Module 6. Calculus of Variation

* Euler - Lagrange equation :

let $y = f(x)$ then the definite integral

$$I = \int_{x_1}^{x_2} F(x, y, y')\, dx$$ is called as functional

Note that the solution $y(x)$ of the above integral is obtained by solving the equation

$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

then this equation is called as Euler - Lagrange equation

Corollary 1. If F does not contain y explicitly then Euler - Lagrange equation is

$$\boxed{\frac{\partial F}{\partial y'} = c}$$

proof: since the Euler - Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{--- (1)}$$

But F does not contain y

$$\Rightarrow \frac{\partial F}{\partial y} = 0$$

$\therefore$ equation ① becomes.

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial y'} = c}$$ (Integrating both side)

* Another form of Euler - Lagrange equation

$$\boxed{\frac{\partial F}{\partial x} - \frac{d}{dx}\left[F - y'\frac{\partial F}{\partial y'}\right] = 0}$$

<u>corollary</u> : 2 If F does not contain x explicitly then Eulers - Lagrange equation is

$$\boxed{F - y'\frac{\partial F}{\partial y'} = c}$$

Examples:

Ex 1: find the extremals of $\int_{x_1}^{x_2} \frac{y'^2}{x^2} dx$

Solution: here, $F = \frac{y'^2}{x^2}$

clearly, F does not contain y explicitly

therefore, the Euler-Lagrange equation

is
$$\frac{\partial F}{\partial y'} = c \quad \text{——} \quad (1)$$

since, $F = \frac{y'^2}{x^2} \Rightarrow \frac{\partial F}{\partial y'} = \frac{1}{x^2} \frac{\partial}{\partial y'}(y'^2)$

i.e $\frac{\partial F}{\partial y'} = \frac{2y'}{x^2}$

$\therefore$ equation (1) becomes

$$\frac{2y'}{x^2} = c$$

$$\Rightarrow \quad y' = \frac{cx^2}{2}$$

i.e. $\frac{dy}{dx} = \frac{c}{2} x^2$

integrating both side we get.

$$\int \frac{dy}{dx} dx = \int \frac{c}{2} x^2 dx + C_2$$

$$y(x) = \frac{c}{2} \frac{x^3}{3} + C_2$$

$$y(x) = \frac{c}{6} x^3 + C_2.$$

$$\Rightarrow \boxed{y(x) = C_1 x^3 + C_2} \quad \text{where } C_1 = \frac{c}{6}$$

Ex. 2 find the extremals of $\int_{x_1}^{x_2} (1+x^2y')\,y'\,dx$

Solution: here, $F = (1+x^2y')y'$

clearly, F does not contain y explicitly

$\therefore$ the Euler Lagrange equation is

$$\frac{\partial F}{\partial y'} = c \qquad \text{——— (1)}$$

But $F = (1+x^2y')y' = y' + x^2y'^2$

$$\Rightarrow \frac{\partial F}{\partial y'} = 1 + 2x^2y'$$

$\therefore$ equation (1) becomes

$$1+2x^2y' = c$$

$$\Rightarrow 1+2x^2\frac{dy}{dx} = c$$

$$\Rightarrow \frac{dy}{dx} = \frac{c-1}{2x^2}$$

Integrating both side we get

$$\int \frac{dy}{dx}\,dx = \int \left(\frac{c-1}{2}\right)\frac{1}{x^2}\,dx + c_2$$

$$y(x) = \left(\frac{c-1}{2}\right)\int x^{-2}\,dx + c_2$$

$$\Rightarrow y(x) = \left(\frac{c-1}{2}\right)\frac{x^{-1}}{-1} + c_2$$

$$\Rightarrow y(x) = -\left(\frac{c-1}{2}\right)\frac{1}{x} + c_2$$

$$\Rightarrow \boxed{y(x) = \frac{c_1}{x} + c_2}\;,\quad c_1 = -\left(\frac{c-1}{2}\right)$$

Ex.③ find the extremals of $\int_0^{\pi} (y'^2 - y^2)\, dx$ given that when $x = 0$, $y = 0$ and when $x = \pi$, $y = 0$

Solution: here, $F = y'^2 - y^2$ and $y(0) = 0$, $y(\pi) = 0$

clearly, F does not contain x explicitely

$\therefore$ the Eulers lagranges equation is

$$F - y' \cdot \frac{\partial F}{\partial y'} = c \quad \text{——} \quad (1)$$

Since, $F = y'^2 - y^2 \Rightarrow \frac{\partial F}{\partial y'} = 2y'$

$\therefore$ equation ① becomes

$$y'^2 - y^2 - y'(2y') = c$$

$$\Rightarrow y'^2 - y^2 - 2y'^2 = c$$

$$\Rightarrow -y'^2 - y^2 = c$$

$$\Rightarrow y'^2 + y^2 = -c$$

$$\Rightarrow y'^2 + y^2 = c_1$$

$$\Rightarrow y'^2 = c_1 - y^2$$

$$\Rightarrow y' = \sqrt{c_1 - y^2}$$

i.e $$\frac{dy}{dx} = \sqrt{c_1 - y^2}$$

$$\Rightarrow \frac{1}{\sqrt{c_1 - y^2}}\, dy = dx$$

Integrating both side we get

$$\int \frac{1}{\sqrt{c_1^2 - y^2}}\, dy = \int dx + c_2$$

$$\sin^{-1}\left(\frac{y}{c_1}\right) = x + c_2$$

$$\Rightarrow \quad \frac{y}{c_1} = \sin(x+c_2)$$

$$\Rightarrow \quad y(x) = c_1 \sin(x+c_2)$$

But $\quad y(0)=0$

$0 = c_1 \sin(0+c_2) \Rightarrow c_1 \neq 0, \quad \sin(c_2)=0 \Rightarrow c_2 = 0$

Also $\quad y(\pi)=0$

$0 = c_1 \sin(\pi + c_2) \Rightarrow \quad c_2 = 0$

$\therefore$ the solution is

$$\boxed{y(x) = c_1 \sin x}$$

Ex ④ Find the curve on which the functional $\int_0^1 (y'^2 + 12xy)\,dx$ with $y(0)=0$ and $y(1)=1$ is extremal

Solution: here, $F = y'^2 + 12xy$

F contains x, y, y'

$\therefore$ The Eulers Lagranges equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \quad \text{——— ①}$$

But $F = y'^2 + 12xy \Rightarrow \frac{\partial F}{\partial y} = 12x$

and $\frac{\partial F}{\partial y'} = 2y'$

$\therefore$ equation ① becomes

$$12x - \frac{d}{dx}(2y') = 0$$

$$\Rightarrow \quad 12x - 2y'' = 0$$

$$\Rightarrow \quad 2y'' = 12x$$

$$\Rightarrow \quad y'' = 6x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = 6x$$

Integrating both side side, we get

$$\int \frac{d^2y}{dx^2}\,dx = \int 6x\,dx + c_1$$

$$\Rightarrow \quad \frac{dy}{dx} = 6\frac{x^2}{2} + c_1$$

$$\Rightarrow \quad \frac{dy}{dx} = 3x^2 + c_1$$

Integrating both side, we get

$$\int \frac{dy}{dx}\,dx = \int 3x^2\,dx + \int c_1\,dx + c_2$$

$$\Rightarrow \quad y(x) = 3\frac{x^3}{3} + c_1 x + c_2$$

$$\Rightarrow \quad y(x) = x^3 + c_1 x + c_2$$

But $y(0) = 0$, $\quad 0 = 0^3 + c_1(0) + c_2 \Rightarrow c_2 = 0$

Also $y(1) = 1$, $\quad 1 = 1^3 + c_1(1) + 0 \Rightarrow c_1 = 0$

$\therefore$ $\boxed{y(x) = x^3}$ is required solution.