Discrete Time Signals and Systems

Discrete and Digital Signals

The ***discrete signal*** is a function of a discrete independent variable. The independent variable is divided into uniform intervals and each interval is represented by an integer. The letter "n" is used to denote the independent variable. The discrete or digital signal is denoted by x(n). The discrete signal is defined for every integer value of the independent variable "n". The magnitude (or value) of discrete signal can take any discrete value in the specified range. Here both the value of the signal and the independent variable are discrete.

When the independent variable is time t, the discrete signal is called ***discrete time signal***. In discrete time signal, the time is quantized uniformly using the relation t = nT, where T is the sampling time period. (The sampling time period is inverse of sampling frequency). The discrete time signal is denoted by x(n) or x(nT).

Representation of Discrete Time Signals

1. Functional representation

In functional representation the signal is represented as mathematical equation, as shown in the following example.

$$x(n) = \begin{cases} -1 & ; \; n = -2 \\ 2 & ; \; n = -1 \\ 1.5 & ; \; n = 0 \\ -0.9 & ; \; n = 1 \\ 1.4 & ; \; n = 2 \\ 1.6 & ; \; n = 3 \\ 0 & ; \; \text{other } n \end{cases}$$

2. Graphical representation

In graphical representation the signal is represented in a two dimensional plane. The independent variable is represented in the horizontal axis and the value of the signal is represented in the vertical axis as shown in fig 6.2.

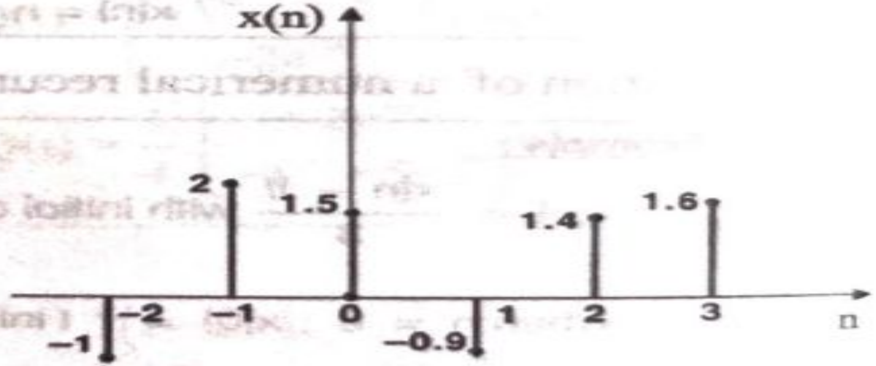

Fig 6.2 : Graphical representation of a discrete time signal.

3. Tabular representation

In tabular representation, two rows of a table are used to represent a discrete time signal. In the first row the independent variable "n" is tabulated and in the second row the value of the signal for each value of "n" are tabulated as shown in the following example.

n	....	–2	–1	0	1	2	3	....
x(n)	....	–1	2	1.5	–0.9	1.4	1.6	....

4. Sequence representation

In sequence representation, the discrete time signal is represented as one dimensional array as shown in the following examples.

An infinite duration discrete time signal with the time origin, $n = 0$, indicated by the symbol ↑ is represented as,

$$x(n) = \{\, \ldots\ -1,\ 2,\ 1.5,\ -0.9,\ \underset{\uparrow}{1.4},\ 1.6,\ \ldots\ \}$$

An infinite duration discrete time signal that satisfies the condition $x(n) = 0$ for $n < 0$ is represented as,

$$x(n) = \{\, \underset{\uparrow}{1.5},\ -0.9,\ 1.4,\ 1.6,\ \ldots\ \} \quad \text{or} \quad x(n) = \{1.5,\ -0.9,\ 1.4,\ 1.6,\ \ldots\ \}$$

A finite duration discrete time signal with the time origin, $n = 0$, indicated by the symbol ↑ is represented as,

$$x(n) = \{\, -1,\ 2,\ \underset{\uparrow}{1.5},\ -0.9,\ 1.4,\ 1.6\, \}$$

A finite duration discrete time signal that satisfies the condition $x(n) = 0$ for $n < 0$ is represented as,

$$x(n) = \{\, \underset{\uparrow}{1.5},\ -0.9,\ 1.4,\ 1.6\, \} \quad \text{or} \quad x(n) = \{\, 1.5,\ -0.9,\ 1.4,\ 1.6\}$$

6.2 Standard Discrete Time Signals

1. Digital impulse signal or Unit sample sequence

Impulse signal, $\delta(n) = 1 \;;\; n = 0$

$= 0 \;;\; n \neq 0$

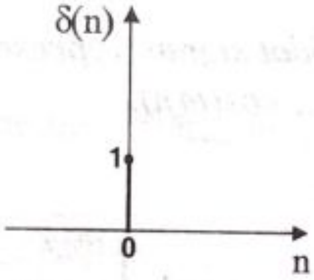

Fig 6.3 : Digital impulse signal.

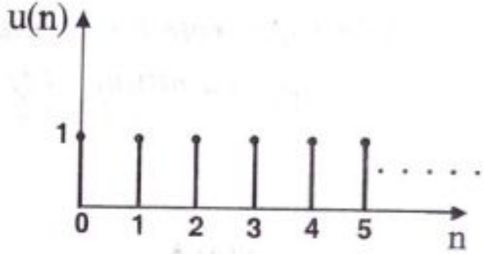

Fig 6.4 : Unit step signal.

2. Unit step signal

Unit step signal, $u(n) = 1 \;;\; n \geq 0$

$= 0 \;;\; n < 0$

3. Ramp signal

Ramp signal, $u_r(n) = n \;;\; n \geq 0$

$= 0 \;;\; n < 0$

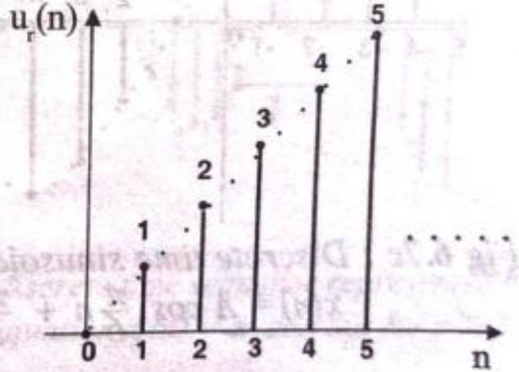

Fig 6.5 : Ramp signal.

4. Exponential signal

Exponential signal, $g(n) = a^n \;;\; n \geq 0$

$$= 0 \;;\; n < 0$$

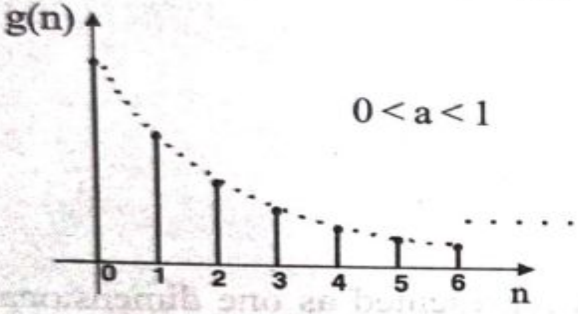

Fig 6.6a : Decreasing exponential signal.

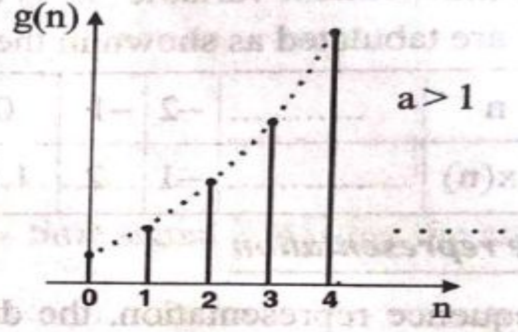

Fig 6.6b : Increasing exponential signal.

Fig 6.6 : Exponential signal.

5. Discrete time sinusoidal signal

The discrete time sinusoidal signal may be expressed as,

$$x(n) = A\cos(\omega_0 n + \theta) \;;\; \text{for n in the range } -\infty < n < +\infty$$

$$x(n) = A\sin(\omega_0 n + \theta) \;;\; \text{for n in the range } -\infty < n < +\infty$$

where, ω_0 = Frequency in radians/sample ; θ = Phase in radians

$$f_0 = \frac{\omega_0}{2\pi} = \text{Frequency in cycles/sample}$$

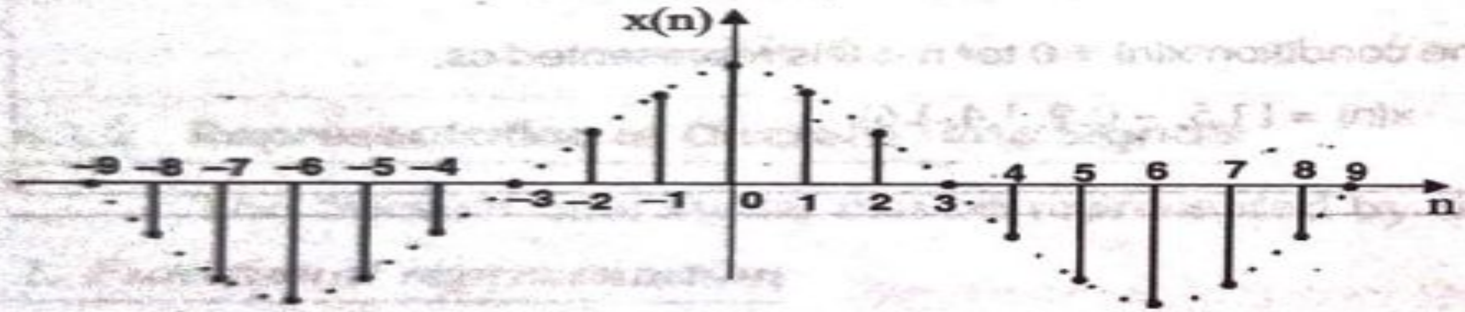

Fig 6.7a : Discrete time sinusoidal signal represented by equation $x(n) = A\cos(\omega_0 n)$.

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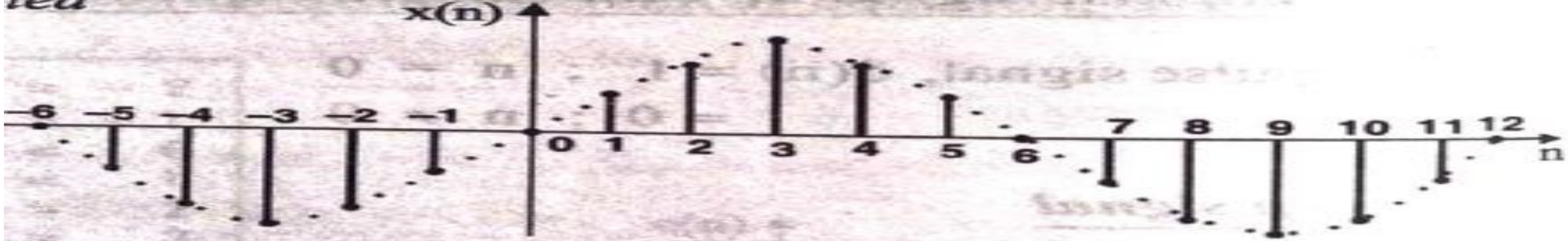


Fig 6.7b : Discrete time sinusoidal signal represented by equation $x(n) = A\sin(\omega_0 n)$.

6. *Discrete time complex exponential signal*

The discrete time complex exponential signal is defined as,

$$x(n) = a^n e^{j(\omega_0 n+\theta)} = a^n[\cos(\omega_0 n + \theta) + j \sin(\omega_0 n + \theta)]$$

$$= a^n \cos(\omega_0 n + \theta) + j\, a^n \sin(\omega_0 n + \theta) = x_r(n) + j\, x_i(n)$$

where, $x_r(n)$ = Real part of $x(n) = a^n \cos(\omega_0 n + \theta)$

$x_i(n)$ = Imaginary part of $x(n) = a^n \sin(\omega_0 n + \theta)$

The real part of x(n) will give an exponentially increasing cosinusoid sequence for $a > 1$ and exponentially decreasing cosinusoid sequence for $0 < a < 1$.

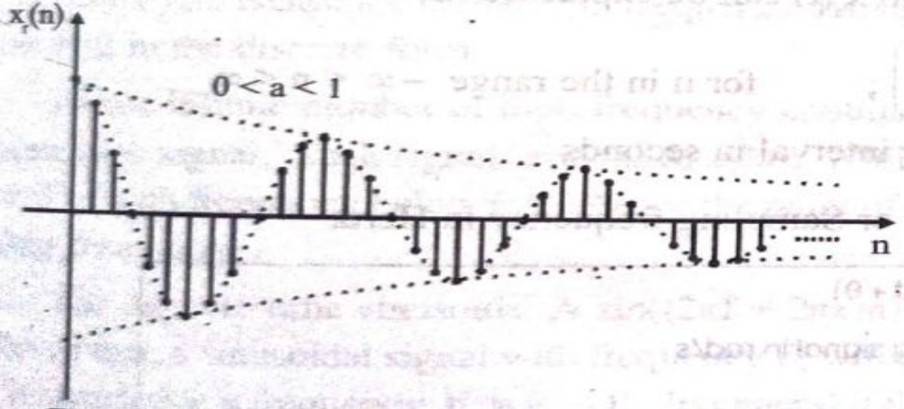


Fig 6.8a : The discrete time sequence represented by the equation, $x_r(n) = a_n \cos \omega_0 n$ for $0 < a < 1$.

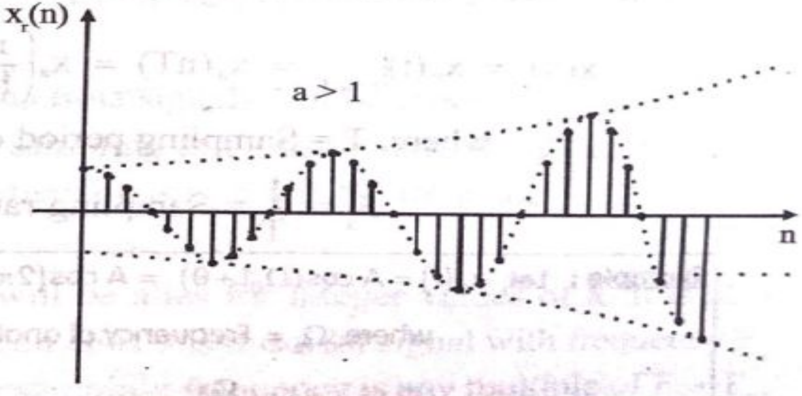


Fig 6.8b : The discrete time sequence represented by the equation $x_r(n) = a_n \cos \omega_0 n$ for $a > 1$.

Fig 6.8 : Real part of complex exponential signal.

The imaginary part of x(n) will give rise to an exponentially increasing sinusoid sequence for a > 1 and exponentially decreasing sinusoid sequence for 0 < a < 1.

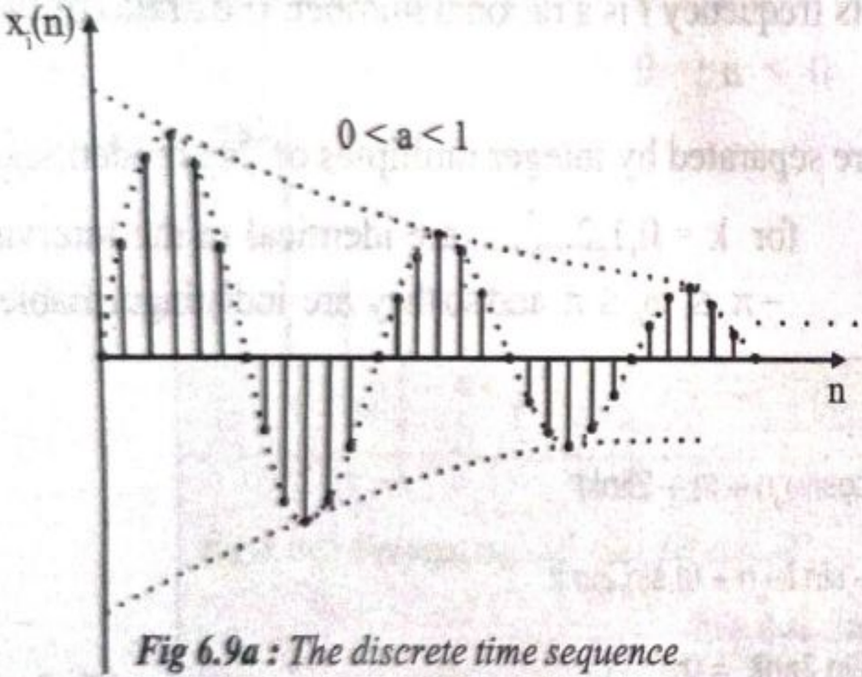


Fig 6.9a : The discrete time sequence represented by the equation, $x_i(n) = a^n \sin \omega_0 n$ for $0 < a < 1$.

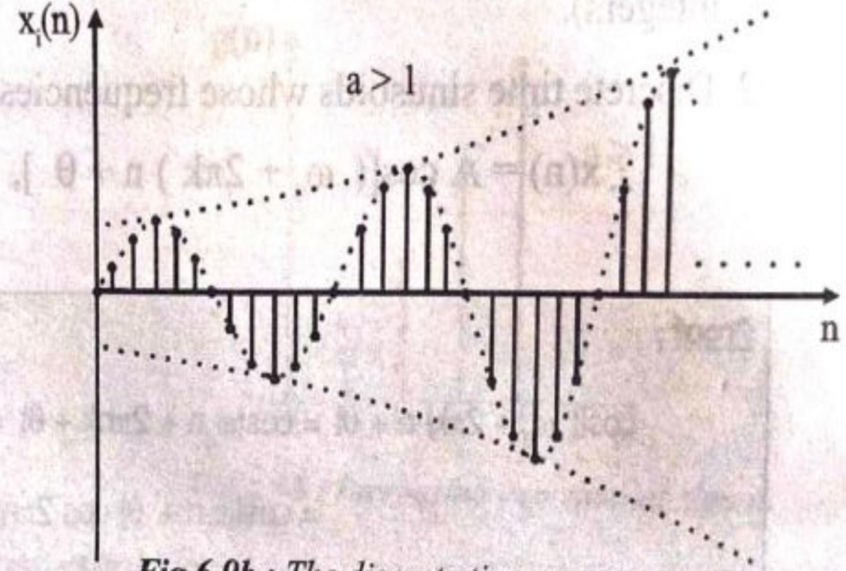


Fig 6.9b : The discrete time sequence represented by the equation $x_i(n) = a^n \sin \omega_0 n$ for $a > 1$.

Fig 6.9 : Imaginary part of complex exponential signal.

Classification of Discrete Time Signals

The discrete time signals are classified depending on their characteristics. Some ways of classifying discrete time signals are,

1. Deterministic and nondeterministic signals
2. Periodic and aperiodic signals
3. Symmetric and antisymmetric signals.
4. Energy signals and power signals
5. Causal and noncausal signals

6.4.1 Deterministic and Nondeterministic Signals

The signals that can be completely specified by mathematical equations are called ***deterministic signals***. The step, ramp, exponential and sinusoidal signals are examples of deterministic signals.

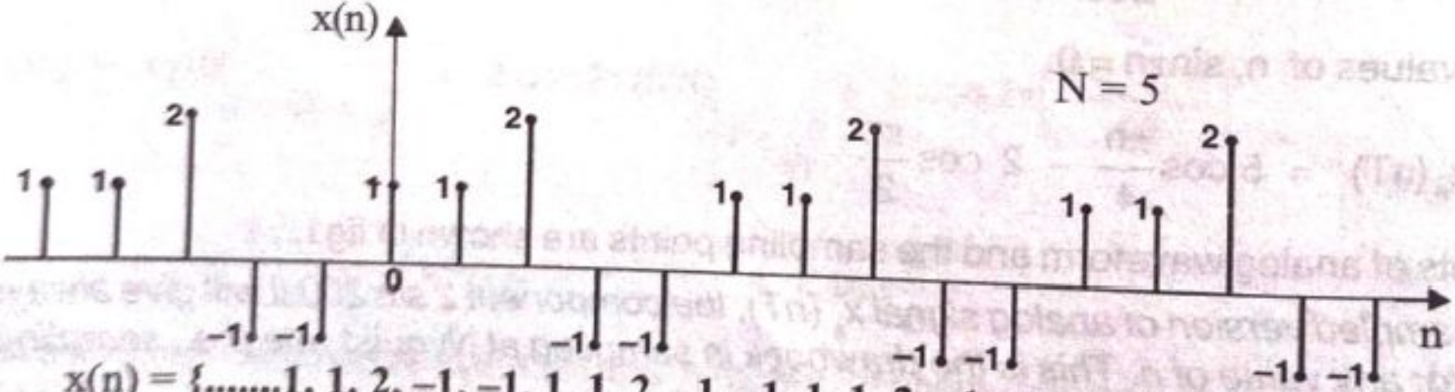
The signals whose characteristics are random in nature are called ***nondeterministic signals***. The noise signals from various sources are best examples of nondeterministic signals.

6.4.2 Periodic and Aperiodic Signals

A signal x(n) is ***periodic*** with periodicity of N samples (where N is an integer) if and only if

$$\boxed{x(n+N) = x(n) \quad ; \text{ for all } n}$$

The smallest value of N for which the above equation is true is called ***fundamental period***. If there is no value of N that satisfies the above equation, then it is called ***aperiodic*** or ***nonperiodic*** signal. When N is the fundamental period, the periodic signals will also satisfy the condition $x(n + kN) = x(n)$, where k is an integer. Periodic signals are power signals. The sinusoidal and complex exponential signals are periodic signals when their fundamental frequency, f_0 is a rational number.



$x(n) = \{.......1, 1, 2, -1, -1, \underset{\uparrow}{1}, 1, 2, -1, -1, 1, 1, 2, -1, -1, ...\}$

Fig 6.10 : Periodic discrete time signal.

Determine whether following signals are periodic or not. If periodic find the fundamental period.

a) $x(n) = \sin\left(\frac{6\pi}{7}n + 1\right)$ b) $x(n) = \cos\left(\frac{n}{8} - \pi\right)$ c) $x(n) = \cos\frac{\pi}{8}n^2$ d) $x(n) = e^{j7\pi n}$

Solution

a) Given that, $x(n) = \sin\left(\frac{6\pi}{7}n + 1\right)$

Let N and M be two integers.

Now, $x(n+N) = \sin\left(\frac{6\pi}{7}(n+N) + 1\right) = \sin\left(\frac{6\pi n}{7} + 1 + \frac{6\pi}{7}N\right)$

Since $\sin(\theta + 2\pi M) = \sin\theta$, for periodicity $\frac{6\pi}{7}N$ should be integral multiple of 2π.

Let, $\frac{6\pi}{7}N = M \times 2\pi$, where M and N are integers.

$$\therefore N = M \times 2\pi \times \frac{7}{6\pi} = \frac{7M}{3}$$

Here N is an integer if, $M = 3, 6, 9, 12, \ldots\ldots$

Let, $M = 3$; $\therefore N = 7$

When $N = 7$; $x(n+N) = \sin\left(\frac{6\pi n}{7} + 1 + \frac{6\pi}{7} \times 7\right) = \sin\left(\frac{6\pi n}{7} + 1 + 6\pi\right) = \sin\left(\frac{6\pi n}{7} + 1\right) = x(n)$

Hence x(n) is periodic with fundamental period of 7 samples.

b) Given that, $x(n) = \cos\left(\frac{n}{8} - \pi\right)$

Let N and M be two integers.

Now, $x(n + N) = \cos\left(\frac{n + N}{8} - \pi\right) = \cos\left(\frac{n}{8} + \frac{N}{8} - \pi\right) = \cos\left(\frac{n}{8} - \pi + \frac{N}{8}\right)$

Since $\cos(\theta + 2\pi M) = \cos\theta$, for periodicity $\frac{N}{8}$ should be equal to integral multiple of 2π.

Let, $\frac{N}{8} = M \times 2\pi$; where M and N are integers.

$\therefore N = 16\pi M$

Here N cannot be an integer for any integer value of M and so x(n) will not be periodic.

c) Given that, $x(n) = \cos\left(\frac{\pi}{8}n^2\right)$

The general form of discrete time cosinusoid is $x(n) = \cos(2\pi f_o n)$.

Let, $2\pi f_o n = \frac{\pi}{8}n^2$

$$\therefore f_o = \frac{\pi}{8}n^2 \times \frac{1}{2\pi n} = \frac{n}{16}$$

Since n is an integer, f_0 is a rational number and so $\cos\left(\frac{\pi}{8}n^2\right)$ is periodic.

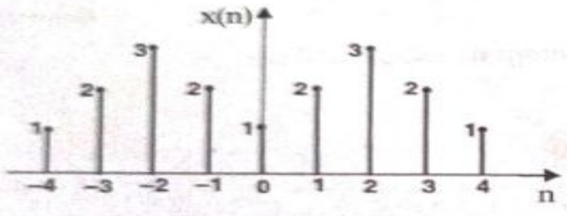
6.4.3 Symmetric (Even) and Antisymmetric (Odd) Signals

The signals may exhibit symmetry or antisymmetry with respect to n = 0. When a signal exhibits symmetry with respect to n = 0 then it is called an ***even signal.*** Therefore the even signal satisfies the condition,

$$x(-n) = x(n)$$

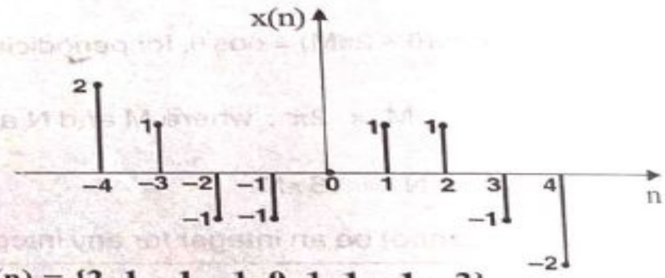
When a signal exhibits antisymmetry with respect to n = 0, then it is called an ***odd signal***. Therefore the odd signal satisfies the condition,

$$x(-n) = -x(n)$$



$$x(n) = \{1, 2, 3, 2, \underset{\uparrow}{1}, 2, 3, 2, 1\}$$

Fig 6.11a : Symmetric (or even) signal.



$$x(n) = \{2, 1, -1, -1, \underset{\uparrow}{0}, 1, 1, -1, -2\}$$

Fig 6.11b : Antisymmetric (or odd) signal.

Fig 6.11 : Symmetric and antisymmetric discrete time signal.

A discrete signal x(n) which is neither even nor odd can be expressed as a sum of even and odd signal.

Let, $\quad x(n) = x_e(n) + x_o(n)$

where, $\quad x_e(n)$ = Even part of x(n) ; $\quad x_o(n)$ = Odd part of x(n)

Note : If x(n) is even then its odd part will be zero. If x(n) is odd then its even part will be zero.

Now, it can be proved that,

$$\boxed{\begin{aligned} x_e(n) &= \frac{1}{2}[x(n) + x(-n)] \\ x_o(n) &= \frac{1}{2}[x(n) - x(-n)] \end{aligned}}$$

Proof :

Let, $x(n) = x_e(n) + x_o(n)$ ......(6.8)

On replacing n by −n in equation (6.8) we get,

$$x(-n) = x_e(-n) + x_o(-n) \quad \text{......(6.9)}$$

Since $x_e(n)$ is even, $x_e(-n) = x_e(n)$

Since $x_o(n)$ is odd, $x_o(-n) = -x_o(n)$

Hence the equation (6.9) can be written as,

$$x(-n) = x_e(n) - x_o(n) \quad \text{......(6.10)}$$

On adding equation (6.8) and (6.10) we get,

$$x(n) + x(-n) = 2\,x_e(n)$$

$$\therefore x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

On subtracting equation (6.10) from equation (6.8) we get,

$$x(n) - x(-n) = 2\,x_o(n)$$

$$\therefore x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

Example 6.5

Determine the even and odd parts of the signals.

a) $x(n) = a^n$ b) $x(n) = 2\,e^{j\frac{\pi}{3}n}$ c) $x(n) = \{4, -4, 2, -2\}$

Solution

a) Given that, $x(n) = a^n$

$$\therefore\ x(-n) = a^{-n}$$

$$\text{Even part, } x_e(n) = \frac{1}{2}[x(n) + x(-n)] = \frac{1}{2}[a^n + a^{-n}]$$

$$\text{Odd part, } x_o(-n) = \frac{1}{2}[x(n) - x(-n)] = \frac{1}{2}[a^n - a^{-n}]$$

b) Given that, $x(n) = 2\, e^{j\frac{\pi}{3}n}$

$$x(n) = 2\, e^{j\frac{\pi}{3}n} = 2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n$$

$$\therefore x(-n) = 2\, e^{-j\frac{\pi}{3}n} = 2\cos\frac{\pi}{3}n - j2\sin\frac{\pi}{3}n$$

$$\text{Even part, } x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

$$= \frac{1}{2}\left[2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n + 2\cos\frac{\pi}{3}n - j2\sin\frac{\pi}{3}n\right] = \frac{1}{2}\left[4\cos\frac{\pi}{3}n\right] = 2\cos\frac{\pi}{3}n$$

$$\text{Odd part, } x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

$$= \frac{1}{2}\left[2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n - 2\cos\frac{\pi}{3}n + j2\sin\frac{\pi}{3}n\right] = \frac{1}{2}\left[j4\sin\frac{\pi}{3}n\right] = j2\sin\frac{\pi}{3}n$$

c) **Given that, $x(n) = \{\underset{\uparrow}{4}, -4, 2, -2\}$**

Given that, $x(n) = \{\underset{\uparrow}{4}, -4, 2, -2\}$, $\therefore\ x(0) = 4 \;;\; x(1) = -4 \;;\; x(2) = 2 \;;\; x(3) = -2$

$x(-n) = \{-2, 2, -4, \underset{\uparrow}{4}\}$, $\therefore\ x(-3) = -2 \;;\; x(-2) = 2 \;;\; x(-1) = -4 \;;\; x(0) = 4$

Even part, $x_e(n) = \frac{1}{2}\,[x(n) + x(-n)]$

At $n = -3$; $x(n) + x(-n) = 0 + (-2) = -2$

At $n = -2$; $x(n) + x(-n) = 0 + 2 = 2$

At $n = -1$; $x(n) + x(-n) = 0 + (-4) = -4$

At $n = 0$; $x(n) + x(-n) = 4 + 4 = 8$

At $n = 1$; $x(n) + x(-n) = -4 + 0 = -4$

At $n = 2$; $x(n) + x(-n) = 2 + 0 = 2$

At $n = 3$; $x(n) + x(-n) = -2 + 0 = -2$

$\therefore x(n) + x(-n) = \{-2, 2, -4, \underset{\uparrow}{8}, -4, 2, -2\}$

$\therefore\ x_e(n) = \frac{1}{2}\,[x(n) + x(-n)]$

$= \{-1, 1, -2, \underset{\uparrow}{4}, -2, 1, -1\}$

Odd part, $x_o(n) = \frac{1}{2}\,[x(n) - x(-n)]$

At $n = -3$; $x(n) - x(-n) = 0 - (-2) = 2$

At $n = -2$; $x(n) - x(-n) = 0 - 2 = -2$

At $n = -1$; $x(n) - x(-n) = 0 - (-4) = 4$

At $n = 0$; $x(n) - x(-n) = 4 - 4 = 0$

At $n = 1$; $x(n) - x(-n) = -4 - 0 = -4$

At $n = 2$; $x(n) - x(-n) = 2 - 0 = 2$

At $n = 3$; $x(n) - x(-n) = -2 - 0 = -2$

$\therefore x(n) - x(-n) = \{2, -2, 4, \underset{\uparrow}{0}, -4, 2, -2\}$

$\therefore\ x_o(n) = \frac{1}{2}\,[x(n) - x(-n)]$

$= \{1, -1, 2, \underset{\uparrow}{0}, -2, 1, -1\}$

6.4.4 Energy and Power Signals

The *energy* E of a discrete time signal x(n) is defined as,

$$\boxed{\text{Energy, } E = \sum_{n=-\infty}^{\infty} |x(n)|^2} \quad .....(6.11)$$

The energy of a signal may be finite or infinite, and can be applied to complex valued and real valued signals.

If energy E of a signal is finite and non-zero, then the signal is called an ***energy signal.*** The exponential signals are examples of energy signals.

The average ***power*** of a discrete time signal x(n) is defined as,

$$\boxed{\text{Power, } P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2} \quad .....(6.12)$$

If power P of a signal is finite and non-zero, then the signal is called a ***power signal***. The periodic signals are examples of power signals.

For energy signals, the energy will be finite and average power will be zero. For power signals the average power is finite and energy will be infinite.

i.e., For energy signal, $0 < E < \infty$ and $P = 0$

For power signal, $0 < P < \infty$ and $E = \infty$

Example 6.6

Determine whether the following signals are energy or power signals.

a) $x(n) = \left(\frac{1}{4}\right)^n u(n)$ b) $x(n) = \cos\left(\frac{\pi}{3}n\right)$ c) $x(n) = u(n)$

$P = \frac{1}{2N+1}\sum_{n=-N}^{N} 1$

Solution

a) Given that, $x(n) = \left(\frac{1}{4}\right)^n u(n)$

Here, $x(n) = \left(\frac{1}{4}\right)^n u(n)$ for all n.

$\therefore\ x(n) = \left(\frac{1}{4}\right)^n = 0.25^n\ ;\quad n \geq 0$

Infinite geometric series sum formula.

$$\sum_{n=0}^{\infty} C^n = \frac{1}{1-C}$$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{\infty} \left|(0.25)^n\right|^2 = \sum_{n=0}^{\infty} (0.25^2)^n$$

$$= \sum_{n=0}^{\infty} (0.0625)^n = \frac{1}{1-0.0625} = 1.067 \text{ joules}$$

Using infinite geometric series sum formula

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left|(0.25)^n\right|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25^2)^n = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{(0.0625)^{N+1} - 1}{0.0625 - 1}$$

Using finite geometric series sum formula

Finite geometric series sum formula.

$$\sum_{n=0}^{\infty} C^n = \frac{C^{N+1} - 1}{C - 1}$$

$$= \frac{1}{\infty} \times \frac{0.0625 - 1}{0.0625 - 1} = 0$$

Here E is finite and P is zero and so x(n) is an energy signal.

b) Given that, $x(n) = \cos\left(\frac{\pi}{3}n\right)$

$$\text{Energy, } E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=-\infty}^{+\infty} \cos^2\left(\frac{\pi}{3}n\right) = \sum_{n=-\infty}^{+\infty} \frac{1 + \cos\frac{2\pi}{3}n}{2}$$

$$= \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty}\left(1 + \cos\frac{2\pi}{3}n\right)\right) = \frac{1}{2}\left(\sum_{n=-\infty}^{+\infty} 1^n + \sum_{n=-\infty}^{+\infty} \cos\frac{2\pi}{3}n\right) = \frac{1}{2}(\infty + 0) = \infty$$

***Note**: Sum of infinite 1's is infinity. Sum of samples of one period of sinusoidal signal is zero.*

$$\text{Power, } P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \cos^2 \frac{\pi n}{3}$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{\left(1 + \cos\frac{2\pi}{3}n\right)}{2}$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2}\left[\sum_{n=-N}^{N} 1^n + \sum_{n=-N}^{N} \cos\frac{2\pi}{3}n\right]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2}\left[\underbrace{1 + 1 + \ldots\ldots + 1}_{N \text{ terms}} + 1 + \underbrace{1 + \ldots\ldots + 1 + 1}_{N \text{ terms}} + 0\right]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{1}{2}[2N+1] = \underset{N\to\infty}{Lt} \frac{1}{2} = \frac{1}{2} \text{ watts}$$

Since P is finite and E is inifinite, x(n) is a power signal.

c) Given that, $x(n) = u(n)$

$$E = \sum_{n=-\infty}^{+\infty} |x(n)|^2 = \sum_{n=0}^{+\infty} (u(n))^2 = \sum_{n=0}^{+\infty} u(n) = 1 + 1 + 1 \,.........\infty = \infty$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} u(n) = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \left(\underbrace{1 + 1 + 1 + ...+1}_{N+1 \text{ terms}} \right)$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} (N+1) = \underset{N\to\infty}{Lt} \frac{N\left(1 + \frac{1}{N}\right)}{N\left(2 + \frac{1}{N}\right)} = \frac{1 + \frac{1}{\infty}}{2 + \frac{1}{\infty}} = \frac{1+0}{2+0} = \frac{1}{2} \text{ watts}$$

Since P is finite and E is infinite, x(n) is a power signal.

6.4.5 Causal, Noncausal and Anticausal signals

A signal is said to be ***causal***, if it is defined for $n \geq 0$. Therefore if $x(n)$ is causal, then $x(n) = 0$ for $n < 0$.

A signal is said to be ***noncausal***, if it is defined for either $n \leq 0$, or for both $n \leq 0$ and $n > 0$. Therefore if $x(n)$ is noncausal, then $x(n) \neq 0$ for $n < 0$. A noncausal signal can be converted to causal signal by multiplying the noncausal signal by a unit step signal, $u(n)$.

When a noncausal signal is defined only for $n \leq 0$, it is called an ***anticausal signal***.

Examples of Causal and Noncausal Signals

$$\left.\begin{array}{l} x(n) = \{\underset{\uparrow}{1}, -1, 2, -2, 3, -3\} \\[2ex] x(n) = \{\underset{\uparrow}{2}, 2, 3, 3, \ldots\ldots\ldots\ldots\} \end{array}\right\} \text{Causal signals}$$

$$\left.\begin{array}{l} x(n) = \{1, -1, 2, -2, 3, \underset{\uparrow}{-3}\} \\[2ex] x(n) = \{\ldots\ldots\ldots\ldots, 2, 2, 3, \underset{\uparrow}{3}\} \\[2ex] x(n) = \{2, 3, 4, \underset{\uparrow}{5}, 4, 3, 2\} \\[2ex] x(n) = \{\ldots\ldots, 2, 3, 4, \underset{\uparrow}{5}, 4, 3, 2, \ldots\ldots\} \end{array}\right\} \text{Noncausal signals}$$

Mathematical Operations on Discrete Time Signals

Scaling of Discrete Time Signals

Amplitude Scaling or Scalar Multiplication

Amplitude scaling of a signal by a constant A is accomplished by multiplying the value of every signal sample by the constant A.

Example :

Let y(n) be amplitude scaled signal of x(n), then $y(n) = A\,x(n)$

Let, $x(n) = 20 \; ; \; n = 0$ and $A = 0.1$,	When $n = 0 \; ; \; y(0) = A\,x(0) = 0.1 \times 20 = 2.0$
$= 36 \; ; \; n = 1$	When $n = 1 \; ; \; y(1) = A\,x(1) = 0.1 \times 36 = 6.6$
$= 40 \; ; \; n = 2$	When $n = 2 \; ; \; y(2) = A\,x(2) = 0.1 \times 40 = 4.0$
$= -15 \; ; \; n = 3$	When $n = 3 \; ; \; y(3) = A\,x(3) = 0.1 \times (-15) = -1.5$

Time Scaling (or Down Sampling and Up Sampling)

There are two ways of time scaling a discrete time signal. They are down sampling and up sampling.

In a signal x(n), if n is replaced by Dn, where D is an integer, then it is called ***down sampling***.

In a signal x(n), if n is replaced by $\frac{n}{I}$, where I is an integer, then it is called ***up sampling***.

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In a signal x(n), if n is replaced by $\frac{n}{I}$, where I is an integer, then it is called ***up sampling***.

Example :

If $x(n) = a^n \; ; \; n \geq 0 \; ; \; 0 < a < 1$

then $x_1(n) = x(2n)$ will be a down sampled version of x(n) and $x_2(n) = x\left(\frac{n}{2}\right)$ will be an up sampled version on x(n).

When $n = 0$; $x_1(0) = x(0) = a^0$

When $n = 1$; $x_1(1) = x(2) = a^2$

When $n = 2$; $x_1(2) = x(4) = a^4$

When $n = 3$; $x_1(3) = x(6) = a^6$ and so on.

When $n = 0$; $x_2(0) = x\left(\frac{0}{2}\right) = x(0) = a^0$

When $n = 1$; $x_2(1) = x\left(\frac{1}{2}\right) = 0$

When $n = 2$; $x_2(2) = x\left(\frac{2}{2}\right) = x(1) = a^1$

When $n = 3$; $x_2(3) = x\left(\frac{3}{2}\right) = 0$

When $n = 4$; $x_2(4) = x\left(\frac{4}{2}\right) = x(2) = a^2$ and so on.

Example :

If $x(n) = a^n \; ; \; n \geq 0 \; ; \; 0 < a < 1$

then $x_1(n) = x(2n)$ will be a down sampled version of x(n) and $x_2(n) = x\left(\frac{n}{2}\right)$ will be an up sampled version on x(n).

When $n = 0 \; ; \quad x_1(0) = x(0) = a^0$

When $n = 1 \; ; \quad x_1(1) = x(2) = a^2$

When $n = 2 \; ; \quad x_1(2) = x(4) = a^4$

When $n = 3 \; ; \quad x_1(3) = x(6) = a^6$ and so on.

When $n = 0 \; ; \; x_2(0) = x\left(\frac{0}{2}\right) = x(0) = a^0$

When $n = 1 \; ; \; x_2(1) = x\left(\frac{1}{2}\right) = 0$

When $n = 2 \; ; \; x_2(2) = x\left(\frac{2}{2}\right) = x(1) = a^1$

When $n = 3 \; ; \; x_2(3) = x\left(\frac{3}{2}\right) = 0$

When $n = 4 \; ; \; x_2(4) = x\left(\frac{4}{2}\right) = x(2) = a^2$ and so on.

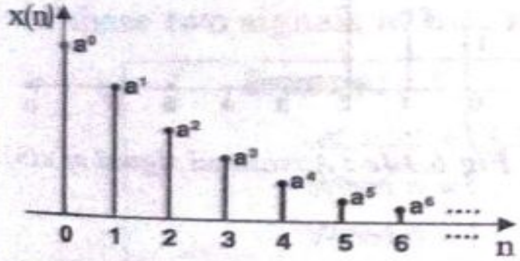


Fig 6.12a : A discrete time signal x(n).

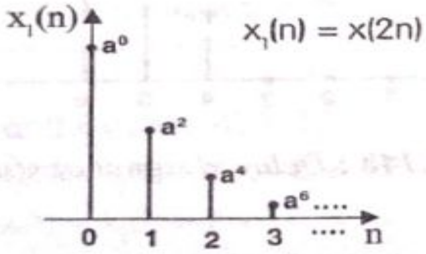


Fig 6.12b : Down sampled signal of x(n).

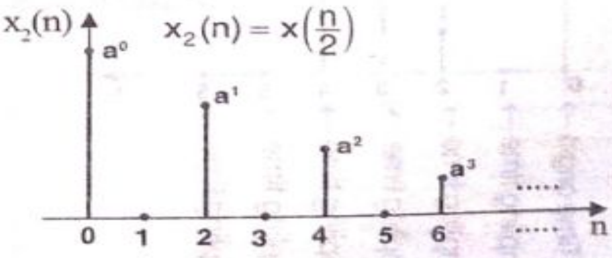


Fig 6.12c : Up sampled signal of x(n).

Fig 6.12 : A discrete time signal and its time scaled version.

Folding (Reflection or transpose) of Discrete Time Signals

The *folding* of a signal x(n) is performed by changing the sign of the time base n in x(n). The folding operation produces a signal x(–n) which is a mirror image of x(n) with respect to time origin n = 0.

Example :

Let $x(n) = n\ ;\ -3 \le n \le 3$. Now the folded signal, $x_1(n) = x(-n) = -n\ ;\ -3 \le n \le 3$

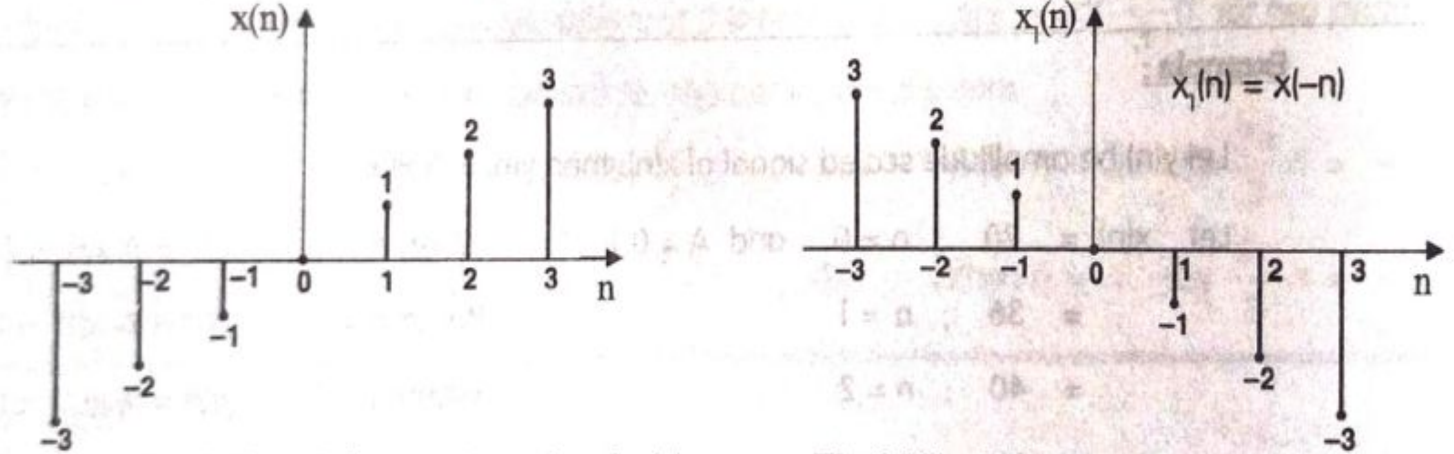


Fig 6.13a : A discrete time signal x(n).

Fig 6.13b : Folded signal of x(n).

Fig 6.13 : A discrete time signal and its folded version.

Time shifting of Discrete Time Signals

A signal x(n) may be shifted in time by replacing the independent variable n by n – m, where m is an integer. If m is a positive integer, the time shift results in a delay by m units of time. If m is a negative integer, the time shift results in an advance of the signal by |m| units in time. The ***delay*** results in shifting each sample of x(n) to the right. The ***advance*** results in shifting each sample of x(n) to the left.

Example :

$$x(n) = \begin{cases} 1 & ; \; n = 2 \\ 2 & ; \; n = 3 \\ 3 & ; \; n = 4 \\ 0 & ; \text{ for other } n \end{cases}$$

Let, $x_1(n) = x(n-2)$, where $x_1(n)$ is delayed signal of $x(n)$

When $n = 4$; $x_1(4) = x(4-2) = x(2) = 1$

When $n = 5$; $x_1(5) = x(5-2) = x(3) = 2$

When $n = 6$; $x_1(6) = x(6-2) = x(4) = 3$

The sample $x(2)$ is available at $n = 2$ in the original sequence $x(n)$, but the same sample is available at $n = 4$ in $x_1(n)$. Similarly every sample of $x(n)$ is delayed by two sampling times.

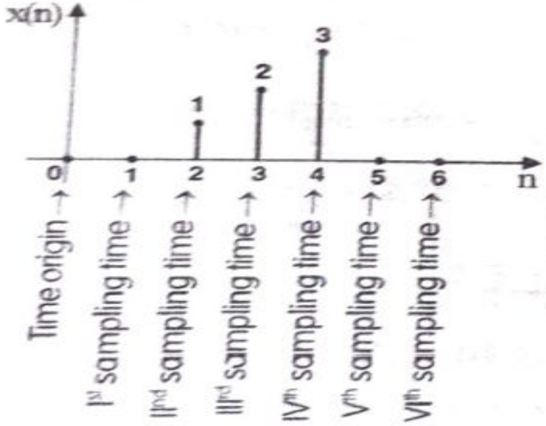
Let, $x_2(n) = x(n+2)$, where $x_2(n)$ is an advanced signal of $x(n)$

When $n = 0$; $x_2(n) = x(0+2) = x(2) = 1$

When $n = 1$; $x_2(n) = x(1+2) = x(3) = 2$

When $n = 2$; $x_2(n) = x(2+2) = x(4) = 3$

The sample $x(2)$ is available at $n = 2$ in the original sequence $x(n)$, but the same sample is available at $n = 0$ in $x_2(n)$. Similarly every sample of $x(n)$ is advanced by two sampling times. Hence the signal $x_2(n)$ is an advanced version of $x(n)$.



g 6.14a : A discrete time signal x(n).

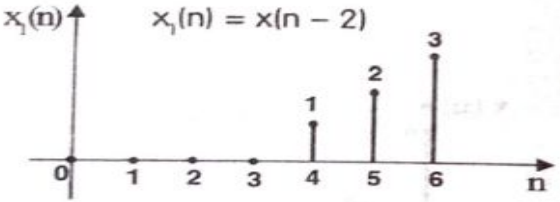


Fig 6.14b : Delayed signal of x(n).

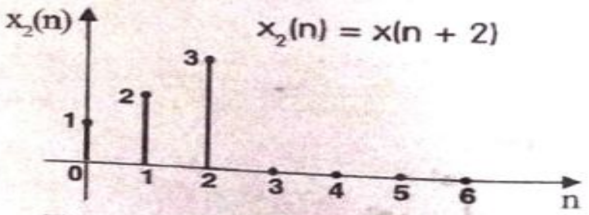


Fig 6.14c : Advanced signal of x(n).

Fig 6.14 : A discrete time signal and its shifted version.

Delayed Unit Impulse Signal

The unit impulse signal is defined as,

$$\delta(n) = 1 \; ; \text{ for } n = 0$$
$$= 0 \; ; \text{ for } n \neq 0$$

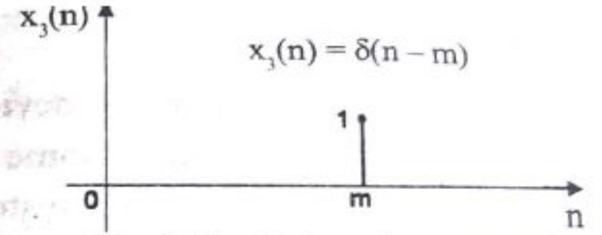


Fig 6.15 : Delayed unit impulse.

The unit impulse signal delayed by m units of time is denoted as $\delta(n - m)$.

Now, $\delta(n - m) = 1 \; ; n = m$
$$= 0 \; ; n \neq m$$

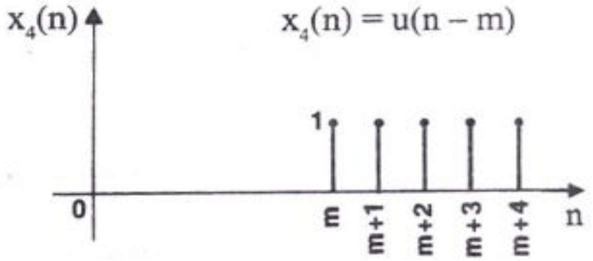


Fig 6.16 : Delayed unit step signal.

Delayed Unit Step Signal

The unit step signal is defined as,

$$u(n) = 1 \; ; \text{ for } n \geq 0$$
$$= 0 \; ; \text{ for } n < 0$$

The unit step signal delayed by m units of time is denoted as $u(n - m)$.

Now, $u(n - m) = 1 \; ; n \geq m$
$$= 0 \; ; n < m$$

Addition of Discrete Time Signals

The *addition* of two signals is performed on a sample-by-sample basis.

The sum of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the sum of the samples of these two signals at that instant.

$$\text{i.e., } y(n) = x_1(n) + x_2(n) \ ; \ -\infty < n < \infty.$$

Example :

Let, $x_1(n) = \{1, 2, -1\}$ and $x_2(n) = \{-2, 1, 3\}$

When $n = 0$; $y(0) = x_1(0) + x_2(0) = 1 + (-2) = -1$

When $n = 1$; $y(1) = x_1(1) + x_2(1) = 2 + 1 = 3$

When $n = 2$; $y(2) = x_1(2) + x_2(2) = -1 + 3 = 2$

$\therefore \ y(n) = x_1(n) + x_2(n) = \{-1, 3, 2\}$

Multiplication of Discrete Time Signals

The ***multiplication*** of two signals is performed on a sample-by-sample basis. The product of two signals $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value at any instant is equal to the product of the samples of these two signals at that instant. The product is also called ***modulation***.

Example :

Let, $x_1(n) = \{1, 2, -1\}$ and $x_2(n) = \{-2, 1, 3\}$

When $n = 0$; $y(0) = x_1(0) \times x_2(0) = 1 \times (-2) = -2$

When $n = 1$; $y(1) = x_1(1) \times x_2(1) = 2 \times 1 = 2$

When $n = 2$; $y(2) = x_1(2) \times x_2(2) = -1 \times 3 = -3$

$\therefore y(n) = x_1(n) \times x_2(n) = \{-2, 2, -3\}$

Discrete Time System

A ***discrete time system*** is a device or algorithm that operates on a discrete time signal, called the input or excitation, according to some well defined rule, to produce another discrete time signal called the output or the response of the system. We can say that the input signal x(n) is transformed by the system into a signal y(n), and the transformation can be expressed mathematically as shown in equation (6.13).The diagrammatic representation of discrete time system is shown in fig 6.17.

$$\text{Response,}\quad y(n) = \mathcal{H}\{x(n)\} \qquad .....(6.13)$$

where, $\mathcal{H}$ denotes the transformation (also called an operator).

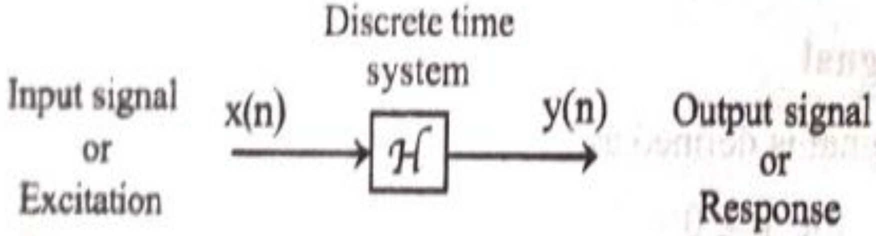


Fig 6.17 : Representation of discrete time system.

Classification of Discrete Time Systems

The discrete time systems are classified based on their characteristics. Some of the classifications of discrete time systems are,

1. Static and dynamic systems
2. Time invariant and time variant systems
3. Linear and nonlinear systems
4. Causal and noncausal systems
5. Stable and unstable systems
6. FIR and IIR systems
7. Recursive and nonrecursive systems

Static and Dynamic Systems

A discrete time system is called ***static*** or ***memoryless*** if its output at any instant n depends at most on the input sample at the same time but not on the past or future samples of the input. In any other case, the system is said to be ***dynamic*** or to have memory.

Example :

$$\left.\begin{array}{l} y(n) = a\,x(n) \\ y(n) = n\,x(n) + 6\,x^3(n) \end{array}\right\} \text{Static systems}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} y(n) = x(n) + 3\,x(n-1) \\ y(n) = \displaystyle\sum_{m=0}^{N} x(n-m) \end{array}\right\} \text{Finite memory is required} \\ \left. y(n) = \displaystyle\sum_{m=0}^{\infty} x(n-m) \right\} \text{Infinite memory is required} \end{array}\right\} \text{Dynamic systems}$$

Time Invariant and Time Variant Systems

A system is said to be *time invariant* if its input-output characteristics do not change with time.

<u>*Definition*</u> : A relaxed system $\mathcal{H}$ is ***time invariant*** or ***shift invariant*** if and only if

$$x(n) \xrightarrow{\mathcal{H}} y(n) \text{ implies that, } x(n-m) \xrightarrow{\mathcal{H}} y(n-m)$$

for every input signal x(n) and every time shift m.

i.e., in time invariant systems, if $\mathbf{y(n) = \mathcal{H}\{x(n)\}}$ then $\mathbf{y(n-m) = \mathcal{H}\{x(n-m)\}}$.

Alternative Definition for Time Invariance

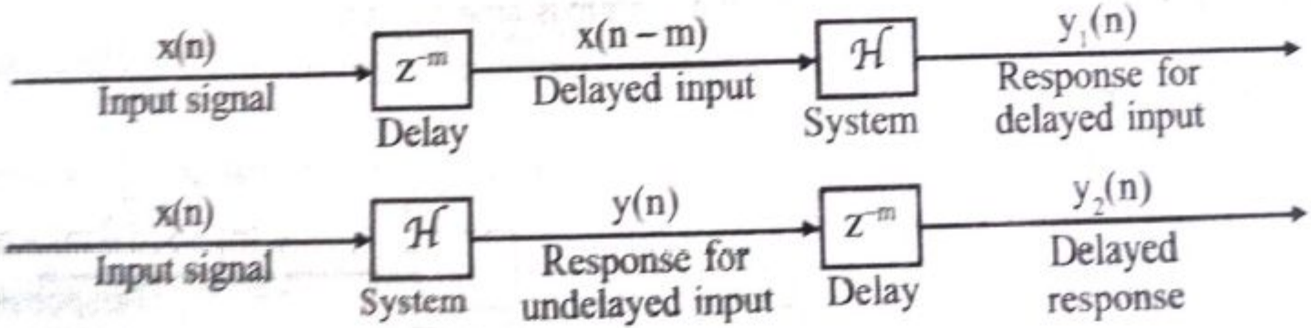
A system $\mathcal{H}$ is ***time invariant*** if the response to a shifted (or delayed) version of the input is identical to a shifted (or delayed) version of the response based on the unshifted (or undelayed) input.

i.e., In a time invariant system, $\mathbf{\mathcal{H}\{x(n-m)\} = z^{-m}\, \mathcal{H}\{x(n)\}}$; for all values of m .....(6.22)

The operator z^{-m} represents a signal delay of m samples.

Procedure to test for time invariance

1. Delay the input signal by m units of time and determine the response of the system for this delayed input signal. Let this response be $y_1(n)$.
2. Delay the response of the system for undelayed input by m units of time. Let this delayed response be $y_2(n)$.
3. Check whether $y_1(n) = y_2(n)$. If they are equal then the system is time invariant. Otherwise the system is time variant.



If, $y_1(n) = y_2(n)$, then the system is time invariant

Fig 6.19 : Diagrammatic explanation of time invariance.

Example 6.10

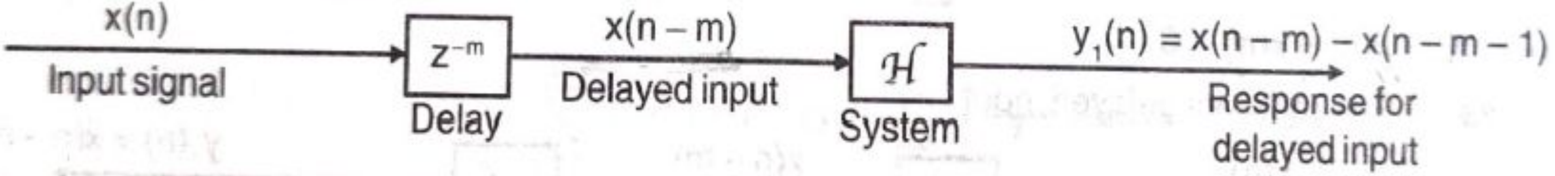
Test the following systems for time invariance.

a) $y(n) = x(n) - x(n-1)$ b) $y(n) = n\,x(n)$ c) $y(n) = x(-n)$ d) $y(n) = x(n) - b\,x(n-1)$

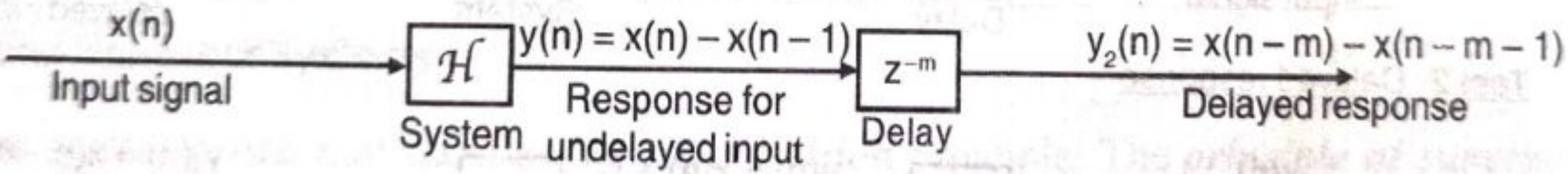
Solution

a) Given that, $y(n) = x(n) - x(n-1)$

<u>**Test 1** : Response for delayed input</u>



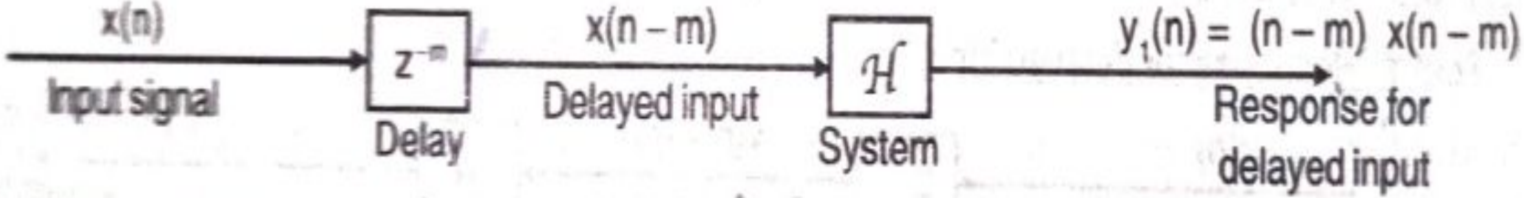
<u>**Test 2** : Delayed response</u>



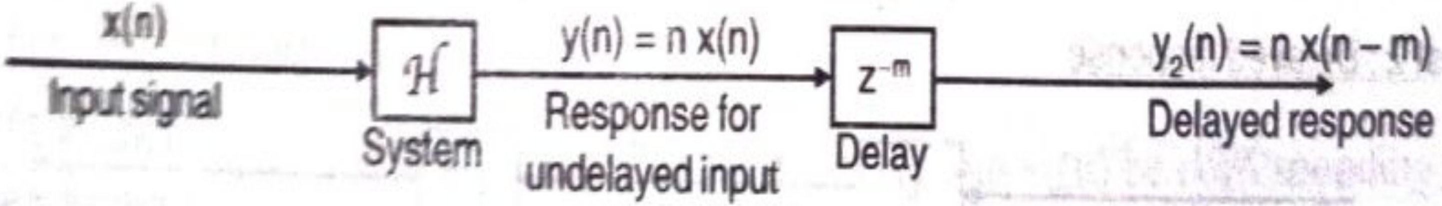
<u>**Conclusion**</u> : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

b) Given that, $y(n) = n\,x(n)$

<u>Test 1 : Response for delayed input</u>



<u>Test 2 : Delayed response</u>

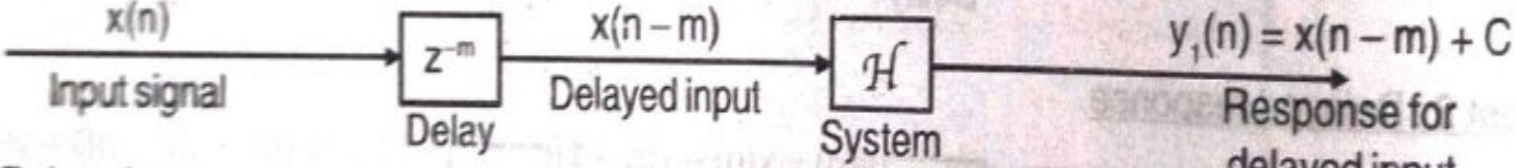


<u>**Conclusion**</u> : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

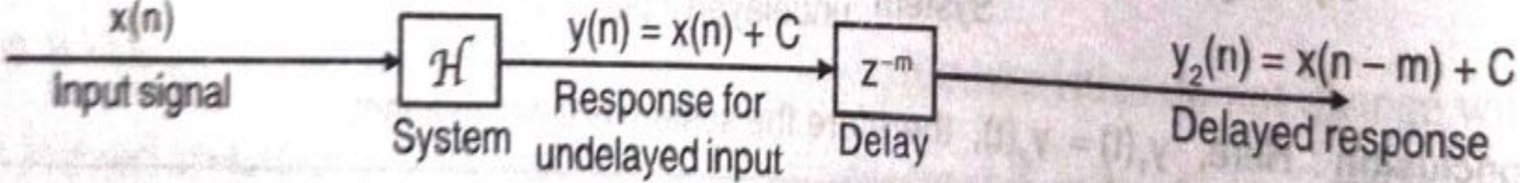
Test the following systems for time invariance.

a) Given that, $y(n) = x(n) + C$

<u>Test 1 : Response for delayed input</u>



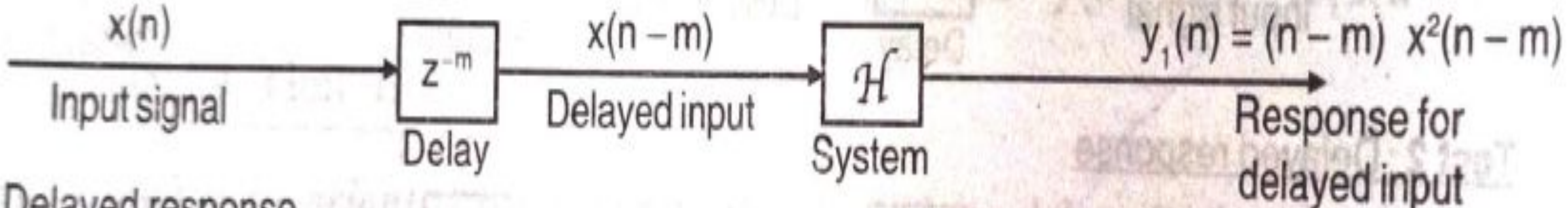
<u>Test 2 : Delayed response</u>



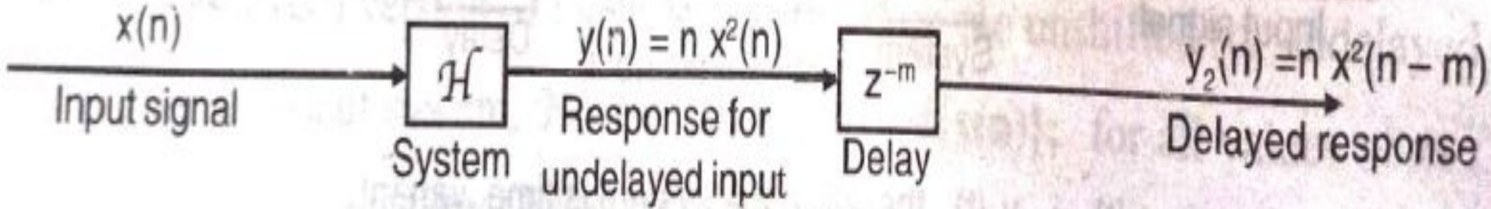
Conclusion : Here, $y_1(t) = y_2(t)$, therefore the system is time invariant.

b) **Given that, $y(n) = n\,x^2(n)$**

<u>**Test 1** : Response for delayed input</u>



<u>**Test 2** : Delayed response</u>



<u>**Conclusion**</u> : Here, $y_1(t) \neq y_2(t)$, therefore the system is time variant.

Linear and Nonlinear Systems

A *linear system* is one that satisfies the superposition principle. The ***principle of superposition*** requires that the response of the system to a weighted sum of the signals is equal to the corresponding weighted sum of the responses of the system to each of the individual input signals.

Definition : A relaxed system $\mathcal{H}$ is ***linear*** if

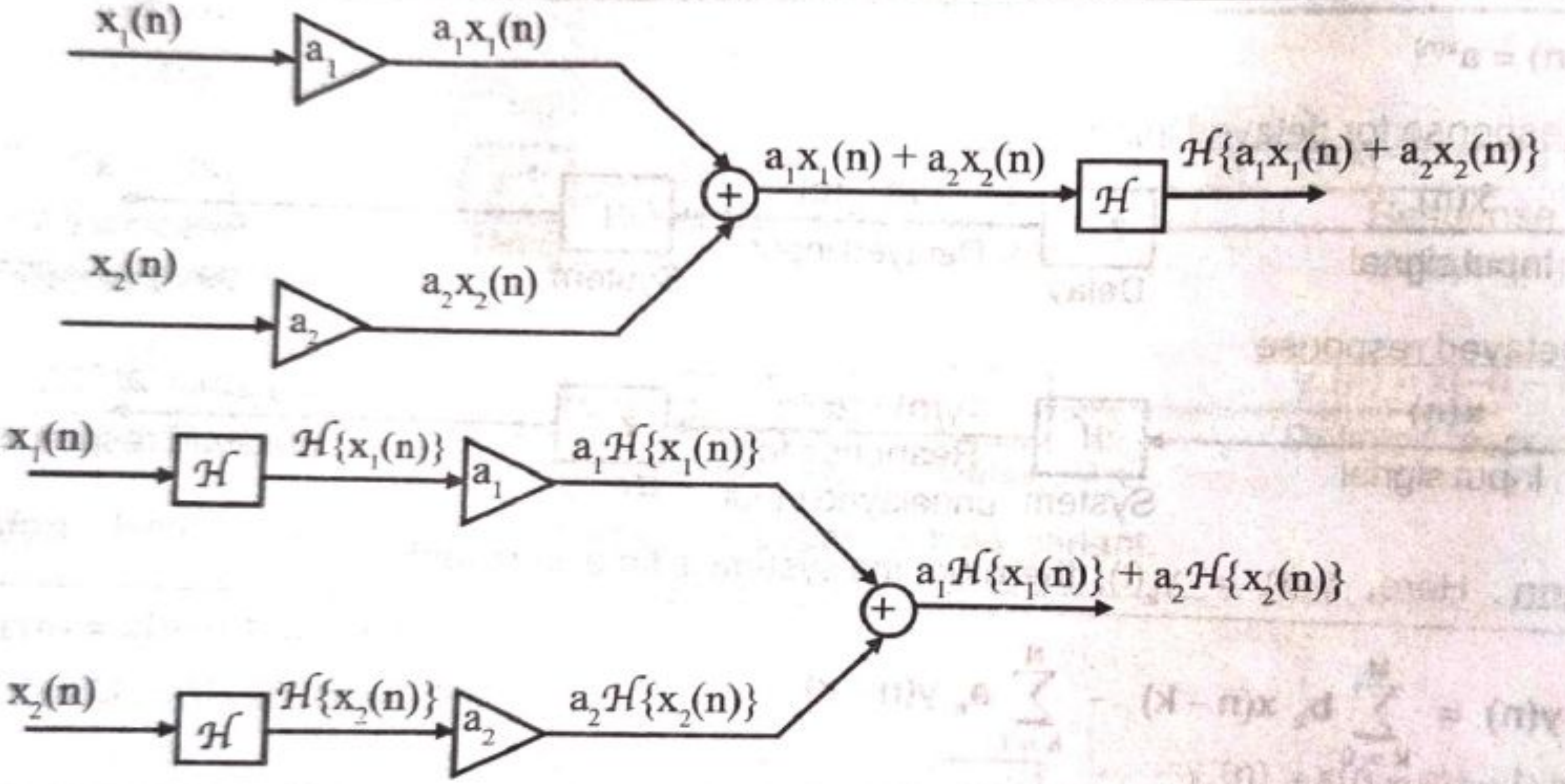
$$\mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, \mathcal{H}\{x_1(n)\} + a_2\, \mathcal{H}\{x_2(n)\} \qquad .....(6.23)$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$ and for any arbitrary constants a_1 and a_2.

If a relaxed system does not satisfy the superposition principle as given by the above definition, the system is ***nonlinear***. The diagrammatic explanation of linearity is shown in fig 6.20.

Procedure to test for linearity

1. Let $x_1(n)$ and $x_2(n)$ be two inputs to system $\mathcal{H}$, and $y_1(n)$ and $y_2(n)$ be corresponding responses.
2. Consider a signal, $x_3(n) = a_1\, x_1(n) + a_2\, x_2(n)$ which is a weighed sum of $x_1(n)$ and $x_2(n)$.
3. Let $y_3(n)$ be the response for $x_3(n)$.
4. Check whether $y_3(n) = a_1\, y_1(n) + a_2\, y_2(n)$. If they are equal then the system is linear, otherwise it is nonlinear.



The system, $\mathcal{H}$ is linear if and only if, $\mathcal{H}\{a_1\, x_1(n) + a_2\, x_2(n)\} = a_1\, \mathcal{H}\{x_1(n)\} + a_2\, \mathcal{H}\{x_2(n)\}$

Fig 6.20 : Diagrammatic explanation of linearity.

Test the following systems for linearity.

a) $y(n) = n\,x(n)$, b) $y(n) = x(n^2)$, c) $y(n) = x^2(n)$, d) $y(n) = A\,x(n) + B$, e) $y(n) = e^{x(n)}$.

Solution

a) Given that, $y(n) = n\,x(n)$

Let H be the system represented by the equation, $y(n) = n x(n)$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = n\,x(n)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore\; y_1(n) = \mathcal{H}\{x_1(n)\} = n\,x_1(n)$$

$y_3(n) = a_1 y_1(n) + a_2 y$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = n\,x_2(n)$$

$x_3(n)$

$$\therefore\; a_1 y_1(n) + a_2 y_2(n) = a_1 n\,x_1(n) + a_2 n\,x_2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = n[a_1 x_1(n) + a_2 x_2(n)] = a_1 n\,x_1(n) + a_2 n\,x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$. Hence the system is linear.

b) **Given that, $y(n) = x(n^2)$**

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = x(n^2)$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = x(n^2)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore y_1(n) = \mathcal{H}\{x_1(n)\} = x_1(n^2)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = x_2(n^2)$$

$$\therefore a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n^2) + a_2 x_2(n^2) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

c) **Given that, $y(n) = x^2(n)$**

Let $\mathcal{H}$ be the system represented by the equation, $y(n) = x^2(n)$ and the system $\mathcal{H}$ operates on $x(n)$ to produce, $y(n) = \mathcal{H}\{x(n)\} = x^2(n)$.

Consider two signals $x_1(n)$ and $x_2(n)$.

Let $y_1(n)$ and $y_2(n)$ be the response of the system $\mathcal{H}$ for inputs $x_1(n)$ and $x_2(n)$ respectively.

$$\therefore y_1(n) = \mathcal{H}\{x_1(n)\} = x_1^2(n)$$

$$y_2(n) = \mathcal{H}\{x_2(n)\} = x_2^2(n)$$

$$\therefore a_1 y_1(n) + a_2 y_2(n) = a_1 x_1^2(n) + a_2 x_2^2(n) \qquad .....(1)$$

Consider a linear combination of inputs, $a_1 x_1(n) + a_2 x_2(n)$. Let the response of the system for this linear combination of inputs be $y_3(n)$.

$$\therefore y_3(n) = \mathcal{H}\{a_1 x_1(n) + a_2 x_2(n)\} = [a_1 x_1(n) + a_2 x_2(n)]^2$$

$$= a_1^2 x_1^2(n) + a_2^2 x_2^2(n) + 2\, a_1 a_2 x_1(n) x_2(n) \qquad .....(2)$$

The condition to be satisfied for linearity is, $y_3(n) = a_1 y_1(n) + a_2 y_2(n)$.

From equations (1) and (2) we can say that, $y_3(n) \neq a_1 y_1(n) + a_2 y_2(n)$. Hence the system is nonlinear.

Causal and Noncausal Systems

Definition : A system is said to be ***causal*** if the output of the system at any time n depends only on the present input, past inputs and past outputs but does not depend on the future inputs and outputs.

If the system output at any time n depends on future inputs or outputs then the system is called ***noncausal*** system.

The causality refers to a system that is realizable in real time. It can be shown that an LTI system is causal if and only if the impulse response is zero for $n < 0$, (i.e., $h(n) = 0$ for $n < 0$).

Let, $x(n)$ = Present input and $y(n)$ = Present output

$\therefore x(n-1), x(n-2), \ldots\ldots,$ are past inputs

$y(n-1), y(n-2), \ldots\ldots,$ are past outputs

In mathematical terms the output of a causal system satisfies the equation of the form

$$y(n) = F\,[x(n), x(n-1), x(n-2), \ldots, y(-1), y(n-2) \ldots]$$

where, $F[\cdot]$ is some arbitrary function.

Test the casuality of the following systems.

a) $y(n) = x(n) - x(n-1)$ b) $y(n) = \sum_{m=-\infty}^{n} x(m)$ c) $y(n) = a\,x(n)$ d) $y(n) = n\,x(n)$

Solution

a) Given that, $y(n) = x(n) - x(n-1)$

When $n = 0$, $y(0) = x(0) - x(-1)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past input $x(-1)$

When $n = 1$, $y(1) = x(1) - x(0)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past input $x(0)$.

From the above analysis we can say that for any value of n, the system output depends on present and past inputs. Hence the system is causal.

b) Given that, $y(n) = \sum_{m=-\infty}^{n} x(m)$

When $n = 0$, $y(0) = \sum_{m=-\infty}^{0} x(m)$

$= \dots x(-2) + x(-1) + x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and past inputs $x(-1)$, $x(-2)$,.....

When $n = 1$, $y(1) = \sum_{m=-\infty}^{1} x(m)$

$= \dots x(-2) + x(-1) + x(0) + x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and past inputs $x(0)$, $x(-1)$, $x(-2)$,..

From the above analysis we can say that for any value of n, the system output depends on present and past inputs Hence the system is causal.

c) Given that, $y(n) = a\,x(n)$

When $n = 0$, $y(0) = a\,x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $n = 1$, $y(1) = a\,x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

d) Given that, $y(n) = n\,x(n)$

When $n=0$, $y(0)=0 \times x(0)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$.

When $n=1$, $y(1)=1 \times x(1)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$.

When $n = 2$, $y(2) = 2 \times x(2)$ $\Rightarrow$ The response at $n = 2$, i.e., $y(2)$ depends on the present input $x(2)$.

From the above analysis we can say that the response for any value of n depends on the present input. Hence the system is causal.

Test the causality of the following systems.

a) $y(n) = x(n) + 3\,x(n+4)$ b) $y(n) = x(n^2)$ c) $y(n) = x(2n)$ d) $y(n) = x(-n)$

Solution

a) Given that, $y(n) = x(n) + 3\,x(n+4)$

When $n = 0$, $y(0) = x(0) + 3\,x(4)$ $\Rightarrow$ The response at $n = 0$, i.e., $y(0)$ depends on the present input $x(0)$ and future input $x(4)$.

When $n = 1$, $y(1) = x(1) + 3\,x(5)$ $\Rightarrow$ The response at $n = 1$, i.e., $y(1)$ depends on the present input $x(1)$ and future input $x(5)$.

From the above analysis we can say that the response for any value of n depends on present and future inputs. Hence the system is noncausal.

b) Givent that, $y(n) = x(n^2)$

When $n = -1$; $y(-1) = x(1)$ $\Rightarrow$ The response at $n = -1$, depends on the future input $x(1)$.

When $n = 0$; $y(0) = x(0)$ $\Rightarrow$ The response at $n = 0$, depends on the present input $x(0)$.

When $n = 1$; $y(1) = x(1)$ $\Rightarrow$ The response at $n = 1$, depends on the present input $x(1)$.

When $n = 2$; $y(2) = x(4)$ $\Rightarrow$ The response at $n = 2$, depends on the future input $x(4)$.

From the above analysis we can say that the response for any value of n (except $n = 0$ and $n = 1$) depends on future inputs. Hence the system is noncausal.

c) Given that, $y(n) = x(2n)$

When $n = -1$; $y(-1) = x(-2)$ $\Rightarrow$ The response at $n = -1$, depends on the past input $x(-2)$.

When $n = 0$; $y(0) = x(0)$ $\Rightarrow$ The response at $n = 0$, depends on the present input $x(0)$.

When $n = 1$; $y(1) = x(2)$ $\Rightarrow$ The response at $n = 1$, depends on the future input $x(2)$.

From the above analysis we can say that the response of the system for $n > 0$, depends on future inputs. Hence the system is noncausal.

d) **Given that,** $y(n) = x(-n)$

When $n = -2$; $y(-2) = x(2)$ $\Rightarrow$ The response at $n = -2$, depends on the future input $x(2)$.

When $n = -1$; $y(-1) = x(1)$ $\Rightarrow$ The response at $n = -1$, depends on the future input $x(1)$.

When $n = 0$; $y(0) = x(0)$ $\Rightarrow$ The response at $n = 0$, depends on the present input $x(0)$.

When $n = 1$; $y(1) = x(-1)$ $\Rightarrow$ The response at $n = 1$, depends on the past input $x(-1)$.

From the above analysis we can say that the response of the system for $n < 0$ depends on future inputs. Hence the system is noncausal.

Stable and Unstable Systems

Definition : An arbitrary relaxed system is said to be ***BIBO stable*** (Bounded Input-Bounded Output stable) if and only if every bounded input produces a bounded output.

Let x(n) be the input of discrete time system and y(n) be the response or output for x(n). The term ***bounded input*** refers to finite value of the input signal x(n) for any value of n. Hence if input x(n) is bounded then there exits a constant M_x such that $|x(n)| \leq M_x$ and $M_x < \infty$, for all n.

Examples of bounded input signal are step signal, decaying exponential signal and impulse signal.
Examples of unbounded input signal are ramp signal and increasing exponential signal.

The term ***bounded output*** refers to finite and predictable output for any value of n. Hence if output y(n) is bounded then there exists a constant M_y such that $|y(n)| \leq M_y$ and $M_y < \infty$, for all n.

In general, the test for stability of the system is performed by applying specific input. On applying a bounded input to a system if the output is bounded then the system is said to be BIBO stable. For LTI (Linear Time Invariant) systems the condition for BIBO stability can be transformed to a condition on impulse response as shown below.

Test the stability of the following systems.

a) $y(n) = \cos[x(n)]$ b) $y(n) = x(-n-2)$ c) $y(n) = n\,x(n)$

Solution

a) Given that, $y(n) = \cos[x(n)]$

The given system is nonlinear system, and so the test for stability should be performed for specific inputs.

The value of $\cos\theta$ lies between -1 to $+1$ for any value of θ. Therefore the output $y(n)$ is bounded for any value of input $x(n)$. Hence the given system is stable.

b) Given that, $y(n) = x(-n-2)$

The given system is time variant system, and so the test for stability should be performed for specific inputs.

The operations performed by the system on the input signal are folding and shifting. A bounded input signal will remain bounded even after folding and shifting. Therefore in the given system, the output will be bounded as long as input is bounded. Hence the given system is BIBO stable.

c) Given that, $y(n) = n\,x(n)$

The given system is time variant system, and so the test for stability should be performed for specific inputs.

Case i: If $x(n)$ tends to infinity or constant, as "n" tends to infinity, then $y(n) = n\,x(n)$ will be infinite as "n" tends to infinity. So the ststem is unstable.

Case ii: If $x(n)$ tends to zero as "n" tends to infinity, then $y(n) = n\,x(n)$ will be zero as "n" tends to infinity. So the system is stable.

Time variant and invariant systems

(d)

(b) The input–output equation for this system is

$$y(n) = \mathcal{T}[x(n)] = nx(n)$$

The response of this system to $x(n-k)$ is

$$y(n,k) = nx(n-k)$$

Now if we delay $y(n)$ in (2.2.18) by k units in time, we obtain

$$\begin{aligned} y(n-k) &= (n-k)x(n-k) \\ &= nx(n-k) - kx(n-k) \end{aligned}$$

This system is time variant, since $y(n,k) \neq y(n-k)$.

(c) This system is described

(c) This system is described by the input–output relation

$$y(n) = \mathcal{T}[x(n)] = x(-n)$$

The response of this system to $x(n-k)$ is

$$y(n, k) = \mathcal{T}[x(n-k)] = x(-n-k)$$

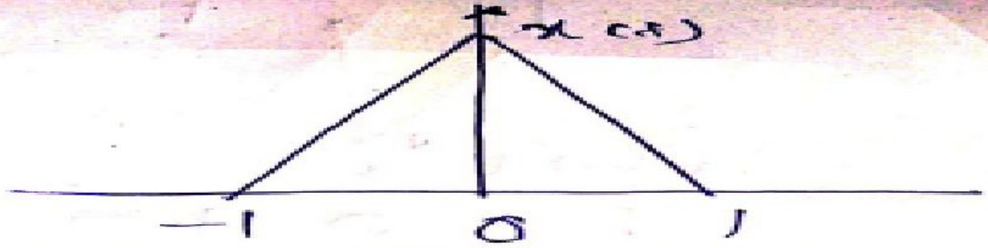
Now, if we delay the output $y(n)$, as given by (2.2.21), by k units in time, the res

$$y(n-k) = x(-n+k)$$

Since $y(n, k) \neq y(n-k)$, the system is time variant.

Operation on signals

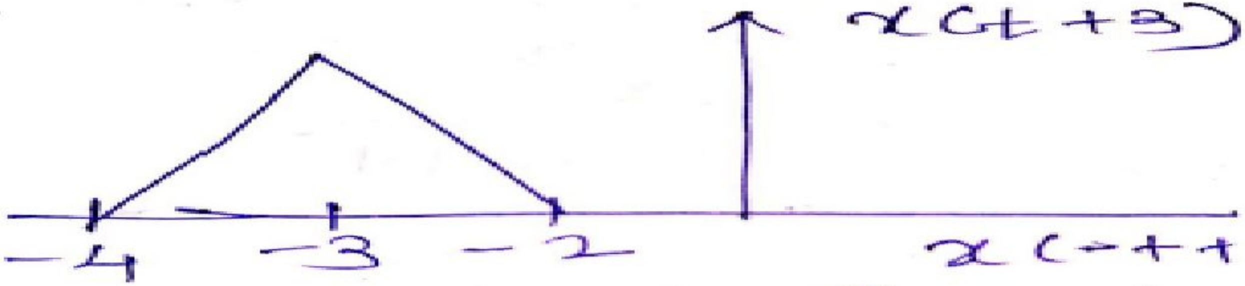
To obtain $f(at-b)$ from $f(t)$,
Perform following operations.
Method 1:
Step1)Time shift $f(t)$ by ‘b’ to obtain $f(t-b)$
Step 2)Time scale by ‘a’ value to obtain $f(at-b)$
Or
Method 2:
Step1)Time scale $f(t)$ by ‘a’ to obtain $f(at)$
Step 2)Time shift by (b/a) to obtain $f(at-b)$
Example 1.6



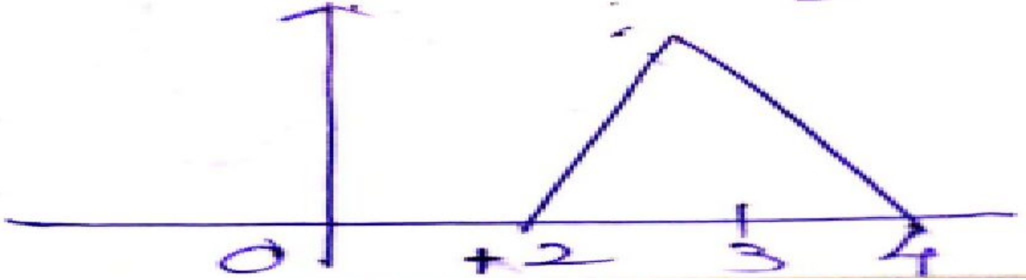
$x(-2t+3)$

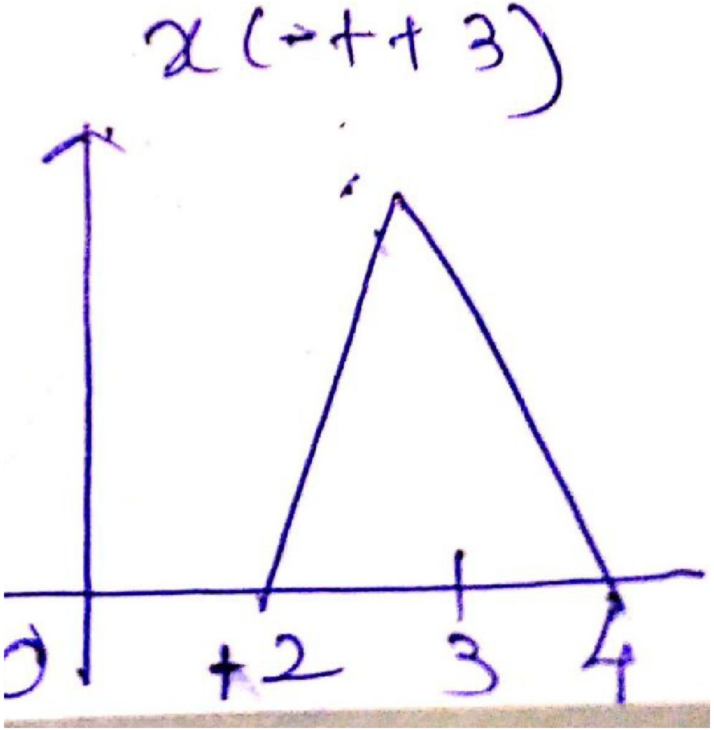
Method 1

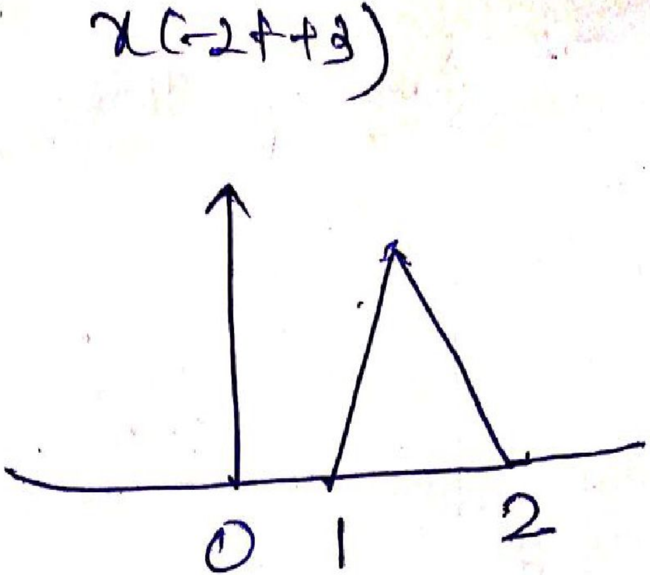
$x(t+3)$



$x(-t+3)$

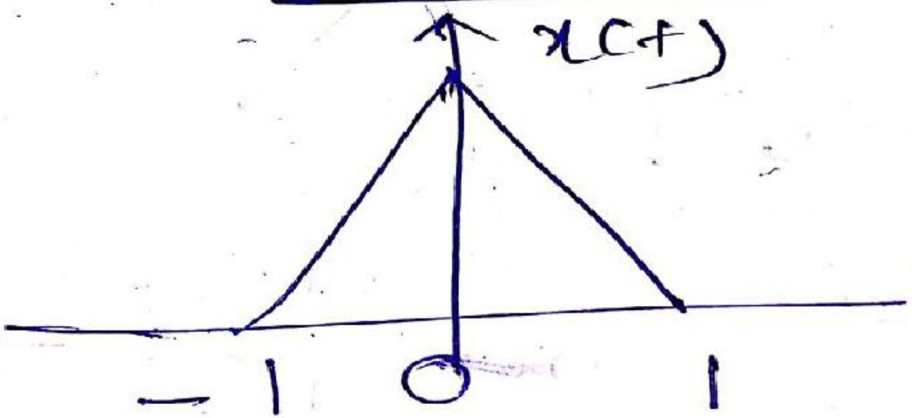


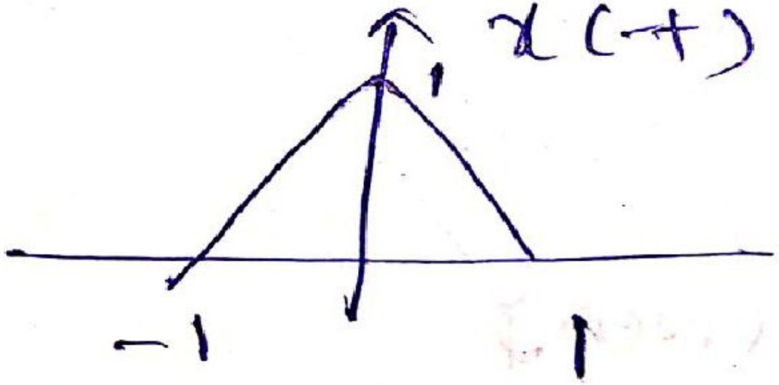

x(-t+3)
0
+2
3
4

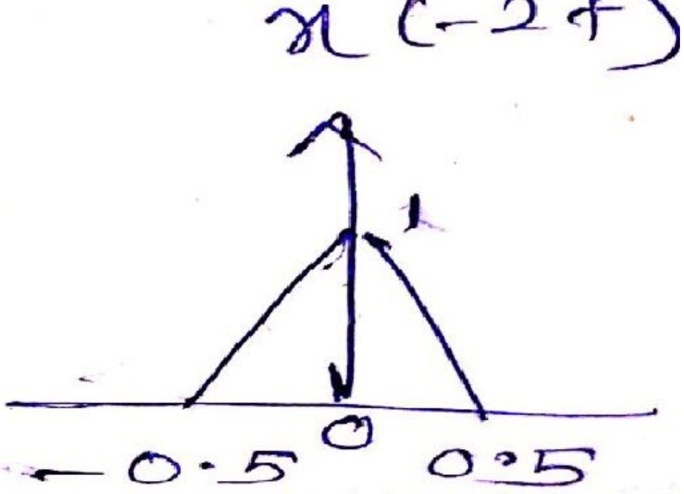

x(-2t+3)
0
1
2

Method 2

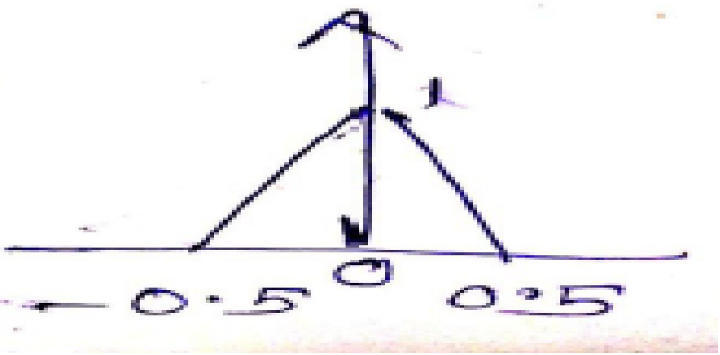
$x(-2t+3)$





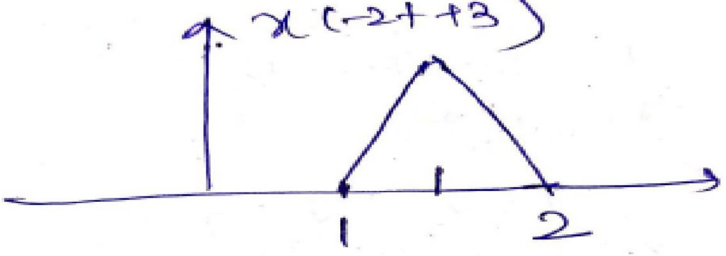


$x(-2t)$

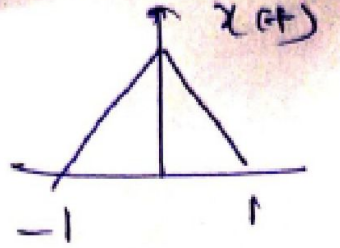


$x(-2t+3)$

shift by $(3/2) = -1.5$

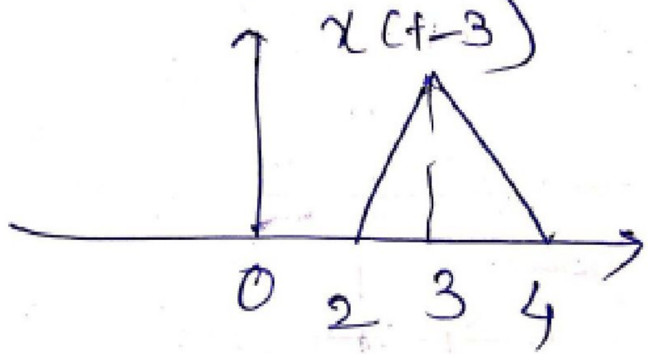


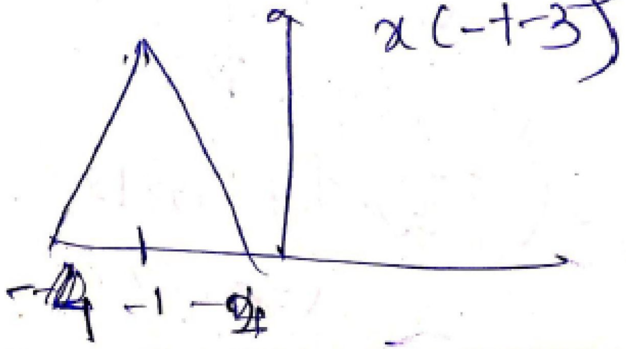
$x(-2t-3)$



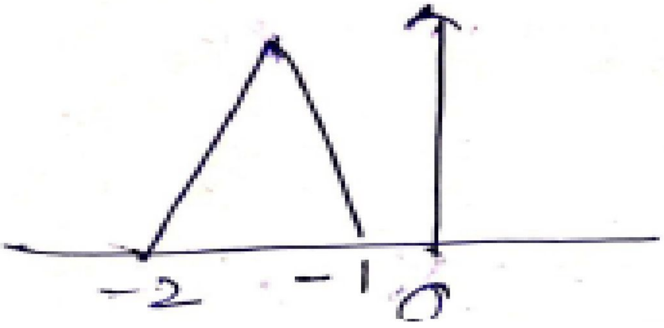
Method 1

Shift by -3



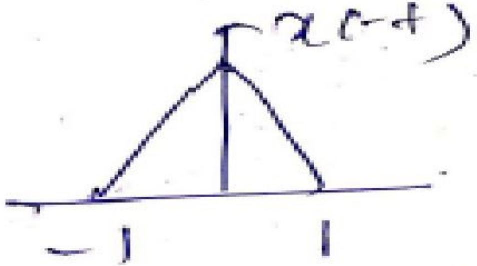


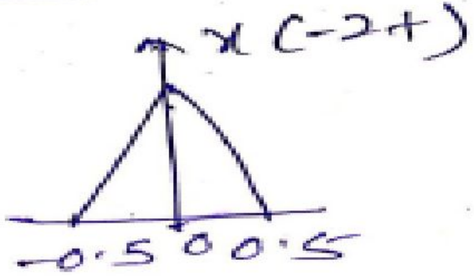
$x(-2t-3)$



Method 2

$x(-2t-3)$





shift by

$3/2 \rightarrow 1.5$