CHAPTER 6

Vector Spaces

1. Introduction

In XI and XII standards you have studied vectors, vector algebra, determinants and matrice to some extent. We are going to generalise these concepts of vectors in two and three dimensior further in this chapter. But before that we shall briefly take a review of what you have studied alread Then we shall study some properties of vectors in general form. Entities that we come across daily life are classified in various ways - one of them is to classify them into two classes as follow : (i) entities that have only magnitude, (ii) entities that have magnitude as well as direction.

Scalars : Entities that have only magnitude are called **scalars**. They are completely describe by numbers. For example, temperature, height, mass, etc. are scalars because they can be ful described by numbers. *e.g.*, Ahemdabad's temperature today is 23° C, Praveen's height is 15 cms, describe the events fully.

Vectors : Entities that have magnitudes as well as direction are called vectors. In order t describe them fully we need to know their magnitude as well as direction. For example, velocit acceleration, force, etc. are vectors. *e.g.*, the velocity of the car is 80 km/hour in the direction of th east. We need to know the magnitude and the direction of the vector to describe it fully.

2. Geometric Representation of a Point or a Vector

A point and a vector have many things in common. Both of them can be represented b coordinates. We shall denote by coordinates a point and a vector. The interpretation will depen upon the context.

We know that a point on a line can be denoted by a single number, a point on a plane can b denoted by two numbers, a point in space can be denoted by three numbers. This is shown in th Fig. 6.1.

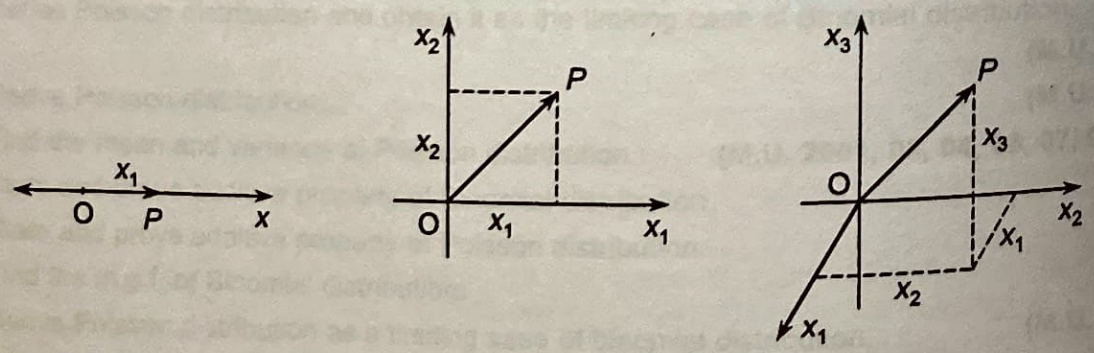

Fig. 6.1

We say that a single number (x_1) represents a point P (or a vector OP) in 1-space, a coupl of two ordered numbers (x_1, x_2) represents a point P (or a vector OP) in 2-space, a triple of th

three ordered numbers (x_1, x_2, x_3) represents a point P (or a vector OP) in 3-space. Although we cannot draw a picture as we go further but there is nothing that prevents us to say that a quadruple (x_1, x_2, x_3, x_4) represents a point P (on a vector OP) in 4-space, a quintuple in 5-space, a sextuple in 6-space, septuple in 7-space, octuple in 8-space, ... Generalising this idea, we say that an **n-tuple of numbers** $(x_1, x_2, x_3, \ldots, x_n)$ represents a point, or a vector in **n-space**. These spaces are alternatively also denoted by $R^1, R^2, R^3, R^4, \ldots, R^n$.

We shall denote a vector in n-space denoted by R^n by $u = (u_1, u_2, u_3, \ldots, u_n)$ and shall call $u_1, u_2, \ldots, u_n$ as its components.

Remark ....

A single point itself can be considered as zero space R^0.

3. Notation

Vectors are denoted in various ways. In 2-space we can denote a vector by the coordinates (x, y) of its end point or by using matrix notation by a row matrix $[x, y]$ or by a column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. We may also denote it by a single letter v or by the directed line segment $\overline{OP}$.

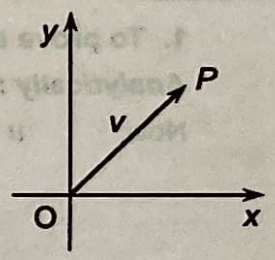

Fig. 6.2

In 3-space, in the same manner, we denote a vector by coordinates (x, y, z) or by a row matrix $[x, y, z]$ or by a column matrix $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. All these natations have the same meaning and hence

$$v = \overline{OP} = (x, y, z) = [x, y, z] = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

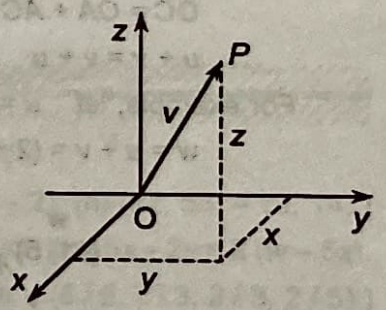

Fig. 6.3

In the same way a vector in R^n can be denoted by

$$u = (u_1, u_2, \ldots, u_n) = [u_1, u_2, \ldots, u_n] = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

In books vectors are denoted by lowercase bold face letters like **u, v, w, ...** etc. But this notation is not suitable while writing by hand. Some authors put bar over the letters $u, v, w, \ldots$ like $\bar{u}, \bar{v}, \bar{w}, \ldots$ but some authors use plane letters $u, v, w, \ldots$ Since in this book we are dealing with vectors and since there is no possibility of confusion, we shall follow the second method and use plane letters $u, v, w, \ldots$ for vectors.

The starting point of a vector is called the **initial point** and the end-point of a vector is called the **terminal point**. If the initial point of a vector, as in the above vectors is the origin, then the **coordinates** of the **terminal point** of the vector are called the **components** of the vector.

4. Vector Arithmetic

Addition, subtraction, the scalar multiplication of vectors in 2 and 3 space follow the following rules. If u, v, w are vectors and k, l are scalars then

(1) $u + v = v + u$	(2) $u + (v + w) = (u + v) + w$	(3) $u + 0 = 0 + u = u$
(4) $u + (-u) = 0$	(5) $k(u + v) = ku + kv$	(6) $(k + l)u = ku + lu$
(7) $k(lu) = (kl)u$	(8) $1 \cdot u = u$.	

Proof : Since a vector can be represented by ordered scalars or by line segments, we have two approaches to prove the above results - one is analytical and the other is geometrical.

We shall prove the first result by two methods in R^3 and leave the others for you as an exercise.

1. To prove $u + v = v + u$

Analytically : Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$

Now,
$$\begin{aligned} u + v &= (u_1, u_2, u_3) + (v_1, v_2, v_3) \\ &= (u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) \\ &= (v_1 + u_1) + (v_2 + u_2) + (v_3 + u_3) \\ &= v + u \end{aligned}$$

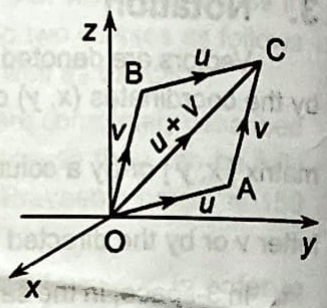

Fig. 6.4

Geometrically : Let $OA = u$ and $OB = v$. Then by completing the parallelogram $OACB$, we see that

$$OC = OA + AC = u + v \quad \text{and} \quad OC = OB + BC = u + v$$

$$\therefore \; u + v = v + u$$

For example, let $u = (2, 3)$ and $v = (-3, 2)$, then

$$w = u + v = (2 + (-3), 3 + 2) = (-1, 5)$$

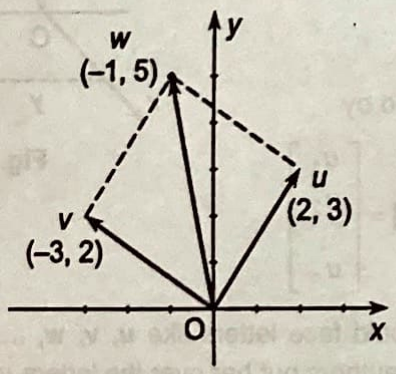

Fig. 6.5 (a)

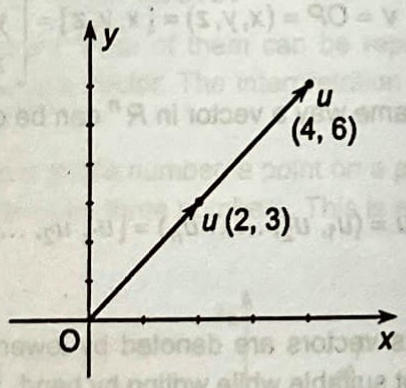

Fig. 6.5 (b)

Similarly, $2u$ means a vector $(2 \times 2, 2 \times 3)$ *i.e.* $(4, 6)$.

Example 1 : If $u = (2, 1, -4)$ and $v = (-1, 3, 5)$, then by the above rules

$$u + v = (2 + (-1), 1 + 3, -4 + (5)) = (1, 4, 1)$$
$$-v = -(-1, 3, 5) = (1, -3, -5)$$
$$u - v = u + (-v) = (2, 1, -4) + (1, -3, -5) = (3, -2, -9)$$
$$2u = 2(2, 1, -4) = (4, 2, -8)$$
$$-3v = -3(-1, 3, 5) = (3, -9, -15)$$

(a) Equality of Vectors

Definition : If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are two vectors in R^n then they are said to be **equal** if their components are equal

i.e.,

$$\boxed{u = v \text{ if } u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n}$$

Zero vector : A vector whose all n-components are zero is called a zero vector and is denoted by O. Thus,

$$\boxed{O = (0, 0, \ldots, 0)}$$

(b) Scalar Multiple

If k is any scalar and u is vector then the scalar multiple of u, denoted by ku, is given by

$$\boxed{ku = (ku_1, ku_2, \ldots, ku_n)}$$

(c) Sum of Two vectors

If u and v are two vectors their sum, denoted by $u + v$ is defined by

$$\boxed{u + v = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n)}$$

(d) Difference of Two Vectors

If u and v are two vectors then their difference, denoted by $u - v$ is defined by

$$\boxed{u - v = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n)}$$

EXERCISE - I

1. If $u = (-3, 2, 1, 0)$ and $v = (4, 7, -3, 2)$, find **(i)** $u - v$, **(ii)** $2u + 7v$.

   [Ans. : (i) (–7, – 5, 4, – 2), (ii) (22, 53, – 19, 14)]

2. If $v = (4, 7, -3, 2)$ and $w = (5, -2, 8, 1)$, find another vector x, such that $5x - 2v = 2(w - 5x)$.

   [Ans. : (6 / 5, 2 / 3, 2 / 3, 2 / 5)]

5. Norm of a Vector

Definition : The **length** of vector $u\ (u_1, u_2)$ or the **magnitude** of the vector $u\ (u_1, u_2)$ is called the **norm** of the vector u and the **norm** is denoted by $\| u \|$.

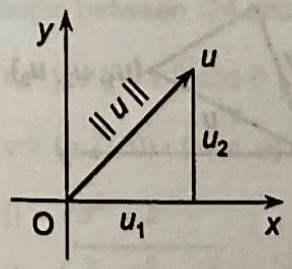

Fig. 6.6 (a)

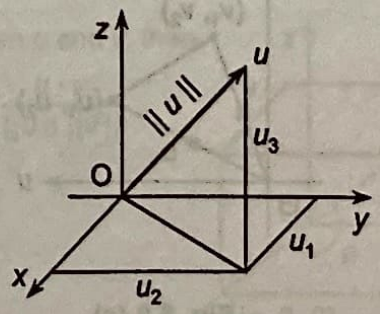

Fig. 6.6 (b)

By Pythagorean Theorem in R^2, $\boxed{\|u\| = \sqrt{u_1^2 + u_2^2}}$

In R^3, the **length** of the vector $u\ (u_1, u_2, u_3)$ or the **magnitude** of the vector *i.e.*, the **norm** of $u\ (u_1, u_2, u_3)$ is given by

$$\boxed{\|u\| = \sqrt{u_1^2 + u_2^2 + u_3^2}}$$

Generalising the above result, we define the norm of a vector $u = (u_1, u_2, \ldots\ldots, u_n)$ in R^n as

$$\boxed{\|u\| = \sqrt{u_1^2 + u_2^2 + \ldots\ldots + u_n^2}}$$

6. Dot Product (Inner Product)

Definition : If u and v are any two non-zero vectors in 2 and 3 space and if θ is the angle between $(0 \le \theta \le \pi)$ them, the **dot product** or the **(Euclidean) inner product** of u and v denoted by $u \cdot v$ or by $\langle u, v \rangle$ is given by

$$\boxed{u \cdot v = \|u\|\,\|v\| \cos\theta}$$

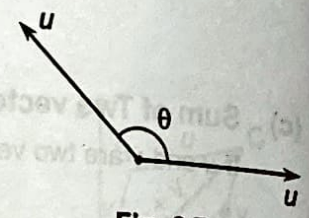

Fig. 6.7

Remark ....

The dot product or the inner product is also called the scalar product because the result is a pure number.

Corollaries : If u, v are two vectors in R^2 or R^3, then

(i) $u \cdot u = \|u\|^2$ or $\|u\| = \sqrt{u \cdot u}$

(ii) If $u \cdot v > 0$, θ is acute,

if $u \cdot v < 0$, θ is obtuse,

if $u \cdot v = 0$, $\theta = \pi/2$.

7. Dot Product OR Inner Product in Terms of Components

If $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are two vectors in R^2 then the **inner product** is given by

$$\boxed{u \cdot v = u_1 v_1 + u_2 v_2}$$

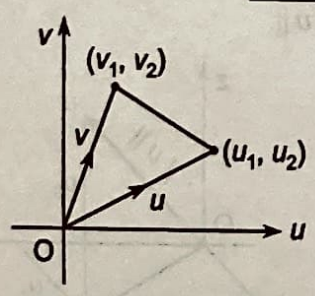

Fig. 6.8 (a)

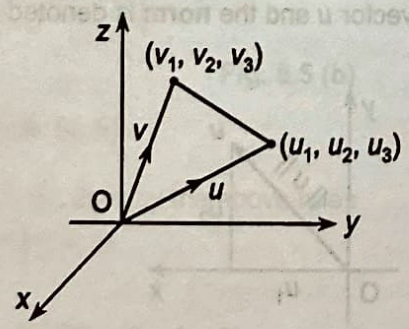

Fig. 6.8 (b)

If $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are two vectors in R^3 then the **inner product** is given by

$$\boxed{u \cdot v = u_1v_1 + u_2v_2 + u_3v_3}$$

Example : If $u = (1, -2, 4)$, $v = (-2, 4, 6)$, $w = (-3, 4, -5)$, find $u \cdot v$, $v \cdot w$.

Sol. :
$$u \cdot v = (1)(-2) + (-2)(4) + (4)(6) = -2 - 8 + 24 = 14$$
$$v \cdot w = (-2)(-3) + (4)(4) + (6)(-5) = 6 + 16 - 30 = -8.$$

8. Angle Between Two Vectors

If θ $(0 \le \theta \le \pi)$ is the angle between two vectors u, v then the angle is given by

$$\boxed{\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}}$$

The result is clear from § 6.

Example 1 : Find the angle between the vectors

(i) $u = (6, 2, 2)$, $v = (3, 0, -9)$; **(ii)** $u = (2, 1, -4)$, $v = (1, -2, 3)$

Sol. : If θ is the angle between two vectors u, v, then $\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$.

(i) Now, $\|u\| = \sqrt{36+4+4} = \sqrt{44}$; $\|v\| = \sqrt{9+0+81} = \sqrt{90}$

$u \cdot v = 18 + 0 - 18 = 0$

$\therefore \cos\theta = \dfrac{0}{\sqrt{44}\,\sqrt{90}} = 0 \quad \therefore \theta = \dfrac{\pi}{2} \quad \therefore$ The vectors are orthogonal.

(ii) $\|u\| = \sqrt{4+1+16} = \sqrt{21}$; $\|v\| = \sqrt{1+4+9} = \sqrt{14}$

$u \cdot v = 2 - 2 - 12 = -12$

$\therefore \cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|} = \dfrac{-12}{\sqrt{21}\,\sqrt{14}} = -\dfrac{12}{7\sqrt{6}} \quad \therefore \theta$ is obtuse.

Example 2 : Find the angle between a diagonal of a cube and one of its edges.

Sol. : Without loss of generality, let one of the vertex of the cube of side a be $A(a, 0, 0)$. Then the end of the diagonal from O is $B(a, a, a)$.

Now, let $u = (a, 0, 0)$ and $v = (a, a, a)$.

If θ is the angle between OA and OB *i.e.*, between u and v, then

$$u \cdot v = \|u\|\,\|v\| \cos\theta \quad \therefore \cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|}$$

Now, $u \cdot v = (a, 0, 0) \cdot (a, a, a) = a^2$

$\|u\| = \sqrt{a^2} = a$

$\|v\| = \sqrt{a^2 + a^2 + a^2} = \sqrt{3} \cdot a$

$\therefore \cos\theta = \dfrac{a^2}{a \cdot \sqrt{3}\,a} = \dfrac{1}{\sqrt{3}} \quad \therefore \theta = \cos^{-1}\left(\dfrac{1}{\sqrt{3}}\right)$

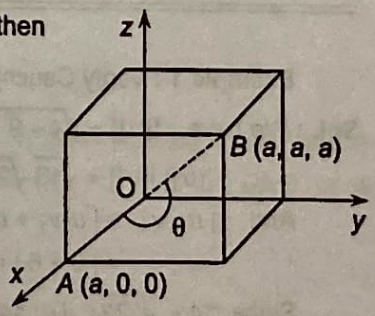

Fig. 6.9

Example 3 : Let R^4 have Euclidean inner product. Find the cosine of the angle between vectors $u = (4, 3, 1, -2)$ and $v = (-2, 1, 2, 3)$.

Sol. : We have

$$u \cdot v = (4, 3, 1, -2) \cdot (-2, 1, 2, 3)$$
$$= (4)(-2) + (3)(1) + (1)(2) + (-2)(3)$$
$$= -8 + 3 + 2 - 6 = -9$$

$$\|u\| = \sqrt{16+9+1+4} = \sqrt{30}; \qquad \|v\| = \sqrt{4+1+4+9} = \sqrt{18}$$

$$\therefore \cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|} = \frac{-9}{\sqrt{30}\,\sqrt{18}} = -\frac{9}{\sqrt{3 \cdot 5 \cdot 2}\,\sqrt{9 \cdot 2}}$$
$$= -\frac{9}{3 \times 2\sqrt{15}} = -\frac{3}{2\sqrt{15}}.$$

9. Cauchy-Schwartz Inequality in R^n

If $u = (u_1, u_2, u_3, \ldots, u_n)$ and $v = (v_1, v_2, v_3, \ldots, v_n)$ are any two vectors in R^n, Cauchy-Schwartz Inequality states that

$$\boxed{|u \cdot v| \le \|u\|\,\|v\|}$$ **(M.U. 2015)**

Proof : The inequality can be proved very easily for vectors in R^2 and R^3.

We know that if u, v are any two vectors in R^2 or R^3 then

$$|u \cdot v| = \|u\|\,\|v\|\,|\cos\theta|$$

But $|\cos\theta| \le 1$ $\quad\therefore\quad |u \cdot v| \le \|u\|\,\|v\|$

Note that the equality bolds good when $u = v$.

Herman Amandus Schwartz (1843-1921)

He was considered as a leading mathematician in the first part of twentieth century in Germany. He was professor of mathematics at Berlin university. He was a devout teacher and took interest in teaching important and trivial facts with equal thoroughness. His techniques were very clever and influenced the work of other mathematicians. The reknowned Cauchy-Schwartz inequality was published in 1885.

Example 1 : Verify Cauchy-Schwartz inequality for the vectors $u = (2, 3, 0)$ and $v = (4, 2, 1)$.

Sol. : We have $\|u\| = \sqrt{4+9+0} = \sqrt{13}$ and $\|v\| = \sqrt{16+4+1} = \sqrt{21}$

$$\therefore \|u\|\,\|v\| = \sqrt{13}\,\sqrt{21} = \sqrt{273}$$

And $|u \cdot v| = |u_1v_1 + u_2v_2 + u_3v_3| = |(2)(4) + (3)(2) + (0)(1)|$

$$= |8 + 6| = 14$$

Since $14 < \sqrt{273}$ $\quad[\because 14^2 = 196 < 273]$

$\therefore$ Cauchy-Schwartz inequality is verified.

Example 2 : Verify Cauchy-Schwartz inequality for the vectors

$$u = (-4, 2, 1) \text{ and } v = (8, -4, -2).$$

(M.U. 2014, 15, 18, 19)

Sol. : We have $\|u\| = \sqrt{16+4+1} = \sqrt{21}$ and $\|v\| = \sqrt{64+16+4} = \sqrt{84}$

$$\therefore \quad \|u\|\,\|v\| = \sqrt{21}\sqrt{84} = \sqrt{21}\sqrt{21}\sqrt{4} = 21 \times 2 = 42$$

And
$$\begin{aligned} |u \cdot v| &= |u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4| \\ &= |(-4)(8) + (2)(-4) + (1)(-2)| \\ &= |-32 - 8 - 2| = 42 \end{aligned}$$

$$\therefore \quad \|u\|\,\|v\| = |u \cdot v|$$

By Cauchy-Schwartz inequality we should have $|u \cdot v| \le \|u\|\,\|v\|$.

Hence, Cauchy-Schwartz inequality holds good for the given vectors.

Example 3 : Verify Cauchy-Schwartz inequality for $u = (2, 3, 1)$ and $v = (3, 0, 4)$. Also find the angle between u and v.

(M.U. 2016)

Sol. : (i) We have $\|u\| = \sqrt{4+9+1} = \sqrt{14}$ and $\|v\| = \sqrt{9+0+16} = \sqrt{25} = 5$

$$\therefore \quad \|u\|\,\|v\| = \sqrt{14}\sqrt{25} = 5\sqrt{14}$$

And
$$\begin{aligned} |u \cdot v| &= |u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4| \\ &= |(2)(3) + (3)(0) + (1)(4)| \\ &= |6 + 0 + 4| = |10| = 10 \end{aligned}$$

Since $10 < 5\sqrt{14} = 2 < \sqrt{14}$, which is true, Cauchy-Schwartz inequality is verified.

(ii) Now angle θ between u and v is given by

$$\cos\theta = \frac{u \cdot v}{\|u\|\,\|v\|} = \frac{10}{5\sqrt{14}} = \frac{2}{\sqrt{14}}$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{2}{\sqrt{14}}\right).$$

Example 4 : For real values of a, b and θ, show that $(a\cos\theta + b\sin\theta)^2 \le a^2 + b^2$ using Cauchy-Schwartz inequality.

(M.U. 2017)

Sol. : Consider two vectors $u = (a, b)$ and $v = (\cos\theta, \sin\theta)$.

Then $u \cdot v = (a, b) \cdot (\cos\theta, \sin\theta) = a\cos\theta + b\sin\theta$

And $\|u\|^2 = a^2 + b^2$ and $\|v\|^2 = \cos^2\theta + \sin^2\theta = 1$

Now, by Cauchy-Schwartz inequality $|u \cdot v| \le \|u\|\,\|v\|$.

$$\therefore \quad |a\cos\theta + b\sin\theta| \le \sqrt{a^2+b^2} \cdot 1$$

Taking squares, we get $(a\cos\theta + b\sin\theta)^2 \le (a^2 + b^2)$.

Example 5 : If $x, y, z > 0$, show that $(x^2 + y^2 + z^2)^{1/2} \ge \frac{1}{13}(3x + 4y + 12z)$ using Cauchy-Schwartz inequality.

(M.U. 2014)

Sol. : Consider two vectors $u = (x, y, z)$ and $v = (3, 4, 12)$.

Then $u \cdot v = (x, y, z) \cdot (3, 4, 12) = 3x + 4y + 12z$

And $\|u\|^2 = x^2 + y^2 + z^2$ and $\|v\|^2 = 3^2 + 4^2 + 12^2 = 169 = 13^2$

Now, by Cauchy-Schwartz Inequality $|u \cdot v| \le \|u\| \, \|v\|$.

$$\therefore \quad (3x + 3y + 12z) \le \sqrt{x^2 + y^2 + z^2} \cdot \sqrt{13^2}$$

$$\therefore \quad (3x + 3y + 12z) \le (x^2 + y^2 + z^2)^{1/2} \cdot 13$$

$$\therefore \quad \frac{1}{13}(3x + 3y + 12z) \le (x^2 + y^2 + z^2)^{1/2}$$

Example 6 : Using Cauchy-Schwartz inequality, prove that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \ge 6$.

Sol. : Consider two vectors $u = (\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right)$.

$$\therefore \quad u \cdot v = (\sqrt{a}, \sqrt{b}, \sqrt{c}) \cdot \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right) = \sqrt{a} \cdot \frac{1}{\sqrt{a}} + \sqrt{b} \cdot \frac{1}{\sqrt{b}} + \sqrt{c} \cdot \frac{1}{\sqrt{c}}$$

$$= 1 + 1 + 1 = 3$$

Now, $\|u\|^2 = a + b + c$ and $\|v\|^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

By Cauchy-Schwartz inequality $(u \cdot v)^2 \le \|u\|^2 \, \|v\|^2$.

$$\therefore \quad 3^2 \le (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

$$\le \frac{a+b+c}{a} + \frac{a+b+c}{b} + \frac{a+b+c}{c}$$

$$\le \frac{b+c}{a} + 1 + \frac{a+c}{b} + 1 + \frac{a+b}{c} + 1$$

$$\therefore \quad 9 \le 3 + \left(\frac{a+b}{c} + \frac{b+c}{c} + \frac{c+a}{b}\right)$$

$$\therefore \quad 6 \le \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b}$$

Example 7 : If a, b, c are three positive real numbers, then using Cauchy-Schwartz inequality prove that $(a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 3^2$.

Sol. : Consider two vectors $u = (\sqrt{a}, \sqrt{b}, \sqrt{c})$ and $v = \left(\frac{1}{\sqrt{a}}, \frac{1}{\sqrt{b}}, \frac{1}{\sqrt{c}}\right)$

then as seen above by Cauchy-Schwartz inequality,

$$3^2 \le (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

Note ....

The result can be generalised further and we can show that

$$(a_1 + a_2 + \ldots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right) \ge n^2$$

The proof is left to you.

10. Relation Between Norm and Inner Product

If u, v are two vectors then

$$\boxed{u \cdot v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2} \quad \text{..... (A)}$$

Proof : By definition

$$\|u+v\|^2 = (u+v)\cdot(u+v)$$
$$= \|u\|^2 + 2(u\cdot v) + \|v\|^2 \quad \text{..... (1)}$$

And
$$\|u-v\|^2 = (u-v)\cdot(u-v)$$
$$= \|u\|^2 - 2(u\cdot v) + \|v\|^2 \quad \text{..... (2)}$$

By subtraction (2) from (1), we get

$$\|u+v\|^2 - \|u-v\|^2 = 4(u\cdot v)$$

Hence, the result.

By adding (1) and (2), we get another result

$$\boxed{\|u+v\|^2 + \|u-v\|^2 = 2[\|u\|^2 + \|v\|^2]}$$

11. Orthogonality

Definition : Two vectors u, v are said to be **orthogonal** if

$$\boxed{u \cdot v = 0}$$

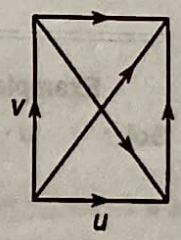

Fig. 6.10

Cor. 1 : From the result (A) proved, we get another condition for orthogonality.

If u and v are two orthogonal vectors, then from (A), we get

$$\|u+v\| = \|u-v\| \qquad (\because\ u\cdot v = 0)$$

In R^2 this means, if u and v are the sides of a rectangle then the diagonals of a rectangle are equal.

Cor. 2 : If u and v are orthogonal vectors then the distance between them is given by

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2$$

This is so because

$$\|u-v\|^2 = (u-v)\cdot(u-v)$$
$$= \|u\|^2 - 2(u\cdot v) + \|v\|^2$$
$$= \|u\|^2 + \|v\|^2 \qquad [\because\ u\cdot v = 0]$$

12. Euclidean Inner Product

We can generalise the concept of inner product given in § 7 to R^n.

Definition : If $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ are any two vectors in R^n then the **(Euclidean) Inner product** of u, v denoted by $u \cdot v$ is defined by

$$\boxed{u \cdot v = u_1v_1 + u_2v_2 + \ldots + u_nv_n}$$

Remark ....

Inner product $u \cdot v$ is also denoted by $\langle u, v \rangle$.

Cor. : Two vectors are orthogonal if $\boxed{u \cdot v = 0}$

Euclid (300?-275? B.C.)

He taught at Alexandria and founded the Alexandrian School of Mathematics. Euclid is called **"the father of geometry"**. When the king of Alexandria (Egypt) Ptolemy I asked Euclid if there is an easy way to learn geometry, he replied "there is no royal road to geometry". Euclid wrote thirteen books on algebra and geometry mostly compiling the works of earlier mathematicians but had made also significant contributions of his own. Algebra and geometry that is taught at school level all over the world is based on his books. It is said that his books are read and studied only next to the Bible. More than 2000 editions of his book are published in various languages since the first print in 1482. Unfortunately very little is known about this great mathematician.

Example 1 : Find the inner product of $u = (-2, 4, 0, -3)$, $v = (3, 4, 6, -4)$.

Sol. :
$$u \cdot v = (-2)(3) + (4)(4) + (0)(6) + (-3)(-4)$$
$$= -6 + 16 + 12 = 22$$

Example 2 : If R^3 has Euclidean inner product and $u = (4, -3, 2)$, $v = (5, 4, -4)$, show that u and v are orthogonal.

Sol. : u, v are orthogonal if $u \cdot v = 0$.

Now, $u \cdot v = 4 \times 5 + (-3) \times 4 + (2) \times (-4)$
$$= 20 - 12 - 8 = 0$$

Hence, u and v are orthogonal.

Example 3 : Let R^3 have the Euclidean inner product. Find the values of k for which $u = (k, k, 1)$ and $v = (k, 5, 6)$ are orthogonal.

Sol. : Two vectors are orthogonal if $u \cdot v = 0$.

$\therefore \ (k, k, 1) \cdot (k, 5, 6) = 0 \qquad \therefore \ k^2 + 5k + 6 = 0$

$\therefore \ (k + 2)(k + 3) = 0$

Hence, $k = -2$ or -3.

Many of the results with which we are familiar in 2-space and 3-space are true in R^n also. For this reason R^n is often referred to as **Euclidean *n*-space.**

Example 4 : Check whether the following vectors are orthogonal

(i) $u = (-2, 3, 4)$, $v = (2, 4, -2)$

(ii) $u = (-4, 6, -10, 1)$, $v = (2, 1, -2, 9)$

Sol. : (i) We have

$$\langle u, v\rangle = u \cdot v = (-2, 3, 4) \cdot (2, 4, -2)$$
$$= -4 + 12 - 8 = 0$$

Hence, u, v are orthogonal.

(ii)
$$\langle u, v\rangle = u \cdot v = (-4, 6, -10, 1) \cdot (2, 1, -2, 9)$$
$$= -8 + 6 + 20 + 9 = 27 \neq 0$$

Hence, u, v are not orthogonal.

Example 5 : If R^3 has the Euclidean inner product, find k such that u, v are orthogonal where

(a) $u = (k, k, 2)$, $v = (k, -5, 3)$; (b) $u = (2, 1, 3)$, $v = (1, 7, k)$

Sol. : Since u, v are orthogonal $\langle u, v\rangle = 0$.

(a)
$$u \cdot v = (k, k, 2) \cdot (k, -5, 3)$$
$$= k(k) + k(-5) + (2)\,3 = 0$$
$$\therefore\ k^2 - 5k + 6 = 0 \qquad \therefore\ (k-3)(k-2) = 0$$
$$\therefore\ k = 3, 2$$

(b)
$$u \cdot v = (2, 1, 3) \cdot (1, 7, k)$$
$$= 2(1) + 1(7) + 3(k) = 0$$
$$\therefore\ 3k + 9 = 0 \qquad \therefore\ k = -3.$$

13. Properties of Euclidean Inner Product

If $u = (u_1, u_2, ..., u_n)$, $v = (v_1, v_2, ..., v_n)$ and $w = (w_1, w_2, ..., w_n)$ are vectors in R^n and k is a scalar, then

(1) $u \cdot v = v \cdot u$	**(2)** $(u + v) \cdot w = u \cdot w + v \cdot w$
(3) $(ku) \cdot v = k(u \cdot v)$	**(4)** $v \cdot v \geq 0$ **if** $v \neq 0$ **and** $v \cdot v = 0$ **if and only if** $v = 0$

14. Unit Vector

A vector whose norm is equal to one is called a **unit vector**. It is denoted by $\hat{u}$. It is easy to see that a unit vector is in the direction of the vector u and is obtained by dividing the vector u by its norm $\|u\|$. Thus, unit vector in the direction u is

$$\boxed{\hat{u} = \frac{1}{\|u\|} u}$$

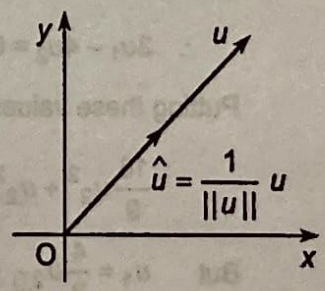

Fig. 6.11

For example, the vectors

$(1/\sqrt{2}, 1/\sqrt{2})$, $(\sqrt{2/3}, 1/\sqrt{3})$,

$(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $(1/2, 1/\sqrt{2}, 1/2)$,

$(1/\sqrt{14}, 2/\sqrt{14}, 3/\sqrt{14})$

are unit vectors. You can write some more unit vectors.

For example, if $u = (2, 4, -5)$, then $\|u\| = \sqrt{4 + 16 + 25} = 3\sqrt{5}$.

The unit vector in the direction of u denoted by $\hat{u}$ is obtained by dividing each component by the norm. Thus, the unit vector

$$\hat{u} = \left(2/3\sqrt{5},\ 4/3\sqrt{5},\ -5/3\sqrt{5}\right)$$

Example 1 : Show that $u = (a, b)$ and $v = (-b, a)$ are orthogonal. Find two vectors that are orthogonal to $(2, -5)$. Generalise your result.

Sol. : For the first part see that

$$u \cdot v = (a, b) \cdot (-b, a) = -ab + ba = 0$$

$\therefore$ u, v are orthogonal.

From this we see that in R^2, a vector orthogonal to $u\ (a, b)$ is obtained by interchanging their positions and changing the sign of one of them.

$\therefore$ Vectors orthogonal to $(2, -5)$ are $(5, 2)$ and $(-5, -2)$ *i.e.* $(5k, 2k)$ for any scalar k.

Example 2 : Find two vectors in R^2 with Euclidean norm 1 where Euclidean inner product with $(3, -1)$ is zero.

Sol. : Let the required vector be $u = (u_1, u_2)$.

Then by data $\sqrt{u_1^2 + u_2^2} = 1$ and $(u_1, u_2) \cdot (3, -1) = (0, 0)$

$\therefore\ 3u_1 - u_2 = 0 \qquad \therefore\ u_2 = 3u_1$

But $\quad u_1^2 + u_2^2 = 1 \qquad \therefore\ u_1^2 + 9u_1^2 = 1$

$\therefore\ 10u_1^2 = 1 \qquad \therefore\ u_1 = \pm\dfrac{1}{\sqrt{10}} \qquad \therefore\ u_2 = 3u_1 = \pm\dfrac{3}{\sqrt{10}}.$

Hence, the two vectors are $\left(\dfrac{1}{\sqrt{10}}, \dfrac{3}{\sqrt{10}}\right)$ and $\left(-\dfrac{1}{\sqrt{10}}, -\dfrac{3}{\sqrt{10}}\right)$.

Example 3 : Find a vector in R^2 with norm 1 and whose inner product with $(3, -4)$ is zero.

Sol. : Let the required vector be $u = (u_1, u_2)$.

By data its norm = 1.

$\therefore\ \sqrt{u_1^2 + u_2^2} = 1 \qquad \therefore\ u_1^2 + u_2^2 = 1$ ..... (1)

Let $v = (3, -4)$.

By data $u \cdot v = 0 \qquad \therefore\ (u_1, u_2) \cdot (3, -4) = 0$

$\therefore\ 3u_1 - 4u_2 = 0 \qquad \therefore\ u_1 = \dfrac{4}{3}u_2.$

Putting these values in (1), we get

$$\frac{16}{9}u_2^2 + u_2^2 = 1 \qquad \therefore\ \frac{25u_2^2}{9} = 1 \qquad \therefore\ u_2 = \pm\frac{3}{5}$$

But $\quad u_1 = \dfrac{4}{3}u_2 \qquad \therefore\ u_1 = \pm\dfrac{4}{5}$

Hence, the required vectors are $\left(\dfrac{4}{5}, \dfrac{3}{5}\right), \left(-\dfrac{4}{5}, -\dfrac{3}{5}\right)$.

Example 4 : Find a unit vector in R^3 orthogonal to both $u = (1, 0, 1)$ and $v = (0, 1, 1)$.

Sol. : Let $x = (x_1, x_2, x_3)$ be the vector orthogonal to both u and v. Then

$$x \cdot u = (x_1, x_2, x_3) \cdot (1, 0, 1) = x_1 + x_3 = 0$$

and $$x \cdot v = (x_1, x_2, x_3) \cdot (0, 1, 1) = x_2 + x_3 = 0$$

$$\therefore \quad x_1 = -x_3 = x_2 \qquad \therefore \quad x = (x_1, x_1, -x_1)$$

$$\therefore \quad ||x|| = \sqrt{x_1^2 + x_1^2 + x_1^2} = \pm\sqrt{3} \cdot x_1$$

$$\therefore \text{ Unit vector orthogonal to both } = \frac{(x_1, x_1, -x_1)}{\pm\sqrt{3} \cdot x_1} = \pm\frac{(1, 1, -1)}{\sqrt{3}}.$$

Example 5 : Find a vector orthogonal to both $u = (-6, 4, 2)$, $v = (3, 1, 5)$.

(M.U. 2014, 15, 17, 18)

Sol. : Let $x = (x_1, x_2, x_3)$ be the vector orthogonal to both u and v. Then

$$x \cdot u = (x_1, x_2, x_3) \cdot (-6, 4, 2) = 0 \qquad \therefore \quad -6x_1 + 4x_2 + 2x_3 = 0$$

and $$x \cdot v = (x_1, x_2, x_3) \cdot (3, 1, 5) = 0 \qquad \therefore \quad 3x_1 + x_2 + 5x_3 = 0$$

By Cramer's rule, (See note at the bottom.)

$$\frac{x_1}{20-2} = \frac{-x_2}{-30-6} = \frac{x_3}{-6-12}$$

$$\therefore \quad \frac{x_1}{18} = -\frac{x_2}{-36} = \frac{x_3}{-18} \qquad \therefore \quad \frac{x_1}{1} = \frac{x_2}{2} = -\frac{x_3}{1}$$

Hence, $(1, 2, -1)$ is orthogonal to both vectors.

Example 6 : Find all vectors in R^3 of Euclidean norm 1 that are orthogonal to the vectors $u_1 = (1, 1, 1)$ and $u_2 = (1, 1, 0)$.

Sol. : Let the required vector be (v_1, v_2, v_3). Since it is orthogonal to u_1 and u_2, we have

$$u_1 \cdot v = (1, 1, 1) \cdot (v_1, v_2, v_3) = v_1 + v_2 + v_3 = 0$$

and $$u_2 \cdot v = (1, 1, 0) \cdot (v_1, v_2, v_3) = v_1 + v_2 + 0 = 0.$$

$\therefore$ By Cramer's Rule,

$$\frac{v_1}{0-1} = -\frac{v_2}{0-1} = \frac{v_3}{1-1} = k \text{ say}$$

$$\therefore \quad v_1 = -k, \; v_2 = k, \; v_3 = 0 \qquad \text{..... (1)}$$

Since the norm of v_1 is 1,

$$v_1^2 + v_2^2 + v_3^2 = 1 \qquad \therefore \quad k^2 + k^2 + 0 = 1$$

$$\therefore \quad 2k^2 = 1 \qquad \therefore \quad k = \pm\frac{1}{\sqrt{2}}.$$

Taking $k = \frac{1}{\sqrt{2}}$, we have from (1) $\quad v_1 = -\frac{1}{\sqrt{2}}, \; v_2 = \frac{1}{\sqrt{2}}, \; v_3 = 0.$

Taking $k = -\frac{1}{\sqrt{2}}$, we have from (1) $\quad v_1 = \frac{1}{\sqrt{2}}, \; v_2 = -\frac{1}{\sqrt{2}}, \; v_3 = 0.$

Example 7 : Find a unit vector orthogonal to both $(1, 1, 0)$ and $(0, 1, 1)$. **(M.U. 2015, 16)**

Sol. : Let the required vector be (v_1, v_2, v_3).

Since it is orthogonal to both the given vectors u_1 and u_2.

$$u_1 \cdot v = (1, 1, 0) \cdot (v_1, v_2, v_3) = v_1 + v_2 + 0 = 0$$

$$u_2 \cdot v = (0, 1, 1) \cdot (v_1, v_2, v_3) = 0 + v_2 + v_3 = 0$$

$\therefore$ By Cramer's Rule,

$$\frac{v_1}{1-0} = -\frac{v_2}{1-0} = \frac{v_3}{1-0} = k \text{ say}$$

$$\therefore \quad v_1 = k, \; v_2 = -k, \; v_3 = k.$$

Since the norm is 1,

$$v_1^2 + v_2^2 + v_3^2 = 1 \qquad \therefore \; k^2 + k^2 + k^2 = 1 \qquad \therefore \; k = \pm \frac{1}{\sqrt{3}}.$$

Taking $k = \frac{1}{\sqrt{3}}$, $\quad v_1 = \frac{1}{\sqrt{3}}, \; v_2 = -\frac{1}{\sqrt{3}}, \; v_3 = \frac{1}{\sqrt{3}}.$

Taking $k = -\frac{1}{\sqrt{3}}$, $\quad v_1 = -\frac{1}{\sqrt{3}}, \; v_2 = \frac{1}{\sqrt{3}}, \; v_3 = -\frac{1}{\sqrt{3}}.$

Example 8 : Find the unit vector in R^3 orthogonal to both $u = (1, 0, 1)$ and $v = (0, 1, 1)$.

(M.U. 2016)

Sol. : Let the required vector be $w = (w_1, w_2, w_3)$.

Since, it is orthogonal to both the given vectors u and v,

$$u \cdot w = (1, 0, 1) \cdot (w_1, w_2, w_3) = w_1 + 0 + w_3 = 0$$

and $\quad v \cdot w = (0, 1, 1) \cdot (w_1, w_2, w_3) = 0 + w_2 + w_3 = 0$

By Cramer's rule,

$$\frac{w_1}{0-1} = -\frac{w_2}{1-0} = \frac{w_3}{1-0} = k \text{ say}$$

$$\therefore \quad w_1 = -k, \; w_2 = -k, \; w_3 = k.$$

Since, the norm of $w = 1$,

$$w_1^2 + w_2^2 + w_3^2 = 1 \quad \therefore \; k^2 + k^2 + k^2 = 1 \quad \therefore \; k = \pm \frac{1}{\sqrt{3}}.$$

Taking $k = \frac{1}{\sqrt{3}}$, $\quad w_1 = -\frac{1}{\sqrt{3}}, \; w_2 = -\frac{1}{\sqrt{3}}, \; w_3 = \frac{1}{\sqrt{3}}$

Taking $k = -\frac{1}{\sqrt{3}}$, $\quad w_1 = \frac{1}{\sqrt{3}}, \; w_2 = \frac{1}{\sqrt{3}}, \; w_3 = -\frac{1}{\sqrt{3}}.$

$$\therefore \quad w = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ or } w = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right).$$

Note ....

Cramer's Rule : We know that the roots of the equations

$a_1x + b_1y = d_1$; $\quad a_2x + b_2x = d_2$ $\quad$ are given by

$$x = \frac{d_1 b_2 - d_2 b_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} d_1 & b_1 \\ d_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_x}{D}; \quad y = \frac{a_1 d_2 - a_2 d_1}{a_1 b_2 - a_2 b_1} = \frac{\begin{vmatrix} a_1 & d_1 \\ a_2 & d_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} = \frac{D_y}{D}$$

This is known as Cramer's rule. The rule is highly convenient to solve simultaneous equations in two and three unknonws.

The rule states that if

(i) D is the determinant of the coefficients.

(ii) D_x = What D becomes when the coefficients of x are replaced by d's.

(iii) D_y = What D becomes when the coefficients of y are replaced by d's then

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}.$$

Remark ....

Before applying Cramer's rule it is necessary to see that the constant terms are transferred on the right hand side.

Three Unknonws : If we are given

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3,$$

then $x = \dfrac{D_x}{D}, \quad y = \dfrac{D_y}{D}, \quad z = \dfrac{D_z}{D}$

where, $$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

i.e. the determinant of the coefficients of x, y, z.

$$D_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

i.e. the determinant obtained from D by replacing a's by d's.

$$D_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$$

i.e. the determinant obtained from D by replacing b's by d's.

$$D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

i.e. the determinant obtained by D by replacing c's by d's.

Modification : If we have $a_1 x + b_1 y + c_1 z = 0$

$$a_2 x + b_2 y + c_2 z = 0$$

Then, by eliminating y first and then eliminating x, we get

$$\frac{x}{b_1 c_2 - b_2 c_1} = -\frac{y}{a_1 c_2 - a_2 c_1} = \frac{z}{a_1 b_2 - a_2 b_1}.$$

Gabriel Cramer (1704 - 1752)

Cramer was a Swiss Mathematician. He has earned a well deserved place for spreading mathematical ideas all over the world. Cramer travelled extensively and met many leading mathematicians of his day.

His most widely known work "Introduction a'l'analyse des lignes courbes alge'briques" was a study of classification of curves. He is known for the rule of solving simultaneous equations. He also wrote on philosophy of law and government, and the history of mathematics. Cramer received numerous honours for his activities.

EXERCISE - II

1. Find the Euclidean norms of the following vectors
   (i) $(3, -4)$; (ii) $(6, 8)$; (iii) $(1, 2, -2)$; (iv) $(2, 3, 0, -6)$
   [Ans. : (i) 5, (ii) 10, (iii) 3, (iv) 7.]
2. Find the Euclidean norms of the following vectors.
   (i) $(1, -2, -2)$; (ii) $(3, -4, 0, -12)$; (iii) $(2, 1, -1, 3, -4)$
   [Ans. : (i) 3, (ii) 13, (iii) $\sqrt{31}$.]
3. Find $u \cdot v$, where (i) $u = (2, 1, 3)$, $v = (-3, 2, 1)$
   (ii) $u = (-2, 2, 3)$, $v = (1, 7, -4)$ **[Ans. : (i) -1, (ii) 0]**
4. Show that the Euclidean norm of $\left(\frac{1}{\|u\|}\right)u$ where u is a non-zero vector in R^n is unity.
5. Find the values of k such that $\| kv \| = 3$ where $v = (2, -3, 0, 6)$. **[Ans. : $\pm 3/7$]**
6. Which of the following sets of vectors are orthogonal with Euclidean inner product.
   (a) (i) $(0, 1), (3, 0)$ (ii) $(1/\sqrt{3}, -1/\sqrt{3}), (1/\sqrt{3}, 1/\sqrt{3})$
   (iii) $(0, 0), (1, 1)$ (iv) $(-1/2, -1/2), (1/2, 1/2)$ in R^2 **[Ans. : All except (iv)]**
   (b) (i) $\left(\frac{1}{\sqrt{3}}, 0, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
   (ii) $\left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{2}}, \frac{1}{3}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, 0\right)$
   (iii) $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$
   (iv) $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1\right), \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{2}\right), \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$ **[Ans. : All except (iv)]**
7. Check whether the following vectors are orthogonal
   (i) $u = (-2, 3, 4)$, $v = (3, -2, 3)$
   (ii) $u = (2, 4, -3, 5)$, $v = (3, 2, 3, -1)$
   (iii) $u = (2, -1, 4, 2, 6)$, $v = (3, 2, 1, 3, -2)$ **[Ans. : (i) Yes, (ii) Yes, (ii) No]**

8. For which values of k are the following vectors orthogonal
   (i) $u = (k, k, 4)$, $v = (k, -7, 3)$
   (ii) $u = (k, -6, k, 8)$, $v = (k, k, -4, 2)$ [**Ans. : (i)** $k = 4, k = 3$; **(ii)** $k = 8, k = 2$]
9. Find the value of k, if $u = (2, 1, 3)$ and $v = (4, 7, k)$ are orthogonal. [**Ans. :** $k = -5$]
10. Find two vectors of unit norm which are orthogonal to (3, 4).
    [**Ans. :** $\left(\frac{4}{5}, -\frac{3}{5}\right), \left(-\frac{4}{5}, \frac{3}{5}\right)$]
11. Find two vectors orthogonal to $u = (a, 0, b)$. [**Ans. : (i)** $(-b, 0, a)$; **(ii)** $(b, 0, -a)$]
12. Find the Euclidean inner product $u \cdot v$ where
    (i) $u = (3, 1, 4, -2)$, $v = (2, 2, 0, 1)$
    (ii) $u = (1, -2, 3, 2)$, $v = (2, 1, 3, 0)$ [**Ans. : (i)** 6, **(ii)** 9]
13. Find two vectors in R^2 with Euclidean norm 1 whose Euclidean inner product with (3, – 4) is zero.
    [**Ans. :** $\left(\frac{4}{5}, \frac{3}{5}\right), \left(-\frac{4}{5}, -\frac{3}{5}\right)$]
14. If $\| u + v \| = 5$ and $\| u - v \| = 1$, find $u \cdot v$. [**Ans. :** 6]
15. Check whether
    (i) $u = (-1, 3, 2)$, $v = (4, 2, -1)$
    (ii) $u = (-4, 6, -10, 1)$, $v = (2, 2, -2, 9)$ are orthogonal. [**Ans. : (i)** Yes, **(ii)** No]
16. Verify the Cauchy-Schwartz inequality
    (i) $u = (4, 5)$, $v = (1, 3)$
    (ii) $u = (2, 1, -3)$, $v = (3, 4, -2)$
    (iii) $u = (2, 1, 3, -1)$, $v = (1, -2, 1, 2)$

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15. Vector Space : Introduction

We have learnt some properties of vectors in R^2, R^3, ... and also in R^n. We shall now consider the vector space generated by vectors in R^n.

We know that a point in two dimensional space is denoted by an ordered pair (x_1, x_2), in three dimensional space by an ordered triple (x_1, x_2, x_3). Similarly a point in four dimensional space is denoted by an ordered quadruple (x_1, x_2, x_3, x_4) in five dimensional space by an ordered quintuple $(x_1, x_2, x_3, x_4, x_5)$ and in n-dimensional space by an ordered n-tuple $(x_1, x_2, x_3, x_4, ..., x_n)$.

We know that a single real number can be used to denote a point on a real line. Conversely to every point on the real line there corresponds a real number. **The set of all points on the line is a one-dimensional space.** We also know that corresponding to any point on a plane (with a coordinate system) there is an ordered pair of real numbers (x, y) and corresponding to every ordered pair of real numbers (x, y), there is a point on the plane. **The set of all ordered pairs (x, y), $-\infty < x < \infty$, $-\infty < y < \infty$ is a two dimensional space.** Similarly, if (x, y, z) is an ordered triplet of real numbers, it corresponds to a point in three dimensional space or in R^3 and conversely for any point in R^3 there is an ordered triplet (x, y, z). Thus, one ordered triplet (x, y, z) gives us one point and **the set of all ordered triplets** (x, y, z) where $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$ **is three dimensional space** consisting of all points in the space.

We also know that an ordered pair (x_1, y_1) or an ordered triplet (x_1, y_1, z_1) have two different geometrical interpretations. It may represent a point in R^2 or R^3 or it may represent a vector in R^2 or R^3.

Definition : If n is a positive integer then the sequence $(x_1, x_2, ..., x_n)$ of n real numbers, called an **ordered n-tuple represents** (a point or) a vector in n-dimensional space denoted by R^n and the set of **all** ordered n-tuples represents n-dimensional space denoted by R^n.

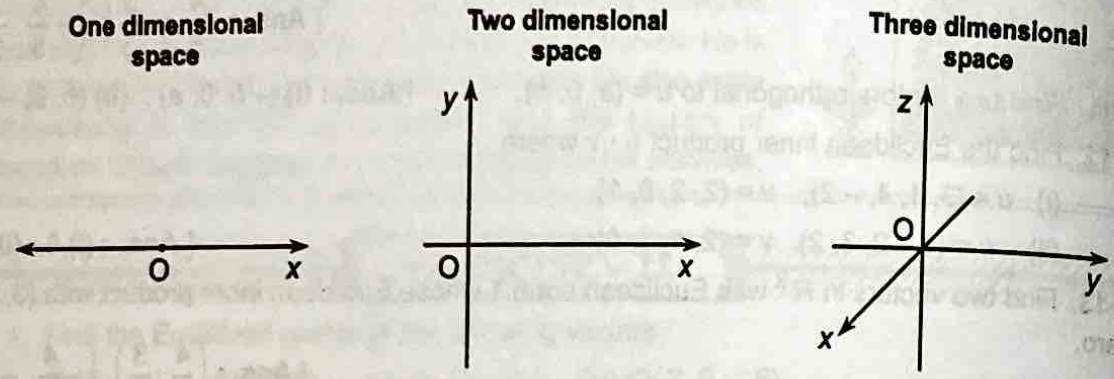

Fig. 6.11

16. Vector Space

Loosely speaking by space, we mean the set of all points which are represented in the same way. The set of the single point 0 is called the zero space, the set of all points on a line (with origin and a 'scale') is called 1-space, the set of points on a plane (with coordinate system defined) is called a 2-space, the set of points in three dimensional space (with a coordinate system) is called a 3-space. Although we are not familiar with spaces of higher dimension, examples of spaces of higher dimensions are not rare. Theory of relativity uses time as the fourth dimension. We can give another example as follows.

Suppose we want to discuss the expenses of constructing a building. We can then denote the cement used in the construction of the building by x_1, land by x_2, rubble by x_3, bricks by x_4, steel by x_5, labour by x_6, supervision by x_7, other expenses by x_8, then we are dealing with 8-space. Total expenditure then can be expressed in terms of these 8 variables.

Definition : Let V be a non-empty set of elements called **vectors** and k, m be scalars. If the set V satisfies the following axioms it is called a **vector space.**

17. Axioms of Vector Space

(a) Closure Axioms

C_1 : If u and v are in V then $u + v$ is in V (V is closed under addition).

C_2 : If k is any scalar then ku is in V. (V is closed under scalar multiplication).

(b) Addition Axioms

A_1 : $u + v = v + u$ [Commutativity]

A_2 : $(u + v) + w = u + (v + w)$ [Associativity]

A_3 : There is an element called **zero** in V such that $u + 0 = u$ for every u in V.

[Existence of additive identity]

A_4 : There is an element $-u$ in V called negative of u corresponding to every in V such that $u + (-u) = 0$. [Existence of additive inverse]

(c) Scalar multiplication Axioms

M_1 : $k(u + v) = ku + kv$ [Distributivity of scalar multiplication]

M_2 : $(k + l)u = ku + lu$ [Distributivity of scalars]

M_3 : $(kl)u = k(lu)$ [Associative law of scalars]

M_4 : There is an element 1 in V called **unity** such that $1 \cdot u = u$ for every u in V. [Existence of multiplicative identity]

Remark ....

Vector spaces in which the scalar $k, m, ...$ are real numbers are called **real vector spaces.** If the scalars $k, m, ...$ can be complex numbers then the vector space is called **complex vector space.**

Uniqueness Theorem : If V is a vector space then show that

(1) additive identity 0 is unique.

(2) additive inverse of a vector u is unique.

Proof : **(1)** If possible let there be two distinct additive identities 0 and 0'.

Then $u + 0 = u$ and $u + 0' = u$.

Putting $u = 0'$ in the first $0' + 0 = 0'$ and putting $u = 0$ in the second $0 + 0' = 0$.

Hence, $0' = 0$ *i.e.* the additive identity is unique.

(2) If possible let u_1 and u_2 be two additive inverses of u.

Then by definition of the inverse,

$$u + u_1 = 0 \quad ..... (1) \qquad \text{and} \qquad u + u_2 = 0 \quad ..... (2)$$

Adding u_2 to both sides of the first equation, we get

$\therefore\ u_2 + (u + u_1) = u_2 + 0$ $\quad\therefore\ u_2 + (u + u_1) = u_2$ [By (A_3)]

$\therefore\ (u_2 + u) + u_1 = u_2$ $\quad\therefore\ 0 + u_1 = u_2$ [By (2)]

$\therefore\ u_1 = u_2$ [By (A_3)]

18. Vector Space in General

We know that the above properties are satisfied by vectors in R^2 and R^3. We shall define the generalised vector space R^n in terms of eight properties given in (b) and (c) and two more called closure properties. These ten properties are called axioms and the elements which satisfy them define a vector space. We shall have 'vectors' of nature different from the vectors in R^2 and R^3 such as matrices or real valued functions. These are called vectors in the sense they satisfy the ten axioms of vectors.

19. Vector Space R^n

If V is the set of n-tuples of real numbers *i.e.* $x_1, x_2, ..., x_n$ are real numbers and if + is usual addition and ku is usual scalar multiplication *i.e.* $(kx_1, kx_2, ..., kx_n)$ then V is a vector space.

We denote this vector space by R^n.

Particular Cases

If $n = 1$ *i.e.* if V is the set of real numbers with usual addition and usual scalar multiplication then V is the vector space of real numbers or space of points on a line. We denote this space by R^1.

If $n = 2$, *i.e.* if V is the set of all ordered pairs of real numbers with usual addition and usual scalar multiplication, then V is the vector space of ordered pairs of real numbers or space of points on a plane. We denote it by R^2.

If $n = 3$, *i.e.* if V is the set of all ordered triplets of real numbers with usual addition and usual scalar multiplication, then V is the vector space of ordered triplets of real numbers or space of points in (usual three dimensional) space. We denote it by R^3.

Example 1 : Check whether the set of all pairs of real numbers of the form $(1, x)$ with operations $(1, y) + (1, y') = (1, y + y')$ and $k(1, y) = (1, ky)$ is a vector space. **(M.U. 2014)**

Sol. : (a) Closure

C_1 : If $u = (1, x)$, $v = (1, y)$ are two elements then

$$u + v = (1, x) + (1, y) = (1, x + y)$$

$\therefore$ V is closed under addition. [$\because$ *e.g.* $(1, 3) + (1, 5) = (1, 8)$]

C_2 : If $u = (1, x)$ then $ku = k(1, x) = (1, kx)$

$\therefore$ V is closed under multiplication. [*e.g.* If $u = (1, 3)$ then $4u = (1, 12)$]

(b) Addition

A_1 :
$$\begin{aligned} u + v &= (1, x) + (1, y) = (1, x + y) \\ &= (1, y + x) = (1, y) + (1, x) \\ &= v + u \end{aligned}$$

A_2 :
$$\begin{aligned} (u + v) + w &= [(1, x) + (1, y)] + (1, z) \\ &= (1, x + y) + (1, z) \\ (u + v) + w &= (1, (x + y) + z) = (1, x + (y + z)) \\ &= (1, x) + [(1, y + z)] \\ &= (1 + x) + [(1 + y) + (1 + z)] \\ &= u + (v + z) \end{aligned}$$

A_3 :
$$\begin{aligned} u + 0 &= (1, x) + (1, 0) = (1, x + 0) \\ &= (1, x) = u \end{aligned}$$

$0 = (1, 0)$ is zero for this space.

A_4 :
$$\begin{aligned} u + (-u) &= (1, x) + (1, -x) \\ &= (1, x - x) = (1, 0) \end{aligned}$$

$\therefore$ $(1, -x)$ is additive inverse $-u$ of $(1, x) = u$.

(c) Scalar Multiplication

M_1 :
$$\begin{aligned} k(u + v) &= k[(1, x) + (1, y)] \\ &= k(1, x + y) = (1, k(x + y)) \\ &= (1, kx + ky) = (1, kx) + (1, ky) \\ &= k(1, x) + k(1, y) = ku + kv \end{aligned}$$

M_2: $(k+l)\,u = (k+l)\,(1, x) = (1, (k+l)\,x)$
$= (1, kx + lx) = (1, kx) + (1, lx)$
$= k\,(1, x) + l\,(1, x)$
$= ku + lu$

M_3: $(kl)\,u = (kl)\,(1, x)$
$= (1, klx) = k\,(1, lx)$
$= k\,[\,l\,(1, x)\,]$
$= k\,(lu)$

M_4: $1 \cdot u = (1, 1 \cdot x)$
$= (1, x) = u$

Hence, 1 is the multiplicative identity for this space.

Since all axioms are satisfied V is vector space.

Example 2 : Let V be a set of positive real numbers with addition and scalar multiplication defined as $x + y = xy$ and $cx = x^c$, show that V is a vector space under this addition and scalar multiplication.

Sol. : (a) Closure **(M.U. 2014)**

C_1 : Since $x + y = xy$ and xy is a real number.

$\therefore$ V is closed under addition. [$\because$ e.g. $2 + 3 = 2 \times 3 = 6$ and 6 is real number]

C_2 : Since $cx = x^c$ and x^c is a real number V is closed under scalar multiplication.

[$\because$ e.g., $c \cdot 3 = 3^c$, 3^c is a real numbers.]

(b) Addition

A_1: $x + y = xy = yx = y + x$

A_2: $(x + y) + z = xy + z = xyz$
$= x\,(yz) = x + (yz)$
$= x + (y + z)$

A_3: $x + 1 = x1 = x$ for all x.

Here, 1 is zero for this operation of addition.

A_4: $x + \dfrac{1}{x} = x\left(\dfrac{1}{x}\right) = 1.$

$\therefore$ $\dfrac{1}{x}$ is the negative of x because 1 is zero here.

(c) Scalar Multiplication

M_1: $k\,(x + y) = k\,xy = (xy)^k$
$= x^k y^k = x^k + y^k$
$= kx + ky$

M_2: $(k + l)\,x = x^{k+l} = x^k \cdot x^l$
$= x^k + x^l = kx + lx$

M_3: $(kl)\,x = x^{kl} = (x^k)^l$
$= k\,(x^l) = k\,(lx)$

M_4: $1 \cdot x = x^1 = x$

$\therefore$ $1 \cdot x = x$ for every x. Hence, V is a vector space.

Note ....

In this example 1 behaves as additive identity as well as multiplicative identity.

Example 3 : Let $V = \{(x, y) \mid x, y \in R, y > 0\}$. Let $(a, b), (c, d) \in V$ and $\alpha \in R$. Defi $(a, b) + (c, d) = (a + c, b \cdot d)$ and $\alpha (a, b) = (\alpha a, b^{\alpha})$.

Examine whether V is a vector space.

Sol. : Left to you.

Example 4 : Consider the set V of all pairs of real numbers (x_1, x_2) and (y_1, y_2) such that th sum and scalar multiplication are defined as follows.

$$x + y = (x_1 + y_1, x_2 + y_2) \text{ and } kx = (kx, 0)$$

Examine whether V is a space.

Sol. : We leave it to you to verify other axioms except the last *viz.* multiplicative identity *i.e.*, M_4.

By data if $u = (3, 4)$ and $v = (2, 5)$ are two elements of V then

$$u + v = (3 + 2, 4 + 5) = (5, 9) \quad \text{and} \quad ku = k(3, 4) = (3k, 0)$$

Note that there does not exist the multiplicative identity.

We have $1 \cdot x = (x, 0) \neq x$

Hence, V is not a space.

Example 5 : Show that the set of real numbers (x, y) with operation

(i) $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and $k(x, y) = (2kx, 2ky)$

(ii) $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (z_1 + z_2, y_1 + y_2, x_1 + x_2)$ and $k(x, y, z) = (kx, ky, kz)$

are not vector spaces in R^2 and R^3 respectively.

Sol. : (i) [Here we have if $u = (2, 3)$, $v = (6, 9)$ then

$$u + v = (2, 3) + (6, 9) = (2 + 6, 3 + 9) = (8, 12)$$

and $5u = 5(2, 3) = (20, 30)$

We shall consider the axiom M_3.

$$m(u) = m(x, y) = (2mx, 2my)$$

$$k(mu) = k(2mx, 2my) = (4kmx, 4kmy)$$

But $(km)(u) = km(x, y) = (2kmx, 2kmy)$

Thus, $k(mu) \neq km\, u$.

Further, we see that $1(x, y) = (2x, 2y) \neq (x, y)$

$\therefore$ There is no multiplicative identity.

Hence, this set is not a vector space.

(ii) For second space consider the axiom A_1.

[Here, we have if $u = (2, 3, 5)$, $v = (6, 9, 12)$, then

$$\begin{aligned} u + v &= (2, 3, 5) + (6, 9, 12) \\ &= (5 + 12, 3 + 9, 2 + 6) \\ &= (17, 12, 8) \ ] \end{aligned}$$

We have $u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (z_1 + z_2, y_1 + y_2, x_1 + x_2)$

$\therefore \quad (u + v) + w = (z_1 + z_2, y_1 + y_2, x_1 + x_2) + (x_3, y_3, z_3)$

$= (x_1 + x_2 + z_3,\ y_1 + y_2 + y_3,\ z_1 + z_2 + x_3)$

And $\quad v + w = (x_2, y_2, z_2) + (x_3, y_3, z_3)$

$= (z_2 + z_3, y_2 + y_3, x_2 + x_3)$

$u + (v + w) = (x_1, y_1, z_1) + (z_2 + z_3, y_2 + y_3, x_2 + x_3)$

$= (z_1 + x_2 + x_3,\ y_1 + y_2 + y_3,\ x_1 + z_2 + z_3)$

Thus, $u + (v + w) \neq (u + v) + w$.

Hence, this set is **not** a vector space.

[For example, if u (2, 3, 6), $v = (3, -1, 2)$ and $w = (-2, 1, 0)$, then

$u + v = (2, 3, 5) + (3, -1, 2)$

$= (5 + 2,\ 3 - 1,\ 2 + 3) = (7, 2, 5)$

$(u + v) + w = (7, 2, 5) + (-2, 1, 0)$

$= (5 + 0, 2 + 1, 7 - 2) = (5, 3, 5)$

Now, consider

$v + w = (3, -1, 2) + (-2, 1, 0)$

$= (2 + 0, -1 + 1, 3 - 2) = (2, 0, 1)$

and $\quad u + (v + w) = (2, 3, 5) + (2, 0, 1)$

$= (5 + 1, 3 + 0, 2 + 2) = (6, 3, 4)$

Thus, $(u + v) + w \neq u + (v + w)$.]

Example 6 : Let $V = R^2$ and define addition and scalar multiplication as follows :

Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$, then $u + v = (u_1 + v_1, u_2 + v_2)$ and $ku = (ku_1, u_2)$.

Prove that it is not a vector space.

Sol. : [Here we have if $u = (3, 5)$, $v = (1, 2)$ then

$u + v = (3 + 1, 5 + 2) = (4, 7)$ and $ku = k\,(3, 5) = (3k, 5)$]

We leave it to you to verify, all axioms except the distributivity of scalar multiplication *i.e.*, M_2.

If $u = (u_1, u_2)$, then

$ku = k\,(u_1, u_2) = (ku_1, u_2)$; $\quad mu = m\,(u_1, u_2) = (mu_1, u_2)$

$\therefore \quad ku + mu = (ku_1, u_2) + (mu_1, u_2) = ((k + m)\,u_1, 2u_2)$

But $\quad (k + m)\,u = ((k + m)\,u_1, u_2)$

$\therefore \quad (k + m)\,u \neq ku + mu.$ Hence, V is not a vector space.

[For example let $u = (3, 5)$, then

$2u = 2\,(3, 5) = (6, 5)$ and $4u = 4\,(3, 5) = (12, 5)$

$\therefore \quad 2u + 4u = (6, 5) + (12, 5) = (18, 10)$

$\therefore \quad (2 + 4)\,u = 6u = 6\,(3, 5) = (18, 5)$

$\therefore \quad (2 + 4)\,u \neq 2u + 4u.$]

Example 7 : Check whether $V = R^3$ is a vector space with respect to the operations

$$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1 - 2, u_2 + v_2 - 3)$$

and $\quad k(u_1, u_2) = (ku_1 + 2k - 2, ku_2 - 3k + 3)$

Sol. : [Here, we have $\quad (2, 5) + (6, 7) = (2 + 6 - 2,\ 5 + 7 - 3) = (6, 9)$

and $\quad k(2, 5) = (k \cdot 2 + 2k - 2,\ k \cdot 5 - 3k + 3)$

$$= (4k - 2, 2k + 3)\ ]$$

We leave it to you to verify all axioms except the distributivity of scalar multiplication *i.e.*, M_2

If $u = (u_1, u_2)$, then

$$ku = (ku_1 + 2k - 2, ku_2 - 3k + 3) \quad \text{and} \quad lu = (lu_1 + 2l - 2, lu_2 - 3l + 3)$$

$$\therefore\ ku + lu = [(k + l)u_1 + (2k + 2l) - 2 - 2 - 2, (k + l)u_2 - 3(k + l) + 3 + 3 - 3]$$

$$= [(k + l)u_1 + 2(k + l) - 6, (k + l)u_2 - 3(k + l) + 3]$$

and $\quad (k + l)u = (k + l)(u_1, u_2)$

$$= [(k + l)u_1 + 2(k + l) - 2, (k + l)u_2 - 3(k + l) + 3]$$

$$\therefore (k + l)u \neq ku + lu.$$

The axiom fails. Hence, V is not a vector space.

20. Vector Space of Real Valued Function F $(-\infty, \infty)$

Let V be the set of all real-valued functions defined for all x in $(-\infty, \infty)$. Let $f(x)$, $g(x)$ be two functions such that their sum is $f(x) + g(x)$. Let k be a scalar such that $(kf)(x) = kf(x)$.

In other words the value of the function $f + g$ at x is the sum of the values of f and g at x. The value of the function kf at x is k times the value of f at x.

The set of all such functions is a vector space V. It is denoted by $F(-\infty, \infty)$. The vector 0 is the function whose value for all x is 0. The graph of this function is the x-axis. The graph of $-f(x)$ is the reflection of $f(x)$ in the x-axis.

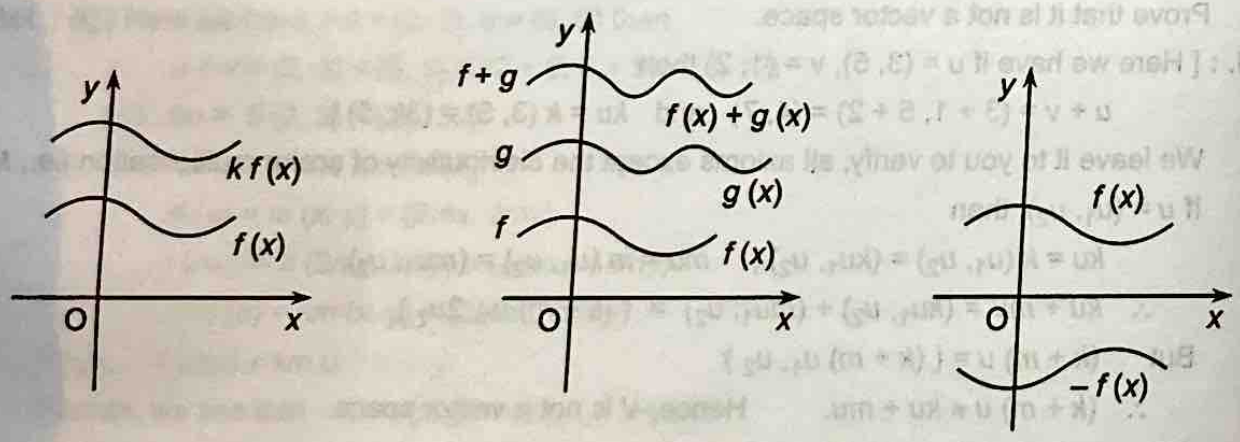

Fig. 6.12

Example 1 : Let $V = F(-\infty, \infty)$ be the set of all real valued functions defined on $(-\infty, \infty)$. For any f and g and for any scalar k, we define

(i) $f = g$ if and only if $f(x) = g(x)$ for all x.

(ii) $(f + g)(x) = f(x) + g(x)$. $\qquad$ (iii) $(kf)(x) = kf(x)$.

Then V is a vector space.

Sol. : (a) Closure

C_1 : If f and g are real valued functions then $f + g$ is also a real valued function.

C_2 : If k is any scalar then kf also is a real valued function.

(b) Addition

A_1 : $(f+g)(x) = f(x) + g(x)$

$= g(x) + f(x) = (g+f)(x)$

$\therefore \quad f + g = g + f$

A_2 : $(f+g)(x) + h(x) = f(x) + g(x) + h(x)$

$= f(x) + (g+h)(x)$

$\therefore \quad (f+g) + h = f + (g+h)$

A_3 : We define 0 as $0(x) = 0$. Then

$(f+0)x = f(x) + 0(x) = f(x) + 0 = f(x)$

A_4 : We define $(-f)(x) = -f(x)$

Now, $f(x) + (-f(x)) = f(x) - f(x) = 0$

$\therefore$ For every f there is $-f$.

(c) Scalar Multiplication

M_1 : $k[f(x) + g(x)] = kf(x) + kg(x) \quad \therefore \; k(f+g) = kf + kg$

M_2 : $(k+l)f(x) = kf(x) + lf(x) \quad \therefore \; (k+l)f = kf + lf$

M_3 : $(kl)f(x) = k[lf(x)] \quad \therefore \; (kl)f = k(lf)$

M_4 : For any function $f(x)$, we define 1 such that $1\,f(x) = f(x)$. $\therefore \; 1 \cdot f = f$.

Thus, all axioms are satisfied.

Remark ....

This space is called **function space.**

21. Vector Space of 2 × 2 Matrices

Under usual addition of matrices and usual scalar multiplication of matrices the set of all 2 × 2 matrices is a vector space.

Example : Prove that the set of 2 × 2 matrices is a vector space with usual addition of matrices and usual scalar multiplication.

Sol. : (a) Closure

C_1 : If $u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ and $v = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}$, then

$$u + v = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} + \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} = \begin{bmatrix} u_{11}+v_{11} & u_{12}+v_{12} \\ u_{21}+v_{21} & u_{22}+v_{22} \end{bmatrix}$$

Since $u + v$ is a 2 × 2 matrix, $u + v$ is also is in V.

C_2 : If k is any scalar then

$$ku = \begin{bmatrix} ku_{11} & ku_{12} \\ ku_{21} & ku_{22} \end{bmatrix} \quad \therefore \; ku \text{ is in } V.$$

(b) Addition Axioms

Axioms A_1 and A_2 can be easily verified.

Axiom A_3 : If $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $u + O = O + u = u$.

Since O is a 2×2 matrix axiom A_3 is satisfied.

Axiom A_4 : If we define $-u = \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix}$, then

$$u + (-u) = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} -u_{11} & -u_{12} \\ -u_{21} & -u_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, the axiom A_4 is satisfied.

(c) Scalar Multiplication Axioms

Axioms M_1, M_2, M_3 can be similarly verified.

Axiom M_4 : We define $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Then, $I \cdot u = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$ for every u in V.

Hence, all the axioms are satisfied.

This space of 2×2 matrices is denoted by M_{22}. The above concept can be generalised. The set of all $m \times n$ matrices also satisfy the above axioms. This space is denoted by M_{mn}.

EXERCISE - III

Examine in each case whether the set of real numbers with operations of addition and scalar multiplication defined as follows is a vector space.

1. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (3kx_1, 3ky_1)$ [**Ans. :** No. M_3 fails.]
2. $(x_1, y_1) + (x_2, y_2) = (x_1, y_1 + y_2)$; $k(x_1, y_1) = (kx_1, ky_1)$ [**Ans. :** No. M_2 fails]
3. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (2kx_1, -ky_1)$
   [**Ans. :** No. $m(x_1, y_1) = (2mx_1, -my_1)$]
   and $k[m(x_1, y_1)] = k(2mx_1, -my_1) = (4km\, x_1, -kmy_1)$
   But $(km)(x_1, y_1) = (2kmx_1, -kmy_1)$]
4. $(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0)$ and $k(x_1, 0) = (kx_1, 0)$. [**Ans. :** Yes]
5. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (kx_1, y_1)$ [**Ans. :** No. M_2 fails.]
6. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (k^2 x_1, k^2 y_1)$ [**Ans. :** No. M_2 fails.]
7. $(x_1, y_1) + (x_2, y_2) = (|x_1 + x_2|, |y_1 + y_2|)$; $k(x_1, y_1) = (kx_1, ky_1)$
   [**Ans. :** No. A_2, A_3, A_4, M_1, M_2 fail.]
8. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, 0)$; $k(x_1, y_1) = (kx_1, ky_1)$
   [**Ans. :** No. There is no additive identity.]
9. $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$; $k(x_1, y_1) = (|k|x_1, |k|y_1)$
   [**Ans. :** No. M_2 fails]

10. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$k(x_1, y_1, z_1) = (k^2x_1, 0, 0)$ [**Ans. :** No. M_2 fails]

11. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$k(x_1, y_1, z_1) = (2kx_1, 2ky_1, 2kz_1)$ [**Ans. :** No. M_4 fails]

12. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2 - 1, y_1 + y_2 - 1, z_1 + z_2 - 1)$
$k(x_1, y_1, z_1) = (kx_1, ky_1, kz_1)$ [**Ans. :** No. M_1, M_2 fail]

13. $(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$
$k(x_1, y_1, z_1) = (k^2 x_1, k^2 y_1, k^2 z_1)$ [**Ans. :** No. M_2 fails]

14. $(x_1, x_2, ..., x_n) + (y_1, y_2, ..., y_n) = (x_1 + y_1, x_2 + y_2, ..., x_n + y_n)$
$k(x_1, x_2, ..., x_n) = (3kx_1, 3kx_2, ..., 3kx_n)$ [**Ans. :** No, M_4 fails]

15. Examine whether the set of matrices of order 2 × 2 as defined below is a vector space.

(i) $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ with usual addition of matrices and scalar multiplication. [**Ans. :** Yes]

(ii) $\begin{bmatrix} a & 1 \\ 1 & b \end{bmatrix}$ with usual addition of matrices and scalar multiplication. [**Ans. :** No. C_1 fails.]

(iii) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with usual matrix addition and $k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

[**Ans. :** No. There is no multiplicative identity.]

(iv) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0$.

[**Ans. :** No. Not closed under addition.

e.g., if $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, then $|A| = 0$ and $|B| = 0$.

But $A + B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $|A + B| = -1 \neq 0$.]

(v) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$.

[**Ans. :** No. Not closed under addition.

If $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A| \neq 0$, $|B| \neq 0$. But $|A + B| = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$.]

16. Prove that the set of all $m \times n$ matrices is a vector space under usual addition of matrices and scalar multiplication.

22. Subspaces

It is possible that one vector space is contained in another vector space. The planes passing through the origin form a vector space which is contained in R^3. (See Ex. 4, 5, page 6-30) Similarly, $F[a, b]$ will be a subspace of $F(-\infty, \infty)$ (See § 20).

Definition : A subset W of a vector space V is called a **sub-space** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

In other words, the subset W of V which satisfies all the ten axioms of V is called a subspace of V. However, in practice if V is a vector space and W is a subset of V, we need not verify all the ten axioms of vector space. All axioms **except c_1 and c_2** *i.e.* (1) for all u, v, $u + v$ is in w and (2) for any scalar k, k is in W, are inherited from V (See page 6-19). We need only to verify the axioms of **closure** (1) under **addition** and (2) **scalar multiplication** only. We state this as a theorem.

Theorem : If W is a non-empty subset of a vector space V then W is a subspace of V if

(i) for all u and v in W, $u + v$ also is in W and

(ii) for any scalar k and a vector u of W, ku is in W.

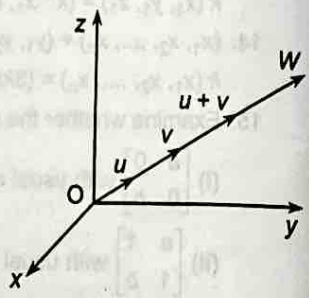

Fig. 6.13

Example 1 : Show that a line through the origin in R^3 is a subspace of R^3.

Sol. : Consider a line through the origin in R^3.

If u and v are any two vectors (with origin as the initial point) on this line the $u + v$ is also a vector on this line. Thus, $u + v$ is closed under addition.

Also if u is a vector on the line and k is any scalar then ku is also a vector on this line.

Thus, W is closed under addition and scalar multiplication.

Hence, a line through the origin is a subspace of R^3.

Remark ....

A line through the origin in R^2 is a subspace of R^2.

Using the result of Ex. 1 above, we get the following list of subspaces of R^2 and R^3.

Subspaces of R^2	**Subspaces of R^3**
1. $\{0\}$	1. $\{0\}$
2. Lines though the origin.	2. Lines through the origin.
	3. Planes through the origin.
3. R^2 itself	4. R^3 itself.

Example 2 : Show that $V = \{(x, y) \mid x = 3y\}$ is a subspace of R^2. State all possible subspaces of R^2. **(M.U. 2016)**

Sol. : Let $u = (x_1, y_1)$, $v = (x_2, y_2)$ be two elements of V.

Then $x_1 = 3y_1$ and $x_2 = 3y_2$

$\therefore\ x_1 + x_2 = 3(y_1 + y_2)$ $\quad\therefore\ u + v = (x_1 + x_2, y_1 + y_2)$ also is in V.

Again $ku = k(x_1, y_1) = (kx_1, ky_1)$

Since $x_1 = 3y_1$, $kx_1 = 3(ky_1)$. Hence, ku is in V.

Since V is closed under addition and scalar multiplication, V is a subspace of R^2.

If k is any real number then $V = \{(x, y) \mid y = kx\}$ gives all possible subspaces of R^2.

Geometrically the given set $V = \{(x, y) \mid x = 3y\}$ is a straight line passing through the origin with slope $m = 1/3$. It is a subspace of the plane R^2. All lines passing through the origin give possible subspaces of R^2. Every line through the origin is a sub-space of R^2.

Example 3 : If W is the set of all points (x, y) in R^2 such that $x \geq 0, y \geq 0$ then W is **not** a subspace of R^2.

Sol. : If (x_1, y_1) and (x_2, y_2) are any two points of W in the first quadrant, then

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2,\ y_1 + y_2)$$

is also a point of W in the first quadrant.

But $k(x, y) \neq (kx, ky)$ for all k.

For instance if $k = -2$,

$$(-2)(x, y) = (-2x, -2y)$$

and this point is **not** in the first quadrant.

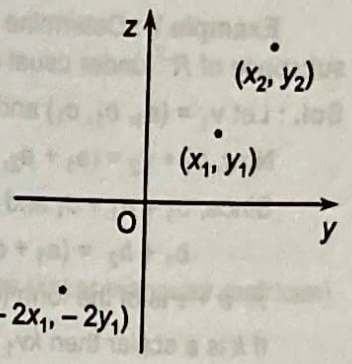

Fig. 6.14

Hence, W is not closed under scalar multiplication and hence is not a subspace of W.

Example 4 : Is $W = \{(a, 1, 1) \mid a \in R\}$ is a subspace of R^3. **(M.U. 2016)**

Sol. : Let $u = \{(a, 1, 1) \mid a \in R\}$ and $v = \{(b, 1, 1) \mid b \in R\}$ be two elements of W.

Then, $u + v = \{a, 1, 1\} + \{b, 1, 1\} = \{a + b, 2, 2\}$

But $u + v$ is not of form $\{a, 1, 1 \mid a \in R\}$ and hence $\notin W$.

$\therefore$ W is not a subsapce of R^3.

Example 5 : Show that any plane through the origin is a sub-space of R^3. **(M.U. 2017)**

Sol. : Let the plane be $ax + by + cz = 0$.

If $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ are any two points on the above plane, then $ax_1 + by_1 + cz_1 = 0$ and $ax_2 + by_2 + cz_2 = 0$.

$\therefore$ $a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = 0$

$\therefore$ $(x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$ is also a point on the plane.

If k is any scalar then we have, $ku = (kx_1,\ ky_1,\ kz_1)$

$\therefore$ $akx_1 + bky_1 + ckz_1 = k(ax_1 + by_1 + cz_1) = k(0) = 0$

$\therefore$ (kx_1, ky_1, kz_1) also lies on the plane.

$\therefore$ Any plane passing through the origin is a subspace of R^3.

Example 6 : Show that the set of solution vectors of the equations

$$x - 2y + 4z = 0,\ 2x - 4y + 8z = 0,\ 3x - 6y + 12z = 0 \text{ is a subspace of } R^3.$$

Sol. : We shall first solve these linear equations.

We have $\begin{bmatrix} 1 & -2 & 4 \\ 2 & -4 & 8 \\ 3 & -6 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ By $\begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & -2 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ $x - 2y + 4z = 0$.

Let $z = t$, $y = s$, then $x = 2s - 4t$ where s and t are parameters. For different values of s and t, we get different solutions.

The solutions are given by $x - 2y + 4z = 0$ which is a plane through the origin. This plane is a subspace of R^3 as seen in Ex. 5.

Example 7 : Determine whether the set of vectors of the form (a, b, c) where $b = a + c$ form a subspace of R^3 under usual addition and scalar multiplication. **(M.U. 2015, 16)**

Sol. : Let $v_1 = (a_1, b_1, c_1)$ and $v_2 = (a_2, b_2, c_2)$ be two vectors in R^3.

Now, $v_1 + v_2 = (a_1 + a_2, b_1 + b_2, c_1 + c_2)$

Since, $b_1 = a_1 + c_1$ and $b_2 = a_2 + c_2$

$$b_1 + b_2 = (a_1 + c_1) + (a_2 + c_2) = (a_1 + a_2) + (c_1 + c_2)$$

$\therefore$ $u + v$ is of the form (α, β, γ) and $\beta = \alpha + \gamma$. $\therefore$ $u + v$ is in R^3.

If k is a scalar then $kv_1 = (ka_1, kb_1, kc_1)$ and since $b_1 = a_1 + c_1$.

$$kb_1 = k(a_1 + c_1) = ka_1 + kc_1$$

$\therefore$ ku is of the form (p, q, r) and $q = p + r$. $\therefore$ ku is in R^3.

Hence, the set of vectors of the form (a, b, c) is a subspace of R^3.

Example 8 : Check whether the following are subspaces of R^3.

(i) $W = \{(a, 0, 0) \mid a \in R\}$ (ii) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$ **(M.U. 2019)**

Sol. : (I) Let $v_1 = (a_1, 0, 0)$ and $v_2 = (a_2, 0, 0)$ be two vectors in R^3.

Now, $v_1 + v_2 = (a_1, 0, 0) + (a_2, 0, 0) = (a_1 + a_2, 0, 0)$

Since $a_1 + a_2 \in R$, $v_1 + v_2$ is in R^3.

If k is any scalar then $kv_1 = k(a_1, 0, 0) = (ka_1, 0, 0)$.

Hence, kv_1 is also in R^3.

Hence, $W = \{(a, 0, 0) \mid a \in R\}$ is a subspace of R^3.

(II) Let $v_1 = (x_1, y_1, z_1)$ where $x_1^2 + y_1^2 + z_1^2 \le 1$.

We consider the second condition of theorem, page 6-29.

If k is any scalar then $kv_1 = (kx_1, kx_2, kx_3)$.

But if $k > 1$ then $(kx_1)^2 + (kx_2)^2 + (kx_3)^2 = k^2(x_1^2 + y_1^2 + z_1^2)$ is not ≤ 1.

Hence, V is not closed under scalar multiplication and hence **not** a subspace of R^3.

Example 9 : Let W be the set of 2×2 matrices of the form $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$. Show that W is a subspace of space V of all 2×2 matrices.

Sol. : Let $P = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix}$, $R = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}$, then $P + R = \begin{bmatrix} a+r & 0 \\ 0 & q+s \end{bmatrix}$ $\therefore$ $P + Q$ is in W.

And $kP = k\begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} kp & 0 \\ 0 & kq \end{bmatrix}$

$\therefore$ kP is in W. $\therefore$ W is a subspace of V.

Example 10 : If W is the set of all symmetric matrices of order $n \times n$, then W is a subspace of all $n \times n$ matrices V.

Sol. : (I) If u and v are two symmetric matrices then their sum $u + v$ is also symmetric and hence in V.

For example, $$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{12} & b_{22} & b_{23} \\ b_{13} & b_{23} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{12}+b_{12} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{13}+b_{13} & a_{23}+b_{23} & a_{33}+b_{33} \end{bmatrix}$$

(ii) If k is a scalar then kA is also symmetric and hence is in V.

For example, $kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{12} & ka_{22} & ka_{23} \\ ka_{13} & ka_{23} & ka_{33} \end{bmatrix}$

Hence, W is a subspace of V.

Example 11 : If W is the set of all $n \times n$ upper triangular matrices (lower triangular matrices) the W is a subspace of all $n \times n$ matrices.

Sol. : Left to you.

Example 12 : If W is the set of all diagonal matrices of order $n \times n$, then W is the subspace of all $n \times n$ matrices.

Sol. : Left to you.

Example 13 : Let W be the set of all functions of the form $p(x) = a_0 + a_1 x + ... + a_n x^n$ where n is any non-negative integer and where $a_0, a_1, ..., a_n$ are real numbers, then W is a subspace of all real valued functions *i.e.*, V.

Sol. : Let $p(x) = a_0 + a_1 x + ... + a_n x^n$ and $q(x) = b_0 + b_1 x + ... + b_n x^n$, then

(i) $p(x) + q(x) = (a_0 + b_0) x + (a_1 + b_1) x^2 + ... + (a_n + b_n) x^n$

$\therefore$ $p(x) + q(x)$ is of the above form.

$\therefore$ $p(x) + q(x)$ is in W.

(ii) $k\,p(x) = k[a_0 + a_1 x + ... + a_n x^n] = ka_0 + ka_1 x + ... + ka_n x^n$

$\therefore$ $k\,p(x)$ is of the above form. $\therefore$ $k\,p(x)$ is in W.

Hence, W is a subspace of V.

Example 14 : Sub spaces of continuous functions defined on $(-\infty, \infty)$.

Sol. : We have seen in § 20, page 6-25 that the set of all real valued functions denoted by $F(-\infty, \infty)$ is a vector space.

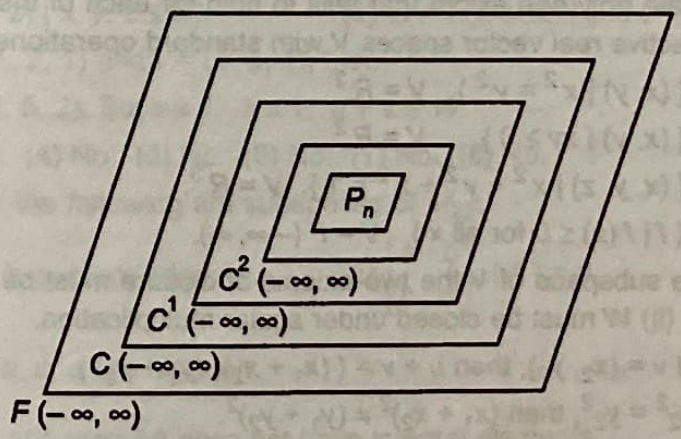

Fig. 6.15

Let $C(-\infty, \infty)$ denote the set of all continuous functions. If f and g are continuous then $f + g$ is also continuous and $k\,f$ where k is any scalar is also continuous. Thus $C(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$.

Let $C^1(-\infty, \infty)$ denote the set of all functions for which first order derivatives exist and are continuous. Then $C^1(-\infty, \infty)$ is a subspace of $F(-\infty, \infty)$. But since a differentiable function is also a continuous function, $C^1(-\infty, \infty)$ is a subspace of $C(-\infty, \infty)$. Similarly, we can show that $C^2(-\infty, \infty)$ *i.e.* the functions having continuous second order derivatives is a subspace of $C^1(-\infty, \infty)$. Thus, we get a hierarchy of differentiable functions $C^1(-\infty, \infty)$, $C^2(-\infty, \infty)$, ..., $C^m(-\infty, \infty)$. If we have set of functions which have continuous derivatives of all orders, we denote the subspace by $C^\infty(-\infty, \infty)$.

The hierarchy is shown diagrammatically in the Fig. 3.5. Further, we know that the set of polynomials P_n has continuous derivative of all orders. Hence, P_n is a subspace of $C^\infty(-\infty, \infty)$.

If instead of taking the open interval $(-\infty, \infty)$, we take closed interval $[a, b]$ then we get the corresponding subspaces $C[a, b]$, $C^m[a, b]$ and $C^\infty[a, b]$. Similarly, if we take open interval (a, b) then the spaces can be denoted by $C(a, b)$, $C^m(a, b)$ and $C^\infty(a, b)$.

Example 15 : Determine whether the set V of all functions f such that $f(0) = 0$ is a subspace of $F(-\infty, \infty)$.

Sol. : If f and g belonging to V are two functions such $f(0) = 0$ and $g(0) = 0$, then $f(0) + g(0) = 0$.

Hence, $f + g$ belong to V.

If $f(0) = 0$, then $k f(0) = 0$.

Hence, $k f$ belongs to V.

$\therefore$ V is a subset of $F(-\infty, \infty)$.

Example 16 : Let V be the vector space of all real valued functions defined on $[0, 1]$. Let W be the set of all real valued functions in V such that $f(c) > 0$ where c is a fixed number in $[0, 1]$. Examine whether W is a subspace of V.

Sol. : If f and g are two functions such that $f(c) > 0$ and $g(c) > 0$, then $f(c) + g(c) > 0$ and hence $f(x) + g(x)$ also is in V.

But if $k < 0$, then $k f(c) < 0$ as $f(c) > 0$.

Hence, $k f(x)$ is not in W. $\therefore$ W is not a subspace of V.

Example 17 : State only one axiom that fails to hold for each of the following sets W to be subspaces of the respective real vector spaces V with standard operations.

(1) $W = \{(x, y) \mid x^2 = y^2\}$, $V = R^2$

(2) $W = \{(x, y) \mid xy \geq 0\}$, $V = R^2$

(3) $W = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$, $V = R^3$.

(4) $W = \{f \mid f(x) \leq 0 \text{ for all } x\}$, $V = F(-\infty, \infty)$.

l. : Since W is to be subspace of V the two axioms of closure must be satisfied (i) W must be ed under addition, (ii) W must be closed under scalar multiplication.

If $u = (x_1, y_1)$ and $v = (x_2, y_2)$, then $u + v = \{(x_1 + x_2), (y_1 + y_2)\}$.

If $x_1^2 = y_1^2$ and $x_2^2 = y_2^2$, then $(x_1 + x_2)^2 \neq (y_1 + y_2)^2$.

W is not closed under addition.

e.g., If $u = (2, 2)$, $v = (4, -4)$, then $u + v = (2 + 4, 2 - 4)$.

But $(2 + 4)^2 \neq (2 - 4)^2$.)

(2) If $u = (x_1, y_1)$ and $v = (x_2, y_2)$, then $u + v = (x_1 + x_2, y_1 + y_2)$.

But $(x_1 + x_2)(y_1 + y_2) \not\geq 0$ for all x_1, x_2, y_1, y_2.

(*e.g.*, if $u = (2, 3)$, $v = (-6, -1)$, then both u, v are in W because

$$2 \times 3 = 6 > 0 \text{ and } (-6) \times (-1) = 6 > 0$$

But $u + v = (2 - 6, 3 - 1) = (-4, 2)$ is not in V because $-4 \times 2 = -8 \not> 0$.)

(3) If $u = (x_1, y_1, z_1)$, then $x_1^2 + y_1^2 + z_1^2 = 1$.

$$ku = (kx_1, ky_1, kz_1)$$

But $k^2 x_1^2 + k^2 x_2^2 + k^2 z_1^2 = k^2 (x_1^2 + y_1^2 + z_1^2) \neq 1$ if $k \neq 1$.

(4) If $f(x) \in W$, then $k f(x) \notin W$ if $k \leq 0$.

e.g. if $f(x) = -x^2$ and $k = -3$, then $3x^2 \not\leq 0$.

Hence, $f(x)$ is not a subspace of W.

EXERCISE - IV

1. Check whether the following are subspaces of R^3.

   (i) $W = \{(a, 0, 0) \mid a \in R\}$

   (ii) $W = \{(a, b, c) \mid b = a + c, a, b, c \in R\}$ **[Ans. : Yes]**

2. Check whether the following are subspaces of R^3.

   (i) $W = \{(a, 1, 1) \mid a \in R\}$ **(M.U. 2014, 16)**

   (ii) $W = \{(a, b, c) \mid b = a + c + 1; a, b, c \in R\}$ **[Ans. : No]**

3. If $V = R^3$ is the vector space of ordered triplets of real numbers with usual addition and scalar multiplication.

   Determine which of the following are subspaces of V.

   (1) $W = \{(x, y, z) \mid x = 1, z = 1\}$ (2) $W = \{(x, y, z) \mid x + y + z = 3\}$

   (3) $W = \{(x, y, z) \mid x^2 - y^2 = 0\}$ (4) $W = \{(x, y, z) \mid y \geq 0\}$

   (5) $W = \{(x, y, z) \mid x + y = 1\}$ (6) $W = \{(x, y, z) \mid x - y = 0 \text{ or } y - z = 0\}$

   (7) $W = \{(x, y, z) \mid x, y, z \leq 0\}$ (8) $W = \{(x, y, z) \mid y \text{ is an integer}\}$

[Ans. : (1) No. if $u = (1, 2, 1)$ and $v = (1, 3, 1)$, then

$u + v = (2, 5, 2)$. But $x \neq 1$, $z \neq 1$, $u + v \notin W$.

(2) No. (3) No. (4) No. (5) No. (6) No. (7) No. (8) No.]

4. Check whether the following are subspaces of M_{22}.

   (1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a, b, c, d are integers. (2) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $|A| = 0$

   (3) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a, b, c, d are rationals.

[Ans. : (1) No. If $k = 1/2$, then kA does not have integral elements.

(2) No. If $A = \begin{bmatrix} 2 & 6 \\ 3 & 9 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$, then $|A| = 0$, $|B| = 0$.

But $A + B = \begin{bmatrix} 6 & 8 \\ 5 & 10 \end{bmatrix}$ and $|A + B| = 60 - 40 = 20 \neq 0$.

(3) No. kA is not matrix of rational numbers if k is irrational.]

5. Determine whether the set V of triplets (x, y, z) of real numbers such that $2x + 3y + 4z = 0$ is a subspace of R^3. [**Ans.** : Yes]

6. Determine whether the set V of triplets (x, y, z) of real numbers such that either $x = 0$ or $z = 0$ is a subspace of R^3. [**Ans.** : No.]

7. Determine whether the set $V = \{ (x, y, z) \mid x = 1, y = 0 \text{ or } z = 0 \}$ is a subset of R^3.

[**Ans.** : No. $(1, y, 0) + (1, 0, z) = (2, y, z) \notin V$]

8. If V is the function space with usual addition of functions and scalar multiplication in R, check whether W is a subspace of V if

(1) $W = \{ f \mid f(1/2) > 0 \}$ (2) $W = \{ f \mid f(1) = 1 + f(2) \}$

(3) $W = \{ f \mid f(x) \leq 0, [-1, 1] \}$

[**Ans.** : (1) No. $k f(1/2) \ngtr 0$, if $k < 0$.

(2) No. $f(1) = 1 + f(2)$, $g(1) = 1 + g(2)$

$f(1) + g(1) = 2 + f(2) + g(2)$ $\therefore$ $f + g \notin W$.

(3) No. If $k < 0$, $k f(x) \nleq 0$ $\therefore$ $k f \notin W$.]

9. Check whether the following set of V polynomials is a subspace of the set of polynomials P_3.

$a_0 + a_1 x + a_2 x^2 + a_3 x^3$ where a_0, a_1, a_2, a_3 are integers.

[**Ans.** : No. If k is irrational then $ku \notin P_3$.]

10. Determine whether the set V of all functions f such that $f(1/2) = 0$ is a subspace of $F(-\infty, \infty)$. [**Ans.** : Yes]

23. Bases and Dimensions

In plane analytic geometry we have learnt that any point in the plane is associated with a pair of ordered numbers (a, b) called its coordinates. Conversely to every pair of ordered pair (a, b), there corresponds a point in the plane. We describe this by saying that there is **one-to-one correspondence** between the points in the plane and ordered pairs of real numbers. Although we generally use rectangular coordinate system, we may as well use oblique coordinate system *i.e.* system given by two non-parallel lines. Coordinates of a point in oblique coordinate system are obtained by drawing lines through P parallel to the axes (and not perpendicular to the axes). Similarly, in R^3 also we may have rectangular system or oblique coordinate system defined by three non-coplanar coordinate axes.

We shall soon extend the concept of coordinate system from R^2 and R^3 to R^n. For this we shall talk of vectors rather than coordinates. Instead of referring to the point P as $P(a, b)$, we shall denote it as a vector in terms of vectors of unit length in the directions of the axes. We have,

$$OP = av_1 + bv_2$$

where v_1 and v_2 are unit vectors in the positive direction of the axes. Thus, we have expressed a vector OP in R^2 as a **linear combination** of unit vectors v_1, v_2. Similarly, a vector OP in R^3 can be expressed as a linear combination of three vectors v_1, v_2, v_3 in the positive direction of the axes as

$$OP = av_1 + bv_2 + cv_3.$$

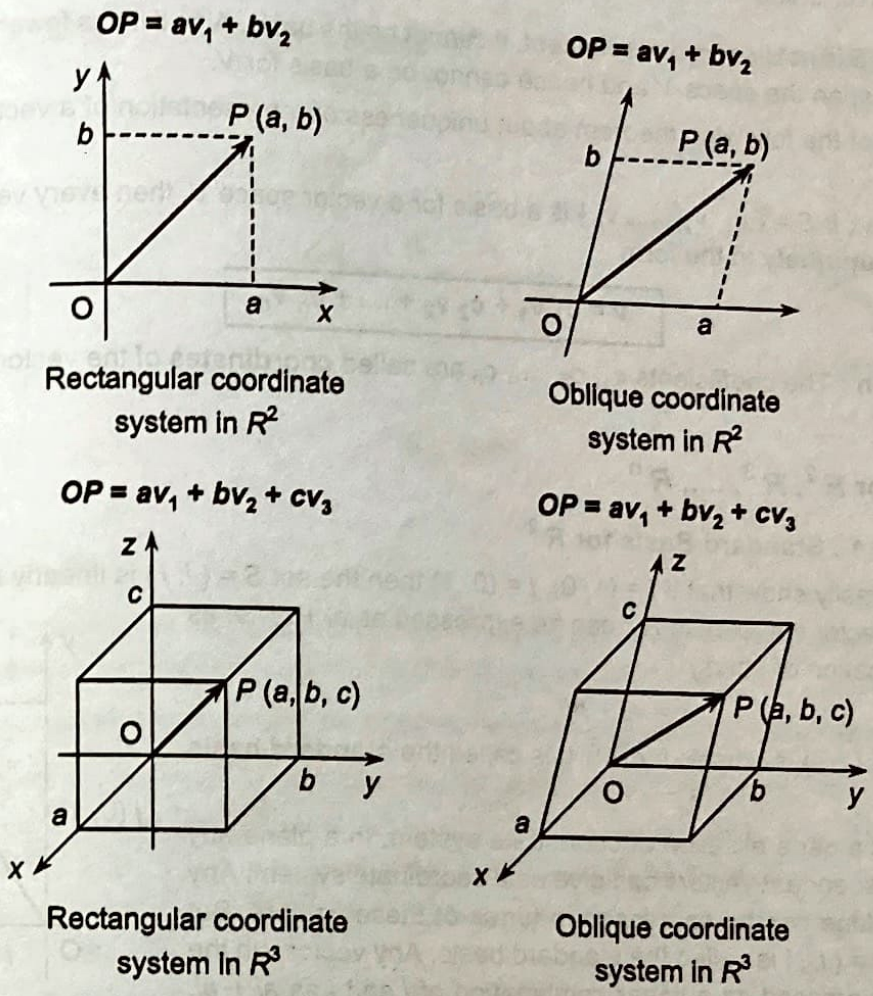

Fig. 6.15

The vectors that specify the coordinate system are called **"basis vectors"**. Any set of vectors can be basis vectors all that we need is that (i) they are linearly independent and (ii) they span the space V.

Although generally basis vectors of length 1 unit are used, it is not always very convenient. For instance, we may use 100 cms on one axes for length and 100 degrees for temperature on the other axis.

We also note that the x and y axes are linearly independent and they span the plane. Also the three coordinate axes x, y, z in R^3 are linearly independent and they span the spare of R^3.

We can now generalise the concept of coordinate system to general vector spaces.

24. Basis

Definition : If V is any vector space and $S = \{ v_1, v_2, ..., v_3 \}$ is a set of vectors in V then S is called a **basis** for V if the following two conditions hold good.

(1) **S is linearly independent,**

(2) **S spans V.**

The definition lays down two conditions for basis.

(I) The set $S = \{ v_1, v_2, ..., v_n \}$ must be linearly independent.

(II) The set S spans the vector space V.

If the set S is not linearly independent, it cannot be the basis. Also if it has fewer than n vectors then it cannot span the space V and hence cannot be a basis for V.

We accept the following theorem about uniqueness of representation of a vector in V without proof.

Theorem : If $S = \{ v_1, v_2, ..., v_n \}$ is a basis for a vector space V, then every vector v in V can be expressed **uniquely** in the form

$$\boxed{v = c_1 v_1 + c_2 v_2 + ... + c_n v_n}$$

Definition : The coefficients $c_1, c_2, ..., c_n$ are called **coordinates** of the vector v with respect to the basis.

(a) Basis for $R^2, R^3, ..., R^n$

Example 1 : Standard Basis for R^2

We can easily show that if $i = (1, 0)$, $j = (0, 1)$ then the set $S = \{ i, j \}$ is linearly independent in R^2. Also any vector $v = (a, b)$ in R^2 can be expressed as $ai + bj$ *i.e.* as a linear combination of i and j.

$$v = ai + bj$$

Thus, $S = \{ i, j \}$ is a basis for R^2. It is called the **standard basis** for R^2.

A basis is a generalisation of coordinate system. In a plane any two linearly independent vectors can give us a coordinate system. Any vector in that plane can be expressed in terms of these vectors. But the above set $S = \{ i, j \}$ is called the standard basis. Any vector v in the plane can be expressed as a linear combination of i and j as $ai + bj$. The coefficients (a, b) are called the coordinates of the point P.

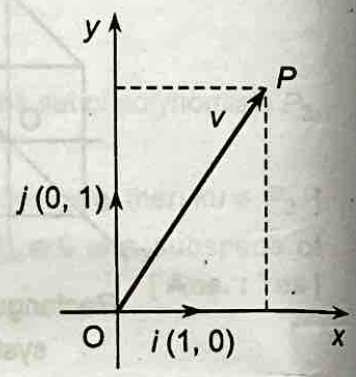

Fig. 6.17

Example 2 : Standard Basis for R^3

We can also show that if

$i = (1, 0, 0)$, $j = (0, 1, 0)$, $k = (0, 0, 1)$

then the set $S = \{ i, j, k \}$ is linearly independent in R^3. Also any vector $v = (a, b, c)$ in R^3 can be expressed as $ai + bj + ck$ *i.e.* as a linear combination of i, j and k.

$$v = ai + bj + ck$$

Thus, $S = \{ i, j, k \}$ is a basis for R^3. It is called the **standard basis** for R^3.

Any vector v can be expressed as $ai + bj + ck$.

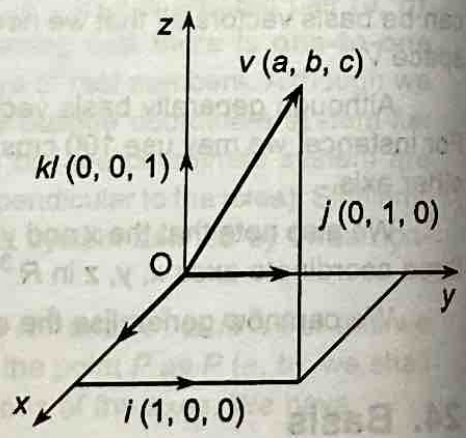

Fig. 6.18

Example 3 : Standard Basis for R^n

We can show that if $e_1 = (1, 0, ..., 0)$, $e_2 = (0, 1, ..., 0)$, ..., $e_n = (0, 0, ..., 1)$ then the set $S = \{ e_1, e_2, ..., e_n \}$ is linearly independent. Also any vector $v = (v_1, v_2, ..., v_n)$ can be expressed as a linear combination of $e_1, e_2, ..., e_n$ *i.e.* as

$$v = v_1 e_1 + v_2 e_2 + ... + v_n e_n$$

Thus, $S = \{ e_1, e_2, ..., e_n \}$ is a basis for R^n. It is called the **standard basis** for R^n.

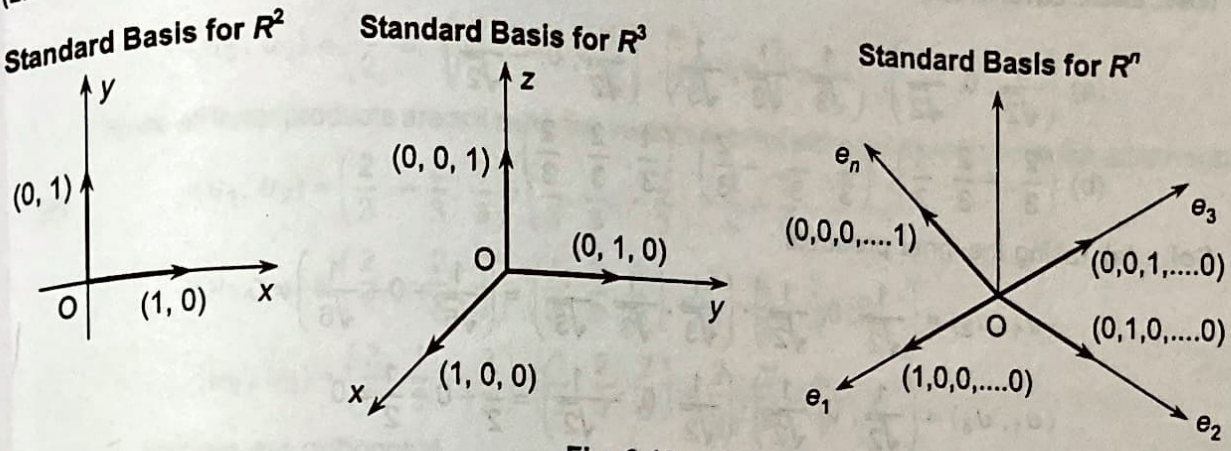

Fig. 6.19

25. Dimension

You might have observed that all basis for R^2 have two vectors, all bases for R^3 have three vectors, all bases for R^n have n vectors. This can be stated as a theorem.

Theorem : All bases for a finite dimensional vector space has the same number of vectors.

The number of vectors in a basis is called the dimension of the vector space.

We define the dimension of zero vector to be zero, the dimension of a line to be one, the dimension of a plane to be two and the dimension of a cube to be three.

Definition : The number of a vectors in a basis for V is called the **dimension** of the vector V and is denoted by dim (V).

We have already seen that the standard basis for vector space R^n is

$$e_1 = (1, 0, ..., 0),\ e_2 = (0, 1, ..., 0),\ ....,\ e_n = (0, 0, ..., 1)$$

Hence, $\boxed{\dim (R^n) = n}$

Hence, $\boxed{\dim (P_n) = n + 1}$

EXERCISE - V

Explain the terms bases and dimensions of a vector space.

26. Orthonormal Basis

In this section we shall learn orthonormal vectors, orthonormal basis. Then we shall learn how to construct orthogonal basis from an ordinary basis of an inner product space by using Gram-Schmidt process.

Definition : A set of vectors in an inner product space is called an **orthogonal set** if all the distinct vectors are orthogonal. If the norm of each vector in the orthogonal set is 1, then it is called **orthonormal set.**

Example 1 : Check whether the following sets of vectors in R^3 are orthogonal with respect to the Euclidean inner product.

(a) $\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$

(b) $\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

Sol. : (a) Taking the inner products

$$\langle u_1, u_2\rangle = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{6}} + 0 + \frac{1}{\sqrt{6}}\right) \neq 0$$

$$\langle u_1, u_3\rangle = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\cdot\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

$$\langle u_2, u_3\rangle = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)\cdot\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$$

Since $\langle u_1, u_2\rangle \neq 0$, the vectors are not orthogonal.

(b)
$$\langle u_1, u_2\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{4}{3} - \frac{2}{3} - \frac{2}{3} = 0$$

$$\langle u_1, u_3\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} - \frac{4}{3} + \frac{2}{3} = 0$$

$$\langle u_2, u_3\rangle = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{3} + \frac{2}{3} - \frac{4}{3} = 0$$

Since all inner products are zero, the vectors are orthogonal.

Example 2 : Check whether the above vectors are orthonormal.

Sol. : (a) Since the vectors are not orthogonal, they are not orthonormal.

(b) $\|u_1\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$; $\quad \|u_2\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$;

$\|u_3\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1.$

Since the vectors u_1, u_2, u_3 are orthogonal and their norms are 1, they are orthonormal.

Example 3 : Which of the following sets are orthonormal with respect to the inner product P_2 defined by

$$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 \quad \text{where } p = a_0 + a_1 x + a_2 x^2,\ q = b_0 + b_1 x + b_2 x^2\ ?$$

(a) $x, \quad \frac{1}{2} + x^2, \quad 1 - x - \frac{1}{2}x^2$

(b) $\frac{2}{3} - \frac{2}{3}x + \frac{1}{3}x^2, \quad \frac{2}{3} + \frac{1}{3}x - \frac{2}{3}x^2, \quad \frac{1}{3} + \frac{2}{3}x + \frac{2}{3}x^2$

Sol. : (a) We have

$$\langle u_1, u_2\rangle = (0, 1, 0)\cdot\left(\frac{1}{2}, 0, 1\right) = 0$$

$$\langle u_1, u_3\rangle = (0, 1, 0)\cdot\left(1, -1, -\frac{1}{2}\right) = -1 \neq 0$$

$$\langle u_2, u_3\rangle = \left(\frac{1}{2}, 0, 1\right)\cdot\left(1, -1, -\frac{1}{2}\right) = \frac{1}{2} + 0 - \frac{1}{2} = 0$$

Since all inner products are not zero, the vectors are not orthogonal and hence not orthonormal.

(b) $$\langle u_1, u_2\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \frac{4}{9} - \frac{2}{9} - \frac{2}{9} = 0$$

$$\langle u_1, u_3\rangle = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{9} - \frac{4}{9} + \frac{2}{9} = 0$$

$$\langle u_2, u_3\rangle = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = \frac{2}{9} + \frac{2}{9} - \frac{4}{9} = 0$$

$\therefore$ Vectors are orthogonal.

Further, $$\|u_1\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1; \quad \|u_2\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1;$$

$$\|u_3\| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1.$$

Since the vectors are orthogonal and have norms 1, they are orthonormal.

[Note that this is the same example as Ex. 1 (b).]

Example 4 : Check whether the following vectors are orthogonal with respect to the inner product of two matrices $u = \begin{bmatrix} u_1 & u_2 \\ u_3 & u_4 \end{bmatrix}$, $v = \begin{bmatrix} v_1 & v_2 \\ v_3 & v_4 \end{bmatrix}$ defined by $\langle u, v\rangle = \text{tr}\,(v' u)$.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 2/3 & -2/3 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2/3 \\ -2/3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 2/3 & 2/3 \end{bmatrix}$

Sol. : (a) We know that

$$\langle u, v\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$$

$\therefore$ $\langle u, v\rangle = 0 + 0 + 0 + 0 = 0$ $\quad\therefore$ u, v are orthogonal.

(b) $\langle u, v\rangle = 0 + \frac{4}{9} - \frac{4}{9} + 0 = 0$ $\quad\therefore$ u, v are orthogonal.

Example 5 : Verify that $v_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $v_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $v_3 = (0, 0, 1)$ are orthonormal and if $u = (1, -1, 2)$ then express u as a linear combination of v_1, v_2, v_3.

Sol. : We have $\langle v_1, v_2\rangle = -\frac{12}{5} + \frac{12}{5} + 0 = 0;$

$\langle v_1, v_3\rangle = 0 + 0 + 0 = 0;$ $\quad\langle v_2, v_3\rangle = 0 + 0 + 0 = 0.$

Hence, v_1, v_2, v_3 are orthogonal.

Further, $$\|v_1\| = \sqrt{\frac{9}{25} + \frac{16}{25} + 0} = 1; \quad \|u_2\| = \sqrt{\frac{16}{25} + \frac{9}{25} + 0} = 1;$$

$$\|u_3\| = \sqrt{0 + 0 + 1} = 1.$$

Since, the vectors are orthogonal and of norms one, they are orthonormal.

Now, $u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \langle u, v_3\rangle v_3$

$$\therefore \quad u = \left[(1, -1, 2)\cdot\left(-\frac{3}{5}, \frac{4}{5}, 0\right)\right]v_1 + \left[(1, -1, 2)\cdot\left(\frac{4}{5}, \frac{3}{5}, 0\right)\right]v_2 + [(1, -1, 2)\cdot(0, 0, 1)]v_3$$

$$= \left(-\frac{3}{5} - \frac{4}{5} + 0\right)v_1 + \left(\frac{4}{5} - \frac{3}{5} + 0\right)v_2 + (0 + 0 + 2)v_3$$

$$= -\frac{7}{5}v_1 + \frac{1}{5}v_2 + 2v_3.$$

(You can verify the equality.)

Example 6 : Find the coordinate vector of u with respect to the orthonormal basis $\{v_1, v_2, v_3\}$ where

$$u = (-1, 0, 2),\ v_1 = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right),\ v_2 = \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right),\ v_3 = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

Sol. : We have already checked that the set $\{v_1, v_2, v_3\}$ is an orthonormal set [Ex. 1 (b), page 6-39]

Now, $u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \langle u, v_3\rangle v_3$

$$\therefore u = \left[(-1, 0, 2)\cdot\left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)\right]v_1 + \left[(-1, 0, 2)\cdot\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)\right]v_2 + \left[(-1, 0, 2)\cdot\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)\right]v_3$$

$$= 0v_1 + \left(-\frac{2}{3} - \frac{4}{3}\right)v_2 + \left(-\frac{1}{3} + \frac{4}{3}\right)v_3$$

$$= 0v_1 - 2v_2 + v_3$$

$$\therefore\ (u)_S = (0, -2, 1).$$

EXERCISE - VI

1. Check whether the following sets of vectors are orthogonal with respect to the Euclidean inner product.

(a) $(0, 1), (2, 0)$ **(b)** $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}\right)$

(c) $(1, 0, 0),\ (0, 1, 0),\ (0, 0, 1)$ **(d)** $\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right), \left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, 0\right)$

[Ans. : All]

2. Check whether the following set of polynomials is orthogonal with respect to the inner product in P_2.

$$1,\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}x^2,\ x^2$$

[Ans. : No]

3. Check whether the following sets of matrices are orthogonal with respect to the inner product on M_{22}.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}, \begin{bmatrix} 0 & 2/3 \\ -2/3 & 1/3 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

[Ans. : Both]

4. Verify that the vectors $v_1 = \left(-\frac{3}{5}, \frac{4}{5}, 0\right)$, $v_2 = \left(\frac{4}{5}, \frac{3}{5}, 0\right)$, $v_3 = (0, 0, 1)$ form an orthonormal basis in R^3 with respect to the Euclidean inner product. Express the vector $(3, -7, 4)$ as a linear combination of v_1, v_2, v_3.

[Ans. : $\left(-\frac{37}{5}v_1 - \frac{9}{5}v_2 + 4v_3\right)$]

5. Check whether the following vectors are orthonormal and then find the coordinates of w with respect to u_1, u_2.

$$u_1 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right),\ u_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),\ w = (3, 7)$$

[Ans. : $(w)_s = \left(-2\sqrt{2}, 5\sqrt{2}\right)$]

6. If $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$, $v_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$, show that the vectors are orthonormal. Express $u = (1, 1, 1)$ as a linear combination of v_1, v_2, v_3 and find the coordinate vectors of u with respect to $S = \{v_1, v_2, v_3\}$.

[Ans. : $u = 1 \cdot u_1 - \frac{1}{5}u_2 + \frac{7}{5}u_3$, $(u)_s = \left(1, -\frac{1}{5}, \frac{7}{5}\right)$]

27. Projection Theorem

We now state the following theorem known as projection theorem.

Projection Theorem

If W is a finite dimensional subspace of an inner product space V, then every vector u in V can be expressed uniquely as

$$\boxed{u = w_1 + w_2}$$

where w_1 is in W and w_2 is in $W^{\perp}$.

($W^{\perp}$ read as "W perp" denotes the orthogonal complement of W.)

Proof : We shall accept this theorem without proof.

Example 1 : Let $W = \text{span}\left\{(0, 1, 0), \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right\}$.

Express, $w = (1, 1, 1)$ in the form $w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^{\perp}$.

Sol. : Let $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Then projection of w on W

$$\text{proj}_W\, w = \langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2$$

$$= [(1, 1, 1)\cdot(0, 1, 0)]v_1 + (1, 1, 1)\cdot\left(-\frac{4}{5}, 0, \frac{3}{5}\right)v_2$$

$$\text{proj}_W\, w = 1\cdot v_1 - \frac{1}{5}v_2 = 1(0, 1, 0) - \frac{1}{5}\left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$w_1 = \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

This is the component of w which is in W.

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w = (1, 1, 1) - \left(\frac{4}{25}, 1, -\frac{3}{25}\right)$$

$$w_2 = \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

This is the component of w which is in $W^\perp$.

Hence, $w = w_1 + w_2$ (w_1 is in W and w_2 is in $W^\perp$)

$$= \left(\frac{4}{25}, 1, -\frac{3}{25}\right) + \left(\frac{21}{25}, 0, \frac{28}{25}\right)$$

(Verify that $w_1 + w_2 = (1, 0, 1)$.)

Example 2 : Let $W = \text{span}\left\{(0, 1, 0), \left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right\}$.

Express $w = (1, 2, 3)$ in the form of $w = w_1 + w_2$, where $w_1 \in W$ and $w_2 \in W^\perp$.

Sol. : Let $v_1 = (0, 1, 0)$, $v_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$.

Then the projection of w on W.

$$\text{proj}_W w = \langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2$$

$$= [(1, 2, 3)\cdot(0, 1, 0)]v_1 + \left[(1, 2, 3)\cdot\left(-\frac{4}{5}, 0, \frac{3}{5}\right)\right]v_2$$

$$= 2v_1 + 1\cdot v_2 = 2(0, 1, 0) + 1\cdot\left(-\frac{4}{5}, 0, \frac{3}{5}\right)$$

$$w_1 = \left(-\frac{4}{5}, 2, \frac{3}{5}\right)$$

This is the component of w in W.

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w = (1, 2, 3) - \left(-\frac{4}{5}, 2, \frac{3}{5}\right)$$

$$w_2 = \left(\frac{9}{5}, 0, \frac{12}{5}\right)$$

This is the component of w in $W^\perp$.

Hence, $w = w_1 + w_2$ (where $w_1 \in W$ and $w_2 \in W^\perp$)

$$= \left(-\frac{4}{5}, 2, \frac{3}{5}\right) + \left(\frac{9}{5}, 0, \frac{12}{5}\right)$$

Example 3 : Let R^4 have the Euclidean inner product. Express $w = (-1, 2, 6, 0)$ in the
$w = w_1 + w_2$ where $w_1 \in W$ and $w_2 \in W^\perp$ where w is spanned by $v_1 = (-1, 0, 1, 2)$ and $v_2 = (0, 1, 0$

Sol. : The project of w on W

$$\text{proj}_W w = \langle w, v_1\rangle v_1 + \langle w, v_2\rangle v_2$$

$$\therefore \quad \text{proj}_W w = [(-1, 2, 6, 0) \cdot (-1, 0, 1, 2)]\, v_1 + [(-1, 2, 6, 0) \cdot (0, 1, 0, 1)]\, v_2$$
$$= 7v_1 + 2v_2$$
$$= 7\,(-1, 0, 1, 2) + 2\,(0, 1, 0, 1)$$
$$\therefore \quad w_1 = (-7, 2, 7, 16)$$

The component of w orthogonal to W is

$$\text{proj}_{W^\perp} w = w - \text{proj}_W w$$
$$= (-1, 2, 6, 0) - (-7, 2, 7, 16)$$
$$\therefore \quad w_2 = (6, 0, -1, -16)$$
$$\therefore \quad w = w_1 + w_2 \quad \text{where } w_1 \in W \text{ and } w_2 \in W^\perp$$
$$= (-7, 2, 7, 16) + (6, 0, -1, -16).$$

28. Gram-Schmidt Process

Gram-Schmidt process gives us a method of finding orthonormal vectors.

Let for convenience V be a non-zero inner product space in R^3 and let $u = \{u_1, u_2, u_3\}$ be any base for V. The following steps lead to an orthogonal basis $\{v_1, v_2, v_3\}$ for V.

Step 1 : Let $v_1 = u_1$.

Step 2 : As shown in the Fig. 4.3 (a), we obtain a vector v_2, orthogonal to v_1.

$v_2 = u_2 - \text{proj } u_2$ which is given by

$$v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$$

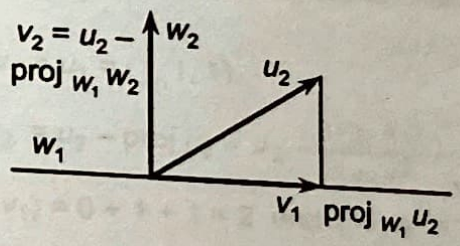

Fig. 6.20 (a)

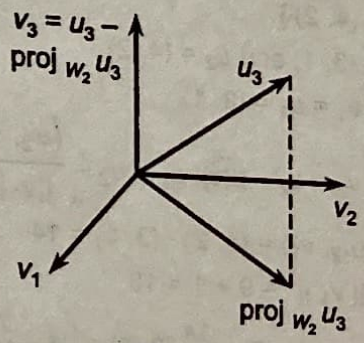

Fig. 6.20 (b)

Step 3 : To find v_3 that is orthogonal to both v_1 and v_2, we find u_3 orthogonal to the space w_2 spanned by v_1 and v_2 which is given by

$$v_3 = v_3 - \text{proj } u_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_2, v_2 \rangle}{\|v_2\|^2} v_2$$

Thus, we get the orthogonal set $\{v_1, v_2, v_3\}$.

Jörgen Pederson Gram (1850-1916)

Jörgen Pederson Gram was an insurance expert in Denmark. He got his master's degree in mathematics with specialisation in the then new developing subject modern algebra. He then joined

the Hafnia Life Insurance Company where he developed mathematical foundations of accident insurance. During this period he got his Ph.D. on "On Series Development Utilising the Least Square Methods". On this thesis Gram-Schmidt process is based. In 1910 he became director of the Danish (Denmark) Insurance Board. Gram eventually developed interest in number theory and won a gold medal from "The Royal Danish Society of Sciences And Letters". He had keen interest in the interplay between applied mathematics and pure mathematics, which led to four treatises on Danish forest management. Unfortunately he died in a an accident while going to attend a meeting of the Royal Danish Society.

Erhardt Schmidt (1876-1959)

Erhardt Schmidt was a reputed German mathematician. He got his Ph.D. degree from Göttingen University, Germany in 1905. He was a student of another giant German mathematician David Hilbert. He then went to teach at Berlin University in 1917 and stayed there for the rest of his life. He made important contributions to various mathematical fields. Schmidt first described 'Gram-Schmidt process' in a paper on integral equations published in 1907. He is credited to have developed the general concept "Hilbert Space" which is fundamental in the study of infinite-dimensional vector spaces.

Example 1 : Construct an orthonormal basis of R^2 by applying Gram-Schmidt orthogonalisation to $S = \{ (3, 1), (4, 2) \}$. **(M.U. 2014)**

Sol. : Let $u_1 = (3, 1)$ and $u_2 = (4, 2)$.

Step 1 : $v_1 = u_1 = (3, 1)$

Step 2 : $v_2 = u_2 - \text{proj. } u_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Now, $\langle u_2, v_1 \rangle = (4, 2) \cdot (3, 1) = 14$

$\| v_1 \|^2 = 9 + 1 = 10$

$$\therefore \quad v_2 = (4, 2) - \frac{14}{10}(3, 1) = \left(4 - \frac{7}{5} \cdot 3,\ 2 - \frac{7}{5} \cdot 1\right) = \left(-\frac{1}{5}, \frac{3}{5}\right)$$

Hence, $v_1 = (3, 1)$, $v_2 = \left(-\dfrac{1}{5}, \dfrac{3}{5}\right)$ form the orthogonal basis for R^2.

Now, the norms of these vectors are

$$\| v_1 \| = \sqrt{9 + 14} = \sqrt{10}$$

$$\| v_2 \| = \sqrt{\frac{1}{25} + \frac{9}{25}} = \sqrt{\frac{10}{25}} = \sqrt{\frac{2}{5}}$$

Hence, the orthonormal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{5} \cdot \sqrt{\frac{5}{2}}, \frac{3}{5} \cdot \sqrt{\frac{5}{2}}\right) = \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$$

Example 2 : Find an orthonormal basis for the subspaces of R^3 by applying Gram-Schmidt process where $S = \{ (1, 2, 0), (0, 3, 1) \}$. **(M.U. 2014, 15)**

Sol. : Let $u_1 = (1, 2, 0)$ and $u_2 = (0, 3, 1)$.

Step 1 : $v_1 = u_1 = (1, 2, 0)$

Step 2 : $v_2 = u_2 - \text{proj. } u_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Now, $\langle u_2, v_1 \rangle = (0, 3, 1) \cdot (1, 2, 0) = 0 + 6 + 0 = 6$

and $\| v_1 \|^2 = 1 + 4 + 0 = 5$

$$\therefore \quad v_2 = (0, 3, 1) - \frac{6}{5}(1, 2, 0) = \left(-\frac{6}{5}, \frac{3}{5}, 1\right)$$

Norms of these vectors are

$$\| v_1 \| = \sqrt{5}, \quad \| v_2 \| = \sqrt{\frac{36}{25} + \frac{9}{25} + 1} = \sqrt{\frac{70}{25}} = \sqrt{\frac{14}{5}}.$$

Hence, the orthonormal basis are

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0\right)$$

$$q_2 = \frac{v_2}{\|v_2\|} = \sqrt{\frac{5}{14}}\left(-\frac{6}{5}, \frac{3}{5}, 1\right) = \left(-\frac{6}{\sqrt{70}}, \frac{3}{\sqrt{70}}, \frac{5}{\sqrt{70}}\right)$$

Example 3 : Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into orthonormal bases where $u_1 = (1, 1, 1)$, $u_2 = (0, 1, 1)$, $u_3 = (0, 0, 1)$. **(M.U. 2016)**

Sol. : Step 1 : $v_1 = u_1 = (1, 1, 1)$.

Step 2 : $v_2 = u_2 - \text{proj } u_2 = u_2 - \dfrac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1$

Now, $\langle u_2, v_1 \rangle = 0 + 1 + 1 = 2$ and $\| v_1 \|^2 = 1 + 1 + 1 = 3$

$$\therefore \quad v_2 = (0, 1, 1) - \frac{2}{3}(1, 1, 1) = \left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3 = u_3 - \dfrac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2$

Now, $\langle u_3, v_1 \rangle = 0 + 0 + 1 = 1$; $\quad \langle u_3, v_2 \rangle = 0 + 0 + \dfrac{1}{3} = \dfrac{1}{3}$

$\| v_1 \|^2 = 3$ as before. $\quad \| v_2 \|^2 = \dfrac{4}{9} + \dfrac{1}{9} + \dfrac{1}{9} = \dfrac{6}{9} = \dfrac{2}{3}$

$$\therefore \quad v_3 = (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1/3}{2/3}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

$$= (0, 0, 1) - \frac{1}{3}(1, 1, 1) - \frac{1}{2}\left(-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}\right) = \left(0, -\frac{1}{2}, \frac{1}{2}\right)$$

Hence, $v_1 = (1, 1, 1)$, $v_2 = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$, $v_3 = \left(0, -\dfrac{1}{2}, \dfrac{1}{2}\right)$ form the orthogonal basis for R^3.

Now, the norms of these vectors are $\|v_1\| = \sqrt{1+1+1} = \sqrt{3}$;

$$\|v_2\| = \sqrt{\frac{4}{9}+\frac{1}{9}+\frac{1}{9}} = \frac{\sqrt{6}}{3}; \qquad \|v_3\| = \sqrt{0+\frac{1}{4}+\frac{1}{4}} = \sqrt{\frac{2}{4}} = \frac{1}{\sqrt{2}}.$$

And hence orthonormal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right); \quad q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right);$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right).$$

Example 4 : Let R^3 have the Euclidean inner product. Use Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into an orthonormal basis where $u_1 = (1, 1, 1)$, $u_2 = (-1, 1, 0)$, $u_3 = (1, 2, 1)$. **(M.U. 2017, 18)**

Sol. : Step 1 : $v_1 = u_1 = (1, 1, 1)$

Step 2 : $v_2 = u_2 - \text{proj } u_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\|v_1\|^2} \cdot v_1$

Now, $\langle u_2, v_1\rangle = -1 + 1 - 0 = 0$ and $\| v_1 \|^2 = 1 + 1 + 1 = 3$

$$\therefore \quad v_2 = (-1, 1, 0) - \frac{0}{3}(1, 1, 1) = (-1, 1, 0)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3 = u_3 - \dfrac{\langle u_3, v_1\rangle}{\|v_1\|^2} \cdot v_1 - \dfrac{\langle u_3, v_2\rangle}{\|v_2\|^2} \cdot v_2$

Now, $\langle u_3, v_1\rangle = 1 + 2 + 1 = 4$; $\quad \langle u_3, v_2\rangle = -1 + 2 + 0 = 1$

$\| v_1 \|^2 = 3$ as before. $\quad \| v_2 \|^2 = 1 + 1 + 0 = 2$

$$\therefore \quad v_3 = (1, 2, 1) - \frac{4}{3}(1, 1, 1) - \frac{1}{2}(-1, 1, 0) = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$$

Hence, $v_1 = (1, 1, 1)$, $v_2 = (-1, 1, 0)$, $v_3 = \left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}\right)$ form orthogonal basis for R^3.

Now, the norms of these vectors are $\|v_1\| = \sqrt{1+1+1} = \sqrt{3}$;

$$\|v_2\| = \sqrt{1+1+0} = \sqrt{2}; \quad \|v_3\| = \sqrt{\frac{1}{36}+\frac{1}{36}+\frac{1}{9}} = \sqrt{\frac{6}{36}} = \frac{1}{\sqrt{6}}$$

Hence, the orthonormal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right); \qquad q_2 = \frac{v_2}{\|v_2\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right);$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}\right) \qquad \left[\because \frac{\sqrt{6}}{3} = \frac{\sqrt{6}\cdot\sqrt{6}}{3\sqrt{6}} = \frac{6}{3\sqrt{6}} = \frac{2}{\sqrt{6}}\right]$$

Example 5 : Let R^3 have the Euclidean inner product. Use the Gram-Schmidt process to transform the basis $\{u_1, u_2, u_3\}$ into orthonormal basis where $u_1 = (1, 0, 0)$, $u_2 = (3, 7, -2)$, $u_3 = (0, 4, 1)$. **(M.U. 2019)**

Sol. : Step 1 : $v_1 = u_1 = (1, 0, 0)$

Step 2 : $v_2 = u_2 - \text{proj } u_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\|v_1\|^2} \cdot v_1$

Now, $\langle u_2, v_1\rangle = 3 + 0 + 0 = 3$ and $\| v_1 \|^2 = 1 + 0 + 0 = 1$

$$\therefore \quad v_2 = (3, 7, -2) - \frac{3}{1}(1, 0, 0) = (0, 7, -2)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3 = u_3 - \dfrac{\langle u_3, v_1\rangle}{\|v_1\|^2}\cdot v_1 - \dfrac{\langle u_3, v_2\rangle}{\|v_2\|^2}\cdot v_2$

Now, $\langle u_3, v_1\rangle = 0 + 0 + 0 = 0$; $\langle u_3, v_2\rangle = 0 + 28 - 2 = 26$

$\| v_1 \|^2 = 1$ as before. $\| v_2 \|^2 = 0 + 49 + 4 = 53$

$$\therefore \quad v_3 = (0, 4, 1) - 0 - \frac{26}{53}(0, 7, -2) = \left(0, \frac{30}{53}, \frac{105}{53}\right)$$

Hence, $v_1 = (1, 0, 0)$, $v_2 = (0, 7, -2)$, $v_3 = \left(0, \dfrac{30}{53}, \dfrac{105}{53}\right)$ form orthogonal basis for R^3.

Now, the norms of these vectors are

$$\|v_1\| = \sqrt{1+0+0} = 1; \qquad \|v_2\| = \sqrt{0+49+4} = \sqrt{53}$$

$$\|v_3\| = \sqrt{0 + \frac{900}{53^2} + \frac{11025}{53^2}} = \frac{15}{\sqrt{53}}$$

And hence, the orthonormal basis for R^3 is

$$q_1 = \frac{v_1}{\|v_1\|} = (1, 0, 0); \quad q_2 = \frac{v_2}{\|v_2\|} = \left(0, \frac{7}{\sqrt{53}}, -\frac{2}{\sqrt{53}}\right);$$

$$q_3 = \frac{v_3}{\|v_3\|} = \left(0, \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}}\right).$$

Example 6 : Construct all orthonormal basis of R^3 using Gram-Schmidt process to

$S = \{ (3, 0, 4), (-1, 0, 7), (2, 9, 11) \}$ **(M.U. 2016)**

Sol. : Step 1 : $v_1 = u_1 = (3, 0, 4)$

Step 2 : $v_2 = u_2 - \text{proj } u_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\|v_1\|^2}\cdot v_1$

Now, $\langle u_2, v_1\rangle = -3 + 0 + 28 = 25$

and $\| v_1 \|^2 = 9 + 0 + 16 = 25$

$$\therefore \quad v_2 = (-1, 0, 7) - \frac{25}{25}(3, 0, 4) = (-4, 0, 3)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3 = u_3 - \dfrac{\langle u_3, v_1\rangle}{\|v_1\|^2}\cdot v_1 - \dfrac{\langle u_3, v_2\rangle}{\|v_2\|^2}\cdot v_2$

Now, $\langle u_3, v_1\rangle = (2, 9, 11)\cdot(3, 0, 4) = 6 + 0 + 44 = 50$;

$\langle u_3, v_2\rangle = (2, 9, 11)\cdot(-4, 0, 3) = -8 + 33 = 25$

$\| v_1 \|^2 = 25$ as above and $\| v_2 \|^2 = 16 + 0 + 9 = 25$

$$\therefore \quad v_3 = (2, 9, 11) - \frac{50}{25}(3, 0, 4) + \frac{25}{25}(-4, 0, 3)$$

$$= (2, 9, 11) - 2\,(3, 0, 4) - 1\,(-4, 0, 3)$$

$$= (0, 9, 0)$$

But $\| v_1 \| = 5$, $\| v_2 \| = 5$ and $\| v_3 \| = \sqrt{0^2 + 9^2 + 0^2} = 9$

$$\therefore \quad q_1 = \frac{v_1}{\|v_1\|} = \frac{1}{5}(3, 0, 4); \quad q_2 = \frac{v_2}{\|v_2\|} = \frac{1}{5}(-4, 0, 3);$$

$$q_3 = \frac{v_3}{\|v_3\|} = \frac{1}{9}(0, 9, 0) = (0, 1, 0).$$

Example 7 : Let R^4 have the Euclidean inner product. Use Gram-Schmidt Process to transform the basis $\{u_1, u_2, u_3\}$ into orthogonal basis, where $u_1 = \{1, 0, 1, 1\}$, $u_2 = \{-1, 0, -1, 1\}$, $u_3 = \{0, -1, 1, 1\}$. **(M.U. 2018)**

Sol. : Step 1 : $v_1 = u_1 = (1, 0, 1, 1)$

Step 2 : $v_2 = u_2 - \text{proj } u_2 = u_2 - \dfrac{\langle u_2, v_1\rangle}{\|v_1\|^2} \cdot v_1$

Now, $\langle u_2, v_1\rangle = -1 + 0 - 1 + 1 = -1$ and $\|v_1\|^2 = 1 + 0 + 1 + 1 = 3$

$$\therefore \quad v_2 = (-1, 0, -1, 1) - \frac{-1}{3}(1, 0, 1, 1) = \left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right)$$

Step 3 : $v_3 = u_3 - \text{proj } u_3 = u_3 - \dfrac{\langle u_3, v_1\rangle}{\|v_1\|^2} \cdot v_1 - \dfrac{\langle u_3, v_2\rangle}{\|v_2\|^2} \cdot v_2$

Now, $\langle u_3, v_1\rangle = 0 + 0 + 1 + 1 = 2$

$$\langle u_3, v_2\rangle = (0, -1, 1, 1) \cdot \left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right) = 0 + 0 - \frac{2}{3} + \frac{4}{3} = \frac{2}{3}$$

$\|v_1\|^2 = 3$ as before. $\quad \|v_2\|^2 = \dfrac{4}{9} + 0 + \dfrac{4}{9} + \dfrac{16}{9} = \dfrac{24}{9} = \dfrac{8}{3}$

$$\therefore \quad v_3 = (0, -1, 1, 1) - \frac{2}{3}(1, 0, 1, 1) - \frac{2/3}{8/3}\left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right)$$

$$\therefore \quad v_3 = (0, -1, 1, 1) - \frac{2}{3}(1, 0, 1, 1) - \frac{1}{4}\left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right)$$

$$= (0, -1, 1, 1) + \left(-\frac{2}{3}, 0, -\frac{2}{3}, -\frac{2}{3}\right) + \left(\frac{1}{6}, 0, \frac{1}{6}, -\frac{1}{3}\right) = \left(\frac{1}{2}, -1, \frac{1}{2}, 0\right)$$

Hence, the required orthogonal set is

$$v_1 = (1, 0, 1, 1), \; v_2 = \left(-\frac{2}{3}, 0, -\frac{2}{3}, \frac{4}{3}\right), \; v_3 = \left(\frac{1}{2}, -1, \frac{1}{2}, 0\right).$$

EXERCISE - VII

1. Construct an orthonormal basis of R^2 using Gram-Schmidt process to
   (I) $S = \{(3, 1), (2, 3)\}$; (II) $S = \{(3, 1), (2, 2)\}$
   [**Ans. :** (I) $\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$, (II) $\left(\frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right), \left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$]
2. Construct an orthonormal basis of R^3 using Gram-Schmidt process to
   $S = \{(3, 0, 4), (-1, 0, 7), (2, 9, 11)\}$
   [**Ans. :** $\left(\frac{3}{5}, 0, \frac{4}{5}\right), \left(-\frac{4}{5}, 0, \frac{3}{5}\right), (0, 1, 0)$]

x	y	y^2	xi
2	1600	2560000	3200

3. Construct an orthonormal basis of R^4 using Gram-Schmidt process to $S = \{(1, 1, 1, 1), (1, 2, 4, 5), (1, -3, -4, -2)\}$

[**Ans. :** $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{-2}{\sqrt{10}}, \frac{-1}{\sqrt{10}}, \frac{1}{\sqrt{10}}, \frac{2}{\sqrt{10}}\right), \left(\frac{16}{\sqrt{910}}, \frac{-17}{\sqrt{910}}, \frac{-13}{\sqrt{910}}, \frac{14}{\sqrt{910}}\right)$]

EXERCISE - VIII

Objective Questions

1. What can we say about the vector u and v if $\| u + v \| = \| u - v \|$?

[**Ans. :** They are orthogonal.]

2. Interpret the following result geometrically in R^2.

$\| u + v \| = \| u - v \|$

[**Ans. :** u, v are the sides of a rectangle.]

3. If u, v and w form a triangle and if $\| u \| = 3$, $\| v \| = 4$, $\| u + v \| = \| u - v \|$, then find $\| w \|$.

[**Ans. :** 5]

4. If u and v are orthogonal unit vectors then find the distance between them. [**Ans. :** $\sqrt{2}$]

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