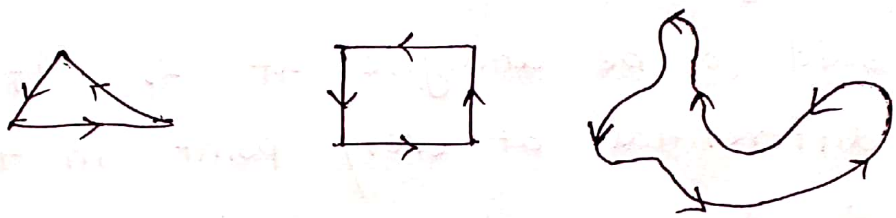
* Cauchy's Integral theorem for simple connected and multiply connected regions

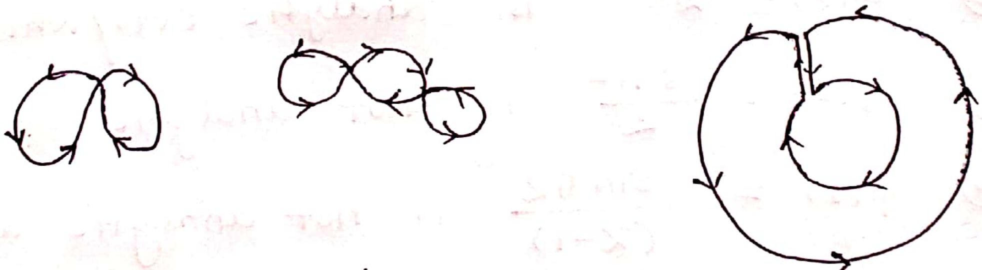
→ * Closed curve : closed curve is one in which end points coincide.

i.e. $\phi(a) = \phi(b)$, for some $a = b$

* simple closed curve :- A closed curve does not intersect itself is called simple closed curve



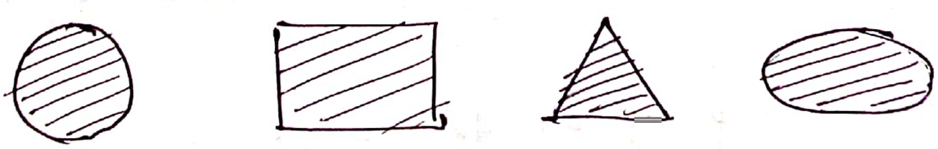
* multiple closed curve : A closed curve intersect itself is called multiple closed curve



* Simply connected domain :-

A domain D is said to be simply connected if every simple closed curve in D contains only points of D

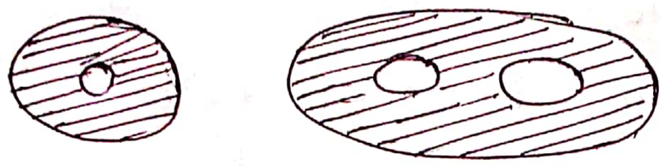
for example : Interior of circle, rectangle



* Multiply connected domain :

A domain which is not simply connected is called multiply connected domain.

for ex. Annulus region, regions with holes.



* Analytic function: A function $f(z)$ is said to be analytic at z_0 if $f(z)$ is differential at every point in the neighbourhood of z_0.

for. ex: ① $f(z) = \frac{1}{z}$ is not analytic at $z=0$

② $f(z) = z^2$ is analytic everywhere

③ $f(z) = \frac{\sin z}{z}$ is not analytic at $z=0$

④ $f(z) = \frac{\sin \pi z}{(z-1)}$ is not analytic at $z=1$

Imp ⇒ Cauchy Integral Theorem for simply connected region (Domain)

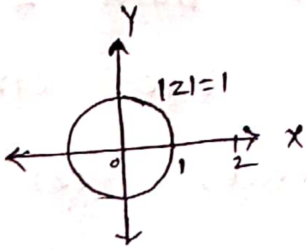
statement :- let $f(z)$ be analytic on and inside a simple closed contour C and let $f'(z)$ also continuous on and inside C,

then
$$\oint_C f(z)\, dz = 0$$

Examples:

Example ① Evaluate $\int_C \frac{1}{z-2}\,dz$, where C is the circle $|z|=1$

Solution: given: $C : |z|=1$



let $f(z) = \frac{1}{z-2}$

Clearly, $f(z)$ is not analytic at $z=2$

But $z=2$ lies outside the circle $|z|=1$

Hence, $f(z)$ is analytic everywhere on and inside C

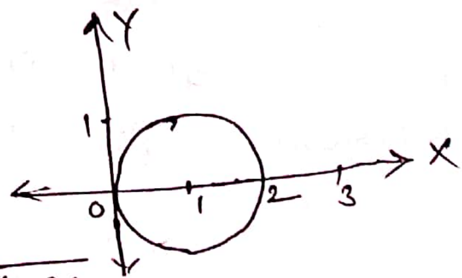
$\therefore$ By cauchy integral theorem,

$$\int_C f(z)\,dz = 0$$

$$\Rightarrow \boxed{\int_C \frac{1}{z-2}\,dz = 0}$$

Example ② Evaluate $\int_C \frac{z+3}{z^2-2z+5}\,dz$, where C is the circle $|z-1|=1$

Solution: given: $C : |z-1|=1$



let $f(z) = \frac{z+3}{z^2-2z+5}$

Note that $z^2-2z+5=0$ gives $z = \frac{2 \pm \sqrt{4-20}}{2}$

i.e. $z = 1 \pm 2i$

$$\therefore \quad f(z) = \frac{z+3}{z^2-2z+5} = \frac{z+3}{[z-(1+2i)][z-(1-2i)]}$$

clearly, $f(z)$ is not analytic at $z = 1+2i$ and $z = 1-2i$

But Both $z = 1+2i$ and $z = 1-2i$ lies Outside the circle $|z-1| = 1$

Hence, $f(z)$ is analytic everywhere on and inside the circle $|z-1|=1$

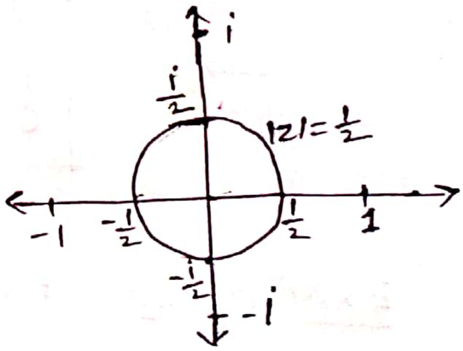
$\therefore$ By cauchy's Integral theorem,

$$\int_C f(z)\, dz = 0$$

$$\Rightarrow \oint_C \frac{z+3}{z^2-2z+5}\, dz = 0$$

Example ③. Evaluate $\int_C \tan z \, dz$, where C is $|z| = \frac{1}{2}$

solution: Given: $C : |z| = \frac{1}{2}$.



let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$

Note that $\cos z = 0 \Rightarrow z = \pm \frac{\pi}{2}$

$\therefore$ $f(z)$ is not analytic at $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$

But Both $z = \frac{\pi}{2}$ and $z = -\frac{\pi}{2}$ lies outside the given circle C i.e. $|z| = \frac{1}{2}$

Hence, $f(z)$ is Analytic everywhere on and inside C

$\therefore$ By cauchy integral theorem,

$$\int_C f(z)\, dz = 0$$

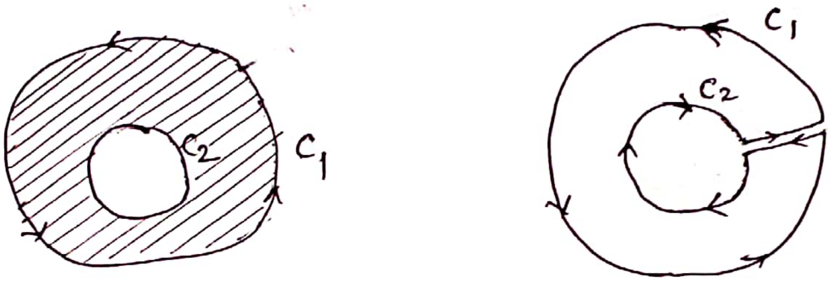
$$\Rightarrow \boxed{\int_C \tan z \, dz = 0}$$

* Cauchy Integral Theorem for Multiply connected region :-

<u>Statement</u> : let C_1 and C_2 be simple closed curves such that C_2 is interior to C_1.

if $f(z)$ is analytic on C_1 and C_2 and $f(z)$ is analytic on each point that is interior to C_1 and exterior to C_2, then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$



<u>Practice Example.</u> Evaluate $\int_C \frac{z^2+z+2}{z^2-7z+2}\,dz$, where

C is the ellipse $25x^2 + 16y^2 = 1$