{V_1(s)}$$

$$G_{21}(s) = \frac{V_1(s)}{V_2(s)}$$

2. ***Current Transfer Function*** It is defined as the ratio of the current transform at one port to the current transform at another port. It is denoted by $\alpha(s)$.

$$\alpha_{12}(s) = \frac{I_2(s)}{I_1(s)}$$

$$\alpha_{21}(s) = \frac{I_1(s)}{I_2(s)}$$

3. ***Transfer Impedance Function*** It is defined as the ratio of the voltage transform at one port to the current transform at another port. It is denoted by $Z(s)$.

$$Z_{12}(s) = \frac{V_2(s)}{I_1(s)}$$

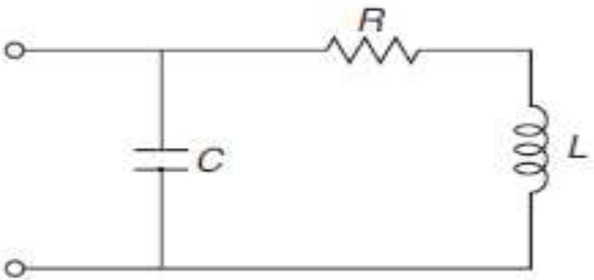
$$Z_{21}(s) = \frac{V_1(s)}{I_2(s)}$$

4. ***Transfer Admittance Function*** It is defined as the ratio of the current transform at one port to the voltage transform at another port. It denoted by $Y(s)$.

$$Y_{12}(s) = \frac{I_2(s)}{V_1(s)}$$

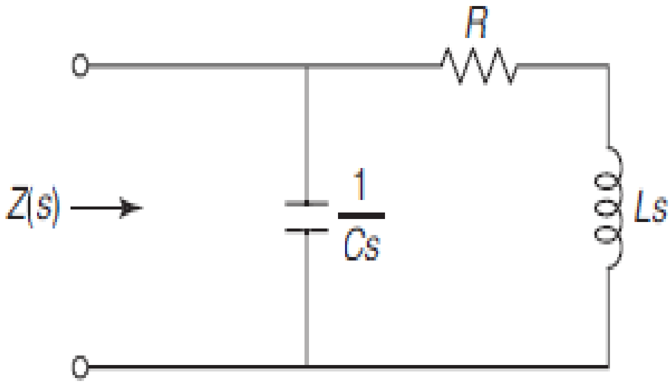
$$Y_{21}(s) = \frac{I_1(s)}{V_2(s)}$$

*Determine the driving-point impedance function of a one-port network shown in Fig.*

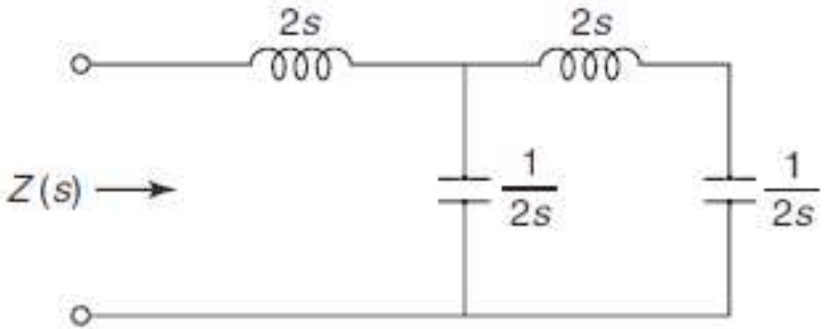



**Solution** The transformed network is shown in Fig.

$$Z(s)=\frac{\frac{1}{Cs}(R+Ls)}{\frac{1}{Cs}+(R+Ls)}=\frac{R+Ls}{LCs^2+RCs+1}=\frac{1}{C}\frac{s+\frac{R}{L}}{s^2+\frac{R}{L}s+\frac{1}{LC}}$$



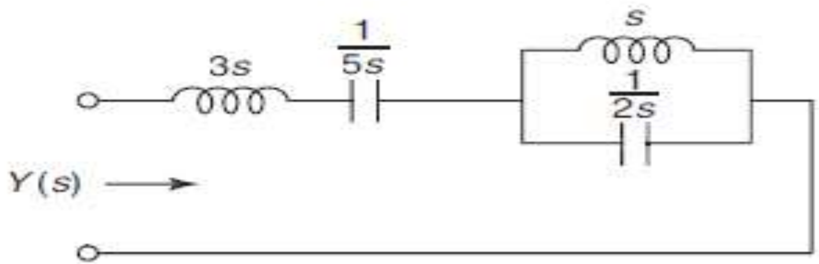
*Determine the driving-point impedance of the network shown in Fig.*




## Solution

$$Z(s) = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{1}{2s}+2s+\frac{1}{2s}} = 2s + \frac{\frac{1}{2s}\left(2s+\frac{1}{2s}\right)}{\frac{2+4s^2}{2s}} = 2s + \frac{2s+\frac{1}{2s}}{2+4s^2} = \frac{4s+8s^3+2s+\frac{1}{2s}}{2+4s^2} = \frac{16s^4+12s^2+1}{8s^3+4s}$$

*Find the driving-point admittance function of the network shown in Fig.*

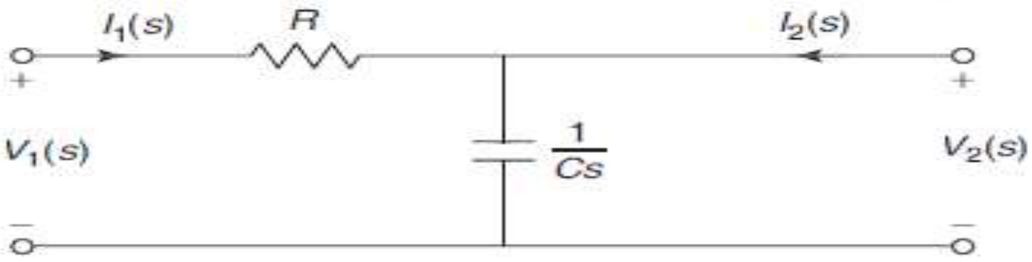



## Solution

$$Z(s) = 3s + \frac{1}{5s} + \frac{s\left(\frac{1}{2s}\right)}{s + \frac{1}{2s}} = 3s + \frac{1}{5s} + \frac{s}{2s^2+1} = \frac{30s^4 + 15s^2 + 2s^2 + 1 + 5s^2}{5s\,(2s^2+1)} = \frac{30s^4 + 22s^2 + 1}{5s\,(2s^2+1)}$$

$$Y(s) = \frac{1}{Z(s)} = \frac{5s\,(2s^2+1)}{30s^4 + 22s^2 + 1}$$

*Find voltage transfer function of the two-port network shown in Fig.*




**Solution** By voltage division rule,

$$V_2(s) = V_1(s)\frac{\frac{1}{Cs}}{R+\frac{1}{Cs}} = V_1(s)\frac{1}{RCs+1} = V_1(s)\frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

Voltage transfer function $$\frac{V_2(s)}{V_1(s)} = \frac{\frac{1}{RC}}{s+\frac{1}{RC}}$$

## POLES AND ZEROS OF NETWORK FUNCTIONS

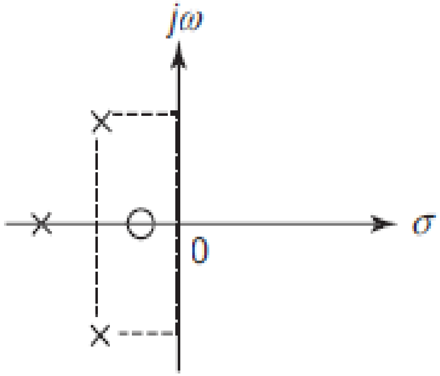
The network function $F(s)$ can be written as ratio of two polynomials.

$$F(s) = \frac{N(s)}{D(s)} = \frac{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}$$

where $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_m$ are the coefficients of the polynomials $N(s)$ and $D(s)$. These are real and positive for networks of passive elements. Let $N(s) = 0$ have $n$ roots as $z_1, z_2, \ldots\ldots, z_n$ and $D(s) = 0$ have m roots as $p_1, p_2, \ldots\ldots, p_m$. Then $F(s)$ can be written as

$$F(s) = H \frac{(s - z_1)(s - z_2)\ldots(s - z_n)}{(s - p_1)(s - p_2)\ldots(s - p_m)}$$

where $\frac{a_n}{b_m}$ is a constant called *scale factor* and $z_1, z_2, \ldots, z_n, p_1, p_2, \ldots, p_m$ are complex frequencies. When the variable $s$ has the values $z_1, z_2, \ldots, z_n$, the network function becomes zero; such complex frequencies are known as the zeros of the network function. When the variable $s$ has values $p_1, p_2, \ldots, p_m$, the network function becomes infinite; such complex frequencies are known as the poles of the network function. A network function is completely specified by its poles, zeros and the scale factor.




Poles and zeros are critical frequencies. The network function becomes infinite at poles, while the network function becomes zero at zeros. The network function has a finite, non-zero value at other frequencies.

Poles and zeros provide a representation of a network function as shown

## RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR DRIVING-POINT FUNCTIONS [COMMON FACTORS IN *N*(*s*) AND *D*(*s*) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real and positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of all poles and zeros, must be negative or zero, i.e., the poles and zeros must lie in left half of $s$ plane.

(4) If the real part of pole or zero is zero, then that pole or zero must be simple.

(5) The polynomials $N(s)$ and $D(s)$ may not have missing terms between those of highest and lowest degree, unless all even or all odd terms are missing.

(6) The degree of $N(s)$ and $D(s)$ may differ by either zero or one only. This condition prevents multiple poles and zeros at $s = \infty$.

(7) The terms of lowest degree in $N(s)$ and $D(s)$ may differ in degree by one at most. This condition prevents multiple poles and zeros at $s = 0$.

## RESTRICTIONS ON POLE AND ZERO LOCATIONS FOR TRANSFER FUNCTIONS [COMMON FACTORS IN *N*(S) AND *D*(S) CANCELLED]

(1) The coefficients in the polynomials $N(s)$ and $D(s)$ must be real, and those for $D(s)$ must be positive.

(2) The poles and zeros, if complex or imaginary, must occur in conjugate pairs.

(3) The real part of poles must be negative or zero. If the real part is zero, then that pole must be simple.

(4) The polynomial $D(s)$ may not have any missing terms between that of highest and lowest degree, unless all even or all odd terms are missing.

(5) The polynomial $N(s)$ may have terms missing between the terms of lowest and highest degree, and some of the coefficients may be negative.

(6) The degree of $N(s)$ may be as small as zero, independent of the degree of $D(s)$.

(7) For voltage and current transfer functions, the maximum degree of $N(s)$ is the degree of $D(s)$.

(8) For transfer impedance and admittance functions, the maximum degree of $N(s)$ is the degree of $D(s)$ plus one.

*Test whether the following represent driving-point immittance functions.*

(a) $\dfrac{5s^4+3s^2-2s+1}{s^3+6s+20}$ (b) $\dfrac{s^3+s^2+5s+2}{s^4+6s^3+9s^2}$ (c) $\dfrac{s^2+3s+2}{s^2+6s+2}$

## Solution

(a) The numerator and denominator polynomials have a missing term between those of highest and lowest degree and one of the coefficient is negative in numerator polynomial. Hence, the function does not represent a driving-point immittance function.

(b) The term of lowest degree in numerator and denominator polynomials differ in degree by two. Hence, the function does not represent a driving-point immittance function.

(c) The function satisfies all the necessary conditions. Hence, it represents a driving-point immittance function.

*Test whether the following represent transfer functions.*

(a) $G_{21} = \dfrac{3s+2}{5s^3+4s^2+1}$ (b) $\alpha_{12} = \dfrac{2s^2+5s+1}{s+7}$ (c) $Z_{21} = \dfrac{1}{s^3+2s}$

## Solution

(a) The polynomial $D(s)$ has a missing term between that of highest and lowest degree. Hence, the function does not represent a transfer function.

(b) The degree of $N(s)$ is greater than $D(s)$. Hence the function does not represent a transfer function.

(c) The function satisfies all the necessary conditions. Hence, it represents a transfer function.

*Obtain the pole-zero plot of the following functions.*

(a) $F(s) = \dfrac{s(s+2)}{(s+1)(s+3)}$

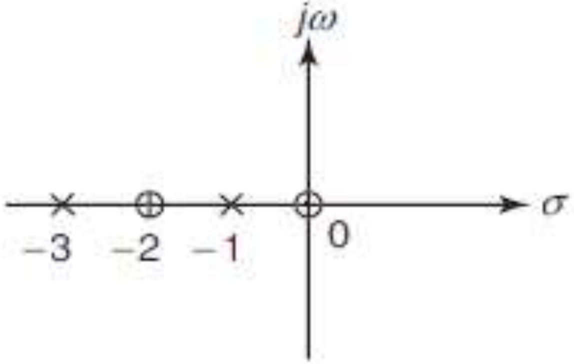
(b) $F(s) = \dfrac{s(s+1)}{(s+2)^2(s+3)}$

(c) $F(s) = \dfrac{s(s+2)}{(s+1+j1)(s+1-j1)}$

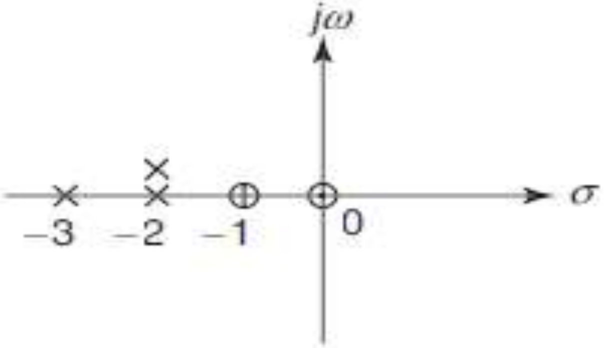
(d) $F(s) = \dfrac{(s+1)^2(s+5)}{(s+2)(s+3+j2)(s+3-j2)}$

(e) $F(s) = \dfrac{s^2+4}{(s+2)(s^2+9)}$

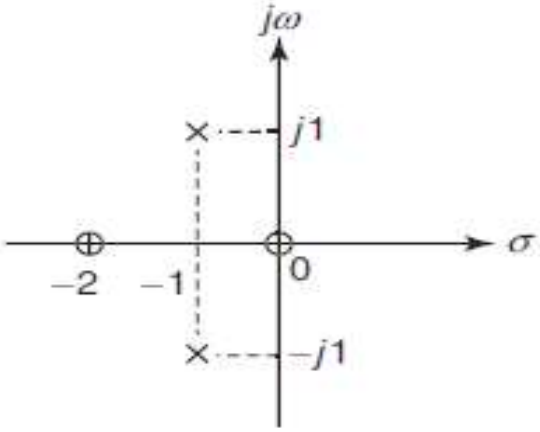
(a) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1$ and $s = -3$. The pole-zero plot is shown in Fig. 8.42.



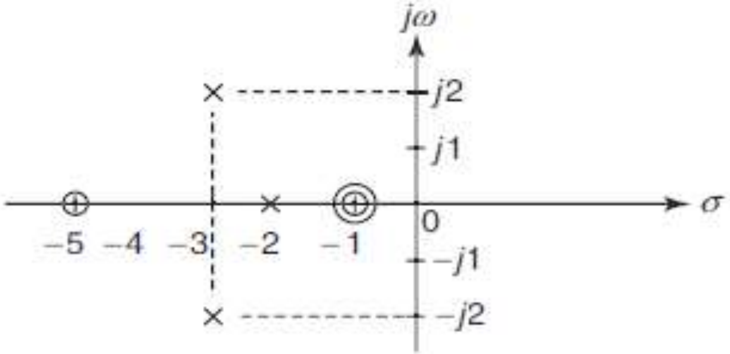

(b) The function $F(s)$ has zeros at $s = 0$ and $s = -1$ and poles at $s = -2, -2$ and $s = -3$. The pole-zero plot is shown in Fig. 8.43.



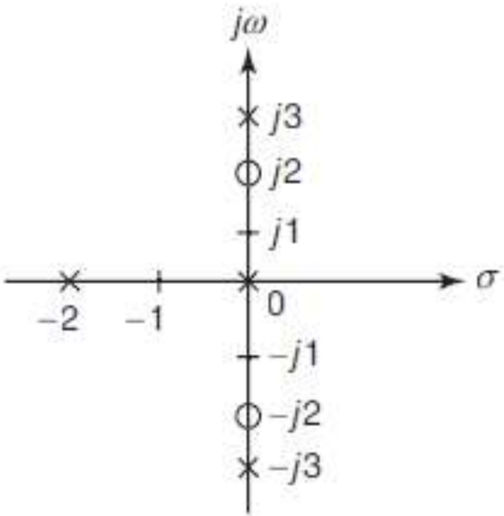
(c) The function $F(s)$ has zeros at $s = 0$ and $s = -2$ and poles at $s = -1 - j1$ and $s = -1 + j1$. The pole-zero plot is shown in Fig. 8.44.



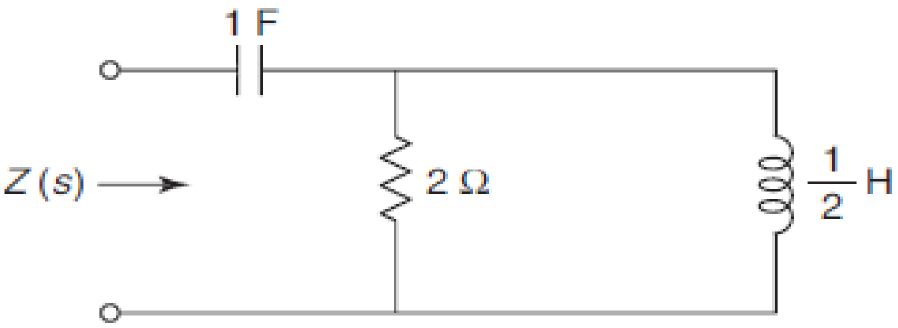

(d) The function $F(s)$ has zeros at $s = -1, -1$ and $s = -5$ and poles at $s = -2$, $s = -3 + j2$ and $s = -3 - j2$. The pole-zero plot is shown in Fig. 8.45.



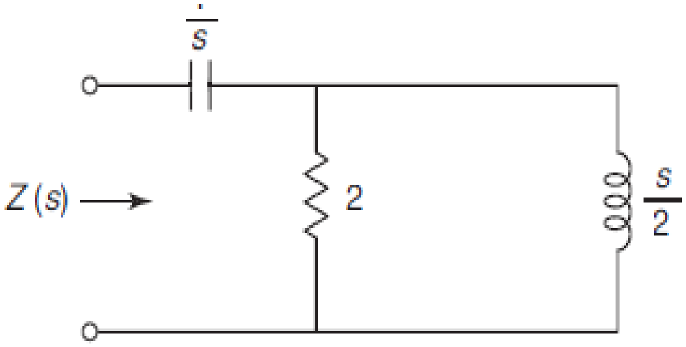
(e) The function $F(s)$ has zeros at $s = j2$ and $s = -j2$ and poles at $s = -2$, $s = j3$ and $s = -j3$. The pole-zero plot is shown in Fig. 8.46.



*Find poles and zeros of the impedance of the network shown in Fig. and plot them on the s-plane.*



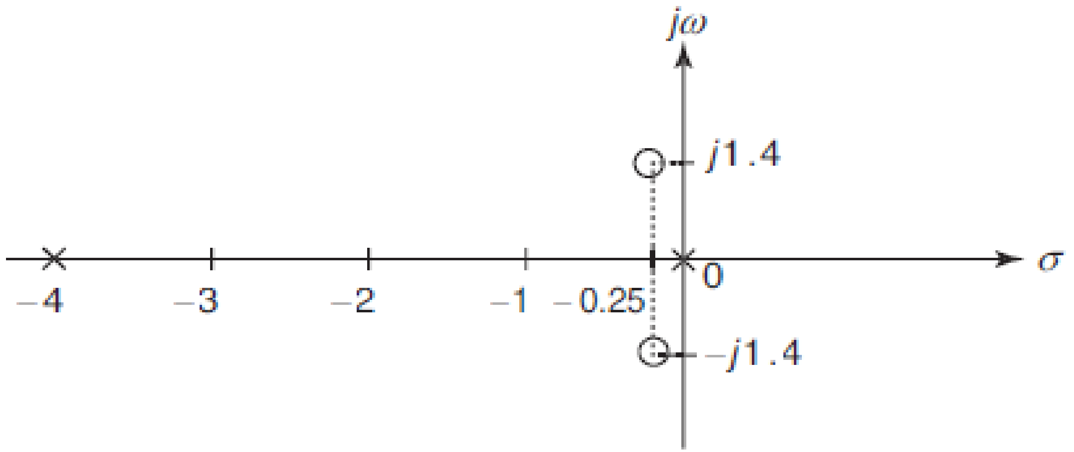

**Solution** The transformed network is shown in Fig.



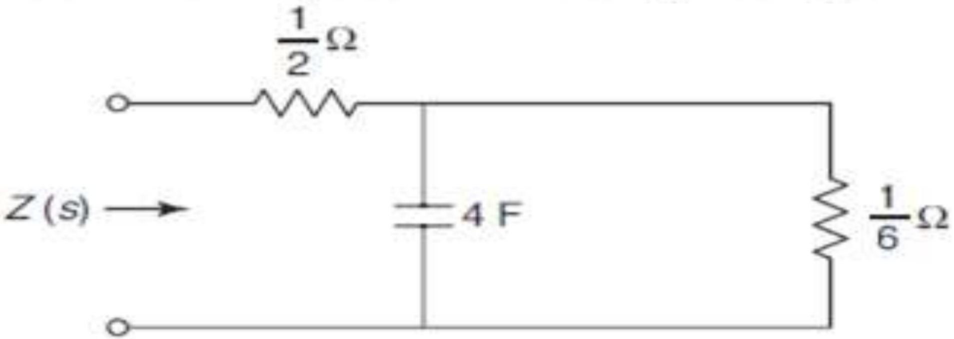

$$Z(s) = \frac{1}{s} + \frac{\frac{s}{2} \times 2}{\frac{s}{2} + 2} = \frac{1}{s} + \frac{2s}{s+4} = \frac{2s^2 + s + 4}{s(s+4)} = \frac{2(s^2 + 0.5s + 2)}{s(s+4)}$$

$$= \frac{2(s + 0.25 + j1.4)(s + 0.25 - j1.4)}{s(s+4)}$$

The function $Z(s)$ has zeros at $s = -0.25 + j1.4$ and $s = -0.25 - j1.4$ and poles at $s = 0$ and $s = -4$ as shown



*Determine the poles and zeros of the impedance function Z (s) in the network shown*




**Solution** The transformed network is shown in Fig. 8.51.

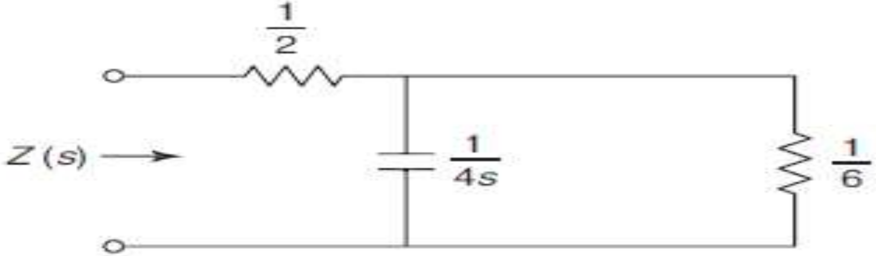
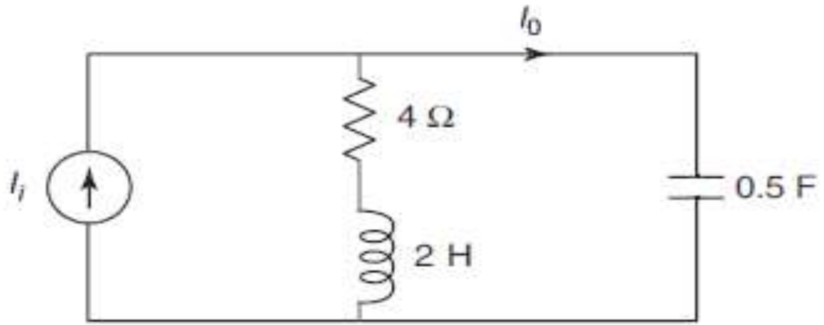



**Fig. 8.51**

$$Z(s) = \frac{1}{2} + \frac{\frac{1}{4s} \times \frac{1}{6}}{\frac{1}{4s} + \frac{1}{6}} = \frac{1}{2} + \frac{1}{4s+6} = \frac{4s+8}{2(4s+6)} = \frac{s+2}{2s+3} = \frac{0.5(s+2)}{s+1.5}$$

The function $Z(s)$ has zero at $s = -2$ and pole at $s = -1.5$.

*For the network shown in Fig. 8.55, plot poles and zeros of function* $\frac{I_0}{I_i}$.




**Solution** The transformed network is shown in Fig. 8.56.
By current-division rule,

$$I_0 = I_i\left(\frac{4+2s}{4+2s+\frac{2}{s}}\right)$$

$$\frac{I_0}{I_i} = \frac{s(4+2s)}{4s+2s^2+2} = \frac{s(s+2)}{s^2+2s+1} = \frac{s(s+2)}{(s+1)(s+1)}$$

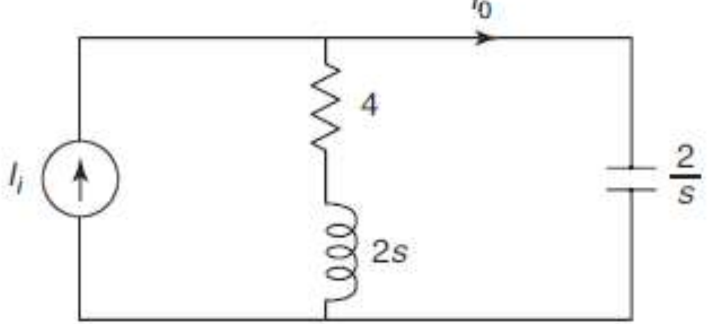
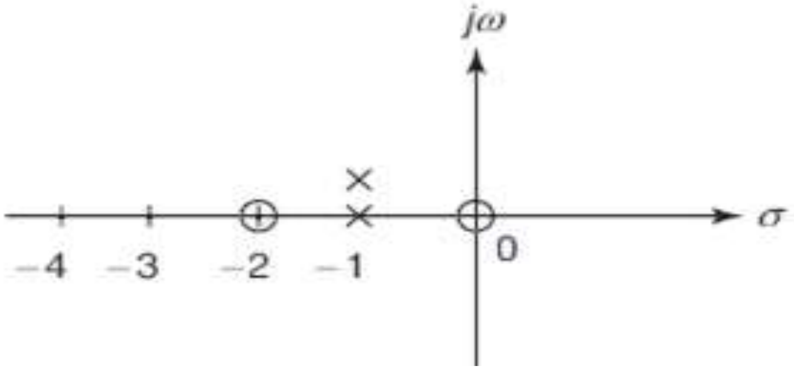
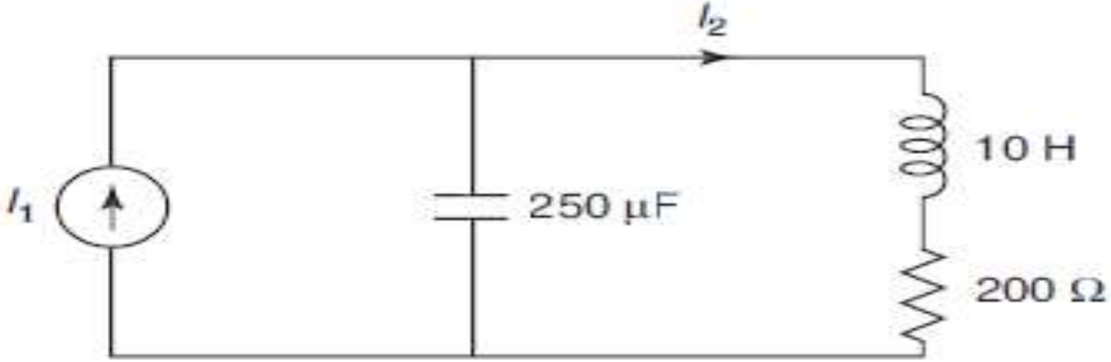



**Fig. 8.56**

The function has zeros at $s = 0$ and $s = -2$ and double poles at $s = -1$.




*Draw the pole-zero diagram of* $\frac{I_2}{I_1}$ *for the network shown in Fig.*



**Solution** The transformed network is shown in Fig. 8.59.

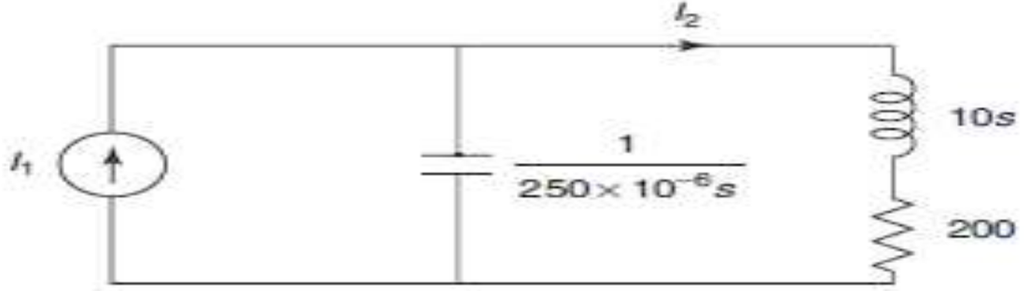


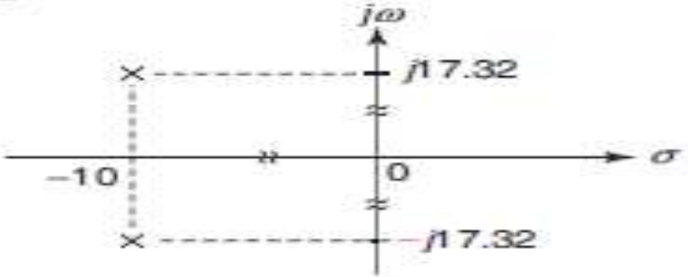

**Fig. 8.59**

By current-division rule,

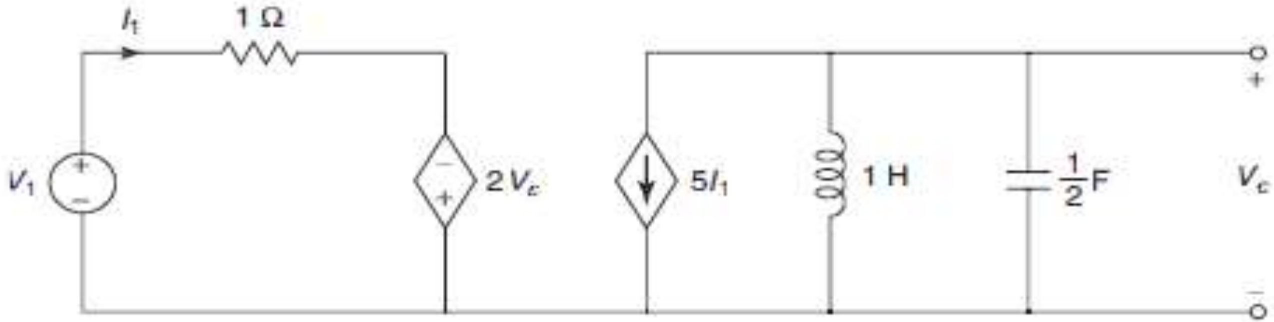
$$I_2 = I_1 \frac{\dfrac{1}{250\times 10^{-6}s}}{\dfrac{1}{250\times 10^{-6}s} + 10s + 200}$$

$$\frac{I_2}{I_1} = \frac{400}{s^2 + 20s + 400} = \frac{400}{(s+10-j17.32)(s+10+j17.32)}$$

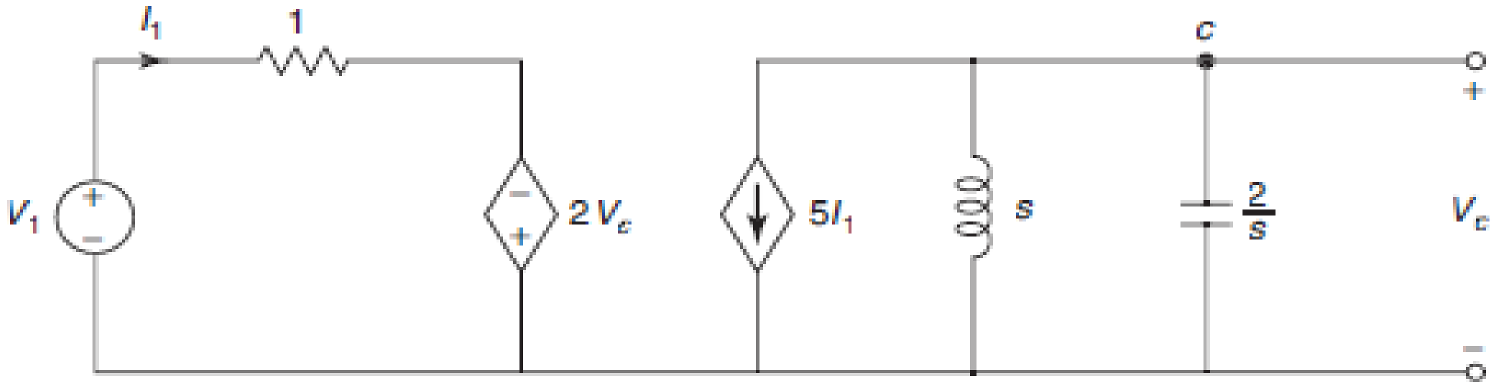
The function has no zero and poles at $s = -10 + j17.32$ and $s = -10 - j17.32$. The pole-zero diagram is shown in Fig. 8.60.



*For the network shown in Fig. 8.61, draw pole-zero plot of* $\frac{V_c}{V_1}$.




**Solution** The transformed network is shown in Fig.



Applying KVL to the left loop,

$$V_1 - 1I_1 + 2V_c = 0$$

$$I_1 = V_1 + 2V_c$$

Applying KCL at Node C,

$$5I_1 + \frac{V_c}{s} + \frac{V_c}{\frac{2}{s}} = 0$$

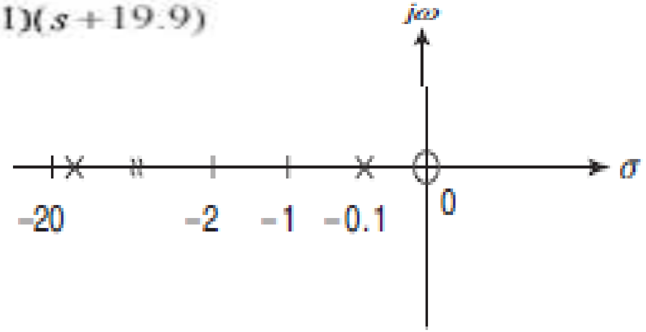
$$5(V_1 + 2V_c) + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$5V_1 + 10V_c + \frac{V_c}{s} + \frac{s}{2}V_c = 0$$

$$V_c\left(\frac{20s + 2 + s^2}{2s}\right) = -5V_1$$

$$\frac{V_c}{V_1} = -\frac{10s}{s^2 + 20s + 2} = -\frac{10s}{(s+0.1)(s+19.9)}$$

The function has zero at $s = 0$ and poles at $s = -0.1$ and $s = -19.9$.



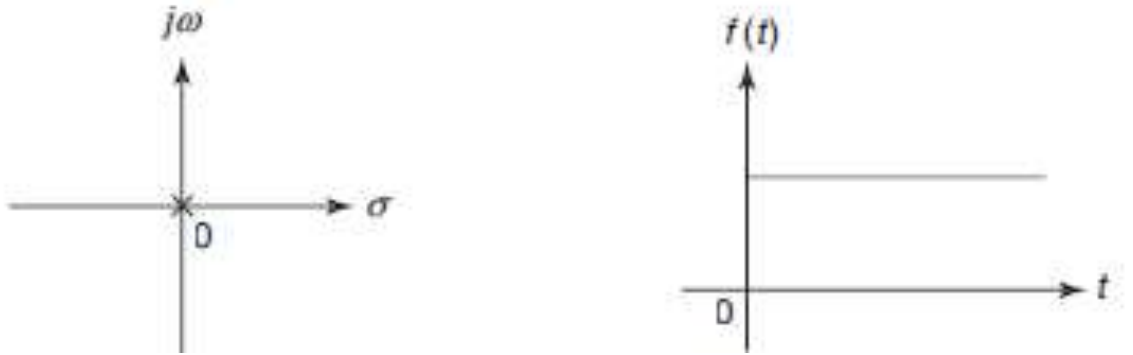
## TIME-DOMAIN BEHAVIOUR FROM THE POLE-ZERO PLOT

The time-domain behaviour of a system can be detemined from the pole-zero plot. Consider a network function

$$F(s) = H\frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_m)}$$

The poles of this function determine the time-domain behaviour of $f(t)$. The function $f(t)$ can be determined from the knowledge of the poles, the zeros and the scale factor $H$. Figure 8.80 shows some pole locations and the corresponding time-domain response.

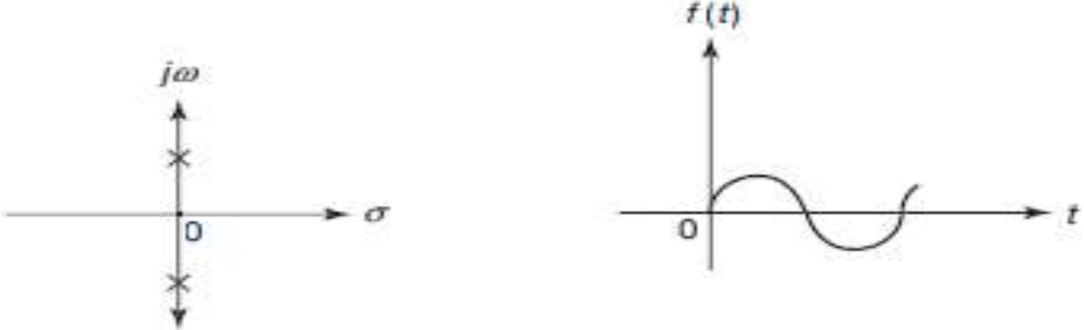
(i) When pole is at origin, i.e., at $s = 0$, the function $f(t)$ represents steady-state response of the circuit i.e., dc value. (Fig. 8.80)




(ii) When pole lies in the left half of the $s$-plane, the response decreases exponentially. (

(iii) When pole lies in the right half of the $s$-plane, the response increases exponentially. A pole in the right-half plane gives rise to unbounded response and unstable system. (Fig. 8.82)


(iv) For $s = 0 + j\omega_n$, the response becomes $f(t) = Ae^{\pm j\omega_n t} = \mathrm{A}(\cos\,\omega_n t \pm j\sin\,\omega_n t)$.The exponential response $e^{\pm j\omega_n t}$ may be interpreted as a rotating phasor of unit length. A positive sign of exponential $e^{j\omega_n t}$ indicates counterclockwise rotation, while a negative sign of exponential $e^{-j\omega_n t}$ indicates clockwise rotation. The variation of exponential function $e^{j\omega_n t}$ with time is thus sinusoidal and hence constitutes the case of sinusoidal steady state. (Fig. 8.83)



## Stability of the Network

Stability of the network is directly related to the location of poles in the $s$-plane.

(i) When all the poles lie in the left half of the $s$-plane, the network is said to be stable.
(ii) When the poles lie in the right half of the $s$-plane, the network is said to be unstable.
(iii) When the poles lie on the $j\omega$ axis, the network is said to be marginally stable.
(iv) When there are multiple poles on the $j\omega$ axis, the network is said to be unstable.
(v) When the poles move away from $j\omega$ axis towards the left half of the $s$-plane, the relative stability of the network improves.

# GRAPHICAL METHOD FOR DETERMINATION OF RESIDUE

Consider a network function,

$$F(s) = H\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

By partial fraction expansion,

$$F(s) = \frac{K_1}{(s-p_1)} + \frac{K_2}{(s-p_2)} + \cdots + \frac{K_m}{(s-p_m)}$$

The residue $K_i$ is given by

$$K_i = (s-p_i)F(s)\big|_{s\to p_i} = H\frac{(p_i-z_1)(p_i-z_2)\cdots(p_i-z_n)}{(p_i-p_1)(p_i-p_2)\cdots(p_i-p_m)}$$

Each term $(p_i - z_i)$ represents a phasor drawn from zero $z_i$ to pole $p_i$.
Each term $(p_i - p_k)$, $i \neq k$, represents a phasor drawn from other poles to the pole $p_i$.

$$K_i = H\frac{\text{Product of phasors (polar form) from each zero to } p_i}{\text{Product of phasors (polar form) from other poles to } p_i}$$

The residues can be obtained by graphical method in the following way:

(1) Draw the pole-zero diagram for the given network function.
(2) Measure the distance from each of the other poles to a given pole.
(3) Measure the distance from each of the other zeros to a given pole.
(4) Measure the angle from each of the other poles to a given pole.
(5) Measure the angle from each of the other zeros to a given pole.
(6) Substitute these values in the required residue equation.

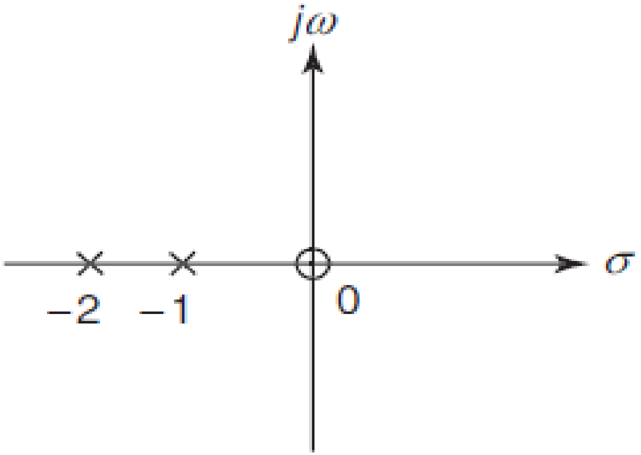
The graphical method can be used if poles are simple and complex. But it cannot be used when there are multiple poles.

*The current I(s) in a network is given by* $I(s) = \dfrac{2s}{(s+1)(s+2)}$. *Plot the pole-zero pattern in the s-plane and hence obtain i(t).*

**Solution** Poles are at $s = -1$ and $s = -2$ and zero is at $s = 0$. The pole-zero plot is shown in Fig.

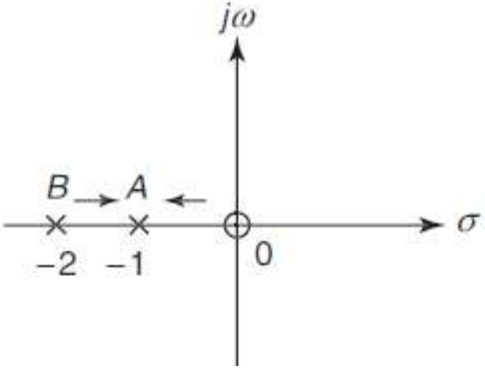
By partial-fraction expansion,

$$I(s) = \frac{K_1}{s+1} + \frac{K_2}{s+2}$$

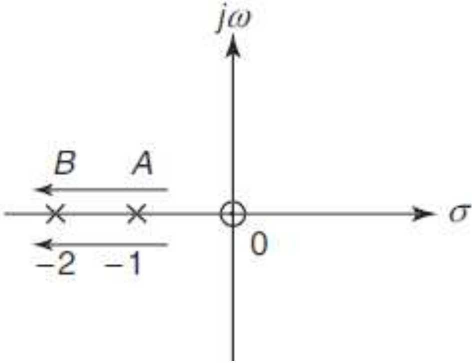


The coefficients $K_1$ and $K_2$, often referred as residues, can be evaluated from he pole-zero diagram. From

$$K_1 = H \frac{\text{Phasor from zero at origin to pole at } A}{\text{Phasor from pole at } B \text{ to pole at } A} = 2\left(\frac{1\angle 180^\circ}{1\angle 0^\circ}\right) = 2\angle 180^\circ = -2$$




$$K_2 = H \frac{\text{Phasor from zero at origin to pole at } B}{\text{Phasor from pole at } A \text{ to pole at } B} = 2\left(\frac{2\angle 180^\circ}{1\angle 180^\circ}\right) = 4$$



$$I(s) = -\frac{2}{s+1} + \frac{4}{s+2}$$

Taking inverse Laplace transform,

$$i\,(t) = -2e^{-t} + 4e^{-2t}$$

*The voltage V(s) of a network is given by*

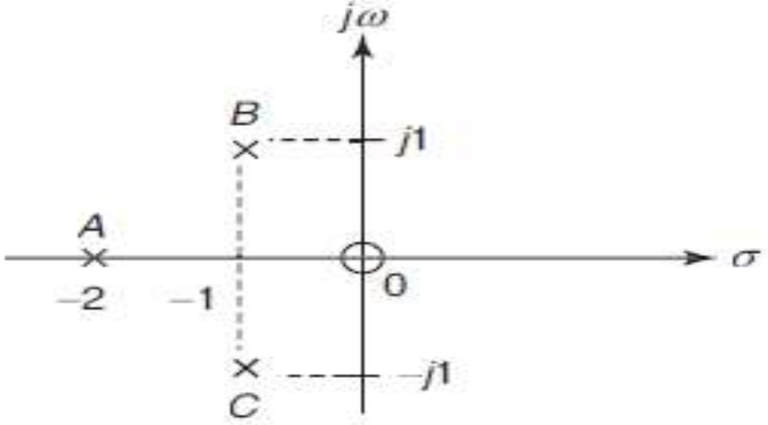
$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)}$$

*Plot its pole-zero diagram and hence obtain v (t).*

**Solution**

$$V(s) = \frac{3s}{(s+2)(s^2+2s+2)} = \frac{3s}{(s+2)(s+1+j1)(s+1-j1)}$$

Poles are at $s = -2$ and $s = -1 \pm j1$ and zero is at $s = 0$ as shown in Fig. 8.89.

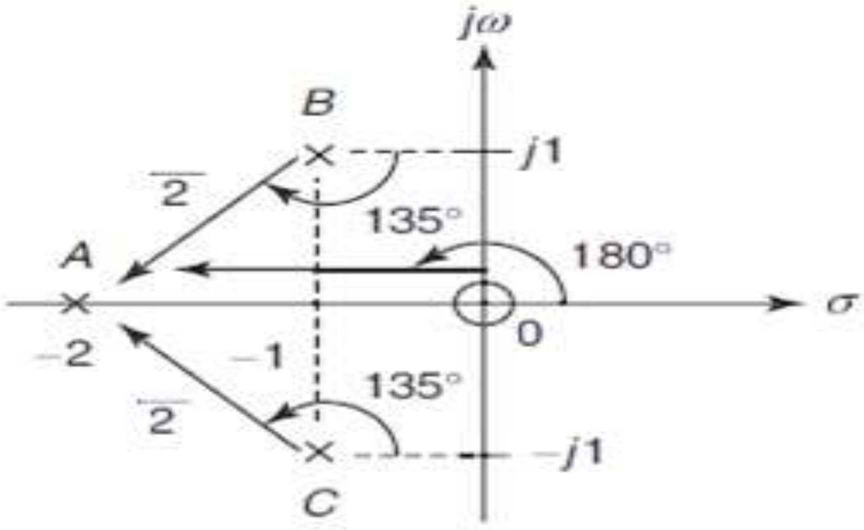


By partial-fraction expansion,

$$V(s) = \frac{K_1}{s+2} + \frac{K_2}{s+1-j1} + \frac{K_2^*}{s+1+j1}$$

The coefficients $K_1$, $K_2$ and $K_2^*$ can be evaluated from the pole-zero diagram. From Fig. 8.90,

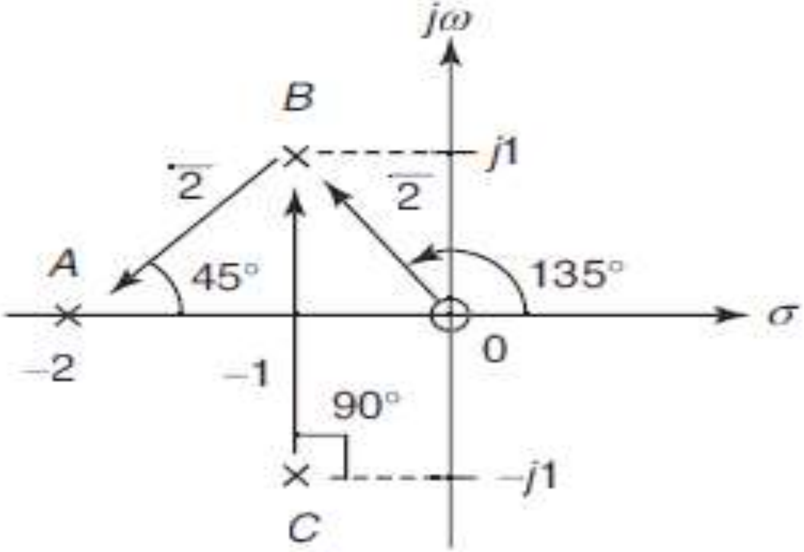
$$K_1 = \frac{3\,(\overline{OA})}{(\overline{BA})\,(\overline{CA})} = 3\left[\frac{2\angle 180^\circ}{(\sqrt{2}\angle -135^\circ)\,(\sqrt{2}\angle 135^\circ)}\right] = 3\sqrt{180^\circ} = -3$$



From Fig. 8.91,

$$K_2 = \frac{3\,(\overline{OB})}{(\overline{AB})\,(\overline{CB})} = 3\left[\frac{(\sqrt{2}\angle 135^\circ)}{(\sqrt{2}\angle 45^\circ)\,(2\angle 90^\circ)}\right] = \frac{3}{2}$$

$$K_2^* = \frac{3}{2}$$



$$V(s) = -\frac{3}{(s+2)} + \frac{\frac{3}{2}}{(s+1-j1)} + \frac{\frac{3}{2}}{(s+1+j1)}$$

Taking inverse Laplace transform,

$$v(t) = -3e^{-2t} + \frac{3}{2}\left[e^{(-1+j1)t} + e^{(-1-j1)t}\right] = -3e^{-2t} + 2\times\frac{3}{2}e^{-t}\left(\frac{e^{j1} + e^{-j1}}{2}\right) = -3e^{-2t} + 3e^{-t}\cos t$$

Find the function v(t) using the pole-zero plot of following function:

$$V(s) = \frac{(s+2)(s+6)}{(s+1)(s+5)}$$

**Solution** If the degree of the numerator is greater or equal to the degree of the denominator, we have to divide the numerator by the denominator such that the remainder can be expanded into partial fractions.

$$V(s) = \frac{s^2+8s+12}{s^2+6s+5} = 1 + \frac{2s+7}{s^2+6s+5} = 1 + \frac{2(s+3.5)}{(s+1)(s+5)}$$

By partial fraction expansion,

$$V(s) = 1 + \frac{K_1}{s+1} + \frac{K_2}{s+5}$$

$K_1$ and $K_2$ can be evaluated from the pole-zero diagram shown in Fig. 8.92 and Fig. 8.93.

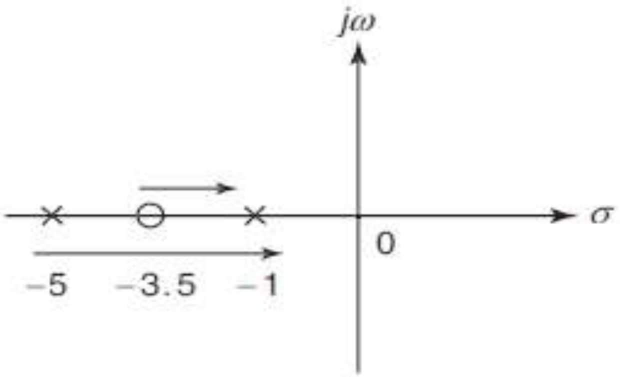



**Fig. 8.92**

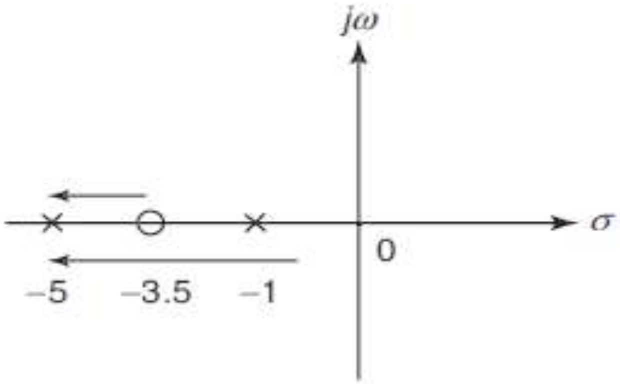



**Fig. 8.93**

From Fig. 8.92

$$K_1 = 2\left(\frac{2.5\angle 0^\circ}{4\angle 0^\circ}\right) = \frac{5}{4}$$

From Fig. 8.93

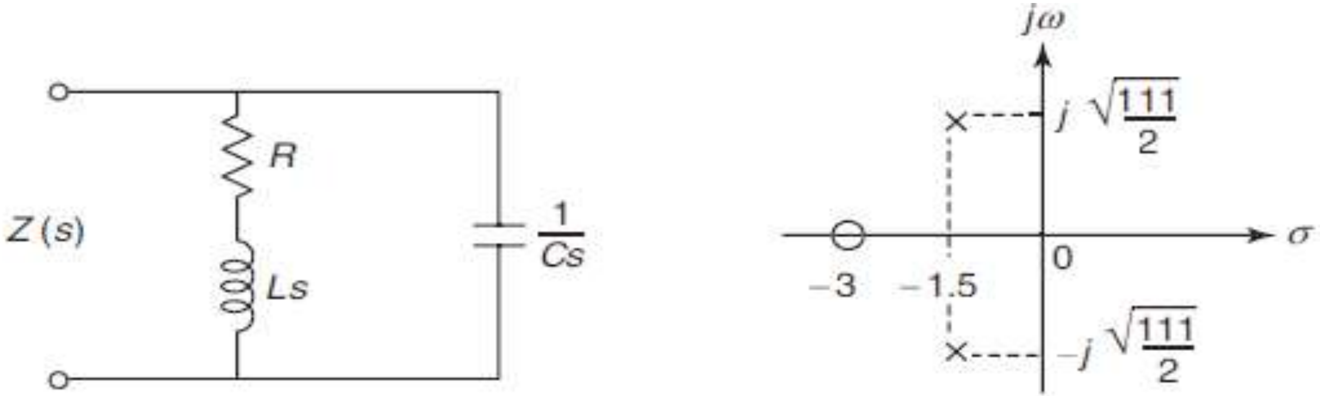
$$K_2 = 2\left(\frac{1.5\angle 180^\circ}{4\angle 180^\circ}\right) = \frac{3}{4}$$

$$V(s) = 1 + \frac{\frac{5}{4}}{s+1} + \frac{\frac{3}{4}}{s+5}$$

Taking inverse Laplace transform,

$$v(t) = \delta(t) + \frac{5}{4}e^{-t} + \frac{3}{4}e^{-5t}$$

*A network and its pole-zero configuration are shown in Fig. 8.74. Determine the values of R, L and C if Z (j0) = 1.*




**Solution**

$$Z(s) = \frac{(Ls+R)\dfrac{1}{Cs}}{(Ls+R)+\dfrac{1}{Cs}} = \frac{Ls+R}{LCs^2+RCs+1} = \frac{\dfrac{1}{C}\left(s+\dfrac{R}{L}\right)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \quad \text{...(i)}$$

From the pole-zero diagram, zero is at $s=-3$ and poles are at $s=-1.5+j\dfrac{\sqrt{111}}{2}$ and $s=-1.5-j\dfrac{\sqrt{111}}{2}$

$$Z(s)=H\frac{s+3}{\left(s+1.5+j\dfrac{\sqrt{111}}{2}\right)\left(s+1.5-j\dfrac{\sqrt{111}}{2}\right)}$$

$$Z(s)=H\frac{s+3}{(s+1.5)^2-\left(j\dfrac{\sqrt{111}}{2}\right)^2}=H\frac{s+3}{s^2+3s+30}$$

When $$Z(j0)=1,$$

$$1=H\left(\frac{3}{30}\right)$$

$$H=10$$

$$Z(s)=\frac{10(s+3)}{s^2+3s+30} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{R}{L} = 3$$

$$\frac{1}{C} = 10$$

$$\frac{1}{LC} = 30$$

Solving the above equations,

$$C = \frac{1}{10}\,\text{F}$$

$$L = \frac{1}{3}\,\text{H}$$

$$R = 1\,\Omega$$

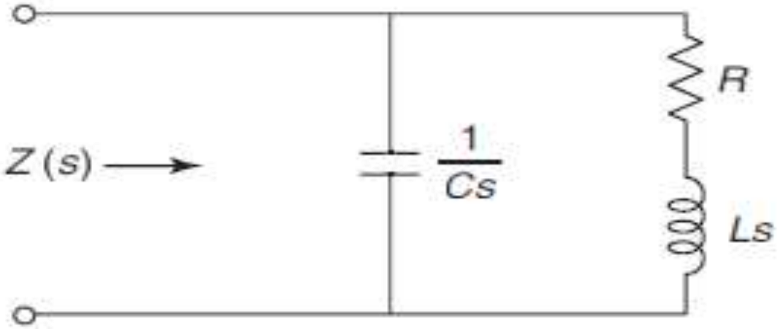
*A network is shown in Fig. 8.75. The poles and zeros of the driving-point function*

*Z(s) of this network are at the following places:*

*Poles at* $-\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$

*Zero at* $-1$

*If* $Z(j0) = 1$*, determine the values of R, L and C.*



**Solution**

$$Z(s)=\frac{(Ls+R)\dfrac{1}{Cs}}{Ls+R+\dfrac{1}{Cs}}=\frac{Ls+R}{LCs^2+RCs+1}=\frac{\dfrac{1}{C}\left(s+\dfrac{R}{L}\right)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \quad \text{...(i)}$$

The poles are at $-\frac{1}{2}\pm j\frac{\sqrt{3}}{2}$ and zero is at $-1$.

$$Z(s)=H\frac{s+1}{\left(s+\dfrac{1}{2}+j\dfrac{\sqrt{3}}{2}\right)\left(s+\dfrac{1}{2}-j\dfrac{\sqrt{3}}{2}\right)}=H\frac{s+1}{\left(s+\dfrac{1}{2}\right)^2-\left(j\dfrac{\sqrt{3}}{2}\right)^2}=H\frac{s+1}{s^2+s+1}$$

When $\quad Z(j0)=1,$

$$1=H\frac{(1)}{(1)}$$

$$H=1$$

$$Z(s)=\frac{s+1}{s^2+s+1} \quad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$C = 1$$

$$\frac{R}{L} = 1$$

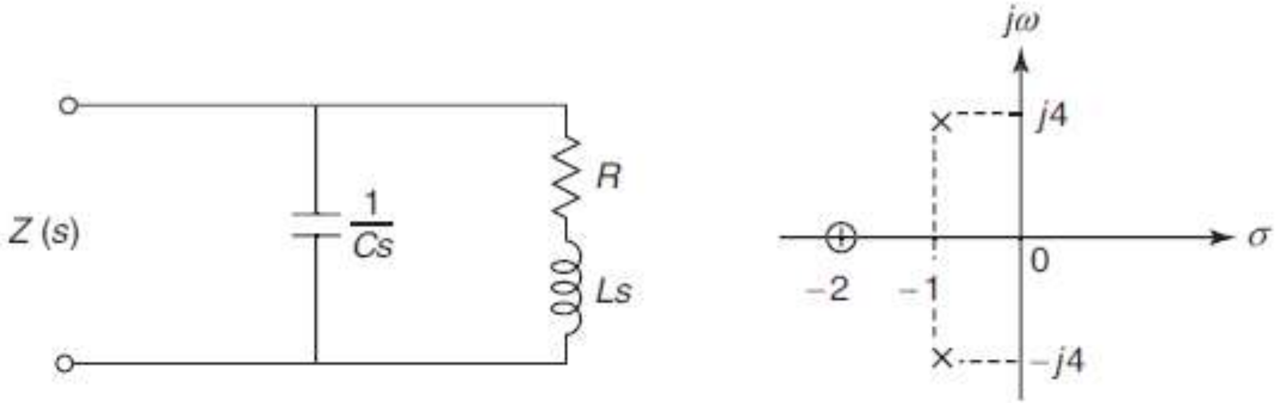
$$\frac{1}{LC} = 1$$

Solving the above equations,

$$C = 1\ \text{F}$$
$$L = 1\ \text{H}$$
$$R = 1\ \Omega$$

*The pole-zero diagram of the driving-point impedance function of the network of is shown below. At dc, the input impedance is resistive and equal to 2 W. Determine the values of*

*R, L and C.*



**Solution**

$$Z(s)=\frac{(Ls+R)\dfrac{1}{Cs}}{Ls+R+\dfrac{1}{Cs}}=\frac{Ls+R}{LCs^2+RCs+1}=\frac{\dfrac{1}{C}\left(s+\dfrac{R}{L}\right)}{s^2+\dfrac{R}{L}s+\dfrac{1}{LC}} \qquad \text{...(i)}$$

From the pole-zero diagram, zero is at $s=-2$ and poles are at $s=-1+j4$ and $s=-1-j4$.

$$Z(s)=H\frac{s+2}{(s+1+j4)(s+1-j4)}=H\frac{s+2}{(s+1)^2-(j4)^2}=H\frac{s+2}{s^2+2s+17}$$

At dc, i.e., $\omega=0, Z(j0)=2$

$$2=H\frac{2}{17}$$

$$H=17$$

$$Z(s)=17\frac{s+2}{s^2+2s+17} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 17$$

$$\frac{R}{L} = 2$$

$$\frac{1}{LC} = 17$$

Solving the above equations,

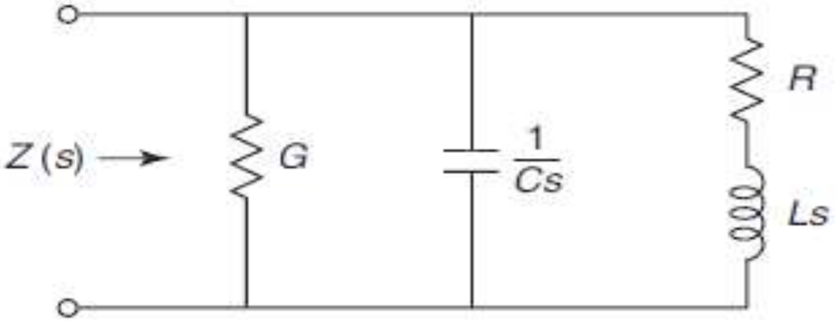
$$C = \frac{1}{17}\,\mathrm{F}$$

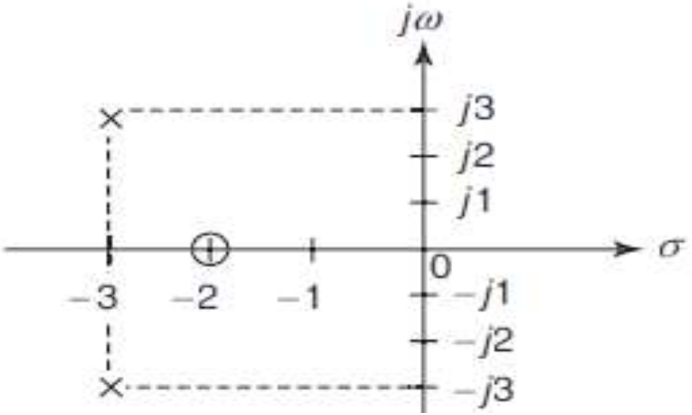
$$L = 1\,\mathrm{H}$$

$$R = 2\,\Omega$$

*A network and pole-zero diagram for driving-point impedance Z(s) are shown in*

*Calculate the values of the parameters R, L, G and C if Z(j0) = 1.*



**Solution** It is easier to calculate $Y(s)$ and then invert it to obtain $Z(s)$.

$$Y(s) = G + Cs + \frac{1}{Ls+R} = \frac{(G+Cs)(Ls+R)+1}{Ls+R} = \frac{LCs^2 + (GL+RC)\,s + 1 + GR}{Ls+R}$$

$$Z(s) = \frac{1}{Y(s)} = \frac{Ls+R}{LCs^2 + (GL+RC)\,s + 1 + GR} = \frac{\frac{1}{C}\left(s+\frac{R}{L}\right)}{s^2 + \left(\frac{G}{C}+\frac{R}{L}\right)s + \left(\frac{1+GR}{LC}\right)} \qquad \text{...(i)}$$

From the pole-zero diagram, zero is at $s = -2$ and poles are at $s = -3 \pm j3$.

$$Z(s) = H\frac{(s+2)}{(s+3-j3)(s+3+j3)} = H\frac{(s+2)}{(s+3)^2-(j3)^2} = H\frac{s+2}{s^2+6s+18}$$

When $$Z(j0) = 1,$$

$$1 = H\frac{2}{18}$$

$$H = 9$$

$$Z(s) = \frac{9\,(s+2)}{(s^2 + 6s + 18)} \qquad \text{...(ii)}$$

Comparing Eq. (ii) with Eq. (i),

$$\frac{1}{C} = 9$$

$$\frac{R}{L} = 2$$

$$\frac{G}{C} + \frac{R}{L} = 6$$

$$\frac{1+GR}{LC} = 18$$

Solving the above equation,

$$C = \frac{1}{9}\,\mathrm{F}$$

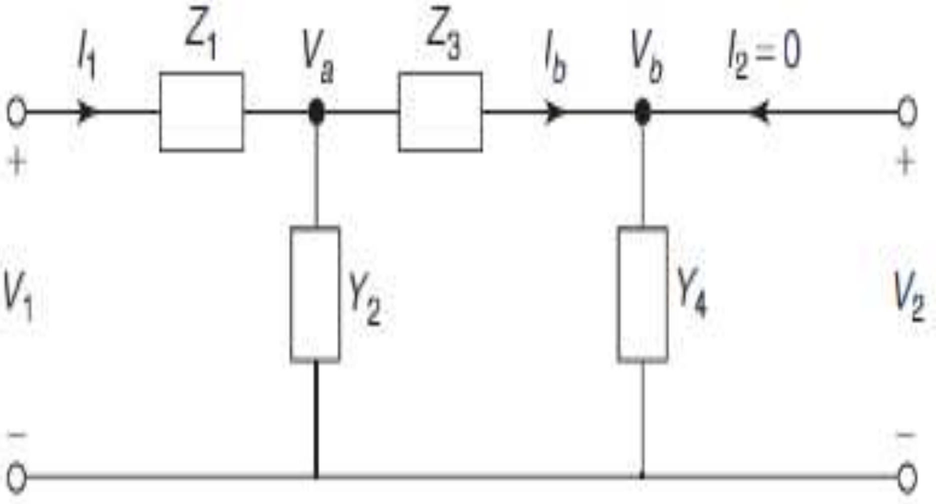
$$L = \frac{9}{10}\,\mathrm{H}$$

$$G = \frac{4}{9}\,\mho$$

$$R = \frac{9}{5}\,\Omega$$

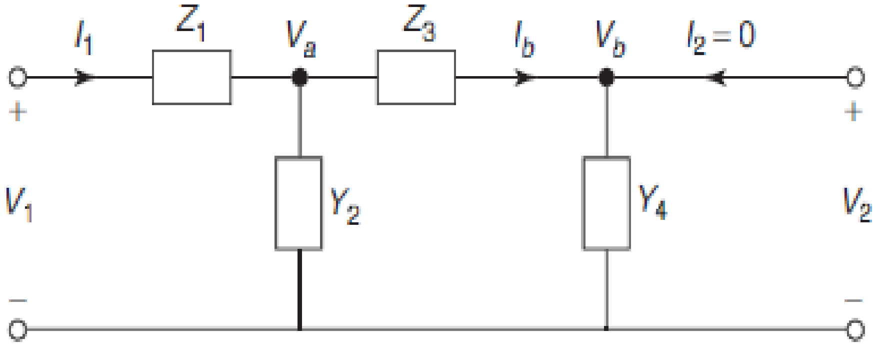
## ANALYSIS OF LADDER NETWORKS

The network functions of a ladder network can be obtained by a simple method. This method depends upon the relationships that exist between the branch currents and node voltages of the ladder network. Consider the network shown in Fig. 8.9 where all the impedances are connected in series branches and all




*Ladder network*

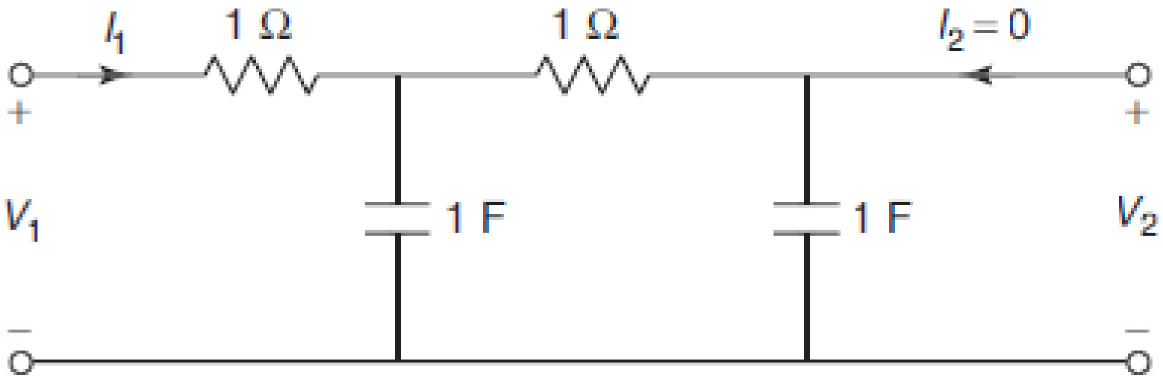
Analysis is done by writing the set of equations. In writing these equations, we begin at the port 2 of the ladder and work towards the port 1.



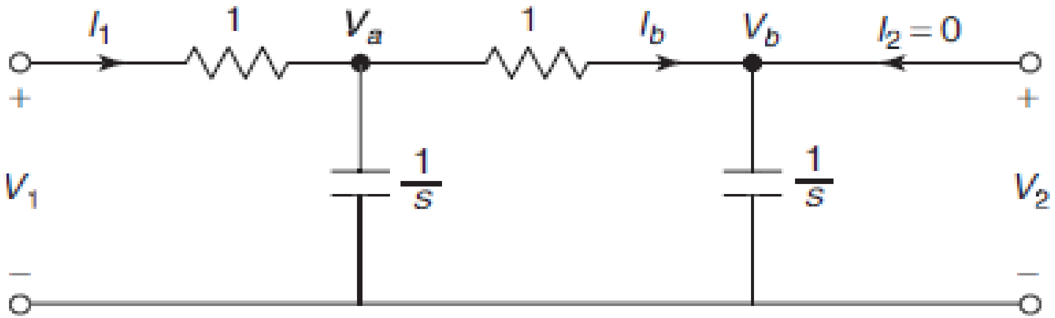

$$V_b = V_2$$
$$I_b = Y_4 V_2$$
$$V_a = Z_3 I_b + V_2 = (Z_3 Y_4 + 1) V_2$$
$$I_1 = Y_2 V_a + I_b = [Y_2 (Z_3 Y_4 + 1) + Y_4] V_2$$
$$V_1 = Z_1 I_1 + V_a = [Z_1 \{Y_2 (Z_3 Y_4 + 1) + Y_4\} + (Z_3 Y_4 + 1)] V_2$$

Thus, each succeeding equation takes into account one new impedance or admittance. Except the first two equations, each subsequent equation is obtained by multiplying the equation just preceding it by imittance (either impedance or admittance) that is next down the line and then adding to this product the equation twice preceding it. After writing these equations, we can obtain any network function.

*For the network shown in Fig. determine transfer function* $\dfrac{V_2}{V_1}$.




**Solution** The transformed network is shown in Fig.




$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = sV_2$$

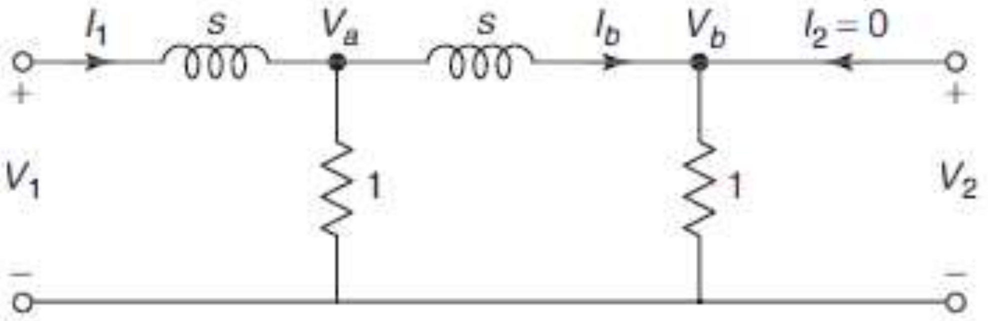
$$V_a = 1\,I_b + V_2$$
$$= sV_2 + V_2 = (s+1)V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s+1)V_2 + sV_2 = (s^2+2s)V_2$$

$$V_1 = 1\,I_1 + V_a = (s^2+2s)V_2 + (s+1)V_2 = (s^2+3s+1)V_2$$

Hence,
$$\frac{V_2}{V_1} = \frac{1}{s^2+3s+1}$$

*For the network shown in Fig. 8.12, determine the voltage transfer function* $\dfrac{V_2}{V_1}$.




**Solution**

$$V_b = V_2$$

$$I_b = \frac{V_2}{1} = V_2$$

$$V_a = s\,I_b + V_2 = sV_2 + V_2 = (s+1)V_2$$

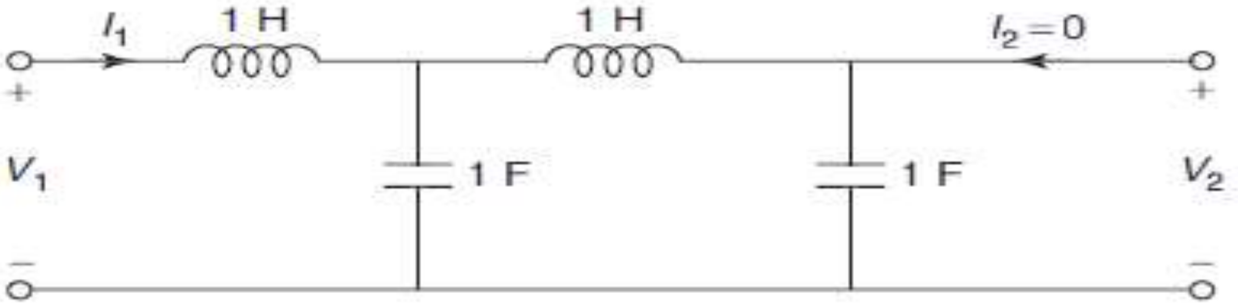
$$I_1 = \frac{V_a}{1} + I_b = (s+1)V_2 + V_2 = (s+2)V_2$$

$$V_1 = s\,I_1 + V_a = s(s+2)V_2 + (s+1)V_2 = (s^2+3s+1)V_2$$

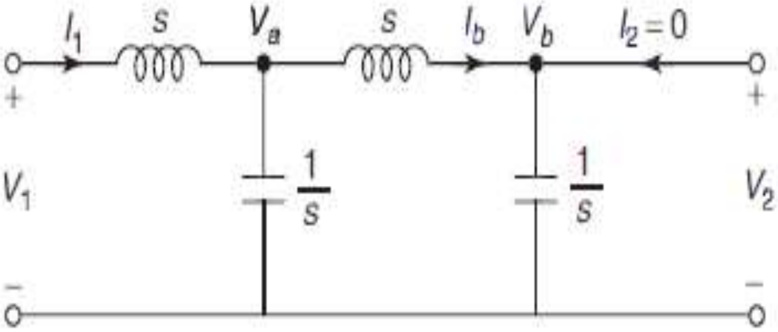
Hence,

$$\frac{V_2}{V_1} = \frac{1}{s^2+3s+1}$$

*Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$ *and* $\frac{V_2}{I_1}$ *for the network shown in Fig.*




**Solution** The transformed network is shown in Fig.




$$V_b = V_2$$

$$I_b = \frac{V_2}{\frac{1}{s}} = s V_2$$

$$V_a = s I_b + V_2 = s(s V_2) + V_2 = (s^2 + 1) V_2$$

$$I_1 = \frac{V_a}{\frac{1}{s}} + I_b = sV_a + I_b = s(s^2+1)V_2 + sV_2 = (s^3+2s)V_2$$

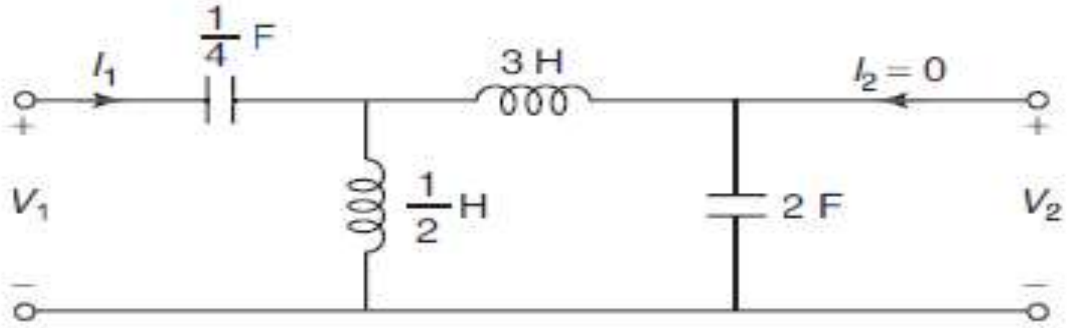
$$V_1 = sI_1 + V_a = s(s^3+2s)V_2 + (s^2+1)V_2 = (s^4+2s^2+s^2+1)V_2 = (s^4+3s^2+1)V_2$$

Hence, $$\frac{V_1}{I_1} = \frac{s^4+3s^2+1}{s^3+2s}$$

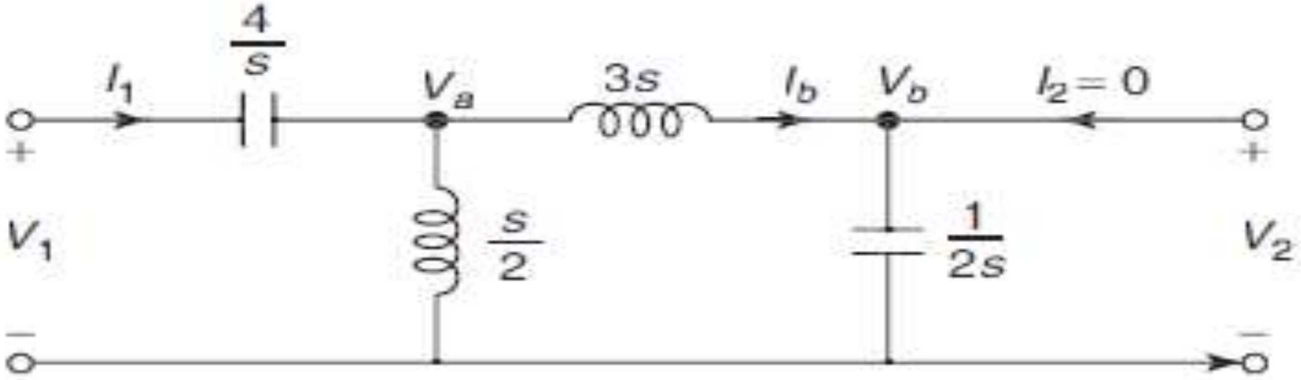
$$\frac{V_2}{V_1} = \frac{1}{s^4+3s^2+1}$$

$$\frac{V_2}{I_1} = \frac{1}{s^3+2s}$$

*Find the network functions* $\frac{V_1}{I_1}, \frac{V_2}{V_1}$, *and* $\frac{V_2}{I_1}$ *for the network in Fig.*




**Solution** The transformed network is shown in Fig.



$$V_b = V_2$$

$$I_b = \frac{V_2}{\dfrac{1}{2s}} = 2sV_2$$

$$V_a = 3s\,I_b + V_2 = 3s\,(2sV_2) + V_2 = (6s^2+1)V_2$$

$$I_1 = \frac{V_a}{\dfrac{s}{2}} + I_b = \frac{2}{s}(6s^2+1)V_2 + 2sV_2 = \left(\frac{14s^2+2}{s}\right)V_2$$

$$V_1 = \frac{4}{s}I_1 + V_a = \frac{4}{s}\left(\frac{14s^2+2}{s}\right)V_2 + (6s^2+1)V_2 = \left(\frac{6s^4+57s^2+8}{s^2}\right)V_2$$

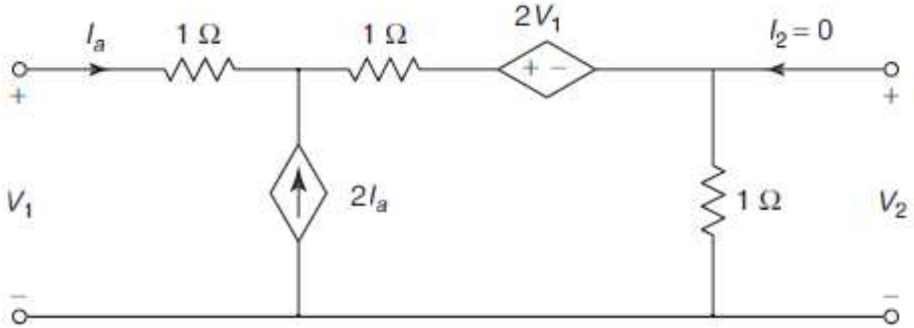
Hence,

$$\frac{V_1}{I_1} = \frac{6s^4+57s^2+8}{14s^3+2s}$$

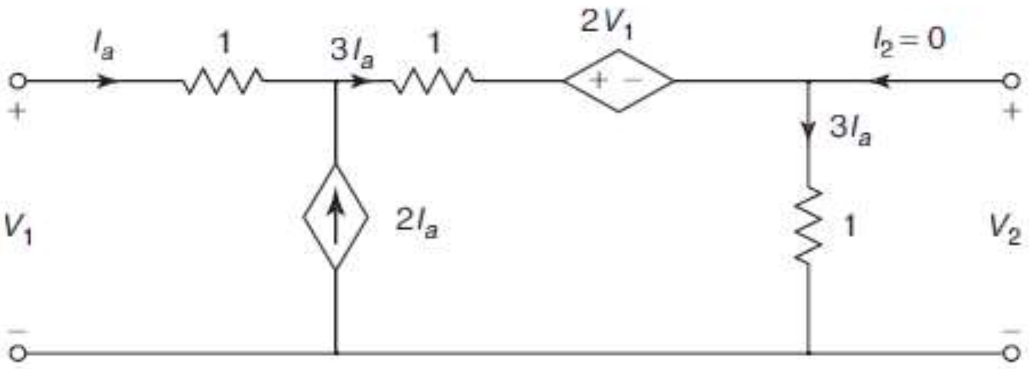
$$\frac{V_2}{V_1} = \frac{s^2}{6s^4+57s^2+8}$$

$$\frac{V_2}{I_1} = \frac{s}{14s^2+2}$$

*Find the network function* $\frac{V_2}{V_1}$ *for the network shown in Fig.*




**Solution** The network is redrawn as shown in Fig. 8.30.



From Fig. 8.30,

$$V_2 = 1\ (3\ I_a) = 3\ I_a$$

Applying KVL to the outermost loop,

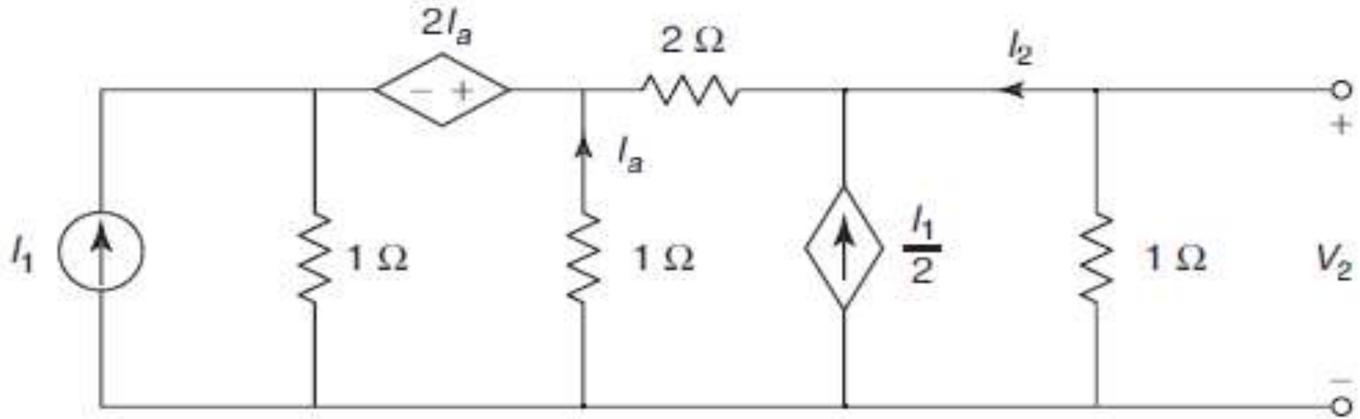
$$V_1 - 1\ (I_a) - 1\ (3\ I_a) - 2\ V_1 - 1\ (3\ I_a) = 0$$

$$V_1 = -7\ I_a$$

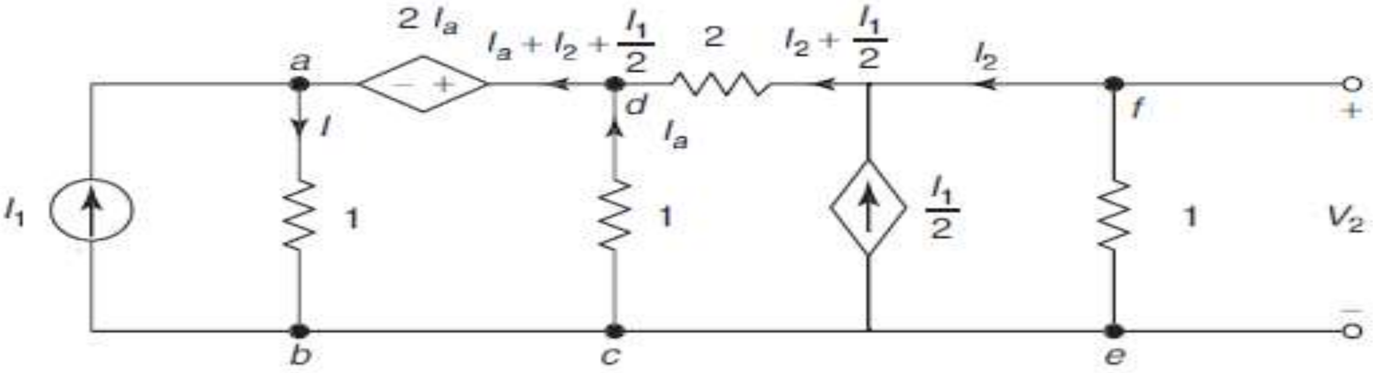
Hence,

$$\frac{V_2}{V_1} = -\frac{3}{7}$$

*Find the network function* $\frac{I_2}{I_1}$ *for the network shown in Fig.*



**Solution** The network is redrawn as shown in Fig. 8.32.




From Fig. 8.32,

$$I = I_1 + I_a + I_2 + \frac{I_1}{2}$$

$$= \frac{3}{2} I_1 + I_a + I_2 \qquad \text{...(i)}$$

Applying KVL to the loop *abcda*,

$$-1I - 1I_a - 2I_a = 0$$

$$-I - 3I_a = 0$$

$$I + 3I_a = 0 \qquad \text{...(ii)}$$

Substituting Eq. (i) in Eq. (ii),

$$\frac{3}{2}I_1 + I_a + I_2 + 3I_a = 0$$

$$\frac{3}{2}I_1 + I_2 + 4I_a = 0 \qquad \text{...(iii)}$$

Applying KVL to the loop *dcefd*,

$$1I_a - 1I_2 - 2\left(I_2 + \frac{I_1}{2}\right) = 0$$

$$I_a - 3I_2 - I_1 = 0$$

$$I_a = 3I_2 + I_1 \qquad \text{...(iv)}$$

Substituting Eq. (iv) in Eq. (iii),

$$\frac{3}{2}I_1 + I_2 + 4(3I_2 + I_1) = 0$$

$$\frac{3}{2}I_1 + I_2 + 12I_2 + 4I_1 = 0$$

$$\frac{11}{2}I_1 + 13I_2 = 0$$

$$13I_2 = -\frac{11}{2}I_1$$

Hence,

$$\frac{I_2}{I_1} = -\frac{11}{26}$$

# HURWITZ POLYNOMIALS

A polynomial $P(s)$ is said to be Hurwitz if the following conditions are satisfied:

(a) $P(s)$ is real when $s$ is real.

(b) The roots of $P(s)$ have real parts which are zero or negative.

## Properties of Hurwitz Polynomials

1. All the coefficients in the polynomial

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

   are positive. A polynomial may not have any missing terms between the highest and the lowest order unless all even or all odd terms are missing.
2. The roots of odd and even parts of the polynomial $P(s)$ lie on the $j\omega$-axis only.
3. If the polynomial $P(s)$ is either even or odd, the roots of polynomial $P(s)$ lie on the $j\omega$-axis only.
4. All the quotients are positive in the continued fraction expansion of the ratio of odd to even parts or even to odd parts of the polynomial $P(s)$.

5. If the polynomial $P(s)$ is expressed as $W(s)\,P_1(s)$, then $P(s)$ is Hurwitz if $W(s)$ and $P_1(s)$ are Hurwitz.
6. If the ratio of the polynomial $P(s)$ and its derivative $P'(s)$ gives a continued fraction expansion with all positive coefficients then the polynomial $P(s)$ is Hurwitz.

This property helps in checking a polynomial for Hurwitz if the polynomial is an even or odd function because in such a case, it is not possible to obtain the continued fraction expansion.

*State for each case, whether the polynomial is Hurwitz or not. Give reasons in each case.*

(a) $s^4 + 4s^3 + 3s + 2$

(b) $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$

**Solution** (a) In the given polynomial, the term $s^2$ is missing and it is neither an even nor an odd polynomial. Hence, it is not Hurwitz.

(b) Polynomial $s^6 + 5s^5 + 4s^4 - 3s^3 + 2s^2 + s + 3$ is not Hurwitz as it has a term $(-3s^3)$ which has a negative coefficient.

*Test whether the polynomial $P(s) = s^4 + s^3 + 5s^2 + 3s + 4$ is Hurwitz.*

**Solution** Even part of $P(s) = m(s) = s^4 + 5s^2 + 4$

Odd part of $P(s) = n(s) = s^3 + 3s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$s^3 + 3s\,)\,s^4 + 5s^2 + 4\,(\,s$$

$$\underline{s^4 + 3s^2}$$

$$2s^2 + 4\,\Big)\,s^3 + 3s\,\Big(\,\frac{1}{2}s$$

$$\underline{s^3 + 2s}$$

$$s\,)\,2s^2 + 4\,(\,2s$$

$$\underline{2s^2}$$

$$4\,\Big)\,s\,\Big(\,\frac{1}{4}s$$

$$\underline{s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

*Test whether the polynomial* $P(s) = s^3 + 4s^2 + 5s + 2$ *is Hurwitz.*

**Solution** Even part of $P(s) = m(s) = 4s^2 + 2$

Odd part of $P(s)$ $n(s) = s^3 + 5s$

The continued fraction expansion can be obtained by dividing $n(s)$ by $m(s)$ as $n(s)$ is of higher order than $m(s)$.

$$Q(s) = \frac{n(s)}{m(s)}$$

$$4s^2 + 2 \Big) \, s^3 + 5s \Big( \, \frac{1}{4}s$$

$$\underline{s^3 + \frac{2}{4}s}$$

$$\frac{9}{2}s \Big) \, 4s^2 + 2 \Big( \, \frac{8}{9}s$$

$$\underline{4s^2}$$

$$2 \Big) \, \frac{9}{2}s \Big( \, \frac{9}{4}s$$

$$\underline{\frac{9}{2}s}$$

$$0$$

Since all the quotient terms are positive, $P(s)$ is Hurwitz.

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4$$

**Solution** Even part of $P(s) = m(s) = s^6 + 8s^4 + 17s^2 + 4$

Odd part of $P(s) = n(s) = 3s^5 + 15s^3 + 12s$

$$Q(s) = \frac{m(s)}{n(s)}$$

By continued fraction expansion,

$$3s^5 + 15s^3 + 12s \,\Big)\, s^6 + 8s^4 + 17s^2 + 4 \,\Big(\frac{1}{3}s$$

$$\underline{s^6 + 5s^4 + 4s^2}$$

$$3s^4 + 13s^2 + 4 \,\Big)\, 3s^5 + 15s^3 + 12s \,\Big(s$$

$$\underline{3s^5 + 13s^3 + 4s}$$

$$2s^3 + 8s \,\Big)\, 3s^4 + 13s^2 + 4 \,\Big(\frac{3}{2}s$$

$$\underline{3s^4 + 12s^2}$$

$$s^2 + 4 \,\Big)\, 2s^3 + 8s \,\Big(2s$$

$$\underline{2s^3 + 8s}$$

$$0$$

The division has terminated abruptly (i.e., the number of partial quotients (that is four) is not equal to the order of polynomial (that is six) with common factor $(s^2 + 4)$.

$$P(s) = s^6 + 3s^5 + 8s^4 + 15s^3 + 17s^2 + 12s + 4 = (s^2 + 4)(s^4 + 3s^3 + 4s^2 + 3s + 1)$$

If both the factors are Hurwitz, $P(s)$ will be Hurwitz.

Let $$P_1(s) = s^2 + 4$$

Since it contains only even functions, we have to find the continued fraction expansion of $\dfrac{P_1(s)}{P_1'(s)}$.

$$P_1'(s) = 2s$$

$$\frac{P_1(s)}{P_1'(s)} = \frac{s^2+4}{2s} = \frac{s^2}{2s} + \frac{4}{2s} = \frac{s}{2} + \frac{1}{\frac{s}{2}}$$

Since all the quotient terms are positive, $P_1(s)$ is Hurwitz.

Now, let
$$P_2(s) = s^4 + 3s^3 + 4s^2 + 3s + 1$$
$$m_2(s) = s^4 + 4s^2 + 1$$
$$n_2(s) = 3s^3 + 3s$$

By continued fraction expansion,

$$3s^3 + 3s\,\big)\, s^4 + 4s^2 + 1 \left(\frac{1}{3}s\right.$$
$$\underline{s^4 + s^2}$$
$$3s^2 + 1)\; 3s^3 + 3s\; (s$$
$$\underline{3s^3 + s}$$
$$2s\,\big)\, 3s^2 + 1 \left(\frac{3}{2}s\right.$$
$$\underline{3s^2}$$
$$1)\; 2s\; (2s$$
$$\underline{2s}$$
$$0$$

Since all the quotient terms are positive, $P_2(s)$ is Hurwitz.

Hence, $P(s) = (s^2+4)(s^4+3s^3+4s^2+3s+1)$ is Hurwitz.