CHAPTER

3

Probability

1. Introduction

You were introduced probability in the X-th standard and you have studied laws of probability, Bayes' theorem probability distributions, particularly Binomial distribution in XII-th standard. In this chapter we shall briefly review probability and then study once again Bays' Theorem in detail. Then we shall study probability distributions in general.

2. Terms used in Axiomatic Theory

The limitations of classical theory of probability are removed by putting the theory in axiomatic form. This approach was suggested by **A. N. Kolmogorov**, a Russian mathematician in 1933.

It should be noted that as in any other axiomatic theory (*e.g.* Geometry) we start with certain undefined terms, state certain axioms about them and then deduce theorems about these terms strictly from the axioms by rules of logic. Before developing the axiomatic probability theory we explain below certain peculiar terms used therein.

Andrey Nikolaevich Kolmogorov (1903 - 1987)

A well-known Russian mathema-tician known for his great contributions to the fields of probability theory, topology, intuitionistic logic, classical mechanics and others. In 1922 he constructed a Fourier Series that diverges almost everywhere. His pioneering work "About The Analytical Methods Of Probability Theory" was published in 1931, in which year he become professor at Moscow University. In 1933 he published "Foundations of The Theory of Probability" laying the foundation of modern axiomatic theory. In 1939 he become a member of U.S.S.R. Academy of Sciences. He along with British mathematician Chapman developed "The Chapman-Kolmogorov Equations" in Random processes. He was a founder of algorithm complexity theory known as "Kolmogorov Complexity Theory". A quotation of Kolmogorov. "Every mathematician believes that he is ahead of all others. The reason why they do not say this in public is they are intelligent people".

(1) Sample space

The aggregate or the set of all possible outcomes of an experiment is called *sample space*, the outcomes *i.e.* the members of the set themselves being called *sample points*. Sample space is denoted by S or Ω and sample points by $w_1, w_2, w_3, ...$ Thus, $\Omega = \{w_1, w_2, w_3, ...\}$.

Example 1 : In a random experiment of tossing a coin the sample points are its outcomes a head (H) and a tail (T) and the sample space is

$$\Omega = \{H, T\}.$$

Example 2 : In a random experiment of tossing a dice the sample points are 1, 2, 3, 4, 5 and 6 the sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

Example 3 : In a throw of two coins the sample points are (H, H), (H, T), (T, H), (T, T), and the sample space is

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}.$$

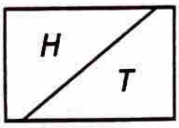
(2) Event

A subset of a sample space is called an *event.*

(i) In a throw of a coin $\Omega = \{H, T\}$. We have shown two events

$A_1 = \{H\}$,

$A_2 = \{T\}$.

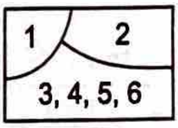


(ii) In a throw of a die $\Omega = \{1, 2, 3, 4, 5, 6\}$, there are a number of events.

$A_1 = \{1\},\ A_2 = \{2\},\ A_3 = \{3, 4, 5, 6\}, \ldots\ldots$

$B_1 = \{1, 2\},\ B_2 = \{1, 3\} \ldots\ldots$

$C_1 = \{1, 2, 3\},\ C_2 = \{1, 2, 4\}, \ldots\ldots$ etc.



We have shown the first three.

(iii) In a throw of a coin and a die there are again a number of events

$A_1 = \{(H, 1), (H, 2)\}$

$A_2 = \{(H, 1), (H, 3), (H, 5),\} \ldots\ldots$

$B_1 = \{(T, 1), (T, 2)\}$

$B_2 = \{(T, 1), (T, 3), (T, 5)\} \ldots\ldots$ etc.

3. Axiomatic Definition of Probability

Let E be a random experiment and S be the sample space. We define the probability $P(A)$ for every element A of S (*i.e.* for every subset A of the sample space S) satisfying the following axioms :

1. $P(A) \geq 0$ (Axiom 1)

2. $P(S) = 1$ (Axiom 2)

3. $P(A \cup B) = P(A) + P(B)$ (Axiom 3)

if A and B are any two exclusive events (*i.e.* they are disjoint sets).

Explanation : The first axiom states that the probability of any event is greater than zero, which means the probability of any event cannot be negative.

The second axiom stales that the **sum** of all the probabilities is equal to 1. This together with the first axiom states that the probability of an event must be less than or equal to 1 *i.e.* $P(A) \leq 1$.

The third axiom states that if two events are mutually exclusive, then the probability that either of them will occur is equal to the some of their probabilities.

For our study the classical definition is sufficient. We give that definition in terms of sets again for ready reference.

Definition : If a sample space S has n points which are equally likely and mutually exclusive and an event A has m points then the ratio m / n is called probability of A and is denoted by $P(A)$. [Fig. 3.1 (a)]

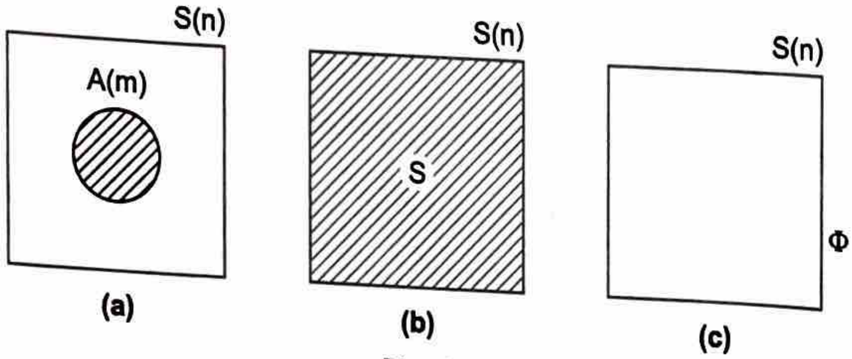


Fig. 3.1

Thus,

$$\boxed{P(A) = \frac{m}{n} = \frac{\text{number of points in } A}{\text{number of point in } S}}$$

Note ....

We shall be concerned with only such random experiments in which all sample points are equally likely. And to find the probability of an event in such a random experiment we need to know the number of points in sample space S and the number of points in the event A.

4. Laws of Probability

There are two laws. The probability of **A or B** *i.e.* $P(A \cup B)$ is given by the *law of addition* and the probability of **A and B** *i.e.* $P(A \cap B)$ is given by the *law of multiplication.*

Theorem 1 : For any two events A and B the probability that exactly B will occur is given by

$$\boxed{P(B \cap \overline{A}) = P(B) - P(A \cap B)}$$

and that exactly A will occur is given by

$$\boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$

where $\overline{A}$ denotes the complement of A and $\overline{B}$ denotes the complement of B.

Proof : To prove this result we first express the event B as the union of two exclusive events $A \cap B$ and $\overline{A} \cap B$.

$$\therefore \quad B = (A \cap B) \cup (\overline{A} \cap B)$$

Since the events on the r.h.s. are exclusive

$$P(B) = P(A \cap B) + P(\overline{A} \cap B) \qquad \text{[By Axiom 3]}$$

$$\therefore \quad P(B \cap \overline{A}) = P(B) - P(A \cap B)$$

Similarly, we can prove that

$$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

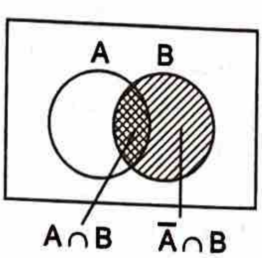


Fig. 3.2

Theorem 2 : (Additive Theorem) (Two Events)

Probability that atleast one of the events A and B will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof : We first express the event $A \cup B$ is the union of two exclusive events A and $\bar{A} \cap B$.

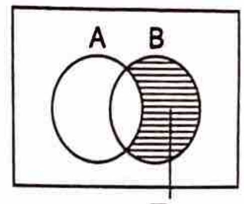


Fig. 3.3

$$A \cup B = A \cup (\bar{A} \cap B)$$

$$\therefore \quad P(A \cup B) = P[A \cup (\bar{A} \cap B)]$$

But the events on the r.h.s. are mutually exclusive.

$$\therefore \quad P(A \cup B) = P(A) + P(\bar{A} \cap B) \quad \text{[By Axiom 3]}$$

$$= P(A) + P(B) - P(A \cap B) \quad \text{[By Theorem 2]}$$

Corollary 1 : If A and B are two mutually exclusive events then the probability that either A or B will happen is the sum of the probabilities of A and B *i.e.*

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

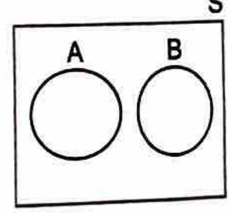


Fig. 3.4

Proof : Since events are exclusive $A \cap B = \Phi$.

$$\therefore \quad P(A \cap B) = P(\Phi) = 0 \quad \text{[By Theorem 1]}$$

Hence, from the above theorem,

$$P(A \cup B) = P(A) + P(B)$$

Corollary 2 : If $A, B, C, \ldots.., K$ are mutually exclusive events such that their union is the whole of sample space then

$$\boxed{P(A) + P(B) + P(C) + \ldots.. + P(K) = 1}$$

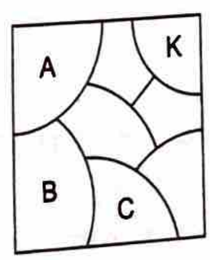


Fig. 3.5

Proof : Since $A \cup B \cup C \cup \ldots.. \cup K = S$

$$\therefore \quad P(A \cup B \cup C \cup \ldots.. \cup K) = P(S)$$

Since, events are mutually exclusive, we get, from above,

$$P(A) + P(B) + P(C) + \ldots.. + P(K) = P(S)$$

But $P(S) = 1$ [By Axiom 2]

$$\therefore \quad P(A) + P(B) + P(C) + \ldots.. + P(K) = 1$$

Note ....

The group of all possible events $A, B, C, \ldots.., K$ of the sample space S such that

$$P(A) + P(B) + P(C) + \ldots.. + P(K) = 1$$

is called **exhaustive.** *e.g.* in the toss of a coin H, T; in the toss of a dice 1, 2, 3, 4, 5, 6 are exhaustive events.

Theorem 3 : *Addition Theorem (Three Events)* : If A, B, C are any three events then the probability that *at least one* of them will occur is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Proof : We prove this theorem by considering the union of A and $(B \cup C)$ and applying the above theorem.

$$\therefore \quad P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

But $P(B \cup C) = P(B) + P(C) - P(B \cap C)$

And $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$\therefore \; P(A \cap (B \cup C)) = P[(A \cap B) \cup (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$
$$\therefore \; P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C)$$
$$- P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Corollary : If A, B, C are three events which are pairwise exclusive then

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C)}$$

Proof : Since, A, B are exclusive $A \cap B = \Phi$. Since, $P(\Phi) = 0$, $P(A \cap B) = 0$. Similarly, $P(B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$.

Hence, from the above result,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

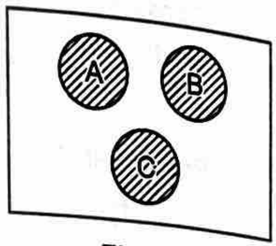


Fig. 3.6

Example : Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. What are the probabilities of their winning ?

Sol. : Let $P(C) = x$ then $P(B) = 2x$ and $P(A) = 4x$.

But the sum of all probabilities is 1.

$$\therefore \; P(A) + P(B) + P(C) = 1 \quad \therefore \; 4x + 2x + x = 1 \quad \therefore \; x = \frac{1}{7}$$

$$\therefore \; P(A) = \frac{4}{7}, \; P(B) = \frac{2}{7}, \; P(C) = \frac{1}{7}.$$

Complementary Events

It is clear that $A \cup \overline{A} = S$ *i.e.* the union of the event and its complement is the whole of sample space S and $\overline{A} \cap A = \Phi$ *i.e.* the event A and its complement are exclusive.

Theorem 4 : $\boxed{P(\overline{A}) = 1 - P(A)}$

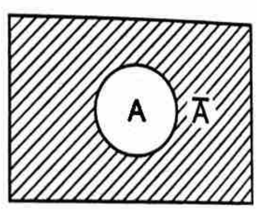


Fig. 3.7

Proof : Since $A \cup \overline{A} = S$, $P(A \cup \overline{A}) = P(S)$. But $P(S) = 1$ and A, $\overline{A}$ are exclusive.

By Axiom 2, $P(A \cup \overline{A}) = P(A) + P(\overline{A})$

$\therefore \; P(A) + P(\overline{A}) = 1 \quad \therefore \; P(\overline{A}) = 1 - P(A)$

Note ....

Since A and $\overline{A}$ are exclusive, we have $A \cup \overline{A} = S$ and $A \cap \overline{A} = \Phi$.

Corollary : Probability of an event is always less than or equal to one.

i.e. $\boxed{P(A) \leq 1}$

Proof : $P(A) = 1 - P(\overline{A})$. But $P(\overline{A}) \geq 0$ by axiom 1.

$\therefore \quad P(A) \leq 1.$

De Morgan's Laws

Since an event is a subset of sample space, De Morgan's are laws are applicable to events. Thus, we get

$$\boxed{\begin{aligned} P(\overline{A \cup B}) &= P(\bar{A} \cap \bar{B}) \\ P(\overline{A \cap B}) &= P(\bar{A} \cup \bar{B}) \end{aligned}}$$

Augustus De Morgan (1806 - 1871)

Augustus De Morgan was born in Madurai, Tamil Nadu. His father was a colonel in the Indian army. His family returned to England when he was 7 months old. When in schools he mastered Latin, Greek and Hebrew and developed strong interest in mathematics.

He was a fellow of the Astronomical Society and a founder of London Mathematical Society. De Morgan greatly influenced the development of mathematics in the 19th century. He was a prolific writer and wrote over 1000 articles in more than 15 journals, in addition to a number of books known for clarity, logical presentation and minute details. He made original contributions to analysis and logic. He coined the term mathematical induction and gave first precise definition of limit in his book "The Differential And Integral Calculus".

Example 1 : Find the probability that a card drawn will be black or picture ?

Sol. : Let A = the card is picture and B = the card is black.

Now A and B are not mutually exclusive events. There are 6 black pictures

$$P(A) = \frac{12}{52}, \quad P(B) = \frac{26}{52}, \quad P(A \cap B) = \frac{6}{52}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{12}{52} + \frac{26}{52} - \frac{6}{52} = \frac{32}{32} = \frac{8}{13}.$$

Note ....

Many examples of this type can be more easily solved by counting the number of points in the desired event. In the above example $A \cup B$ = {Black or Picture Cards} has 26 + 12 − 6 = 32 points.

$$\therefore \quad P(A \cup B) = \frac{32}{52} = \frac{8}{13}.$$

Example 2 : Two cards are drawn from a pack of cards. Find the probability that they will be both red or both pictures.

Sol. : Let A = { both red } and B = { both pictures }

$$P(A) = \frac{^{26}C_2}{^{52}C_2}, \quad P(B) = \frac{^{12}C_2}{^{52}C_2}$$

But A and B are not exclusive. There are six red picture cards.

$$\therefore \quad P(A \cap B) = \frac{{}^{6}C_2}{{}^{52}C_2},$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{{}^{26}C_2}{{}^{52}C_2} + \frac{{}^{12}C_2}{{}^{52}C_2} - \frac{{}^{6}C_2}{{}^{52}C_2} = \frac{203}{663}.$$

5. Conditional Probability

Consider general case of conditional probability (See Fig. 3.8 on page 3-8). Let n = the number of points in S, m_1 = number of points in A, m_2 = number of points in B, m_{12} = number of points in A and B both.

Then, $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$ and $P(A \cap B) = \frac{m_{12}}{n}$.

Now, suppose in a trial we know the result partially *i.e.* we know that A has occurred. What is the probability **now** that B has occurred along with A ?

Since we know that A has occurred the outcome of the trial is one of those points in A *i.e.* one of m_1 (and not n) And of these, m_{12} are in B. Hence, the probability that B will occur, when A has already occurred is (m_{12} / m_1). This is called conditional probability of B under the condition that A has occurred. It is denoted by $P(B / A)$.

Thus, $$P(B/A) = \frac{m_{12}}{m_1}.$$

Definition : Let A and B be any two events in a sample space S. The probability that B will occur, given that A has already occurred is called the **conditional probability of B** and is denoted by $P(B / A)$.

Similarly the probability that A will occur given that B has already occurred is called the **conditional probability of A** and is denoted by $P(A / B)$.

Example : Suppose there are 100 students in a class and the results of an examination of the class are given in the following table.

	Passed	Failed	Total
Boys	28	32	60
Girls	26	14	40
Total	54	46	100

Let us define two events A, B as follows :

A = a student has passed, B = a student is a male student.

Suppose a student is selected at random and is known to be a male student. What is the probability that **this** student has passed ? In symbols, we want to find $P(A / B)$.

Since the student is a male student, the sample space of B has 60 points. Of these 28 have passed *i.e.* A has now 28 points. Hence,

$$\therefore \quad P(A/B) = \frac{28}{60} = \frac{7}{15}$$

But, from the table we see that 28 is the number of points in $A \cap B$ and 60 is the number of points in B

$$\boxed{P(A/B) = \frac{\text{No. of points in } A \cap B}{\text{No. of points in } B}} \quad \text{.......... (1)}$$

or $$\boxed{P(A/B) = \frac{\text{No. of points in } A \text{ out of } B}{\text{No. of points in } B}} \quad \text{.......... (2)}$$

Similarly, we can find that

$$\boxed{P(B/A) = \frac{\text{No. of points in } B \cap A}{\text{No. of points in } A}} \quad \text{.......... (3)}$$

or $$\boxed{P(B/A) = \frac{\text{No. of points in } B \text{ out of } A}{\text{No. of points in } A}} \quad \text{.......... (4)}$$

Further, we see that, (1) can be written as

$$P(A/B) = \frac{(\text{No. of points in } A \cap B) / \text{No. of points in } S}{(\text{No. of points in } B) / \text{No. of points in } S}$$

$$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, from (3) we can get $P(B/A) = \dfrac{P(A \cap B)}{P(A)}$.

From these we get $P(A \cap B) = P(A/B) \times P(B)$ and $P(A \cap B) = P(B/A) \times P(A)$ which is called the law of multiplication of probability.

Theorem 5 : *Multiplication Theorem*

If A and B are two events and neither is null then the probability that both of them will occur is given by

$$\boxed{P(A \cap B) = P(A) \times P(B/A)} \quad \text{or} \quad \boxed{P(A \cap B) = P(B) \times P(A/B)}$$

where $P(A/B)$ and $P(B/A)$ denote the conditional probabilities which are greater than zero.

Proof : Let the number of points in A be m_1 and those in B be m_2. Let the number of points in $(A \cap B)$ be m_{12} and let n be the total number of points in S.

$$P(A) = \frac{m_1}{n}, \quad P(B) = \frac{m_2}{n}$$

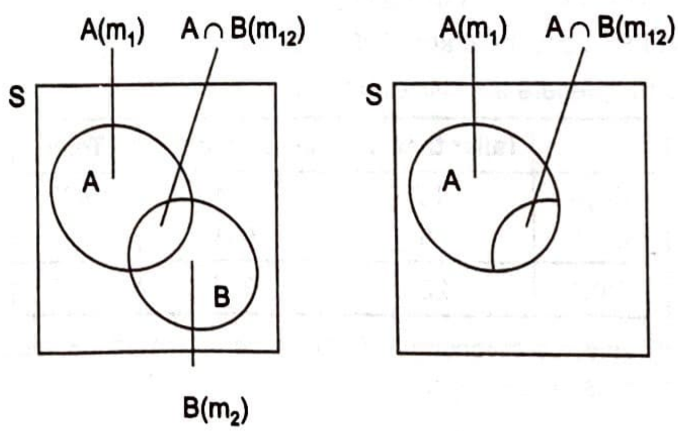


Fig. 3.8

To find the probability of B when A has happened, we have to consider the sample space of A which has m_1 points. In this sample space only B can occur (along with A) which has m_{12} points.

$$\therefore \quad P(B/A) = \frac{\text{No. of points in } A \cap B}{\text{No. of points in } A} = \frac{m_{12}}{m_1}$$

Now, $P(A \cap B) = \dfrac{m_{12}}{n} = \dfrac{m_1}{n} \times \dfrac{m_{12}}{m_1}$

$$\therefore \quad P(A \cap B) = P(A) \times P(B/A)$$

Similarly, we can prove that $P(A \cap B) = P(B) \times P(A/B)$.

Example 1 : There are 11 tickets in a box bearing numbers 1 to 11. Three tickets are drawn one after the other without replacement. Find the probability that they are drawn in the order bearing (i) even, odd, even number, (ii) odd, odd, even number. **(M.U. 1997)**

Sol. : This is an example on conditional probability. Out of 11 tickets, 5 are even and 6 are odd.

$$P(\text{even, odd, even}) = P(\text{even}) \cdot P(\text{odd}) \cdot P(\text{even})$$

$$= \frac{5}{11} \cdot \frac{6}{10} \cdot \frac{4}{9} = \frac{4}{33}$$

$$P(\text{odd, odd, even}) = P(\text{odd}) \cdot P(\text{odd}) \cdot P(\text{even})$$

$$= \frac{6}{11} \cdot \frac{5}{10} \cdot \frac{5}{9} = \frac{5}{33}$$

Example 2 : In a bag there are 4 white and 3 black balls. If four balls are drawn one by one at random without replacement, what is probability that the balls so drawn are alternately of different colours ?

Sol. : Try.

$$P(A) = \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{35}; \qquad P(B) = \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{35}$$

$$P(\text{Alternate colour}) = \frac{3}{35} + \frac{3}{35} = \frac{6}{35}.$$

Example 3 : In a certain college 4% of the boys and 1% of the girls are taller than 1·8 mts. Furthermore 60% of the students are girls. If a student selected at random is taller than 1·8 mts, what is the probability that the student was a boy ? **(M.U. 2002, 03)**

Sol. : For convenience suppose there are 1000 students in the college.

Then we can easily prepare the following table.

	Taller than 1·8	Less than 1·8	Total
Boys	16	384	400
Girsl	6	594	600
Total	22	978	1000

Since the student selected at random is found to be taller than 1·8, the student is one of 22. But out of these 22 students 16 are boys.

$$\therefore \quad \text{Required Probability} = \frac{16}{22} = \frac{8}{11}.$$

6. Partition of Sample Space

Definition : The events $A_1, A_2, \ldots, A_n$ are said to represent a partition of sample space S if

(i) $A_i \cap A_j = 0$ for all $i \neq j$

and (ii) $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$.

In words : The events $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive *i.e.* when the experiment is performed *one and only one* of the events A_i occurs and must occur.

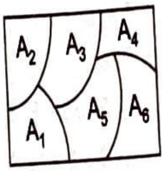


Fig. 3.9

For example, in tossing of a die the events $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, $A_3 = \{6\}$ represent a partition of the sample space S. However. $B_1 = \{1, 2, 3\}$, $B_2 = \{3, 4\}$, $B_3 = \{4, 5, 6\}$ do not represent a partition (why ?). Also, $C_1 = \{1\}$, $C_2 = \{3, 4\}$, $C_3 = \{6\}$ do not represent a partition. (why ?)

Example 1 : A, B and C are bidding for a contract. It is believed that A has exactly half the chance than B has. B in turn is 4 / 5th as likely as C to get the contract. What is the probability for each to get the contract ? **(M.U. 2004)**

Sol. : Let $P(A)$, $P(B)$, $P(C)$ denote the probabilities that A, B, C will get the contract respectively. Further, let $P(C) = x$. Then,

$$P(B) = \frac{4}{5}x \text{ and } P(A) = \frac{1}{2} \cdot \frac{4}{5}x = \frac{2}{5}x$$

Since one of A, B, C must get the contract, the three events partition the sample space

$$x + \frac{4}{5}x + \frac{2}{5}x = 1 \quad \therefore \quad 11x = 5 \quad \therefore \ x = \frac{5}{11}.$$

$$\therefore \quad P(A) = \frac{2}{11}, \ P(B) = \frac{4}{11}, \ P(C) = \frac{5}{11}$$

Example 2 : If the events A, B and C form a partition of the sample space and $P(A) = 2P(B) = 3P(C)$, find (i) $P(A \cup B)$, (ii) $P(B \cup C)$.

Sol. : Let $P(A) = x$, then $P(B) = \frac{x}{2}$ and $P(C) = \frac{x}{3}$.

Since A, B and C form partition S. These are exhaustive events.

$$\therefore P(A) + P(B) + P(C) = 1$$

$$\therefore \quad x + \frac{x}{2} + \frac{x}{3} = 1 \quad \therefore \quad \frac{11}{6}x = 1 \quad \therefore \quad x = \frac{6}{11}$$

$$\therefore \quad P(A) = \frac{6}{11}, \quad P(B) = \frac{3}{11}, \quad P(C) = \frac{2}{11}.$$

Since, the events are mutually exclusive.

$$P(A \cup B) = P(A) + P(B) = \frac{6}{11} + \frac{3}{11} = \frac{9}{11}$$

and $$P(B \cup C) = P(B) + P(C) = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}.$$

Example 4 : If the events A, B and C form a partition of the sample space S and $3P(A) = 2P(B) = 6P(C)$, find $P(A \cup B)$.

Please do it. **[Ans. : 5 / 6]**

Example 5 : If the events A, B, C and D form a partition of the sample space S and the probabilities of A and C are equal, the event B is twice as likely as A and the event D is twice as likely as B, find $P(A \cup B \cup C)$.

Please do it. **[Ans. :** 1 / 2]

Theorem on Total Probability

Let $A_1, A_2, A_3, \ldots, A_n$ be a partition of S and B be some event defined on S. Then,

$$P(B) = P(B/A_1) \times P(A_1) + P(B/A_2) \times P(A_2) + \ldots + P(B/A_n) \times P(A_n)$$

We accept this theorem without proof.

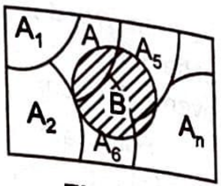


Fig. 3.10

If we denote the probabilities $P(A_i)$ by p_i and the conditional probabilities $P(B/A_i)$ by p'_i then, the above theorem can be stated as

$$\boxed{P(B) = p_1 p'_1 + p_2 p'_2 + p_3 p'_3 + \ldots p_n p'_n}$$

Remark ....

The above theorem is known as the theorem on **total probability**. In some problems it may be difficult to evaluate $P(B)$ directly. But when additional information that A_i has occurred is known, we may evaluate $P(B/A)$ and use the above theorem to find $P(B)$.

Example 1 : Four roads lead away from a jail. A prisoner trying to escape from the jail selects a road at random. If road A is selected, the probability of escaping is 1/ 8, for road B, it is 1/ 6, for road C it is 1/ 4 and for road D it is 9 / 10.

What is the probability that a prisoner will succeed in escaping from the jail ? **(M.U. 2005)**

Sol. : Let E = Success in Escaping.

p_1 = selecting road A, p_2 = selecting road B,

p_3 = selecting road C, p_4 = selecting road D.

$p_1 = P(A) = 1/4$, $p_2 = P(B) = 1/4$,

$p_3 = P(C) = 1/4$, $p_4 = P(D) = 1/4$.

$p_1' = P(E/A) = 1/8$, $p_2' = P(E/B) = 1/6$,

$p_3' = P(E/C) = 1/4$, $p_4' = P(E/D) = 9/10$.

$$\therefore \quad P(E) = p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'$$

$$\therefore \quad P(E) = \frac{1}{4} \cdot \frac{1}{8} + \frac{1}{4} \cdot \frac{1}{6} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{9}{10}$$

$$= \frac{1}{4}\left(\frac{1}{8} + \frac{1}{6} + \frac{1}{4} + \frac{9}{10}\right) = \frac{1}{4} \cdot \frac{163}{120} = \frac{163}{480}.$$

Example 2 : In a box there are four tags numbered 1 and 6 tags numbered 2. There are two urns U_1 and U_2 containing 3 red and 7 black balls and 8 red and 2 black balls respectively. One tag is drawn from the box and one ball is drawn from the urn whose number is found on the tag drawn. Find the probability that a red ball is drawn.

Sol. : Try. **[Ans. :** 0·6]

EXERCISE - I

1. The probabilities that three students A, B and C will pass the common entrance test for engineering are $4/9$, $2/9$ and $1/3$ respectively. The probabilities that they will get admission in the same engineering college are $3/10$, $1/2$ and $4/5$ respectively.

Find the probability that they will get admission in the same engineering college.

[Ans. : 23 / 45]

2. The chances that A, B and C will be the Education Minister of Government of India are in the ratio $4 : 1 : 2$. The probabilities that they will introduce reservations in professional colleges for backward classes are 0·3, 0·8 and 0·5 respectively.

Find the probability that the bill for reservation will be introduced. **[Ans. : 3 / 7]**

3. In a factory an article is produced on three machines. Their respective productions are 300 units by A, 250 units by B and 450 units by C. It is found that the percentages of defective articles for A, B, C are 1, 1·2 and 2 selected at random from a days production (which are mixed).

Find the probability that the selected article is defective. **[Ans. : 0·015]**

We now state an important theorem known as Bayes' Theorem. It enables us to evaluate what may be called **reverse probabilities**. Suppose there are two boxes (I and II) which contain 2 white and 3 black balls; and 3 white and 4 black balls. If a box is chosen at random and a ball is drawn from it, what is the probability that the ball drawn is white ? We know how to calculate this probability. Now, consider the question : If the ball drawn is known to be white, what is the probability that it was drawn from the 1st box ? The question can be answered by Bayes' Theorem. If the 'result' is known, Bayes' Theorem enables us to find the probability of the 'cause'. For this reason it is also sometimes known as the formula for the **"probability of causes"**.

7. Bayes' Theorem

Let the events $A_1, A_2, \ldots\ldots, A_n$ represent a partition of the sample space S. Let B be any other event defined on S. If $P(A_i) \neq 0$, $i = 1, 2, \ldots\ldots, n$ and $P(B) \neq 0$ then

$$P(A_i / B) = \frac{P(A_i) \times P(B/A_i)}{\Sigma P(A_i) \times P(B/A_i)}$$

If we write $p_1 = P(A_1),\ p_2 = P(A_2),\ p_3 = P(A_3) \ldots\ldots$ etc.

and $p'_1 = P(B/A_1),\ p'_2 = P(B/A_2),\ p'_3 = P(B/A_3) \ldots\ldots$ etc.

then Bayes' theorem can be stated as

$$\boxed{P(A_i / B) = \frac{p_i\, p'_i}{p_1 p'_1 + p_2 p'_2 + \ldots\ldots + p_n p'_n}}$$

Proof : We have by conditional probability

$$P(A_i / B) = \frac{P(A_i \cap B)}{P(B)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

But $$P(B/A_i) = \frac{P(B \cap A_i)}{P(A_i)} \quad \therefore \quad P(A_i \cap B) = P(A_i) \cdot P(B/A_i) \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

From (i) and (ii), we get

$$\therefore \quad P(A_i / B) = \frac{P(A_i) \cdot P(B / A_i)}{P(B)} = \frac{p_i \, p'_i}{P(B)} \qquad \text{.................... (iii)}$$

But $\quad B = (B \cap A_1) \cup (B \cap A_2) ..... (B \cap A_n)$

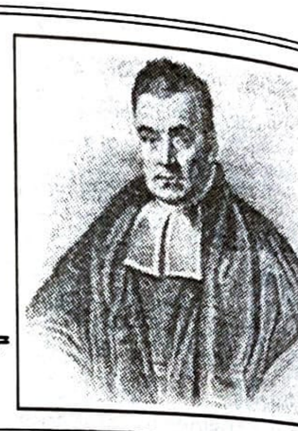
$$\therefore \quad P(B) = P(B \cap A_1) + P(B \cap A_2) + ... + P(B \cap A_n)$$

But $\quad P(B \cap A_i) = P(A_i) \cdot P(B / A_i)$ etc. $= p_1 \, p'_1$ etc.

Hence, from (iii) $\quad P(A_j / B) = \dfrac{p_i \, p'_i}{p_1 \, p'_1 + p_2 \, p'_2 + ... + p_n \, p'_n}$

Thomas Bayes (1701 - 1761)

Thomas Bayes was an English mathematician known for the theorem that bears his name. This theorem was published after his death by Richar Price. He published two works in his lifetime, one theological and one mathematical. He became a Fellow of the Royal Society in 1742. It is said that he learned mathematics and probability from a book by De Moivre. In his later years he took deep interest in probability.



Example 1 : There are in a bag three true coins and one false coin with head on both sides. A coin is chosen at random and tossed four times. If head occurs all the four times, what is the probability that the false coin was chosen and used ? **(M.U. 2003)**

Sol. : $\quad P$ (selecting true coin) $= p_1 = \dfrac{3}{4}$. $\quad P$ (selecting false coin) $= p_2 = \dfrac{1}{4}$.

$$p'_1 = P(\text{getting all four heads with true coin}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}.$$

$$p'_2 = P(\text{getting all four heads with false coin}) = 1 \cdot 1 \cdot 1 \cdot 1 = 1.$$

$$\therefore \quad \text{Required Probability} = \frac{p_2 \, p'_2}{p_1 \, p'_1 + p_2 \, p'_2} = \frac{(1/4) \cdot 1}{(3/4) \cdot (1/16) + (1/4) \cdot 1} = \frac{16}{19}.$$

Example 2 : A coin is tossed. If it turns up heads two balls are drawn from urn A otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. What is the probability that urn A was used, given that both balls drawn are black ? **(M.U. 2001, 04)**

Sol. : We have $p_1 = P(H) = \dfrac{1}{2}$, $p_2 = P(T) = \dfrac{1}{2}$

$$p'_1 = P(\text{two black balls from } A) = \frac{{}^3C_2}{{}^8C_2}$$

$$p'_2 = P(\text{two black balls from } B) = \frac{{}^7C_2}{{}^8C_2}$$

$$\therefore \quad \text{Required Probability} = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{\frac{1}{2}\cdot\frac{{}^3C_2}{{}^8C_2}}{\frac{1}{2}\cdot\frac{{}^3C_2}{{}^8C_2}+\frac{1}{2}\cdot\frac{{}^7C_2}{{}^8C_2}}$$

$$= \frac{{}^3C_2}{{}^3C_2 + {}^7C_2} = \frac{3\cdot 2/2\cdot 1}{(3\cdot 2/2\cdot 1)+(7\cdot 6/2\cdot 1)}$$

$$= \frac{6}{6+42} = \frac{6}{48} = \frac{1}{8}.$$

Example 3 : In a certain test there are multiple choice questions. There are four possible answers to each question and one of them is correct. An intelligent student can solve 90% questions correctly by reasoning and for the remaining 10% questions he gives answers by guessing. A weak student can solve 20% questions correctly by reasoning and for the remaining 80% questions he gives answers by guessing. An intelligent students gets the correct answer, what is the probability that he was guessing ? **(M.U. 2004)**

Sol. : Consider the intelligent student.

Let p_1 = answering by reasoning $= \frac{90}{100} = \frac{9}{10}$.

p_2 = answering by guessing $= \frac{10}{100} = \frac{1}{10}$.

p_1' = answer is correct (by reasoning) = 1

p_2' = answering is correct (by guessing) $= \frac{1}{4}$.

$$\text{Required Probability} = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2'} = \frac{(1/10)\cdot(1/4)}{(9/10)\cdot 1 + (1/10)\cdot(1/4)}$$

$$= \frac{1/40}{37/40} = \frac{1}{37}.$$

Example 4 : A certain test for a particular cancer is known to be 95% accurate. A person submits to the test and the result is positive. Suppose that a person comes from a population of 100,000 where 2000 people suffer from that disease. What can we conclude about the probability that the person under test has that particular cancer ? **(M.U. 2006)**

Sol. : We have

p_1 = probability a person has the cancer $= \frac{2000}{100{,}000} = \frac{2}{100} = 0.02$

p_2 = probability that a person does not have the cancer = 1 – 0·02 = 0·98

p_1' = test is positive when a person has cancer $= \frac{95}{100} = 0.95$

p_2' = test is positive when person does not have a cancer $= \frac{5}{100} = 0.05$

$$\therefore \quad \text{Required probability} = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2'}$$

$$\therefore \quad \text{Required probability} = \frac{(2/100)\cdot(95/100)}{(2/100)\cdot(95/100)+(98/100)\cdot(5/100)}$$

$$= \frac{190}{680} = 0.279$$

Example 5 : A bag contains 7 red and 3 black balls and another bag contains 4 red and 5 black balls. One ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. If this ball happens to be red, find the probability that a black ball was transferred. **(M.U. 2002)**

Sol. : We have

$$p_1 = \text{Probability of transferring black ball} = \frac{3}{10}$$

$$p'_1 = \text{Probability of now drawing a red ball} = \frac{4}{10}$$

$$p_2 = \text{Probability of transferring red ball} = \frac{7}{10}$$

$$p'_2 = \text{Probability of now drawing red ball} = \frac{5}{10}$$

$$\therefore \quad \text{Required Probability} = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{(3/10)\cdot(4/10)}{(3/10)(4/10)+(7/10)(5/10)}$$

$$= \frac{12}{47}.$$

Example 6 : A man speaks truth 3 times out of 5. When a die is thrown, he states that it gave an ace. What is the probability that this event has actually happened ? **(M.U. 2019)**

Sol. : We have

$$p_1 = \text{Probability he speaks truth} = \frac{3}{5} \;;\; p'_1 = \text{Probability of an ace} = \frac{1}{6}$$

$$p_2 = \text{Probability he speaks a lie} = \frac{2}{5} \;;\; p'_2 = \text{Probability of not ace} = \frac{5}{6}$$

$$P\,(\text{he speaks truth when ace has occured}) = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2}$$

$$= \frac{(3/5)(1/6)}{(3/5)(1/6)+(2/5)(5/6)} = \frac{3}{3+10} = \frac{3}{13}.$$

Example 7 : A lot of IC chips is known to contain 3% defective chips. Each chip is tested before delivery but the tester is not completely reliable. It is known that :

P (Tester says the chip is good / The chip is actually good) = 0·95 and P (Tester says the chip is defective / The chip is actually defective) = 0·96.

If a tested chip is declared defective by the tester. What is the probability that it is actually defective ?

Sol. : Let p_1 = Chip is defective = 0·03

p'_1 = Tester says the chip is defective / The chip is defective = 0·96

p_2 = Chip is good = 0·97

p'_2 = Tester says the chip is defective / The chip is good = 0·05.

By Bayes' Theorem,

P (Chip is defective / Tester says it is defective)

$$= \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{0.03 \times 0.96}{0.03 \times 0.96 + 0.97 \times 0.05} = 0.37.$$

Example 8 : A binary communication transmitter sends data as one of two types of signals denoted by 0 or 1. Due to noise, sometimes a transmitted 1 is received as 0 and vice versa.

If the probability that a transmitted 0 is correctly received as 0 is 0·9 and the probability that a transmitted 1 is correctly received as 1 is 0·8 and if the probability of transmitting 0 is 0·45, find the probability that (i) a 1 is received, (ii) a 0 is received, (iii) a 1 was transmitted given that 1 was received, (iv) a 0 was transmitted given that a 0 was received, (v) the error has occurred.

Sol. : We are given that

$P(T_0)$ = a 0 is transmitted = 0·45

$P(T_1)$ = a 1 is transmitted = $1 - P(T_0) = 1 - p_1 = 0.55$

$P(R_0 / T_0)$ = a 0 is received when a 0 was transmitted = 0·9

$P(R_1 / T_0)$ = a 1 received when a 0 was transmitted

= 1 – 0·9 = 0·1

$P(R_1 / T_1)$ = a 1 is received when a 1 was transmitted = 0·8

$P(R_0 / T_1)$ = a 0 is received when a 1 was transmitted

= 1 – 0·8 = 0·2.

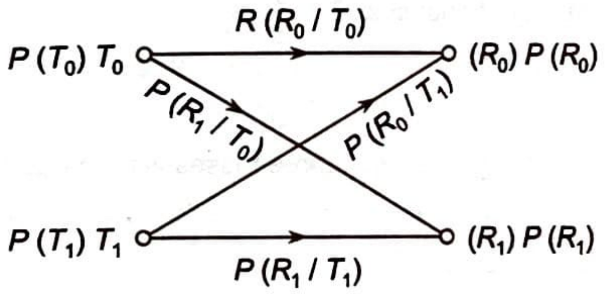


Fig. 3.11

Now, we calculate the required probabilities as follows :

(i) P (1 is received) = P (1 is received when 1 is transmitted)
+ P (1 is received when 0 is transmitted)

$$\therefore \quad P(R_1) = P(R_1 / T_1) \cdot P(T_1) + P(R_1 / T_0) \cdot P(T_0)$$
$$= 0.8 \times 0.55 + 0.1 \times 0.45$$
$$= 0.485$$

(ii) P (0 is received) = P (0 is received when 0 is transmitted)
+ P (0 is received when 1 is transmitted)

$$\therefore \quad P(R_0) = P(R_0 / T_0) \cdot P(T_0) + P(R_0 / T_1) \cdot P(T_1)$$
$$= 0.9 \times 0.45 + 0.2 \times 0.55$$
$$= 0.515$$

Now, by Bayes' Theorem

(iii) P (1 was transmitted given that 1 was received) *i.e.*

$$P(T_1/R_1) = \frac{P(R_1/T_1)\ P(T_1)}{P(R_1)} = \frac{0.8 \times 0.55}{0.485} = 0.907$$

(iv) P (0 was transmitted given that 0 was received) *i.e.*

$$P(T_0/R_0) = \frac{P(R_0/T_0) \cdot P(T_0)}{P(R_0)} = \frac{0.9 \times 0.45}{0.515} = 0.786$$

(v) $P(\text{Error}) = P$ (0 was received when 1 is transmitted given that 1 was transmitted) + P (1 was received when 0 was transmitted given that 0 was transmitted)

$$= P(R_0/T_1) \cdot P(T_1) + P(R_1/T_0) \cdot P(T_0)$$

$$= 0.2 \times 0.55 + 0.1 \times 0.45 = 0.155.$$

Example 9 : A box contains three biased coins A, B and C. The probability that a head will result when A is tossed is $1/3$, when B is tossed, it is $2/3$ and when C is tossed, it is $3/4$.

(a) If one of the coins is chosen at random and is tossed 3 times, head resulted twice and tail once. What is the probability that the coin chosen was A ?

(b) What is the probability of getting head when a coin selected at random is tossed once ?

(c) What is the probability that we would get two heads in the first three tosses and a head again in the fourth toss with the same coin ?

Sol. : We have p_1 = Probability of choosing $A = \frac{1}{3}$.

p_2 = Probability of choosing $B = \frac{1}{3}$.

p_3 = Probability of choosing $C = \frac{1}{3}$.

p_1' = Probability of getting 2 heads in three tosses with the coin A

$$= {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = 3 \cdot \frac{1}{9} \cdot \frac{2}{3} = \frac{2}{9}.$$

p_2' = Probability of getting 2 heads in three tosses with the coin B

$$= {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 3 \cdot \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{9}.$$

p_3' = Probability of getting 2 heads in three tosses with the coin C

$$= {}^3C_2\left(\frac{3}{4}\right)^2 \cdot \left(\frac{1}{4}\right) = 3 \cdot \frac{9}{16} \cdot \frac{1}{4} = \frac{27}{64}.$$

$\therefore$ Required Probability

$$= \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2 + p_3 p_3'}$$

$$= \frac{(1/3)(2/9)}{(1/3) \cdot (2/9) + (1/3) \cdot (4/9) + (1/3) \cdot (27/64)}$$

$$= \frac{(2/9)}{(2/9) + (4/9) + (27/64)} = \frac{128}{627}.$$

(b) We do not know which coin was tossed.

If the coin was A, the probability that it will give head

$$= (\text{Prob. of choosing } A) \times (\text{Prob. of giving head})$$

$$= \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

If the coin was B, probability of getting head.

$$= (\text{Prob. of choosing } B) \times (\text{Prob. of giving head})$$

$$= \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}.$$

If the coin was C, probability of getting head

$$= (\text{Prob. of choosing } C) \times (\text{Prob. of giving head})$$

$$= \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

$\therefore$ The required probability $= \frac{1}{9} + \frac{2}{9} + \frac{1}{4} = \frac{7}{12}.$

(c) Now, getting two heads in the first three tosses and a head in the fourth toss with A

$$= P(\text{Choosing } A) \times P(\text{Getting two heads in three tosses with } A)$$
$$\times P(\text{Getting a head in the fourth toss with } A)$$

$$= p_1 \cdot p_1' \cdot \frac{1}{3} = \frac{1}{3} \cdot \frac{2}{9} \cdot \frac{1}{3} = \frac{2}{81}.$$

Similarly, getting two heads in the first three tosses and a head in the fourth toss with B

$$= p_2 \cdot p_2' \cdot \frac{2}{3} = \frac{1}{3} \cdot \frac{4}{9} \cdot \frac{2}{3} = \frac{8}{81}$$

And getting two heads in the first three tosses and a head in the fourth toss with C

$$= p_3\, p_3' \cdot \frac{3}{4} = \frac{1}{3} \cdot \frac{27}{64} \cdot \frac{3}{4} = \frac{27}{256}$$

$\therefore$ Required Probability $= \frac{2}{81} + \frac{8}{81} + \frac{27}{256} = 0.23.$

Example 10 : A bag contains five balls, the colours of which are not known. Two balls were drawn from the bag and they were found to be white. What is the probability that all balls are white ? **(M.U. 2005)**

Sol. : Since two balls drawn are white, the bag may contain 2 white or 3 white or 4 white or 5 white balls.

Let these events be denoted by A_1, A_2, A_3, A_4 respectively. We can assume that the probabilities of these events are equal.

Let $p_1 = P(A_1),\ p_2 = P(A_2),\ p_3 = P(A_3),\ p_4 = P(A_4).$

$$\therefore\ p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$$

Now, two balls out of 5 can be drawn in ${}^5C_2$ ways.

$$\therefore\ p_1' = P(\text{drawing two balls when two balls are white}) = \frac{{}^2C_2}{{}^5C_2} = \frac{2 \cdot 1}{5 \cdot 4} = \frac{2}{20}.$$

$p_2' = P$ (drawing two white balls when 3 balls are white) $= \dfrac{{}^3C_2}{{}^5C_2} = \dfrac{3 \cdot 2}{5 \cdot 4} = \dfrac{6}{20}.$

$p_3' = P$ (drawing 2 white balls when 4 balls are white) $= \dfrac{{}^4C_2}{{}^5C_2} = \dfrac{4 \times 3}{5 \times 4} = \dfrac{12}{20}.$

$p_4' = P$ (drawing 2 white balls when 5 balls are white) $= \dfrac{{}^5C_2}{{}^5C_2} = \dfrac{5 \cdot 4}{5 \cdot 4} = \dfrac{20}{20}.$

$\therefore$ By Baye's theorem Required Probability

$$= \frac{p_4 p_4'}{p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'}$$

$$= \frac{(1/4) \cdot (20/20)}{(1/4) \cdot (2/20) + (1/4) \cdot (6/20) + (1/4) \cdot (12/20) + (1/4) \cdot (20/20)}$$

$$= \frac{20}{2 + 6 + 12 + 20} = \frac{20}{40} = \frac{1}{2}.$$

Example 11 : There are three boxes containing respectively 1 white 2 red, 3 black balls; 2 white, 3 red and 1 black ball; 3 white, 1 red and 2 black balls. A box is chosen at random and two balls are drawn from it. The two balls are found to be one red and one white. Find the probability that those have come from box 1, box 2 and box 3.

Sol. : Since there are three boxes, say, A_1, A_2, A_3, then

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3} \qquad \therefore \quad p_1 = p_2 = p_3 = \frac{1}{3}$$

Let B be the event that one ball is white and the other is red. Then,

$$p_1' = P(B/A_1) = \frac{{}^1C_1 \cdot {}^2C_1}{{}^6C_2} = \frac{1 \cdot 2 \cdot 2}{6 \cdot 5} = \frac{2}{15}$$

$$p_2' = P(B/A_2) = \frac{{}^2C_1 \cdot {}^3C_1}{{}^6C_2} = \frac{2 \cdot 3 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

$$p_3' = P(B/A_3) = \frac{{}^3C_1 \cdot {}^1C_1}{{}^6C_2} = \frac{3 \cdot 1 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

By Baye's Theorem,

$$P(A_1/B) = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2' + p_3 p_3'}$$

$$\therefore \quad P(A_1/B) = \frac{(1/3)(2/15)}{(1/3)(2/15) + (1/3)(6/15) + (1/3)(3/15)} = \frac{2/45}{11/45} = \frac{2}{11}$$

$$P(A_2/B) = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{(1/3)(6/15)}{11/45} = \frac{6}{11}$$

$$P(A_3/B) = \frac{p_3 p_3'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{(1/3)(3/15)}{11/45} = \frac{3}{11}.$$

Example 12 : Three factories A, B, C produce 30 %, 50 % and 20 % of the total production of an item. Out of their production 80 %, 50 % and 10 % are defective. An item is chosen at random and found to be defective. Find the probability that it was produced by the factory A. **(M.U. 2019)**

Sol. : $p_1 = P$ (item is produced by A) = 0·3

$p_2 = P$ (item is produced by B) = 0·5

$p_3 = P$ (item is produced by C) = 0·2

Let D be the event that the item is defective then

$$p_1' = P(D/A) = 0 \cdot 8, \quad p_2' = P(D/B) = 0 \cdot 5, \quad p_3' = P(D/C) = 0 \cdot 1$$

Now, the required event is A/D.

$$\therefore \quad P(A/D) = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{0 \cdot 3 \times 0 \cdot 8}{0 \cdot 3 \times 0 \cdot 8 + 0 \cdot 5 \times 0 \cdot 5 + 0 \cdot 2 \times 0 \cdot 1}$$

$$= \frac{0 \cdot 24}{0 \cdot 24 + 0 \cdot 25 + 0 \cdot 02} = \frac{0 \cdot 24}{0 \cdot 51} = 0 \cdot 47.$$

EXERCISE - II

1. If $P(A) = 0 \cdot 7$, $P(B) = 0 \cdot 6$, $P(A/B) = 0 \cdot 4$, find $P(B/A)$. [**Ans. :** 12 / 35]

2. If A_1 and A_2 are two mutually exclusive and exhaustive events of sample space S and if B is another event defined on S, such that $P(A_1) = 0 \cdot 6$, $P(A_2) = 0 \cdot 4$ and $P(B/A_1) = 0 \cdot 4$, $P(B/A_2) = 0 \cdot 5$, find $P(B)$. [**Ans. :** 0·44]

3. A bag contains two coins one of which is a false coin with head on both sides and the other is a true coin. When a coin, taken at random from the bag, is tossed, it gave a head. What is the probability that the true coin was taken and tossed ? [**Ans. :** 1 / 3]

4. A bag contains two dice, one of which is regular and fair and the other is false with number 6 on all its faces. A dice was drawn from the bag and tossed. It gave 6. What is the probability that the dice obtained was the false one ? [**Ans. :** 6 / 7]

5. In a bolt factory, machines A, B, C produce respectively 25 %, 35 % and 40 %. Of their output 5 %, 4 % and 2 % are defective. A bolt is drawn at random from a days production and is found defective. What is the probability that it was produced by machines A, B, C ? **(M.U. 1997)**

[**Ans. :** 25 / 69, 28 / 69, 16 / 69]

6. A manufacturing firm produces steel pipes in three plants with daily production of 500, 1000 and 2000 units. According to past experience it is known that the fraction of defective output produced by the three plants are respectively 0·005, 0·008 and 0·010.

If a pipe is selected at random from a day's output and was found to be defective, what is the probability that it came from the first plant ? [**Ans. :** 0·08]

7. A given lot of I.C. chips contains 3 % defective chips. Each chip is tested before delivery. The tester itself is not totally reliable, such that P (tester shows the chip is good / chip is actually good) = 0·96 and P (tester shows chip is defective / chip is actually defective) = 0·94. If a tested chip is shown to be defective, what is the probability that it is actually defective ? [**Ans. :** 0·42]

8. A binary communication system transmits and also receives data as '0' and '1'. Due to noise a transmitted '0' is sometimes received as '1' and a transmitted '1' is some times received as '0'. The probability that a transmitted '0' is correctly received as '0' is 0·95 and the probability that a transmitted '1' is correctly received as '1' is 0·90, probability of transmitting '0' is 0·45. If a signal is sent find :

(i) Probability that a '1' is received
(ii) Probability that a '0' is received.
(iii) Probability that a '1' is transmitted, given that a '1' is received.
(iv) Probability that a '0' is transmitted, given that a '0' is received.

[**Ans.** : **(i)** 0·5175, **(ii)** 0·4825, **(iii)** 0·9565, **(iv)** 0·8860, **(v)** 0·0775]

9. A newly constructed flyover is likely to collapse. The chance that the design is faulty is 0·5. The chance that the flyover will collapse if the design is faulty is 0·95 otherwise it is 0·30. The flyover collapsed. What is the probability that it collapsed because of faulty design ? **(M.U. 2004)**

[**Ans.** : 0·76]

10. Three identical urns have the following composition of black and white balls :

First urn : 2 black, 1 white

Second urn : 1 black, 2 white

Third urn : 2 black, 2 white

One of the urns is selected at random and one ball is drawn. What is the probability of drawing a white ball ? [**Ans.** : 1/ 2]

11. There are three urns having the following compositions of black and white balls :

Urn 1 : 7 white, 3 black balls,

Urn 2 : 4 white, 6 black balls,

Urn 3 : 2 white, 8 black balls.

One of the urns is chosen at random with probabilities 0·2, 0·6 and 0·2 respectively. From the chosen urn two balls are drawn at random without replacement. Calculate the probabilities that both these balls are white. [**Ans.** : 8 / 45]

12. Three boxes A, B, C contain 30, 50, 20 bulbs respectively. The percentages of defective bulbs in these boxes are 4, 3 and 5 respectively. A box is selected at random and a bulb is drawn. What is the probability that the bulb will be defective ? [**Ans.** : 0·037]

13. A man is equally likely to choose any one of three routes C_1, C_2, C_3 from his house to the railway station. The probabilities of missing the train by the routes C_1, C_2 and C_3 are 2 / 5, 3 / 10, 1 / 20. He sets out on a day and misses the train. What is the probability that the route C_3 was selected ? [**Ans.** : 1/ 15]

14. A factory manufacturing television sets gets its supply of components from 4 units A, B, C, D in the following proportion 15%, 20%, 30% and 35% respectively. It was found that of the supply received 1%, 2%, 2% and 3% articles are defective. A television set was chosen at random from the total output and was found to be defective. What is the probability that it came from unit D ?

(M.U. 2002) [**Ans.** : 21 / 44]

15. A man has three coins A, B, C. A is unbiased. The probability that head will result when B is tossed is 2 / 3. The probability that head will result when C is tossed is 1/ 3. If one of the coins is chosen at random and is tossed three times it gave two heads and one tail. Find the probability that the coin A was chosen. **(M.U. 1997, 2002)** [**Ans.** : 9 / 25]

16. Bag I contains 6 blue and 4 red balls.

Bag II contains 2 blue and 6 red balls.

Bag III contains 1 blue and 8 red balls.

A bag is chosen at random and two balls are drawn from it. If both the balls were found to be blue, find the probability that bag II was chosen. **(M.U. 2004)** [**Ans.** : 3 / 31]

17. A coin is tossed. If it turns up head two balls are drawn from urn A , otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. In both cases selection is made with replacement. What is the probability that A was chosen given that both the balls drawn are black.

(M.U. 2002) [Ans. : 9 / 58 **]**

18. An urn A contains ten red and three black balls. Another urn B contains three red and five black balls. Two balls are transferred from urn A to the urn B without noticing their colour. One ball now is drawn from the urn B and it is found to be red. What is the probability that one red and one black ball have been transferred ?

[Ans. : 20 / 59 **]**

19. There are three boxes A, B and C. The probability of getting a white ball from the box A is 1/ 3, from the box B is 2 / 3 and from the box C is 3 / 4.

A box is chosen at random and three balls are drawn from it (without replacement) and it was found that two of them were white.

(a) What is the probability that the box A was chosen ?

(b) If a ball is drawn from a box selected at random, what is the probability that it will be white ?

(c) What is the probability that we would get two white balls in the first three draws and a white ball again in the fourth draw ?

(**Hint :** See Solved Ex. 9)

[Ans. : (a) 128 / 627, **(b)** 7 / 12, **(c)** 0·23 **]**

20. For a certain binary communication channel, the probability that a transmitted '0' is received as a '0' is 0·95 while the probability that a transmitted '1' is received as '1' is 0.90. If the probability of transmitting a '0' is 0·4, find the probability that (i) a '1' is received, (ii) a '1' was transmitted given that '1' was received.

[Ans. : (i) 0·56, **(ii)** 27 / 28 **]**

8. Random Variable

In the previous chapter we learnt how to find the probability of an outcome and the laws of probability. We saw that the outcome of an experiment can be anything : it may be colour (black-white-red .....) of a ball, a gender (male-female) of a child, a suit (club-diamond-spade-heart) of a card or a number (1 – 2 – 3 – 4 – 5 – 6) of a die or a logical answer (yes - no) of a question or result of a toss (head - tail) of a coin. In most of the problems the outcome of an experiment is a number *e.g.* the salary of a person, the height of a student, the temperature at a place, the rainfall on a particular day. However, when the outcome is not a number we can express these outcomes in numbers by agreeing to denote,

(a) head by 1 and tail by 0 (b) boy by 1 and girl by 0

(c) yes by 1 and no by 0 (d) club by 1, diamond by 2, spade by 3 and heart by 4

(e) red by 1, white by 2 and black by 3, etc.

In probability problems it is found convenient to think of a variable and consider the values of the variable which describe the outcomes of the experiment. In the toss of a coin the variable takes values 1 and 0; in the selection of a child it takes values 1 and 0, in the answers of the question it takes the values 1 and 0, in drawing a card it takes values 1, 2, 3, 4, in drawing a ball it takes values 1, 2, 3 etc. This variable may take discrete values or may take any value in a range continuously *e.g.* the arrival time of a bus at a stop, say, between 9 a.m. and 9.10 a.m. may be any number between the range. Such a variable is called a **random variable**. Actually random variable is a misnomer. It is a function which assigns a real number to the outcome of an experiment. A random variable is denoted by X and a particular value of X is denoted by x. In the toss of a coin X assigns the value 1 to H and 0 to T, in the selection of a ball X assigns value 1 to red, 2 to white and 3 to black.

In these cases X takes discrete values and is called a discrete random variable. But in the case of arrival time of a bus X takes continuously any value between 9 am. and 9.10 am. and hence X is a continuous random variable. In this way, we consider X as a function from sample space S to the set of real numbers R. Thus we get the following definition.

(a) Definition

Let E be an experiment and S be the sample space associated with it. A function X assigning to every element s of S one and only one real number $x = X(s)$ of R is called a *random variable*.

Since X is a function whose domain is the set of outcomes of an experiment and whose range is a part or the whole of real line $(-\infty < x < \infty)$, it can be shown pictorially as a mapping from the sample space to the real line.

The random variable X can be discrete or continuous depending upon the nature of its domain.

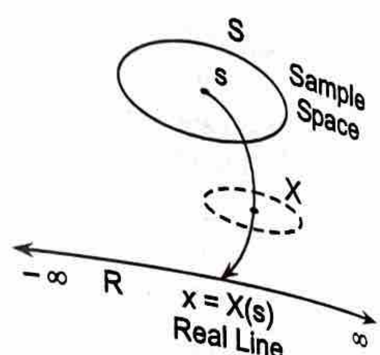


Fig. 3.12

Remarks ....

1. In simple words a variable used to denote the numerical value of the outcome of an experiment is called the random variable, abbreviated as r.v.
2. X is a *function* and still we call it a variable.
3. We are not interested in the *functional nature* of X but in the *values of* X.
4. X must be single valued *i.e.* for every s of S there corresponds *exactly one value* of X. Different elements of S may lead to the same value of X (See Example 2). But two values of X cannot be assigned to the same sample point.
5. We shall denote *random variables* by capital letters X, Y, Z, ... and shall denote the *unknown values* of these random variables by small letters $x, y, z, ..., x_1, x_2, ... y_1, y_2, ... z_1, z_2$, ... etc. This is an important distinction and students should note it carefully. With this notation it is meaningless to write $P(x \geq 10)$ say since x being a value of X either is or is not ≥ 10. Instead we should write $P(X \geq 10)$.

Example 1 : Suppose the experiment E is to toss a fair coin.

Then $S = \{H, T\}$. If X is the random variable denoting the number of heads then we have $X(H) = 1$ and $X(T) = 0$. [See Fig. 3.13]

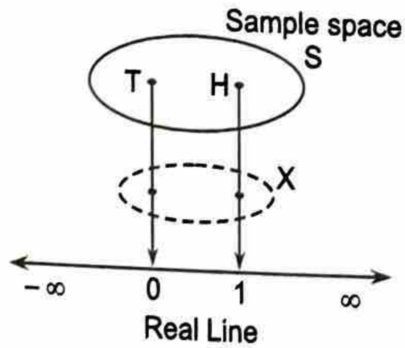


Fig. 3.13

Example 2 : Suppose the experiment E is to toss two fair coins.

Then $S = \{(H, H), (H, T), (T, H), (T, T)\}$. If X is the random variable denoting the number of heads then $X(H, H) = 2$, $X(H, T) = 1$, $X(T, H) = 1$, $X(T, T) = 0$.

[See Fig. 3.14]

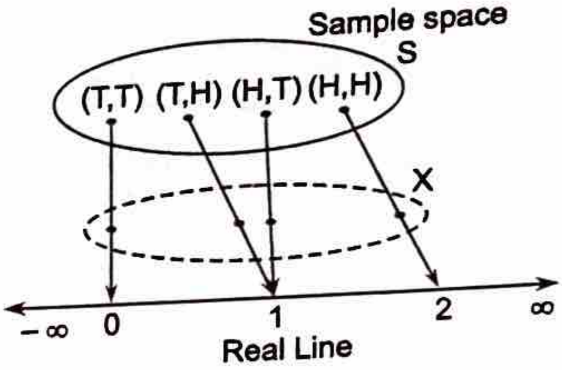


Fig. 3.14

Example 3 : Suppose the experiment is to record the temperature at a place.

If X denoted the temperature then X can take any value from -273° F to 212° F say. [See Fig. 3.15]

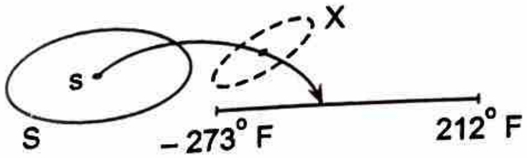


Fig. 3.15

Example 4 : Suppose the experiment is to record the time required to complete a software project.

If X denotes the time then X can take any value (theoretically) from 0 to ∞. [See Fig. 3.16]

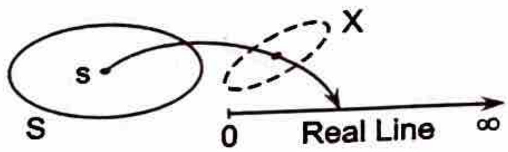


Fig. 3.16

In examples 1 and 2, X takes discrete values and in examples 3 and 4, X takes continuously all values between a specified interval. In the first case X is called a ***discrete random variable*** in the second case it is called ***continuous random variable***.

(b) Definition

Let X be a random variable. If X takes finite or countably infinite values $x_0, x_1, x_2, \ldots$ then X is called a ***discrete random variable***.

(c) Definition

Let X be a random variable. If X takes uncountably infinite values in a given interval then X is called a ***continuous random variable***.

9. Probability Distribution of a Discrete Random Variable

As we know already, with every possible outcome of an experiment there will be associated its probability. We shall be interested in the values of the random variable X along with their probabilities. If x_i is the value of X and $P(x_i)$ is the probability of x_i then set of pairs $\{x_i, P(x_i)\}$ is called the **probability distribution**.

Definition : Let X be a **discrete random variable**. Let $x_1, x_2, \ldots x_n \ldots$ be the possible values of X. With each possible outcome x_i we associate a number $p(x_i) = P(X = x_i)$ called the probability of x_i. The numbers $p(x_i)$, $i = 1, 2, \ldots n \ldots$ must satisfy the following conditions :

1. $p(x_i) \geq 0$ for all i
2. $\sum_{i=1} p(x_i) = 1$

The function p is called the **probability function** or **probability mass function (p.m.f)** or **probability density function (p.d.f.)** of the random variable X and the set of pairs (x_i, p_i) is called the probability distribution of X.

The probability distribution of a discrete random variable X taking values $x_1, x_2, x_3 \ldots\ldots x_n \ldots$ with probabilities $p_1, p_2, p_3 \ldots\ldots p_n \ldots\ldots$ where $p_i \geq 0$ and $\Sigma\, p_i = 1$ can be given in tabular form as

X	x_1	x_2	x_3	.....	x_n	.....
$P(x_i)$	p_1	p_2	p_3	.....	p_n	.....

Example 1 : State true or false with justification :

(a) A random variable X takes values 0, 1, 2 and 3 then $p(X = x) = \dfrac{x-1}{2}$ can be its probability distribution.

(b) A random variable takes values 0, 1, 2 and $p(x) = \dfrac{x+1}{3}$ is its probability distribution.

Sol. : As seen above for a probability distribution, two conditions must be satisfied.

(i) Each probability must be equal to or greater than zero but less than one.

(ii) The sum of all probabilities must be equal to unity.

Putting $x = 0, 1, 2, 3$ in (a), we get

$$P(0) = -\frac{1}{2},\ P(1) = 0,\ P(2) = \frac{1}{2},\ P(3) = 1$$

Since, the probability cannot be negative, $P(X = x) = \dfrac{x-1}{2}$ cannot be a probability distribution.

Putting $x = 0, 1, 2, 3$, in (b), we get

$$P(0) = \frac{1}{3},\ P(1) = \frac{2}{3},\ P(2) = 1.$$

Although all probabilities are positive, the sum of all the probabilities is 2, (greater than 1). Hence, $P(X = x) = \dfrac{x+1}{3}$ also cannot be a probability distribution.

Example 2 : From the past experience it was found that the daily demand at an autogarage was as under.

Daily Demand	:	5	6	7
Probability	:	0·25	0·65	0·10

Check if this is a probability distribution. Find also the probability that over a period of two days the number of demands would be 11 or 12.

Sol. : Since the sum of all probabilities = 0·25 + 0·65 + 0·10 = 1, it is a probability distribution.

P (11 requests over two days)

$= P$ (5 requests on the first day and 6 on the second)

$+ P$ (6 request on the first day and 5 on the second)

$= (0{\cdot}25 \times 0{\cdot}65) + (0{\cdot}65 \times 0{\cdot}25)$

$= 0{\cdot}1625 + 0{\cdot}1625 = 0{\cdot}325$

P (12 requests over two days)

$= P$ (5 requests on the first day and 7 on the second)

$\quad + P$ (6 requests on the first day and 6 on the second)

$\quad + P$ (7 requests on the first day and 5 on the second)

$= (0{\cdot}25 \times 0{\cdot}10) + (0{\cdot}65 \times 0{\cdot}65) + (0{\cdot}10 \times 0{\cdot}25)$

$= 0{\cdot}025 + 0{\cdot}4224 + 0{\cdot}025 = 0{\cdot}7975.$

Example 3 : Find the probability distribution of number of heads (X) obtained when a fair coin is tossed 4 times.

Sol. : When a coin is tossed 4 times, there are $2^4 = 16$ outcomes which are listed below.

HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,

THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT.

(Write first H, T, H, T alternately for 16 times. Then 2 H's and 2 T's alternately, then 4 H's and 4 T's alternatively and lastly 8 H's and 8 T's alternately). Hence, the probability distribution of X is

X	0	1	2	3	4
$P(X = x)$	1 / 16	4 / 16	6 / 16	4 / 16	1 / 16

Example 4 : Write down the probability distribution of the sum of numbers appearing on the toss of two unbiased dice. **(M.U. 2004, 06)**

Sol. : When two dice are thrown, we get the sum of numbers as shown below by slant dotted lines.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

It is easy to see that the sum 2 appears once, 3 twice, 4 thrice etc. The probability of each single event is 1/ 36.

The probability distribution obtained is shown below.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Example 5 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9. **(M.U. 1997)**

Sol. : $P(X \text{ is odd}) = P(X = 3, 5, 7, 9 \text{ or } 11)$

$= P(X = 3) + P(X = 5) + P(X = 7) + P(X = 9) + P(X = 11)$

$$\therefore\ P(X \text{ is odd}) = \frac{2}{36}+\frac{4}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36}=\frac{18}{36}=\frac{1}{2}$$

$$\begin{aligned} P(3 \le X \le 9) &= P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9) \\ &= P(X=3)+P(X=4)+\ldots\ldots+P(X=9) \\ &= \frac{2}{36}+\frac{3}{36}+\ldots\ldots+\frac{4}{36}=\frac{29}{36}. \end{aligned}$$

Example 6 : Two unbiased dice are thrown. Let x_1 and x_2 denote the scores obtained and y denote the maximum of them *i.e.* $y = \max.\ (x_1, x_2)$. Find the probability distribution of Y.

Sol. : Refer to the chart of Ex. 4 above. It is easy to see that y takes values, 2, 3, 4, 5, 6 with probabilities 1/36, 3/36, 5/36, 7/36, 9/36, 11/36 respectively as shown by thick arrows.

Y	1	2	3	4	5	6
$P(Y = y)$	1/36	3/36	5/36	7/36	9/36	11/36

Example 7 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9.

Sol. :

$$\begin{aligned} P(X \text{ is odd}) &= P(X = 3, 5, 7, 9 \text{ or } 11) \\ &= P(X=3)+P(X=5)+P(X=7)+P(X=9)+P(X=11) \\ &= \frac{2}{36}+\frac{4}{36}+\frac{6}{36}+\frac{4}{36}+\frac{2}{36}=\frac{18}{36}=\frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(3 \le X \le 9) &= P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9) \\ &= P(X=3)+P(X=4)+\ldots\ldots+P(X=9) \\ &= \frac{2}{36}+\frac{3}{36}+\ldots\ldots+\frac{4}{36}=\frac{29}{36}. \end{aligned}$$

Example 8 : The probability mass function of a random variable X is zero except at the points $X = 0, 1, 2$. At these points it has the values $P(0) = 3c^3$, $P(1) = 4c - 10c^2$, $P(2) = 5c - 1$.

(i) Determine c, (ii) Find $P(X < 1)$, $P(1 < X \le 2)$, $P(0 < X \le 2)$. **(M.U. 2001)**

Sol. : Since $\sum p_i = 1$, we have, $P(0) + P(1) + P(2) = 1$.

$$\therefore\ 3c^3 - 10c^2 + 4c + 5c - 1 = 1 \qquad \therefore\ 3c^3 - 10c^2 + 9c - 2 = 0$$

$$(3c-1)(c-2)(c-1) \qquad \therefore\ c = 1/3$$

(The other values are not admissible. Why?)

$\therefore$ The probability distribution is

X	0	1	2
$P(X = x)$	1/9	2/9	2/3

$$\therefore\ P(X<1) = P(X=0) = \frac{1}{9}; \quad P(1 < X \le 2) = P(X=2) = \frac{2}{3};$$

$$P(0 < X \le 2) = P(X=1)+P(X=2) = \frac{2}{9}+\frac{2}{3}=\frac{8}{9}.$$

Example 9 : A random variable X has the following probability distribution

X	:	0	1	2	3	4	5	6	7
$P(X = x)$	:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) $P\left(\dfrac{1.5 < X < 4.5}{X > 2}\right)$, (iii) The smallest value of λ for which $P(X \le \lambda) > \dfrac{1}{2}$.

Sol. : (i) Since $\Sigma\, p\,(x_i) = 1$, we get

$$10k^2 + 9k = 1 \qquad \therefore\ 10k^2 + 9k - 1 = 0$$

$$\therefore\ 10k^2 + 10k - k - 1 = 0 \qquad \therefore\ 10k\,(k+1) - 1\,(k+1) = 0$$

$$\therefore\ (10k - 1)\,(k + 1) = 0 \qquad \therefore\ k = 1/10.\ \ k \text{ cannot be } -1.$$

$\therefore$ The probability distribution of X is

X	:	0	1	2	3	4	5	6	7
$P(X = x)$	:	0	1/10	2/10	2/10	3/10	1/100	2/100	17/100

(ii) Now, $P(A/B) = \dfrac{P(A \cap B)}{P(B)}$

$$\therefore\ P\left(\frac{1.5 < X < 4.5}{X > 2}\right) = \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P(X > 2)} = \frac{P(2 < X < 4.5)}{P(X > 2)}$$

$$= \frac{P(X = 3,\ 4)}{P(X = 3,\ 4,\ 5,\ 6,\ 7)} = \frac{5/10}{70/100} = \frac{5}{7}.$$

Now from the table we find that

$$P\,(X \le 3) = P\,(X = 0) + P\,(X = 1) + P\,(X = 2) + P\,(X = 3)$$

$$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}.$$

Hence, $P(X \le 4) = \dfrac{8}{10} > \dfrac{1}{2}$. Hence, $\lambda = 4$.

Example 10 : An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws made successively with replacement from the urn. **(M.U. 2007)**

Sol. : We get the following probabilities.

No. of red balls	**Probability**
0	$\frac{4}{7} \cdot \frac{4}{7} \cdot \frac{4}{7}$
1	$\left\{ \begin{array}{l} \frac{3}{7} \cdot \frac{4}{7} \cdot \frac{4}{7} \\ \frac{4}{7} \cdot \frac{3}{7} \cdot \frac{4}{7} \\ \frac{4}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} \end{array} \right.$
2	$\left\{ \begin{array}{l} \frac{3}{7} \cdot \frac{3}{7} \cdot \frac{4}{7} \\ \frac{3}{7} \cdot \frac{4}{7} \cdot \frac{3}{7} \\ \frac{4}{7} \cdot \frac{3}{7} \cdot \frac{3}{7} \end{array} \right.$
3	$\frac{3}{7} \cdot \frac{3}{7} \cdot \frac{3}{7}$

$\therefore$ Probability distribution is

x	0	1	2	3	Total
$p(x)$	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$	1

Example 11 : A random variable X has the following probability function :

X	:	1	2	3	4	5	6	7
$P(X = x)$	:	k	$2k$	$3k$	k^2	$k^2 + k$	$2k^2$	$4k^2$

Find **(i)** k, **(ii)** $P(X < 5)$, **(iii)** $P(X > 5)$, **(iv)** $P\left(\frac{X < 5}{2 < X \le 6}\right)$, **(v)** $P\left(\frac{X = 4}{3 \le X \le 5}\right)$.

(M.U. 2019)

Sol. : Since $\Sigma\, p(x_i) = 1$,

$$k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2 = 1$$

$$\therefore\ 8k^2 + 7k - 1 = 0 \quad \therefore\ (8k - 1)(k + 1) = 0$$

$$\therefore\ k = 1/8 \quad \text{or} \quad k = -1 \text{ which is impossible (why ?)}$$

Thus, we have the following probability distribution.

X	:	1	2	3	4	5	6	7
$P(X = x)$	:	1/8	2/8	3/8	1/64	9/64	2/64	4/64

(i)
$$P(X < 5) = P(X = 1, 2, 3, 4,)$$
$$= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$$
$$= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{1}{64} = \frac{49}{64}.$$

(ii)
$$P(X > 5) = P(X = 6, 7) = P(X = 6) + P(X = 7)$$
$$= \frac{2}{64} + \frac{4}{64} = \frac{6}{64} = \frac{3}{32}.$$

(iii)
$$P\left(\frac{X < 5}{2 < X \le 6}\right) = \frac{P(X < 5 \cap 2 < X \le 6)}{P(2 < X \le 6)} = \frac{P(2 < X < 5)}{P(2 < X \le 6)}$$
$$= \frac{P(X = 3, 4)}{P(X = 3, 4, 5, 6)} = \frac{25/64}{36/64} = \frac{25}{36}.$$

(iv)
$$P\left(\frac{X = 4}{3 \le X \le 5}\right) = \frac{P(X = 4 \cap 3 \le X \le 5)}{P(3 \le X \le 5)}$$
$$= \frac{P(X = 4)}{P(X = 3, 4, 5)} = \frac{1/64}{34/64} = \frac{1}{64}.$$

Example 12 : The probability of a man hitting the target is 1/4. How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3? **(M.U. 2016)**

Sol. : We are given that the probability of hitting the target $p = 1/4$.

Probability of not hitting the target, $q = 1 - \frac{1}{4} = \frac{3}{4}$.

$\therefore$ Probability of not hitting the target in n trials

$$= \left(\frac{3}{4}\right)\left(\frac{3}{4}\right) \ldots\ldots (n \text{ times}) = \left(\frac{3}{4}\right)^n$$

$\therefore$ Probability of hitting the target at least once in n trials

$$= 1-\left(\frac{3}{4}\right)^n$$

We want this probability to be greater 2/3.

$$\therefore \quad 1-\left(\frac{3}{4}\right)^n > \frac{2}{3} \qquad \therefore \quad 1-\left(\frac{2}{3}\right) > \left(\frac{3}{4}\right)^n$$

$$\therefore \quad \frac{1}{3} > \left(\frac{3}{4}\right)^n \qquad \therefore \quad \left(\frac{3}{4}\right)^n < \frac{1}{3}$$

Taking logarithms of both sides

$$n\,(\log 3 - \log 4) < -\log 3$$

$$\therefore \quad n\,(0{\cdot}4771 - 0{\cdot}6021) < -\log 3 \qquad \therefore \quad -n\,(0{\cdot}1250) < -0{\cdot}4771$$

$$\therefore \quad n > \frac{0{\cdot}4771}{0{\cdot}1250} = 3{\cdot}81 \qquad \therefore \quad n > 4$$

$\therefore$ He must fire at least 4 times, so that he will hit the target at least once, with probability of success greater 2/3.

EXERCISE - III

1. If $P(X = x) = x/25$, $x = 1, 3, 5, 7, 9$, find $P(X = 1 \text{ or } 3)$ and $P(4 < X < 8)$.

[Ans. : (i) 4 / 25, **(ii)** 12 / 25 **]**

2. Verify whether the following functions can be considered as p.m.f. and if so find $P(X = 1 \text{ or } 3)$. Give reasons.

(i) $P(X = x) = \frac{1}{5}$, $x = 0, 1, 2, 3, 4$. **(ii)** $P(X = x) = \frac{x^2+1}{18}$, $x = 0, 1, 2, 3$.

(iii) $P(X = x) = \frac{x^2-2}{8}$, $x = 1, 2, 3$. **(iv)** $P(X = x) = \frac{2x+1}{18}$, $x = 0, 1, 2, 3$.

[Ans. : (i) Yes, 2 / 5 ; **(ii)** Yes, 2 / 3 ; **(iii)** No ; **(iv)** No **]**

3. If the p.m.f. $P(X = x)$ of a discrete random variate which assumes values x_1, x_2, x_3 such that $P(x_1) = 2P(x_2) = 3P(x_3)$, obtain the probability distribution of X.

[Ans. : (i) $P(x_1) = 2/11$, **(ii)** $P(x_2) = 3/11$, **(iii)** $P(x_3) = 6/11$ **]**

4. Presuming the daily demand to be independent, find the probability that over a two -days period the number of requests at a service station will be (i) 9, (ii) 10, if the past record show that the demand was 4, 5 or 6 with probabilities 0·50, 0·40 or 0·10 respectively.

(**Hint :** The event can occur as **(i)** (4, 5), (5, 4); **(ii)** (4, 6), (6, 4), (5, 5).)

[Ans. : (i) 0.40, **(ii)** 0·26 **]**

5. Find the probability distribution and the probability mass function of the number of points obtained when a fair die is tossed. **[Ans. :** $P(X = x_i) = 1/6$, $i = 1, 2, 3, 4, 5, 6$ **]**

6. Find the probability distribution and the p.m.f. of the number of heads obtained when an unbiased coin is tossed three times.

$$\left[\textbf{Ans. : } P(X = x) = \frac{{}^3C_x}{2^3},\ x = 0, 1, 2, 3\right]$$

7. The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X=x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X<4)$, $P(3<X\leq 6)$. **(M.U. 2001, 05, 10, 15, 19)**

[**Ans. :** $k = 1/49$, $16/49$, $33/49$]

8. A random variable X has the following probability function

X	1	2	3	4	5	6	7
$P(X=x)$	k	$2k$	$3k$	k^2	k^2+k	$2k^2$	$4k^2$

Find **(i)** k, **(ii)** $P(X<5)$, **(iii)** $P(X>5)$, **(iv)** $P(0\leq X\leq 5)$.

[**Ans. : (i)** $k = 1/8$, **(ii)** $49/64$, **(iii)** $3/32$, **(iv)** $29/32$]

9. A discrete random variable X has the following probability distribution

X	-2	-1	0	1	2	3
$P(X=x)$	0·1	k	0·2	$2k$	0·3	$3k$

Find **(i)** k, **(ii)** $P(X\geq 2)$, **(iii)** $P(-2<X<2)$. **(M.U. 2009)**

[**Ans. : (i)** $k = 1/15$, **(ii)** $1/2$, **(iii)** $2/5$]

10. Given the following probability function of a discrete random variable X

X	0	1	2	3	4	5	6	7
$P(X=x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2+c$

(i) Find c, **(ii)** Find $P(X\geq 6)$, **(iii)** $P(X<6)$, **(iv)** Find k if, $P(X\leq k)>1/2$, where k is a positive integer, **(v)** $P(1{\cdot}5<X<4{\cdot}5 / X>2)$. **(M.U. 1996, 2003, 05)**

[**Ans. : (i)** $c = 0{\cdot}1$, **(ii)** 0·19, **(iii)** 0·81, **(iv)** $k = 4$, **(v)** $5/7$]

11. In the example 9 above, find

(i) $P(-1\leq X\leq 1 / -2\leq X\leq 3)$, **(ii)** $P(X\leq 2 / 0\leq X\leq 4)$, **(iii)** $P(X\leq 1 / X\leq 2)$.

[**Ans. : (i)** $\frac{1}{2}$, **(ii)** $\frac{19}{26}$, **(iii)** $\frac{15}{24}$.]

12. A random variable X takes values $-2, -1, 0, 1, 2$ such that $P(X>0) = P(X=0) = P(X<0)$; $P(X=-2) = P(X=-1)$, $P(X=1) = P(X=2)$. Obtain the probability distribution of X.

Also find (i) $P\left(\dfrac{-1\leq X\leq 1}{-2\leq X\leq 0}\right)$, (ii) $P\left(\dfrac{X=1}{0\leq X\leq 2}\right)$.

[**Ans. :**

X :	-2	-1	0	1	2
$P(X=x)$:	1/6	1/6	1/3	1/6	1/6

(i) 4/5, **(ii)** 1/5]

13. A random variable X takes values 0, 1, 2, ... n with probabilities proportional to ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... ${}^nC_n$. Find the proportionality constant.

(**Hint :** $k[{}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n] = 1 \quad \therefore \ k\cdot 2^n = 1$) [**Ans. :** $k = 2^{-n}$]

14. A random variable X assumes four values with probabilities $(1+3x)/4$, $(1-x)/4$, $(1+2x)/4$ and $(1-4x)/4$. For what value of x do these values represent the probability distribution of X? **(M.U. 2004)**

[**Ans. :** $\Sigma p_i = 1$. But $\dfrac{1+3x}{4}\geq 0$ if $x\geq -\dfrac{1}{3}$ and also $\dfrac{1-4x}{4}\geq 0$ if $x\leq \dfrac{1}{4} \quad \therefore \ -\dfrac{1}{3}\leq x\leq \dfrac{1}{4}$]

15. The amount of bread X (in hundred kgs) that a certain bakery is able to sell in a day is a random variable with probability density function is given by

$$f_X(x) = \begin{cases} Ax, & 0 < x < 5 \\ A(10-x), & 5 < x < 10 \\ 0, & \text{elsewhere} \end{cases}$$

Find (i) A, (ii) the probabilities of the events : B, the amount of bread sold in a day is more than 500 kgs, C : the amount of bread sold in a day is less than 500 kgs, D : the amount of bread sold in a day is between 250 kgs and 750 kgs. (iii) Are the events B and C exclusive ? (iv) Are the events B and D exclusive. **(M.U. 2005) [Ans. : (i)** $A = 1/25$, **(ii)** $1/2$, $1/2$, 0.75, **(iii)** Yes, **(iv)** No **]**

10. Distribution Function of a Discrete Random Variable X

Probability distribution of X gives us the probability $p(x)$ that X will take a particular value x. Sometimes we need to know the probability that X will take a value less than or equal to a given value x. This probability is obtained by adding the probabilities of all values less than or equal to x.

Suppose, X is a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p(x_i)$, $i = 1, 2, \ldots, n \ldots$ such that

(i) $p(x_i) \geq 0$ for all i,

(ii) $\Sigma\, p(x_i) = 1$ and consider the following table.

X	x_1	x_2	x_3	.....	x_n	.....
$\mathbf{F(x_i) = F(X = x_i)}$	$p(x_i)$	$\sum_1^2 p(x_i)$	$\sum_1^3 p(x_i)$	.....	$\sum_1^n p(x_i)$	.....

The table states that

$$\begin{aligned} F(x_1) &= P(a \leq X < x) = p(x_1) \\ F(x_2) &= P(X \leq x_2) = p(a \leq X \leq x_1) + p(x_1 \leq X \leq x_2) \\ &= p(x_1) + p(x_2) \\ F(x_3) &= P(X \leq x_3) = p(a \leq X \leq x_1) + p(x_1 \leq X \leq x_2) + p(x_2 \leq X \leq x_3) \\ &= p(x_1) + p(x_2) + p(x_3). \end{aligned}$$

And so on.

The graph shown in the Fig. 3.17 shows this diagrammatically.

The function F is called the **distribution function**. We have more precise definition as follows.

(a) Definition

Let X be a discrete random variable taking values $x_1, x_2, \ldots$ such that $x_1 < x_2 < x_3 \ldots$ with probabilities $p(x_1), p(x_2), \ldots$ such that $p(x_i) \geq 0$ for all i and $\Sigma\, p(x_i) = 1$.

Consider F defined by $\quad F(x_i) = p(X \leq x_i),\ i = 1, 2, 3, \ldots$

i.e. $\quad F(x_i) = p(x_1) + p(x_2) + \ldots + p(x_i)$

then the function F is called the **cumulative distribution function** or simply **distribution function** and the set of pairs $\{x_i, F(x_i)\}$ is called the **cumulative probability distribution**.

Note ....

The distribution function is to the probability mass function as cumulative frequency distribution is to frequency distribution.

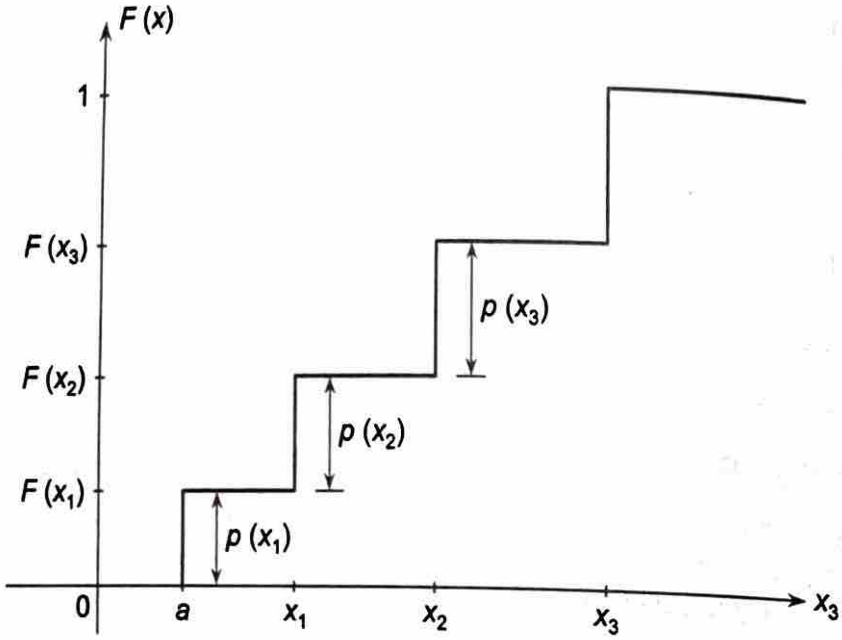


Fig. 3.17 : Graph of Distribution Function.

(b) Important Properties of Distribution function

The distribution function F of a random variable X has the following important properties.

1. $0 \leq F(x) \leq 1$

Proof : Since, $F(x_1) = p(x_1)$, $F(x_2) = p(x_1) + p(x_2)$

.... $F(x_n) = p(x_1) + p(x_2) + ... + p(x_n)$

and $0 \leq p(x_i) \leq 1$ for every i and $\Sigma\, p(x_i) = 1$, it is clear that $0 \leq F(x) \leq 1$.

2. $F(x) = 0$ for $x < a$ and $F(x) = 1$ for $x > b$ where $a < x_1 < x_2 < ..... x_n < b$.

Proof : If $x < a$ where $a < x_1$, $p(x) = 0$ and hence, $F(x) = 0$ for $x < a$.

If $x > b$ then $F(x) = p(x_1) + p(x_2) + ... + p(x_n) = 1$.

3. $F(x)$ is a step function

Proof : By definition, $F(x_1) = F(X < x_1) = p(x_1)$

$F(x_2) = F(X < x_2) = p(x_1) + p(x_2)$

i.e. $F(X)$ has the same value $p(x_1)$ for $x_1 \leq x \leq x_2$,

and the same value $p(x_1) + p(x_2)$ for $x_2 \leq x \leq x_3$ .....

Hence, the graph of $F(x)$ is made up of horizontal line segments taking "jumps" at the possible values x_i of X. The jump is of magnitude $p(x_i) = P(X = x_i)$.

Hence, $F(x)$ is a step functions.

Example 1 : A random variable X has the probability function given below :

$f(x) = k$ if $x = 0$; $f(x) = 2k$ if $x = 1$;

$f(x) = 3k$ if $x = 2$; $f(x) = 0$ otherwise.

(i) Determine the value of k, (ii) Evaluate $P(X < 2)$, $P(X \leq 2)$, $P(0 < X < 2)$, (iii) Obtain the distribution function.

Sol. : The probability distribution can be tabulated as

X	0	1	2
$p(x)$	k	$2k$	$3k$

(i) Since $\Sigma p_i = 1$, $k + 2k + 3k = 1$ $\quad \therefore \; k = 1/6.$

(ii) $P(X < 2) = P(X = 0) + P(X = 1) = k + 2k = 3/6 = 1/2$

$P(X \leq 2) = P(X = 0) + P(X = 1) + P(X = 2) = 6k = 1.$

$P(0 < X < 2) = P(X = 1) = 2k = 1/3.$

(iii) Distribution function of X is

X	0	1	2
$F(x)$	1/6	1/2	1

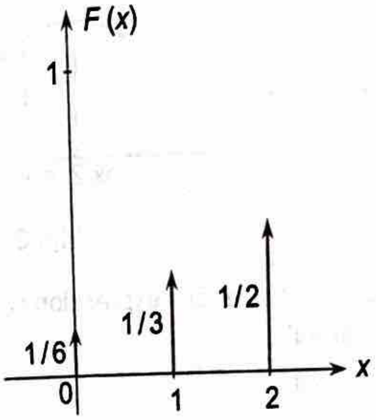


Fig. 3.18 (a)
Probability Density Function.

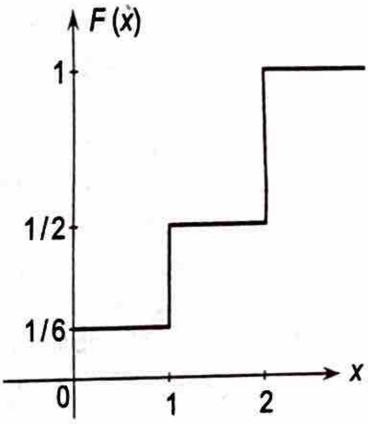


Fig. 3.18 (b)
Distribution Function.

EXERCISE - IV

1. A random variable takes values 1, 2, 3, 4 such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$. Find the probability distribution and the cumulative distribution function. **(M.U. 2004)**

[**Ans. :**

x	1	2	3	4
$P(x)$	15/61	10/61	30/61	6/61
$F(x)$	15/61	25/61	55/61	1

2. A shipment of 8 computers contains 3 that are defective. If a college makes a random purchase of 2 of these computers, find the probability distribution of the defective computers. Find also distribution function. **(M.U. 2005)**

[**Ans. :**

x	0	1	2
$P(x)$	10/28	15/28	3/28
$F(x)$	10/28	25/28	1

3. The probability density function of a random variable X is

X	:	0	1	2	3	4	5	6
$P(X = x)$	:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X < 4)$, $P(3 < X \leq 6)$. [**Ans. :** k = 1/49, 16/49, 33/49]

11. Continuous Random Variable

Definition : A random variable is called a **continuous random variable** if it takes all values between an interval (a, b).

For example, age, height, weight are continuous random variables.

12. Probability Density Function of A Continuous Random Variable

Let $y = f(x)$ be a continuous function of x such that the area $f(x)\,\delta x$ represents the probability that X will lie in the interval $(x, x + \delta x)$. Symbolically,

$$P(x \le X \le x + \delta x) = f_X(x)\,\delta x$$

where, $f_X(x)$ denotes the value of $f(x)$ at x.

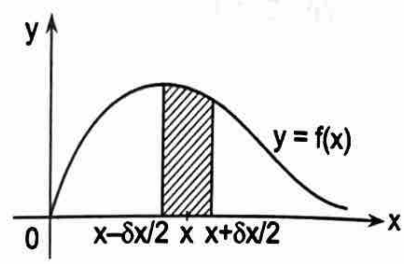


Fig. 3.19

The adjoining figure denotes the curve $y = f(x)$ and the area under the curve in the interval $(x, x + \delta x)$. The function satisfying certain conditions giving the probability that x will lie between certain limits is called **probability density function,** or simply **density function** of a continuous random variable X and is abbreviated as p.d.f. The curve given by $y = f(x)$ is called the **probability density curve** or simply **probability curve**. The expression $f(x)\,dx$ is usually denoted by $df(x)$ and is known as **probability differential**.

Definition : A continuous function $y = f(x)$ such that

(i) $f(x)$ is integrable. (ii) $f(x) \ge 0$

(iii) $\int_a^b f(x)\,dx = 1$ if X lies in $[a, b]$ and

(iv) $\int_\alpha^\beta f(x)\,dx = P(\alpha \le X \le \beta)$ where $a < \alpha < \beta < b$

is called **probability desnity function** of a continuous random variable X.

Thus, for a continuous random variale X,

$$\boxed{P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx}$$

Clearly $\int_\alpha^\beta f(x)\,dx$ represents the area under the curve $y = f(x)$, the x-axis and the ordinates at $x = \alpha$ and $x = \beta$. Further since, the total probability is one, if X lies in the interval $[a, b]$ then $\int_a^b f(x)\,dx = 1$. For a continuous random variable X the range may be finite $[a, b]$ or infinite $[-\infty, \infty]$.

Properties of Probability Density Function

The probability density function $f(x)$ has the following properties.

(i) $f(x) \ge 0,\ -\infty < x < \infty$ (*i.e.* the curve $y = f(x)$ lies above the x-axis in the first and second quadrants only)

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (*i.e.* the total area under the curve and the x-axis is one.)

(iii) The probability that $\alpha \le X \le \beta$ is given by $P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx$.

Notes ....

1. The property (1) and the property (2) can be used to verify whether a given function $f(x)$ can be a probability density function.

2. You know that for A discrete random variable the probability at $X = c$ may not be zero. But, in a continuous random variable $P(X = c)$ is always zero because $P(X=c)=\int_c^c f_X(x)\,dx$ and this definite integral is zero. Hence, for a continuous random variable X,

$$P(\alpha \le X \le \beta) = P(\alpha < X < \beta) = P(\alpha < X \le \beta) = P(\alpha < X < B)$$

In other words we may include or may not include the end-points in the interval.

3. Any function $f(x)$ of a real variable x can be a probability density function if it satisfies the first two properties given above *viz.* $f(x)$ is non-negative for all value of x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Sometimes $\int_{-\infty}^{\infty} f(x)\,dx$ is not equal to 1 but $\int_{-\infty}^{\infty} k f(x) = 1$ for some value of k.

In such cases k is called the ***normalising factor*** or ***normalisation constant***.

Example 1 : Find the normalising factor, k if the following function is a probability density function

$$f(x) = \begin{cases} k(1-x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Also find $P(0{\cdot}1 < X < 0{\cdot}2)$ and $P(X > 0{\cdot}5)$.

Sol. : Since $0 < x < 1$, $f(x) \ge 0$ for all x.

Now, $\int_0^1 f(x)\,dx = k\int_0^1 (1-x^2)\,dx = k\left[x - \frac{x^3}{3}\right]_0^1 = k\frac{2}{3}$

But this must be equal to 1.

$$\therefore \quad \frac{2k}{3} = 1 \quad \therefore\ k = \frac{3}{2}$$

(i) $P(0{\cdot}1 < X < 0{\cdot}2) = \int_{0\cdot1}^{0\cdot2} \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0\cdot1}^{0\cdot2}$

$$= \frac{3}{2}\left[\left\{0{\cdot}2 - \frac{(0{\cdot}2)^3}{3}\right\} - \left\{0{\cdot}1 - \frac{(0{\cdot}1)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0{\cdot}0977] = 0{\cdot}146$$

(ii) $P(X > 0{\cdot}5) = \int_{0\cdot5}^{1} \frac{3}{2}(1-x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0\cdot5}^{1}$

$$= \frac{3}{2}\left[\left\{1 - \frac{1}{3}\right\} - \left\{0{\cdot}5 - \frac{(0{\cdot}5)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0{\cdot}667 - 0{\cdot}458] = 0{\cdot}313$$

Example 2 : A continuous random variable X has the following probability law

$$f(x) = kx^2, \quad 0 \le x \le 2$$

Determine k and find the probabilities that (i) $0.2 \le X \le 0.5$, (ii) $X \ge 3/4$ given that $X \ge 1/2$.

(M.U. 2005)

Sol. : Since the total probability *i.e.* the total area is unity

$$\int_a^b f(x)\,dx = \int_0^2 kx^2 dx = 1$$

$$k\left[\frac{x^3}{3}\right]_0^2 = 1 \quad \therefore\ k \cdot \frac{8}{3} = 1 \quad \therefore\ k = \frac{3}{8}$$

(i) $$P(0{\cdot}2 \le X \le 0{\cdot}5) = \frac{3}{8}\int_{0\cdot2}^{0\cdot5} x^2 dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot2}^{0\cdot5} = \frac{1}{8}\left[0{\cdot}5^3 - 0{\cdot}3^3\right] = 0{\cdot}0123$$

(ii) Let $A = (X \ge 1/2)$, $B = (X \ge 3/4)$

$$\therefore\ P(A) = P(X \ge 1/2) = \frac{3}{8}\int_{0\cdot5}^{2} x^2 dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot5}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0{\cdot}5^3\right] = 0{\cdot}984$$

$$P(B) = P(X \ge 3/4) = \frac{3}{8}\int_{0\cdot75}^{2} x^2 dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot75}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0{\cdot}75^3\right] = 0{\cdot}947$$

$$P(A \cap B) = P(B) = 0{\cdot}947$$

$$\therefore\ P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0{\cdot}947}{0{\cdot}984} = 0{\cdot}96$$

Example 3 : Let X be a continuous random variable with probability distribution

$$p(x) = \begin{cases} \dfrac{x}{6} + k & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and find $P(1 \le x \le 2)$.

(M.U. 1998, 2004)

Sol. : Since the total probability is one

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^3 \left(\frac{x}{6} + x\right)dx = \left[\frac{x^2}{12} + kx\right]_0^3 = \frac{3}{4} + 3k = 1$$

$$\therefore\ 3\left(\frac{1}{4} + k\right) = 1 \quad \therefore\ \frac{1}{4} + k = \frac{1}{3} \quad \therefore\ k = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore\ p(x) = \begin{cases} \dfrac{x}{6} + \dfrac{1}{12} & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore \quad P(1 \le x \le 2) = \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx = \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}[(4+2) - (1+1)] = \frac{1}{12}(4) = \frac{1}{3}.$$

Example 4 : Let X be a continuous random variable with p.d.f. $f(x) = kx(1-x)$, $0 \le x \le 1$. Find k and determine a number b such that $P(X \le b) = P(X \ge b)$. **(M.U. 2003, 11, 15, 19)**

Sol. : Since $\int_{-\infty}^{\infty} f(x)\, dx = 1$, we have

$$k \int_0^1 (x - x^2)\, dx = 1 \quad \therefore \quad k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore \quad k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore \quad k = 6.$$

Since, the total probability is 1 and $P(x \le b) = P(x \ge b)$, $P(x \le b) = 1/2$.

$$\therefore \quad \int_0^b f(x)\, dx = \frac{1}{2}$$

$$\therefore \quad 6\int_0^b (x - x^2)\, dx = \frac{1}{2} \quad \therefore \quad \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = \frac{1}{12}$$

$$\therefore \quad \frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{12} \quad \therefore \quad 6b^2 - 4b^3 = 1 \quad \therefore \quad 4b^3 - 6b^2 - 1 = 0$$

$$\therefore \quad 4b^3 - 2b^2 - 4b^2 - 2b + 2b - 1 = 0$$

$$\therefore \quad (2b-1)(2b^2 - 2b + 1) = 0 \quad \therefore \quad b = 1/2.$$

Example 5 : The probability that a person will die in the time interval (t_1, t_2) is given by

$$P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)\, dt$$

where, $$f(t) = \begin{cases} 3 \times 10^{-9}(100t - t^2)^2, & 0 < t < 100 \\ 0, & \text{elsewhere.} \end{cases}$$

Find (i) the probability that Mr. X will die between the ages 60 and 70. (ii) the probability that he will die between the ages 60 and 70, given that he has survived upto age 60. **(M.U. 2005)**

Sol. : (i) $P(60 \le t \le 70) = \int_{60}^{70} 3 \times 10^{-9}(100t - t^2)^2\, dt$

$$= 3 \times 10^{-9} \int_{60}^{70} (100^2 t^2 - 200t^3 + t^4)\, dt$$

$$= 3 \times 10^{-9}\left[100^2 \frac{t^3}{3} - 200\frac{t^4}{4} + \frac{t^5}{5}\right]_{60}^{70}$$

$$= 3 \times 10^{-9}\left[27.89 \times 10^8 - 22.75 \times 10^8\right]$$

$$= 0{\cdot}1542.$$

(ii) $$P\left(\frac{60 \le T \le 70}{T \ge 60}\right) = \frac{P(60 \le t \le 70 \cap t \ge 60)}{P(t \ge 60)} = \frac{P(60 \le t \le 70)}{P(60 \le t \le 100)}$$

$$= \frac{\int_{60}^{70} f(t)\,dt}{\int_{60}^{100} f(t)\,dt} = \frac{0.1542}{0.3174} = 0.4858.$$

EXERCISE - V

1. A function is defined as

$$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ \dfrac{2x+3}{18} & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$$

Show that $f(x)$ is a probability density function and find the probability that $2 < x < 3$.

(M.U. 2005) [**Ans. :** 5 / 18]

2. A random variable X has the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find $P(1 \le X \le 3)$, $P(X \ge 0.5)$. [**Ans. : (i)** 0·133, **(ii)** 0·368]

3. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \end{cases}$$

Find (i) the value of k, (ii) $P(x \le 1.5)$. [**Ans. : (i)** $k = 2/3$, **(ii)** 2 / 3]

4. A continuous random variable has the following probability density function.

$$f(x) = \begin{cases} (x/4) + k & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and $P(1 \le X \le 2)$. [**Ans. :** $k = 1/4$, 5 / 8]

5. Find the value of k such that the following will be the probability density function. Find also $P(x \le 1.5)$.

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \\ k(3-x) & 2 \le k \le 3 \end{cases}$$

(M.U. 2003) [**Ans. :** $k = \frac{1}{2}, \frac{1}{2}$]

13. Continuous Distribution Function

Probability distribution of X or the probability density function of X helps us to find the probability that X will be within a given interval $[a, b]$ *i.e.* $P(a \le X \le b) = \int_a^b f(x)\,dx$, other conditions being satisfied.

However, sometimes we need to know that probability that X will be less than a given value x. For a continuous random variable X, this probability is obtained by integrating $f(x)$ from $-\infty$ (or the lower limit of the interval) to x. The function so obtained is called **distribution function.**

Definition : If X is a continuous random variable X, having the probability density function $f(x)$ then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < X < \infty$$

is called **distribution function** or **cumulative distribution function** of the random variable X.

Some Important Properties of Distribution Function $F(x)$ of a Continuous Random Variable

1. The function $F(x)$ is defined for every real number x.
2. Since $F(x)$ denotes probability and probability of X lies between 0 and 1,
   $0 \le F(x) \le 1.$
3. $F(x)$ is a non-decreasing function which means if $x_1 \le x_2$, then $F(x_1) \le F(x_2)$.
4. **The derivative of $F(x)$** *i.e.* $F'(x)$ exists at all points (except perhaps at a finite number of points) and **is equal to the probability density function $f(x)$.**

$$F'(x) = \frac{d}{dx}F(x) = f(x) \ge 0 \text{ provided the derivative exists.}$$

5. If $F(x)$ is a distribution function of a continuous random variable then
   $P(a \le X \le b) = F(b) - F(a)$.

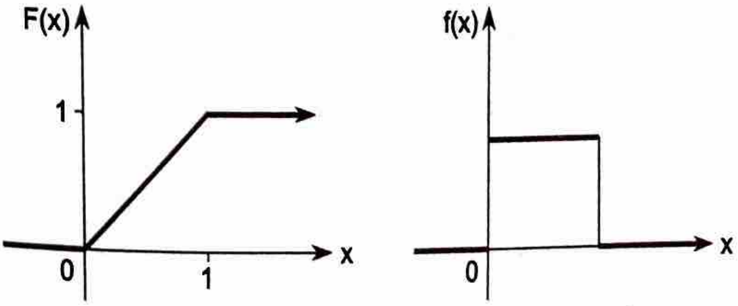
Example 1 : A continuous variable X has the following distribution function

$$F(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 \le x \end{cases} \qquad \text{.................... (1)}$$

Find the probability density function and draw the graphs of both p.d.f. and c.d.f.

Sol. : The p.d.f. is

$$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & 1 < x \end{cases}$$



Graph of c.d.f.　　Graph of p.d.f.

Fig. 3.20

From the graph of $F(x)$ we see that $F(x)$ is continuous at all points including $x = 0$ and $x = 1$. $f(x)$ is obtained by differentiating $F(x)$.

Example 2 : For the distribution function given below, find p.d.f.

$$F(x) = \begin{cases} 0 \quad , & x < 0 \\ 1 - e^{-x/4}, & x \geq 0 \end{cases}$$

Also find the probabilities : $P(X \leq 4)$, $P(X \geq 8)$, $P(4 \leq X \leq 8)$.

Sol. : $F(x)$ satisfies all the conditions of a distribution function. If $f(x)$ is the corresponding probability density function

$$f(x) = F'(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x \geq 0 \\ 0 \quad , & x < 0 \end{cases}$$

Now, we have to verify that $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

$$\therefore \quad \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{\infty} \frac{1}{4} e^{-x/4}\, dx = 0 + \int_{0}^{\infty} \frac{1}{4} e^{-x/4} dx = \frac{1}{4}\left[\frac{e^{-x/4}}{-1/4}\right]_0^{\infty}$$

$$= -\left[e^{-x/4}\right]_0^{\infty} = -[0 - 1] = 1$$

Hence, $F(x)$ is a distribution function.

Now, $P(X \leq 4) = F(4) = 1 - e^{-1} = 1 - \frac{1}{e} = \frac{e-1}{e}$

$$P(X \geq 8) = 1 - P(x \leq 8) = 1 - F(8)$$

$$= 1 - \left[1 - e^{-2}\right] = e^{-2} = 1/e^2$$

$$P(4 \leq X \leq 8) = F(8) - F(4) = (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2} = \frac{e-1}{e^2}$$

Example 3 : If $f(x) = \begin{cases} x e^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$

(i) Show that $f(x)$ is a probability density function. (ii) Find its distribution function.

Sol. : If $f(x)$ is a p.d.f., we must have $f(x) \geq 0$ where $x \geq 0$ and $\int_0^{\infty} f(x)\, dx = 1$.

Clearly $f(x) = xe^{-x^2/2} \geq 0$ for $x \geq 0$.

Now, $\int_0^{\infty} x e^{-x^2/2}\, dx = \int_0^{\infty} e^{-t}\, dt \qquad \left[t = x^2/2\right]$

$$= \left[-e^{-t}\right]_0^{\infty} = -[0 - 1] = 1$$

$\therefore$ $f(x)$ is a probability density function.

Now, its distribution function is given by

$$F(x) = \int_0^{x} f(x)\, dx = \int_0^{x} x e^{-x^2/2}\, dx = -\left[e^{-x^2/2}\right]_0^{x} \qquad \text{[As above]}$$

$$= -\left[e^{-x^2/2} - 1\right] = 1 - e^{-x^2/2}, \quad x \geq 0.$$

EXERCISE - VI

1. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 \;, & x < -1 \\ \dfrac{x+1}{4}, & -1 \le x \le 3 \\ 1 \;, & 3 < x \end{cases}$$

Find the probability density function. Draw the graphs of both p.d.f. and c.d.f.

2. The c.d.f. of a continuous random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

Find the p.d.f. Draw the graphs of both p.d.f. and c.d.f.

Also find $P\left(\dfrac{1}{2} \le X \le \dfrac{4}{5}\right)$. [**Ans.** : 0·195]

3. Find the distribution functions corresponding to the following probability density functions.

(i) $f(x) = \begin{cases} \dfrac{1}{2} x^2 e^{-x}, & 0 \le x < \infty \\ 0 \;, & \text{otherwise} \end{cases}$ (ii) $f(x) = \begin{cases} x \;, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0 \;, & \text{otherwise} \end{cases}$

(iii) $f(x) = \begin{cases} \lambda (x-1)^4, & 1 \le x \le 3, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$

[**Ans.** : (i) $F(x) = \begin{cases} 1 - e^{-x}\left(1 + x + \dfrac{x^2}{2}\right), & x \ge 0 \\ 0 \;, & \text{otherwise} \end{cases}$

(ii) $F(x) = \begin{cases} 0 \;, & x < 0 \\ x^2/2 \;, & 0 \le x \le 1 \\ 2x - 0{\cdot}5x^2 - 1, & 1 \le x \le 2 \\ 1 \;, & x > 2 \end{cases}$

(iii) $\lambda = \dfrac{5}{32}$; $F(x) = \begin{cases} 0 \;, & x \le 1 \\ \dfrac{(x-1)^5}{32}, & 1 \le x \le 3 \\ 1 \;, & x \ge 3 \end{cases}$]

4. A continuous random variable X has the following probability density function

$$f(x) = \frac{a}{x^5}, \quad 2 \le x \le 10$$

Determine the constant a, distribution function of X and find the probability of the event

[**Ans.** : (i) $a = \dfrac{2500}{39}$, (ii) $\dfrac{625}{39}\left[\dfrac{1}{16} - \dfrac{1}{x^4}\right]$, (iii)

5. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1-(1+x)e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find the probability density function and find $P(0 \leq X \leq 1)$.

[Ans. : (i) $f(x) = x\,e^{-x}$, **(ii)** $P(0 \leq X \leq 1) = F(1) - F(0) = \frac{e-}{e}$

EXERCISE - VII

Theory

1. State and prove Bayes' Theorem.
2. Define the following terms giving suitable examples
   (a) Random variable
   (b) Discrete random variable
   (c) Continuous random variable
   (d) Probability density function
   (e) Distribution function
3. State the properties of probability density function.
4. State the properties of distribution function.

☆

CHAPTER 4

Mathematical Expectation

1. Introduction

Suppose two coins are tossed twenty times. Let X be the number of heads obtained in a toss. X then, takes values 0, 1 and 2. Suppose further that no heads, one head and two heads were obtained 4, 10, 6 times respectively. Then, the average number of heads per toss

$$= \frac{4(0) + 10(1) + 6(2)}{6 + 10 + 4} = 1 \cdot 1$$

This is the average value and is not necessarily a possible outcome of the toss.

The ratios 4 / 20, 10 / 20, 6 / 20 of 0, 1, 2 heads to the total number of tosses are the relative frequencies of $X = 0, 1, 2$. If the experiment is repeated very large number of times, we know that, these relative frequencies tend to the probabilities 1 / 4, 1 / 2, 1 / 4 of 0, 1, 2 heads because in the toss of two coins we have the following.

Sample space	:	HH	$\underbrace{HT, TH}$	TT
Probability	:	1/4	1/2	1/4

The average calculated with probabilities in place of relative frequencies above is called **expected value** or **mathematical expectation** and is denoted by $E(X)$. Thus,

$$E(X) = \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(2) = 1$$

$$E(X) = \text{sum of the products of the values and their probabilities}$$

$$= p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots\ldots$$

This means, a person who throws two coins over and over again will get one head per toss on the average. This suggests us that the expected value of X can be obtained by multiplying the values of X by their respective probabilities and taking the sum. This leads us to the following definition of expectation of a discrete random variable X.

2. Expectation of a Random Variable

(a) Definition : If a **discrete random variable X** assumes values $x_1, x_2, \ldots\ldots x_n \ldots\ldots$ with probabilities $p_1, p_2, \ldots\ldots p_n \ldots\ldots$ respectively then the **mathematical expectation of X** denoted by $E(X)$ (if it exists) is defined by

$$\boxed{E(X) = p_1 x_1 + p_2 x_2 + \ldots\ldots + p_n x_n + \ldots\ldots}$$

i.e. $\boxed{E(X) = \Sigma p_i x_i}$ where $\Sigma p_i = 1$.

if $\Sigma p_i x_i$ is absolutely convergent.

This value is also referred to as *mean value* of X. It is also denoted by μ_1'. $\therefore \mu_1' = E(X)$.