* Zeros of an analytic function :-

Definition: let $f(z)$ be the analytic function then a point z_0 is said to be zeros of $f(z)$ if $f(z_0) = 0$

Note that :

① If $f(z_0) = 0$, $f'(z_0) \neq 0$ then z_0 is called simple zero or zero of order 1

② If $f(z_0) = 0$, $f'(z_0) = 0$, $f''(z_0) \neq 0$ then z_0 is called zero of order 2

③ If $f(z_0) = 0$, $f'(z_0) = 0, \dots, f^{(n-1)}(z_0) = 0$, $f^{(n)}(z_0) \neq 0$ then z_0 is called zero of order 'n'

Example ① find zeros of $f(z) = (z-1)e^z$

solution: Given: $f(z) = (z-1)e^z$

clearly, $f(z) = 0$ if $(z-1)e^z = 0$ if $z=1$

$\Rightarrow z = 1$

$\therefore z = 1$ is zero of $f(z)$

Now, $f'(z) = (z-1)e^z + e^z$

$\therefore f'(1) = (1-1)e^1 + e^1 = e$

i.e. $f'(1) \neq 0$

Hence, $z = 1$ is simple zero of $f(z)$

Example ② find the zeros and their order of

$$f(z) = z^2 \sin z$$

Solution: consider $f(z) = 0$

$\Rightarrow \quad z^2 \sin z = 0$

$\Rightarrow \quad z = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

Therefore, $z = 0, \pm\pi, \pm 2\pi, \dots$ are all the zeros of $f(z)$

Now, $f'(z) = z^2 \cos z + 2z \sin z$

$\Rightarrow \quad f'(z) = 0$, for only $z = 0$

But $f'(z) \neq 0$ for every $z = \pm\pi, \pm 2\pi, \pm 3\pi, \dots$

$\therefore \quad z = \pm\pi, \pm 2\pi, \pm 3\pi$ are zeros of order 1 or simple

Now, $f''(z) = 2z\cos z - z^2 \sin z + 2z \cos z + 2\sin z$

i.e. $f''(z) = 4z\cos z - z^2 \sin z + 2\sin z$

$\Rightarrow \quad f''(z) = 0$ for $z = 0$

Now, $f'''(z) = 4\cos z - 4z\sin z - z^2 \cos z - 2z\cos z + 2\cos z$

i.e. $f'''(z) = 6\cos z - 4z\sin z - z^2\cos z - 2z\cos z$

$\Rightarrow \quad f'''(z) \neq 0$, for $z = 0$

therefore, $z = 0$ is zero of order 3

Homework:

* find the zeros and its order of following function

① $f(z) = z \tan z$ ② $(z^2 - 1)(z^3 + 3z + 2)$

* Singular point:

The point z_0 is said to be singular point of $f(z)$ if

i) $f(z)$ is not analytic at z_0

ii) $f(z)$ is analytic at every point in the neighbourhood of z_0

for example: ① $f(z) = \frac{z^2}{z-2}$

$\therefore$ $z = 2$ is singular point of $f(z)$

② $f(z) = \frac{z}{z(z+1)}$

$\therefore$ $z=0$ and $z=-1$ are singular points of $f(z)$

→ Negative term in Laurent's Series:-

Note that $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$

that is $f(z) = \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\right]$

$+ \left[b_1 \frac{1}{z-z_0} + b_2 \frac{1}{(z-z_0)^2} + b_3 \frac{1}{(z-z_0)^3} + \cdots \right]$

In the above power series the terms

$\frac{1}{z-z_0}$, $\frac{1}{(z-z_0)^2}$, $\frac{1}{(z-z_0)^3}$, ... are called as Negative terms.

* Types of singularity :

(1) **Pole :-** The singular point z_0 of $f(z)$ is said to be pole If the Laurent's series of $f(z)$ around $z = z_0$ contains only finite number of Negative terms

Note that: If z_0 is pole of $f(z)$ of order 1 then z_0 is called simple pole of $f(z)$

(2) Removable singularity :

The singular point z_0 of $f(z)$ is said to be Removable If The Laurent's series of $f(z)$ around $z=z_0$ does not contains a Negative term

(3) Essential singularity :

The singular point z_0 of $f(z)$ is said to be Essential If The Laurent's series of $f(z)$ around $z=z_0$ contains infinite number of Negative term

Examples:

Determine the nature of singularities of following function.

(1) $\dfrac{e^z}{z^3}$ (2) $\dfrac{\sin z}{z}$ (3) $e^{\frac{1}{z}}$

(4) $\dfrac{\cot \pi z}{(z-a)^3}$

Solution: (1) Given $f(z) = \dfrac{e^z}{z^3}$

$$\therefore \quad f(z) = \frac{1}{z^3}\left[e^z\right]$$

$$= \frac{1}{z^3}\left[1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right]$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{2!\cdot z} + \frac{1}{3!} + \frac{z}{4!} + \frac{z^2}{5!} + \cdots$$

that is Laurent's series expansion of $f(z)$ contains only finite Negative terms (3 Negative terms)

Hence, $z=0$ is pole of order 3

(2) Given: $f(z) = \dfrac{\sin z}{z}$

clearly, $z=0$ is singularity of $f(z)$

Now, $f(z) = \dfrac{1}{z}\left[\sin z\right]$

$$= \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

that is Laurents series of $f(z)$ contains no Negative term

Hence, $z=0$ is Removable singularity.

(3) Given: $f(z) = e^{\frac{1}{z}}$

clearly, $z=0$ is singular point of $f(z)$

since, $e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$

$\therefore f(z) = e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \frac{1}{4!\,z^4} + \cdots$

that is laurent's series of $f(z)$ contains infinite number of Negative terms

Hence, $z=0$ is a essential singularity.

(4) Given: $f(z) = \frac{\cot \pi z}{(z-a)^3}$

$\Rightarrow f(z) = \frac{\cos \pi z}{\sin \pi z \cdot (z-a)^3}$

$\therefore$ The points $z=a$, $z = 0, \pm 1, \pm 2, \cdots$ are singular points.

$\therefore$ here, $z=a$ is pole of order 3

and $z = 0, \pm 1, \pm 2, \cdots$ are simple Poles

<u>Homework</u>:

Determine the Nature of singularity

(1) $z e^{\frac{1}{z^2}}$ (2) $\frac{1-e^{2z}}{z^3}$ (3) $\frac{1-e^{z}}{z}$

(4) $\frac{1-\cos 2z}{z}$

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