INTRODUCTION

Whenever a network containing energy storage elements such as inductor or capacitor is switched from one condition to another, either by change in applied source or change in network elements, the response current and voltage change from one state to the other state. The time taken to change from an initial steady state to the final steady state is known as the *transient period*. This response is known as *transient response* or *transients*. The response of the network after it attains a final steady value is independent of time and is called the steady-state response. The complete response of the network is determined with the help of a differential equation.

INITIAL CONDITIONS

In solving the differential equations in the network, we get some arbitary constant. Initial conditions are used to determine these arbitrary constants. It helps us to know the behaviour of elements at the instant of switching.

To differentiate between the time immediately before and immediately after the switching, the signs '–' and '+' are used. The conditions existing just before switching are denoted as $i\,(0^-)$, $v\,(0^-)$, etc. Conditions just after switching are denoted as $i\,(0^+)$, $v\,(0^+)$.

Sometimes conditions at $t = \infty$ are used in the evaluation of arbitrary constants. These are known as *final conditions*.

In solving the problems for initial conditions in the network, we divide the time period in the following ways:

1. Just before switching (from $t = -\infty$ to $t = 0^-$)
2. Just after switching (at $t = 0^+$)
3. After switching (for $t > 0$)

If the network remains in one condition for a long time without any switching action, it is said to be under steady-state condition.

1. *Initial Conditions for the Resistor* For a resistor, current and voltage are related by $v(t) = Ri(t)$. The current through a resistor will change instantaneously if the voltage changes instantaneously. Similarly, the voltage will change instantaneously if the current changes instantaneously.

2. *Initial Conditions for the Inductor* For an inductor, current and voltage are related by,

$$v(t) = L\frac{di}{dt}$$

Voltage across the inductor is proportional to the rate of change of current. It is impossible to change the current through an inductor by a finite amount in zero time. This requires an infinite voltage across the inductor. An inductor does not allow an abrupt change in the current through it.

The current through the inductor is given by,

$$i(t) = \frac{1}{L}\int_0^t v(t)dt + i(0)$$

where $i(0)$ is the initial current through the inductor.

If there is no current flowing through the inductor at $t = 0^-$, the inductor will act as an open circuit at $t = 0^+$. If a current of value I_0 flows through the inductor at $t = 0^-$, the inductor can be regarded as a current source of I_0 ampere at $t = 0^+$.

3. *Initial Conditions for the Capacitor* For the capacitor, current and voltage are related by,

$$i(t) = C\frac{dv(t)}{dt}$$

Current through a capacitor is proportional to the rate of change of voltage. It is impossible to change the voltage across a capacitor by a finite amount in zero time. This requires an infinite current through the capacitor. A capacitor does not allow an abrupt change in voltage across it.

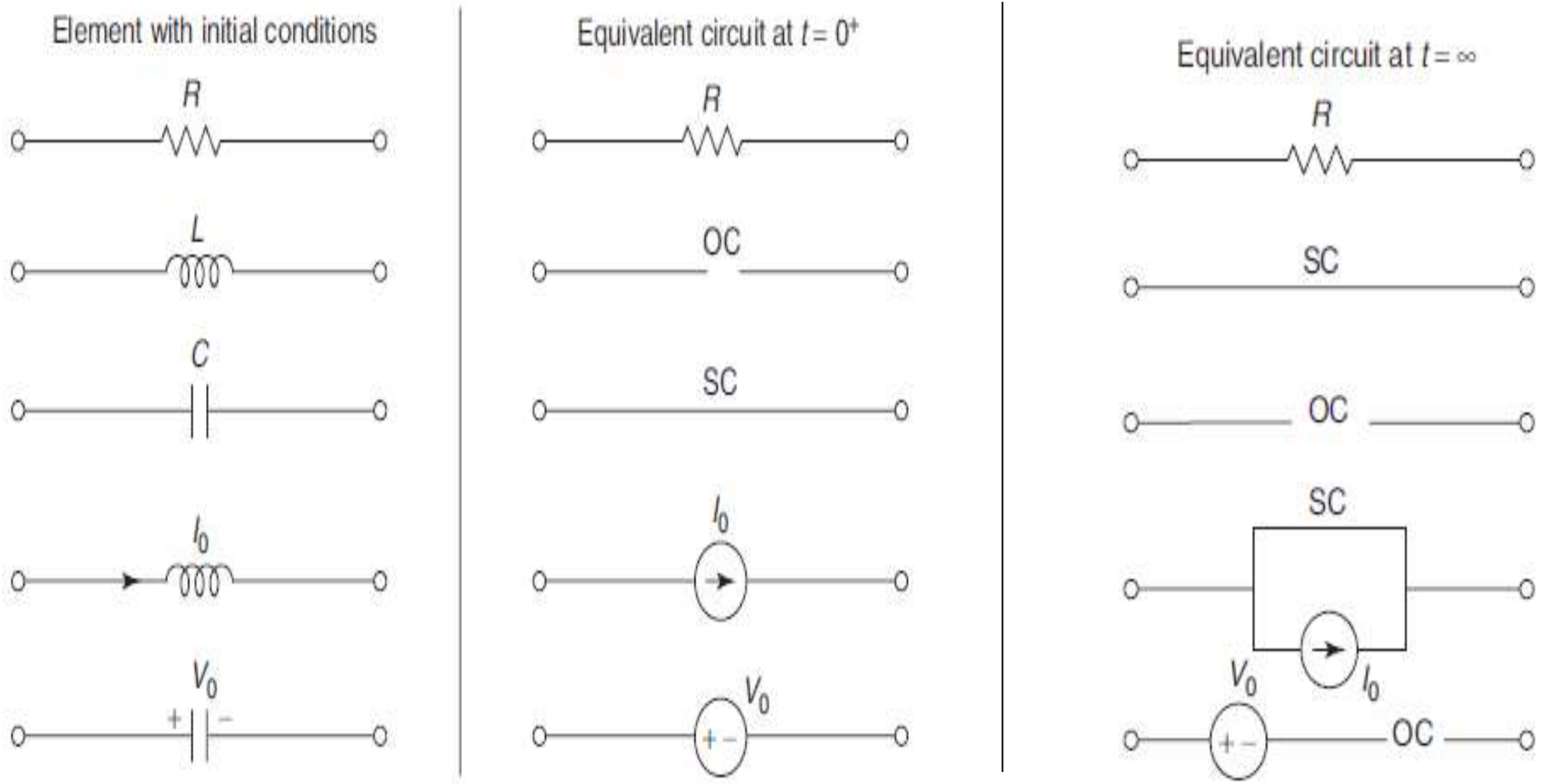
The voltage across the capacitor is given by,

$$v(t) = \frac{1}{C}\int_0^t i(t)dt + v(0)$$

where $v(0)$ is the initial voltage across the capacitor.

If there is no voltage across the capacitor at $t = 0^-$, the capacitor will act as a short circuit at $t = 0^+$. If the capacitor is charged to a voltage V_0 at $t = 0^-$, it can be regarded as a voltage source of V_0 volt at $t = 0^+$. These

Initial conditions *Final conditions*



Procedure for Evaluating Initial Conditions

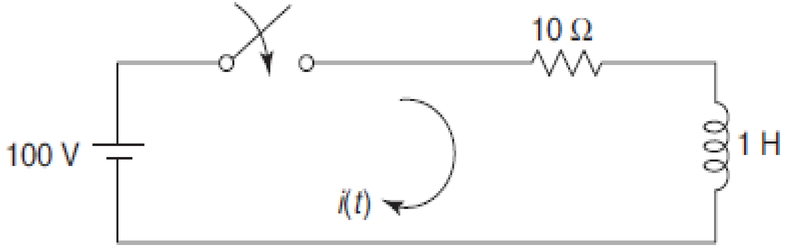
(a) Draw the equivalent network at $t = 0^-$. Before switching action takes place, i.e., for $t = -\infty$ to $t = 0^-$, the network is under steady-state conditions. Hence, find the current flowing through the inductors $i_L(0^-)$ and voltage across the capacitor $v_C(0^-)$.

(b) Draw the equivalent network at $t = 0^+$, i.e., immediately after switching. Replace all the inductors with open circuits or with current sources $i_L(0^+)$ and replace all capacitors by short circuits or voltage sources $v_C(0^+)$. Resistors are kept as it is in the network.

(c) Initial voltages or currents in the network are determined from the equivalent network at $t = 0^+$.

(d) Initial conditions, i.e., $\frac{di}{dt}(0^+), \frac{dv}{dt}(0^+), \frac{d^2i}{dt^2}(0^+), \frac{d^2v}{dt^2}(0^+)$ are determined by writing integro-differential equations for the network for $t > 0$, i.e., after the switching action by making use of initial condition.

In the given network of Fig. the switch is closed at $t = 0$. With zero current in the inductor, find i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.

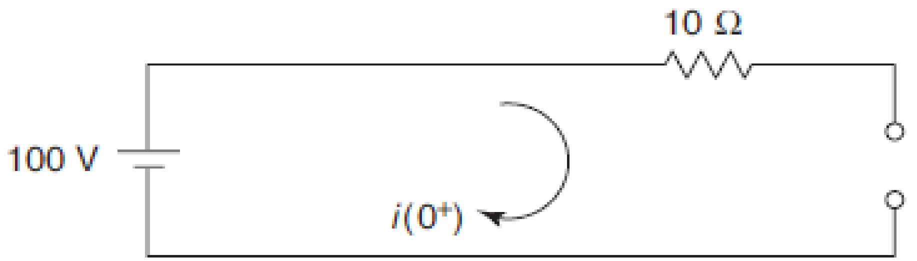


Solution

At $t = 0^-$, no current flows through the inductor.

$$i(0^-) = 0$$

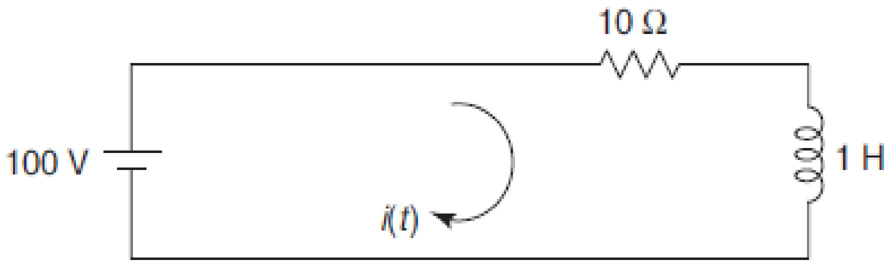
At $t = 0^+$, the network is shown in Fig.



At $t = 0^+$, the inductor acts as an open circuit.

$$i(0^+) = 0$$

For $t > 0$, the network is shown in Fig.
Writing the KVL equation for $t > 0$,



$$100 - 10i - 1\frac{di}{dt} = 0 \qquad \ldots\text{(i)}$$

$$\frac{di}{dt} = 100 - 10i \qquad \ldots\text{(ii)}$$

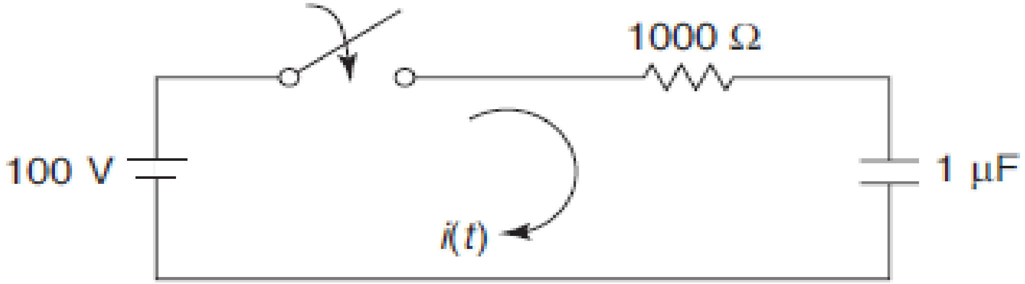
At $t = 0^+$, $$\frac{di}{dt}(0^+) = 100 - 10i(0^+) = 100 - 10(0) = 100 \text{ A/s}$$

Differentiating Eq. (ii),

$$\frac{d^2i}{dt^2} = -10\frac{di}{dt}$$

At $t = 0^+$, $$\frac{d^2i}{dt^2}(0^+) = -10\frac{di}{dt}(0^+) = -10(100) = -1000 \text{ A/s}^2$$

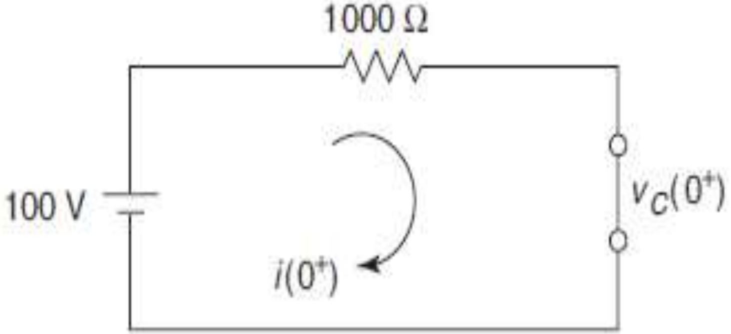
In the network of Fig. the switch is closed at $t = 0$. With the capacitor uncharged, find value for i, $\frac{di}{dt}$ and $\frac{d^2 i}{dt^2}$ at $t = 0^+$.



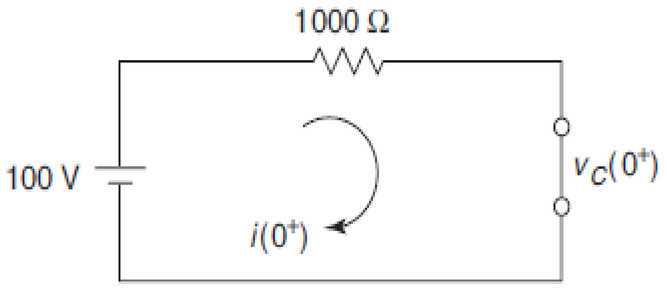
Solution

At $t = 0^-$, the capacitor is uncharged.

$$v_C(0^-) = 0$$
$$i(0^-) = 0$$



At $t = 0^+$, the network is shown in Fig.



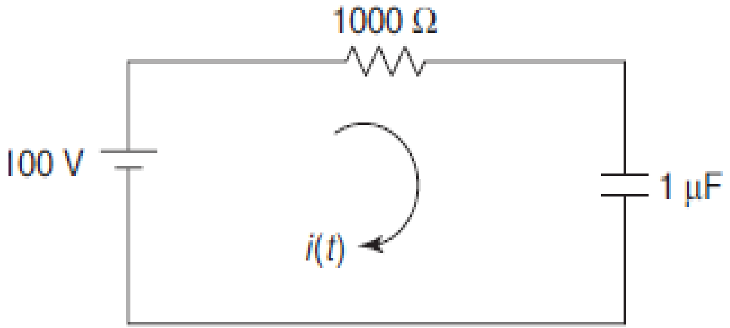
At $t = 0^+$, the capacitor acts as a short circuit.

$$v_C(0^+) = 0$$

$$i(0^+) = \frac{100}{1000} = 0.1 \text{ A}$$

For $t > 0$, the network is shown in Fig.

Writing the KVL equation for $t > 0$,



$$100 - 1000i - \frac{1}{1 \times 10^{-6}} \int_0^t i \, dt = 0 \qquad \ldots \text{(i)}$$

Differentiating Eq. (i),

$$0 - 1000\frac{di}{dt} - 10^6 i = 0$$

$$\frac{di}{dt} = -\frac{10^6}{1000} i$$

At $t = 0^+$,

$$\frac{di}{dt}(0^+) = -\frac{10^6}{1000} i(0^+) = -\frac{10^6}{1000}(0.1) = -100 \text{ A/s}$$

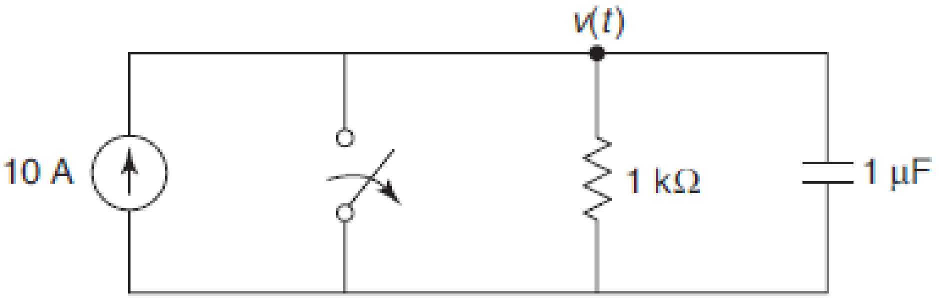
Differentiating Eq. (ii),

$$\frac{d^2 i}{dt^2} = -\frac{10^6}{1000}\frac{di}{dt}$$

At $t = 0^+$,

$$\frac{d^2 i}{dt^2}(0^+) = -\frac{10^6}{1000}\frac{di}{dt}(0^+) = -\frac{10^6}{1000}(-100) = 10^5 \text{ A/s}^2$$

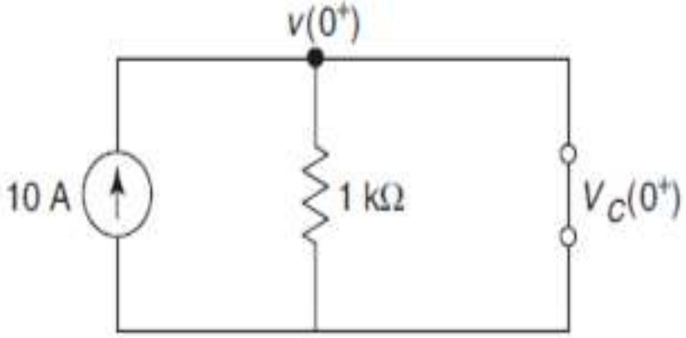
In the given network of Fig. the switch is opened at $t = 0$. Solve for v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ at $t = 0^+$.



Solution At $t = 0^-$, switch is closed. Hence, the voltage across the capacitor is zero.

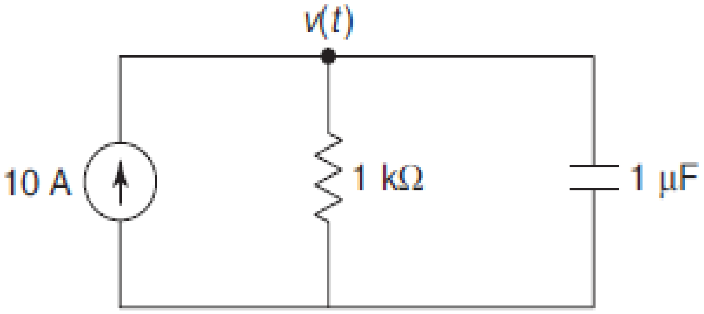
$$v(0^-) = v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.16. At $t = 0^+$, the capacitor acts as a short circuit.



$$v(0^+) = v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.17.
Writing the KCL equation for $t > 0$,



$$\frac{v}{1000} + 10^{-6}\frac{dv}{dt} = 10$$

At $t = 0^+$,
$$\frac{v(0^+)}{1000} + 10^{-6}\frac{dv}{dt}(0^+) = 10$$

$$\frac{dv}{dt}(0^+) = \frac{10}{10^{-6}} = 10 \times 10^6 \text{ V/s}$$

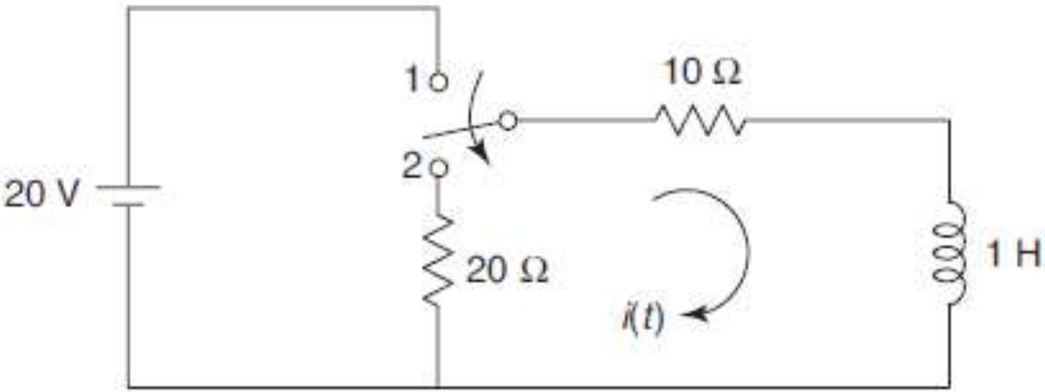
Differentiating Eq. (i),

$$\frac{1}{1000}\frac{dv}{dt} + 10^{-6}\frac{d^2v}{dt^2} = 0$$

At $t = 0^+$,
$$\frac{1}{1000}\frac{dv}{dt}(0^+) + 10^{-6}\frac{d^2v}{dt^2}(0^+) = 0$$

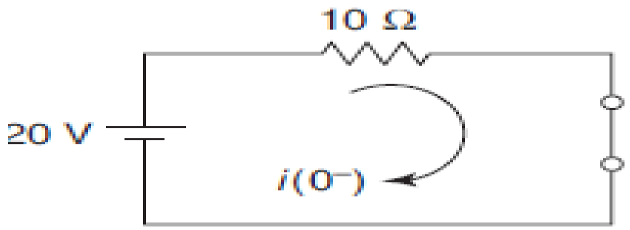
$$\frac{d^2v}{dt^2}(0^+) = -\frac{1}{1000 \times 10^{-6}} \times 10 \times 10^6 = -10 \times 10^9 \text{ V/s}^2$$

In the network shown in Fig. the switch is changed from the position 1 to the position 2 at $t = 0$, steady condition having reached before switching. Find the values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.



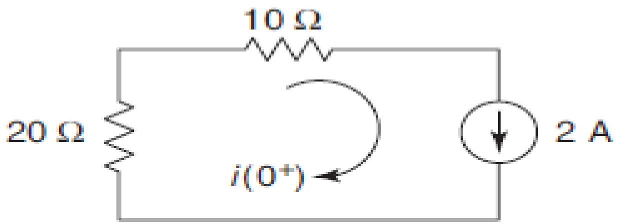
Solution At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{20}{10} = 2\ \text{A}$$



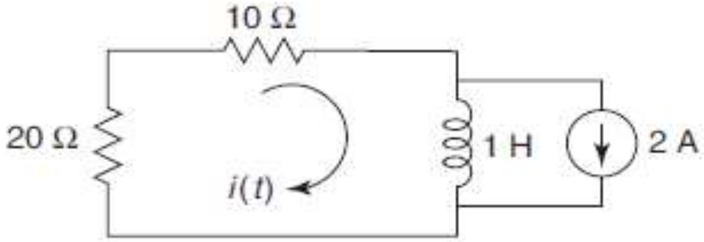
At $t = 0^+$, the network is shown in Fig.

At $t = 0^+$, the inductor acts as a current source of 2 A.



$$i(0^+) = 2\ \text{A}$$

For $t > 0$, the network is shown in Fig.
Writing the KVL equation for $t > 0$,



$$-20i - 10i - 1\frac{di}{dt} = 0 \qquad \text{...(i)}$$

At $t = 0^+$,
$$-30i(0^+) - \frac{di}{dt}(0^+) = 0$$

$$\frac{di}{dt}(0^+) = -30 \times 2 = -60 \text{ A/s}$$

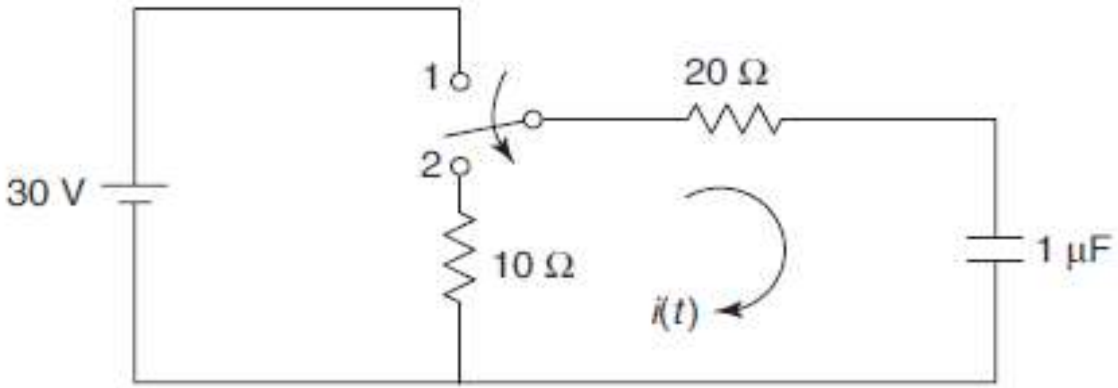
Differentiating Eq. (i),

$$-30\frac{di}{dt} - \frac{d^2i}{dt^2} = 0$$

At $t = 0^+$,
$$-30\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$

$$\frac{d^2i}{dt^2}(0^+) = 1800 \text{ A/s}^2$$

In the network shown in Fig. the switch is changed from the position 1 to the position 2 at $t = 0$, steady condition having reached before switching. Find the values of i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.



Solution At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 30 \text{ V}$$

$$i(0^-) = 0$$

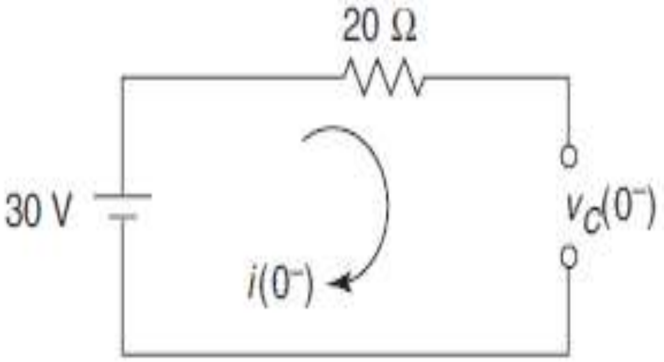


Fig. 6.26

At $t = 0^+$, the network is shown in Fig. 6.27.

At $t = 0^+$, the capacitor acts as a voltage source of 30 V.

$$v_C(0^+) = 30 \text{ V}$$

$$i(0^+) = -\frac{30}{30} = -1 \text{ A}$$

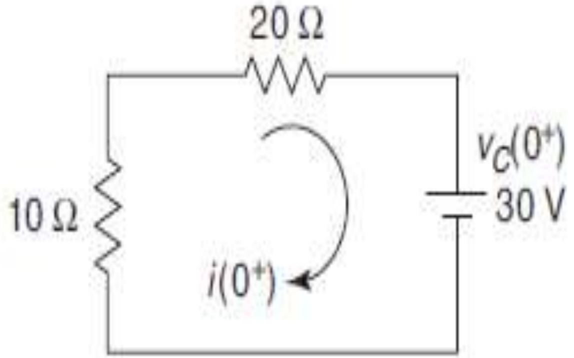


Fig. 6.27

For $t > 0$, the network is shown in Fig. 6.28.

Writing the KVL equation for $t > 0$,

$$-10i - 20i - \frac{1}{1\times10^{-6}}\int_0^t i\,dt - 30 = 0 \qquad \ldots\text{(i)}$$

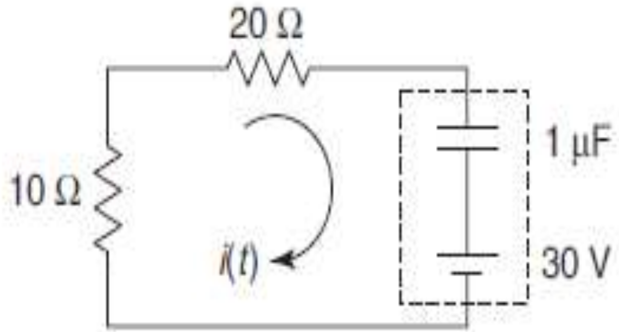


Fig. 6.28

Differentiating Eq. (i),

$$-30\frac{di}{dt} - 10^6 i = 0 \qquad \ldots\text{(ii)}$$

At $t = 0^+$,

$$-30\frac{di}{dt}(0^+) - 10^6 i(0^+) = 0$$

$$\frac{di}{dt}(0^+) = \frac{10^6(-1)}{30} = 0.33\times10^5 \text{ A/s}$$

Differentiating Eq. (ii),

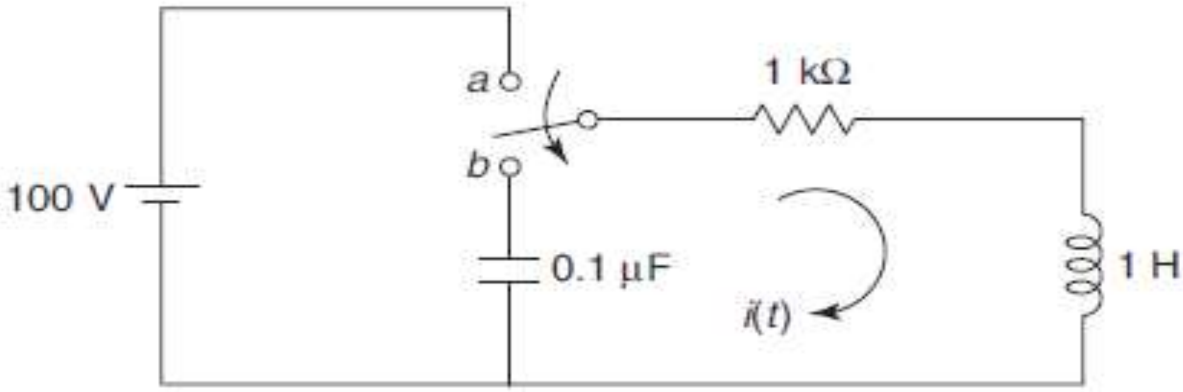
$$-30\frac{d^2i}{dt^2}-10^6\frac{di}{dt}=0$$

At $t=0^+$,

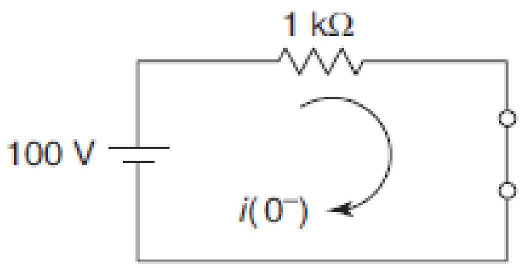
$$-30\frac{d^2i}{dt^2}(0^+)-10^6\frac{di}{dt}(0^+)=0$$

$$\frac{d^2i}{dt^3}(0^+)=-\frac{10^6\times 0.33\times 10^5}{30}=-1.1\times 10^9\ \mathrm{A/s^2}$$

In the network of Fig. the switch is changed from the position 'a' to 'b' at $t = 0$. Solve for i, $\frac{di}{dt}$ and $\frac{d^2i}{dt^2}$ at $t = 0^+$.



Solution At $t = 0^-$, the network attains steady condition. Hence, the inductor acts as a short circuit.

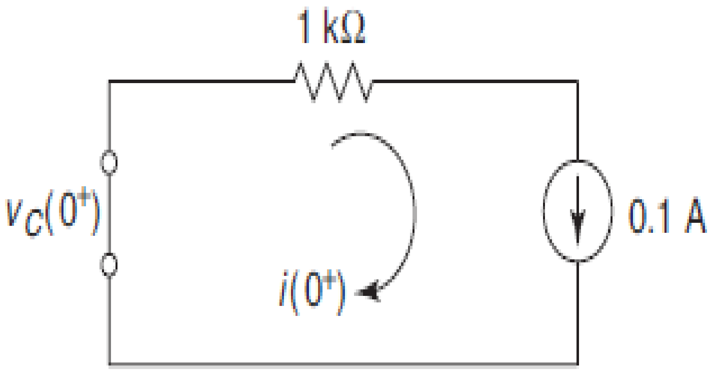


$$i(0^-) = \frac{100}{1000} = 0.1 \text{ A}$$

$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig.

At $t = 0^+$, the inductor acts as a current source of 0.1 A and the capacitor acts as a short circuit.

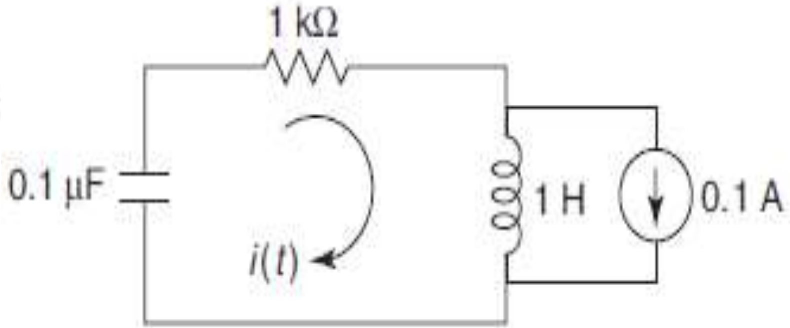


$$i(0^+) = 0.1 \text{ A}$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig.
Writing the KVL equation for $t > 0$,

$$-\frac{1}{0.1\times10^{-6}}\int_0^t i\,dt - 1000i - 1\frac{di}{dt} = 0 \qquad \ldots\text{(i)}$$



At $t = 0^+$,
$$-0 - 1000i(0^+) - \frac{di}{dt}(0^+) = 0$$

$$\frac{di}{dt}(0^+) = -1000\,i(0^+) = -1000\times0.1 = -100 \text{ A/s}$$

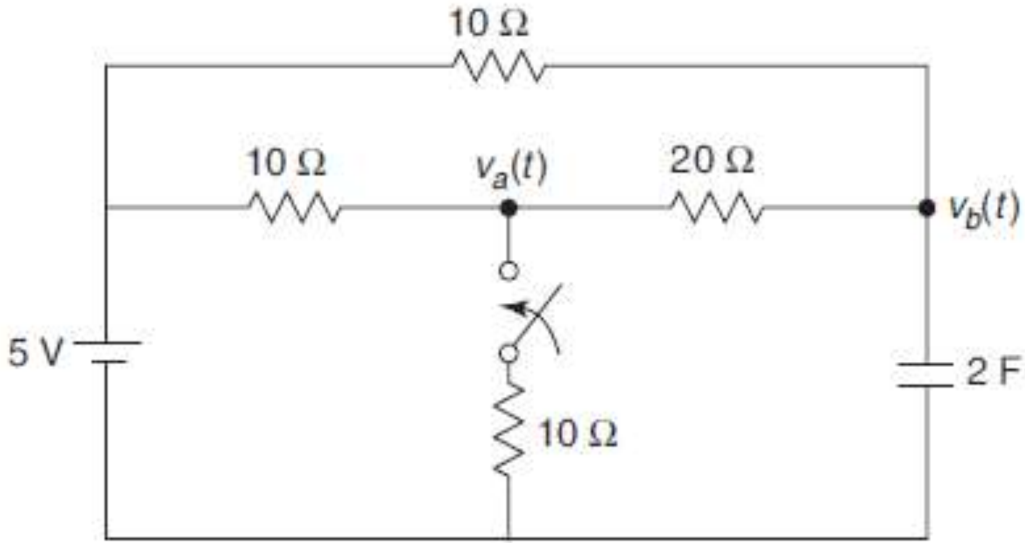
Differentiating Eq. (i),

$$-\frac{1}{10^{-7}}i - 1000\frac{di}{dt} - \frac{d^2i}{dt^2} = 0$$

At $t = 0^+$, $$-10^7\, i(0^+) - 1000\frac{di}{dt}(0^+) - \frac{d^2i}{dt^2}(0^+) = 0$$

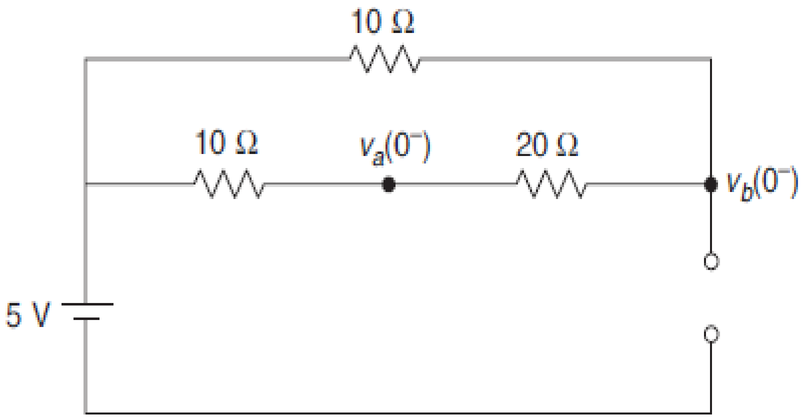
$$\frac{d^2i}{dt^2}(0^+) = -10^7(0.1) - 1000(-100) = -9\times 10^5\ \text{A/s}^2$$

In the accompanying Fig. is shown a network in which a steady state is reached with switch open. At $t = 0$, switch is closed. Determine $v_a(0^-)$, $v_a(0^+)$, $v_b(0^-)$ and $v_b(0^+)$.



Solution At $t = 0^-$, the network is shown

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

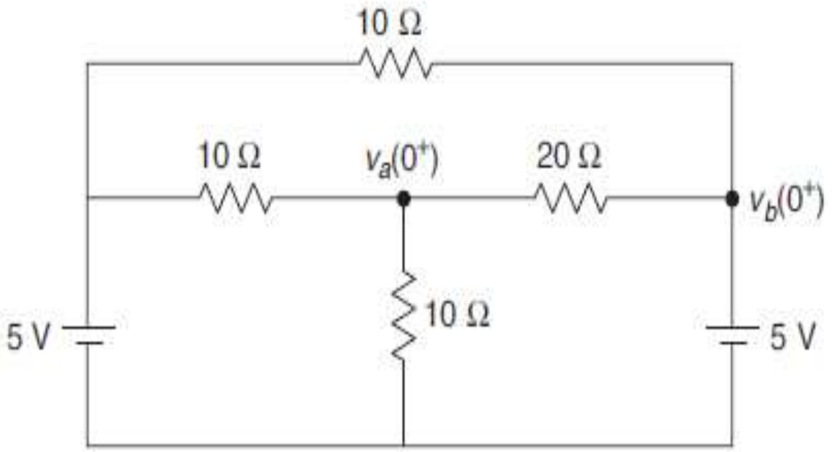


$$v_a(0^-) = 5\text{ V}$$

$$v_b(0^-) = 5\text{ V}$$

At $t = 0^+$, the network is shown in Fig.

At $t = 0^+$, the capacitor acts as a voltage source of 5 V.



$$v_b(0^+) = 5\text{ V}$$

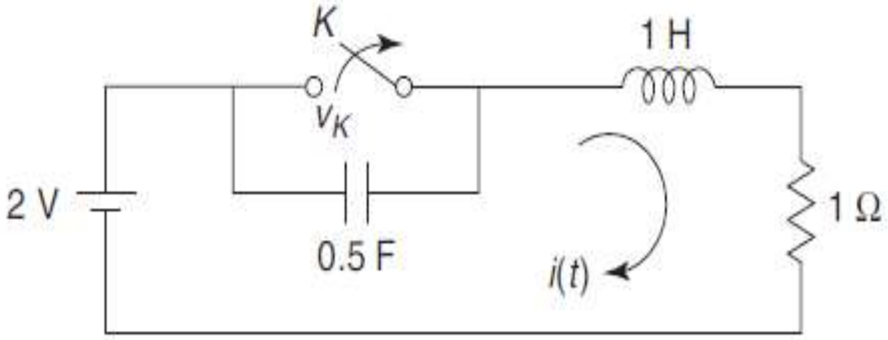
Writing the KCL equation at $t = 0^+$,

$$\frac{v_a(0^+)-5}{10}+\frac{v_a(0^+)}{10}+\frac{v_a(0^+)-5}{20}=0$$

$$0.25v_a(0^+)=0.75$$

$$v_a(0^+)=3\ \text{V}$$

The network of Fig. 6.37 attains steady-state with the switch closed. At $t = 0$, the switch is opened. Find the voltage across the switch v_K and $\frac{dv_K}{dt}$ at $t = 0^+$.



Solution: At $t = 0^-$, the network is shown in Fig. 6.38. At $t = 0^-$, the network attains steady-state condition. The capacitor acts as an open circuit and the inductor acts as a short circuit.

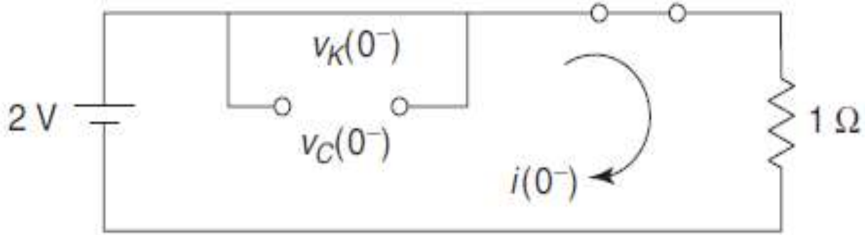


Fig. 6.38

$$i(0^-) = \frac{2}{1} = 2\ \text{A}$$

$$v_C(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.39.

At $t = 0^+$, the capacitor acts as a short circuit and the inductor acts as a current source of 2 A.

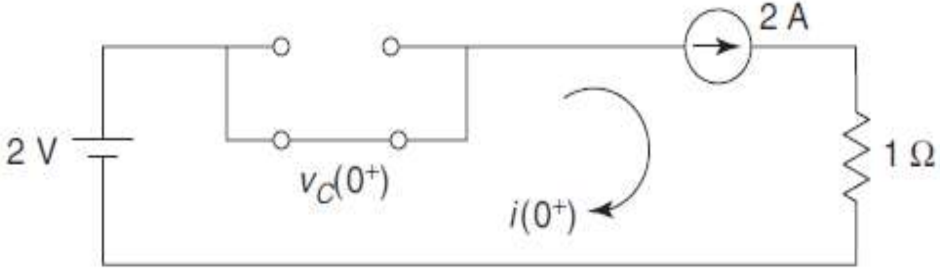


Fig. 6.39

$$i(0^+) = 2\ \text{A}$$

$$v_C(0^+) = 0$$

$$v_K(0^+) = 0$$

Also,

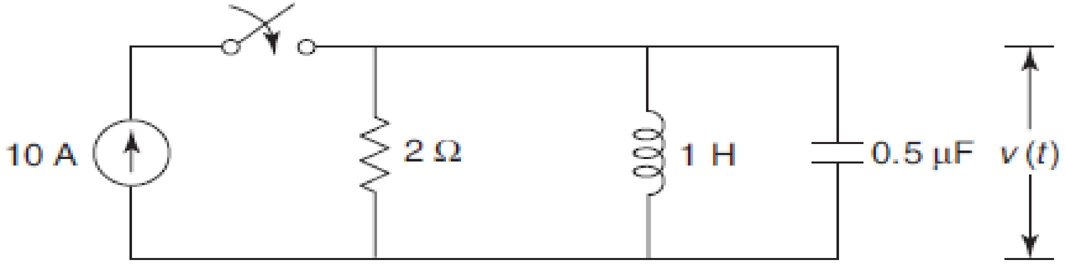
$$v_K = \frac{1}{C}\int i\,dt$$

$$\frac{dv_K}{dt} = \frac{i}{C}$$

At $t = 0^+$,

$$\frac{dv_K}{dt}(0^+) = \frac{i(0^+)}{C} = \frac{2}{0.5} = 4\ \text{A/s}$$

For the network shown in Fig. 6.18, the switch is closed at $t = 0$, determine v, $\frac{dv}{dt}$ and $\frac{d^2v}{dt^2}$ at $t = 0^+$.



Solution At $t = 0^-$, no current flows through the inductor and there is no voltage across the capacitor.

$$i_L(0^-) = 0$$

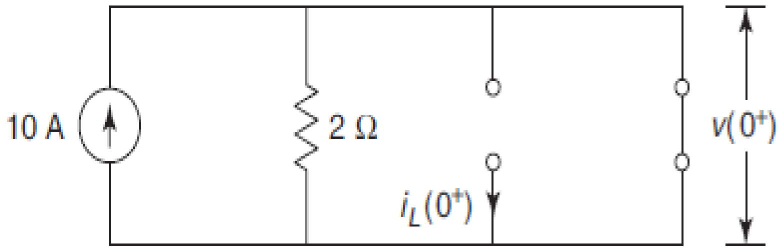
$$v(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.19.

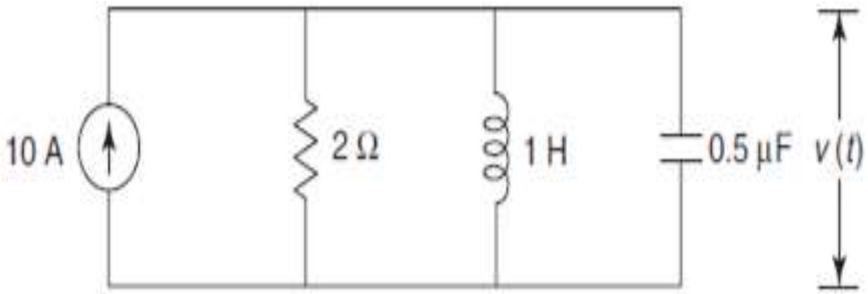
At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

$$i_L(0^+) = 0$$

$$v(0^+) = 0$$



For $t > 0$, the network is shown in Fig. 6.20.



Writing the KCL equation for $t > 0$,

$$\frac{v}{2} + \frac{1}{1}\int_{1}^{t} v dt + 0.5 \times 10^{-6} \frac{dv}{dt} = 10 \qquad \ldots\text{(i)}$$

At $t = 0^+$,

$$\frac{v(0^+)}{2} + 0 + 0.5 \times 10^{-6} \frac{dv}{dt}(0^+) = 10$$

$$\frac{dv}{dt}(0^+) = 20 \times 10^6 \ \text{V/s}$$

Differentiating Eq. (i),

$$\frac{1}{2}\frac{dv}{dt} + v + 0.5 \times 10^{-6} \frac{d^2 v}{dt^2} = 0$$

At $t = 0^+$,

$$\frac{1}{2}\frac{dv}{dt}(0^+) + v(0^+) + 0.5 \times 10^{-6} \frac{d^2 v}{dt^2}(0^+) = 0$$

$$\frac{d^2 v}{dt^2}(0^+) = -20 \times 10^{12} \ \text{V/s}^2$$

In the network shown in Fig. 6.40, assuming all initial conditions as zero, find $i_1(0^+)$, $i_2(0^+)$, $\frac{di_1}{dt}(0^+)$, $\frac{di_2}{dt}(0^+)$, $\frac{d^2i_1}{dt^2}(0^+)$ *and* $\frac{d^2i_2}{dt^2}(0^+)$.

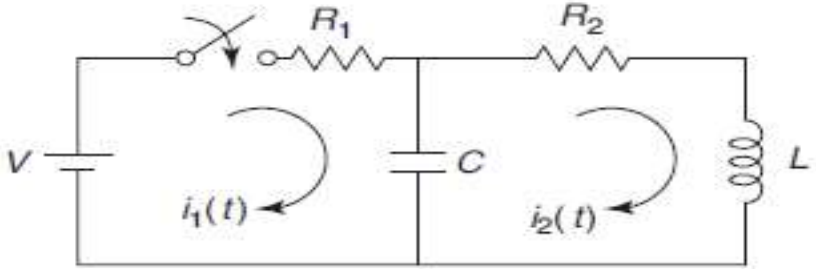


Fig. 6.40

Solution At $t = 0^-$, all initial conditions are zero.

$$v_C(0^-) = 0$$

$$i_1(0^-) = 0$$

$$i_2(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig. 6.41.

At $t = 0^+$, the inductor acts as an open circuit and the capacitor acts as a short circuit.

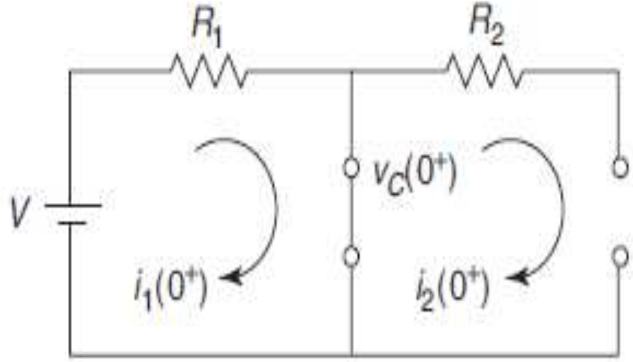


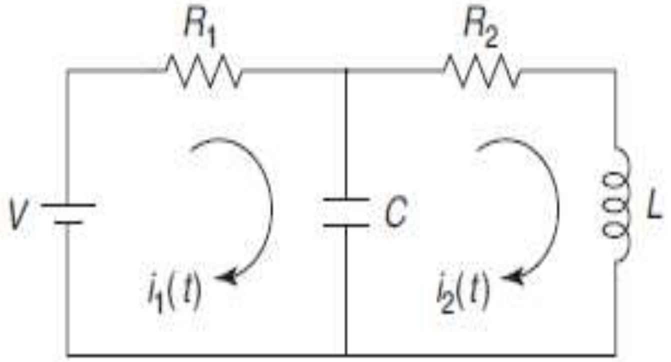
Fig. 6.41

$$i_1(0^+) = \frac{V}{R_1}$$

$$i_2(0^+) = 0$$

$$v_C(0^+) = 0$$

For $t > 0$, the network is shown in Fig. 6.42.



Writing the KVL equations for two meshes for $t > 0$,

$$V - R_1 i_1 - \frac{1}{C}\int_0^t (i_1 - i_2)\,dt = 0 \qquad \ldots\text{(i)}$$

and $$-\frac{1}{C}\int(i_2-i_1)dt-R_2i_2-L\frac{di_2}{dt}=0 \qquad \ldots\text{(ii)}$$

From Eq. (ii), at $t=0^+$,

$$-\frac{1}{C}\int_0^{0^+}(i_2-i_1)dt-R_2i_2(0^+)-L\frac{di_2}{dt}(0^+)=0$$

$$\frac{di_2}{dt}(0^+)=0$$

Differentiating Eq. (i),

$$0-R_1\frac{di_1}{dt}-\frac{1}{C}(i_1-i_2)=0 \qquad \ldots\text{(iii)}$$

At $t=0^+$, $$0-R_1\frac{di_1}{dt}(0^+)-\frac{1}{C}i_1(0^+)+\frac{1}{C}i_2(0^+)=0$$

$$R_1\frac{di_1}{dt}(0^+)+\frac{1}{C}\frac{V}{R_1}=0$$

$$\frac{di_1}{dt}(\mathbf{0}^+)=-\frac{V}{R_1^2C}$$

Differentiating Eq. (iii),

$$-R_1 \frac{d^2 i_1}{dt^2} - \frac{1}{C}\frac{di_1}{dt} + \frac{1}{C}\frac{di_2}{dt} = 0$$

At $t = 0^+$,
$$-R_1 \frac{d^2 i_1}{dt^2}(0^+) - \frac{1}{C}\frac{di_1}{dt}(0^+) + \frac{1}{C}\frac{di_2}{dt}(0^+) = 0$$

$$\frac{d^2 i_1}{dt^2}(0^+) = \frac{V}{R_1^3 C^2}$$

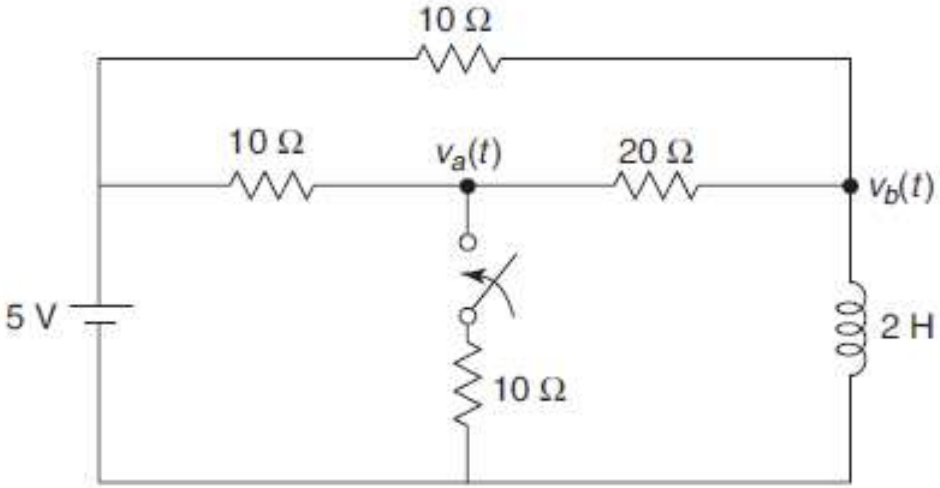
Differentiating Eq. (ii),

$$-\frac{1}{C}(i_2 - i_1) - R_2 \frac{di_2}{dt} - L\frac{d^2 i_2}{dt^2} = 0$$

At $t = 0^+$,
$$\frac{d^2 i_2}{dt^2}(0^+) = -\frac{R_2}{L}\frac{di_2}{dt}(0^+) - \frac{1}{LC}[i_2(0^+) - i_1(0^+)] = \frac{V}{R_1 LC}$$

In the network shown in Fig. 6.46, a steady state is reached with the switch open.

At $t = 0$, the switch is closed. For the element values given, determine the value of $v_a(0^-)$, $v_b(0^-)$, $v_a(0^+)$ and $v_b(0^+)$.



Solution At $t = 0^-$, the network is shown in Fig. 6.47.

At $t = 0^-$, the network attains steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = \frac{5}{(10 \,||\, 30)} = \frac{5}{7.5} = \frac{2}{3}\text{ A}$$

$$v_b(0^-) = 0$$

$$v_a(0^-) = 5 \times \frac{20}{30} = 3.33\text{ V}$$

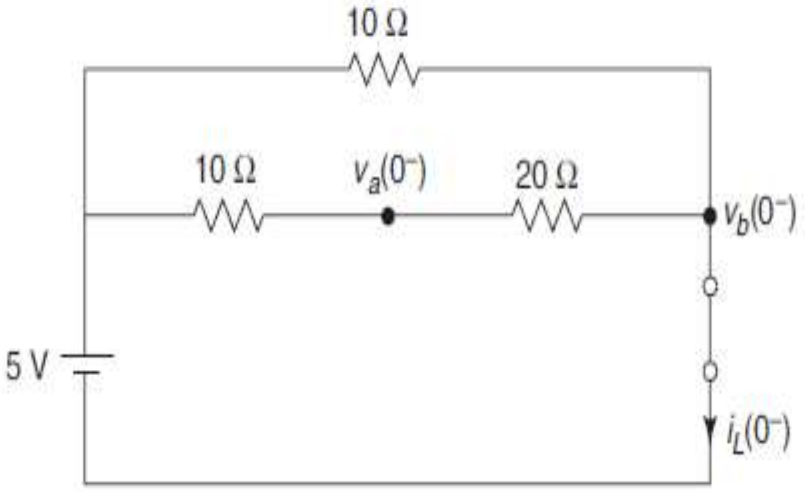


Fig. 6.47

At $t = 0^+$, the network is shown in Fig. 6.48.

At $t = 0^+$, the inductor acts as a current source of $\frac{2}{3}$ A.

$$i_L(0^+) = \frac{2}{3}\text{ A}$$

Writing the KCL equations at $t = 0^+$,

$$\frac{v_a(0^+) - 5}{10} + \frac{v_a(0^+)}{10} + \frac{v_a(0^+) - v_b(0^+)}{20} = 0$$

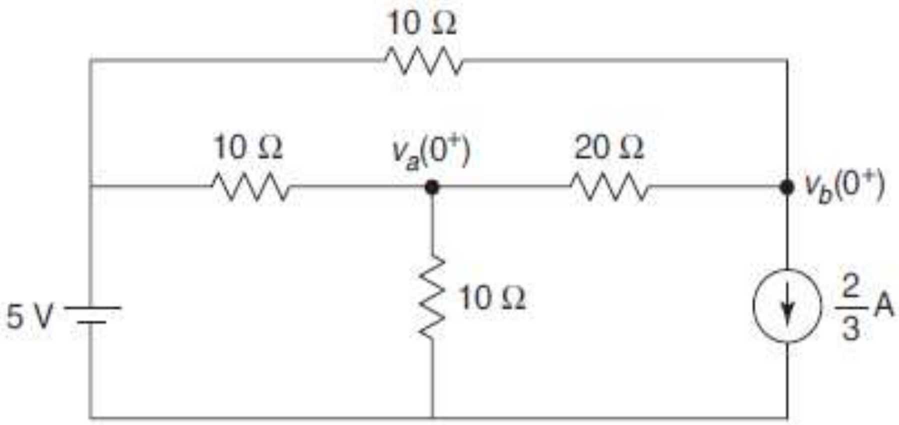


Fig. 6.48

and
$$\frac{v_b(0^+) - v_a(0^+)}{20} + \frac{v_b(0^+) - 5}{10} + \frac{2}{3} = 0$$

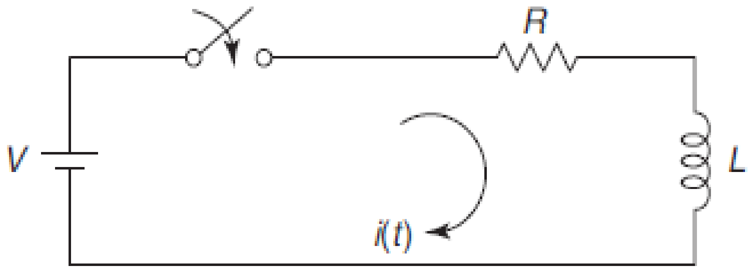
Solving these two equations,

$$v_a(0^+) = 1.9\text{ V}$$

$$v_b(0^+) = -0.477\text{ V}$$

RESISTOR-INDUCTOR CIRCUIT

Consider a series RL circuit as shown in Fig. The switch is closed at time $t = 0$. The inductor in the circuit is initially un-energised.



Series RL circuit

Applying KVL to the circuit for $t > 0$,

$$V - Ri - L\frac{di}{dt} = 0$$

This is a linear differential equation of first order. It can be solved if the variables can be separated.

$$(V - Ri)\, dt = L\, di$$

$$\frac{L\, di}{V - Ri} = dt$$

Integrating both the sides,

$$-\frac{L}{R} l_n(V - Ri) = t + k$$

where l_n denotes that the logarithm is of base e and k is an arbitrary constant. k can be evaluated from the initial condition. In the circuit, the switch is closed at $t = 0$, i.e., just before closing the switch, the current in the inductor is zero. Since the inductor does not allow sudden change in current, at $t = 0^+$, just after the switch is closed, the current remains zero.

Setting $i = 0$ at $t = 0$,

$$-\frac{L}{R} l_n V = k$$

$$-\frac{L}{R} l_n (V - Ri) = t - \frac{L}{R} l_n V$$

$$-\frac{L}{R}[l_n (V - Ri) - l_n V] = t$$

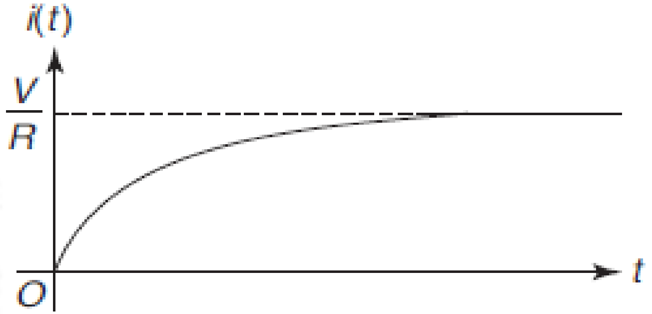
$$\frac{V - Ri}{V} = e^{-\frac{R}{L}t}$$

$$V - Ri = Ve^{-\frac{R}{L}t}$$

$$Ri = V - Ve^{-\frac{R}{t}t}$$

$$i = \frac{V}{R} - \frac{V}{\boldsymbol{R}} e^{-\frac{R}{L}t} \qquad \text{for } t > 0$$

The complete response is composed of two parts, the steady-state response or forced response or zero state response $\frac{V}{R}$ and transient response or natural response or zero input response $\frac{V}{R}e^{-\frac{R}{L}t}$



The natural response is a characteristic of the circuit. Its form may be found by considering the source-free circuit. The forced response has the characteristics of forcing function, i.e., applied voltage. Thus, when the switch is closed, response reaches the steady-state value after some time interval as shown in Fig. 6.75.

Note:

1. Consider a homogeneous equation,

$$\frac{di}{dt} + Pi = 0 \quad \text{where } P \text{ is a constant.}$$

The solution of this equation is given by,

$$i(t) = k\,e^{-Pt}$$

The value of k is obtained by putting $t = 0$ in the equation for $i(t)$.

2. Consider a non-homogeneous equation,

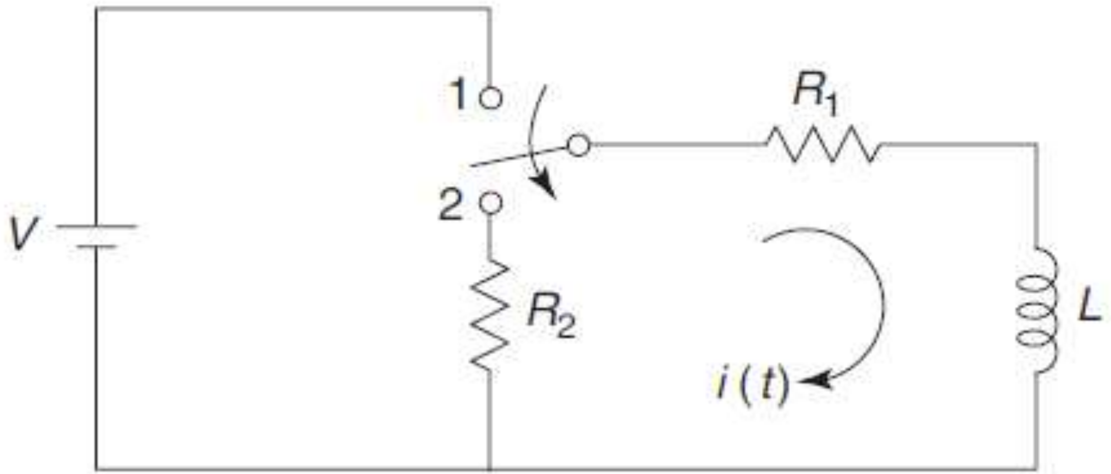
$$\frac{di}{dt} + Pi = Q$$

where P is a constant and Q may be a function of the independent variable t or a constant. The solution of this equation is given by,

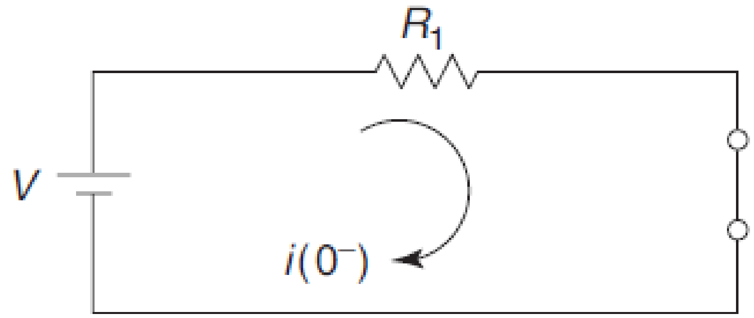
$$i(t) = e^{-Pt} \int Q\,e^{Pt}\,dt + k\,e^{-Pt}$$

The value of k is obtained by putting $t = 0$ in the equation of $i(t)$.

In the network of Fig. the switch is initially at the position 1. On the steady state having reached, the switch is changed to the position 2. Find current i(t).



Solution At $t = 0^-$, the network is shown in Fig.



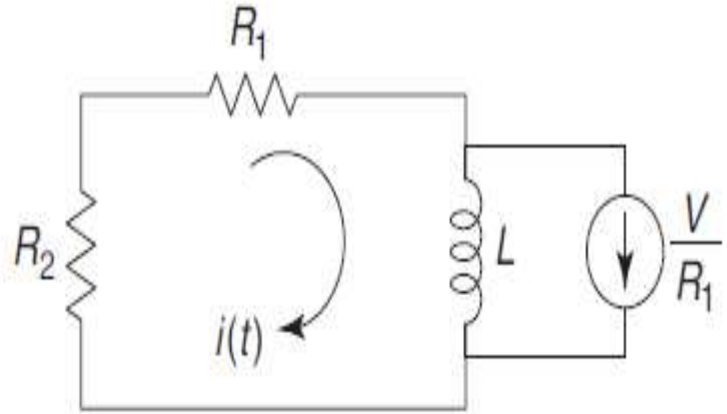
At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{V}{R_1}$$

Since the inductor does not allow sudden change in current,

$$i(0^+) = \frac{V}{R_1}$$

For $t > 0$, the network is shown in Fig.
Writing the KVL equation for $t > 0$



$$-R_2 i - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{(R_1 + R_2)}{L} i = 0$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = 0$,

$$P = \frac{R_1 + R_2}{L}$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt}$$

$$i(t) = k\,e^{-\left(\frac{R_1+R_2}{L}\right)t}$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt}$$

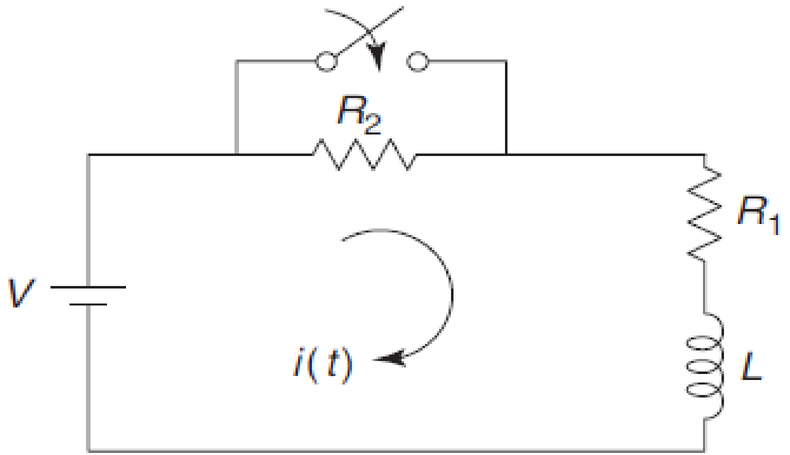
$$i(t) = k\,e^{-\left(\frac{R_1+R_2}{L}\right)t}$$

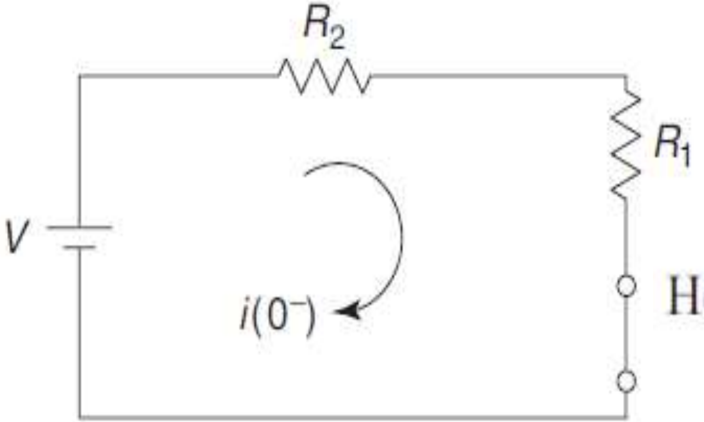
At $t = 0$, $i(0) = \frac{V}{R_1}$

$$\frac{V}{R_1} = k\,e^{0} = k$$

$$i(t) = \frac{V}{R_1} e^{-\left(\frac{R_1+R_2}{L}\right)t} \qquad \text{for } t > 0$$

In the network shown in Fig. the switch is closed at t = 0, a steady state having previously been attained. Find the current i (t).





Solution At $t = 0^-$, the network is shown in Fig.

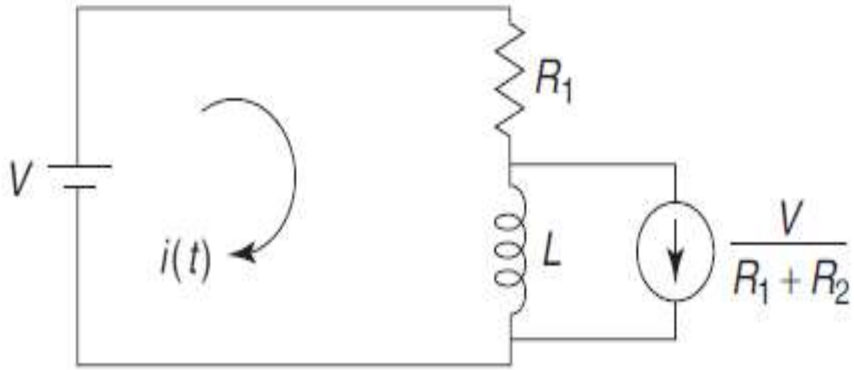
At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{V}{R_1 + R_2}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = \frac{V}{R_1 + R_2}$$

For $t > 0$, the network is shown in Fig.
Writing the KVL equation for $t > 0$,



$$V - R_1 i - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} + \frac{R_1}{L} i = \frac{V}{L}$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = \frac{R_1}{L}, \; Q = \frac{V}{L}$$

The solution of this differential equation is given by,

$$
\begin{aligned}
i(t) &= e^{-Pt}\int Qe^{Pt}\,dt + k\,e^{-Pt} \\
&= e^{-\frac{R_1}{L}t}\int \frac{V}{L}\,e^{\frac{R_1}{L}t}\,dt + k\,e^{-\frac{R_1}{L}t} \\
&= \frac{V}{R_1} + k\,e^{-\frac{R_1}{L}t}
\end{aligned}
$$

At $t = 0,\ i(0) = \dfrac{V}{R_1 + R_2}$

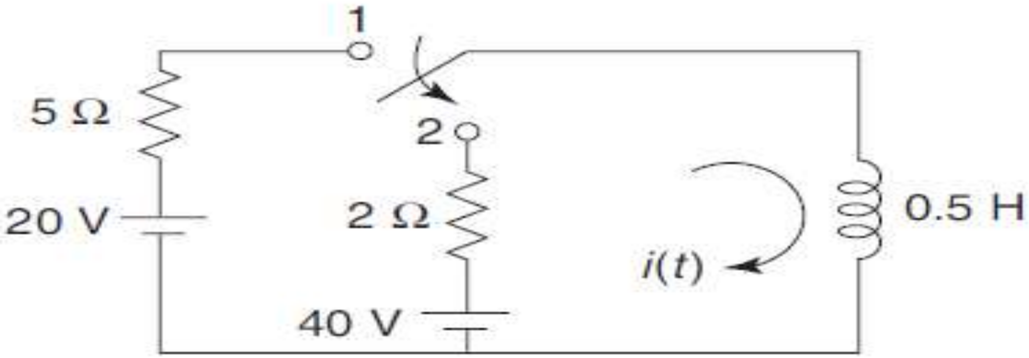
$$\frac{V}{R_1 + R_2} = \frac{V}{R_1} + k$$

$$k = -\frac{VR_2}{R_1(R_1 + R_2)}$$

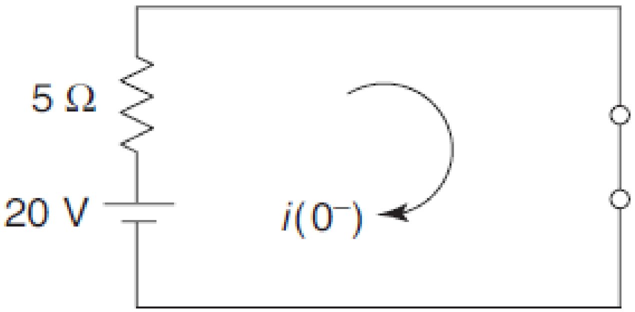
$$i(t) = \frac{V}{R_1} - \frac{VR_2}{R_1(R_1 + R_2)} e^{-\frac{R_1}{L}t}$$

$$= \frac{V}{R_1}\left(1 - \frac{R_2}{R_1 + R_2} e^{-\frac{R_1}{L}t}\right) \quad \text{for } t > 0$$

In the network of Fig. the switch is moved from 1 to 2 at $t = 0$. Determine $i(t)$.



Solution At $t = 0^-$, the network is shown in Fig.



At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

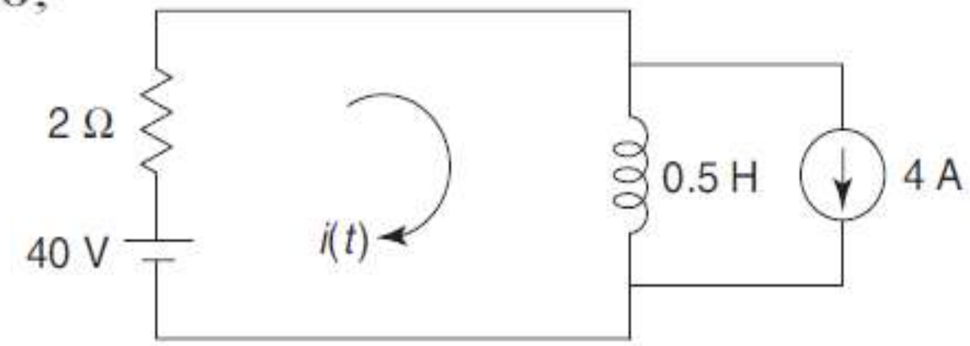
$$i(0^-) = \frac{20}{5} = 4 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 4 \text{ A}$$

For $t > 0$, the network is shown in Fig.

Writing the KVL equation for $t > 0$,



$$40 - 2i - 0.5\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 4i = 80$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 4, \quad Q = 80$$

The solution of this differential equation is given by,

$$\begin{aligned}
i(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\
&= e^{-4t} \int 80\, e^{4t}\, dt + k\, e^{-4t} \\
&= \frac{80}{4} + k\, e^{-4t} \\
&= 20 + k\, e^{-4t}
\end{aligned}$$

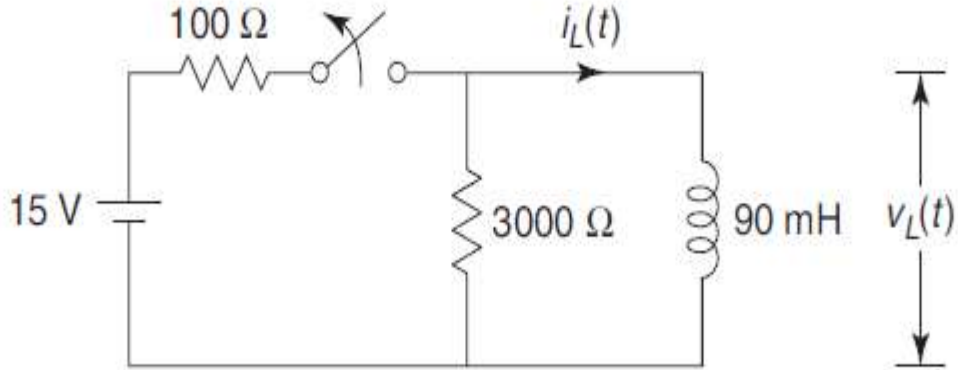
At $t = 0$, $i(0) = 4$ A

$$4 = 20 + k$$

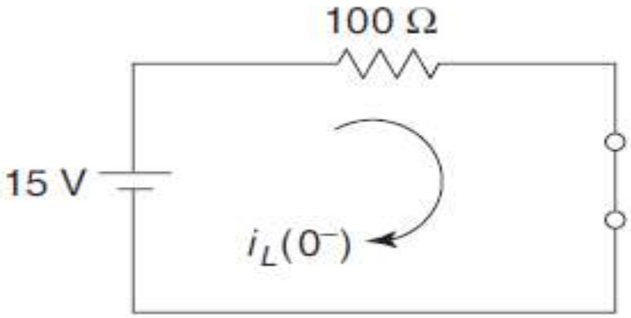
$$k = -16$$

$$i(t) = 20 - 16 e^{-4t} \qquad \text{for } t > 0$$

For the network shown in Fig. steady state is reached with the switch closed. The switch is opened at $t = 0$. Obtain expressions for $i_L(t)$ and $v_L(t)$.



Solution At $t = 0^-$, the network is shown in Fig.



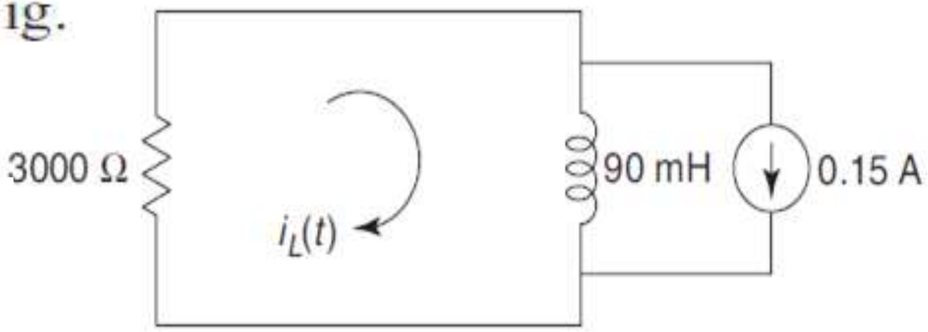
At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i_L(0^-) = \frac{15}{100} = 0.15 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 0.15 \text{ A}$$

For $t > 0$, the network is shown in Fig.



Writing the KVL equation for $t > 0$,

$$-3000 i_L - 90 \times 10^{-3} \frac{di_L}{dt} = 0$$

$$\frac{di_L}{dt} + 33.33 \times 10^3 \, i_L = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 33.33 \times 10^3$$

The solution of this differential equation is given by,

$$i_L(t) = k\, e^{-Pt}$$

$$i_L(t) = k\, e^{-33.33 \times 10^3 t}$$

At $t = 0$, $i_L(0) = 0.15$ A

$$0.15 = k$$

$$i_L(t) = 0.15 e^{-33.33 \times 10^3 t} \qquad \text{for } t > 0$$

Also,

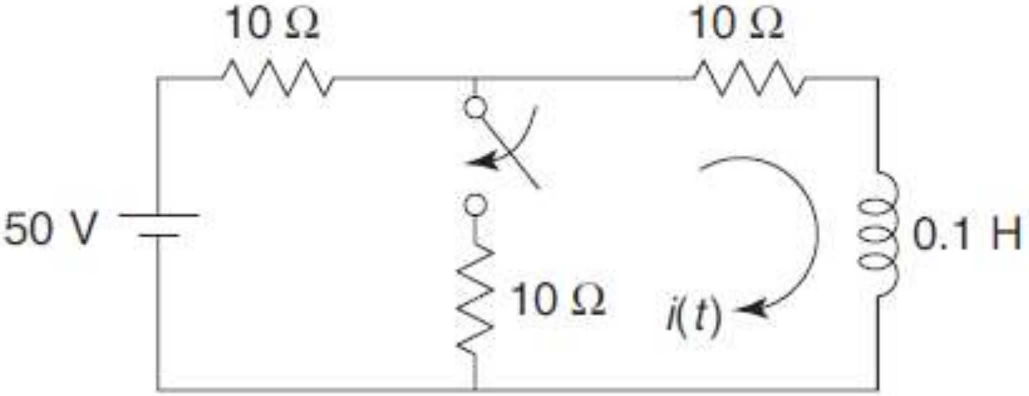
$$v_L(t) = L \frac{di_L}{dt}$$

$$= 90 \times 10^{-3} \frac{d}{dt}(0.15\, e^{-33.33 \times 10^3 t})$$

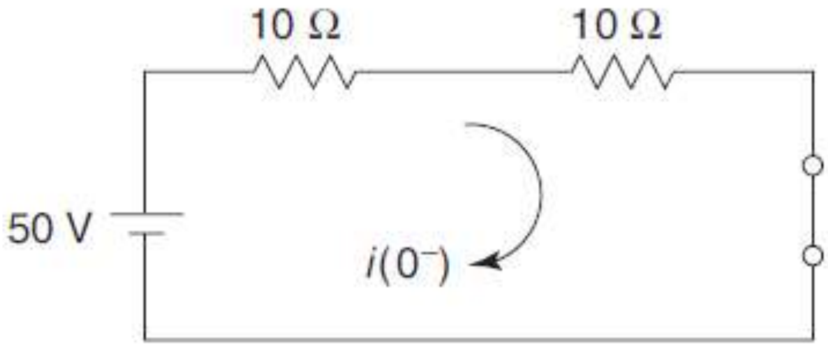
$$= -90 \times 10^{-3} \times 0.15 \times 33.33 \times 10^3 \times e^{-33.33 \times 10^3 t}$$

$$= -450 e^{-33.33 \times 10^3 t} \qquad \text{for } t > 0$$

In the network of Fig. the switch is open for a long time and it closes at $t = 0$. Find $i\ (t)$.



Solution At $t = 0^-$, the network is shown in Fig.



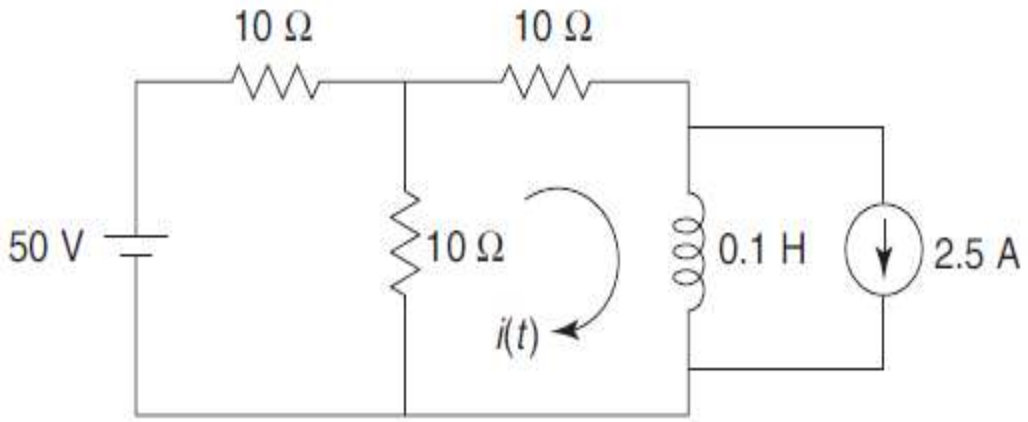
At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{50}{10+10} = 2.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig.

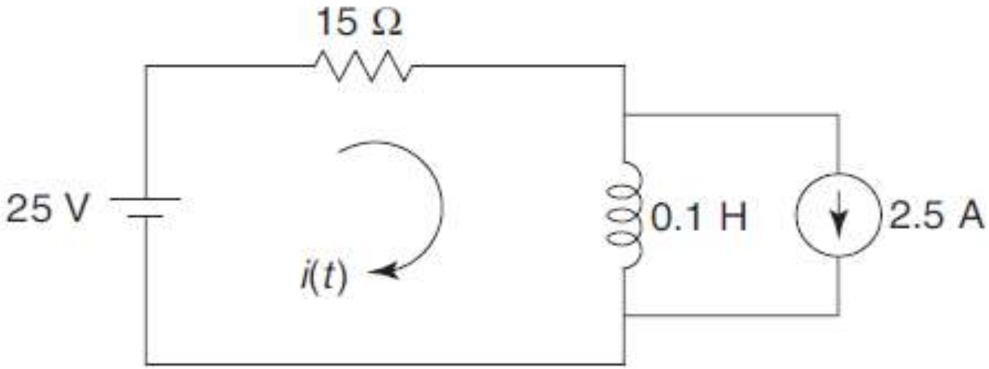


For $t > 0$, representing the network to the left of the inductor by Thevenin's equivalent network,

$$V_{eq} = 50 \times \frac{10}{10+10} = 25 \text{ V}$$

$$R_{eq} = (10 \,||\, 10) + 10 = 15 \text{ }\Omega$$

Writing the KVL equation for $t > 0$,



$$25 - 15i - 0.1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 150i = 250$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 150, \quad Q = 250$$

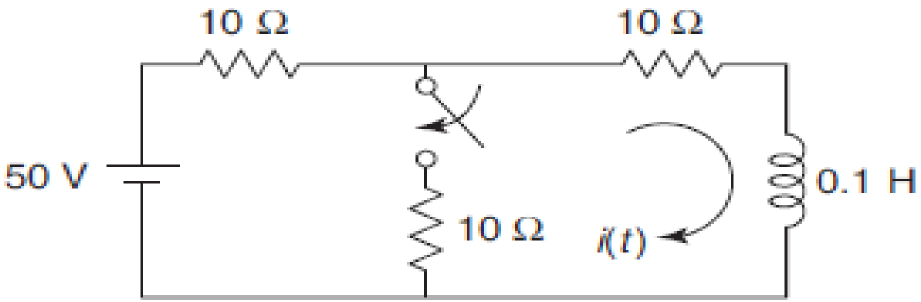
The solution of this differential equation is given by,

$$
\begin{aligned}
i(t) &= e^{-Pt}\int Q\, e^{Pt}\; dt + k\, e^{-Pt} \\
&= e^{-150t}\int 250\, e^{150t}\; dt + k\, e^{-Pt} \\
&= \frac{250}{150} + k\, e^{-150t} \\
&= 1.667 + k\, e^{-150t}
\end{aligned}
$$

At $t = 0,\ i(0) = 2.5$ A

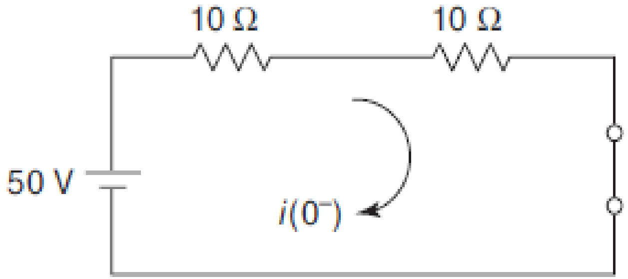
$$
\begin{aligned}
2.5 &= 1.667 + k \\
k &= 0.833 \\
i(t) &= 1.667 + 0.833\, e^{-150t} \qquad \text{for } t > 0
\end{aligned}
$$

In the network of Fig.the switch is open for a long time and it closes at $t = 0$. Find i (t).



Solution At $t = 0^-$, the network is shown in Fig.

At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

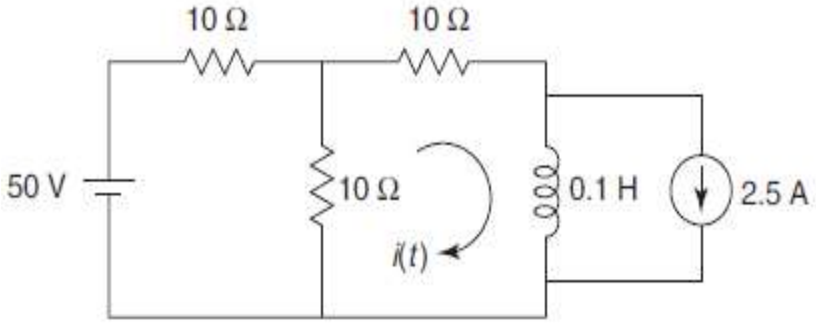


$$i(0^-) = \frac{50}{10+10} = 2.5 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i(0^+) = 2.5 \text{ A}$$

For $t > 0$, the network is shown in Fig.

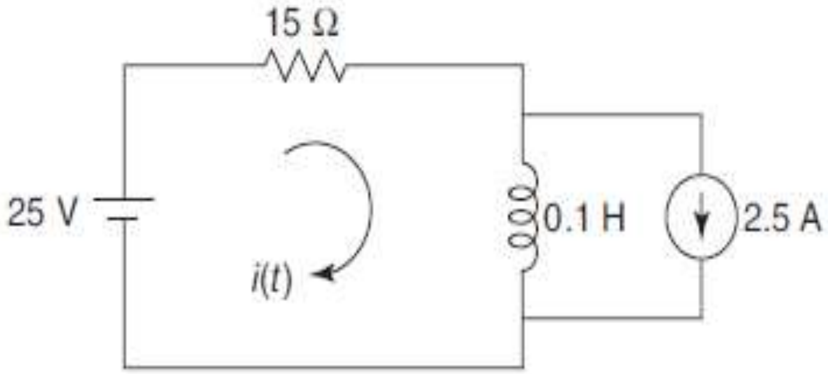


For $t > 0$, representing the network to the left of the inductor by Thevenin's equivalent network,

$$V_{eq} = 50 \times \frac{10}{10+10} = 25 \text{ V}$$

$$R_{eq} = (10 \,||\, 10) + 10 = 15 \ \Omega$$

For $t > 0$, Thevenin's equivalent network is shown in Fig.



Writing the KVL equation for $t > 0$,

$$25 - 15i - 0.1\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 150i = 250$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$, F

$$P = 150, \qquad Q = 250$$

The solution of this differential equation is given by,

$$i(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-150t} \int 250\, e^{150t}\, dt + k\, e^{-Pt}$$

$$= \frac{250}{150} + k\, e^{-150t}$$

$$= 1.667 + k\, e^{-150t}$$

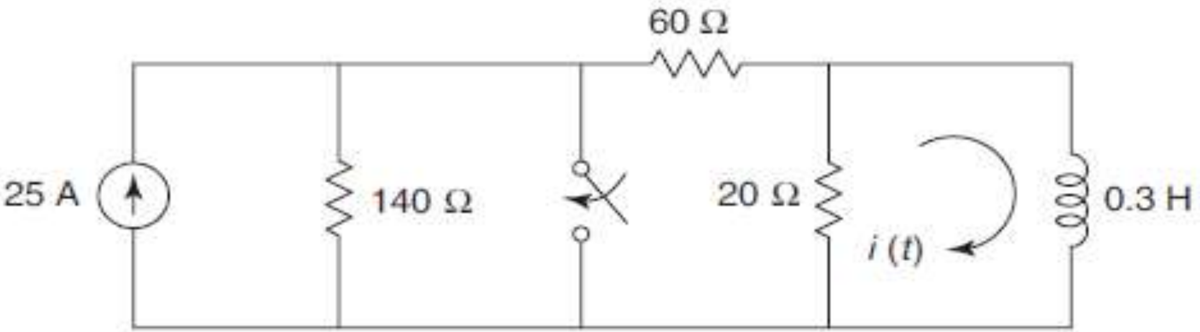
At $t = 0$, $i(0) = 2.5$ A

$$2.5 = 1.667 + k$$

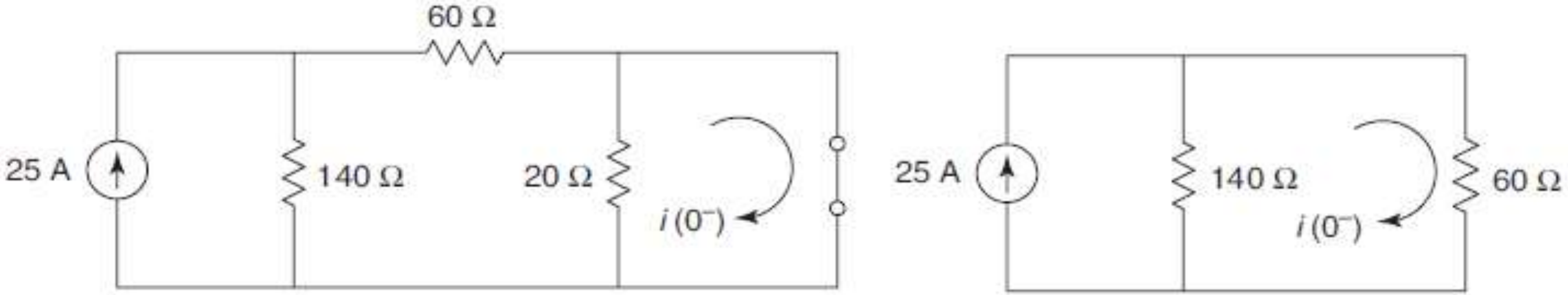
$$k = 0.833$$

$$i(t) = 1.667 + 0.833\, e^{-150t} \qquad \text{for } t > 0$$

Find the current i (t) for t > 0.



Solution At $t = 0^-$, the inductor acts as a short circuit. Simplifying the network as shown in Fig.



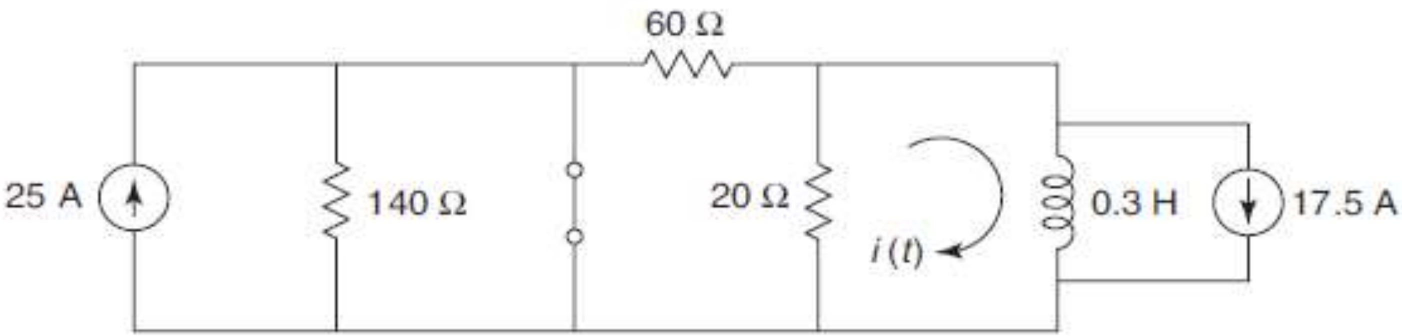
(a) (b)

$$i(0^-) = 25 \times \frac{140}{140+60} = 17.5 \text{ A}$$

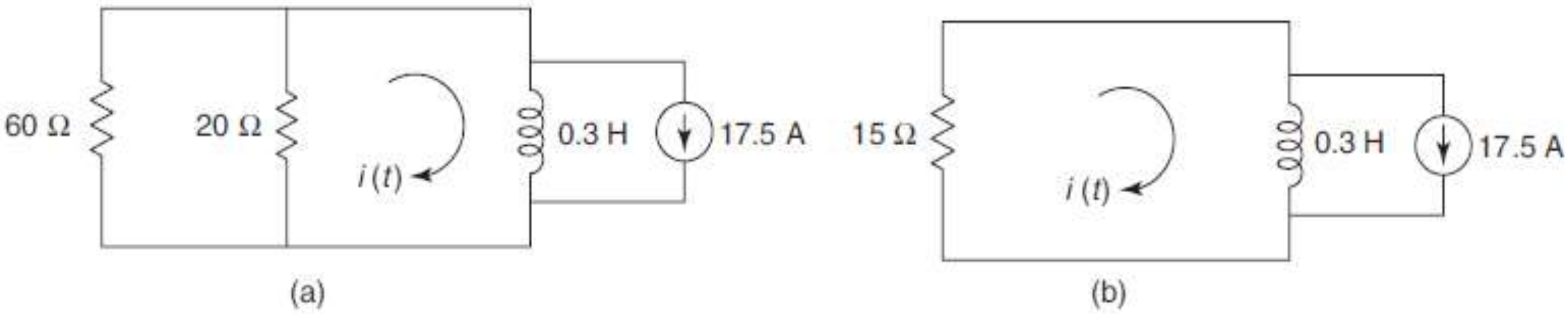
Since current through the inductor cannot change instantaneously,

$$i(0^+) = 17.5 \text{ A}$$

For $t > 0$, the network is shown in Fig.



Simplifying the network by source transformation as shown in Fig.



Writing the KVL equation for $t > 0$,

$$-15i - 0.3\frac{di}{dt} = 0$$

$$\frac{di}{dt} + 50i = 0$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = 0$,

$$P = 50$$

The solution of this differential equation is given by,

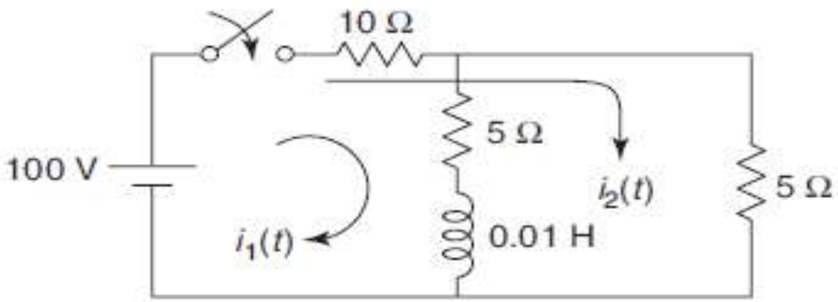
$$i(t) = k\,e^{-Pt} = k\,e^{-50t}$$

At $t = 0$, $i\,(0) = 17.5$ A

$$k = 17.5$$

$$i(t) = 17.5\,e^{-50t} \qquad \text{for } t > 0$$

In the network of Fig. determine currents $i_1(t)$ and $i_2(t)$ when the switch is closed at $t = 0$.



Solution At $t = 0^-$,

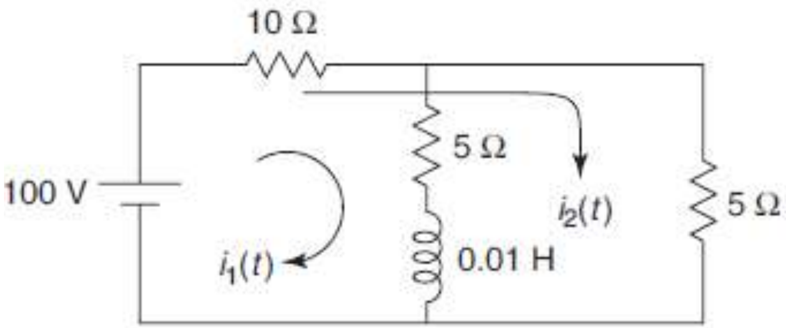
$$i_1(0^-) = i_2(0^-) = 0$$

At $t = 0^+$,

$$i_1(0^+) = 0$$

$$i_2(0^+) = \frac{100}{15} = 6.67 \text{ A}$$

For $t > 0$, the network is shown in Fig.
Writing the KVL equations for $t > 0$,



$$100 - 10(i_1 + i_2) - 5i_1 - 0.01\frac{di_1}{dt} = 0 \qquad \ldots\text{(i)}$$

and
$$100 - 10(i_1 + i_2) - 5i_2 = 0 \qquad \ldots\text{(ii)}$$

From Eq. (ii),

$$i_2 = \frac{100 - 10i_1}{15}$$

Substituting in Eq. (i),

$$\frac{di_1}{dt} + 833i_1 = 3333$$

Comparing with the differential equation $\frac{di}{dt} + Pi = Q$,

$$P = 833, Q = 3333$$

The solution of this differential equation is given by,

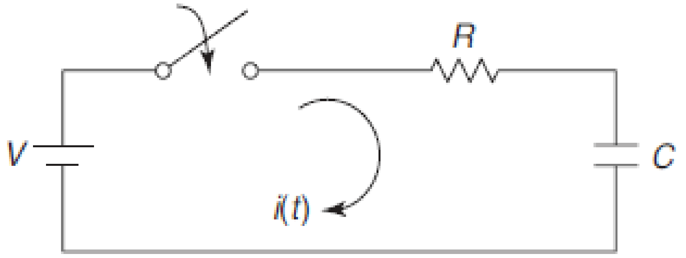
$$
\begin{aligned}
i_1(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\
&= e^{-833t} \int 3333\, e^{833t}\, dt + k\, e^{-833t} \\
&= \frac{3333}{833} + k\, e^{-833t} \\
&= 4 + k\, e^{-833t}
\end{aligned}
$$

At $t = 0$, $i_1(0) = 0$

$$
\begin{aligned}
0 &= 4 + k \\
k &= -4 \\
i_1(t) &= 4 - 4\, e^{-833t} \\
&= 4(1 - e^{-833t}) \qquad \text{for } t > 0 \\
i_2(t) &= \frac{100 - 10 i_1}{15} \\
&= \frac{100 - 10(4 - 4\, e^{-833t})}{15} \\
&= 4 + 2.67\, e^{-833t} \qquad \text{for } t > 0
\end{aligned}
$$

RESISTOR–CAPACITOR CIRCUIT

Consider a series RC circuit as shown in Fig. The switch is closed at time $t = 0$. The capacitor is initially uncharged. Applying KVL to the circuit for $t > 0$,



Series RC circuit

$$V - Ri - \frac{1}{C}\int_0^t i\,dt = 0$$

Differentiating the above equation,

$$0 - R\frac{di}{dt} - \frac{i}{C} = 0$$

$$\frac{di}{dt} + \frac{1}{RC}i = 0$$

This is a linear differential equation of first order. The variables may be separated to solve the equation.

$$\frac{di}{i} = -\frac{dt}{RC}$$

Integrating both the sides,

$$l_n i = -\frac{1}{RC}t + k$$

The constant k can be evaluated from initial condition. In the circuit shown, the switch is closed at $t = 0$. Since the capacitor never allows sudden change in voltage, it will act as short circuit at $t = 0^+$. Hence, current in the circuit at $t = 0^+$ is $\frac{V}{R}$.

$$l_n \frac{V}{R} = k$$

Setting $i = \frac{V}{R}$ at $t = 0$,

$$l_n i = -\frac{1}{RC}t + l_n \frac{V}{R}$$

$$l_n i - l_n \frac{V}{R} = -\frac{1}{RC}t$$

$$l_n \frac{i}{\left(\frac{V}{R}\right)} = -\frac{1}{RC}t$$

$$\frac{i}{\frac{V}{R}} = e^{-\frac{1}{RC}t}$$

$$i = \frac{V}{R}e^{-\frac{1}{RC}t} \quad \text{for } t > 0$$

The term RC is called time constant and is denoted by T.

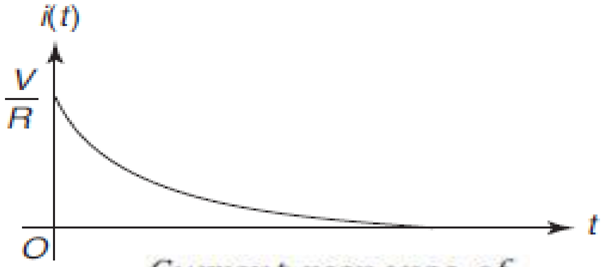
$$T = RC$$

After 5 time constants, the current drops to 99 per cent of its initial value.

The voltage across the resistor is

$$v_R = Ri = R\frac{V}{R}e^{-\frac{1}{RC}t}$$

$$= Ve^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$



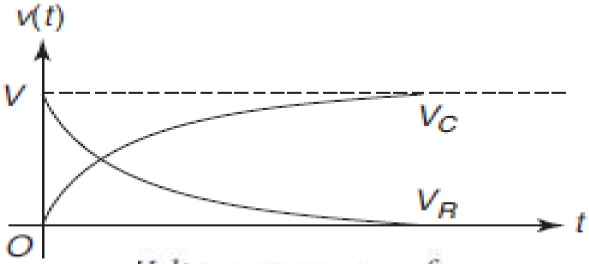
Current response of series RC circuit

Similarly, the voltage across the capacitor is

$$v_C = \frac{1}{C}\int_0^t i\,dt$$

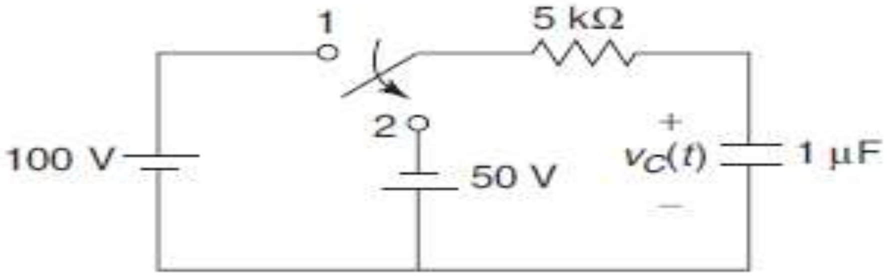
$$= \frac{1}{C}\int_0^t \frac{V}{R}e^{-\frac{1}{RC}t}$$

$$= -Ve^{-\frac{1}{RC}t} + k$$



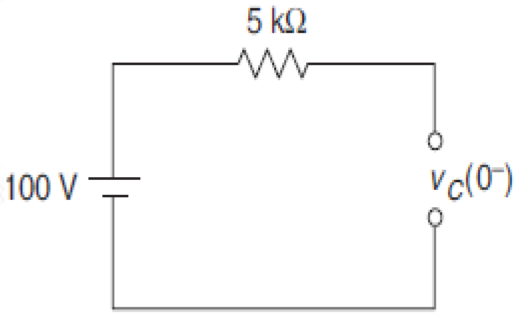
Voltage response of series RC circuit

The switch in the circuit of Fig. is moved from the position 1 to 2 at $t = 0$. Find $v_C(t)$.



Solution At $t = 0^-$, the network is shown in Fig.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

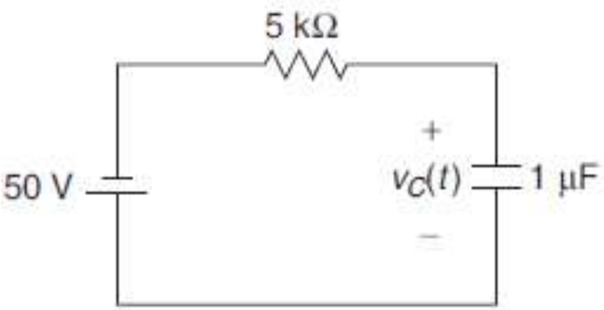


$$v_C(0^-) = 100 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 100 \text{ V}$$

For $t > 0$, the network is shown in Fig.
Writing the KCL equation for $t > 0$,



$$1 \times 10^{-6} \frac{dv_C}{dt} + \frac{v_C + 50}{5000} = 0$$

$$\frac{dv_C}{dt} + 200 v_C = 10^4$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 200, \quad Q = 10^4$$

Solution of this differential equation is given by,

$$v_C(t) = e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

Solution of this differential equation is given by,

$$v_C(t) = e^{-Pt}\int Q\, e^{Pt}\, dt + k\, e^{-Pt}$$

$$= e^{-200t}\int 10^4\, e^{200t}\, dt + k\, e^{-200t}$$

$$= \frac{10^4}{200} + k\, e^{-200t}$$

$$= -50 + k\, e^{-200t}$$

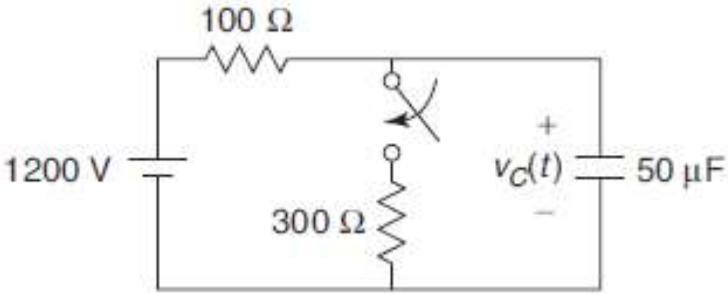
At $t = 0$, $v_C(0) = 100$ V

$$100 = -50 + k$$

$$k = 150$$

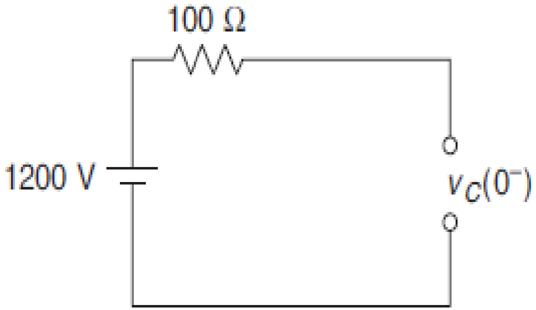
$$v_C(t) = -50 + 150\, e^{-200t} \qquad \text{for } t > 0$$

For the network shown in Fig. the switch is open for a long time and closes at $t = 0$. Determine $v_C(t)$.



Solution At $t = 0^-$, the network is shown in Fig.

At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.

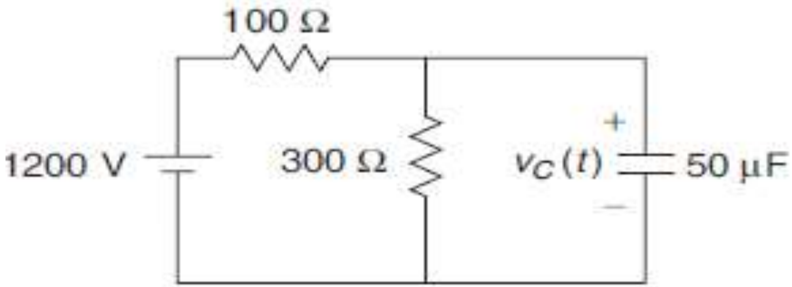


$$v_C(0^-) = 1200 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 1200 \text{ V}$$

For $t > 0$, the network is shown in Fig.
Writing the KCL equation for $t > 0$,



$$50\times10^{-6}\frac{dv_C}{dt}+\frac{v_C}{300}+\frac{v_C-1200}{100}=0$$

$$\frac{dv_C}{dt}+266.67\,v_C=0.24\times10^6$$

Comparing with the differential equation $\frac{dv}{dt}+Pv=Q$,

$$P=266.67,\quad Q=0.24\times10^6$$

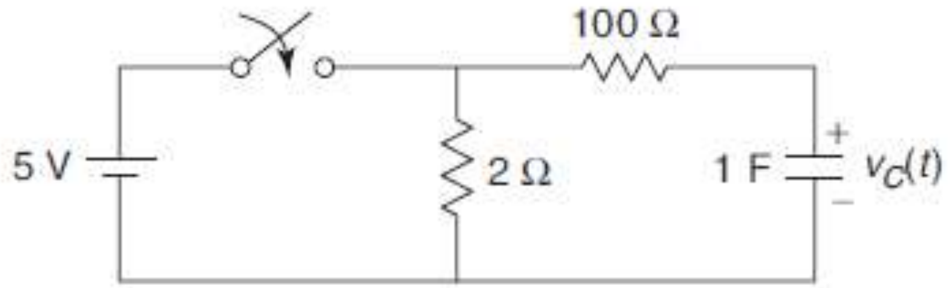
The solution of this differential equation is given by,

$$\begin{aligned}
v_C(t) &= e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt} \\
&= e^{-266.67t}\int 0.24\times 10^{6}\,e^{266.67t}\,dt + k\,e^{-266.67t} \\
&= \frac{0.24\times 10^{6}}{266.67} + k\,e^{-266.67t} \\
&= 900 + k\,e^{-266.67t}
\end{aligned}$$

At $t = 0, v_C(0) = 1200$ V

$$\begin{aligned}
1200 &= 900 + k \\
k &= 300 \\
v_C(t) &= 900 + 300\,e^{-266.67t} \qquad \text{for } t > 0
\end{aligned}$$

In Fig. 6.136, the switch is closed at $t = 0$ Find $v_C(t)$ for $t > 0$.

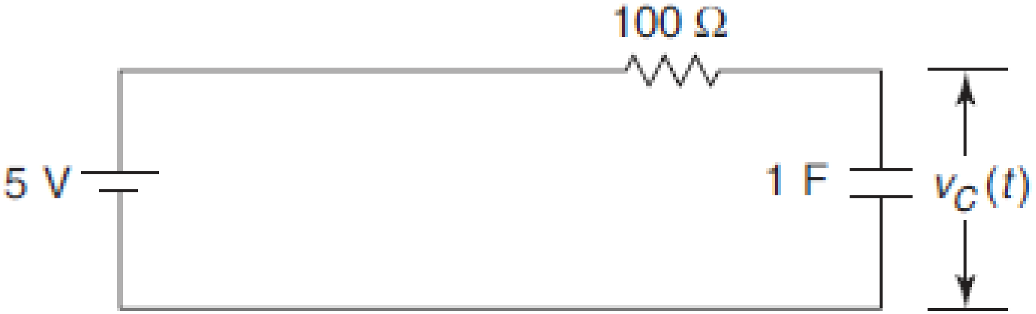


Solution At $t = 0^-$, $\quad v_C(0^-) = 0$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

Since the resistor of 2 Ω is connected in parallel with the voltage source of 5 V, it becomes redundant.



For $t > 0$, the network is as shown in Fig.
Writing KCL equation for $t > 0$,

$$\frac{v_C - 5}{100} + 1\frac{dv_C}{dt} = 0$$

$$100\frac{dv_C}{dt} + v_C = 5$$

$$\frac{dv_C}{dt} + 0.01v_C = 0.05$$

Comparing with the differential equation $\dfrac{dv}{dt} + Pv = Q$,

$$P = 0.01, \quad Q = 0.05$$

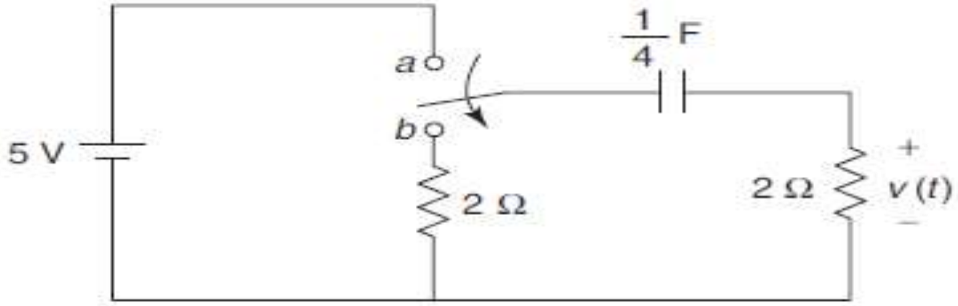
The solution of this differential equation is given by,

$$\begin{aligned}
v_C(t) &= e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt}\\
&= e^{-0.01}\int 0.05\,e^{0.01t}\,dt + k\,e^{-0.01t}\\
&= \frac{0.05}{0.01} + k\,e^{-0.01t}\\
&= 5 + k\,e^{-0.01t}
\end{aligned}$$

At $t = 0, v_C(0) = 0$

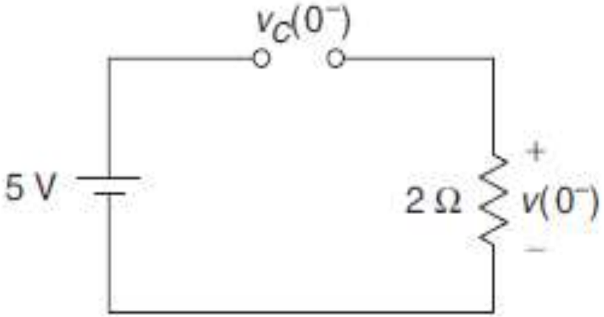
$$\begin{aligned}
0 &= 5 + k\\
k &= -5\\
v_C(t) &= 5 - 5\,e^{-0.01t}\\
&= 5(1 - e^{-0.01t}) \qquad \text{for } t > 0
\end{aligned}$$

In the network shown, the switch is shifted to position b at t = 0. Find v (t) for t > 0.



Solution At $t = 0^-$, the network is shown in Fig.

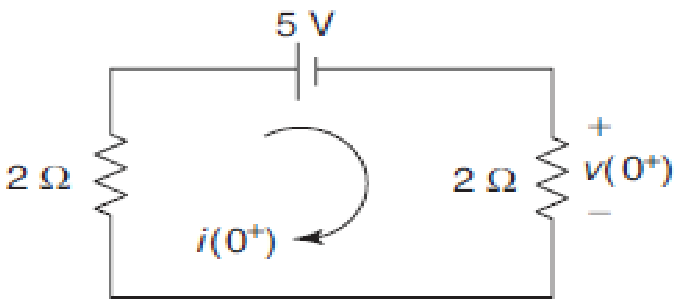
At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor acts as an open circuit.



$$v_C(0^-) = 5 \text{ V}$$

$$v(0^-) = 0$$

At $t = 0^+$, the network is shown in Fig.

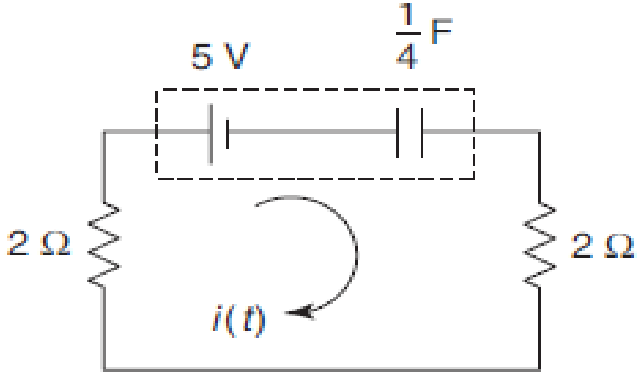


At $t = 0^+$, the capacitor acts as a voltage source of 5 V.

$$i(0^+) = -\frac{5}{4} = -1.25 \text{ A}$$

$$v(0^+) = -1.25 \times 2 = -2.5 \text{ V}$$

For $t > 0$, the network is shown in Fig.



Writing the KVL equation for $t > 0$,

$$-2i - 5 - \frac{1}{\frac{1}{4}}\int_0^t i\,dt - 2i = 0 \qquad \ldots\text{(i)}$$

Differentiating Eq. (i),

$$-4\frac{di}{dt} - 4i = 0$$

$$\frac{di}{dt} + i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 1$$

The solution of this differential equation is given by,

$$i(t) = ke^{-Pt} = ke^{-t}$$

At $t = 0, i(0) = -1.25$ A

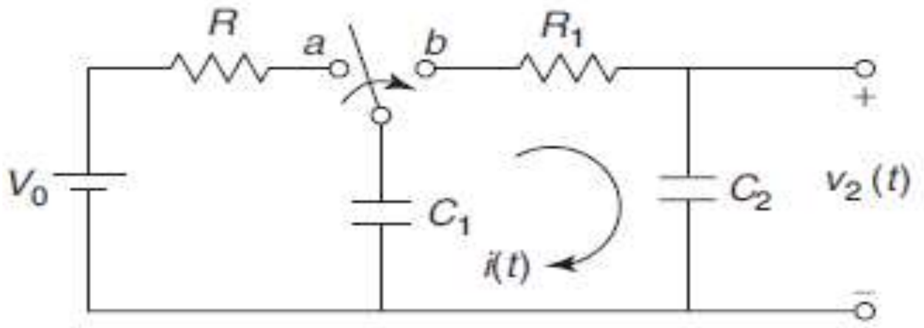
$$k = -1.25$$

$$i(t) = -1.25\, e^{-t} \quad \text{for } t > 0$$

$$v(t) = 2i(t)$$

$$= -2.5e^{-t} \qquad \text{for } t > 0$$

The switch is moved from the position a to b at $t = 0$, having been in the position a for a long time before $t = 0$. The capacitor C_2 is uncharged at $t = 0$. Find $i(t)$ and $v_2(t)$ for $t > 0$.



Solution At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor C_1 acts as an open circuit and it will charge to V_0 volt.

$$v_{C_1}(0^-) = V_0$$

$$v_{C_2}(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

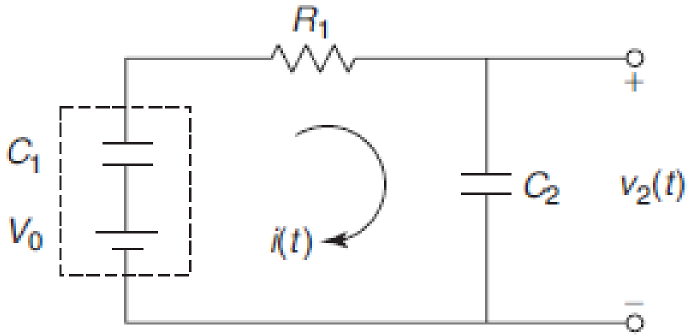
$$v_{C_1}(0^+) = V_0$$

$$v_{C_2}(0^+) = 0$$

$$i(0^+) = \frac{V_0}{R_1}$$

For $t > 0$, the network is shown in Fig.
Writing the KVL equation for $t > 0$,

$$V_0 - \frac{1}{C_1}\int_0^t i\,dt - R_1 i - \frac{1}{C_2}\int_0^t i\,dt = 0 \qquad \text{...(i)}$$



Differentiating Eq. (i),

$$-\frac{i}{C_1}-R_1\frac{di}{dt}-\frac{i}{C_2}=0$$

$$\frac{di}{dt}+\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)i=0$$

Comparing with the differential equation $\frac{di}{dt}+Pi=0$,

$\therefore$

$$P=\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)$$

The solution of this differential equation is given by,

$$i(t)=k\,e^{-Pt}=k\,e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1 C_2}\right)t}$$

At $t = 0$, $i(0) = \dfrac{V_0}{R_1}$

$$k = \frac{V_0}{R_1}$$

$$i(t) = \frac{V_0}{R_1} e^{-\frac{1}{R_1}\left(\frac{C_1+C_2}{C_1C_2}\right)t}$$

$$= \frac{V_0}{R_1} e^{-\frac{1}{R_1C}t} \qquad \text{where, } C = \frac{C_1C_2}{C_1+C_2}$$

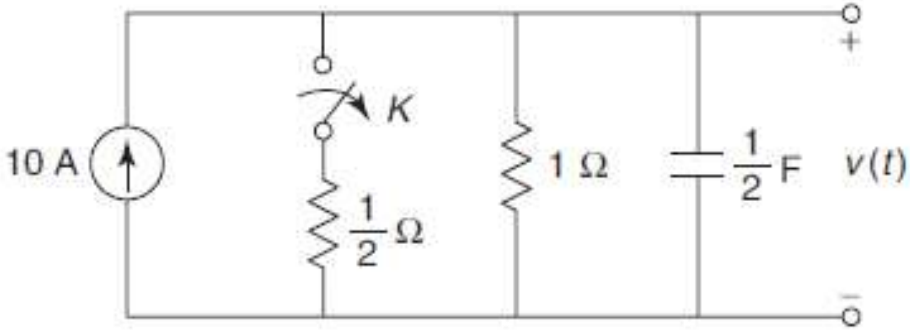
$$v_2(t) = \frac{1}{C_2}\int_0^t i\, dt$$

$$= \frac{1}{C_2}\int_0^t \frac{V_0}{R_1} e^{-\frac{t}{R_1C}} dt$$

$$= \frac{V_0}{R_1C_2} R_1C\left(1 - e^{-\frac{1}{R_1C}t}\right)$$

$$= \frac{V_0C_1}{C_1+C_2}\left(1 - e^{-\frac{t}{R_1C}t}\right) \qquad \text{for } t > 0$$

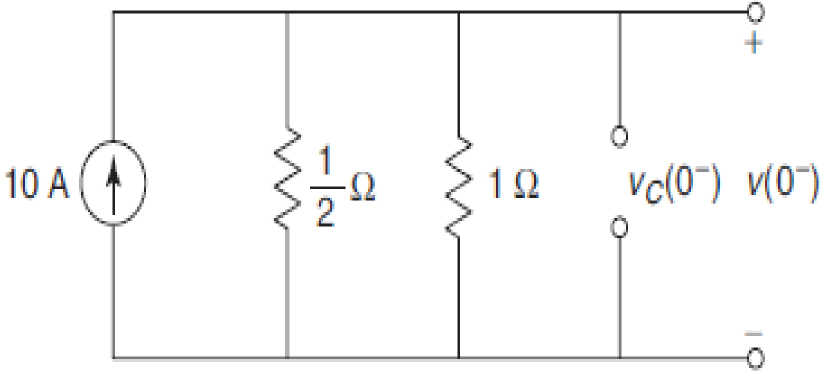
For the network shown in Fig. the switch is opened at $t = 0$. Find $v(t)$ for $t > 0$.



Solution At $t = 0^-$, the network is shown in Fig.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.

$$v_C(0^-) = 0$$



Writing the KCL equation at $t = 0^-$,

$$\frac{v(0^-)}{1} + \frac{v(0^-)}{\frac{1}{2}} = 10$$

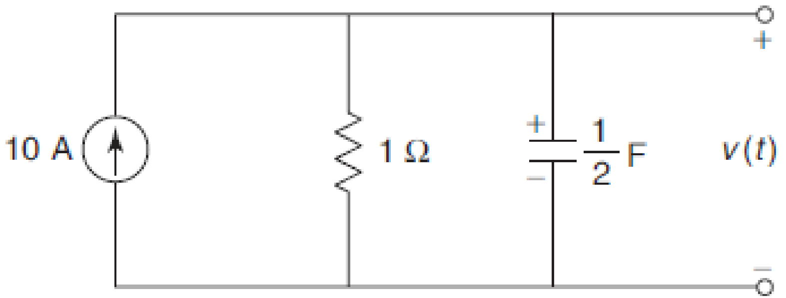
$$3v(0^-) = 10$$

$$v(0^-) = 3.33 \text{ V}$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = v(0^+) = 3.33 \text{ V}$$

For $t > 0$, the network is shown in Fig.



Writing the KCL equation for $t > 0$,

$$\frac{1}{2}\frac{dv}{dt} + \frac{v}{1} = 10$$

$$\frac{dv}{dt} + 2v = 20$$

Comparing with the differential equation $\frac{dv}{dt} + Pv = Q$,

$$P = 2, \quad Q = 20$$

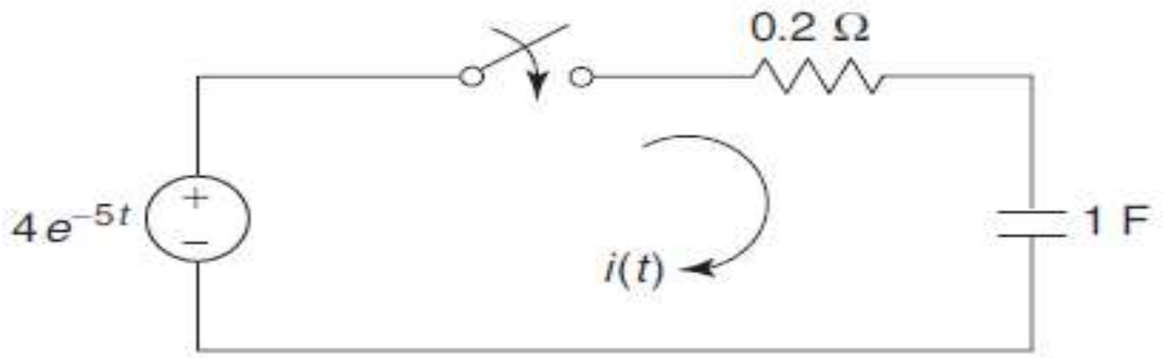
The solution of this differential equation is given by,

$$\begin{aligned} v(t) &= e^{-Pt} \int Q\, e^{Pt}\, dt + k\, e^{-Pt} \\ &= e^{-2t} \int 20\, e^{-2t}\, dt + k\, e^{-2t} \\ &= \frac{20}{2} + k\, e^{-2t} \\ &= 10 + k\, e^{-2t} \end{aligned}$$

At $t = 0, \quad v(0) = 3.33$ V

$$\begin{aligned} 3.33 &= 10 + k \\ k &= 6.67 \\ v(t) &= 10 + 6.67\, e^{-2t} \end{aligned}$$

In the network of Fig. an exponential voltage $4e^{-5t}$ is applied at time $t=0$. Find the expression for current $i(t)$. Assume zero voltage across the capacitor at $t=0$.

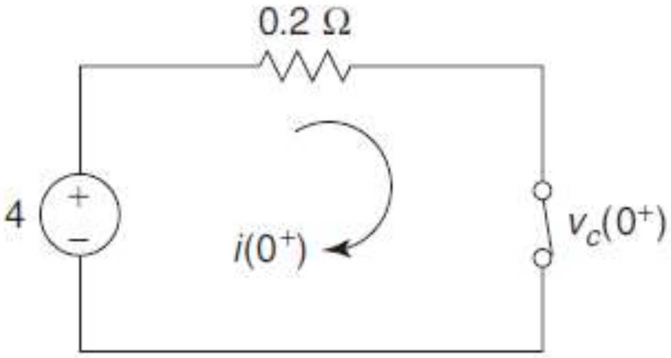


Solution At $t=0^-$,

$$v_C(0^-)=0$$

$$i(0^-)=0$$

At $t = 0^+$, the network is shown in Fig.



Since voltage across the capacitor cannot change instantaneously,

$$v_C(0^+) = 0$$

$$i(0^+) = \frac{4}{0.2} = 20 \text{ A}$$

Writing the KVL equation for $t > 0$,

$$4e^{-5t} - 0.2i - \frac{1}{1}\int_0^t i\,dt = 0$$

Differentiating Eq. (i),

$$-20e^{-5t} - 0.2\frac{di}{dt} - i = 0$$

$$\frac{di}{dt} + 5i = -100e^{-5t}$$

Comparing with the differential equation $\dfrac{di}{dt} + Pi = Q$,

$$P = 5, \quad Q = -100\,e^{-5t}$$

The solution of this differential equation is given by,

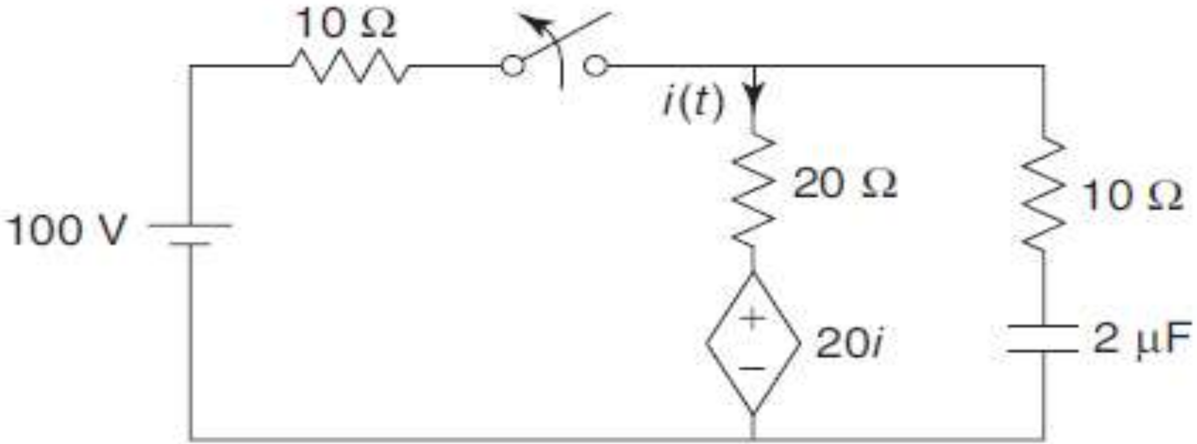
$$
\begin{aligned}
i(t) &= e^{-Pt}\int Q\,e^{Pt}\,dt + k\,e^{-Pt} \\
&= e^{-5t}\int -100\,e^{-5t}\,e^{5t}\,dt + k\,e^{-5t} \\
&= -100t\,e^{-5t} + \boldsymbol{k}\,e^{-5t}
\end{aligned}
$$

At $t = 0, \quad i(0) = 20$ A

$$20 = k$$

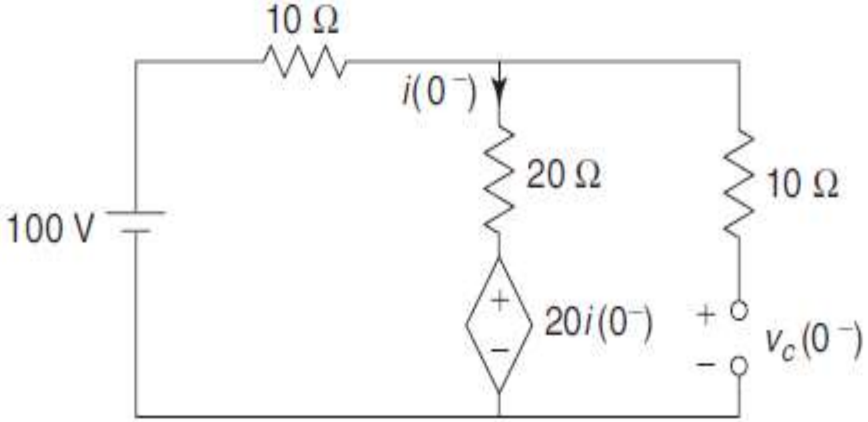
$$i(t) = -100\,t\,e^{-5t} + 20\,e^{-5t} \qquad \text{for } t > 0$$

For the network shown in Fig. find the current i(t) when the switch is opened at t = 0.



Solution At $t = 0^-$, the network is shown in Fig.

At $t = 0^-$, the network attains steady-state condition. Hence, the capacitor acts as an open circuit.



Writing the KVL equation at $t = 0^-$,

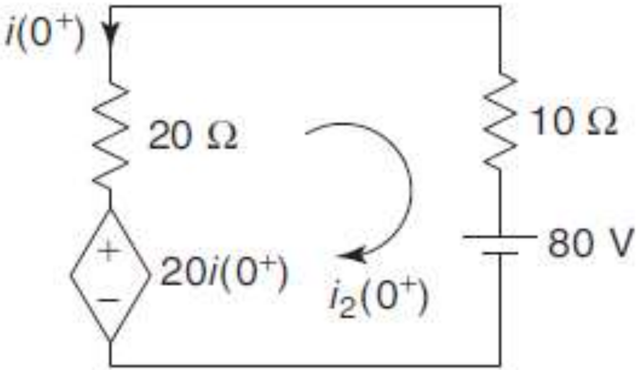
$$100 - 10i(0^-) - 20i(0^-) - 20i(0^-) = 0$$

$$i(0^-) = 2 \text{ A}$$

Also, $$20\,i(0^-) + 20\,i(0^-) - 0 - v_C(0^-) = 0$$

$$v_C(0^-) = 40\,i(0^-) = 40(2) = 80 \text{ V}$$

At $t = 0^+$, the network is shown in Fig.



From Fig. $i(0^+) = -i_2(0^+)$

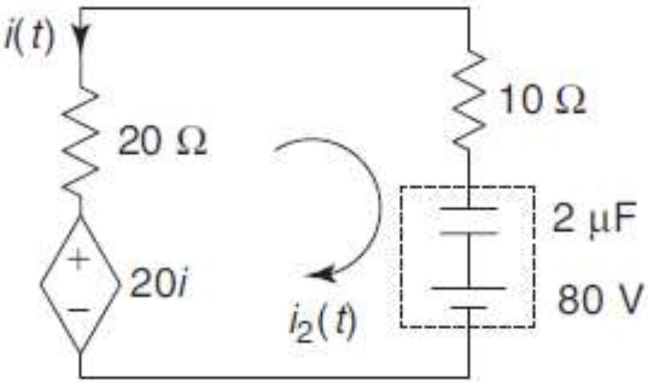
$$20i(0^+) - 20i_2(0^+) - 10i_2(0^+) - 80 = 0$$

$$20i(0^+) + 20i(0^+) + 10i(0^+) - 80 = 0$$

$$i(0^+) = 1.6 \text{ A}$$

$$v_C(0^+) = 80 \text{ V}$$

For $t > 0$, the network is shown in Fig.



From Fig. $i(t) = -i_2(t)$

Writing the KVL equation for $t > 0$,

$$20i - 20i_2 - 10i_2 - \frac{1}{2\times10^{-6}}\int_0^t i_2\, dt - 80 = 0$$

$$20i + 20i_2 + 10i_2 + \frac{1}{2\times10^{-6}}\int_0^t i\,dt - 80 = 0$$

$$50i + \frac{1}{2\times10^{-6}}\int i\,dt - 80 = 0$$

Differentiating Eq. (i),

$$50\frac{di}{dt} + 5\times10^5 i = 0$$

$$\frac{di}{dt} + 1\times10^4 i = 0$$

Comparing with the differential equation $\frac{di}{dt} + Pi = 0$,

$$P = 1\times10^4$$

The solution of this differential equation is given by,

$$i(t) = k\,e^{-Pt} = k\,e^{-1\times10^4 t}$$

At $t = 0$, $i(0) = 1.6$ A

$$1.6 = k$$

$$i(t) = 1.6e^{-1\times10^4 t} \qquad \text{for } t > 0$$

Frequency Domain Analysis of RLC Circuits

LAPLACE TRANSFORMATION

The Laplace transform of a function $f(t)$ is defined as

$$F(s) = L\{f(t)\} = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t$$

where s is the complex frequency variable.

$$s = \sigma + j\omega$$

The function $f(t)$ must satisfy the following condition to possess a Laplace transform,

$$\int_0^\infty |f(t)|e^{-\sigma t}\,\mathrm{d}t < \infty$$

where σ is real and positive.

The inverse Laplace transform $L^{-1}\{F(s)\}$ is

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} \, ds$$

LAPLACE TRANSFORMS OF SOME IMPORTANT FUNCTIONS

Constant Function k

The Laplace transform of a constant function is

$$L\{k\} = \int_0^\infty k e^{-st} dt = k \left[\frac{e^{-st}}{-s} \right]_0^\infty = \frac{k}{s}$$

Function t^n

The Laplace transform of $f(t)$ is

$$L\{t^n\} = \int_0^\infty t^n e^{-st}\,dt$$

Putting $st = x,\ dt = \dfrac{dx}{s}$

$$L\{t^n\} = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x}\,\frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty x^n e^{-x}\,dx = \frac{\sqrt{n+1}}{s^{n+1}}, s > 0, n+1 > 0$$

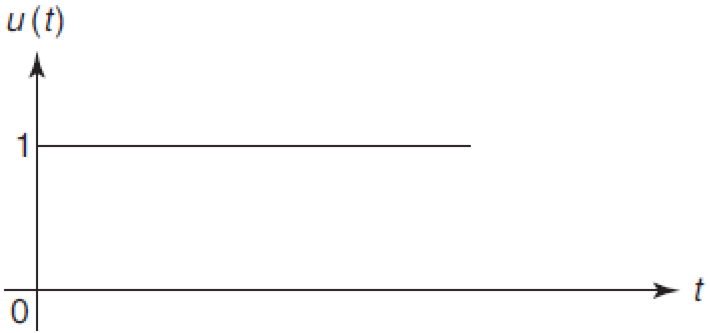
If n is a positive integer, $\sqrt{n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

Unit-Step Function

The unit-step function (Fig 7.1) is defined by the equation,

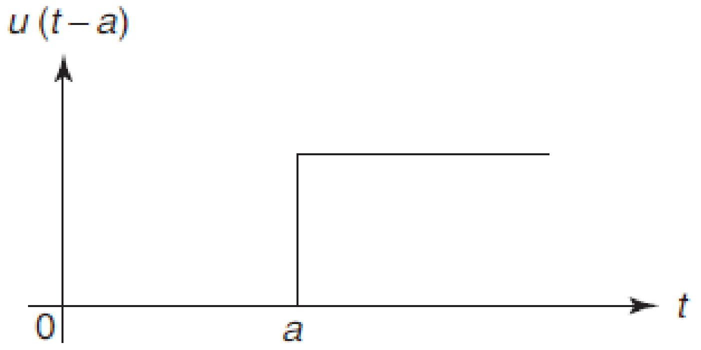
$$u(t) = 1 \qquad t > 0$$
$$= 0 \qquad t < 0$$



The Laplace transform of unit step function is

$$L\{u(t)\} = \int_0^\infty 1 \cdot e^{-st}\, dt = \left[-\frac{e^{-st}}{s} \right]_0^\infty = \frac{1}{s}$$

Delayed or Shifted Unit-Step Function



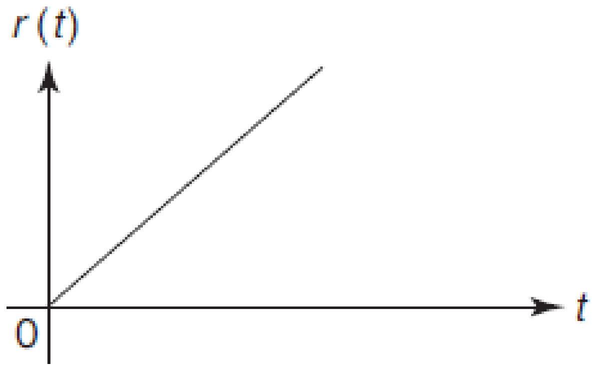
The delayed or shifted unit-step function (Fig 7.2) is defined by the equation

$$u(t-a) = 1 \qquad t > a$$
$$= 0 \qquad t < a$$

The Laplace transform of $u(t-a)$ is

$$L\{u(t-a)\} = \int_a^{\infty} 1 \cdot e^{-st}\, dt = \left[-\frac{e^{-st}}{s}\right]_a^{\infty} = \frac{e^{-as}}{s}$$

Unit-Ramp Function



The unit-ramp function (Fig 7.3) is defined by the equation

$$r(t) = t \qquad t > 0$$
$$= 0 \qquad t < 0$$

The Laplace transform of the unit-ramp function is

$$L\{r(t)\} = \int_0^{\infty} t\, e^{-st} = \frac{1}{s^2}$$

Delayed Unit-Ramp Function

The delayed unit-ramp function (Fig 7.4) is defined by the equation

$$r(t-a) = t \qquad t > a$$
$$= 0 \qquad t < a$$

The Laplace transform of $r(t-a)$ is

$$L\{r(t-a)\} = \int_a^{\infty} t\, e^{-st}\, dt = \frac{e^{-as}}{s^2}$$

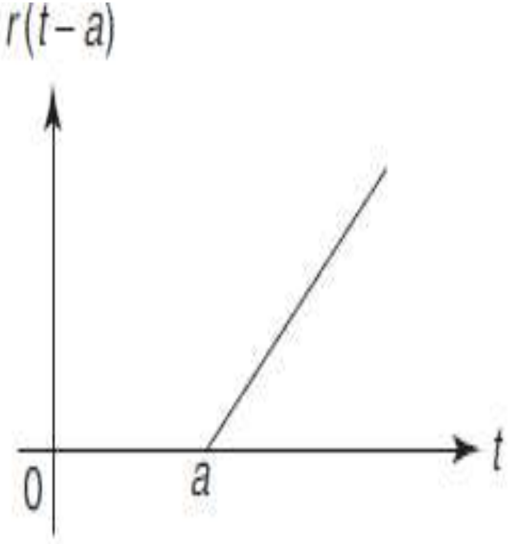


Fig. 7.4 *Delayed unit-ramp function*

Unit-Impulse Function

The unit-impulse function (Fig 7.5) is defined by the equation

$$\delta(t) = 0 \qquad t \neq 0$$

and
$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad t = 0$$

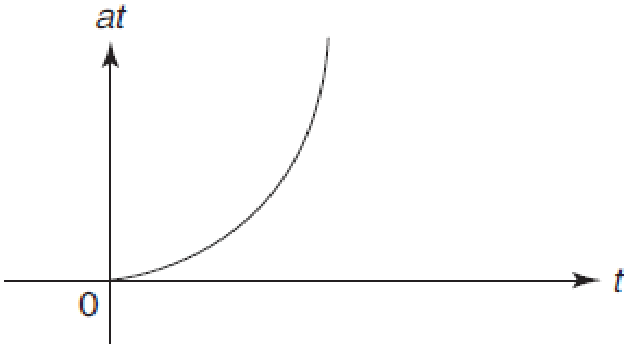
The Laplace transform of the unit-impulse function is

$$L\{\delta(t)\} = \int_{0}^{\infty} \delta(t) e^{-st}\, dt = 1$$

Exponential Function (e^{at})

The Laplace transform of the exponential function (Fig 7.6) is

$$L\{e^{at}\} = \int_0^\infty e^{at} e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt = \left[-\frac{e^{-(s-a)t}}{s-a}\right]_0^\infty = \frac{1}{s-a}$$



Sine Function

We know that $\sin \omega t = \frac{1}{2j}\left[e^{j\omega t} - e^{-j\omega t}\right]$.

The Laplace transform of the sine function is

$$L\{\sin \omega t\} = L\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\} = \frac{1}{2j}\left[L\{e^{j\omega t}\} - L\{e^{-j\omega t}\}\right] = \frac{1}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{\omega}{s^2+\omega^2}$$

Cosine Function

We know that $\cos \omega t = \frac{1}{2}\left[e^{j\omega t} + e^{-j\omega t}\right]$.

The Laplace transform of the cosine function is

$$L\{\cos \omega t\} = L\left\{\frac{1}{2}(e^{j\omega t} + e^{-j\omega t})\right\} = \frac{1}{2}\left[L\{e^{j\omega t}\} + L\{e^{-j\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right] = \frac{s}{s^2+\omega^2}$$

Hyperbolic sine function

We know that $\sinh \omega t = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic sine function is

$$L\{\sinh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} - e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} - L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} - \frac{1}{s+\omega}\right] = \frac{\omega}{s^2-\omega^2}$$

Hyperbolic cosine function

We know that $\cosh \omega t = \frac{1}{2}\left(e^{\omega t} + e^{-\omega t}\right)$.

The Laplace transform of the hyperbolic cosine function is

$$L\{\cosh \omega t\} = L\left\{\frac{1}{2}(e^{\omega t} + e^{-\omega t})\right\} = \frac{1}{2}\left[L\{e^{\omega t}\} + L\{e^{-\omega t}\}\right] = \frac{1}{2}\left[\frac{1}{s-\omega} + \frac{1}{s-\omega}\right] = \frac{s}{s^2 - \omega^2}$$

Exponentially Damped Function

Laplace transform of an exponentially damped function $e^{-at} f(t)$ is

$$L\left\{e^{-at} f(t) dt\right\} = \int_{0^-}^{\infty} f(t) e^{-at} e^{-st}\, dt = \int_{0^-}^{\infty} f(t) e^{-(s+a)t}\, dt = F(s+a)$$

Thus, the transform of the function $e^{-at} f(t)$ is obtained by putting $(s+a)$ in place of s in the transform of $f(t)$.

$$L\left\{e^{-at} \sin \omega t\right\} = \frac{\omega}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \sinh \omega t\right\} = \frac{\omega}{(s+a)^2 - \omega^2}$$

$$L\left\{e^{-at} \cos \omega t\right\} = \frac{s+a}{(s+a)^2 + \omega^2} \qquad L\left\{e^{-at} \cosh \omega t\right\} = \frac{s+a}{(s+a)^2 - \omega^2}$$

PROPERTIES OF LAPLACE TRANSFORM

Linearity

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$ then $L\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)$

where a and b are constants.

Proof

$$L\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

$$L\{af_1(t) + bf_2(t)\} = \int_0^\infty \{af_1(t) + bf_2(t)\} e^{-st} dt = a\int_0^\infty f_1(t) e^{-st} dt + b\int_0^\infty f_2(t) e^{-st}\, dt = aF_1(s) + bF_2(s)$$

Time Scaling

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

Proof

$$L\{f(t)\} = \int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$L\{f(at)\} = \int_0^{\infty} f(at) e^{-st}\, dt$$

Putting $at = x,\ dt = \frac{dx}{a}$

$$L\{(f(at)\} = \int_0^{\infty} f(x)\, e^{-s\left(\frac{x}{a}\right)} \frac{dx}{a} = \frac{1}{a}\int_0^{\infty} f(x)\, e^{-\left(\frac{s}{a}\right)x}\, dx = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Frequency-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{e^{-at} f(t)\} = F(s+a)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t)t\, e^{-st}\, d$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-at} f(t) e^{-st}\, dt = \int_0^\infty f(t) e^{-(s+a)t}\, dt = F(s+a)$$

Time-Shifting Theorem

If $L\{f(t)\} = F(s)$ then $L\{f(t-a)\} = e^{-as} F(s)$

Proof

$$L\{f(t)\} = \int_0^\infty f(t) e^{-st} \, dt$$

$$L\{f(t-a)\} = \int_0^\infty f(t-a)\, e^{-st} \, dt$$

Putting $\quad t - a = x, \quad dt = dx$

When $\quad t = a, \quad x = 0$

$t \to \infty, \quad x \to \infty$

$$L\{f(t-a)\} = \int_0^\infty f(x) e^{-s(a+x)} \, dx = e^{-as} \int_0^\infty f(x) e^{-sx} \, dx = e^{-as} \int_0^\infty f(t)\, e^{-st} \, dt = e^{-as} F(s)$$

Multiplication by *t* (Frequency-Differentiation Theorem)

If $L\{f(t)\} = F(s)$ then $L\{t\,f(t)\} = -\dfrac{d}{ds}F(s)$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

Differentiating both the sides w.r.t s using DUIS,

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = \int_0^\infty \frac{\partial}{\partial s} f(t)e^{-st}\,dt$$

$$= \int_0^\infty (-t)\,f(t)e^{-st}\,dt = \int_0^\infty \{-t\,f(t)\}e^{-st}dt = -L\{t\,f(t)\}$$

$$L\{t\,f(t)\} = (-1)\frac{d}{ds}F(s)$$

Division by t (Frequency-Integration Theorem)

If $L\{f(t)\} = F(s)$, then $L\left\{\dfrac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$

Proof

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$$

Integrating both the sides w.r.t s from s to ∞,

$$\int_s^\infty F(s)\,ds = \int_s^\infty \int_0^\infty f(t)e^{-st}\,dt\,ds$$

Since s and t are independent variables, interchanging the order of integration,

$$\int_s^\infty F(s)ds = \int_0^\infty \left[\int_s^\infty f(t)e^{-st}\,ds\right]dt = \int_0^\infty \left[\frac{1}{-t}f(t)e^{-st}\right]_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}\,dt$$

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds$$

Time-Differentiation Theorem: Laplace Transform of Derivatives

If $L\{f(t)\} = F(s)$ then

$$L\{f'(t)\} = sF(s) - f(0)$$

$$L\{f''(t)\} = s^2 F(s) - sf(0) - f'(0)$$

In general,

$$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{(n-1)}(0)$$

Proof

$$L\{f'(t)\} = \int_0^\infty f'(t)\, e^{-st}\, dt$$

Integrating by parts,

$$L\{f'(t)\} = \left[f(t) e^{-st} \right]_0^\infty - \int_0^\infty (-s) f(t) e^{-st} dt = -f(0) + s \int_0^\infty f(t) e^{-st} dt = -f(0) + sL\{f(t)\}$$

Similarly,

$$L\{f''\ t\} = -f' + sL\{f'\ t\} = -f'(0) + s[-f(0) + sL\{f(t)\}] = -f'(0) - sf(0) + \mathbf{s}^2 L\{f(t)\}$$

In general, $$L\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - s^{n-3} f''(0) \ldots - f^{n-1}(0)$$

Time-Integration Theorem: Laplace Transform of Integral

If $L\{f(t)\} = F(s)$ then $L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}$

Proof

$$L\left\{\int_0^t f(t)dt\right\} = \int_0^\infty \left\{\int_0^t f(t)\,dt\right\} e^{-st}\,dt$$

Integrating by parts,

$$L\left\{\int_0^t f(t)dt\right\} = \left[\int_0^t f(t)dt\left(\frac{e^{-st}}{-s}\right)\right]_0^\infty - \int_0^\infty \left[\left(\frac{e^{-st}}{-s}\right)\left(\frac{d}{dt}\int_0^t f(t)dt\right)\right]dt = \int_0^\infty \frac{1}{s} f(t)e^{-st}\,dt = \frac{1}{s}L\{f(t)\} = \frac{F(s)}{s}$$

Initial Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim\limits_{t \to 0} f(t) = \lim\limits_{s \to \infty} sF(s)$

Proof We know that,

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\, e^{-st} dt + f(0)$$

$$\lim_{s \to \infty} sF(s) = \lim_{s \to \infty} \int_0^\infty f'(t) e^{-st} dt + f(0) = \int_0^\infty \lim_{s \to \infty} [f'(t) e^{-st}] dt + f(0) = 0 + f(0) = f(0) = \lim_{t \to 0} f(t)$$

Final Value Theorem

If $L\{f(t)\} = F(s)$ then $\lim\limits_{t \to \infty} f(t) = \lim\limits_{s \to \infty} sF(s)$

Proof We know that

$$L\{f'(t)\} = sF(s) - f(0)$$

$$sF(s) = L\{f'(t)\} + f(0) = \int_0^\infty f'(t)\, e^{-st} dt + f(0)$$

$$\lim_{s \to \infty} sF(s) = \lim_{s \to 0} \int_0^\infty f'(t)\, e^{-st} dt + f(0) = \int_0^\infty \lim_{s \to 0} \left[f'(t) e^{-st} \right] dt + f(0) = \int_0^\infty f'(t) dt + f(0)$$

$$= \left| f(t) \right|_0^\infty + f(0) = \lim_{t \to \infty} f(t) - f(0) + f(0) = \lim_{t \to \infty} f(t)$$

INVERSE LAPLACE TRANSFORM

If $L\{f(t)\} = F(s)$ then $f(t)$ is called inverse Laplace transform of $F(s)$ and symbolically written as

$$f(t) = L^{-1}\{F(s)\}$$

where L^{-1} is called the inverse Laplace transform operator.

Inverse Laplace transform can be found by the following methods:

(i) Standard results
(ii) Partial fraction expansion

Standard Results

Inverse Laplace transforms of some simple functions can be found by standard results and properties of Laplace transform.

Example

Find the inverse Laplace transform of $\dfrac{s^2-3s+4}{s^3}$.

Solution

$$F(s)=\frac{s^2-3s+4}{s^3}=\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}$$

$$L^{-1}\{F(s)\}=1-3t+2t^2$$

Example

Find the inverse Laplace transform of $\frac{3s+4}{s^2+9}$.

Solution

$$F(s)=\frac{3s+4}{s^2+9}=\frac{3s}{s^2+9}+\frac{4}{s^2+9}$$

$$L^{-1}\{F(s)\}=3\cos 3t+\frac{4}{3}\sin 3t$$

Example *Find the inverse Laplace transform of* $\frac{2s+2}{s^2+2s+10}$.

Solution

$$F(s)=\frac{2s+2}{s^2+2s+10}=\frac{2(s+1)}{(s+1)^2+9}$$

$$L^{-1}\{F(s)\}=2e^{-t}L^{-1}\left\{\frac{s}{s^2+9}\right\}=2e^{-t}\cos 3t$$

Partial Fraction Expansion

Find the inverse Laplace transform of $\dfrac{s+2}{s(s+1)(s+3)}$.

Solution

$$F(s)=\frac{s+2}{s(s+1)(s+3)}$$

By partial-fraction expansion,

$$F(s)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+3}$$

$$A=sF(s)\Big|_{s=0}=\frac{s+2}{(s+1)(s+3)}\Bigg|_{s=0}=\frac{2}{3}$$

$$B = (s+1)F(s)\Big|_{s=-1} = \left.\frac{s+2}{s(s+3)}\right|_{s=-1} = -\frac{1}{2}$$

$$C = (s+3)F(s)\Big|_{s=-3} = \left.\frac{s+2}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

$$F(s) = \frac{2}{3}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{6}\cdot\frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{2}{3}L^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{6}L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{2}{3} - \frac{1}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

Find the inverse Laplace transform of $\dfrac{s+2}{s^2(s+3)}$.

Solution

$$F(s) = \frac{s+2}{s^2(s+3)}$$

By partial-fraction expansion,

$$F(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

$$s+2 = As(s+3) + B(s+3) + Cs^2$$

$$= As^2 + 3As + Bs + 3B + Cs^2$$

$$= (A+C)s^2 + (3A+B)s + 3B$$

Comparing coefficients of s^2, s^1 and s^0,

$$A + C = 0$$

$$3A + B = 1$$

$$3B = 2$$

Solving these equations,

$$A = \frac{1}{9},\ B = \frac{2}{3},\ C = -\frac{1}{9}$$

$$F(s) = \frac{1}{9} \cdot \frac{1}{s} + \frac{2}{3} \cdot \frac{1}{s^2} - \frac{1}{9} \cdot \frac{1}{s+3}$$

$$L^{-1}\{F(s)\} = \frac{1}{9} L^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3} L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{9} L^{-1}\left\{\frac{1}{s+3}\right\} = \frac{1}{9} + \frac{2}{3} t - \frac{1}{9} e^{-3t}$$

FREQUENCY DOMAIN REPRESENTAION OF RLC CIRCUITS

Voltage–current relationships of network elements can also be represented in the frequency domain.

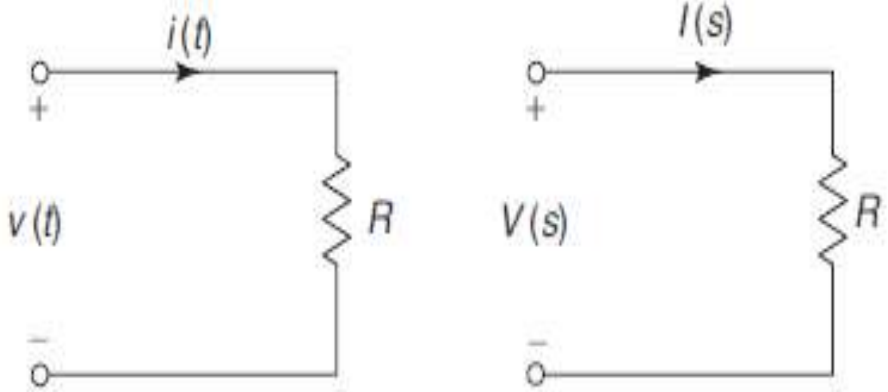
1. Resistor For the resistor, the v–i relationship in time domain is

$$v(t) = R\,i(t)$$

The corresponding frequency–domain relation are given as

$$V(s) = RI(s)$$

The transformed network is shown in Fig 7.7.



2. Inductor For the inductor, the v–i relationships in time domain are

$$v(t) = L\frac{di}{dt}$$

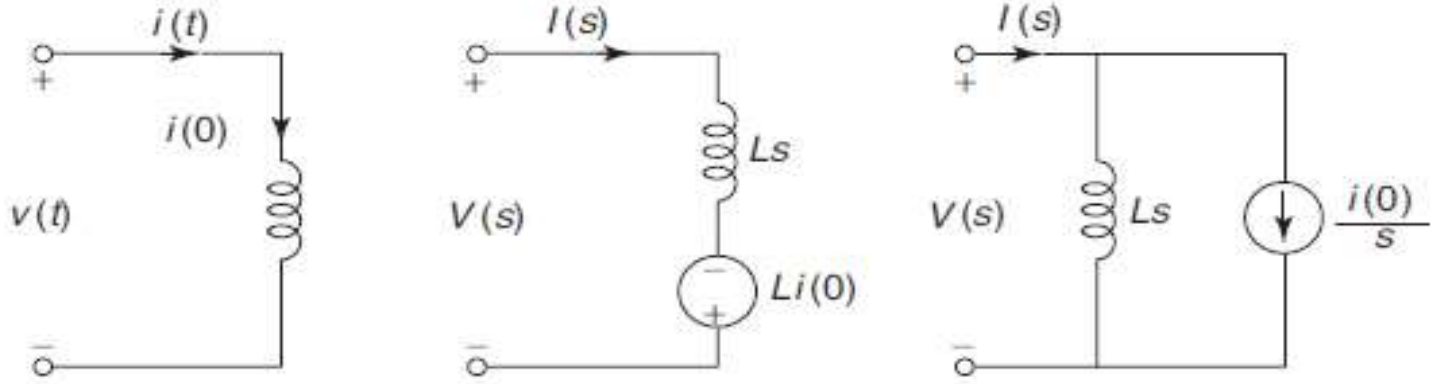
$$i(t) = \frac{1}{L}\int_0^t v(t)\,dt + i(0)$$

The corresponding frequency-domain relation are given as

$$V(s) = Ls\,I(s) - L\,i(0)$$

$$I(s) = \frac{1}{Ls}V(s) + \frac{i(0)}{s}$$

The transformed network is shown in Fig 7.8.



3. Capacitor For capacitor, the v–i relationships in time domain are

$$v(t) = \frac{1}{C}\int_0^t i(t)\,dt + v(0)$$

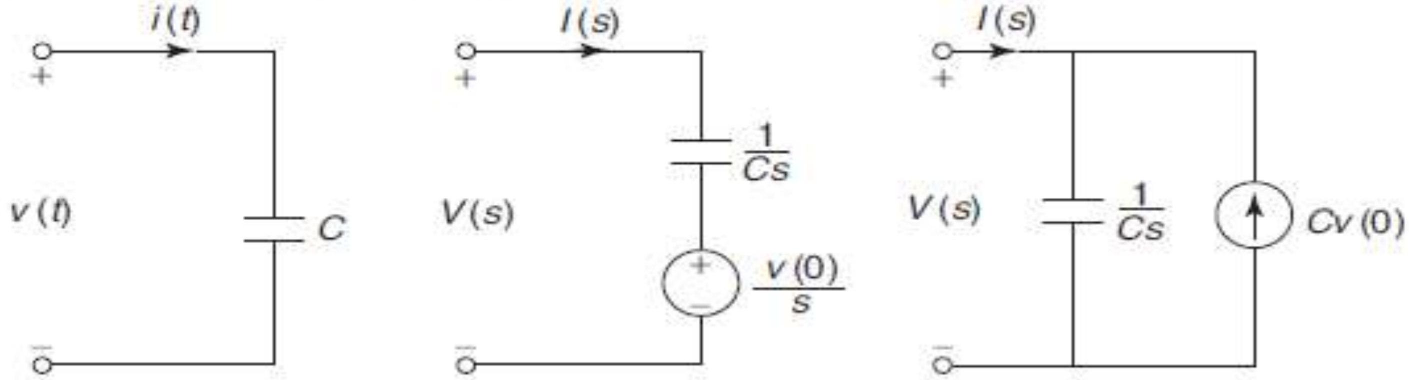
$$i(t) = C\frac{dv}{dt}$$

The corresponding frequency–domain relations are given as

$$V(s) = \frac{1}{Cs}I(s) + \frac{v(0)}{s}$$

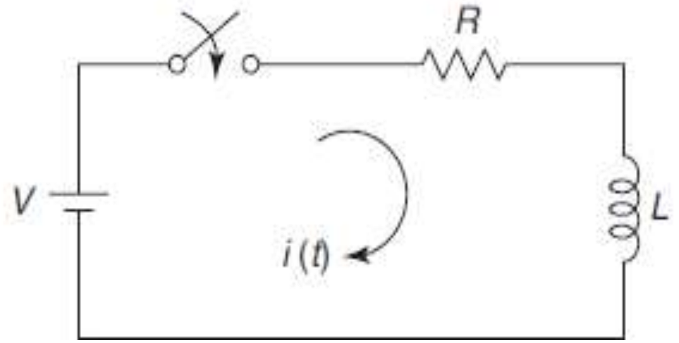
$$I(s) = CsV(s) - Cv(0)$$

The transformed network is shown in Fig 7.9.



RESISTOR–INDUCTOR CIRCUIT

Consider a series RL circuit as shown in Fig. 7.10. The switch is closed at time $t = 0$.



For $t > 0$, the transformed network is shown in Fig. 7.11.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - Ls\, I(s) = 0$$

$$I(s) = \frac{\frac{V}{L}}{s\left(s + \frac{R}{L}\right)}$$

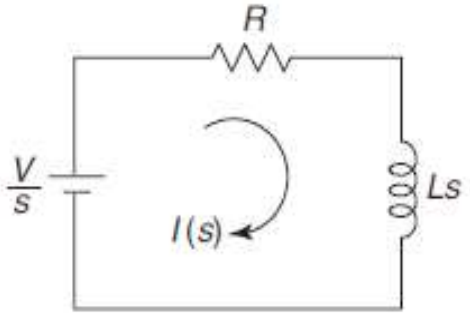


Fig. 7.11 *Transformed network*

By partial-fraction expansion,

$$I(s)=\frac{A}{s}+\frac{B}{s+\frac{R}{L}}$$

$$A=sI(s)\big|_{s=0}=s\times\left.\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\right|_{s=0}=\frac{V}{R}$$

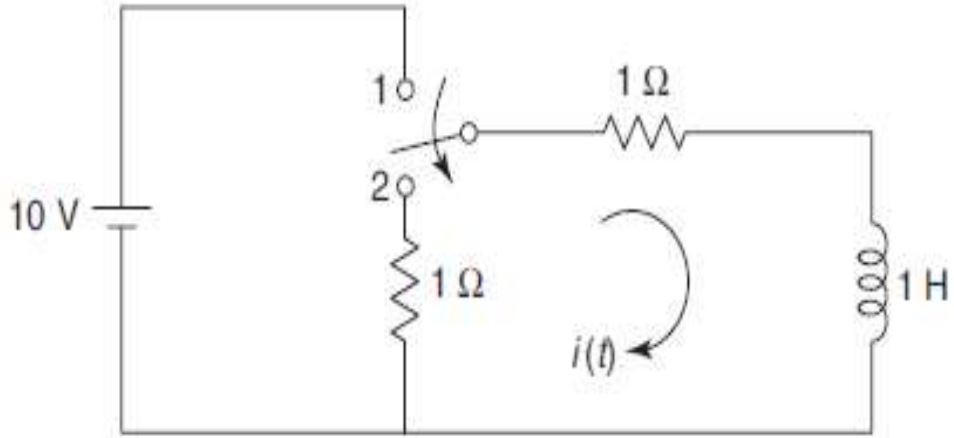
$$B=\left(s+\frac{R}{L}\right)I(s)\bigg|_{s=-\frac{R}{L}}=\left(s+\frac{R}{L}\right)\times\left.\frac{\frac{V}{L}}{s\left(s+\frac{R}{L}\right)}\right|_{s=-\frac{R}{L}}=-\frac{V}{R}$$

$$I(s)=\frac{\frac{V}{R}}{s}+\frac{\left(-\frac{V}{R}\right)}{s+\frac{R}{L}}$$

Taking the inverse Laplace transform,

$$
\begin{aligned}
i(t) &= \frac{V}{R} - \frac{V}{R} e^{-\frac{R}{L}t} \\
&= \frac{V}{R}\left[1 - e^{-\frac{R}{L}t}\right] \qquad \text{for } t > 0
\end{aligned}
$$

In the network of Fig. 7.12, the switch is moved from the position 1 to 2 at t = 0, steady-state condition having been established in the position 1. Determine i (t) for t > 0.



Solution At $t = 0^-$, the network is shown in Fig 7.13. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor acts as a short circuit.

$$i(0^-) = \frac{10}{1} = 10 \text{ A}$$

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = 10 \text{ A}$$

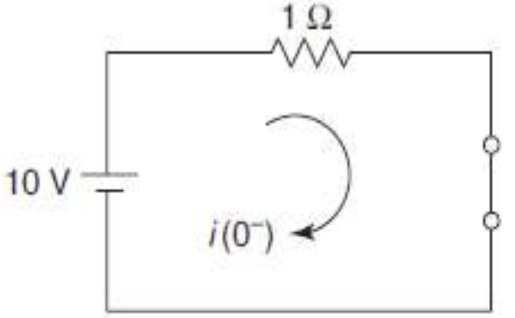
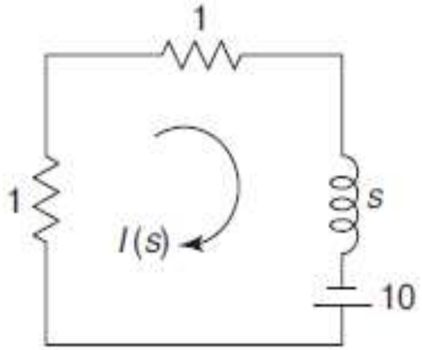


Fig. 7.13

For $t > 0$, the transformed network is shown in Fig. 7.14.
Applying KVL to the mesh for $t > 0$,



$$-I(s) - I(s) - sI(s) + 10 = 0$$

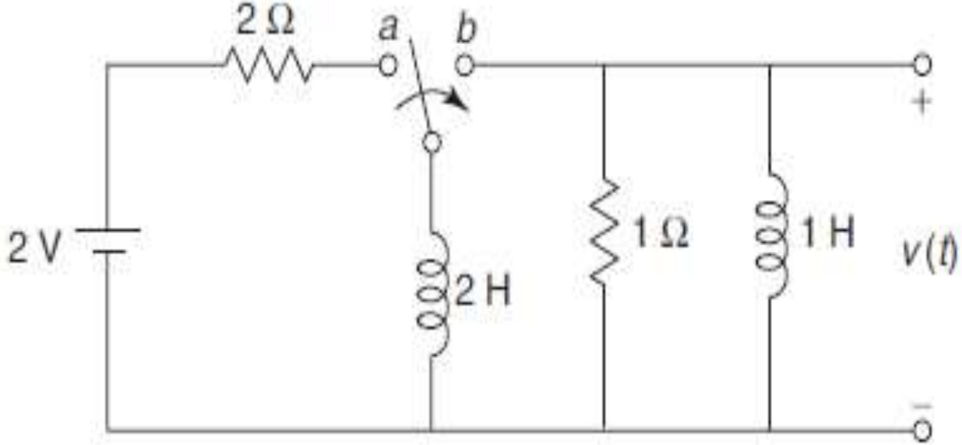
$$I(s)(s+2) = 10$$

$$I(s) = \frac{10}{s+2}$$

Taking inverse Laplace transform,

$$i(\text{t}) = 10e^{-2t} \quad \text{for } t > 0$$

The network of Fig. 7.15 was initially in the steady state with the switch in the position a. At $t = 0$, the switch goes from a to b. Find an expression for voltage $v(t)$ for $t > 0$.



Solution At $t = 0^-$, the network is shown in Fig 7.16. At $t = 0^-$, the network has attained steady-state condition. Hence, the inductor of 2H acts as a short circuit.

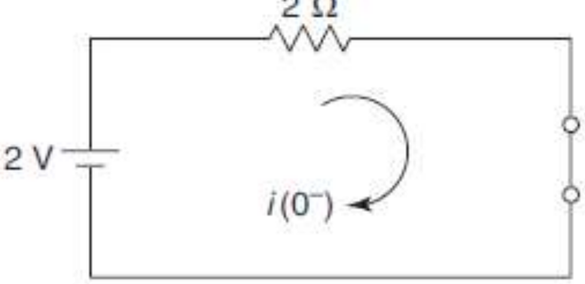


Fig. 7.16

$$i(0^-) = \frac{2}{2} = 1 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i\,(0^+) = 1 \text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.17.

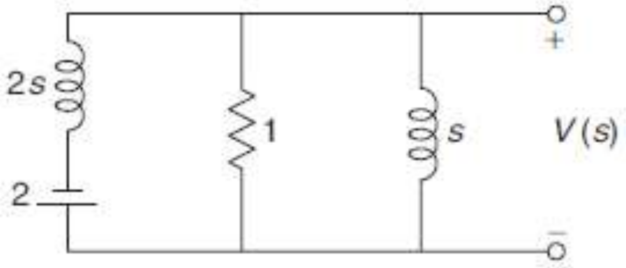


Fig. 7.17

Applying KCL at the node for $t > 0$,

$$\frac{V(s)+2}{2s} + \frac{V(s)}{1} + \frac{V(s)}{s} = 0$$

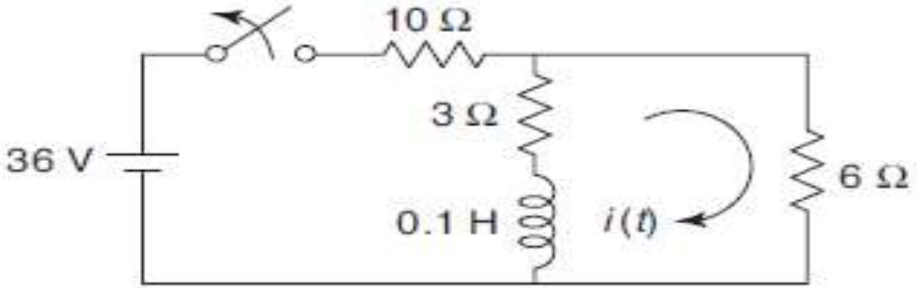
$$V(s)\left(1 + \frac{3}{2s}\right) = -\frac{1}{s}$$

$$V(s) = \frac{-\frac{1}{s}}{\frac{2s+3}{2s}} = -\frac{2}{2s+3} = -\frac{1}{s+1.5}$$

Taking the inverse Laplace transform,

$$v\,(t) = -\,e^{-1.5\,t} \qquad \text{for } t > 0$$

In the network of Fig. 7.18, the switch is opened at $t = 0$. Find $i(t)$.



Solution At $t = 0^-$, the network is shown in Fig. 7.19. At $t = 0^-$, he switch is closed and steady-state condition is reached. Hence, he inductor acts as a short circuit.

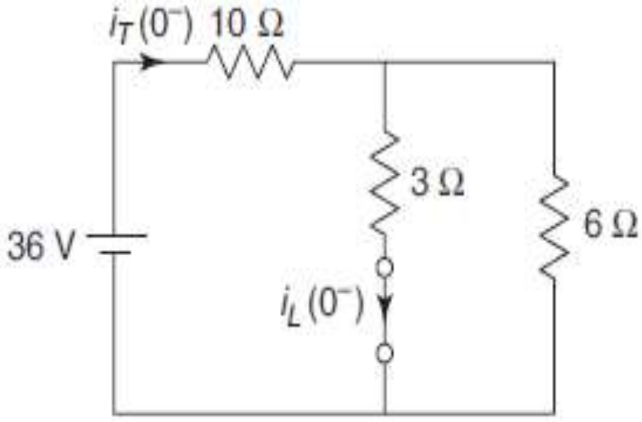


Fig. 7.19

$$i_T(0^-) = \frac{36}{10 + (3 \parallel 6)} = \frac{36}{10 + 2} = 3 \text{ A}$$

$$i_L(0^-) = 3 \times \frac{6}{6 + 3} = 2 \text{ A}$$

Since current through the inductor cannot change instantaneously,

$$i_L(0^+) = 2\text{ A}$$

For $t > 0$, the transformed network is shown in Fig. 7.20

Applying KVL to the mesh for $t > 0$,

$$-0.2 - 0.1s\, I(s) - 3I(s) - 6I(s) = 0$$

$$0.1sI(s) + 9I(s) = -0.2$$

$$I(s) = \frac{-0.2}{0.1s+9} = \frac{-2}{s+90}$$

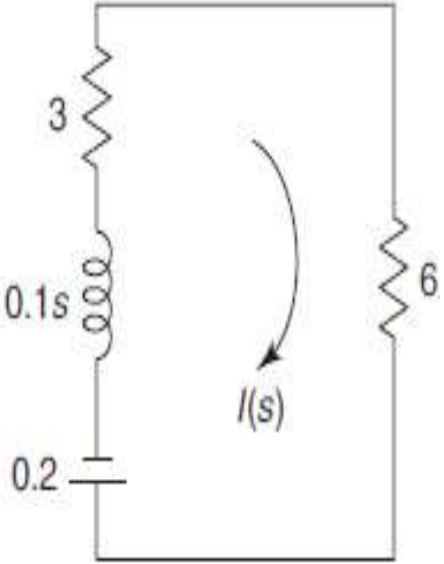
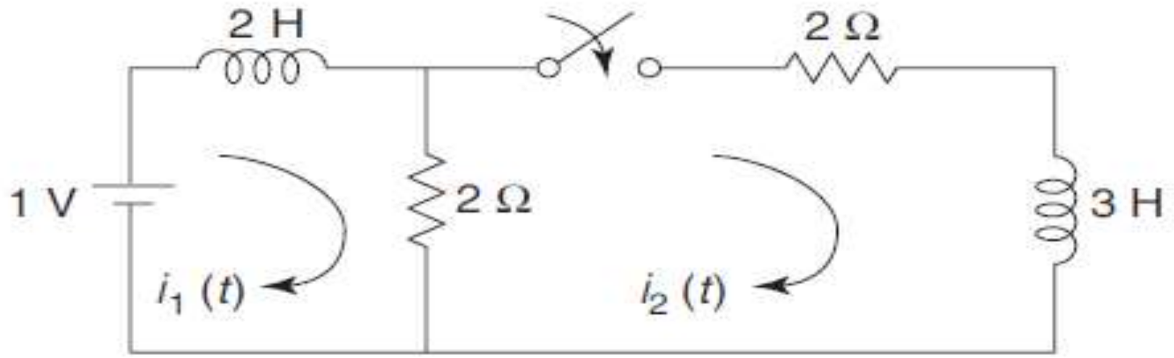


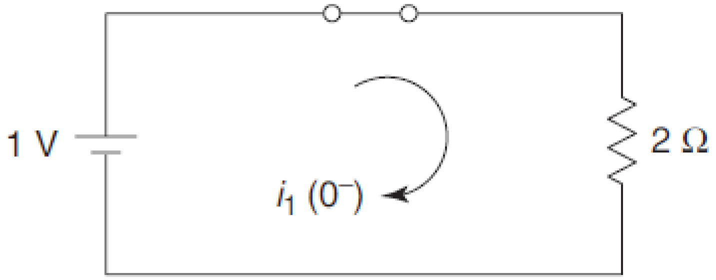
Fig. 7.20

Taking inverse Laplace transform,

$$i(t) = -2e^{-90t} \qquad \text{for } t > 0$$

In the network shown in Fig. the switch is closed at $t = 0$, the steady-state being reached before $t = 0$. Determine current through inductor of 3 H.





Solution At $t = 0^-$, the network is shown in Fig. 7.25. At $t = 0^-$, steady-state condition is reached. Hence, the inductor of 2 H acts as a short circuit.

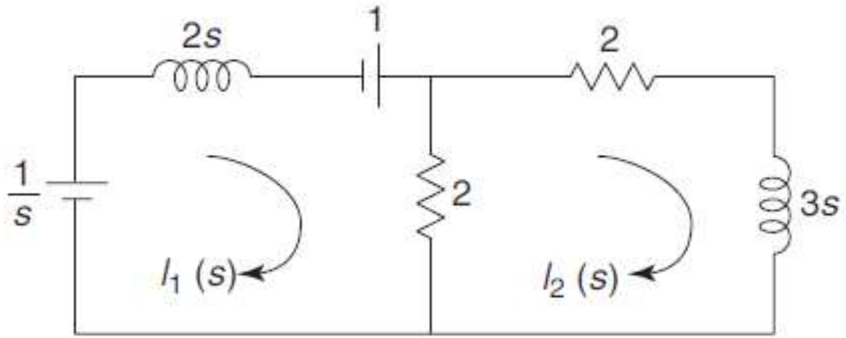
$$i_1(0^-) = \frac{1}{2} \text{ A}$$

$$i_2(0^-) = 0$$

Since current through the inductor cannot change instantaneously,

$$i_1(0^+) = \frac{1}{2} \text{ A}$$

$$i_2(0^+) = 0$$



For $t > 0$, the transformed network is shown in Fig. 17

Applying KVL to Mesh 1,

$$\frac{1}{s} - 2s\,I_1(s) + 1 - 2[I_1(s) - I_2(s)] = 0$$

$$(2 + 2s)\,I_1(s) - 2I_2(s) = 1 + \frac{1}{s}$$

Applying KVL to Mesh 2,

$$-2\,[I_2(s) - I_1(s)] - 2I_2(s) - 3s\,I_2\,(s) = 0$$

$$-2I_1(s) + (4 + 3s)\,I_2(s) = 0$$

By Cramer's rule,

$$I_2(s) = \frac{\begin{vmatrix} 2+2s & 1+\frac{1}{s} \\ -2 & 0 \end{vmatrix}}{\begin{vmatrix} 2+2s & -2 \\ -2 & 4+3s \end{vmatrix}} = \frac{\frac{2}{s}(s+1)}{(2+2s)(4+3s)-4} = \frac{s+1}{s(3\boldsymbol{s}^2+7s+2)} = \frac{s+1}{3s\left(s+\frac{1}{3}\right)(s+2)} = \frac{\frac{1}{3}(s+1)}{s(s+2)\left(s+\frac{1}{3}\right)}$$

By partial-fraction expansion,

$$I_2(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+\frac{1}{3}}$$

$$A = s\,I_2(s)\Big|_{s=0} = \left.\frac{\frac{1}{3}(s+1)}{(s+2)\left(s+\frac{1}{3}\right)}\right|_{s=0} = \frac{1}{2}$$

$$B = (s+2)I_2(s)\Big|_{S=-2} = \left.\frac{\frac{1}{3}(s+1)}{s\left(s+\frac{1}{3}\right)}\right|_{s=-2} = -\frac{1}{10}$$

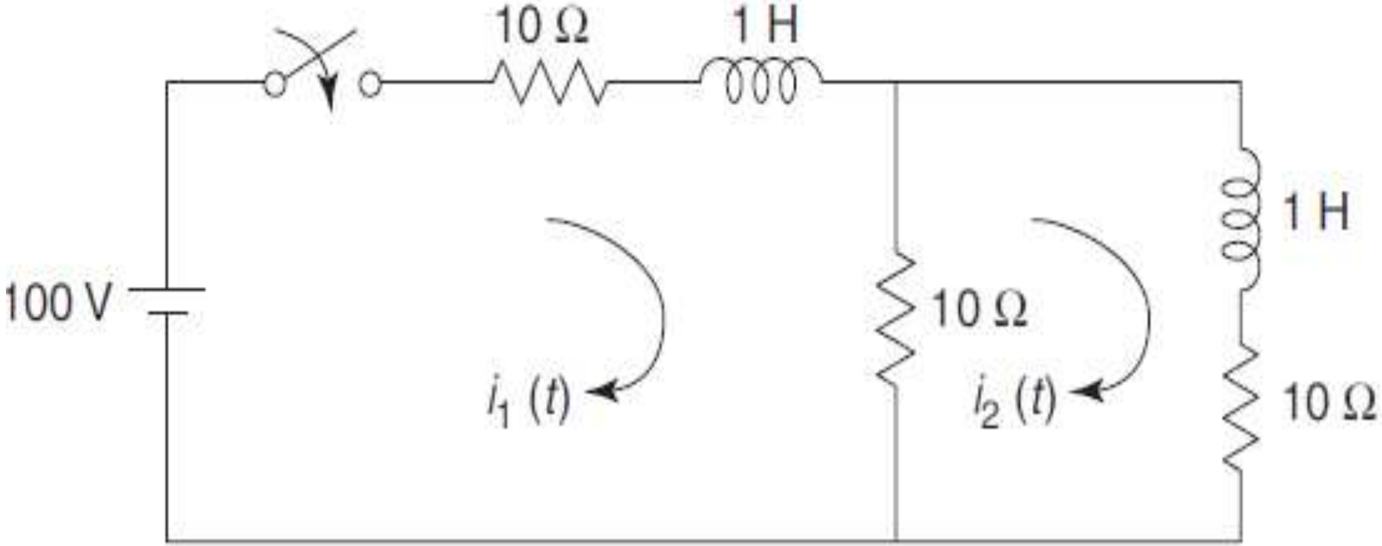
$$C = \left(s + \frac{1}{3}\right) I_2(s)\Big|_{s=-\frac{1}{3}} = \left.\frac{\frac{1}{3}(s+1)}{s(s+2)}\right|_{s=-\frac{1}{3}} = -\frac{2}{5}$$

$$I_2(s) = \frac{1}{2}\frac{1}{s} - \frac{1}{10}\frac{1}{s+2} - \frac{2}{5}\frac{1}{s+\frac{1}{3}}$$

Taking inverse Laplace transform

$$i_2(t) = \frac{1}{2} - \frac{1}{10}e^{-2t} - \frac{2}{5}e^{-\frac{1}{3}t} \quad \text{for } t > 0$$

In the network of Fig. the switch is closed at $t = 0$ with the network previously unenergised. Determine currents $i_1(t)$.



Solution For $t > 0$, the transformed network is shown in Fig. 7.28.

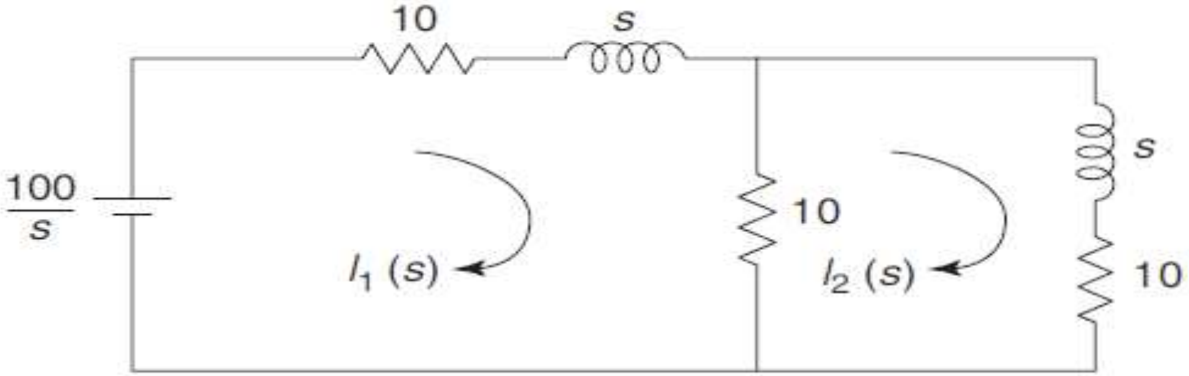


Fig. 7.28

Applying KVL to Mesh 1,

$$\frac{100}{s} - 10I_1(s) - sI_1(s) - 10\left[I_1(s) - I_2(s)\right] = 0$$

$$(s+20)I_1(s) - 10I_2(s) = \frac{100}{s}$$

Applying KVL to Mesh 2,

$$-10\left[I_2(s) - I_1(s)\right] - s\,I_2(s) - 10\,I_2(s) = 0$$

$$-10\,I_1(s) + (s+20)I_2(s) = 0$$

By Cramer's rule,

$$I_1(s) = \frac{\begin{vmatrix} \frac{100}{s} & -10 \\ 0 & s+20 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}} = \frac{\frac{100}{s}(s+20)}{(s+20)^2 - 100} = \frac{100(s+20)}{s(s^2+40s+300)} = \frac{100(s+20)}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_1(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{s+30}$$

$$A = s\, I_1(s)\,|_{s=0} = \left.\frac{100(s+20)}{(s+10)(s+30)}\right|_{s=0} = \frac{20}{3}$$

$$B = (s+10) I_1(s)\,|_{s=-10} = \left.\frac{100(s+20)}{s(s+30)}\right|_{s=-10} = -5$$

$$C = (s+30) I_1(s)\,|_{s=-30} = \left.\frac{100(s+20)}{s(s+10)}\right|_{s=-30} = -\frac{5}{3}$$

$$I_1(s) = \frac{20}{3}\frac{1}{s} - \frac{5}{s+10} - \frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_1(t) = \frac{20}{3} - 5e^{-10t} - \frac{5}{3}e^{-30t}$$

Similarly,

$$I_2(s)=\frac{\begin{vmatrix} s+20 & \dfrac{100}{s} \\ -10 & 0 \end{vmatrix}}{\begin{vmatrix} s+20 & -10 \\ -10 & s+20 \end{vmatrix}}=\frac{\dfrac{1000}{s}}{(s+20)^2-100}=\frac{1000}{s(s^2+40s+300)}=\frac{1000}{s(s+10)(s+30)}$$

By partial-fraction expansion,

$$I_2(s)=\frac{A}{s}+\frac{B}{s+10}+\frac{C}{s+30}$$

$$A=sI_2(s)\,|_{s=0}=\left.\frac{1000}{(s+10)(s+30)}\right|_{s=0}=\frac{10}{3}$$

$$B=(s+10)I_2(s)\,|_{s=-10}=\left.\frac{1000}{s(s+30)}\right|_{s=-10}=-5$$

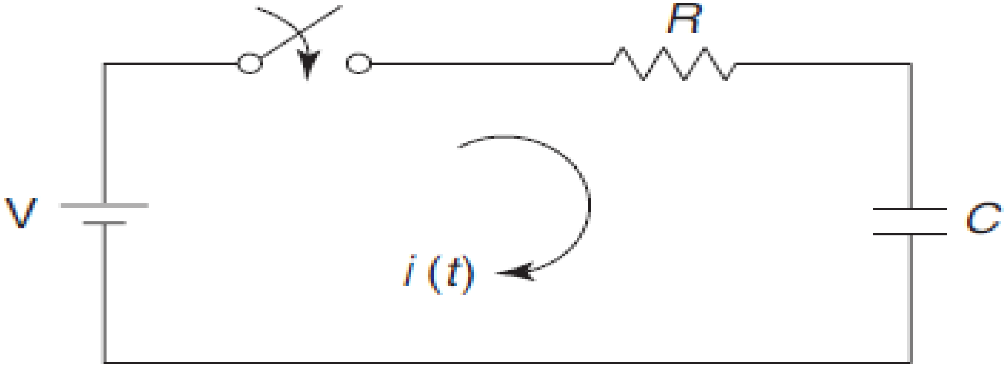
$$C=(s+30)I_2(s)\,|_{s=-30}=\left.\frac{1000}{s(s+10)}\right|_{s=-30}=\frac{5}{3}$$

$$I_2(s)=\frac{10}{3}\frac{1}{s}-\frac{5}{s+10}+\frac{5}{3}\frac{1}{s+30}$$

Taking inverse Laplace transform,

$$i_2(t) = \frac{10}{3} - 5e^{-10t} + \frac{5}{3}e^{-30t}$$

RESISTOR–CAPACITOR CIRCUIT



Consider a series RC circuit as shown in Fig. 7.29. The switch is closed at time $t = 0$.

For $t > 0$, the transformed network is shown in Fig. 7.30.

Applying KVL to the mesh,

$$\frac{V}{s} - RI(s) - \frac{1}{Cs} I(s) = 0$$

$$\left(R + \frac{1}{Cs}\right) I(s) = \frac{V}{s}$$

$$I(s) = \frac{\frac{V}{s}}{R + \frac{1}{Cs}} = \frac{\frac{V}{s}}{\frac{RCs+1}{Cs}} = \frac{\frac{V}{R}}{s + \frac{1}{RC}}$$

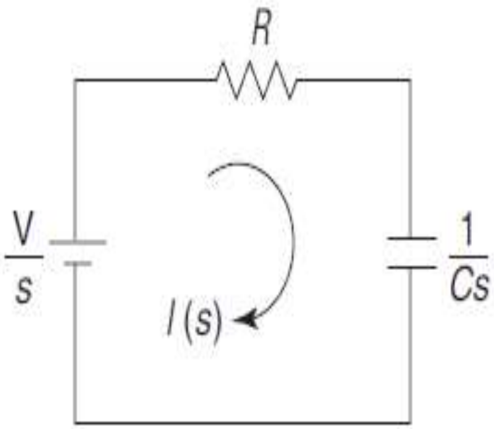
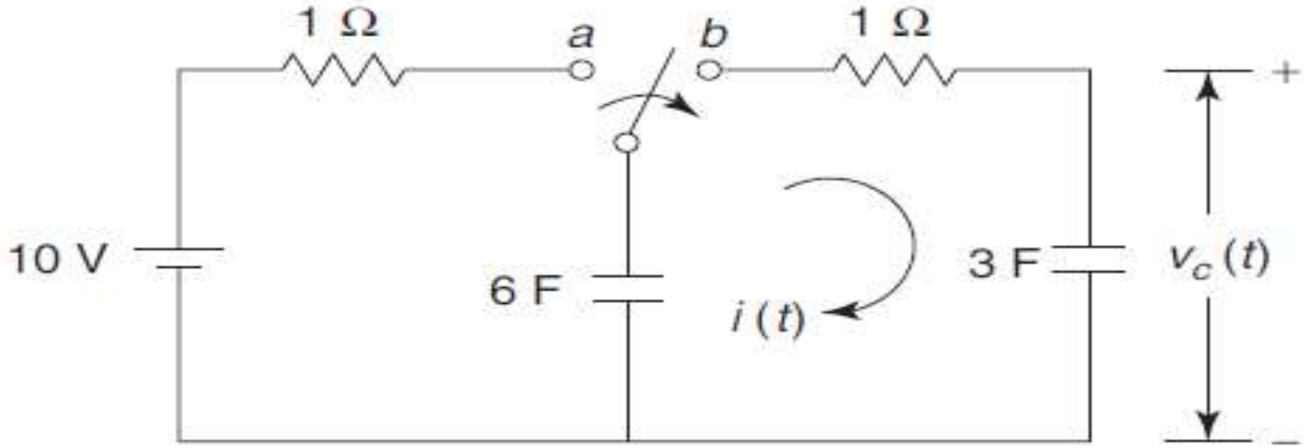


Fig. 7.30 *Transformed network*

Taking the inverse Laplace transform,

$$i(t) = \frac{V}{R} e^{-\frac{1}{RC}t} \qquad \text{for } t > 0$$

In the network of Fig. 7.31, the switch is moved from a to b at $t = 0$. Determine $i(t)$ and $v_c(t)$.

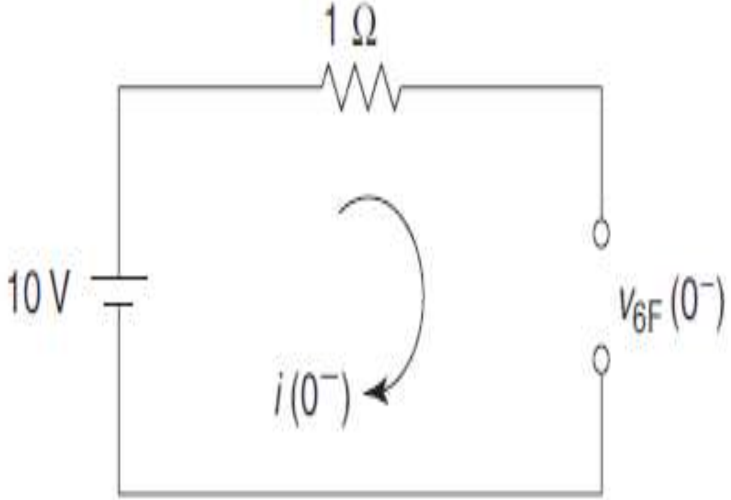


Solution At $t = 0^-$, the network is shown in Fig. 7.32. At $t = 0^-$, the network has attained steady-state condition. Hence, the capacitor of 6 F acts as an open circuit.

$$v_{6\,F}(0^-) = 10\ V$$

$$i(0^-) = 0$$

$$v_{3\,F}(0^-) = 0$$



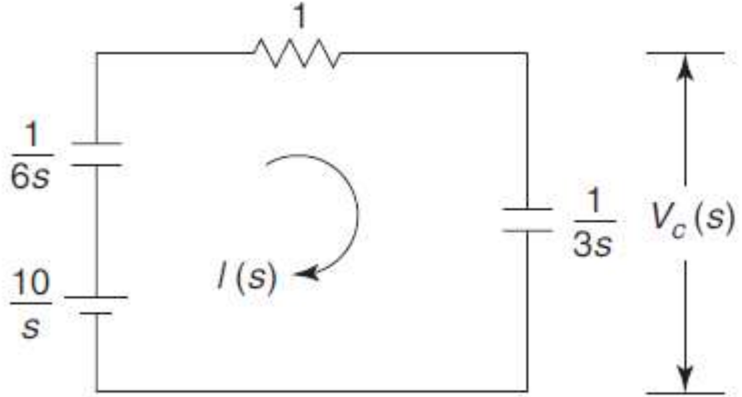
Since voltage across the capacitor cannot change instantaneously,

$$v_{6\,F}(0^+) = 10\ V$$

$$v_{3\,F}(0^+) = 0$$

For $t > 0$, the transformed network is shown in 7.33.

Applying KVL to the mesh for $t > 0$,



$$\frac{10}{s} - \frac{1}{6s}I(s) - I(s) - \frac{1}{3s}I(s) = 0$$

$$\frac{1}{6s}I(s) + I(s) + \frac{1}{3s}I(s) = \frac{10}{s}$$

$$I(s) = \frac{10}{s\left(1 + \frac{1}{6s} + \frac{1}{3s}\right)} = \frac{60}{6s+3} = \frac{10}{s+0.5}$$

Taking the inverse Laplace transform,

$$i(t) = 10e^{-0.5t} \qquad \text{for } t > 0$$

Voltage across the 3 F capacitor is given by

$$V_c(s) = \frac{1}{3s} I(s) = \frac{10}{3s(s+0.5)}$$

By partial-fraction expansion,

$$V_c(s) = \frac{A}{s} + \frac{B}{s+0.5}$$

$$A = sV_c(s)\Big|_{s=0} = \frac{10}{3(s+0.5)}\Bigg|_{s=0} = \frac{20}{3}$$

$$B = (s+0.5)V_c(s)\Big|_{s=-0.5} = \frac{10}{3s}\Bigg|_{s=-0.5} = -\frac{20}{3}$$

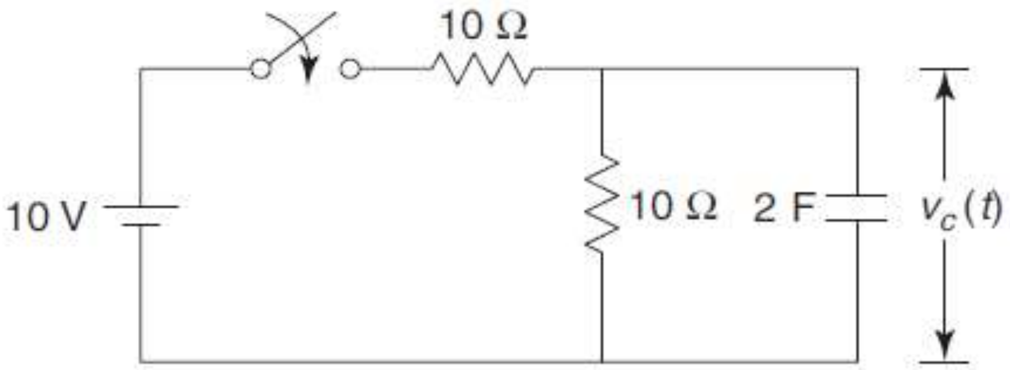
$$V_c(s) = \frac{20}{3}\frac{1}{s} - \frac{20}{3}\frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v_c(t) = \frac{20}{3} - \frac{20}{3}e^{-0.5t}$$

$$= \frac{20}{3}(1 - e^{-0.5t}) \qquad \text{for } t > 0$$

The switch in the network shown in Fig. 7.34 is closed at t =0. Determine the voltage cross the capacitor.



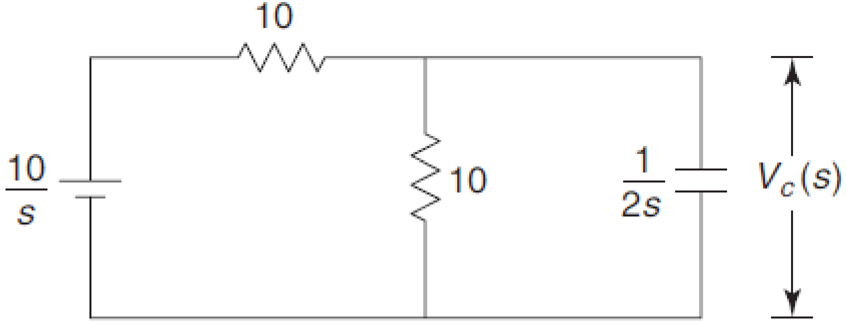
Solution At $t = 0^-$, the capacitor is uncharged.

$$v_c(0^-) = 0$$

Since the voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig.
Applying KCL at the node for $t > 0$,



$$\frac{V_c(s) - \frac{10}{s}}{10} + \frac{V_c(s)}{10} + \frac{V_c(s)}{\frac{1}{2s}} = 0$$

$$2sV_c(s) + 0.2V_c(s) = \frac{1}{s}$$

$$V_c(s) = \frac{1}{s(2s+0.2)} = \frac{0.5}{s(s+0.1)}$$

By partial-fraction expansion,

$$V_c(s)=\frac{A}{s}+\frac{B}{s+0.1}$$

$$A=sV_c(s)\Big|_{s=0}=\frac{0.5}{s+0.1}\Big|_{s=0}=\frac{0.5}{0.1}=5$$

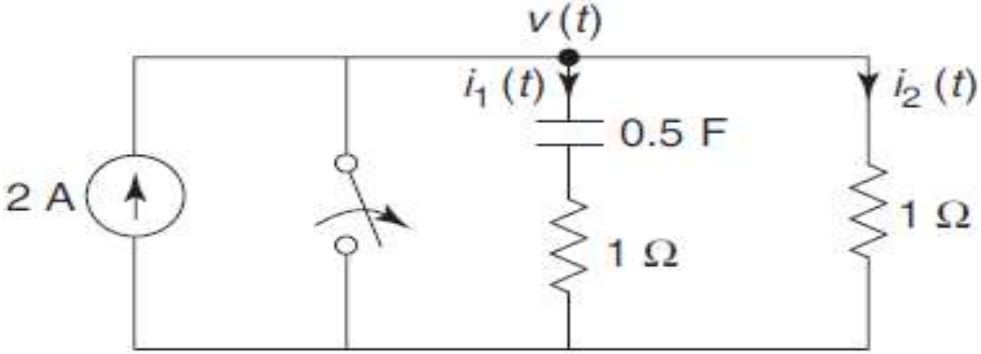
$$B=(s+0.1)V_c(s)\Big|_{s=-0.1}=\frac{0.5}{s}\Big|_{s=-0.1}=-\frac{0.5}{0.1}=-5$$

$$V_c(s)=\frac{5}{s}-\frac{5}{s+0.1}$$

Taking inverse Laplace transform,

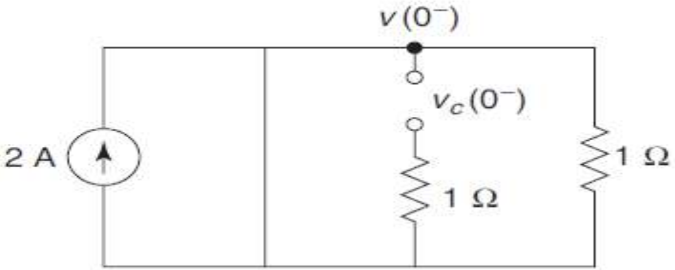
$$v_c(t)=5-5e^{-0.1t} \qquad \text{for } t>0$$

In the network of Fig. the switch is closed for a long time and at t =0, the switch is opened. Determine the current through the capacitor.



Solution At $t = 0^-$, the network is shown in Fig. 7.37. At $t = 0^-$, the switch is closed and steady-state condition is reached. Hence, the capacitor acts as an open circuit.

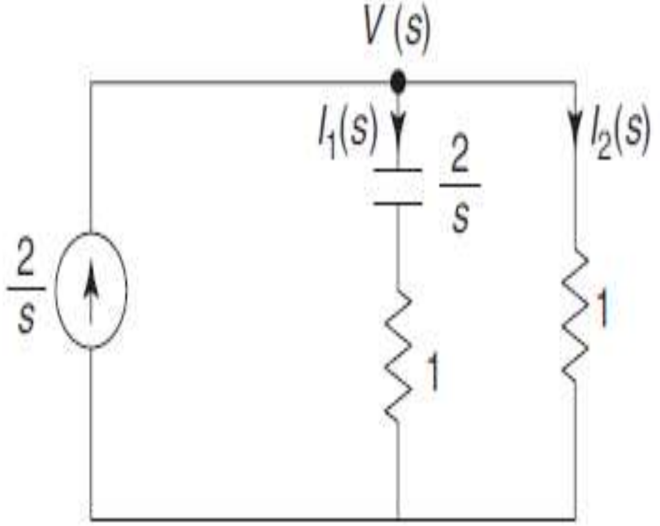
$$v_c(0^-) = 0$$



Since voltage across the capacitor cannot change instantaneously,

$$v_c(0^+) = 0$$

For $t > 0$, the transformed network is shown in Fig. 7.38.



Applying KVL to two parallel branches,

$$\frac{2}{s}I_1(s) + I_1(s) = I_2(s)$$

Applying KCL at the node for $t > 0$,

$$\frac{2}{s} = I_1(s) + I_2(s)$$

$$\frac{2}{s}I_1(s)+I_1(s)=\frac{2}{s}-I_1(s)$$

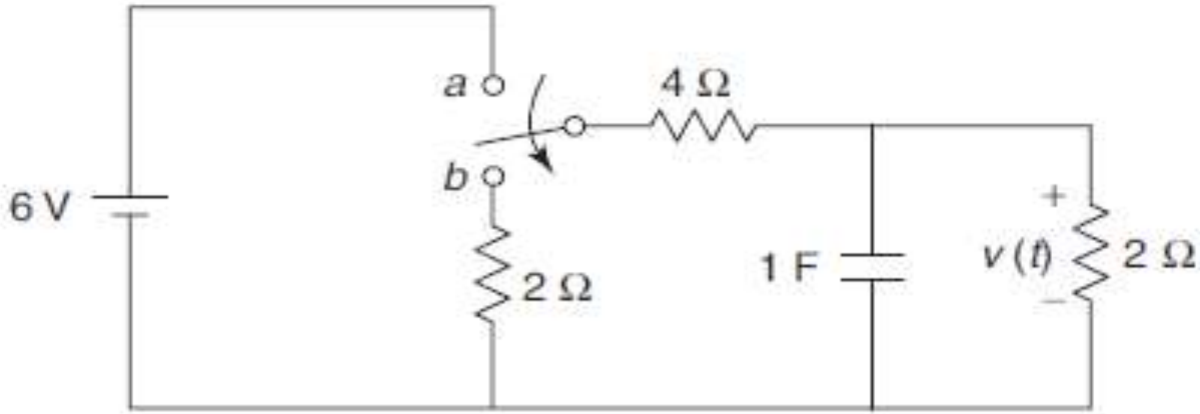
$$\frac{2}{s}I_1(s)+2I_1(s)=\frac{2}{s}$$

$$I_1(s)=\frac{\frac{2}{s}}{\frac{2}{s}+2}=\frac{1}{s+1}$$

Taking the inverse Laplace transform,

$$i_1(t)=e^{-t} \qquad \text{for } t>0$$

In the network of Fig. 7.39, the switch is moved from a to b, at $t = 0$. Find $v(t)$.



Solution At $t = 0^-$, the network is shown in Fig 7.40. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor acts as an open circuit.

$$v(0^-) = 6 \times \frac{2}{4+2} = 2 \text{ V}$$

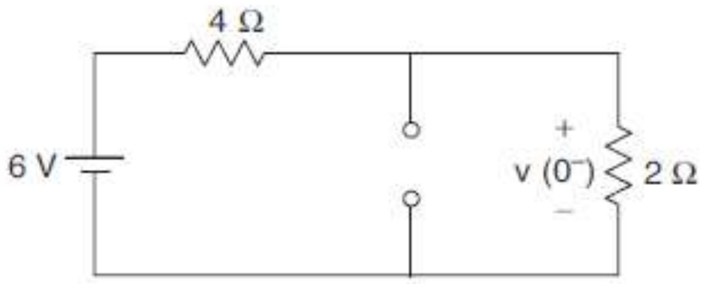


Fig. 7.40

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

For $t > 0$, the transformed network is shown in Fig. 7.41.

Applying KCL at the node for $t > 0$,

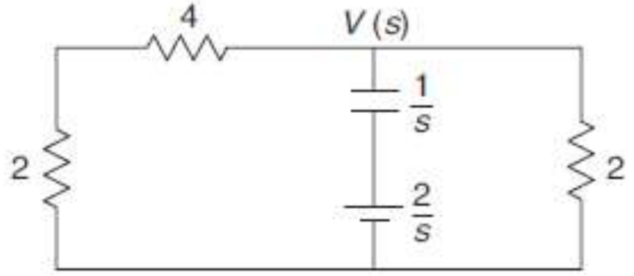


Fig. 7.41

$$\frac{V(s)}{6} + \frac{V(s) - \frac{2}{s}}{\frac{1}{s}} + \frac{V(s)}{2} = 0$$

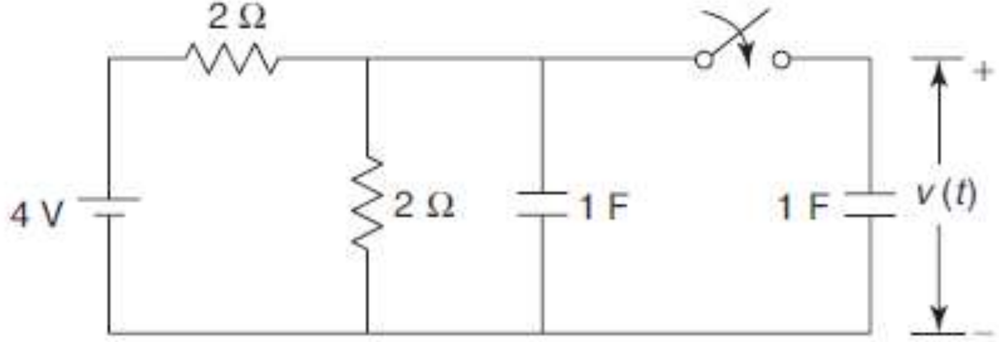
$$V(s)\left(\frac{2}{3} + s\right) = 2$$

$$V(s) = \frac{2}{s + \frac{2}{3}}$$

Taking the inverse Laplace transform,

$$v(t) = 2e^{-\frac{2}{3}t} \qquad \text{for } t > 0$$

The network shown in Fig. 7.42 has acquired steady-state at $t < 0$ with the switch open. The switch is closed at $t = 0$. Determine v (t).



Solution At $t = 0^-$, the network is shown in Fig 7.43. At $t = 0^-$, steady-state condition is reached. Hence, the capacitor of 1 F acts as an open circuit.

$$v(0^-) = 4 \times \frac{2}{2+2} = 2 \text{ V}$$

Since voltage across the capacitor cannot change instantaneously,

$$v(0^+) = 2 \text{ V}$$

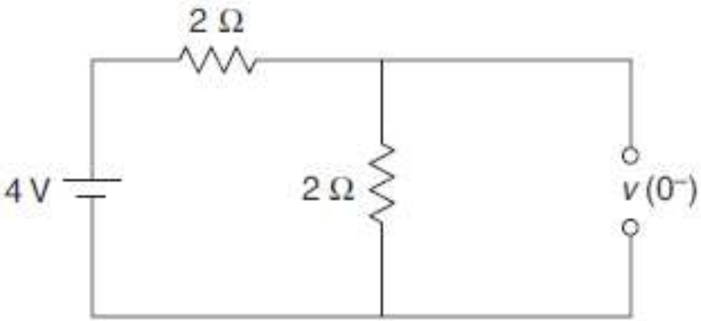
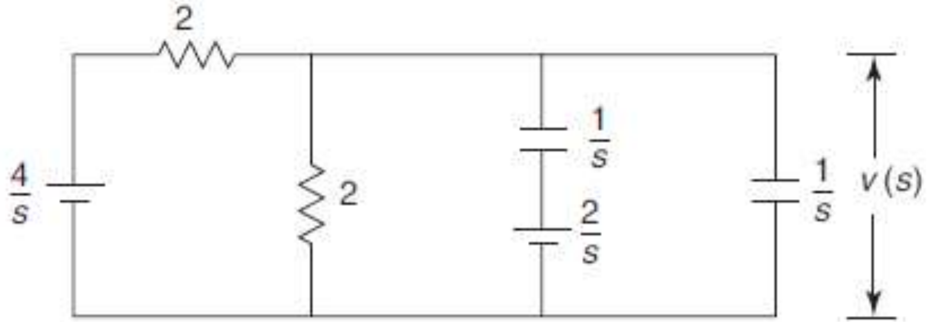


Fig. 7.43

For $t > 0$, the transformed network is shown in Fig. 7.44.



Applying KCL at the node for $t > 0$,

$$\frac{V(s)-\frac{4}{s}}{2}+\frac{V(s)}{2}+\frac{V(s)-\frac{2}{s}}{\frac{1}{s}}+\frac{V(s)}{\frac{1}{s}}=0$$

$$2sV(s)+V(s)=\frac{2}{s}+2$$

$$V(s)=\frac{\frac{2}{s}+2}{2s+1}=\frac{2s+2}{s(2s+1)}=\frac{2}{s}-\frac{2}{2s+1}=\frac{2}{s}-\frac{1}{s+0.5}$$

Taking the inverse Laplace transform,

$$v(t)=2-e^{-0.5t} \qquad \text{for } t>0$$