* Cauchy Residue Theorem:

Statement: let $z_1, z_2, \cdots, z_n$ be the singular point of $f(z)$ lies inside a simple closed curve C

If $f(z)$ is analytic on and inside a simple closed curve C except the points $z_1, z_2, \cdots, z_n$ then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \left[\text{sum of residues at } z_1, z_2, \cdots z_n\right]}$$

In perticular,

① If z_1 is only the singular point of $f(z)$ lie inside C then

$$\boxed{\oint_C f(z)\,dz = 2\pi i \left[\text{Residues of } f(z) \text{ at } z_1\right]}$$

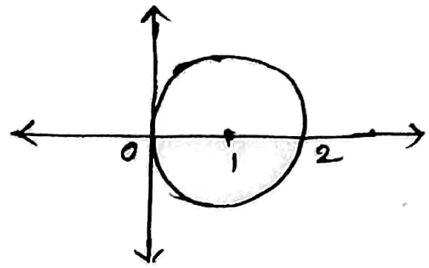
Examples: Using Cauchy residue theorem evaluate $\oint_C \frac{z^2+3}{z^2-1}\,dz$ where C is the circle $|z-1| = 1$

Solution: let $f(z) = \frac{z^2+3}{z^2-1} = \frac{z^3+3}{(z+1)(z-1)}$

Note that $z^2-1=0 \Rightarrow z = 1, -1$

$\therefore$ clearly, $z=1$ and $z=-1$ are the simple pole of $f(z)$

since, $C: |z-1| = 1$ which equation of circle with centre $(1,0)$ and radius 1



$\therefore$ only a pole $z=1$ lies inside C while $z=-1$ lies outside C

Now, Residue at $(z=1)$

$$= \lim_{z \to 1} (z-1) \cdot f(z)$$

$$= \lim_{z \to 1} (z-1) \frac{z^3+3}{(z+1)(z-1)}$$

$$= \lim_{z \to 1} \frac{z^3+3}{z+1}$$

$$= \frac{(1)^3+3}{1+1}$$

$$= 2$$

$\therefore$ By Cauchy Residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \left[\text{Residue of } f(z) \text{ at } z_1 \right]$$

$$\Rightarrow \oint_C \frac{z^3+1}{z^2-1}\,dz = 2\pi i\,[2] = 4\pi i$$

i.e $$\oint_C \frac{z^3+1}{z^2-1}\,dz = 4\pi i$$

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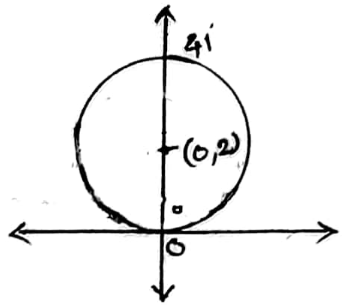
Ex. 2 Using residue theorem evaluate

$$\oint_C \frac{e^{2z}}{(z-\pi i)^3}\, dz \quad \text{where } C \text{ is } |z-2i| = 2$$



Solution: let $f(z) = \dfrac{e^{2z}}{(z-\pi i)^3}$

Given: $C = |z-2i| = 2$ which is circle with centre $(0,2)$ and radius 2

Note that $(z-\pi i)^3 = 0 \Rightarrow z = \pi i$

clearly, $z = \pi i$ is pole of $f(z)$ of order 3

and $z = \pi i = (0, \pi)$ lies inside C

Now, Residue at $z = z_0 = \pi i$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m . f(z) \right] \quad \left(\begin{array}{c} \text{if } z_0 \text{ is} \\ \text{Pole of order} \\ \text{'m'} \end{array} \right)$$

$$= \frac{1}{2!} \lim_{z \to \pi i} \frac{d^2}{dz^2} \left[(z-\pi i)^3 . \frac{e^{2z}}{(z-\pi i)^3} \right]$$

$$= \frac{1}{2} \lim_{z \to \pi i} \frac{d^2}{dz^2} \left[e^{2z} \right]$$

$$= \frac{1}{2} \lim_{z \to \pi i} \frac{d}{dz} \left(2 e^{2z} \right)$$

$$= \frac{1}{2} \lim_{z \to \pi i} 4 e^{2z} = \frac{1}{2}\, 4\, e^{2(\pi i)} = 2$$

$$(\because e^{2\pi i} = 1)$$

$\therefore$ By cauchy residue theorem.

$$\int_C f(z)\, dz = 2\pi i \left[\text{Residue of } f(z) \text{ at } z_0 \right]$$

$$\Rightarrow \int_C f(z)\, dz = 2\pi i\, (2) = 4\pi i$$