Electronics and Telecommunication Engineering

FH2022

Engineering Mathematics IV

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RISE WITH EDUCATION

Syllabus Content

Module 01: Complex Integration [06 Lectures]

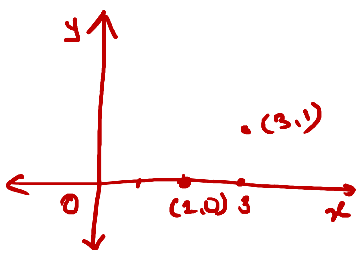
1.1 Line Integral, Cauchy's Integral theorem for simple connected and multiply connected regions (without proof), Cauchy's Integral formula (without proof).
1.2 Taylor's and Laurent's series (without proof).
1.3 Definition of Singularity, Zeroes, poles of $f(z)$, Residues, Cauchy's Residue Theorem (without proof).
Self-learning Topics: Application of Residue Theorem to evaluate real integrations

Course Outcome

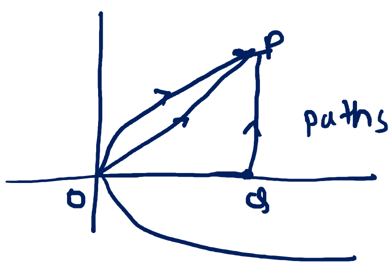
- To understand line and contour integrals and expansion of complex valued function in a power series.

1.1. Line Integral

a. Cartesian form ← $z = x+iy \equiv (x,y)$ **b. Polar form**



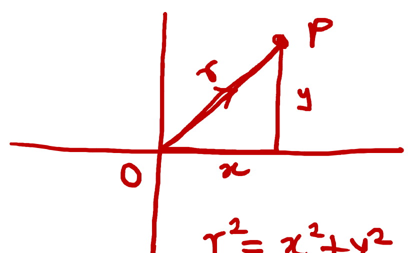
xy- plane
z-plane



$\checkmark\ 2 = 2+i0 \equiv (2,0)$

$\checkmark\ i = 0+1i \equiv (0,1)$

$\checkmark\ 3+i = (3,1)$



$r^2 = x^2+y^2$

$r = \sqrt{x^2+y^2}$

OR $x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

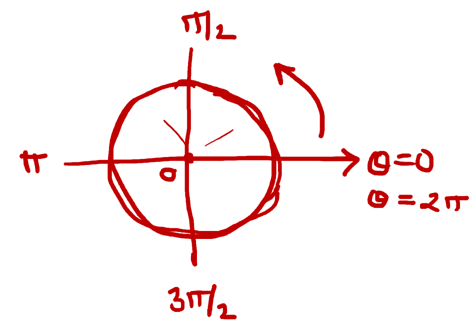
$r = \sqrt{x^2+y^2}$

$x = r\cos\theta, \quad y = r\sin\theta$

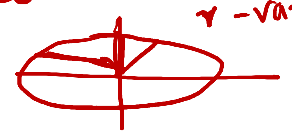
$z = x+iy = r\cos\theta + i\sin\theta \rightarrow$

$= r(\cos\theta + i\sin\theta) \longrightarrow$ polar

$= re^{i\theta} \rightarrow$ exponential



$r = \cos 2\theta$



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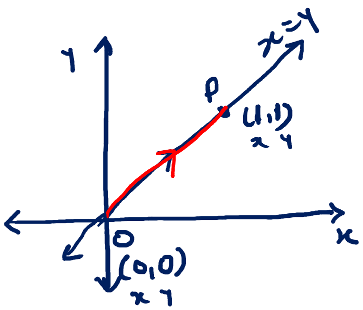
1.1. Line Integral [Cartesian form] $\int f(z)\,dz$

Example 01: Evaluate $\int_0^{1+i} z^2 \cdot dz$, *along i) a line* $y = x$ *ii) the parabola* $x = y^2$. Is the line integral independent of path?

$\equiv (1,1)$ $\quad$ $f(z)$

$z = x+iy, \quad dz = dx + i\,dy$

i) $\boxed{y=x}$: Fix x :



$y = x$
$dy = dx$

i) x : 0 to 1

ii) $f(z) = z^2 = (x+iy)^2$
$= (x+ix)^2$
$= x^2(1+i)^2$
$= x^2[1^2 + 2i + i^2]$
$= x^2[1 + 2i - 1]$
$= 2ix^2$

iii) $dz = dx + i\,dy$
$= dx + i\,dx$
$= (1+i)\,dx$

$$I = \int_0^1 2ix^2(1+i)\,dx$$
$$= 2i(1+i)\left[\frac{x^3}{3}\right]_0^1$$
$$= 2i(1+i)\left[\frac{1}{3} - 0\right]$$
$$= \frac{2}{3}i(1+i)$$
$$= \frac{2}{3}(-1+i) \quad \#.$$

$y = x^2$, $x = \sqrt{y}$

$f(z) = x + iy + y^2$
$= x + ix^2 + x^4$

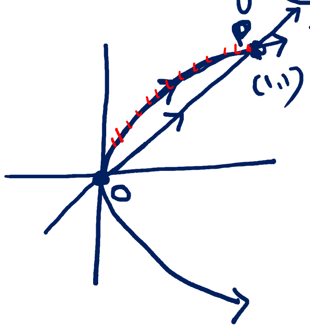
$f(z) = \sqrt{y} + iy + y^2$

$y = 0$
$\boxed{dy = 0}$
$\int dy = 0$

1.1. Line Integral [Cartesian form]

Example 01: Evaluate $\int_0^{1+i} z^2 \cdot dz, along\ i)\ a\ line\ y = x\ \ ii)\ the\ parabola\ \ x = y^2$. Is the line integral independent of path?

→ ii) Along $\boxed{x=y^2}$



$x=y^2$; $x=y$

$y=y^2$

$y^2-y=0$

$y(y-1)=0$

$y=0,\ y=1$

$y=0,\ x=0 \Rightarrow (0,0)$

$y=1;\ x=1 \Rightarrow (1,1)$

$\boxed{\text{Fix } y}$

i) y : 0 to 1

ii) $z^2 = (x+iy)^2$

$= (y^2+iy)^2$

$= y^4 + 2iy^3 - y^2$

iii) $dz = dx + i\,dy$, $x = y^2$

$dx = 2y\,dy$

$= 2y\,dy + i\,dy$

$= (2y+i)\,dy$

$\therefore I = \int_0^1 (y^4+2iy^3-y^2)(2y+i)\,dy$ ✓

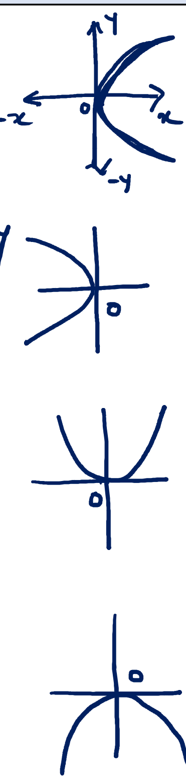
$= \int_0^1 \left[2y^5 + iy^4 + 4iy^4 - 2y^3 - 2y^3 - iy^2\right] dy$

$= \int_0^1 (2y^5 + 5iy^4 - 4y^3 - iy^2)\,dy$

$= \left[\frac{\cancel{2}y^6}{\cancel{6}3} + \cancel{5}i\frac{y^5}{\cancel{5}} - \frac{\cancel{4}y^4}{\cancel{4}} - \frac{iy^3}{3}\right]_0^1$

$= \frac{1}{3} + i - 1 - \frac{i}{3}$

$= \boxed{-\frac{2}{3} + \frac{2i}{3}}$ #



$x = y^2$

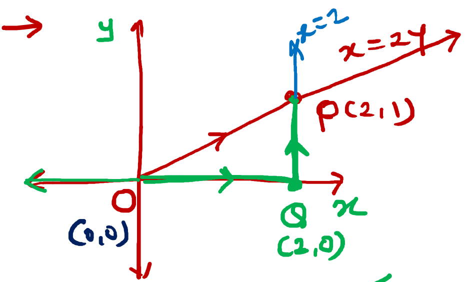
$-x = y^2$

$y = x^2$

$-y = x^2$

Example 02: Evaluate $\int_{0+i0}^{2+i}(\bar{z})^2\,dz$ along, i) the line $x = 2y$ $r = dx + idy$ (0,0), (2,1) $z = x + iy$
ii) the real axis from 0 to 2 and then vertically to 2+i (0,0) to (2,0) 2+i0
iii) the parabola $x = 2y^2$ $\int dx$

1.1. Line Integral [Cartesian]



i) Along OP; $\boxed{x = 2y} \rightarrow dx = 2dy$

$\boxed{\text{fix } y}$

a) y: 0 to 1

b) $f(z) = \bar{z}^2 = (x - iy)^2$
$= (2y - iy)^2$
$= 4y^2 - 4iy^2 - y^2$
$= (3 - 4i)y^2$

c) $dz = dx + idy$
$= 2dy + idy = (2+i)dy$

$$I = \int_0^1 (3-4i)y^2 (2+i)\,dy$$

$$= (3-4i)(2+i)\left[\frac{y^3}{3}\right]_0^1$$

$$= (6 + 3i - 8i + 4)\left[\frac{1}{3}\right]$$

$$\boxed{I = \frac{10}{3} - \frac{5i}{3}}$$ #

ii) Along OQ: $\boxed{y = 0}$
$dy = 0$
we can not solve for y
we solve for x

a) x: 0 to 2

b) $f(z) = \bar{z}^2 = (x - iy)^2$
$= x^2$

c) $dz = dx + idy = dx$

$$I_1 = \int_0^2 x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 = \frac{8}{3}$$

Now Along QP; $\boxed{x = 2}$, $dx = 0$
we have to solve for y.

a) y: 0 to 1

b) $f(z) = (x - iy)^2$
$= (2 - iy)^2$
$= 4 - 4iy - y^2$

$dz = dx + idy = idy$

$$I_2 = \int_0^1 (4 - 4iy - y^2)\,i\,dy$$

$$= i\left[4y - 4i\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= i\left[4 - 2i - \frac{1}{3}\right]$$

$$= i\left[\frac{11}{3} - 2i\right]$$

$$= \boxed{2 + i\frac{11}{3} = I_2}$$

$\therefore I = I_1 + I_2$

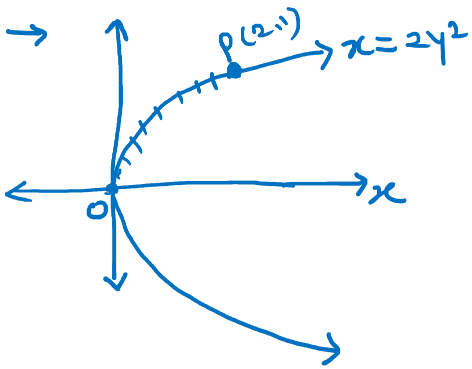
$$= \frac{8}{3} + 2 + i\frac{11}{3}$$

$$\boxed{I = \frac{14}{3} + i\frac{11}{3}}$$

Example 02: Evaluate $\int_0^{2+i} (\overline{z})^2 \, dz$ *along , i) the line* $x = 2y$
ii) the real axis from 0 to 2 and then vertically to 2+i
iii) the parabola $x = 2y^2$

$(0,0) \to (2,1)$

1.1. Line Integral [Cartesian]



Along OP: $\boxed{x = 2y^2}$

$\boxed{\text{Fix } y}$

a) $y: 0$ to 1

b) $f(z) = (x-iy)^2$

$= (2y^2 - iy)^2$

$= 4y^4 - 4iy^3 - y^2$

c) $dz = dx + i\,dy \Rightarrow dz = 4y\,dy + i\,dy = (4y+i)dy$

$x = 2y^2 \Rightarrow dx = 4y\,dy$

$$\therefore I = \int_0^1 (4y^4 - 4iy^3 - y^2)(4y+i)\,dy$$

$$= \int_0^1 \left[16y^5 + 4iy^4 - 16iy^4 + \cancel{4y^3} - \cancel{4y^3} - iy^2\right] dy$$

$$= \int_0^1 (16y^5 - 12iy^4 - iy^2)\,dy$$

$$= \left[16\frac{y^6}{6} - 12i\frac{y^5}{5} - i\frac{y^3}{3}\right]_0^1$$

$$= \frac{16}{6} - \frac{12i}{5} - \frac{i}{3} = \frac{8}{3} - i\left(\frac{12}{5} + \frac{1}{3}\right)$$

$$= \frac{8}{3} - i\left(\frac{36+5}{15}\right)$$

$$= \boxed{\frac{8}{3} - i\frac{41}{15}}$$

Example 03: Evaluate $\int \overline{z}\, dz$ *from* $z = 0$ *to* $z = 4 + 2i$ *along the curve* $z = t^2 + it$

1.1. Line Integral [Cartesian]

$(0,0) \longrightarrow (4,2)$ (x, y)

$$\begin{tikzcd} & z \arrow[dl, dash] \arrow[dr, dash] & \\ x \arrow[dr, dash] & & y \arrow[dl, dash] \\ & t & \end{tikzcd}$$

$z = t^2 + it$

$x + iy = t^2 + it$

$\boxed{x = t^2}$, $\boxed{y = t}$

a)

x	0	4
y	0	2
t	0	2

b) $f(z) = \overline{z} = x - iy$

$= t^2 - it$

c) $dz = dx + i\,dy$; $x = t^2$, $dx = 2t \cdot dt$ | $y = t$, $dy = dt$

$= 2t\,dt + i\,dt$

$= (2t + i)\,dt$

$$I = \int_0^2 (t^2 - it)(2t + i)\,dt$$

$$= \int_0^2 (2t^3 + \underline{it^2 - 2it^2} + t)\,dt$$

$$= \int_0^2 (2t^3 - it^2 + t)\,dt$$

$$= \left[2\frac{t^4}{4} - i\frac{t^3}{3} + \frac{t^2}{2} \right]_0^2$$

$$= \left[\frac{1}{2} 16 - \frac{8i}{3} + 2 \right] - [0]$$

$$= 8 - \frac{8i}{3} + 2 = \boxed{10 - \frac{8i}{3}} \quad \#$$

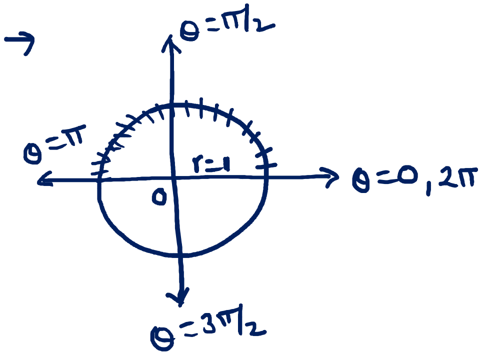
Example 04: Evaluate $\int z^2 \, dz$ along the path PM, where $P\ (0,0)$ and $M\ (3,1)$

1.1. Line Integral [Cartesian]

1.1. Line Integral [Polar Form]

Use $z = re^{i\theta}, dz = ire^{i\theta} d\theta$

Example 05: Evaluate $\int_c \bar{z}\, dz$, *where c is upper half of the circle* $|z| = 1$ $\boxed{|z| = r}$



θ: 0 to π

$f(z) = \bar{z} = \overline{(e^{i\theta})} = e^{-i\theta}$

$dz = ire^{i\theta}.d\theta$

$\boxed{dz = ie^{i\theta}.d\theta}$

$z = re^{i\theta}$, $r = 1$

$\boxed{z = e^{i\theta}}$

$$I = \int_0^\pi e^{-i\theta} . ie^{i\theta}.d\theta$$

$$= i\int_0^\pi e^{i\theta - i\theta}.d\theta \qquad \{e^a.e^b = e^{a+b}\}$$

$$= i\int_0^\pi e^0\, d\theta$$

$$= i\int_0^\pi 1\, d\theta \qquad \{\because e^0 = 1\}$$

$$= i[\theta]_0^\pi$$

$$= i(\pi - 0)$$

$$\boxed{I = \pi i}$$

$z = r(\cos\theta + i\sin\theta)$

$z = r\, e^{i\theta}$

$|z| = 1$

$|r(\cos\theta + i\sin\theta)| = 1$

$|x+iy| = \sqrt{x^2+y^2}$

$r^2\cos^2\theta + r^2\sin^2\theta = 1$

$r^2 = 1$

$\boxed{r = 1}$

$\overline{2+i} = 2-i$

1.1. Line Integral [Polar Form]

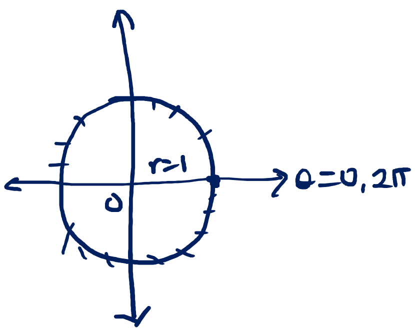
Use $z = re^{i\theta}, dz = ire^{i\theta}d\theta$

Example 06: Show that $\int_c \log z \, dz = 2\pi i$, *where C is the unit circle in the z − plane* , $|z|=1$

$\rightarrow \therefore$ on circle, $\theta : 0$ to 2π

$z = e^{i\theta}$

$dz = i e^{i\theta} . d\theta$



$f(z) = \log z = \log(e^{i\theta})$

$= i\theta \cdot \log e$ $\{\because \log x^m = m \log x\}$

$= i\theta$ $\{\because \log e = 1\}$

$dz = i e^{i\theta} . d\theta$

$I = \int_0^{2\pi} i\theta \cdot i e^{i\theta} . d\theta$ $\left\{ \int u \cdot v = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots \right.$ ↓ algebraic

$= -\int_0^{2\pi} \theta \cdot e^{i\theta} . d\theta$

$= -\left[\theta \cdot \left(\frac{e^{i\theta}}{i}\right) - 1\left(\frac{e^{i\theta}}{i^2}\right) + 0\right]_0^{2\pi}$ $\quad i^2 = -1$

$= -\left[\left[\frac{2\pi}{i} e^{2\pi i} + e^{2\pi i}\right] - \left[0 + e^0\right]\right]$ $\left\{ \because e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1 + 0 = 1; \; e^{i\theta} = \cos\theta + i \sin\theta \right.$

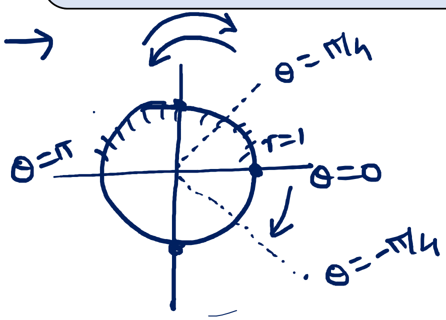
$= -\left[\frac{2\pi}{i} + \cancel{1} - \cancel{1}\right]$

$= -\frac{2\pi}{i} \times \frac{i}{i} = -\frac{2\pi i}{i^2} = \boxed{2\pi i}$ #

1.1. Line Integral [Polar Form]

Use $z = re^{i\theta}, dz = ire^{i\theta}d\theta$

Example 07: $Evaluate\ \int_c (z - z^2)dz, where\ C\ is\ the\ upper\ half\ of\ the\ unit\ circle.$, $|z|=1$
$What\ is\ the\ value\ of\ the\ integral\ for\ lower\ half\ of\ the\ same\ circle?$



$\theta : 0$ to π

$f(z) = z - z^2 = e^{i\theta} - e^{2i\theta}$

$dz = ie^{i\theta}\cdot d\theta$

$\therefore I = \int_0^{\pi} (e^{i\theta} - e^{2i\theta})\cdot ie^{i\theta}\cdot d\theta$

$= i\int_0^{\pi} [e^{2i\theta} - e^{3i\theta}]d\theta \quad \text{— (1)}$

$= i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi}$

$= \left[\frac{e^{2\pi i}}{2} - \frac{e^{3\pi i}}{3}\right] - \left[\frac{e^0}{2} - \frac{e^0}{3}\right]$

$e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$

$e^{3\pi i} = \cos 3\pi + i\sin 3\pi = -1$

$= \cancel{\frac{1}{2}} + \frac{1}{3} - \cancel{\frac{1}{2}} + \frac{1}{3}$

$= \frac{2}{3}$ #

Now, on lower half of circle

$\theta : \pi$ to 2π, from (1)

$I = i\int_{\pi}^{2\pi} (e^{2i\theta} - e^{3i\theta})d\theta$

$= i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_{\pi}^{2\pi}$

$= \left[\frac{e^{4\pi i}}{2} - \frac{e^{6\pi i}}{3}\right] - \left[\frac{e^{2\pi i}}{2} - \frac{e^{3\pi i}}{3}\right]$

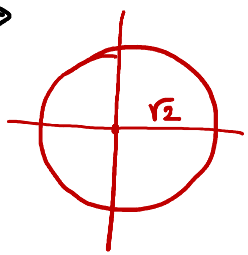
$=$ $e^{4\pi i} = \cos 4\pi + i\sin 4\pi = 1$

$= \cancel{\frac{1}{2}} - \frac{1}{3} - \cancel{\frac{1}{2}} - \frac{1}{3}$

$= -\frac{2}{3}$ #

Example 08: Evaluate $\int_c (z^2 - 2\,\bar{z} + 1)dz$, *where c is circle* $x^2 + y^2 = 2$ $\leftarrow (x-a)^2 + (y-b)^2 = r^2$

centre (a, b)



θ: 0 to 2π, $z = re^{i\theta} = \sqrt{2}\cdot e^{i\theta}$

$$f(z) = z^2 - 2\bar{z} + 1$$

$$= 2e^{2i\theta} - 2\sqrt{2}\,e^{-i\theta} + 1$$

$$dz = \sqrt{2}\cdot i e^{i\theta}.d\theta$$

$$\therefore\ I = \int_0^{2\pi} \left[2e^{2i\theta} - 2\sqrt{2}\,e^{-i\theta} + 1\right]\cdot\sqrt{2}\,i\,e^{i\theta}\cdot d\theta$$

$$= i\sqrt{2}\int_0^{2\pi}\left[2e^{3i\theta} - 2\sqrt{2} + e^{i\theta}\right]d\theta$$

$$= i\sqrt{2}\left[2\cdot\frac{e^{3i\theta}}{3i} - 2\sqrt{2}\cdot\theta + \frac{e^{i\theta}}{i}\right]_0^{2\pi}$$

$$= i\sqrt{2}\left[\left[\frac{2}{3i}e^{6\pi i} - 4\pi\sqrt{2} + \frac{e^{2\pi i}}{i}\right] - \left[\frac{2}{3i} - 0 + \frac{1}{i}\right]\right]$$

$$= i\sqrt{2}\left[\cancel{\frac{2}{3i}} - 4\pi\sqrt{2} + \cancel{\frac{1}{i}} - \cancel{\frac{2}{3i}} - \cancel{\frac{1}{i}}\right]$$

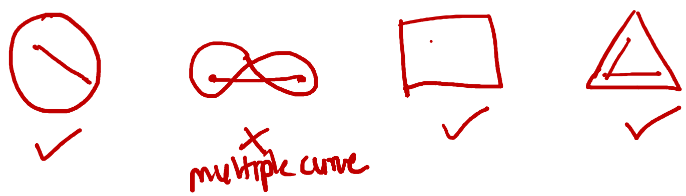
$$\boxed{I = -8\pi i}$$

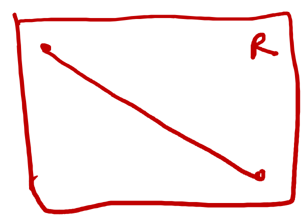
Example 09:Evaluate $\int_c f(z)dz$, $where\ f(z) = x^2 + ixy\ from\ A(1,1)\ to\ B\ (2,4) along\ the\ curve\ x = t, y = t^2$

1.1 Cauchy’s Integral Theorem [CIT]

Simply Closed curve: If a closed curve does not intersect itself then it is called a simple closed curve. If a closed curve intersect itself it is called multiple curve

Simply Connected domain: If R is a region and every closed curve in R encloses the points of R only, then R is called Simply connected domain.



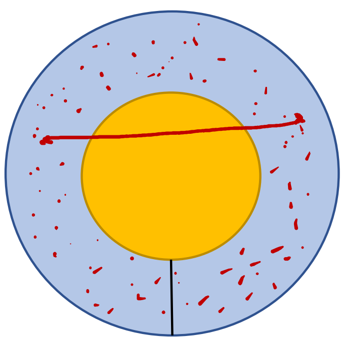


1.1 Cauchy's Integral Theorem [CIT]

CIT Statement: If f(z) is analytic function and if its derivative f'(z) is continuous at each point within and on a simple closed curve C then the integral of f(z) along the closed curve C is zero i.e

$$\boxed{\int_c \boldsymbol{f(z)dz = 0}}$$

Extended Cauchy's Integral Theorem: If f(z) is analytic function in R between two simple close curves C_1 *and* C_2 then $\int_{c_1} \boldsymbol{f(z)dz} = \int_{c_2} \boldsymbol{f(z)dz}$



1.1 Cauchy's Integral Formula [CIF]

CIT Statement: If f(z) is analytic inside and on a simple closed curve C of simply connected region R and if z_0 is any point within C, then

except z_0,

$$\int_c \frac{f(z)}{z - z_0} dz = 2\pi i \cdot f(z_0): \quad \boldsymbol{For\ nonrepeated\ z_0\ [n = 1]}$$

$$\int_c \frac{f(z)}{(z - z_0)^n} dz = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(z_0): \quad \boldsymbol{For\ repeated\ z_0\ \ n - times}$$

i) $\int_c \frac{z}{(z-1)} dz = 2\pi i\, f(z)\big|_{z=1}$, $z = f(z)$

$z = z_0 = 1$

$= 2\pi i\,(1)$

$= 2\pi i$

ii) $\int_c \frac{z}{(z-1)^2} dz = \frac{2\pi i}{(2-1)!} f'(z)\big|_{z=z_0}$

$= 2\pi i\,(1)\big|_{z=1}$

$= 2\pi i$

$$\int \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

CIT: If all points are outside the curve then I=0 by CIT

CIF: 1) If all points are inside C then **use Partial fraction** to separate the denominator to have single point at denominator.

2) If all points are outside C , then **by CIT, I=0**

3) If one point inside and one point outside C, then **we shift outside point to the numerator and modify the $f(z)$** and keep only inside point at denominator

i) $\int_C \frac{f(z)}{(z-a)(z-b)}\,dz$, a,b are in C $= \int_C \frac{A f(z)}{z-a}\,dz + \int_C \frac{B f(z)}{z-b}\,dz = 2\pi i\, f(a) + 2\pi i\, f(b)$

ii) $\int_C \frac{f(z)}{(z-a)(z-b)}\,dz$, a,b are outside C $= 0$ by CIT

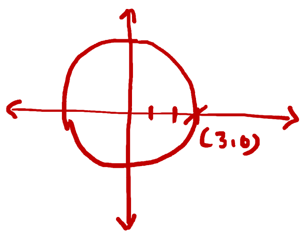
iii) $\int_C \frac{f(z)}{(z-a)(z-b)}\,dz$, $z=a$ is inside C, $z=b$ is outside C $= \int_C \frac{[f(z)/z-b]}{z-a}\,dz = 2\pi i \left[\frac{f(z)}{z-b}\right]_{z=a}$.

Example 01: Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}\,dz$, *where c is the curve* $|z| = 3 = r$

$\boxed{|z| = r}$ circle

$\rightarrow (z-1)(z-2) = 0 \Rightarrow \boxed{z = 1, 2} = (1,0), (2,0)$

$|z| = 3 \Rightarrow$ circle $(0,0)$, $r = 3$



$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} \quad (1)$$

$$= \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

Equating Numerators.

$$1 = A(z-2) + B(z-1)$$

$z = 1$: $1 = A(-1) \Rightarrow \boxed{A = -1}$

$z = 2$: $1 = B(1) \Rightarrow \boxed{B = 1}$

$(1) \Rightarrow$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

$$= \frac{1}{z-2} - \frac{1}{z-1}$$

$$\int_c \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_c \frac{e^{2z}}{z-2}\,dz - \int_c \frac{e^{2z}}{z-1}\,dz$$

$\boxed{z_0 = 2}$ $\boxed{z_0 = 1}$

$$\equiv \int_c \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

By CIF

$$I = 2\pi i\, f(2) - 2\pi i\, f(1)$$

Here

$f(z) = e^{2z}$

$\therefore f(2) = e^4, \quad f(1) = e^2$

$\therefore I = 2\pi i e^4 - 2\pi i e^2$

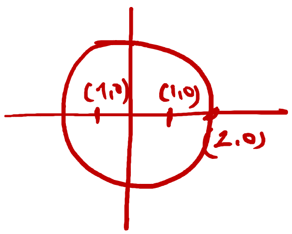
$= 2\pi i (e^4 - e^2)$

#

Example 02: Evaluate $\int_c \frac{3z^2+z}{z^2-1}\,dz$, where c is the curve $|z| = 2$

$\rightarrow z^2-1=0 \Rightarrow (z-1)(z+1)=0 \Rightarrow z=1,-1$

$|z|=2 \Rightarrow$ circle $(0,0)$, $r=2$



$\therefore$ Both $z=1,-1$ are Inside c

we use partial fraction ①

$$\frac{1}{(z+1)(z-1)} = \frac{A}{z+1}+\frac{B}{z-1} = \frac{A(z-1)+B(z+1)}{(z+1)(z-1)}$$

$$1 = A(z-1)+B(z+1)$$

$z=1$: $1=2B \Rightarrow B=1/2$

$z=-1$: $1=-2A \Rightarrow A=-1/2$

① $\Rightarrow$ $$\frac{1}{(z+1)(z-1)} = -\frac{1}{2}\cdot\frac{1}{z+1} + \frac{1}{2}\cdot\frac{1}{z-1}$$

$$= \frac{1}{2}\cdot\frac{1}{z-1} - \frac{1}{2}\cdot\frac{1}{z+1}$$

$$\int_c \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_c \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\int_c \frac{3z^2+z}{z+1}\,dz \equiv \int_c \frac{f(z)}{z-z_0}\,dz$$

$\boxed{z_0=1}$ $\boxed{f(z)=3z^2+z}$ $\boxed{z_0=-1}$

$$=\frac{1}{\cancel{2}}\,\cancel{2}\pi i\, f(1) - \frac{1}{\cancel{2}}\cdot\cancel{2}\pi i\, f(-1)$$

$$=\pi i\,[f(1)-f(-1)]$$

$f(1)=3(1)^2+1=4$

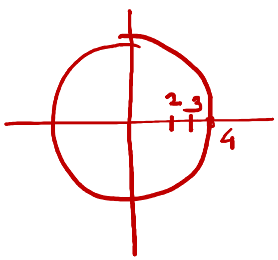
$f(-1)=3(-1)^2-1=2$

$$=\pi i\,[4-2]$$

$$=\boxed{2\pi i}$$

Example 03: Evaluate $\int_c \frac{\sin\pi z^2+\cos\pi z^2}{(z-2)(z-3)}dz$, *where c is the curve* $|z| = 4$

→ $z = 2, 3$ are inside $c: |z| = 4$



$$\frac{1}{(z-2)(z-3)} = -\left[\frac{1}{z-2} - \frac{1}{z-3}\right]$$

$$= \frac{1}{z-3} - \frac{1}{z-2}$$

$$I = \int_c \frac{\sin\pi z^2+\cos\pi z^2}{z-3}dz - \int_c \frac{\sin\pi z^2+\cos\pi z^2}{z-2}dz$$

$\boxed{z_0=3}$ $\boxed{z_0=2}$

By CIF

$= 2\pi i\, f(3) - 2\pi i\, f(2)$

$= 2\pi i\left[\sin 9\pi + \cos 9\pi - \sin 4\pi - \cos 4\pi\right]$

$= 2\pi i\left[0 - 1 - 0 - 1\right] = \boxed{-4\pi i}$

$$\frac{5}{(z-3)(z-7)} = \frac{5}{-4}\left[\frac{1}{z-3} - \frac{1}{z-7}\right]$$

$(z-7) - (z-3)$

$\cancel{z} - 7 - \cancel{z} + 3$

-4

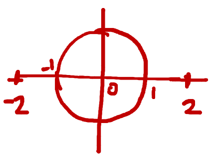
$$\frac{10}{(z-1)(z-8)} = \frac{10}{-7}\left[\frac{1}{z-1} - \frac{1}{z-8}\right]$$

$$= \frac{-10}{7(z-1)} + \frac{10}{7}\frac{1}{(z-8)}$$

Example 04: Evaluate $\int_c \frac{z+6}{z^2-4} dz$, where c is the circle $i)\ |z|=1,\quad ii)\ |z-2|=1,\quad iii)\ |z+2|=1$

$\rightarrow\ z^2-4=0 \Rightarrow (z+2)(z-2)=0 \Rightarrow z=2,-2$

i) $|z|=1$: Both $z=2,-2$ are outside



$\therefore$ By CIT, $I=0$

2) $|z-2|=1$, $(2,0)$, $r=1$

$z=2$ is inside c ; $(z-2)=0$

$z=-2$ is outside c ; $(z+2)\neq 0$

$|z|=r$: $(0,0)$

$|z-a|=r$: $(a,0)$

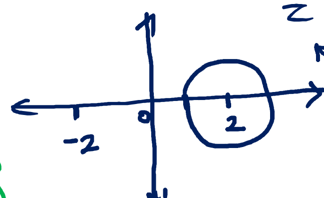
$|z-ai|=r$: $(0,a)$

$|z+a|=r$; $(-a,0)$

$|z+ai|=r$; $(0,-a)$

$|z-a-bi|$; (a,b)

$|z+a+bi|$; $(-a,-b)$



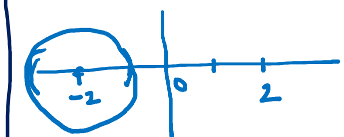
$$I=\int_c \frac{z+6}{(z+2)(z-2)}dz=\int_c \frac{\left[\frac{z+6}{z+2}\right]}{z-2}dz \equiv \int_c \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

$z-2=0$

• $f(z)=\frac{z+6}{z+2}$, $z_0=2$

By CIF, $I=2\pi i\left[\frac{z+6}{z+2}\right]_{z=2}=2\pi i\left[\frac{8}{4}\right]=\boxed{4\pi i}$

iii) $|z+2|=1$; $(-2,0)$, $r=1$



$\therefore z=-2$ is inside c $\{z+2\}$

$z=2$ is outside c ; $(z-2)$

$$\therefore I=\int_c \frac{z+6}{(z-2)(z+2)}dz=\int_c \frac{\left[\frac{z+6}{z-2}\right]}{z+2}dz$$

By CIF, $f(z)=\frac{z+6}{z+2}$, $z_0=-2$

$$=2\pi i\, f(-2)=2\pi i\left[\frac{z+6}{z-2}\right]_{z=-2}$$

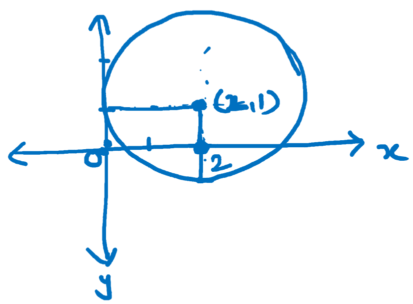
$$=2\pi i\left[\frac{4}{-4}\right]$$

$$=\boxed{-2\pi i}$$

Example 05: Evaluate $\int_c \frac{z+2}{z^3-2z^2}dz, where\ c\ is\ the\ circle \quad |z-2-i| = 2$

$\rightarrow \quad z^3 - 2z^2 = 0 \Rightarrow z^2(z-2) = 0 \Rightarrow z = 0, 0, 2$

$|z-2-i| = 2; \quad (2,1), \ r = 2$



$z = 0$ is outside c

$z = 2$ is inside c

$\therefore \quad I = \int_c \frac{z+2}{z^2(z-2)}dz = \int_c \frac{\left[\frac{z+2}{z^2}\right]}{z-2}dz = \int_c \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$

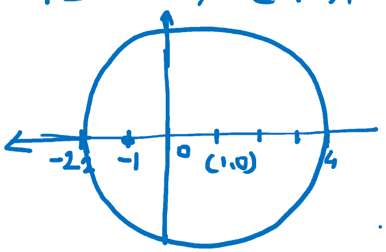
$f(z) = \frac{z+2}{z^2}, \quad z_0 = 2, \quad$ By CIF

$I = 2\pi i \left[\frac{z+2}{z^2}\right]_{z=2} = 2\pi i \left[\frac{4}{4}\right] = \boxed{2\pi i}$ #

Example 06: Evaluate $\int_c \frac{e^{2z}}{(z+1)^4} dz$, where c is the circle $|z-1| = 3$

$\rightarrow (z+1)^4 = 0 \Rightarrow z = -1, -1, -1, -1$

$|z-1| = 3$, $(1,0)$, $r = 3$



$z = -1$ is inside c, & repeated $n = 4$ times.

$$\int_c \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$\therefore$ By CIF

$$I = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{\pi i}{3} f'''(-1) \quad \text{— (1)}$$

$f(z) = e^{2z}$

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$f'''(-1) = 8e^{-2}$

Putting in (1)

$$I = \frac{\pi i}{3} \cdot 8e^{-2}$$

$$\boxed{I = \frac{8\pi i}{3e^2}}$$

#

Example 07: Evaluate $\int_c \frac{1}{z^3(z+4)} dz$, *where c is the circle* $|z| = 2$

$\rightarrow$ $z^3(z+4)=0 \Rightarrow z = 0,0,0,-4$

$z=0$ is inside c, is a centre of $|z|=2$

$z=-4$ is outside c.

$$I = \int_c \frac{1}{z^3(z+4)} dz = \int_c \frac{\left[\frac{1}{z+4}\right]}{z^3} dz$$

Here $n=3$, $f(z) = \frac{1}{z+4}$, $z_0 = 0$

By CIF

$$= \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$= \frac{2\pi i}{2!} \cdot f''(0)$$

$$= \pi i \, f''(0) \quad \text{--- (1)}$$

$$\frac{d}{dx}\left(\frac{1}{f(x)}\right) = -\frac{f'(x)}{(f(x))^2}$$

$$f(z) = \frac{1}{z+4}$$

$$f'(z) = -\frac{1}{(z+4)^2} = -(z+4)^{-2}$$

$$f''(z) = 2(z+4)^{-3} = \frac{2}{(z+4)^3}$$

put $z=0$

$$f''(0) = \frac{2}{4^3} = \frac{2}{64} = \frac{1}{32}$$

(1) $\Rightarrow$ $\boxed{I = \frac{\pi i}{32}}$

#

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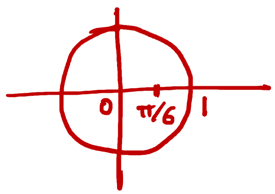
Example 08: Evaluate $\int_c \frac{\sin^6 z}{\left(z-\frac{\pi}{6}\right)^3} dz$, *where c is the circle* $|z| = 1$

$\rightarrow$ $z = \frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}$, $\Rightarrow$ $\frac{\pi}{6} = \frac{22}{7}\cdot\frac{1}{6} = \frac{11}{21} = 0.52$



$z = \pi/6$ is inside point & $n = 3$

By CIF

$$I = \int_c \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} f''(\pi/6)$$

$$= \pi i\, f''(\pi/6) \quad \text{--- (1)}$$

Now $f(z) = \sin^6 z$

$f'(z) = 6\sin^5 z\cdot\cos z$

$f''(z) = 6\left[\sin^5 z(-\sin z) + \cos z\,(5\sin^4 z\cdot\cos z)\right]$

$= 6\left[-\sin^6 z + 5\cdot\cos^2 z\cdot\sin^4 z\right]$

put $z = \pi/6$

$\therefore f''(\pi/6) = 6\left[-\left(\frac{1}{2}\right)^6 + 5\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)^4\right]$

$= 6\left[-\frac{1}{2^6} + \frac{15}{2^6}\right] = \frac{\cancel{6}^{\,3}\times\cancel{14}^{\,7}}{\cancel{2^6}\;2^4}$

$= \frac{21}{16}$

(1) $\Rightarrow$

$I = \pi i\cdot\frac{21}{16} = \boxed{\frac{21\pi i}{16}}$ #

Example 09: Evaluate $\int_c \frac{z^2+4}{(z-2)(z+3i)} dz, where\ c\ is\ the\ circle\ i) \quad |z+1| = 2, \quad ii)\ |z-2| = 2$

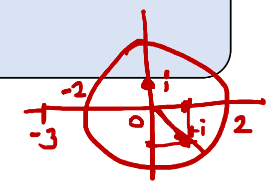
Examples on ξ [xi]

$x+iy \equiv (x,y)$

$1-i \equiv (1,-1)$

Example 10: If $f(\xi) = \int_C \frac{3z^2+7z+}{z-\xi}\, dz,$ $where\ C\ is\ a\ circle\ |z| = 2, find\ the\ values\ of$

$f(i),\quad f'(1-i),\quad f''(1-i),\quad f(-3)$



→ Assume ξ is inside C.

then by CIF

$$f(\xi) = \int_C \frac{3z^2+7z+1}{z-\xi}\,dz = 2\pi i\left[3\xi^2+7\xi+1\right]$$

$f(\xi) = 2\pi i\left[3\xi^2+7\xi+1\right]$ — (1)

$f'(\xi) = 2\pi i\left[6\xi+7\right]$ — (2)

$f''(\xi) = 2\pi i(6) = 12\pi i$ — (3)

i) $f(i) \Rightarrow \xi = i \equiv (0,1)$ inside $|z|=2$

eqn (1) $\Rightarrow f(i) = 2\pi i\left[3i^2+7i+1\right] = 2\pi i\left[-3+7i+1\right] = 2\pi i\left[-2+7i\right]$

ii) $f'(1-i) \Rightarrow \xi = 1-i \equiv (1,-1)$ is inside

eqn (2) $\Rightarrow f'(1-i) = 2\pi i\left[6(1-i)+7\right] = 2\pi i\left[13-6i\right]$

iii) $f''(1-i)$; $\xi = 1-i$ is inside C.

eqn (3) $\Rightarrow f''(1-i) = 12\pi i$ #

iv) $f(-3)$: $\xi = -3 \equiv (-3,0)$ is outside

$\therefore$ By CIT, $\boxed{I = f(-3) = 0}$

Example 11: If $f(\xi) = \int_c \frac{4z^2+z+5}{z-\xi}\,dz,\ \ where\ C\ is\ the\ ellipse\ \frac{x^2}{4} + \frac{y^2}{9} = 1, find\ the\ values\ of$

$f(i),\qquad f'(-1),\qquad f''(-i),\qquad f(3),$

Examples on ξ $[xi]$

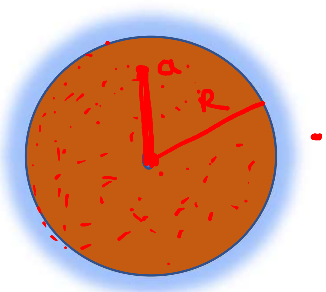
1.2 Taylor's and Laurent's series

A series of the type $c_0 + c_1(z-a) + c_2(z-a)^2 + \cdots + c_n(z-a)^n + \cdots$ *is called Series in powers of* $(z-a)$ *then it can be written as* $\sum_{\{n=1\}}^{\infty} c_n(z-a)^n$.

$c_0 + c_1 z + c_2 z^2 + c_3 z^3 + ---$

Where z *is a complex number. The constants* $c_0, c_1, \ldots$ *are called the coefficients. The constant a is called the centre of the series.*

There exists R such that $|z-a| < R$ *then the series is convergent, if* $|z-a| > R$ *then the series is divergent whereas at* $|z-a| = R$*, the series may or may not be convergent, R is called region or radius of convergence*



1.2 Taylor's and Laurent's series

Power series are important in complex analysis because every power series is analytic and every analytic function can be represented as power series such series are called ***Taylor series***.

Analytic functions can also be represented by another type of series containing positive as well as negative powers of $(z - a)$ *such series are called* ***Laurent series***. *They are useful for evaluating real and complex integrals*

Examples: e^z, $sinz$, $cosz$,

Taylor

$f(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$

Laurent

$f(z) = \frac{1}{z} + 1 + z + z^2 + z^3 + z^4 + \cdots$

Example 01: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{2}{(z-1)(z-2)}$, indicate the region of convergence ROC

$\rightarrow (z-1)(z-2)=0 \Rightarrow z=1,2$

∴ ROC : i) $|z|<1$ — 2

ii) $1<|z|<2$ — 2

iii) $|z|>2$ — 2

Now $\frac{2}{(z-1)(z-2)} = \frac{2}{-1}\left[\frac{1}{z-1} - \frac{1}{z-2}\right]$

$= 2\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$

i) $\boxed{|z|<1}$:

$f(z) = 2\left[\frac{1}{-2\left[1-\frac{z}{2}\right]} - \frac{1}{-1\left[1-z\right]}\right]$

$= 2\left[-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}\right]$

$(1-z)^{-1} = 1+z+z^2+z^3+z^4+\cdots$

$(1+z)^{-1} = 1-z+z^2-z^3+z^4-\cdots$

$f(z) = 2\left[-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\frac{z^4}{16}+\cdots\right)+\left(1+z+z^2+z^3+\cdots\right)\right]$

This is Taylor's series convergent in $|z|<1$

ii) $1<|z|<2$: $f(z) = 2\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$

$f(z) = 2\left[\frac{1}{-2\left[1-\frac{z}{2}\right]} - \frac{1}{z\left(1-\frac{1}{z}\right)}\right] = 2\left[-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right]$

$= 2\left[-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)+\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\right]$

This is Laurent's series convergent in $1<|z|<2$

iii)

Example 01: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{2}{(z-1)(z-2)}$, *indicate the region of convergence*

iii) $f(z) = 2\left[\frac{1}{z-2} - \frac{1}{z-1}\right]$

$\boxed{|z|>2}$

$$f(z) = 2\left[\frac{1}{z\left(1-\frac{2}{z}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)}\right]$$

$$= 2\left[\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}\right]$$

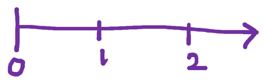
$$= \frac{2}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+\cdots - \left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)\right]$$

This is Laurent series convergent in $|z|>2$

#

Example 02: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{1}{z(z+1)(z-2)}$, *indicate the region of convergence*

$\rightarrow z(z+1)(z-2)=0 \Rightarrow z = 0, -1, 2$



ROC: i) $|z|<1$
ii) $1<|z|<2$
iii) $|z|>2$

$$f(z) = \frac{1}{z(z+1)(z-2)} = \frac{A}{z} + \frac{B}{z+1} + \frac{C}{z-2}$$

$$1 = A(z+1)(z-2) + Bz(z-2) + Cz(z+1)$$

$z=0$: $1 = A(1)(-2) \Rightarrow A = -1/2$

$z=-1$: $1 = B(-1)(-3) \Rightarrow B = 1/3$

$z=2$: $1 = C(2)(3) \Rightarrow C = 1/6$

$$f(z) = -\frac{1}{2z} + \frac{1}{3}\cdot\frac{1}{z+1} + \frac{1}{6}\cdot\frac{1}{z-2}$$

Do not expand.

$z \neq 0$

i) $|z|<1$

$$f(z) = -\frac{1}{2z} + \frac{1}{3}\frac{1}{(1+z)} + \frac{1}{6}\frac{1}{-2(1-\frac{z}{2})}$$

$|z|<1$
$\parallel$
$(-1<z<1)$

$$= -\frac{1}{2z} + \frac{1}{3}(1+z)^{-1} - \frac{1}{12}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3}\left(1 - z + z^2 - z^3 + z^4 - \dots\right) - \frac{1}{12}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right)$$

This is Laurent's expansion convergent in $|z|<1$

ii) $1<|z|<2$

$$f(z) = -\frac{1}{2z} + \frac{1}{3}\frac{1}{z(1+\frac{1}{z})} + \frac{1}{6}\cdot\frac{1}{-2(1-\frac{z}{2})}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{12}\left(1-\frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right) - \frac{1}{12}\left(1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right)$$

is Laurent's expansion is convergent in $1<|z|<2$.

Example 04: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{z-1}{(z+1)(z-3)}$, *in the region of convergence* i) $|z| < 1$, ii) $1 < |z| < 3$, iii) $|z| > 3$

$\boxed{z = -1, 3}$

$$f(z) = \frac{z-1}{(z+1)(z-3)} = \frac{A}{z+1} + \frac{B}{z-3}$$

$$z-1 = A(z-3) + B(z+1)$$

$z=-1$: $-2 = A(-4) \Rightarrow A = 1/2$

$z=3$: $2 = B(4) \Rightarrow B = 1/2$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} + \frac{1}{z-3}\right]$$

i) $|z|<1$

$$f(z) = \frac{1}{2}\left[\frac{1}{1(1+z)} + \frac{1}{-3(1-\frac{z}{3})}\right]$$

$$= \frac{1}{2}\left[(1+z)^{-1} - \frac{1}{3}(1-\frac{z}{3})^{-1}\right]$$

$$= \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots - \frac{1}{3}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \cdots\right)\right]$$

This is Taylor's series, convergent in $|z|<1$

2) $1<|z|<3$

$$f(z) = \frac{1}{2}\left[\frac{1}{1+z} + \frac{1}{z-3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z(1+\frac{1}{z})} + \frac{1}{-3(1-\frac{z}{3})}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}(1+\frac{1}{z})^{-1} - \frac{1}{3}(1-\frac{z}{3})^{-1}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{3}\left(1 + \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \left(\frac{z}{3}\right)^3 + \cdots\right)\right]$$

This is Laurent's Series, convergent in $1<|z|<3$

iii) $|z|>3$

$$f(z) = \frac{1}{2}\left[\frac{1}{1+z} + \frac{1}{z-3}\right]$$

$$= \frac{1}{2}\left[\frac{1}{z(1+\frac{1}{z})} + \frac{1}{z(1-\frac{3}{z})}\right] = \frac{1}{2z}\left[(1+\frac{1}{z})^{-1} + (1-\frac{3}{z})^{-1}\right]$$

Laurent. in $|z|>3$.

Example 04: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{z-1}{(z+1)(z-3)}$, *in the region of convergence* *i)* $|z| < 1$, $\quad ii)\ 1 < |z| < 3$, $\quad iii)\ |z| > 3$

Example 05: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{2z-3}{z^2-4z+3}$, in the powers of $(z-4)$ indicating the region of convergence

or {about $z=4$}

$\rightarrow f(z) = \frac{2z-3}{(z-1)(z-3)} = \frac{2(z-4)+5}{[(z-4)+3][(z-4)+1]} = \frac{2u+5}{(u+3)(u+1)}, \quad u = z-4$

$\therefore f(u) = \frac{2u+5}{(u+3)(u+1)}, \quad (u+3)(u+1)=0$

$u = -1, -3$

ROC:
i) $|u|<1 \; : |z-4|<1$
ii) $1<|u|<3 : 1<|z-4|<3$
iii) $|u|>3 : |z-4|>3$

$f(u) = \frac{2u+5}{(u+3)(u+1)} = \frac{A}{u+3} + \frac{B}{u+1}$

$2u+5 = A(u+1) + B(u+3)$

$u=-3: \; -1 = A(-2) \Rightarrow A = 1/2$

$u=-1: \; 3 = 2B \Rightarrow B = 3/2$

$f(u) = \frac{1}{2}\cdot\frac{1}{u+3} + \frac{3}{2}\cdot\frac{1}{u+1}$

i) $|u|<1$

$\rightarrow f(u) = \frac{1}{2}\cdot\frac{1}{3(1+\frac{u}{3})} + \frac{3}{2}\cdot\frac{1}{(1+u)}$

$= \frac{1}{6}(1+\frac{u}{3})^{-1} + \frac{3}{2}(1+u)^{-1}$

$= \frac{1}{6}\left[1 - \frac{u}{3} + (\frac{u}{3})^2 - (\frac{u}{3})^3 + \cdots\right]$

$+ \frac{3}{2}\left[1 - u + u^2 - u^3 + \cdots\right]$

put $u = z-4$

$f(z) = \frac{1}{6}\left[1 - \frac{(z-4)}{3} + (\frac{z-4}{3})^2 - (\frac{z-4}{3})^3 + \cdots\right] + \frac{3}{2}\left[1-(z-4)+(z-4)^2-(z-4)^3+\cdots\right]$

is Taylor's series in $|z-4|<1$

ii) $f(u) = \frac{1}{2}\cdot\frac{1}{u+3} + \frac{3}{2}\cdot\frac{1}{u+1}$

$1<|u|<3$:

$\therefore f(u) = \frac{1}{2}\cdot\frac{1}{3(1+\frac{u}{3})} + \frac{3}{2}\cdot\frac{1}{u(1+\frac{1}{u})}$

$= \frac{1}{6}\left[1+\frac{u}{3}\right]^{-1} + \frac{3}{2u}\left(1+\frac{1}{u}\right)^{-1}$

$= \frac{1}{6}\left[1 - \frac{u}{3} + (\frac{u}{3})^2 - \cdots\right] + \frac{3}{2u}\left[1 - \frac{1}{u} + \frac{1}{u^2} - \cdots\right]$

$\boxed{z-4=u}$

$= \frac{1}{6}\left[1 - \frac{z-4}{3} + \frac{(z-4)^2}{9} + \cdots\right] + \frac{3}{2(z-4)}\left[1 - \frac{1}{z-4} + \frac{1}{(z-4)^2} - \cdots\right]$

is Laurent series in $1<|z-4|<3$

Example 05: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{2z-3}{z^2-4z+3}$*, in the powers of* $(z-4)$ *indicating the region of convergence*

Example 06: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$, *about* $z = -1$ *indicating the region of convergence*

$z = 0, 1, 2$ $(z+1)$

$$f(z) = \frac{7(z+1)-9}{(z+1-1)(z+1-3)(z+1)} = \frac{7u-9}{(u-1)(u-3)u}, \quad \boxed{u = z+1}$$

$u(u-1)(u-3) = 0 \Rightarrow u = 0, 1, 3.$

ROC: i) $|u| < 1$

ii) $1 < |u| < 3$

iii) $|u| > 3$

$$f(u) = \frac{7u-9}{u(u-1)(u-3)} = \frac{A}{u} + \frac{B}{u-1} + \frac{C}{u-3}$$

$$7u - 9 = A(u-1)(u-3) + B(u)(u-3) + C u(u-1)$$

$u = 0$; $-9 = A(-1)(-3) \Rightarrow \boxed{A = -3}$

$u = 1$; $-2 = B(1)(-2) \Rightarrow \boxed{B = 1}$

$u = 3$; $12 = C\,3 \cdot 2 \Rightarrow \boxed{C = 2}$

$$f(u) = \frac{-3}{u} + \frac{1}{u-1} + \frac{2}{u-3}$$

i) $|u| < 1$: $|z+1| < 1$

$$f(u) = -\frac{3}{u} + \frac{1}{-(1-u)} + \frac{2}{-3(1-\frac{u}{3})}$$

$$= -\frac{3}{u} - (1-u)^{-1} - \frac{2}{3}(1-\frac{u}{3})^{-1}$$

$$= \frac{-3}{u} - \left[1 + u + u^2 + u^3 + \cdots\right] - \frac{2}{3}\left[1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{3}\right)^3 + \cdots\right]$$

$$\therefore f(z) = \frac{-3}{z+1} - \left[1 + (z+1) + (z+1)^2 + \cdots\right] - \frac{2}{3}\left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \cdots\right]$$

This is Laurent's series, convergent in $|z+1| < 1$

ii) $1 < |u| < 3$

$$f(u) = -\frac{3}{u} + \frac{1}{u(1-\frac{1}{u})} + \frac{2}{-3(1-\frac{u}{3})}$$

Laurent.

iii) $|u| > 3$

$$f(u) = \frac{-3}{u} + \frac{1}{u(1-\frac{1}{u})} + \frac{2}{u(1-\frac{3}{u})}$$

Laurent

Example 06: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{7z-2}{z(z-2)(z+1)}$, *about* $z = -1$ *indicating the region of convergence*

Example 07: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{z}{(z-1)(z-2)}$, *about* $z = -2$ *indicating the region of convergence*

Example 07: Find the all possible Taylor's and Laurent's expansion of $f(z) = \frac{z}{(z-1)(z-2)}$, *about* $z = -2$ *indicating the region of convergence*

1.3 Residues

$f(z)=z-2$
$f(2)=0$, 2 is zero of $f(z)$

Zero of f(z): Let $f(z)$ is an analytic in R and z_0 is a point in R, then if $f(z_0)=0$, z_0 is called zero of $f(z)$ in R.

If $f(z_0)=0$ *and* $f'(z_0)\neq 0$, *then* z_0 *is called simple zero or a zero of order one*

If $f(z_0)=0$ *and* $f'(z_0)=0$, *but* $f''(z_0)\neq 0$ *then* z_0 *is called a zero of order two* and so on.

Examples: $sinz, (z-1)e^z$ *are functions with simple zero*

$f(z)=\sin z$, $f(0)=0$
$f'(z)=\cos z$, $f'(0)\neq 0$

$f(z_0)=0$
$f'(z_0)=0$
$f''(z_0)=0$
$\vdots$
$f^{n-1}(z_0)=0$
$f^{n}(z_0)\neq 0$

$f(z)=z^3$, $f(0)=0$
$f'(z)=3z^2$, $f'(0)=0$
$f''(z)=6z$, $f''(0)=0$
$\boxed{f'''(z)=6}$, $f'''(0)\neq 0$
$\Rightarrow$ a zero of order $\underline{3}$

$f(z)=(z-1)e^z$
$f'(z)=(z-1)e^z+e^z$.
$f'(1)=0+e\neq 0$

1.3 Residues

Singularity: *If $f(z)$ is analytic at every point except z_0, then z_0 is called the singularity of $f(z)$.*

Example: $\frac{z^2}{z-3}, \frac{1}{z+1}$

$z=3$ $z=-1$

Isolated Singularity: *If there is no other singular point inside the neighbourhood of z_0 other than z_0* : $\frac{z^2}{z-2}$

$\frac{z^2}{z-2}$, $\boxed{z=2}$

(1)

1.85 2 2.73

(1.85, 2.73)

1.3 Residues

Pole: *Let f(z) can be expanded as a Laurent Series convergent in R as follows;*

$f(z) = \sum_{\{n=0\}}^{\{\infty\}} a_n (z - z_0)^n + \sum_{\{n=1\}}^{\{\infty\}} b_n (z - z_0)^{-n}$, *where first part is called Analytic part and second part is called Principal part.*

$+ \frac{b_1}{(z-z_0)} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \frac{b_{n+1}}{(z-z_0)^{n+1}} + \cdots \to 0$

$\boldsymbol{If\ b_{n+1} = b_{n+2} = \cdots = 0, then\ z = z_0\ is\ called\ a\ pole\ of\ order\ n}$

Residue

$\boldsymbol{For\ a\ pole\ of\ Order\ 1\ or\ simple\ pole\ we\ have};$

1.3 Residues

Removable Singularity:

If $\lim_{z\to z_0} f(z)$ *exists or if expansion of* $f(z)$ *does not contain negative powers of z then* z_0 *is called removable singularity. Example:* $f(z) = \frac{\sin z}{z}$ *has removable singularity at* $z = 0$

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= \boxed{1} - \underbrace{\frac{z^2}{3!} + \frac{z^4}{5!} - \cdots}_{\to 0}$$

$\boxed{z=0}$

$\boxed{f(0)=1}$

$\lim_{x\to 0} \frac{\sin x}{x} \quad \left(\frac{0}{0}\right)$

L-Hopital rule.

$\lim_{x\to 0} \frac{\cos x}{1} = 1$ #

1.3 Residues

$f(z) = \frac{z^2}{(z-4)(z-7)^2}$, $(z-4)(z-7)^2 = 0$

$z = 4, 7, 7$

$z=4$ is simple pole $(n=1)$

$z=7$ is a pole of order 2

$Res(z=4) = \lim_{z\to 4} \cancel{(z-4)} \cdot \frac{z^2}{\cancel{(z-4)}(z-7)^2} = \boxed{\frac{16}{9}}$

Residue: If $f(z) = \sum_{\{n=0\}}^{\{\infty\}} a_n (z-z_0)^n + \sum_{\{n=1\}}^{\{\infty\}} b_n (z-z_0)^{-n}$, *then*

b_1 *, the coefficient of* $\frac{1}{z-z_0}$*, is called*

the residue of $f(z)$ *at* $z = z_0$.

$Res(z=7) = \frac{1}{(2-1)!} \cdot \lim_{z\to 7} \frac{d}{dz} \cancel{(z-7)^2} \cdot \frac{z^2}{(z-4)\cancel{(z-7)^2}} = C$

Formulae:

1.Residue at simple pole: $Res(z = z_0) = \lim_{\{z\to z_0\}} (z-z_0) f(z)$

2.Residue at a pole of order n: $Res(z = z_0) = \frac{1}{(n-1)!} \lim_{\{z\to z_0\}} [\frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n f(z)]$

Example 01: Find the pole of the function $f(z) = \frac{z^2}{(z-1)^2(z+2)}$, *and find the residue at each pole*

1.Residue at simple pole:
$Res(z = z_0) = \lim_{\{z \to z_0\}} (z - z_0) f(z)$

$\rightarrow (z-1)^2(z+2)=0 \Rightarrow z=1,1,-2$

$z=-2$ is a simple pole

$z=1$ is a pole of order $n=2$

$$Res(z=-2) = \lim_{z \to -2} \cancel{(z+2)} \cdot \frac{z^2}{(z-1)^2\cancel{(z+2)}} = \boxed{\frac{4}{9}}$$

$$Res(z=1) = \frac{1}{(2-1)!} \lim_{z \to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \frac{z^2}{\cancel{(z-1)^2}(z+2)}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz}\left[\frac{z^2}{z+2}\right]$$

$$= \lim_{z \to 1} \left[\frac{(z+2)2z - z^2 \cdot 1}{(z+2)^2}\right] = \frac{6-1}{9} = \boxed{\frac{5}{9}}$$

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Example 02: Find the residues of the function $f(z) = \frac{\sin \pi z}{(z-1)^2(z-2)}$, *at each pole*

1.Residue at simple pole:
$Res(z = z_0) = \lim_{\{z \to z_0\}} (z - z_0) f(z)$

$\rightarrow (z-1)^2(z-2) = 0 \Rightarrow z = 1, 1, 2$

$z = 2$ is a simple pole

$z = 1$ is a pole of order $n = 2$

$$Res(z=2) = \lim_{z \to 2} \cancel{(z-2)} \frac{\sin \pi z}{(z-1)^2 \cancel{(z-2)}} = \frac{\sin 2\pi}{1} = 0$$

$$Res(z=1) = \frac{1}{(n-1)!} \lim_{z \to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right]$$

$$= \lim_{z \to 1} \frac{d}{dz} \left[\cancel{(z-1)^2} \cdot \frac{\sin \pi z}{\cancel{(z-1)^2}(z-2)} \right]$$

$$= \lim_{z \to 1} \frac{(z-2)\cos \pi z \cdot \pi - \sin \pi z \cdot 1}{(z-2)^2}$$

$$= \frac{-(-1)\pi - 0}{1}$$

$$= \pi \quad \#$$

1.3 Cauchy's Residue Theorem

Statement: If f(z) is analytic inside and on a simple closed curve C except at a finite number of isolated singular points inside C then $\int_c f(z)dz = 2\pi i$ [*sum of the residues at inside points*]

$\int_c$ in/out

0 i) out

shift ii) in/out

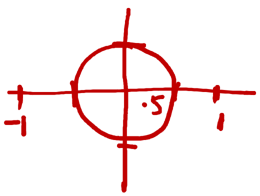
P.F. iii) in

Example 03: Evaluate $\int_c \frac{z^2}{(z+1)(z-1)^2}dz$, *where i) c:* $|z| = 0.5$, *ii)* $|z| = 2$

$\int_c f(z)dz = 2\pi i\,[sum\ of\ the\ residues]$

$\rightarrow (z+1)(z-1)^2 = 0 \Rightarrow z = -1, 1, 1$

i) $|z| = 0.5$, $(0,0)$, $r = 0.5$



both points are outside

$\therefore$ By CIT; $\boxed{I = 0}$

ii) $|z| = 2$: Both are inside

$z = -1$ is a simple pole

$z = 1$ is a pole of order $n = 2$

$$Res(z=-1) = \lim_{z \to -1} \cancel{(z+1)} \cdot \frac{z^2}{\cancel{(z+1)}(z-1)^2} = \frac{1}{4}$$

$$Res(z=1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \cdot \frac{z^2}{\cancel{(z-1)^2}(z+1)}\right]$$

$$= \lim_{z \to 1} \frac{(z+1)2z - z^2(1)}{(z+1)^2} = \frac{4-1}{4} = \frac{3}{4}.$$

$\therefore$ By CRT

$I = 2\pi i\,[\text{Sum of Residues}]$

$= 2\pi i \left[\frac{1}{4} + \frac{3}{4}\right]$

$= 2\pi i$

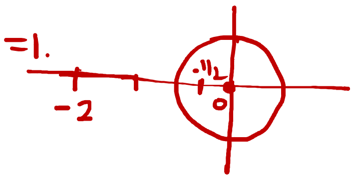
Example 04: Evaluate $\int_c \frac{2z-1}{z(2z+1)(z+2)}dz$ *, using residue theorem,* $c: |z| = 1$

$$\int_c f(z)dz = 2\pi i\,[\text{sum of the residues}]$$

$\rightarrow$ $z(2z+1)(z+2)=0 \Rightarrow z=0, -\frac{1}{2}, -2$

✓ ✓ ✗

$C: |z|=1.$



$z=0$ is a simple pole

$z=-\frac{1}{2}$ is a simple pole

$\therefore \text{Res}(z=0) = \lim_{z\to 0} (\cancel{z}-0)\frac{2z-1}{\cancel{z}(2z+1)(z+2)} = \frac{-1}{2}$

$\text{Res}(z=-\frac{1}{2}) = \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2})\cdot\frac{2z-1}{z(2z+1)(z+2)}$

$= \lim_{z\to -\frac{1}{2}} \cancel{(z+\frac{1}{2})}\cdot\frac{2z-1}{z\cdot 2\cancel{(z+\frac{1}{2})}(z+2)} = \frac{-2}{(-1)(-\frac{1}{2}+2)} = \frac{-2}{-3/2} = \frac{4}{3}$

$\therefore$ By C.R.T

$I = 2\pi i\left[-\frac{1}{2}+\frac{4}{3}\right]$

$= 2\pi i\left[\frac{-3+8}{6}\right]$

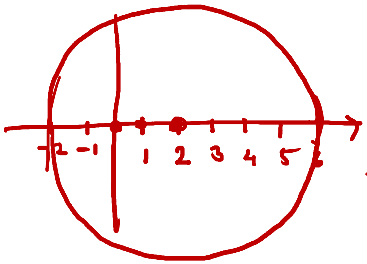
$\boxed{I = 5\pi i/3}$

Example 05: Evaluate $\int_c \frac{\cos\pi z^2+\sin\pi z^2}{z-z^2}\,dz$ *, using residue theorem,* $c: |z-2| = 4$

$$\int_c f(z)dz = 2\pi i\,[\text{sum of the residues}]$$

→ $z - z^2 = z(1-z) = 0 \Rightarrow z = 0, 1$

$|z-2| = 4$, $(2,0)$, $r = 4$



All points are inside & simple poles

$$\therefore \text{Res}(z=0) = \lim_{z\to 0} \cancel{(z-0)} \cdot \frac{\cos\pi z^2+\sin\pi z^2}{\cancel{z}(1-z)} = \frac{1+0}{1} = 1$$

$$\text{Res}(z=1) = \lim_{z\to 1} (z-1) \cdot \frac{\cos\pi z^2+\sin\pi z^2}{z(1-z)}$$

$$= \lim_{z\to 1} \cancel{(z-1)} \cdot \frac{\cos\pi z^2+\sin\pi z^2}{-z\cancel{(z-1)}}$$

$$= \frac{-1+0}{-1} = 1$$

By CRT

$$I = 2\pi i(1+1)$$

$$\boxed{I = 4\pi i}$$

1.3 Application of Residue $[f(\theta)]$

Let us consider the unit circle $|z| = 1$, *Use* $z = e^{i\theta}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}, d\theta = \frac{dz}{iz}$

Self Learning

Example 01: Evaluate $\int_0^{2\pi} \frac{d\theta}{5+3sin}$

$z = e^{i\theta}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}, d\theta = \frac{dz}{iz}$

Example 02: Evaluate $\int_0^{\pi} \frac{d\theta}{3+2co}$

$z = e^{i\theta}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}, d\theta = \frac{dz}{iz}$

Example 02: Evaluate $\int_0^{\pi} \frac{d\theta}{3+2co}$

$z = e^{i\theta}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}, d\theta = \frac{dz}{iz}$

Example 03: Show that $\int_0^{2\pi} \frac{cos2\theta d\theta}{5+4co} = \frac{\pi}{6}$

$$z = e^{i\theta}, sin\theta = \frac{z^2 - 1}{2iz}, cos\theta = \frac{z^2 + 1}{2z}, d\theta = \frac{dz}{iz}$$

Example 03: Show that $\int_0^{2\pi} \frac{cos2\theta d\theta}{5+4cos\theta} = \frac{\pi}{6}$

$z = e^{i\theta}, sin\theta = \frac{z^2-1}{2iz}, cos\theta = \frac{z^2+1}{2z}, d\theta = \frac{dz}{iz}$

Example 04: Show that $\int_0^{2\pi} \frac{\cos 3\theta \, d\theta}{5-4\cos\theta} = \frac{\pi}{12}$

$$z = e^{i\theta}, \sin\theta = \frac{z^2-1}{2iz}, \cos\theta = \frac{z^2+1}{2z}, d\theta = \frac{dz}{iz}$$

1.3 Application of Residue $[f(x)]$

- *Consider* $\int_{-\infty}^{\infty} f(x)dx$

  -- *Consider The curve consisting of large semi circle with centre at origin in the upper half of the plane and its diameter on the real axis*

  -- $zf(z) \to 0 \; as \; |z| \to \infty$

  -- *Find the poles of f(z) lying in the upper half of the plane*

  -- *Find the residues at these poles*

  -- *Then by Cauchy's residue theorem find out integral value*

Example 01: Evaluate $\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9}\,dx$ *using contour integration*

Example 01: Evaluate $\int_{-\infty}^{\infty} \frac{x^2+x+2}{x^4+10x^2+9}\,dx$ *using contour integration*

Example 02: Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$

Example 02: Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx$

Example 03: Show that $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)^3} dx = \frac{3\pi}{8}$