CHAPTER 2

Complex Integration

1. Introduction

We have already studied differentiability of complex functions. Now, we shall take up the reverse process *i.e.* integration of complex functions. As in real variables, in complex variables also we have definite and indefinite integrals. An indefinite integral of a complex variable is a function whose derivative is equal to a given analytic function in a region. The indefinite integrals of many elementary functions can be obtained by mere inversion of known derivatives. However, the theory of definite integral of real variables cannot be used straight way for complex variables. The definite integral of a complex variable may depend upon the path of integration in the complex plane.

2. Path of Integration

In the case of real variables, the path of integration of $\int_a^b f(x)\,dx$ is always along the real axis from $x = a$ to $x = b$. But in the case of complex variables the path of the definite integral $\int_a^b f(z)\,dz$ may be any curve joining the points $z = a$ and $z = b$. Generally, the value of this integration depends upon the path. However, as we shall see later the value remains the same in some special cases. Our approach to this topic for obvious reasons will be practical rather than analytical.

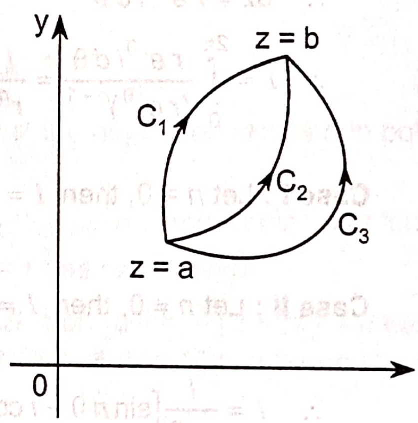


Fig. 2.1

3. Definition

Let $f(z)$ be a continuous function of the complex variable $z = x + iy$ defined at every point of a curve C whose end points are A and B. Let us divide the curve C into n parts by points

$$A = P_0(z_0),\ P_1(z_1),\ P_2(z_2)\ldots\ldots P_i(z_i)\ldots\ldots P_n(z_n) = B$$

Let $\delta z_i = z_i - z_{i-1}$ and let ξ_i be a point on the arc $P_{i-1} - P_i$. Then the limit of the sum $\sum_{i=1}^{n} f(\xi_i)\,\delta z_i$ as $n \to \infty$ in such a way that each $\delta z_i \to 0$, if it exits is called the **line integral** of $f(z)$ along C and is denoted by

$$\int_C f(z)\,dz.$$

If C is a closed curve *i.e.* if P_0 and P_n coincide the integral is called the **contour integral** and is denoted by

$$\oint_C f(z)\,dz.$$

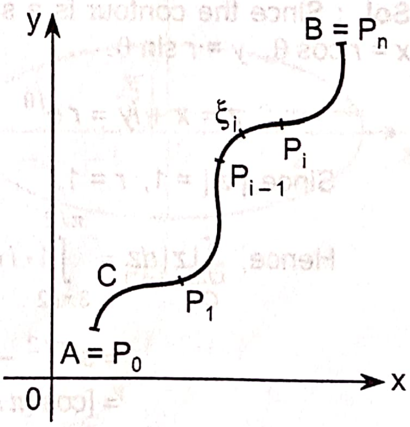


Fig. 2.2

4. Evaluation of Line Integral

In practice, the evaluation of a line integral is reduced to the evaluation of two real line integrals as follows.

Since $z = x + iy$, $dz = dx + i\,dy$.

If $f(z) = u + iv$.

$$\int_C f(z)\,dz = \int_C (u + iv)(dx + i\,dy)$$

$$\boxed{\int_C f(z)\,dz = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)}$$

Thus, the integral on the l.h.s. is converted into two integrals on r.h.s.

(a) When the contour is a circle

Example 1 : Evaluate $\int_C \frac{dz}{(z-z_0)^{n+1}}$, where n is an integer and C is a circle $|z - z_0| = r$. **(M.U. 2003)**

Sol. : Let $z - z_0 = r\,e^{i\theta}$, so that θ varies from 0 to 2π as z describes the circle C.

$$\therefore \quad dz = r\,e^{i\theta} i\,d\theta$$

$$\therefore \quad I = \int_0^{2\pi} \frac{r\,e^{i\theta} i\,d\theta}{(r\,e^{i\theta})^{n+1}} = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta} d\theta$$

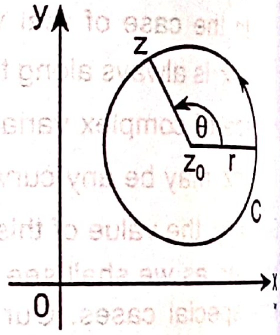


Fig. 2.3

Case I : Let $n = 0$, then $I = i\int_0^{2\pi} d\theta = 2\pi i$

Case II : Let $n \neq 0$, then $I = \frac{i}{r^n}\int_0^{2\pi} (\cos n\theta - i\sin n\theta)\,d\theta$

$$\therefore \quad I = \frac{i}{nr^n}\left[\sin n\theta + i\cos n\theta\right]_0^{2\pi} = 0$$

Example 2 : Evaluate $\int_C |z|\,dz$, where C is the left half of unit circle $|z| = 1$ from $z = -i$ to $z = i$.

(M.U. 1993, 2001, 03, 05, 06)

Sol. : Since the contour is a semi-circle and $f(z) = |z| = \sqrt{x^2 + y^2}$, we use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$.

$$z = x + iy = r\,e^{i\theta} \qquad \therefore \sqrt{x^2 + y^2} = r^2$$

Since, $|z| = 1$, $r = 1$, $\quad \therefore \quad z = e^{i\theta} \quad \therefore \quad dz = i\,e^{i\theta} d\theta$.

Hence, $\int_C |z|\,dz = \int_{3\pi/2}^{\pi/2} 1 \cdot i\,e^{i\theta} d\theta = \left[e^{i\theta}\right]_{3\pi/2}^{\pi/2}$

$$= e^{i\pi/2} - e^{3i\pi/2}$$

$$= [\cos(\pi/2) + i\sin(\pi/2)] - [\cos(3\pi/2) + i\sin(3\pi/2)]$$

$$= (0 + i) - (0 - i) = 2i$$

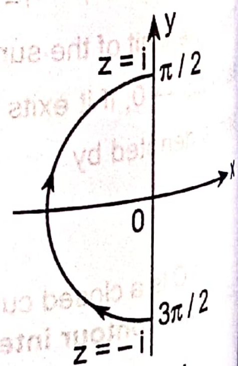


Fig. 2.4

Remark ....

When the contour is a circle it is better to use polar form $z = r e^{i\theta}$.

Note ....

We shall often need the curves represented by

(i) $|z| = r$, (ii) $|z - z_0| = r$, (iii) $|z - c| + |z + c| = k$.

(i) Since $z = x + iy$, $|z| = \sqrt{x^2 + y^2}$.

$\therefore$ $|z| = r$ gives $\sqrt{x^2 + y^2} = r$ i.e. $x^2 + y^2 = r^2$.

Thus, $|z| = r$ represents a circle with centre at the origin and radius r.

(ii) Since $z - z_0 = (x + iy) - (x_0 + iy_0) = (x - x_0) + i(y - y_0)$,

$$|z - z_0| = \sqrt{[x - x_0)^2 + (y - y_0)^2}.$$

$$\therefore \ |z - z_0| = r \text{ gives } = \sqrt{x - x_0)^2 + (y - y_0)^2} = r$$

i.e. $\quad (x - x_0)^2 + (y - y_0)^2 = r^2$.

Thus, $|z - z_0| = r$ represents a circle with centre at (x_0, y_0) and radius r.

Further, since parametric equations of the circle with centre at the origin and radius r are $x = r\cos\theta$, $y = r\sin\theta$, we have

$$z = x + iy = r\cos\theta + i\,r\sin\theta = r(\cos\theta + i\sin\theta) = r e^{i\theta}$$

Thus, $z = r e^{i\theta}$, $0 \le \theta \le \pi$ represents the circle with centre at the origin and radius r, in polar form.

Similarly, $(z - z_0) = r e^{i\theta}$, represents a circle with centre at z_0 (x_0, y_0) and radius r in polar form. This is so because $|z - z_0| = r$ is the circle $(x - x_0)^2 + (y - y_0)^2 = r^2$ as seen above.

(iii) If we denote points (x, y) by P, $(c, 0)$ by A and $(-c, 0)$ by B, then $|z - c| + |z + c| = k$ means $l(PA) + l(PB) = k$. But by definition of ellipse, this is an ellipse with foci at A $(c, 0)$ and B $(-c, 0)$ and major axis equal to k.

Now, for the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, we know that the foci are at $(\pm ae, 0)$ and also the semi-minor axis is given by $b^2 = a^2(1 - e^2) = a^2 - a^2e^2$. Hence, for the above ellipse semi-minor axis is given by $b^2 = \left(\dfrac{k}{2}\right)^2 - c^2$.

For example, $|z - 2| + |z + 2| = 6$ represents an ellipse with foci at $(2, 0)$ and $(-2, 0)$ and major axis equal to 6 i.e. semi-major axis equal to 3. The semi-minor axis is given by

$$b^2 = \left(\frac{k}{2}\right)^2 - c^2 = \left(\frac{6}{2}\right)^2 - (2)^2 = 9 - 4 = 5.$$

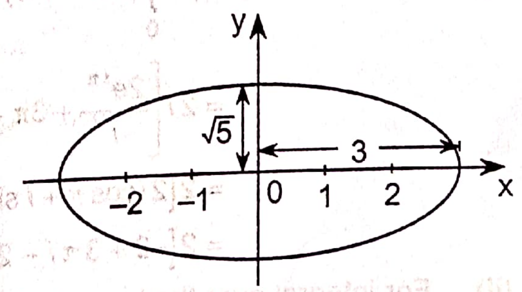


Fig. 2.5

The ellipse is shown in the adjoining figure. Thus, the ellipse $|z - 2| + |z + 2| = 6$ is $\dfrac{x^2}{9} + \dfrac{y^2}{5} = 1$.

Example 3 : Evaluate $\int_C \bar{z}\, dz$, where C is the upper half of the circle $r = 1$. **(M.U. 201**

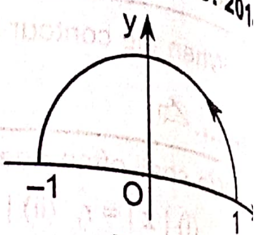


Fig. 2.6

Sol. : Let us put $z = r e^{i\theta}$.

Since $r = 1$, $z = e^{i\theta}$, $dz = e^{i\theta} . i . d\theta$.

$$\therefore \quad \int_C \bar{z}\, dz = \int_1^{-1} \bar{z}\, dz = \int_0^{\pi} e^{-i\theta} . e^{i\theta} . i . d\theta = \int_0^{\pi} i\, d\theta = i\left[\theta\right]_0^{\pi} = i\pi.$$

Example 4 : Evaluate $\int_C (\bar{z} + 2z)\, dz$ along the circle $x^2 + y^2 = 1$. **(M.U. 2014, 1**

Sol. : Let us put $z = r e^{i\theta}$.

$$\because \quad r = 1,\ z = e^{i\theta},\ \bar{z} = e^{-i\theta} \qquad \therefore \quad dz = i e^{i\theta}\, d\theta$$

$$\therefore \quad \int_C (\bar{z} + 2z)\, dz = \int_0^{2\pi} (e^{-i\theta} + 2e^{i\theta})\, i \cdot e^{i\theta}\, d\theta$$

$$= \int_0^{2\pi} i\, d\theta + 2i \int_0^{2\pi} e^{2i\theta}\, d\theta = i\left[\theta\right]_0^{2\pi} + 2i\left[\frac{e^{2i\theta}}{2i}\right]_0^{2\pi}$$

$$= 2i\pi + e^{4i\pi} - 1$$

Example 5 : Evaluate $\int_C \frac{2z+3}{z}\, dz$, where C is

(i) the upper half of the circle $|z| = 2$.

(ii) the lower half of the circle $|z| = 2$.

(ii) the whole circle in anticlock-wise direction. **(M.U. 200**

Sol. : Let us put $z = 2\, e^{i\theta}$.

$$\therefore \quad dz = 2i e^{i\theta}\, d\theta \qquad \therefore \quad \frac{2z+3}{z} = 2 + \frac{3}{z} = 2 + 3e^{-i\theta}$$

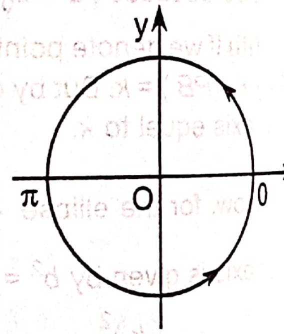


Fig. 2.7

(i) For integral over the upper half

$$\int_C f(z)\, dz = \int_C \frac{2z+3}{z}\, dz = \int_0^{\pi} (2 + 3e^{-i\theta})\, 2i e^{i\theta}\, d\theta$$

$$= 2i \int_0^{\pi} (2e^{i\theta} + 3)\, d\theta = 2i\left[\frac{2e^{i\theta}}{i} + 3\theta\right]_0^{\pi}$$

$$= 2i\left[\frac{2e^{i\pi}}{i} + 3\pi - \frac{2}{i}\right] = 2\left[2e^{i\pi} + 3\pi i - 2\right]$$

$$= 2\left[2(\cos\pi + i\sin\pi) + 3\pi i - 2\right]$$

$$= 2\left[-2 + 3\pi i - 2\right] = 2(3\pi i - 4).$$

(ii) For integral over the lower half.

$$\int_C f(z)\, dz = 2i \int_{\pi}^{0} (2e^{i\theta} + 3)\, d\theta = 2i\left[\frac{2e^{i\theta}}{i} + 3\theta\right]_{\pi}^{0} = 2i\left[\frac{2}{i} - \frac{2e^{i\pi}}{i} - 3\pi\right]$$

$$= 2\left[2 - 2(\cos\pi + i\sin\pi) - 3\pi i\right] = 2\left[4 - 3\pi i\right]$$

(iii) For the whole circle

$$\int_C f(z)\,dz = 2i\left[\frac{2e^{i\theta}}{i} + 3\theta\right]_0^{2\pi} = 2i\left[\frac{2}{i}e^{2i\pi} + 6\pi - \frac{2}{i}\right] = 2\left[2e^{2i\pi} + 6\pi i - 2\right]$$

$$= 2\left[2(\cos 2\pi + i\sin 2\pi) + 6\pi i - 2\right] = 2\left[6\pi i\right] = 12.$$

Example 6 : Show that $\int_C \log z\,dz = 2\pi i$, where C is the unit circle in the z-plane.

(M.U. 2000, 06, 17, 18)

Sol. : Since the contour is a circle we use polar coordinates.

We put $z = e^{i\theta}$ $\because r = 1$. $\therefore$ $dz = i\,e^{i\theta}d\theta$; θ varies from 0 to 2π.

$$\therefore\ I = \int_0^{2\pi} (\log e^{i\theta})\cdot i\,e^{i\theta}d\theta = i\int_0^{2\pi} i\,\theta e^{i\theta}d\theta = -\int_0^{2\pi} e^{i\theta}\theta\,d\theta$$

$$= -\left[\theta\cdot\frac{e^{i\theta}}{i} - \int\frac{e^{i\theta}}{i}\cdot 1\cdot d\theta\right]_0^{2\pi} = -\left[\theta\cdot\frac{e^{i\theta}}{i} - \frac{e^{i\theta}}{-1}\right]_0^{2\pi} = -\left[\theta\cdot\frac{e^{i\theta}}{i} + e^{i\theta}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{2i\pi}}{i} + e^{2i\pi} - 0 - 1\right] = -\left[\frac{2\pi}{i} + 1 - 1\right] \qquad \left[\because e^{2i\pi} = 1\right]$$

$$= -\frac{2\pi i}{i^2} = 2\pi i.$$

Example 7 : Evaluate $\int_C z^2 dz$, where C is the circle $x = r\cos\theta$, $y = r\sin\theta$, from $\theta = 0$ to $\theta = \pi/3$.

(M.U. 2005, 06)

Sol. : As above $I = \int_C z^2\,dz = \int_0^{\pi/3} r^2 e^{2i\theta}\cdot r e^{i\theta}\cdot i\,d\theta = r^3 i\int_0^{\pi/3} e^{3i\theta}\,d\theta = r^3 i\left[\frac{e^{3i\theta}}{3i}\right]_0^{\pi/3}$

$$= \frac{r^3}{3}\left[e^{i\pi} - 1\right] = \frac{r^3}{3}\left[\cos\pi + i\sin\pi - 1\right] = -\frac{2r^3}{3}.$$

Example 8 : Evaluate $\int_C (z - z^2)\,dz$, where C is the upper half of the circle $|z| = 1$. What is the value of the integral for the lower half of the same circle ?

(M.U. 1997, 2004, 06)

Sol. : Let us put $z = e^{i\theta}$ $\therefore$ $dz = e^{i\theta}d\theta$. And θ varies from 0 to π.

$$\therefore\ \int_C (z - z^2)\,dz = \int_0^{\pi} (e^{i\theta} - e^{2i\theta})e^{i\theta}\cdot i\,d\theta = i\int_0^{\pi}(e^{2i\theta} - e^{3i\theta})\,d\theta$$

$$= i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_0^{\pi} = \left[\frac{e^{2i\pi}}{2} - \frac{e^{3i\pi}}{3} - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \left[\frac{1}{2}(\cos 2\pi + i\sin 2\pi) - \frac{1}{3}(\cos 3\pi + i\sin 3\pi) - \frac{1}{2} + \frac{1}{3}\right]$$

$$= \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{2} + \frac{1}{3}\right] = \frac{2}{3}.$$

The value of the integral for the lower half of the same circle in the same positive direction i.e. when θ varies from π to 2π.

$$= \int_C (z - z^2)\,dz = i\left[\frac{e^{2i\theta}}{2i} - \frac{e^{3i\theta}}{3i}\right]_{\pi}^{2\pi} = i\left[\frac{e^{4i\pi}}{2i} - \frac{e^{6i\pi}}{3i} - \frac{e^{2i\pi}}{2i} + \frac{e^{3i\pi}}{3i}\right]$$

$$= \left[\frac{\cos 4\pi + i\sin 4\pi}{2} - \frac{\cos 6\pi + i\sin 6\pi}{3} - \frac{\cos 2\pi + i\sin 2\pi}{2} + \frac{\cos 3\pi + i\sin 3\pi}{2}\right]$$

$$= \left[\frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3}\right] = -\frac{2}{3}.$$

Alternatively : Since for the closed curve C, the total integral is zero, for the remaining part the circle the integral = $-2/3$.

(b) When the contour is a straight line or a parabola

Example 1 : Evaluate the integral $\int_0^{1+i} (x - y + ix^2)\,dz$.

(i) along the line from $z = 0$ to $z = 1 + i$. **(M.U. 2002, (**

(ii) along the real axis from $z = 0$ to $z = 1$ and then along the line parallel to the imagina axis from $z = 1$ to $z = 1 + i$. **(M.U. 19**

(iii) along the imaginary axis from $z = 0$ to $z = i$ and then along the line parallel to the real a from $z = i$ to $z = i + 1$.

(iv) along the parabola $y^2 = x$. **(M.U. 19**

Sol. : (i) Let OA be the line from $z = 0$ *i.e.*, $(0, 0)$ to $z = 1 + i$. *i.e.*, $(1, 1)$. Equation of the line OA $y = x$ *i.e.*, on OA, $y = x$ $\therefore$ $dy = dx$.

$\therefore$ $dz = dx + i\,dy = dx + i\,dx = (1 + i)\,dx$

and x varies from 0 to 1.

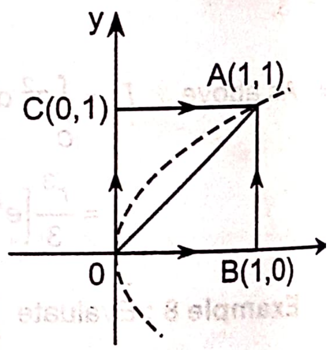


Fig. 2.8

$$\therefore\ I = \int_0^{1+i} (x - y + ix^2)\,dz$$

$$= \int_0^1 (x - x + ix^2)(1 + i)\,dx = \int_0^1 ix^2(1 + i)\,dx$$

$$= \int_0^1 (i - 1)\,x^2\,dx = (i - 1)\left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}(i - 1)$$

(ii) Here, the contour is first the segment OB and then the segment BA.

On OB, $y = 0$. $\therefore$ $dy = 0$ $\therefore$ $dz = dx + i\,dy = dx$ and x varies from 0 to 1.

$$\therefore\ \int_{OB} (x - y + ix^2)\,dz = \int_0^1 (x + ix^2)\,dx = \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + i\frac{1}{3}$$

On BA, $x = 1$ $\therefore$ $dx = 0$ $\therefore$ $dz = dx + i\,dy = i\,dy$ and y varies from 0 to 1.

$$\therefore\ \int_{BA} (x - y + ix^2)\,dz = \int_0^1 (1 - y + i)\,i\,dy = \int_0^1 [(-1 + i) - iy]\,dy$$

$$\therefore \quad \int_{BA} (x - y + ix^2)\, dz = \left[(-1+i)y - \frac{iy^2}{2}\right]_0^1 = -1 + \frac{i}{2}$$

Hence, adding the two results.

$$I = \frac{1}{2} + i\frac{1}{3} - 1 + \frac{i}{2} = -\frac{1}{2} + \frac{5}{6}i$$

(iii) Here, the contour is first the segment OC and then the segment CA.

On OC, $x = 0$ $\therefore$ $dx = 0$ $\therefore$ $dz = dx + i\,dy = i\,dy$ and y varies from 0 to 1.

$$\therefore \quad \int_{OC} (x - y + ix^2)\, dz = \int_0^1 (-y)\, i\, dy = -i\left[\frac{y^2}{2}\right]_0^1 = -\frac{i}{2}$$

On CA, $y = 1$ $\therefore$ $dy = 0$ $\therefore$ $dz = dx + i\,dy = dx$ and x varies from 0 to 1.

$$\therefore \quad \int_{CA} (x - y + ix^2)\, dz = \int_0^1 (x - 1 + ix^2)\, dx = \left[\frac{x^2}{2} - x + i\frac{x^3}{3}\right]_0^1 = -\frac{1}{2} + \frac{i}{3}$$

Hence, adding the two results.

$$I = -\frac{i}{2} - \frac{1}{2} + \frac{i}{3} = -\frac{1}{2} - \frac{i}{6}$$

(iv) Here, the contour is the arc OA of the parabola $y^2 = x$.

Hence, $z = x + iy = y^2 + iy$

$\therefore$ $dz = 2y\,dy + i\,dy = (2y + i)\,dy$ and y varies from 0 to 1.

$$\therefore \quad I = \int_0^{1+i} (x - y + ix^2)\, dz = \int_0^1 (y^2 - y + iy^4)(2y + i)\, dy$$

$$= \int_0^1 (2y^3 - 2y^2 + i\,2y^5 + iy^2 - iy - y^4)\, dy$$

$$= \left[\frac{1}{2}y^4 - \frac{2y^3}{3} + i\frac{1}{3}y^6 + \frac{iy^3}{3} - \frac{iy^2}{2} - \frac{y^5}{5}\right]_0^1$$

$$= \frac{1}{2} - \frac{2}{3} + \frac{i}{3} + \frac{i}{3} - \frac{i}{2} - \frac{1}{5} = -\frac{11}{30} + \frac{i}{6}.$$

Note ....

Note that the line integral depends upon the path.

Example 2 : Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\, dz$, along (i) the straight line joining $(1 - i)$ to $(2 + i)$, (ii) $x = t + 1$, $y = 2t^2 - 1$ a parabola. **(M.U. 1993, 2005)**

Sol. : (i) We first find the equation of the line through the given points $(1, -1)$ and $(2, 1)$.

The equation is $\dfrac{y+1}{-1-1} = \dfrac{x-1}{1-2}$ *i.e.* $\dfrac{y+1}{2} = \dfrac{x-1}{1}$

$\therefore$ $y + 1 = 2x - 2$ $\therefore$ $y = 2x - 3$

$\therefore \; dy = 2\,dx \qquad \therefore \; dz = dx + i\,dy = (1 + 2i)\,dx$

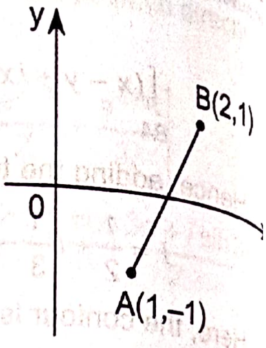


Fig. 2.9

Hence, the integral becomes

$$\int_{1-i}^{2+i} (2x + iy + 1)\,dz = \int_1^2 [2x + i(2x-3) + 1](1+2i)\,dx$$

$$= (1+2i)\left[x^2 + i(x^2 - 3x) + x\right]_1^2$$

$$= (1+2i)\left[\{4 + i(4-6) + 2\} - \{1 + i(1-3) + 1\}\right]$$

$$= (1+2i)\left[(6-2i) - (2-2i)\right] = 4(1+2i)$$

(ii) If $x = t + 1,\; y = 2t^2 - 1,\; z = (t+1) + i(2t^2 - 1) \quad \therefore \; dz = (1 + 4it)\,dt.$

When $z = 1 - i,\; t = 0$ and when $z = 2 + i,\; t = 1.$

$$\therefore \; I = \int_0^1 \left[2(t+1) + i(2t^2 - 1) + 1\right](1 + 4it)\,dt$$

$$= \int_0^1 \left\{2(t+1) + i(2t^2 - 1) + 1\right\} + \left\{8i(t^2 + t) - 4(2t^3 - t) + 4it\right\} dt$$

$$= \left[\left\{2\left(\frac{t^2}{2} + t\right) + i\left(\frac{2t^3}{3} - t\right) + t\right\} + \left\{8i\left(\frac{t^3}{3} + \frac{t^2}{2}\right) - 4\left(\frac{2t^4}{4} - \frac{t^2}{2}\right) + 2it^2\right\}\right]_0^1$$

$$= \left[2\left(\frac{3}{2}\right) + i\left(-\frac{1}{3}\right) + 1 + 8i\left(\frac{5}{6}\right) - 4(0) + 2i\right]$$

$$= 4 + \frac{25}{3}i.$$

Example 3 : Evaluate $\int_0^{1+i} z^2 dz$, along (i) the line $y = x$, (ii) the parabola $x = y^2$. Is the line integral independent of the path ? Explain. **(M.U. 1992, 98, 2013, 16,**

Sol. : Let OA be the line from $z = 0$ to $z = 1 + i$.

(i) On the line OA i.e. $y = x,\; dy = dx$

$\therefore \; dz = dx + i\,dy = (1 + i)\,dx$

and x varies from 0 to 1.

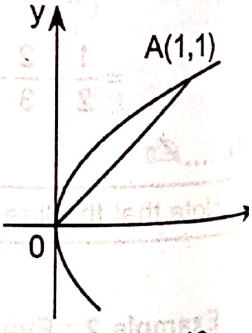


Fig. 2.10

$$\therefore \; I = \int_0^{1+i} (x + iy)^2 dz = \int_0^1 (x^2 - y^2 + 2ixy)(1+i)\,dx$$

$$= \int_0^1 (x^2 - x^2 + 2ix^2)(1+i)\,dx \qquad [\because y = x]$$

$$= 2i(1+i)\int_0^1 x^2\,dx = 2i(1+i)\left[\frac{x^3}{3}\right]_0^1$$

$$= \frac{2}{3}i(1+i) = \frac{2}{3}(i - 1)$$

(II) On the arc OA of the parabola $x = y^2$, $dx = 2y\,dy$

$\therefore\ dz = dx + i\,dy = (2y + i)\,dy$

$$\therefore\ I = \int_0^1 (x^2 - y^2 + 2ixy)(2y + i)\,dy$$

$$= \int_0^1 (y^4 - y^2 + 2i\,y^3)(2y + i)\,dy \quad [\because x = y^2]$$

$$= \int_0^1 (2y^5 - 2y^3 + 4i\,y^4 + iy^4 - iy^2 - 2y^3)\,dy$$

$$= \int_0^1 \left[(2y^5 - 4y^3) + i\,(5y^4 - y^2)\right]dy$$

$$= \left[\left(\frac{y^6}{3} - y^4\right) + i\left(y^5 - \frac{y^3}{3}\right)\right]_0^1 = \left[\left(\frac{1}{3} - 1\right) + i\left(1 - \frac{1}{3}\right)\right]$$

$$= -\frac{2}{3} + \frac{2}{3}i = \frac{2}{3}(i - 1).$$

The two integrals are equal *i.e.* the integral is independent of path because $f(z) = z^2$ is an **analytic function**. (See the corollary given on page 2-18)

Example 4 : Integrate the function $f(z) = x^2 + i\,xy$ from A (1, 1) to B (2, 4) along the curve $x = t$, $y = t^2$. **(M.U. 1993, 98, 2003, 04)**

(Or integrate xz along the line from A (1, 1) to B (2, 4) in the complex plane.) **(M.U. 2011)**

Sol. : Putting $x = t$, $y = t^2$ in $f(z)$, we get $f(z) = x^2 + i\,xy = t^2 + i\,t^3$ and $dz = dx + i\,dy = dt + 2i\,t\,dt = (1 + 2i\,t)\,dt$. And t varies from 1 to 2.

$$\therefore\ \int_A^B f(z)\,dz = \int_1^2 (t^2 + it^3)(1 + 2i\,t)\,dt = \int_1^2 (t^2 + 2it^3 + it^3 - 2t^4)\,dt$$

$$= \int_1^2 \left[(t^2 - 2t^4) + 3i\,t^3\right]dt = \left[\frac{t^3}{3} - \frac{2t^5}{5} + 3i \cdot \frac{t^4}{4}\right]_1^2$$

$$= \left[\left(\frac{8}{3} - \frac{64}{5} + 3i \cdot \frac{16}{4}\right) - \left(\frac{1}{3} - \frac{2}{5} + \frac{3i}{4}\right)\right]$$

$$= -\frac{151}{15} + i \cdot \frac{45}{4}.$$

Example 5 : Evaluate $\int \bar{z}\,dz$ from $z = 0$ to $z = 4 + 2i$ along the curve $z = t^2 + i\,t$. **(M.U. 1993)**

Sol. : The curve $z = t^2 + it$ can be given by $x = t^2$, $y = t$. When z varies from $z = 0$ to $z = 4 + 2i$, t varies from $t = 0$ to $t = 2$.

Further $\bar{z} = t^2 - it$ and $dz = (2t + i)\,dt$ since $z = t^2 + i\,t$.

$$\therefore \int_A^B \bar{z}\,dz = \int_0^2 (t^2 - it)(2t + i)\,dt = \int_0^2 (2t^3 + it^2 - 2it^2 + t)\,dt$$

$$= \int_0^2 \left[(2t^3 + t) - it^2\right] dt = \left[\frac{t^4}{2} + \frac{t^2}{2} - \frac{it^3}{3}\right]_0^2$$

$$= \left[8 + 2 - i \cdot \frac{8}{3}\right] = \frac{30 - 8i}{3}.$$

Example 6 : If 0 is the origin, L is the point $z = 3$, M is the point $z = 3 + i$, evaluate $\int z^2 dz$ alo (i) the path OM, (ii) the path OLM, (iii) the path $OLMO$. **(M.U. 1992,**

Sol. : Now $z^2 = (x + iy)^2 = x^2 - y^2 + 2i\,xy$

(i) Along the path OM : The equation of the line OM is

$$\frac{y - 1}{1 - 0} = \frac{x - 3}{3 - 0} \quad i.e.\ y = \frac{x}{3}$$

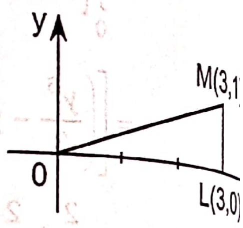


Fig. 2.11

Also $dz = dx + i\,dy = dx + \dfrac{i}{3}dx$.

Along OM, x varies from 0 to 3.

$$\therefore \int z^2 dz = \int (x^2 - y^2 + 2i\,xy)(dx + i\,dy)$$

$$= \int_0^3 \left(x^2 - \frac{x^2}{9} + 2i\,x \cdot \frac{x}{3}\right)\left(1 + \frac{i}{3}\right) dx = \left(1 + \frac{i}{3}\right)\int_0^3 \left(\frac{8x^2}{9} + \frac{2i}{3}x^2\right) dx$$

$$= \left(1 + \frac{i}{3}\right)\left[\frac{8x^3}{27} + \frac{2i}{9}x^3\right]_0^3 = \left(1 + \frac{i}{3}\right)[8 + 6i]$$

$$= 8 + \frac{8i}{3} + 6i - \frac{6}{3} = \frac{18 + 26i}{3}$$

(ii) Along the path OLM : Along OL, $y = 0$

$\therefore$ $z^2 = x^2$, $dy = 0$ and $dz = dx$, x varies from 0 to 3.

$$\int z^2 dz = \int_0^3 x^2 dx = \left[\frac{x^3}{3}\right]_0^3 = 9$$

Along LM, $x = 3$, $\therefore$ $z^2 = 9 - y^2 + 6iy$, $dx = 0$ $\therefore$ $dz = i\,dy$ and y varies from 0 to 1.

$$\therefore \int z^2 dz = \int_0^1 (9 - y^2 + 6iy)\,i\,dy = i\left[9y - \frac{y^3}{3} + 3iy^2\right]_0^1 = i\left(9 - \frac{1}{3} + 3i\right)$$

$$= \left(\frac{26}{3} + 3i\right)i = -3 + \frac{26}{3}i.$$

$\therefore$ Integral along OLM = Integral along OL + integral along LM

$$= 9 + \left(-3 + \frac{26}{3}i\right) = 6 + \frac{26i}{3}$$

(III) Along the path $OLMO$

Integral along $OLMO$ = integral along OL + integral along LM + integral along MO.

$$= 9 + \left(-3 + \frac{26i}{3}\right) + \left[-\left(\frac{18 + 26i}{3}\right)\right] = 0$$

Example 7 : Find $\int \text{Im}(z)\,dz$ along (i) the unit circle described once in positive direction from $z = 1$ to $z = 1$. (ii) the straight line from $P(z_1)$ to $Q(z_2)$. **(M.U. 1991)**

Sol. : Since $z = x + iy$, $\text{Im}(z) = y$ and $dz = dx + i\,dy$

(i) Along the unit circle $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\,d\theta$

$$\therefore \quad I = \int_C \text{Im}(z)\,dz = \int_C y\,(dx + i\,dy) = \int_0^{2\pi} \sin\theta(-\sin\theta + i\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} (-\sin^2\theta + i\sin\theta\cos\theta)\,d\theta = \int_0^{2\pi}\left[-\left\{\frac{1-\cos 2\theta}{2}\right\} + i\sin\theta\cos\theta\right]d\theta$$

$$= \left[-\frac{1}{2}\left\{\theta - \frac{\sin 2\theta}{2}\right\} + i\,\frac{\sin^2\theta}{2}\right]_0^{2\pi} = -\frac{1}{2}(2\pi) = -\pi.$$

(ii) The equation of the line PQ is $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

$$\therefore \quad (x_1 - x_2)(y - y_1) = (y_1 - y_2)(x - x_1)$$

$$\therefore \quad (x_1 - x_2)\,dy = (y_1 - y_2)\,dx$$

$$\therefore \quad I = \int_C \text{Im}(z)\,dz = \int_{y_1}^{y_2} y\,(dx + i\,dy) = \int_{y_1}^{y_2} (y\,dx + iy\,dy)$$

$$= \int_{y_1}^{y_2}\left[\left(\frac{x_1 - x_2}{y_1 - y_2}\right)y + iy\right]dy = \frac{x_1 - x_2}{y_1 - y_2}\left\{\frac{y^2}{2}\right\}_{y_1}^{y_2} + i\left\{\frac{y^2}{2}\right\}_{y_1}^{y_2}$$

$$= \frac{x_1 - x_2}{y_1 - y_2}\cdot\frac{(y_2^2 - y_1^2)}{2} + i\,\frac{(y_2^2 - y_1^2)}{2}$$

$$= \frac{(x_2 - x_1)(y_2 + y_1)}{2} + i\,\frac{(y_2 + y_1)(y_2 - y_1)}{2}$$

$$= \left[\frac{(x_2 - x_1) + i(y_2 - y_1)}{2}\right](y_2 + y_1)$$

$$= \frac{1}{2}(z_2 - z_1)\,\text{Im}(z_2 + z_1)$$

Example 8 : Evaluate $\int_0^{1+i} (x^2 + iy)\,dz$, along the path (i) $y = x$, (ii) $y = x^2$.

Is the line integral independent of the path ? **(M.U. 1996, 2009, 14)**

Sol. : (i) Along the path $y = x$: $\because$ $y = x$, $dy = dx$

$\therefore$ $dz = dx + i\,dy = dx + i\,dx = (1 + i)\,dx$. And x varies from 0 to 1.

$$\int_0^{1+i}(x^2+iy)\,dz=\int_0^1(x^2+ix)(1+i)\,dx=(1+i)\left[\frac{x^3}{3}+\frac{ix^2}{2}\right]_0^1$$

$$=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)=(1+i)\frac{(2+3i)}{6}$$

$$=\frac{(2+2i+3i-3)}{6}=\frac{-1+5i}{6}$$

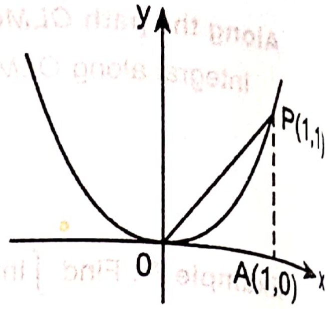


Fig. 2.12

(ii) **Along the path $y=x^2$:** $y=x^2,\ dy=2x\,dx$

$$\int_0^1(x^2+ix^2)(dx+2ix\,dx)=(1+i)\int_0^1 x^2(1+2ix)\,dx$$

$$=(1+i)\int_0^1(x^2+2ix^3)\,dx$$

$$=(1+i)\left[\frac{x^3}{3}+i\cdot\frac{x^4}{2}\right]_0^1=(1+i)\left(\frac{1}{3}+\frac{i}{2}\right)=\frac{-1+5i}{6}$$

The two line integrals are equal.

(iii) Now, consider the integral along a third path, say, along OA and then along AP. Along OA, x varies from 0 to 1 and $y=0$

$\therefore\ dy=0 \qquad \therefore\ dz=dx.$

$$\therefore\ \int_{OA}(x^2+iy)\,dz=\int_0^1 x^2\,dx=\left[\frac{x^3}{3}\right]_0^1=\frac{1}{3}$$

Along AP, $x=1$ $\therefore$ $dx=0$ and y varies from 0 to 1. $\therefore$ $dz=i\,dy$.

$$\therefore\ \int_{AP}(x^2+iy)\,dz=\int_0^1(1+iy)\,i\,dy=i\left[y+\frac{iy^2}{2}\right]_0^1=i\left(1+\frac{i}{2}\right)=i-\frac{1}{2}$$

$$\therefore\ \int_0^{1+i}(x^2+iy)\,dz=\frac{1}{3}+i-\frac{1}{2}=-\frac{1}{6}+i$$

Thus, the third integral is not equal to the first two. Hence, the integral is not independent of the path.

(iv) Again let $f(z)=x^2+iy=u+iv$. $\therefore$ $u=x^2,\ v=y$.

$\therefore\ u_x=2x,\ u_y=0,\ v_x=0,\ v_y=1.$

Hence, Cauchy-Rieman equations $u_x=v_y$ and $u_y=-v_x$ are not satisfied. $f(z)$ is not analytic and hence, integral is not independent of the path.

(See the corollary given on page 2-18)

Example 9 : Evaluate $\int f(z)\,dz$ along the parabola $y=2x^2$ from $z=0$ to $z=3+18i$ where $f(z)=x^2-2iy$.

(M.U. 1996, 2015)

Sol. : $\because\ y=2x^2$ $\therefore\ dy=4x\,dx$ $\therefore\ dz=dx+i\,dy=dx+i\,4x\,dx=(1+4ix)\,dx$

$$\therefore\ \int_c f(z)\,dz=\int_0^3(x^2-2i\cdot 2x^2)(1+4ix)\,dx=\int_0^3(x^2-4ix^2+4ix^3+16x^3)\,dx$$

$$\therefore \int_C f(z)\,dz = \left[\frac{x^3}{3} - 4i\cdot\frac{x^3}{3} + 4i\cdot\frac{x^4}{4} + 16\cdot\frac{x^4}{4}\right]_0^3$$

$$= [9 - 4i\cdot 9 + i\cdot 81 + 4\cdot 81] = 333 + 45i.$$

Example 10 : Evaluate $\int f(z)\,dz$, along the parabola $y = 2x^2$ from $z = 0$ to $z = 3 + 18i$ where $f(z) = x^2 - 2i\,xy$. **(M.U. 1996, 2014)**

Sol. : $\because \quad y = 2x^2,\ dy = 4x\,dx$

$$\therefore \ dz = dx + i\,dy = dx + i\,4x\,dx = (1 + 4ix)\,dx$$

$$\int_C f(z)\,dz = \int_0^3 (x^2 - 2ix\cdot 2x^2)(1 + 4ix)\,dx = \int_0^3 (x^2 - 4ix^3 + 4ix^3 + 16x^4)\,dx$$

$$= \int_0^3 (x^2 + 16x^4)\,dx = \left[\frac{x^3}{3} + \frac{16x^5}{5}\right]_0^3 = \left[9 + 16\cdot\frac{243}{5}\right] = \frac{3933}{5}.$$

Example 11 : State true or false with proper justification.

If $f(z) = (x^2 + 2x - y^2) + 2iy\,(x + 1)$ then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$, where C_1 and C_2 are respectively $y^2 = x^3$ and $y = x^2$ joining the points (0, 0) and (1, 1). **(M.U. 1996)**

Sol. : Let $f(z) = u + iv$ where $u = x^2 + 2x - y^2$, $v = 2y\,(x + 1) = 2xy + 2y$

$$\therefore \ u_x = 2x + 2, \qquad u_y = -2y$$

$$v_x = 2y, \qquad v_y = 2x + 2 \qquad \therefore \ u_x = v_y \text{ and } u_y = -v_x.$$

Cauchy-Rieman equations are satisfied. Hence, $f(z)$ is analytic.

Hence, $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ because for an analytic function the integral is independent of the path. (See the corollary given on page 2-18)

EXERCISE - I

1. Evaluate

(a) $\int_0^{1+i} (x^2 - iy)\,dz$ along the path (i) $y = x$, (ii) $y = x^2$. $\left[\textbf{Ans. : (i)}\ \frac{5}{6} - \frac{i}{6},\ \textbf{(ii)}\ \frac{5}{6} + \frac{i}{6}\right]$

(M.U. 2015)

(b) $\int_{1-i}^{2+i} (2x + iy + 1)\,dz$ along the curve $x = t + 1$, $y = 2t^2 - 1$. $\left[\textbf{Ans. :}\ 4 + \frac{25}{3}i\right]$

2. Evaluate $\int_0^{3+i} z^2 dz$

(i) along the real axis from 0 to 3 and then vertically to $3 + i$.

(ii) along the imaginary axis from 0 to i and then horizontally to $3 + i$.

(iii) along the parabola $x = 3y^2$.

(M.U. 2006) [**Ans. :** $6 + \frac{26}{3}i$ in each case.]

3. Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along **(M.U. 2001, 02, 03)**

(i) the line $x = 2y$.

(ii) the real axis from 0 to 2 and then vertically to $2 + i$. **(M.U. 2006)**

(iii) the parabola $2y^2 = x$.

(**Hint :** $(\bar{z})^2 = (x - iy)^2$) [**Ans. : (i)** $\frac{10}{3} - \frac{5}{3}i$, **(ii)** $\frac{14}{3} + \frac{11}{3}i$, **(iii)** $\frac{8}{3} - \frac{41}{15}i$]

4. Evaluate $\int_C (y - x - 3x^2 i)\, dz$, where C is a straight line from $z = 0$ to $z = 1 + i$.

[**Ans. :** $1 - i$]

5. Evaluate $\int_C \frac{dz}{z}$, where C is the circle $|z| = r$ in the positive sense.

[**Ans. :** $2\pi i$. Put $z = re^{i\theta}$, $dz = r i e^{i\theta} d\theta$]

6. Evaluate $\int (z - z^2)\, dz$ along the upper half of the circle $|z| = 1$. **(M.U. 1994)**

[**Ans. :** $\frac{2}{3}$. Put $z = re^{i\theta}$]

7. Evaluate $\int_C z^2 dz$ from $P(1, 1)$ to $Q(2, 4)$ where

(i) C is the curve $y = x^2$, (ii) C is the line $y = 3x - 2$,

(iii) C is the curve $x = t$, $y = t^2$. [**Ans. :** $-\frac{86}{3} - 6i$ in each case.]

8. Evaluate $\int_C |z|^2 dz$ where C is the boundary of the square C with vertices (0, 0), (1, 0), (1, 1), (0, 1).

[**Ans. :** $-1 + i$]

9. Evaluate $\int_C (z + 1)\, dz$, where C is the boundary of the square whose vertices are $z = 0$, $z = 1$, $z = 1 + i$, $z = i$.

[**Ans. :** 0]

10. Evaluate $\int_C \frac{z+2}{z} dz$, where C is the semi-circle $z = 2e^{i\theta}$, $0 \le \theta \le \pi$.

[**Ans. :** $-4 + 2\pi i$. Put $z = 2e^{i\theta}$]

11. Evaluate $\int_C (x - iy)\, dz$ from (0, 0) to (4, 2) where C is first, the line segment joining (0, 0) to (0, 2) and then the line segment joining (0, 2) to (4, 2).

[**Ans. :** $10 - 8i$]

12. Evaluate $\int_C (2z^3 + 8z + 2)\, dz$, where C is the arc of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ between the points (0, 0) and $(2\pi a, 0)$. **(M.U. 1991, 93)**

(**Hint :** Find $\int_0^{2\pi a} f(z)\, dz$.) [**Ans. :** $4\pi a [2\pi^3 a^3 + 4\pi a + 1]$]

13. Evaluate $\int_0^{1+2i} z^2 dz$ along the curve $2x^2 = y$. **(M.U. 1998, 2003)** $\left[\textbf{Ans. :} -\frac{11}{3}-\frac{2i}{3}\right]$

14. Evaluate $\int_C (3z^2 + 2z + 1)\, dz$, where C is the arc of the cycloid $x = a\,(\theta + \sin\theta)$, $y = a\,(1 - \cos\theta)$ between $\theta = 0$ to $\theta = 2\pi$. **(M.U. 1997)** $\left[\textbf{Ans. :}\ 2\pi a\left[4\pi^2 a^2 + 2\pi a + 1\right]\right]$

15. Evaluate $\int_C z^2 dz$, where C is the arc of the circle $x = r\cos\theta$, $y = r\sin\theta$ from $\theta = 0$ to $\theta = \pi/3$. [**Ans. :** $-2r^3/3$]

16. Evaluate $\int f(z)\, dz$ along the square whose vertices are (1, 1), (2, 1), (2, 2), (1, 2) in anti-clockwise direction where $f(z) = x - 2iy$. **(M.U. 1990)** [**Ans. :** $3i$]

17. Evaluate $\int_C (z^2 - 2\bar{z} + 1)\, dz$, where C is the circle $x^2 + y^2 = 2$.

(**Hint :** Put $z = \sqrt{2}\, e^{i\theta}$, $\bar{z} = \sqrt{2}\, e^{-i\theta}$.) [**Ans. :** $-8\pi i$]

18. Evaluate $\int_C (z^2 + 3z^{-4})\, dz$, where C is the upper half of the unit circle from (1, 0) to (–1, 0). **(M.U. 2000)** [**Ans. :** 4 / 3]

19. Evaluate $\int_0^{2+i} z^2 dz$

(i) along the line $x = 2y$. **(M.U. 2005)**

(ii) along the real axis from $z = 0$ to $z = 2$ and then along the line parallel to the imaginary axis from $z = 2$ to $z = 2 + i$.

(iii) along the imaginary axis from $z = 0$ to $z = i$ and then along the line parallel to the real axis from $z = i$ to $z = 2 + i$.

(iv) along the parabola $2y^2 = x$. [**Ans. :** $\frac{1}{3}(2 + 11i)$ in each case.]

20. Evaluate $\int_C \bar{z}\, dz$ from $z = 0$ to $z = 4 + 2i$, where C is

(i) the curve given by $z = t^2 + i\,t$.

(ii) the line from $z = 0$ to $z = 2i$ and then the line from $z = 2i$ to $z = 4 + 2i$.

[**Ans. : (i)** $10 - \frac{8}{3}i$, **(ii)** $10 - 4i$]

21. Evaluate $\int_C z\, dz$ from $z = 0$ to $z = 1 + i$ along the curve $z = t^2 + it$. [**Ans. :** i]

22. Evaluate $\int_{1-i}^{1+i} (ix + iy + 1)\, dx$ along straight line joining $(1 - i)$ to $(1 + i)$. [**Ans. :** $2\,(i - 1)$]

5. Cauchy's Theorem : Introduction

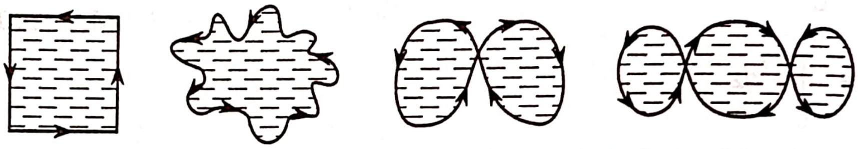
Now, we shall study a very important theorem in the field of complex integration due to Cauchy. He proved a very beautiful and apparently very simple theorem about integration of an analytic function around a closed contour.

Augustin Louis (Baron de) Cauchy (Pronounced as Coshy) (1789 - 1857)

A French mathematician of great repute who contributed various branches of mathematics. He wanted to be an engineer b because of poor health he was advised to pursue mathematics. H mathematical work began in 1811 when he gave briliant solutio to some difficult problems of that time. In the next 35 years published 700 papers in various branches of mathematics. He supposed to have initiated the era of modern analysis.

6. Simply and Multiply Connected Regions

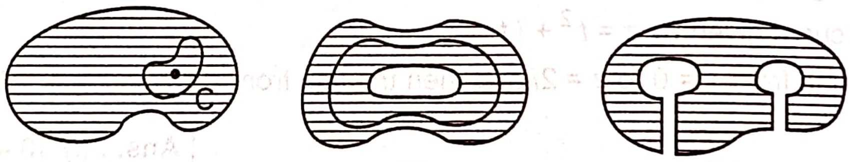
If a closed curve does not intersect itself, it is called a **simple closed curve** or a **Jorda Curve** [See Fig. 2.13 (a), (b)]. If a closed curve intersects itself it is called a **multiple curve** (Se Fig. 2.13 (c), (d)].



(a) (b) (c) (d)

Fig. 2.13

A region R is called a **simply connected region** if every closed curve in the region enclose points of the region R only. In other words this means that every closed curve which lies in R can b contracted to a point without leaving R [See Fig. 2.14 (a)]. A region which is not simply connected i called multiply connected [See Fig. 2.14 (b)]. In simple terms a simply connected region is one which has no holes in it. A multiply connected region can be converted into simply connected regio by giving it one or more cuts [See Fig. 2.14 (c)].



(a) (b) (c)

Fig. 2.14

7. Theorem

If $f(z)$ is continuous on a closed curve C of length l where, $|f(z)| \le M$, then $\left|\int_C f(z)\,dz\right| \le Ml$.

Proof : We first note that the integral is the limit of infinite sums and that the modulus of a sum is less than or equal to sum of their modulie.

$$\therefore \quad \left|\int_C f(z)\,dz\right| \le \int_C |f(z)\,dz| = \int_C |f(z)|\,|dz|$$

$$\therefore \quad \left|\int_C f(z)\,dz\right| \le \int_C M\,|dz| \qquad [\because |f(z)| \le M]$$

$$\le M \int_C \sqrt{dx^2 + dy^2} \qquad [\because dz = dx + i\,dy]$$

$$\le M \int_C ds \qquad \left[\because ds = \sqrt{dx^2 + dy^2}\right]$$

$$\le M\,l \qquad \left[\because \int_C ds = l\right]$$

Hence, we get $\left|\int_C f(z)\,dz\right| \le M\,l$. .... (1)

8. Cauchy's Integral Theorem

If $f(z)$ is an analytic function and if its derivative $f'(z)$ is continuous at each point within and on a simple closed curve C then the integral of $f(z)$ along the closed curve C is zero *i.e.*

$$\boxed{\oint f(z)\,dz = 0}$$ **(M.U. 2005)**

Proof : Let the region enclosed by the curve C be denoted by R.

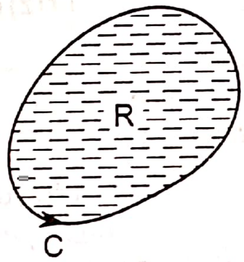


Fig. 2.15

Let $f(z) = u(x, y) + iv(x, y) = u + iv$

$dz = dx + i\,dy$

$$\therefore \quad \oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy)$$

Since, $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R and on C.

Hence, we can apply Green's theorem

$$\int_C (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$ to the above integrals. .... (2)

$$\therefore \quad \oint f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \qquad \text{..................... (3)}$$

But $f(z)$ is analytic at each point of the region R, hence, by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{..................... (4)}$$

Putting these values from (4) in (3) the r.h.s. of (3) becomes zero.

Hence, $\oint_C f(z)\,dz = 0$.

Note ....

The French mathematician Goursat proved the theorem without assuming that $f'(z)$ is continuous. Hence, the theorem now becomes, **"if $f(z)$ is analytic in and on a closed counter C then $\oint_C f(z)\,dz = 0$."**

This theorem is known as Cauchy-Goursat Theorem.

(a) Corollary

If $f(z)$ is analytic in R then the line integral of $f(z)$ along any curve in R joining any two points of R is the same if the curve wholly lies in R *i.e.* the line integral is independent of the path joining the two points.

(b) Extension of Cauchy's Integral Theorem

Cauchy's theorem can be applied even if the region is multiply connected.

Theorem : If $f(z)$ is analytic in R between two simple closed curves C_1 and C_2 then,

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$ (M.U. 2001)

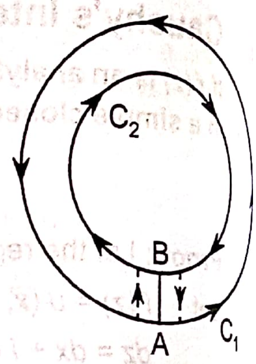


Fig. 2.16

Proof : To prove the theorem we introduce a cross-cut AB as shown in the figure. Then by Cauchy's Theorem

$$\int_C f(z)\,dz = 0$$

where, C is the curve indicated by arrows in the Fig. 2.16 *viz.* the path along $\vec{AB}$, then the curve C_2 in clockwise direction, then the path $\vec{BA}$ and then C_1 in anti-clockwise direction.

$$\therefore \quad \int_{AB} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{C_1} f(z)\,dz = 0$$

Now, the integrals along AB and BA cancel out

$$\therefore \quad \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz = 0$$

Reversing the direction of the integral along C_2, we get

$$\int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz = 0.$$ And transposing, we get

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$

where the curves C_1 and C_2 are taken in the same anti-clockwise direction.

Generalisation

The line integral of a single valued analytic function $f(z)$ over the outer contour C of a multiply connected region is equal to the sum of the integrals over the inner contours $C_1, C_2, C_3, \ldots C_n$ where $C_1, C_2, C_3, \ldots C_n$ are the boundaries of the multiply connected region.

Example 1 : Evaluate $\int_C \dfrac{z+3}{z^2-2z+5}\,dz$, where C is the circle $|z-1| = 1$.

Sol. : C is a closed curve *i.e.* a circle with centre (1, 0) and radius 1.

Now, $z^2 - 2z + 5 = 0$ gives $z = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$.

These points are outside C.

Hence, $f(z) = \dfrac{z+3}{z^2-2z+5}$ is analytic in and on C.

Hence, by Cauchy's theorem

$$\int_C f(z)\,dz = 0 \qquad \therefore \quad \int_C \frac{z+3}{z^2-2z+5}\,dz = 0.$$

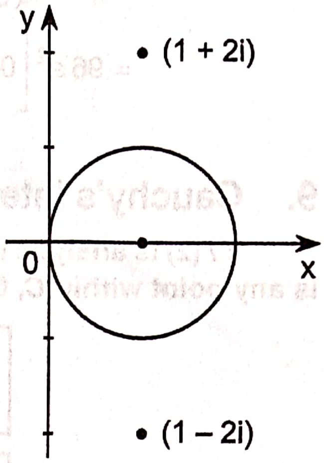


Fig. 2.17

Example 2 : Evaluate $\int_C \tan z\,dz$, where C is $|z| = 1/2$.

Sol. : The contour C is a circle with centre (0, 0) and radius 1/ 2.

Now $\quad \tan z = \dfrac{\sin z}{\cos z}$

and $\quad \cos z = 0$ for $z = \pm \pi / 2$.

But $z = \pm \pi/2$ lies outside C. Hence, $f(z)$ is analytic in and on C.

Hence, by Cauchy's theorem

$$\int_C f(z)\,dz = 0 \qquad \therefore \quad \int_C \tan z\,dz = 0.$$

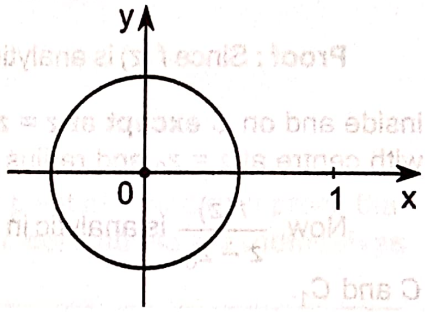


Fig. 2.18

Example 3 : Evaluate $\int_C (8\bar{z} + 3z)\,dz$ around the curve $x^{2/3} + y^{2/3} = a^{2/3}$. **(M.U. 2004)**

Sol. : Since C is a closed curve and the part $f(z) = 3z$ is analytic by Cauchy's Theorem $\int_C 3z\,dz = 0$.

Hence, $I = \int 8\bar{z}\,dz = \int_C 8(x - iy)(dx + i\,dy) = 8\int \{(x\,dx + y\,dy) + i\,(x\,dy - y\,dx)\}$

Now, we put $\quad x = a\cos^3\theta,\ y = a\sin^3\theta$

and $\quad dx = -3a\cos^2\theta\sin\theta\,d\theta,$

$\quad dy = 3a\sin^2\theta\cos\theta\,d\theta.$

$$\therefore \quad I = 8 \cdot 4 \cdot 3a^2 \int_0^{\pi/2} [(-\cos^5\theta\sin\theta) + \sin^5\theta\cos\theta] + i\,[(\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)]\,d\theta$$

[For reduction formulae. See App. Maths. - II.]

$$= 96\,a^2 \left\{ \left[\frac{\cos^6\theta}{6}\right]_0^{\pi/2} + \left[\frac{\sin^6\theta}{6}\right]_0^{\pi/2} + \int_0^{\pi/2} i\,[\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta]\,d\theta \right\}$$

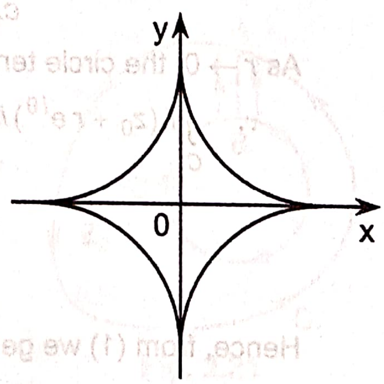


Fig. 2.19

$$\therefore \quad I = 96\,a^2\left\{\left(0-\frac{1}{6}\right)+\left(\frac{1}{6}-0\right)+i\left[\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2}+\frac{3\cdot1\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2}\right]\right\}$$

$$= 96\,a^2\left[0+i\left(\frac{\pi}{32}+\frac{\pi}{32}\right)\right] = \frac{96\,a^2 i}{16}\pi = 6\pi a^2 i.$$

9. Cauchy's Integral Formula (Fundamental Formula)

If $f(z)$ is analytic inside and on a closed curve C of a simply connected region R and if z_0 is any point within C, then

$$\boxed{f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz}$$

i.e.
$$\boxed{\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)}$$

(M.U. 1999, 2002, 03, ...)

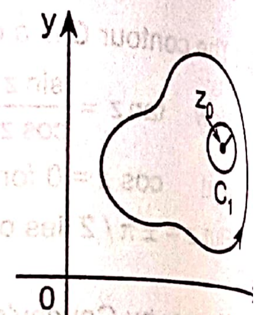


Fig. 2.20

Proof : Since $f(z)$ is analytic inside and on C, $\dfrac{f(z)}{z-z_0}$ is also analytic inside and on C **except at $z = z_0$**. We draw a small circle C_1 around z_0 with centre at $z = z_0$ and radius r lying wholly inside C.

Now, $\dfrac{f(z)}{z-z_0}$ is analytic in the region enclosed between the curves C and C_1.

Hence, by Cauchy's extended theorem,

$$\int_C \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z)}{z-z_0}\,dz \qquad \ldots\ldots\ldots\ldots(1)$$

Putting $z - z_0 = r e^{i\theta}$, $dz = r i e^{i\theta} d\theta$, we get on C_1,

$$\int_{C_1} \frac{f(z)}{z-z_0}\,dz = \int_{C_1} \frac{f(z_0 + r e^{i\theta})}{r e^{i\theta}} \cdot r i e^{i\theta} d\theta$$

$$= \int_{C_1} f(z_0 + r e^{i\theta})\, i\, d\theta$$

As $r \to 0$, the circle tends to the point z_0. Hence, by taking the limit, we get

$$\therefore \quad \int_{C_1} f(z_0 + r e^{i\theta})\, i\, d\theta = \int_{C_1} f(z_0)\, i\, d\theta = i f(z_0)\int_{C_1} d\theta$$

$$= i f(z_0)\int_0^{2\pi} d\theta = 2\pi i\, f(z_0)$$

Hence, from (1) we get $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0).$

Corollary : Cauchy's Integral Formula for derivatives

Differentiating under the integral sign with respect to the parameter z_0, we get

$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

$$f''(z_0) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^3}dz$$

....

....

$$f^{n-1}(z_0) = \frac{(n-1)!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^n}dz$$

The formula can be remembered as

$$\boxed{\int_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!}f^{n-1}(z_0)}$$

Note ....

These formulae give us derivatives of a function at any interior point of R and also prove that **an analytic function possesses derivatives of all orders and these derivatives are themselves analytic.**

10. Theorem (Extension of Cauchy's Integral Formula To Multiply Connected Regions)

If $f(z)$ is analytic in a region R bounded by two closed curves C_1 and C_2 one within the other (as shown in the figure), **if z_0 is any point in the region R then**

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz$$

Proof : We introduce a cross-cut AB. They by Cauchy's Integral formula

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

where C is the curve indicated by arrows in the figure *viz.* the path along $\overrightarrow{AB}$, then the curve C_2 in clockwise sense, then the path along $\overrightarrow{BA}$, then the curve C_1 in anti-clockwise direction.

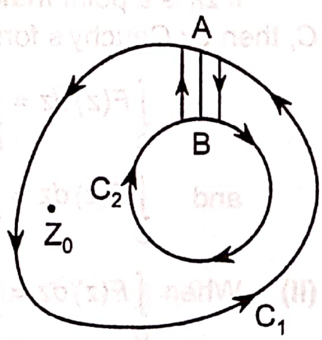


Fig. 2.21

$$\therefore \quad f(z_0) = \frac{1}{2\pi i}\int_{AB} \frac{f(z)}{z-z_0}dz + \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z-z_0}dz$$

$$+ \frac{1}{2\pi i}\int_{BA} \frac{f(z)}{z-z_0}dz + \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z-z_0}dz$$

Now, since the paths along $\vec{AB}$ and $\vec{BA}$ are in opposite directions the integrals along AB, BA cancel out

$$\therefore \quad f(z_0) = \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z - z_0}\,dz + \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z - z_0}\,dz$$

Reversing the direction of the integral along C_2, we get

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z - z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z - z_0}\,dz$$

Particular Case : If C_1 and C_2 are two concentric circles and z_0 is any point in the annulus region, then we have

$$f(z_0) = \frac{1}{2\pi i}\int_{C_1} \frac{f(z)}{z - z_0}\,dz - \frac{1}{2\pi i}\int_{C_2} \frac{f(z)}{z - z_0}\,dz$$

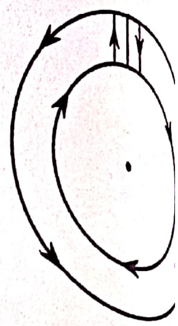


Fig. 2.22

11. Converse of Cauchy's Theorem (Morera's Theorem)

If $f(z)$ is continuous in a region R and if $\int_C f(z)\,dz = 0$ for every simple closed curve C w... can be drawn in R then $f(z)$ is analytic in R.

We accept this theorem without proof.

12. Procedure to Find The Integral

$\int_C F(z)\,dz$ where C is a closed curve in a given region R.

(i) When $\int_C F(z)\,dz = \int_C \frac{\Phi(z)\,dz}{z - z_0}$ or $\int \frac{\Phi(z)}{(z - z_0)^n}\,dz.$

If z_0 is a point **outside** C, $F(z)$ is analytic in and on C. We put $F(z) = f(z)$ and then by Cauc... theorem of § 8, page 2-17.

$$\int_C f(z)\,dz = 0 \qquad \text{[See Ex.1(a)]}$$

If z_0 is a point **inside** C, $F(z)$ is **not** analytic in C. We put $\Phi(z) = f(z)$ which may be analy... C, then by Cauchy's formula of § 9, page 2-20.

$$\int_C F(z)\,dz = \int_C \frac{f(z)}{z - z_0}\,dz = 2\pi i\, f(z_0)$$

and $\int_C F(z)\,dz = \int \frac{f(z)\,dz}{(z - z_0)^n} = \frac{2\pi i}{(n-1)!} r^{n-1}(z_0)$ [See Ex. 1(b), 2, 3 and Ex. 5]

(ii) When $\int_C F(z)\,dz = \int \frac{\Phi(z)}{(z - z_1)(z - z_2)}\,dz$

If z_1 and z_2 are **both inside** C, the $F(z)$ is **not** analytic in C. We then express $\frac{1}{(z - z_1)(z - \ldots}$ as partial fraction $\frac{k_1}{z - z_1} + \frac{k_2}{z - z_2}$. We put $\Phi(z) = f(z)$ which *may be* analytic in C. Then by Cauc... formula.

$$\therefore \quad \int_C F(z)\,dz = k_1 \int_C \frac{f(z)}{z-z_1}\,dz + k_2 \int_C \frac{f(z)}{z-z_2}\,dz$$

$$= k_1 \cdot 2\pi i\, f(z_1) + k_2 \cdot 2\pi i\, f(z_2)$$ [See Ex. 1, 2, 3 pages 2-27]

(iii) When $\int_C F(z)\,dz = \int \frac{\Phi(z)}{(z-z_1)(z-z_2)}$

If z_1 and z_2 are **both outside C**, $F(z)$ is analytic in C. We put $F(z) = f(z)$. Then by Cauchy's theorem

$$\int_C F(z)\,dz = \int_C f(z)\,dz = 0$$ [See Ex. 1, 2 pages 2-28]

(iv) If z_1 **is outside** and z_2 **is inside C**, $F(z)$ is **not** analytic in C. We then write $\frac{\Phi(z)}{z-z_1} = f(z)$. Then $f(z)$ *may be* analytic in C. By Cauchy's integral formula

$$\int_C f(z)\,dz = \int_C \frac{f(z)}{z-z_2}\,dz = 2\pi i\, f(z_2).$$ [See Ex. 1, 2, 3, 4 pages 2-28, 2-29, 2-30]

Type I : $\int_C F(z)\,dz = \int \frac{\Phi(z)\,dz}{z-z_0}$ **or** $\int \frac{\Phi(z)\,dz}{(z-z_0)^n}$ **and z_0 is a point inside or outside C**

Example 1 : Find (a) $\int_C \frac{dz}{z-z_0}$, (b) $\int \frac{dz}{(z-z_0)^n}$, $n \neq 1$ where C is a simple closed curve and $z = z_0$ is a point (a) outside C, (b) inside C. **(M.U. 2003, 04)**

Sol. : (a) Since $z = z_0$ is a point **outside** C, both $\frac{1}{z-z_0}$ and $\frac{1}{(z-z_0)^n}$ are analytic in and on C.

Hence, by Cauchy's Theorem.

$$\int_C \frac{dz}{z-z_0} = 0 \quad \text{and} \quad \int_C \frac{dz}{(z-z_0)^n} = 0$$

(b) Since $z = z_0$ is **inside** C, both $\frac{1}{z-z_0}$ and $\frac{1}{(z-z_0)^n}$ are **not** analytic in C. We, therefore, put $f(z) = 1$ which is analytic in and on C. Hence, by Cauchy's formula.

$$\int_C \frac{dz}{z-z_0} = 2\pi i\, f(z_0) = 2\pi i \qquad [\because f(z) = 1,\ f(z_0) = 1]$$

and $$\int_C \frac{f(z)\,dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(z_0)$$

$$\int_C \frac{dz}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(z_0) = 0$$

$$\left[\because f(z) = 1,\ f'(z) = 0,\ f''(z) = 0, \ldots \text{ etc. } f^{n-1}(z_0) = 0\right]$$

Example 2 : Evaluate $\int_C \frac{1}{z} \cdot \cos z\,dz$ where C is the ellipse $9x^2 + 4y^2 = 1$. **(M.U. 2003, 19)**

Sol. : The point $z = 0$ lies **inside** the ellipse $9x^2 + 4y^2 = 1$.

Hence, by Cauchy's integral formula,

$$\int_C \frac{1}{z} \cdot \cos z \, dz = 2\pi i \cos(0) = 2\pi i.$$

Example 3 : Evaluate $\int_C \frac{\cot z}{z} dz$ where C is the ellipse $9x^2 + 4y^2 = 1$.

Sol. : We have $\int_C \frac{\cot z}{z} dz = \int_C \frac{\cos z}{z \sin z} dz$

The point $z = 0$ lies **inside** the ellipse.

Hence, by Cauchy's integral formula

$$\int_C \frac{\cot z}{z} dz = \int_C \frac{\cos z}{z \sin z} dz = 2\pi i \cos(0) = 2\pi i.$$

Example 4 : Evaluate $\oint_C \frac{e^3 z}{z - \pi i} dz$ where C is the curve $|z-2| + |z+2| = 6$. **(M.U. 2003)**

Sol. : As seen in Fig. 2.5, page 2-3, $|z-2| + |z+2| = 6$ is an ellipse with semi-major axis $a = 6/2 = 3$ and semi-minor axis given by $b^2 = 3^2 - 2^2 = 5$ as shown in the Fig. 2.23. (For alternative explanation see the next example.)

The point $z = \pi i$ *i.e.* (0, 3.14) lies **outside** the ellipse.

Hence, by Cauchy's Theorem,

$$\oint_C \frac{e^3 z}{z - \pi i} dz = 0.$$

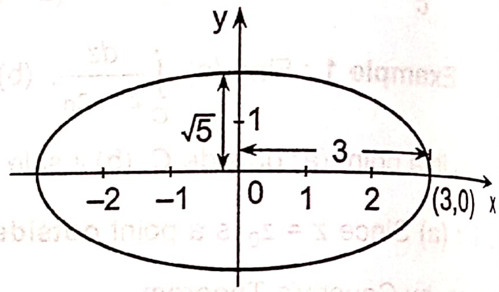


Fig. 2.23

Example 5 : Evaluate $\int_C \frac{e^{3z}}{z - i} dz$ where C is the curve $|z-2| + |z+2| = 6$.

(M.U. 1994, 2003)

Sol. : We can write $|z-2| + |z+2| = 6$ as $|x - 2 + iy| + |x + 2 + iy| = 6$

i.e. $\sqrt{(x-2)^2 + y^2} + \sqrt{(x+2)^2 + y^2} = 6$

Putting $y = 0$, $x - 2 + x + 2 = 6$ $\quad \therefore \; x = 3$.

Putting $x = 0$, $\sqrt{y^2 + 4} + \sqrt{y^2 + 4} = 6$

$\therefore \; 2\sqrt{y^2 + 4} = 6 \quad \therefore \; y^2 + 4 = 9$

$\therefore \; y = \pm\sqrt{5}$

The curve $|z-2| + |z+2| = 6$ is an ellipse with foci at (−2, 0), (2, 0) and intersecting the real axis in (−3, 0), (3, 0) and imaginary axis in $(0, \sqrt{5})$, $(0, -\sqrt{5})$.

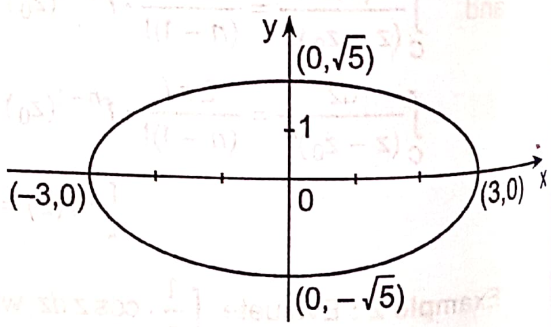


Fig. 2.24

The point $z = i$ *i.e.* (0, 1) lies **inside** C and $f(z) = e^{3z}$ is analytic in and on C. Hence, by Cauchy's integral formula.

$$\int_C \frac{e^{3z}}{z-i}\,dz = 2\pi i\, e^{3i} = 2\pi i\,(\cos 3 + i \sin 3)$$

Example 6 : Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz$ where C is $|z| = 1$. **(M.U. 1994, 95, 2000, 04)**

Sol. : The contour $|z| = 1$ is a circle with centre at the origin and radius unity. Hence the point $z = \pi/6$ lies **inside** C. $f(z) = \sin^6 z$ is analytic in and on C. Hence, by corollary of Cauchy's Integral formula.

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

where $f(z) = \sin^6 6$ and $z_0 = \pi/6$, $n = 3$.

Now, $f(z) = \sin^6 z$

$$\therefore \quad f'(z) = 6\sin^5 z \cos z$$

$$f''(z) = 6(5\sin^4 z \cos^2 z - \sin^6 z)$$

$$f''(\pi/6) = 6(\sin^4 \pi/6 \cos^2 \pi/6 - \sin^6 \pi/6)$$

$$= 6\left(5 \cdot \frac{1}{16} \cdot \frac{3}{4} - \frac{1}{64}\right) = \frac{6 \cdot 14}{64} = \frac{21}{16}$$

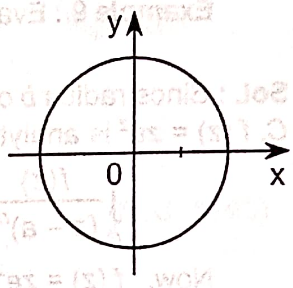


Fig. 2.25

$$\therefore \quad \int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i}{2!} \cdot \frac{21}{16} = \frac{21\pi i}{16}$$

Example 7 : Evaluate $\int_C \frac{\sin^6 z}{[z-(\pi/2)]^3}\,dz$ where C is the circle $|z| = 2$. **(M.U. 1997, 2014)**

Sol. : The contour $|z| = 2$ is a circle with centre at the origin and radius 2. Here, the point $z = \frac{\pi}{2} = \frac{3 \cdot 14}{2} = 1 \cdot 57$ lies inside the circle. The root is repeated thrice. Hence, by the formula given in corollary on page 2-21.

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0) \qquad \text{where, } f(z) = \sin^6 z \text{ and } z_0 = \frac{\pi}{2},\ n = 3.$$

As seen in the above example, $f' = 6\sin^5 z \cos z$

$$f''(z) = 6(5\sin^4 z \cos^2 z - \sin^6 z)$$

$$f''(\pi/2) = 6\left(5\sin^4 \frac{\pi}{2} \cos^2 \frac{\pi}{2} - \sin^6 \frac{\pi}{2}\right) = -6$$

$$\therefore \quad \int_C \frac{\sin^6 z}{[z-(\pi/2)]^3}\,dz = \frac{2\pi i}{2!}(-6) = -6\pi i.$$

Example 8 : Evaluate $\int_C \frac{e^{2z}}{(z-1)^4} dz$ where C is the circle $|z| = 2$. **(M.U. 2016)**

Sol. : Since the point $z = 1$ lies inside the circle $|z| = 2$, $f(z) = e^{2z}$ is analytic in and on C. Hence, by corollary of Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n} = \frac{2\pi i}{(n-1)!} \cdot f^{n-1}(a)$$

Now, $f(z) = e^{2z}$ $\quad \therefore f'(z) = 2e^{2z}, \quad f''(z) = 4e^{2z}, \quad f'''(z) = 8e^{2z}$.

$\therefore f'''(1) = 8e^2$.

$$\therefore \int_C \frac{e^{2z}}{(z-1)^4} = \frac{2\pi i}{3!} \cdot 8e^2 = \frac{8\pi i}{3} \cdot e^2$$

Example 9 : Evaluate $\int_C \frac{ze^z}{(z-a)^3} dz$ where C is a circle with centre at the origin and radius $b > a$.

Sol. : Since radius b of the circle with centre at the origin is greater than a, the point $z = a$ lies inside C. $f(z) = ze^z$ is analytic in and on C. Hence, by corollary of Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a)$$

Now, $f(z) = ze^z$

$\therefore f'(z) = ze^z + e^z$

$f''(z) = ze^z + e^z + e^z = ze^z + 2e^z$

$f''(a) = ae^a + 2e^a = (a+2)e^a$

$$\therefore \int_C \frac{ze^z}{(z-a)^3} dz = \frac{2\pi i}{2!} f''(a) = \frac{2\pi i}{2!}(a+2)e^a = (a+2)e^a \pi i$$

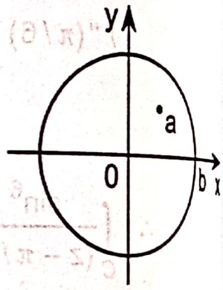


Fig. 2.26

Example 10 : Evaluate $\int_C \frac{\tan(z/2)}{(z-a)^2} dz$, $(-2 < a < 2)$ where C is the boundary of the square with centre at the origin and sides of length 2.

Sol. : Since $-2 < a < 2$ and the sides of the square are of length 2, $z = a$ lies **inside** the square.

If $f(z) = \tan\frac{z}{2}$, $f'(z) = \frac{1}{2}\sec^2\frac{z}{2}$

does not exist at $z = \pm\pi, \pm 3\pi$, ... etc. But these points lie outside C. Hence, by corollary of Cauchy's integral formula

$$\int_C \frac{f(z)}{(z-a)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a)$$

$$\therefore \int_C \frac{\tan(z/2)}{(z-a)^2} dz = \frac{2\pi i}{1!}\left(\frac{1}{2}\sec^2\frac{a}{2}\right) = \pi i \sec^2\frac{a}{2}$$

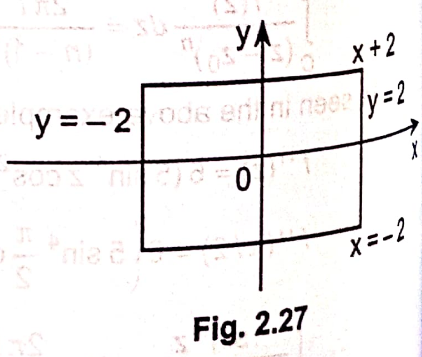


Fig. 2.27

Type II : $\int_C F(z)\,dz = \int_C \frac{\Phi(z)}{(z-z_1)(z-z_2)}\,dz$ **and z_1 and z_2 are both inside C.**

Example 1 : Evaluate $\int_C \frac{3z^2+z}{z^2-1}\,dz$, where C is the circle $|z| = 2$. **(M.U. 1998, 2005)**

Sol. : The circle $|z| = 2$ has centre at (0, 0) and radius 2. Now, $z^2 - 1 = 0$ gives $(z-1)(z+1) = 0$. $\therefore z = 1, -1$, Both these points lie **inside** the circle. Hence, we use partial fractions and write $\frac{1}{z^2-1} = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$ and write $f(z) = 3z^2 + z$, which is analytic in and on C.

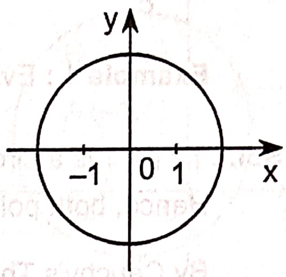


Fig. 2.28

$$\therefore \int_C \frac{3z^2+z}{z^2-1}\,dz = \frac{1}{2}\int_C \frac{3z^2+z}{z-1}\,dz - \frac{1}{2}\int_C \frac{3z^2+z}{z+1}\,dz$$

$$= \frac{1}{2}\cdot 2\pi i\, f(1) - \frac{1}{2}\cdot 2\pi i\, f(-1) \quad \text{where } f(z) = 3z^2 + z$$

$$= \pi i\,(4) - \pi i\,(2) = 2\pi i$$

Example 2 : Evaluate $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$, where C is the circle $|z| = 3$. **(M.U. 1993)**

Sol. : The circle $|z| = 3$ has centre at (0, 0) and radius 3. The points $z = 1$ and $z = 2$ lie **inside** the circle. Hence, $f(z)$ is not analytic in C. Hence, we use the method of partial fractions and write $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ and write $f(z) = e^{2z}$ which is analytic in C.

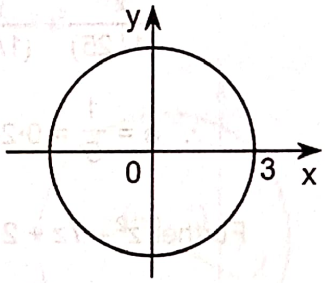


Fig. 2.29

$$\therefore \int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz = \int_C \frac{e^{2z}}{z-2}\,dz - \int_C \frac{e^{2z}}{z-1}\,dz$$

$$= 2\pi i\, f(2) - 2\pi i\, f(1) \quad \text{where } f(z) = e^{2z}$$

$$= 2\pi i\, e^4 - 2\pi i\, e^2 = 2\pi i\, e^2(e^2 - 1)$$

Example 3 : Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)}\,dz$, where C is the circle $|z| = 4$.

Sol. : $|z| = 4$ is a circle with centre at the origin and radius 4. Hence $z = 2$, $z = 3$ lie **inside** the circle. Hence, the given function $f(z)$ is not analytic in C. Hence we use the method of partial fractions and write $\frac{1}{(z-2)(z-3)} = \frac{1}{z-3} - \frac{1}{z-2}$ and write $f(z) = \sin \pi z^2 + \cos \pi z^2$ which is analytic in C.

By Cauchy's integral formula,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)}\,dz$$

$$= \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-3}\,dz - \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}\,dz$$

$$= 2\pi i\, f(3) + 2\pi i\, f(2) \quad \text{where } f(z) = \sin \pi z^2 + \cos \pi z^2$$

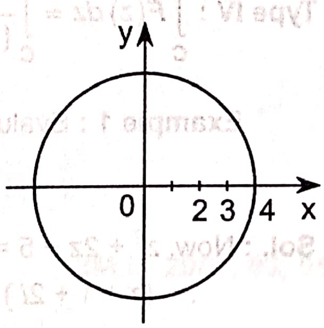


Fig. 2.30

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)(z-3)} dz = 2\pi i(\sin 9\pi + \cos 9\pi) - 2\pi i(\sin 4\pi + \cos 4\pi)$$

$$= -2\pi i - 2\pi i = -4\pi i$$

Type III : $\int_C F(z)\,dz = \int \frac{\Phi(z)}{(z-z_1)(z-z_2)} dz$ **and z_1 and z_2 are both outside C.**

Example 1 : Evaluate $\int_C \frac{z+2}{(z-3)(z-4)} dz$, where C is the circle $|z| = 1$. **(M.U. 20**

Sol. : $|z| = 1$ is a circle with centre at the origin and radius 1.

Hence, both points $z = 3$ and $z = 4$ lie **outside** the circle C and $f(z)$ is analytic in C.

By Cauchy's Theorem, $\int_C \frac{z+2}{(z-3)(z-4)} dz = 0$.

Example 2 : Evaluate $\int_C \frac{z^2+2+1}{z^2-7z+2} dz$, where C is the ellipse $25x^2 + 16y^2 = 1$.

(M.U. 1992, 93,

Sol. : The equation of the ellipse can be written as

$$\frac{x^2}{(1/25)} + \frac{y^2}{(1/16)} = 1 \quad i.e. \quad \frac{x^2}{(1/5)^2} + \frac{y^2}{(1/4)^2} = 1$$

$$\therefore \quad a = \frac{1}{5} = 0{\cdot}2, \quad b = \frac{1}{4} = 0{\cdot}25.$$

Further $z^2 - 7z + 2 = 0$ gives $z = \dfrac{7 \pm \sqrt{49-8}}{2} = \dfrac{7 \pm \sqrt{41}}{2}$.

The root $\dfrac{7+\sqrt{41}}{2} = \dfrac{7+6{\cdot}4}{2} \simeq \dfrac{13{\cdot}4}{2} = 6{\cdot}7 > 0{\cdot}2$ and $0{\cdot}25$.

The root $\dfrac{7-\sqrt{41}}{2} = \dfrac{7-6{\cdot}4}{2} \simeq \dfrac{0{\cdot}6}{2} = 0{\cdot}3 > 0{\cdot}2$ and $0{\cdot}25$.

Since, both roots are outside the contour $f(z)$ is analytic in and on C. By Cauchy's Theore

$$\int_C f(z)\,dz = 0$$

Type IV : $\int_C F(z)\,dz = \int_C \frac{\Phi(z)}{(z-z_1)(z-z_2)} dz$ **and z_1 is inside and z_2 is outside C.**

Example 1 : Evaluate $\int_C \frac{z+3}{z^2+2z+5} dz$, where C is the circle (i) $|z| = 1$, (ii) $|z+1-i| =$

(M.U. 1997, 20

Sol. : Now, $z^2 + 2z + 5 = 0$ gives $(z+1)^2 + 2^2 = 0$

$\therefore \quad (z+1+2i)(z+1-2i) = 0 \qquad \therefore \quad z = -1-2i, \; z = -1+2i.$

(I) Now, $|z| = 1$ is a circle with centre at the origin and radius 1. Hence, both the points lie **outside** the circle C and $f(z)$ is analytic in C.

By Cauchy's Theorem $\int_C \frac{z+3}{z^2+2z+5}\, dz = 0$.

(II) Now, $|z+1-i| = 2$, is a circle with centre at $A(-1+i)$ *i.e.* $(-1, 1)$ and radius 2. The point $B(-1, 2)$ lies **inside** and the point $C(-1, -2)$ (not shown in the figure) lies **outside** the circle. Hence, we put $f(z) = \frac{z+3}{z+1+2i}$ which is analytic in C. By Cauchy's formula

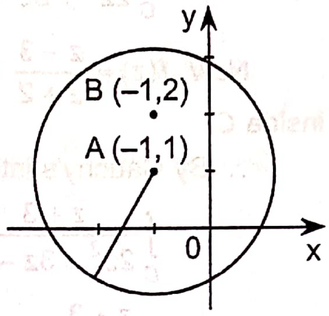


Fig. 2.31

$$\int_C \frac{z+3}{z^2+2z+5}\, dz = \int_C \frac{(z+3)/(z+1+2i)}{z+1-2i}\, dz$$

$$= \int_C \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0) \text{ where } z_0 = -1+2i$$

$$= 2\pi i \cdot \frac{-1+2i+3}{-1+2i+1+2i} = 2\pi i \cdot \frac{(2+2i)}{4i} = \pi(1+i)$$

Example 2 : Evaluate $\int_C \frac{z+4}{z^2+2z+5}\, dz$ where C is the circle $|z+1-i| = 2$.

(M.U. 2003, 05, 16)

Sol. : The circle $|z+1-i| = 2$ has centre at $-1+i$ *i.e.*, $(-1, 1)$ and radius 2.

Now, $z^2+2z+5 = 0$ gives $(z+1)^2+2^2 = 0$

$\therefore \quad (z+1+2i)(z+1-2i) = 0$

$\therefore \quad z = -1-2i$ and $z = -1+2i$.

The root $z = -1+2i$ *i.e.*, $(-1, 2)$ lies **inside C** and $z = -1-2i$ *i.e.*, $(-1, -2)$ lies **outside C**.

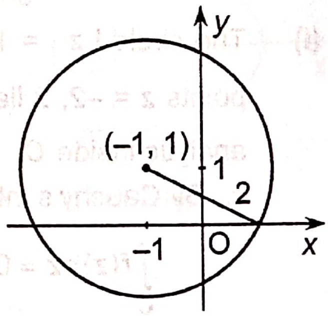


Fig. 2.32

Hence, we put $f(z) = \frac{z+4}{z+1+2i}$ which is analytic in C.

By Cauchy's formula,

$$\int_C \frac{z+4}{z^2+2z+5}\, dz = \int_C \frac{(z+4)/(z+1+2i)}{(z+1-2i)}\, dz$$

$$= \int_C \frac{f(z)}{z+1-2i}\, dz = 2\pi i f(z_0) \quad \text{where } z_0 = -1+2i$$

$$\therefore \quad \int_C \frac{z+4}{z^2+2z+5}\, dz = 2\pi i \cdot \frac{-1+2i+4}{-1+2i+1+2i}$$

$$= 2\pi i \cdot \frac{3+2i}{4i} = (3+2i)\frac{\pi}{2}$$

Example 3 : Evaluate $\int_C \frac{z+3}{2z^2+3z-2}\, dz$ where C is the circle $|z-i| = 2$.

(M.U. 2001, 03, 05)

Sol. : The circle $|z - i| = 2$ has centre at (0, 1) and radius 2. Now, $2z^2 + 3z - 2 = (2z - 1)(z + 2)$. Hence, $z = 1/2$, lies **inside** the circle and $z = -2$ lies **outside** the circle.

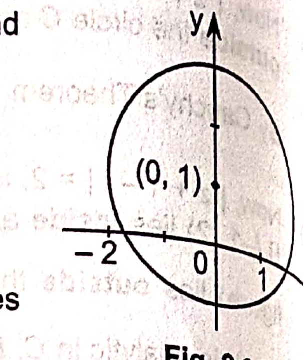


Fig. 2.33

Hence, we write

$$\int_C \frac{z+3}{2z^2+3z-2}\,dz = \int_C \frac{(z+3)/(z+2)}{2z-1}\,dz$$

Now, $f(z) = \dfrac{z+3}{z+2}$ is analytic in and on C and the point $z = 1/2$ lies inside C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+3}{2z^2+3z-2}\,dz = \int_C \frac{(z+3)/(z+2)}{2z-1}\,dz = 2\pi i\, f(z_0)$$

where, $f(z) = \dfrac{z+3}{z+2}$ and $z_0 = \dfrac{1}{2}$.

$$\therefore\ f(z_0) = \frac{(1/2)+3}{(1/2)+2} = \frac{7}{5} \qquad \therefore\ \int_C \frac{z+3}{2z^2+3z-2}\,dz = 2\pi i\cdot\frac{7}{5} = \frac{14}{5}\pi i.$$

Example 4 : Evaluate $\displaystyle\int_C \frac{z+6}{z^2-4}\,dz$ where C is the circle

(i) $|z| = 1$, (ii) $|z - 2| = 1$, (iii) $|z + 2| = 1$. **(M.U. 2001, 10,**

Sol. : Now $z^2 - 4 = 0$ gives $(z+2)(z-2) = 0 \quad \therefore\ z = -2, 2.$

(i) The circle $|z| = 1$ has centre at the origin and radius 1. The points $z = -2, 2$ lie outside the circle. Hence, $f(z) = \dfrac{z+6}{z^2-4}$ is analytic inside C.

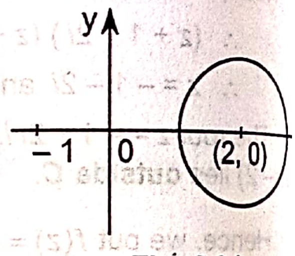


Fig. 2.34

$\therefore$ By Cauchy's integral formula

$$\int_C f(z)\,dz = 0 \qquad \therefore\ \int_C \frac{z+6}{z^2-4}\,dz = 0$$

(ii) The circle $|z - 2| = 1$ has centre at (2, 0) and radius 1. The point (2, 0) lies **inside** C and point (– 2, 0) lies **outside** C. Hence, we write

$$\int_C \frac{z+6}{(z-2)(z+2)}\,dz = \int_C \frac{(z+6)/(z+2)}{z-2}\,dz$$

Now, $f(z) = \dfrac{z+6}{z+2}$ is analytic in and on C and the point $z = 2$ lies **inside** C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+6}{z^2-4}\,dz = \int_C \frac{(z+6)/(z+2)}{z-2}\,dz = 2\pi i\, f(z_0) \quad \text{where, } f(z) = \frac{z+6}{z+2} \text{ and } z_0 = 2.$$

Now, $f(z_0) = \dfrac{2+6}{2+4} = \dfrac{8}{4} = 2$

$$\therefore\ \int_C \frac{z+6}{z^2-4}\,dz = 2\pi i\,(2) = 4\pi i.$$

(iii) The circle $|z+2|=1$ has centre at $z=-2$ and radius 1. The point $(-2, 0)$ lies **inside** C and the point $(2, 0)$ lies **outside** C. Hence, we write

$$\int_C \frac{z+6}{(z-2)(z+2)}dz = \int_C \frac{(z+6)/(z+2)}{z+2}dz$$

Now, $f(z) = \dfrac{z+6}{z-2}$ is analytic in and on C and the point $z=-2$ lies **inside** C.

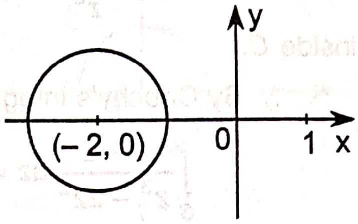


Fig. 2.35

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+6}{z^2-4}dz = \int_C \frac{(z+6)/(z-2)}{z+2}dz = 2\pi i\, f(z_0) \quad \text{where, } f(z) = \frac{z+6}{z-2} \text{ and } z_0 = -2.$$

Now, $f(z_0) = \dfrac{-2+6}{-2+4} = \dfrac{4}{-4} = -1.$

$$\therefore \int_C \frac{z+6}{z^2-4}dz = 2\pi i(-1) = -2\pi i.$$

Example 5 : Evaluate $\displaystyle\int_C \frac{4z-1}{z^2-3z-4}dz$, where C is the ellipse $x^2+4y^2=4$. **(M.U. 2003)**

Sol. : The ellipse $x^2+4y^2=4$ *i.e.* $\dfrac{x^2}{4}+\dfrac{y^2}{1}=1$ has centre at the origin and major axis 2 and minor axis 1.

Now, $z^2-3z-4=(z-4)(z+1)$. Hence, $z=-1$ lies **inside** C and $z=4$ lies **outside** C. Hence, we write

$$\int_C \frac{4z-1}{z^2-3z-4}dz = \int \frac{(4z-1)/(z-4)}{z+1}dz$$

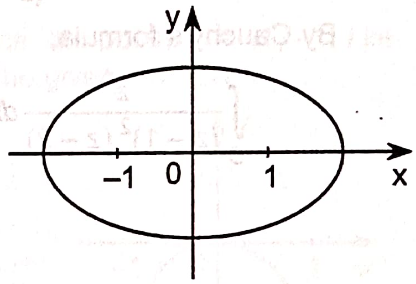


Fig. 2.36

Now, $f(z) = \dfrac{4z-1}{z-4}$ is analytic in and on C and the point $z=-1$ lies **inside** C.

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{4z-1}{z^2-3z-4}dz = \int_C \frac{(4z-1)/(z-4)}{z+1}dz$$

$$= 2\pi i\, f(z_0) \qquad \text{where } f(z) = \frac{4z-1}{z-4} \text{ and } z_0 = -1$$

$$\therefore f(z_0) = \frac{-4-1}{-1-4} = \frac{-5}{-5} = 1 \qquad \therefore \int_C \frac{4z-1}{z^2-3z-4}dz = 2\pi i(1) = 2\pi i$$

Example 6 : Evaluate $\displaystyle\int_C \frac{z+2}{z^3-2z^2}dz$, where C is the circle $|z-2-i|=2$. **(M.U. 2009)**

Sol. : The circle $|z-2-i|=2$ has centre at $2+i$ *i.e.* $(2, 1)$ and radius 2. The point $z=0$ lies **outside** the circle but $z=2$ *i.e.* $(2, 0)$ lies **inside** the circle.

Hence, we write $\displaystyle\int \frac{z+2}{z^2(z-2)}dz = \int \frac{(z+2)/z^2}{z-2}dz.$

Now, $f(z) = \dfrac{z+2}{z^2}$ is analytic in and on C and the point $z = 2$ lies inside C.

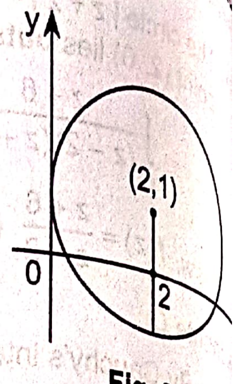


Fig. 2.37

$\therefore$ By Cauchy's integral formula

$$\int_C \frac{z+2}{z^3 - 2z^2}\, dz = \int_C \frac{(z+2)/z^2}{(z-2)}\, dz = 2\pi i\, f(z_0)$$

where, $f(z) = \dfrac{z+2}{z^2}$ and $z_0 = 2$.

$$\therefore \quad f(z_0) = \frac{2+2}{4} = \frac{4}{4} = 1 \quad \therefore \quad \int_C \frac{z+2}{z^2(z-2)}\, dz = 2\pi i\,(1) = 2\pi i.$$

Example 7 : Evaluate $\displaystyle\int_C \frac{z\, dz}{(z-1)^2\,(z-2)}$ where C is the circle $|z - 2| = 0.5$. **(M.U. 20**

Sol. : Here C is the circle with centre at (2, 0) and radius 0·5.

Now, $(z-1)^2\,(z-2) = 0$ gives $z = 1$ and $z = 0$ *i.e.*, z is (1, 0) and z is (2, 0).

From the figure we see that $z = (2, 0)$ lies **inside C** and $z = (1, 0)$ lies **outside C.**

Hence, we put $f(z) = \dfrac{z}{(z-1)^2}$ which is analytic in C.

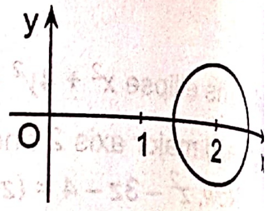


Fig. 2.38

By Cauchy's formula,

$$\int_C \frac{z}{(z-1)^2\,(z-2)}\, dz = \int_C \frac{z/(z-1)^2}{z-2}\, dz$$

$$= \int_C \frac{f(z)}{z-2}\, dz = 2\pi i\, f(z_0) \quad \text{where } z_0 = 2$$

$$= 2\pi i \cdot \frac{2}{(2-1)^2} = 4\pi i.$$

Example 8 : Evaluate $\displaystyle\int_C \frac{z^2}{z^4 - 1}\, dz$, where C is the circle

(i) $|z| = 1/2$, (ii) $|z - 1| = 1$, (iii) $|z + i| = 1$. **(M.U. 1994, 2005**

Sol. : We first express the denominator in terms of linear factors.

Let $\dfrac{1}{z^4 - 1} = \dfrac{a}{z-1} + \dfrac{b}{z+1} + \dfrac{c}{z-i} + \dfrac{d}{z+i}$

$\therefore \quad 1 = a\,(z+1)\,(z^2+1) + b\,(z-1)\,(z^2+1) + c\,(z+i)\,(z^2-1) + d\,(z-i)\,(z^2-1)$

When $z = 1$, $\quad 1 = a \cdot 4 \quad \therefore\ a = 1/4.$

When $z = -1$, $\quad 1 = -4b \quad \therefore\ b = -1/4.$

When $z = i$, $\quad 1 = -4ic \quad \therefore\ c = -1/4i.$

When $z = -i$, $\quad 1 = 4id \quad \therefore\ d = 1/4i.$

$$\therefore \quad \frac{z^2}{z^4 - 1} = \frac{1}{4} \cdot \frac{z^2}{z-1} - \frac{1}{4} \cdot \frac{z^2}{z+1} - \frac{1}{4i} \cdot \frac{z^2}{z-i} + \frac{1}{4i} \cdot \frac{z^2}{z+i}$$

(i) $|z| = 1/2$ is a circle with centre at the origin and radius $1/2$. Hence, all the points $z = 1, -1, i, -i$ lie **outside** the circle $|z| = 1/2$. The function $f(z) = \dfrac{z^2}{z^4 - 1}$ is analytic in and on the C. Hence, by Cauchy's Theorem $\int_C \dfrac{z^2}{z^4-1} dz = 0$.

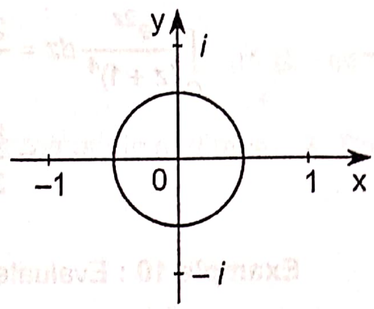


Fig. 2.39

(ii) $|z - 1| = 1$ is a circle with centre at $(1, 0)$ and radius 1. Hence, the points $z = -1, i, -i$ lie **outside** the circle. The only point $z = 1$ lies inside the circle. The given function

$$\Phi(z) = \frac{z^2/(z^2+1)(z+1)}{(z-1)}.$$

Now, $f(z) = \dfrac{z^2}{(z^2+1)(z+1)}$ is analytic in C and on C.

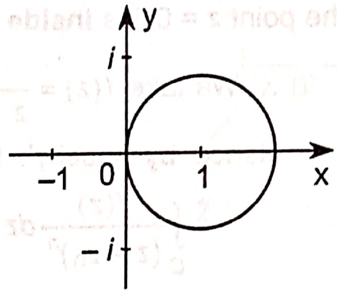


Fig. 2.40

Hence, by Cauchy's formula,

$$\int_C f(z)\,dz = 2\pi i\, f(z_0) \quad \text{where } z_0 = 1$$

$$= 2\pi i \cdot \frac{1}{4}(1) = \frac{\pi i}{2}$$

(iii) $|z + i| = 1$ is a circle with centre at $(0, -i)$ and radius 1. Hence, the points $z = +1, -1, i$ lie **outside** the circle. The only point $z = -i$ lies inside the circle. The given function

$$\Phi(z) = \frac{z^2/(z^2-1)(z-i)}{(z+i)}.$$

Now, $\quad f(z) = \dfrac{z^2}{(z^2-1)(z-i)}$

is analytic in and on C. Hence, by Cauchy's formula,

$$\int_C f(z)\,dz = 2\pi i\, f(z_0) \quad \text{where } z_0 = -i$$

$$= 2\pi i \cdot \frac{1}{4i}(-1) = -\frac{\pi}{2}.$$

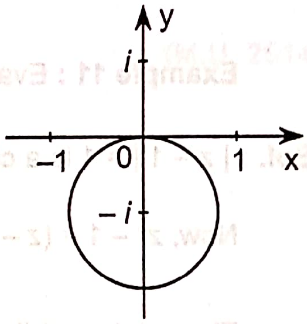


Fig. 2.41

Example 9 : Evaluate $\int_C \dfrac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z - 1| = 3$. **(M.U. 1999)**

Sol. : The circle $|z - 1| = 3$ has centre at C $(1, 0)$ and radius 3. Further, $z + 1 = 0$ gives A. $z = -1$. The point A lies **inside** the circle. Hence, $e^{2z}/(z+1)^4$ is not analytic in C. We take $f(z) = e^{2z}$ which is analytic in C.

By Corollary of Cauchy's Formula

$$\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \quad \int_C \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f^3(z_0)$$

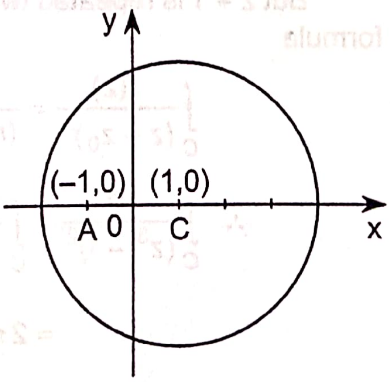


Fig. 2.42

$$\therefore \quad \int_C \frac{e^{2z}}{(z+1)^4}\,dz = \frac{2\pi i}{3!}\cdot\frac{8}{e^2} \qquad \left[\because f(z) = e^{2z},\ f^3(z) = 8e^{2z} \text{ and } z_0 = -1\right]$$

$$= \frac{8\pi i}{3e^2}.$$

Example 10 : Evaluate $\int_C \frac{dz}{z^3(z+4)}$, where C is the circle $|z| = 2$. **(M.U. 1993, 97, 2004, 1**

Sol. : $|z| = 2$ is a circle with centre at the origin and radius 2. $z^3(z+4) = 0$ gives $z = 0$ and $z = -$
The point $z = 0$ lies **inside** the circle and $z = -4$ lies **outside** the circle.

$\therefore$ We take $f(z) = \frac{1}{z+4}$ which is analytic in C.

Hence, by Cauchy's formula

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \quad \int_C \frac{dz/(z+4)}{z^3} = \frac{2\pi i}{2!} f^2(z_0) \ \text{ where } f(z) = \frac{1}{z+4}$$

$$= \pi i\left[\frac{2}{(z+4)^3}\right] \text{ at } z_0 = 0$$

$$= \frac{\pi i}{32}.$$

Example 11 : Evaluate $\oint_C \frac{1}{(z^3-1)^2}\,dz$ where C is $|z-1| = 1$. **(M.U. 1998, 201**

Sol. : $|z-1| = 1$ is a circle with centre at (1, 0) and radius 1.

Now, $z^3 - 1 = (z-1)(z^2+z+1) = 0$ gives $z = 1$ or $z = \frac{-1 \pm \sqrt{3}\cdot i}{2}$.

The point $z = 1$ lies **inside** the circle and the points $z = \frac{-1 \pm \sqrt{3}\cdot i}{2}$ lie **outside** the circle.

Hence, we write $\oint_C \frac{dz}{(z^3-1)^2} = \oint_C \frac{1/(z^2+z+1)^2}{(z-1)^2}\,dz$

But $z = 1$ is repeated twice. Hence, by Corollary of Cauchy's formula

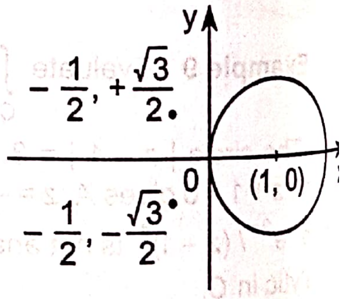


Fig. 2.43

$$\int_C \frac{f(z)}{(z-z_0)^n} = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\therefore \quad \int_C \frac{dz}{(z^3-1)^2} = \int_C \frac{1/(z^2+z+1)^2}{(z-1)^2}\,dz$$

$$= 2\pi i\left[\frac{d}{dz}\cdot\frac{1}{(z^2+z+1)^2}\right]_{z=1} = 2\pi i\left[\frac{-2(2z+z)}{(z^2+z+1)^2}\right]_{z=1}$$

$$= 2\pi i\left[\frac{-2(3)}{3^2}\right] = -\frac{4\pi i}{9}.$$

Example 12 : Evaluate $\oint_C \frac{\sin^6 z}{[z-(\pi/2)]^3} dz$ where C is $|z| = 2$. **(M.U. 1997)**

Sol. : $z - \pi/2 = 0$ gives $z = \pi/2$ and $|z| = 2$ is a circle with center at the origin and radius 2. The point $z = \pi/2$ lies inside C. Hence, by corollary of Cauchy's formula

$$\int_C \frac{f(z)}{z-z_0} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$$

$$\int_C \frac{\sin^6 z}{[z-(\pi/2)]^3} dz = \frac{2\pi i}{2!} f''(z_0)$$

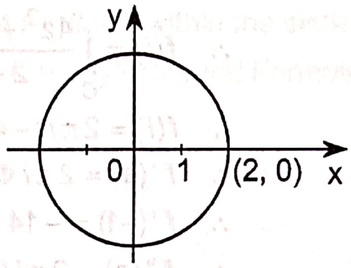


Fig. 2.44

where, $f(z) = \sin^6 z$.

Now, $f'(z) = 6 \sin^5 z \cos z$.

$\therefore \quad f''(z) = 30 \sin^4 z \cos^2 z + 6 \sin^5 z (-\sin z)$

$\therefore \quad f''(z = \pi/2) = 30(0) + 6(-1) = -6$.

$$\therefore \quad \int_C \frac{\sin^6 z}{[z-(\pi/2)]^3} dz = \frac{2\pi i}{2}(-6) = -6\pi i$$

Type V : To find the value of the function $f(\xi) = \int_C \frac{\Phi(z)}{z-\xi}$ at $\xi = \xi_0$.

Example 1 : If $f(\xi) = \int_C \frac{3z^2 + 2z + 1}{z - \xi} dz$, where C is the circle $x^2 + y^2 = 4$, find the values of (i) $f(3)$, (ii) $f'(1-i)$, (iii) $f''(1-i)$. **(M.U. 2014)**

Sol. : The circle $x^2 + y^2 = 4$ is $|z| = 2$.

(i) The point $z = 3$ lies **outside** the circle $|z| = 2$.

$\therefore$ By Cauchy's Integral Theorem

$$f(3) = \int_C \frac{3z^2 + 2z + 1}{z-3} dz = 0$$

(ii) The point $z = 1 - i$ *i.e.* $(1, -1)$ lies **inside** the circle. Hence, we take $\Phi(z) = 3z^2 + 2z + 1$ which is analytic everywhere

$$\therefore \quad \int_C \frac{\Phi(z)}{z-\xi} dz = 2\pi i \, \Phi(\xi) = 2\pi i (3\xi^2 + 2\xi + 1)$$

$$\therefore \quad f(\xi) = \int_C \frac{3z^2 + 2z + 1}{z-\xi} dz = 2\pi i (3\xi^2 + 2\xi + 1)$$

$\therefore \quad f'(\xi) = 2\pi i (6\xi + 2)$ and $f''(\xi) = 2\pi i (6)$

$\therefore \quad f'(1-i) = 2\pi i [6(1-i) + 2] = 2\pi i (8 - 6i)$ and $f''(1-i) = 12\pi i$.

Example 2 : If $f(a) = \int_C \frac{4z^2 + z + 5}{z-a} dz$, where C is $|z| = 2$,

find the values of $f(1)$, $f(i)$, $f'(-1)$, $f''(-i)$.

Sol. : The circle $|z| = 2$ has centre at the origin and radius 2.

The point $z = 1$ lies inside the circle. $f(z) = 4z^2 + z + 5$ is analytic in and on C and $z = 1$ lies **inside** it. Hence, by Cauchy's formula

$$f(1) = \int_C \frac{4z^2 + z + 5}{z - 1} dz = 2\pi i \, \Phi(z_0) \qquad \text{where, } \Phi(z) = 4z^2 + z + 5 \text{ and } z_0 = 1.$$

$$\therefore \quad f(1) = 2\pi i (4 + 1 + 5) = 20\pi i$$

The point $z = i$ also lies **inside** the circle

$$\therefore \quad f(i) = \int_C \frac{4z^2 + z + 5}{z - i} dz = 2\pi i \, \Phi(z_0) \qquad \text{where, } \Phi(z) = 4z^2 + z + 5 \text{ and } z_0 = i.$$

$$\therefore \quad f(i) = 2\pi i(-4 + i + 5) = 2\pi i(1 + i) = 2\pi(i - 1)$$

$$\therefore \quad f'(a) = 2\pi i \, \Phi'(a) = 2\pi i (8z + 1)$$

$$\therefore \quad f'(-1) = -14\pi i$$

$$\therefore \quad f''(a) = 2\pi i (8) = 16\pi i$$

$$\therefore \quad f''(-1) = 16\pi i$$

Example 3 : If $f(\xi) = \int_C \frac{4z^2 + z + 5}{z - \xi} dz$ where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, find the values $f(i)$, $f'(-1)$, $f''(-i)$ and $f(3)$. **(M.U. 2003, 06,**

Sol. : (i) The point $z = i$ *i.e.* (0, 1) lies **inside** the ellipse,

$$f(z) = 4z^2 + z + 5$$

is analytic in and on C. Hence, by Cauchy's formula

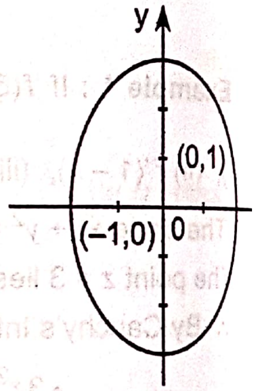


Fig. 2.45

$$f(i) = \int_C \frac{4z^2 + z + 5}{z - i} dz = 2\pi i \; \Phi(z_0)$$

where, $\Phi(z) = 4z^2 + z + 5$ and $z_0 = i$.

$$\therefore \quad f(i) = 2\pi i (4i^2 + i + 5)$$

$$= 2\pi i(-4 + i + 5) = 2\pi i (1 + i)$$

(ii) The point (–1, 0) also lies **inside** the ellipse. Hence, we take $\Phi(z) = 4z^2 + z + 5$, which is analytic in and on C.

$$\therefore \quad f(\xi) = \int_C \frac{4z^2 + z + 5}{z - \xi} dz = 2\pi i (4\xi^2 + \xi + 5)$$

$$\therefore \quad f'(\xi) = 2\pi i (8\xi + 1) \text{ and } f''(\xi) = 2\pi i \cdot 8$$

$$\therefore \quad f'(-1) = 2\pi i \cdot [8(-1) + 1) = -14\pi i$$

(iii) $f''(\xi) = 16\pi i \qquad \therefore \quad f''(-i) = 16\pi i$

(iv) The point (3, 0) lies **outside** the ellipse. Hence,

$$\int_C \frac{4z^2 + z + 5}{z - \xi} dz = 0.$$

Miscellaneous Examples

Example 1 : If C is the circle $|z| = 1$, using the integral $\int_C \frac{e^{kz}}{z} dz$, where k is real, show that $\int_0^{\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta = \pi$.

(M.U. 1993, 2001, 03, 14)

Sol. : We obtain the integral in two different ways.

(I) $|z| = 1$ is a circle with centre at the origin and radius = 1. The point $z = 0$ lies within the circle. Hence, we write $f(z) = e^{kz}$ which is analytic in and on C. Hence, by Cauchy's Integral Formula

$$\int_C \frac{e^{kz}}{z} dz = 2\pi i f(z_0) \text{ where } f(z) = e^{kz} \text{ and } z_0 = 0$$

$$= 2\pi i\, e^{kz_0} \text{ at } z_0 = 0$$

$$= 2\pi i\, e^0 = 2\pi i.$$

(II) Now, if we put $z = e^{i\theta}$, $dz = i e^{i\theta} d\theta$

$$\int_C \frac{e^{kz}}{z} dz = \int_0^{2\pi} \frac{e^{ke^{i\theta}}}{e^{i\theta}} i e^{i\theta}\, d\theta = i \int_0^{2\pi} e^{ke^{i\theta}}\, d\theta$$

$$\therefore \int_C \frac{e^{kz}}{z} dz = i \int_0^{2\pi} e^{k(\cos\theta + i\sin\theta)} d\theta = i \int_0^{2\pi} e^{k\cos\theta} \cdot e^{ik\sin\theta}\, d\theta$$

$$= i \int_0^{2\pi} e^{k\cos\theta} \{\cos(k\sin\theta) + i\sin(k\sin\theta)\}\, d\theta$$

$$\therefore\ 2\pi i = i \int_0^{2\pi} e^{k\cos\theta} \{\cos(k\sin\theta) + i\sin(k\sin\theta)\}\, d\theta$$

Equating imaginary parts

$$2\pi = \int_0^{2\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta$$

$$2\pi = 2\int_0^{\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta \qquad \left[\because \int_0^{a} f(x)\,dx = 2\int_0^{a/2} f(x)\,dx \text{ if } f(a-x) = f(x)\right]$$

$$\therefore \int_0^{\pi} e^{k\cos\theta} \cos(k\sin\theta)\, d\theta = \pi.$$

(For another method see Ex. 16, page 2-122.)

Example 2 : Show that $\int_C \frac{dz}{z+1} = 2\pi i$, where C is the circle $|z| = 2$. Hence, deduce that $\int_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2 + y^2} = 0$ and $\int_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2 + y^2} = 2\pi$.

(M.U. 1991)

Sol. : $|z| = 2$ is a circle with centre at $z = 0$ and radius = 2. The point $z = -1$ lies **inside** the circle. We take $f(z) = 1$, which is analytic in and on C. Hence, by Cauchy's Integral Formula

$$\int_C \frac{dz}{z+1} = 2\pi i\, f(z_0) = 2\pi i \quad \text{.......... (1)}$$

Now $$\int_C \frac{dz}{z+1} = \int_C \frac{dx + i\,dy}{(x+1)+iy} = \int_C \frac{dx + i\,dy}{(x+1)+iy} \cdot \frac{(x+1)-iy}{(x+1)-iy}$$

$$= \int_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2 + y^2} + i\int_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2 + y^2} \quad \text{.......... (2)}$$

From (1) and (2) equating real and imaginary parts

$$\int_C \frac{(x+1)\,dx + y\,dy}{(x+1)^2 + y^2} = 0 \quad \text{and} \quad \int_C \frac{(x+1)\,dy - y\,dx}{(x+1)^2 + y^2} = 2\pi.$$

Example 3 : If $f(z)$ is analytic in and on a simple closed curve C, prove that

$$f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}\,dz.$$

Hence, evaluate $\int_C \frac{e^{iz}}{z^4}\,dz$ where C is the circle $|z| = 2$. **(M.U. 1997)**

Sol. : Since $f(z)$ is analytic in and on C, by Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a)$$

Differentiating under the integral sign, w.r.t. a,

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i\, f'(a)$$

Differentiating twice again w.r.t. a,

$$\int_C \frac{2f(z)}{(z-a)^3}\,dz = 2\pi i\, f''(a) \quad \text{and} \quad \int_C 3!\frac{f(z)}{(z-a)^4}\,dz = 2\pi i\, f'''(a)$$

$$\therefore\quad f'''(a) = \frac{3!}{2\pi i}\int_C \frac{f(z)}{(z-a)^4}\,dz.$$

Now, let $f(z) = e^{iz}$, $a = 0$ and C be $|z| = 2$.

Since, $f(z)$ is analytic in and on C by the above result

$$f'''(0) = \frac{3!}{2\pi i}\int_C \frac{e^{iz}}{z^4}\,dz$$

Now $f(z) = e^{iz}$, $f'(z) = i\,e^{iz}$, $f''(z) = -e^{iz}$, $f'''(z) = -i\,e^{iz}$

$$\therefore\quad f'''(0) = -i\,e^0 = -i \qquad \therefore\quad (-i) = \frac{3!}{2\pi i}\int_C \frac{e^{iz}}{z^4}\,dz$$

$$\therefore\quad \int_C \frac{e^{iz}}{z^4}\,dz = -\frac{2\pi i^2}{3!} = \frac{\pi}{3}.$$

Example 4 : Evaluate $\int_C \frac{z+1}{z^3 - 2z^2} dz$ where C is (a) the circle $|z| = 1$, (b) the circle $|z - 2 - i| = 2$, (c) the circle $|z - 1 - 2i| = 2$. **(M.U. 2002)**

Sol. : (a) $|z| = 1$ is a circle with centre at the origin and radius 1.

Now, $z^3 - 2z^2 = z^2(z-2)$. $\therefore$ $z = 0$ lies **inside** the circle and $z = 2$ lies **outside** it.

Hence, we write $\frac{z+1}{z^3 - 2z^2} = \frac{(z+1)/(z-2)}{z^2}$ and take $f(z) = \frac{z+1}{z-2}$ which is analytic in C.

Then by the corollary of Cauchy's formula,

$$\int_C \frac{f(z)}{(z-z_0)^n} dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0) \quad \text{and} \quad z_0 = 0.$$

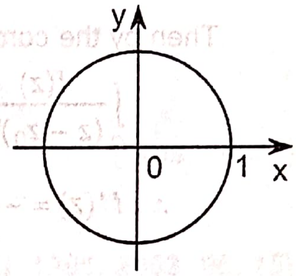


Fig. 2.46

$$\therefore \quad f'(z) = \frac{(z-2)\cdot 1 - (z+1)\cdot 1}{(z-2)^2} = -\frac{3}{(z-2)^2}$$

$$\therefore \quad f'(0) = -\frac{3}{2^2} = -\frac{3}{4}$$

$$\therefore \quad \int_C \frac{z+1}{z^3 - 2z^2} dz = \int_C \frac{(z+1)/(z+2)}{z^2} dz = \frac{2\pi i}{1!}\left(-\frac{3}{4}\right) = -\frac{3\pi}{2}.$$

(b) $|z - 2 - i| = 2$ is a circle with centre at (2, 1) and radius 2.

$\therefore$ The point $z = 0$ *i.e.* (0, 0) lies **outside** the circle and the point $z = 2$ *i.e.* (2, 0) lies **inside** the circle. Hence, we take $f(z) = (z+1)/z^2$ which is analytic in C.

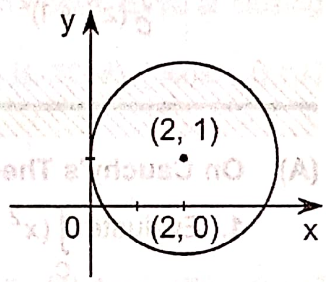


Fig. 2.47

$\therefore$ By Cauchy's formula

$$\int_C \frac{z+1}{z^3 - 2z^2} dz = \int_C \frac{(z+1)/z^2}{z-2} dz$$

$$= \int_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0) \quad \text{where } f(z) = \frac{z+1}{z^2} \text{ and } z_0 = 2$$

$$= 2\pi i \cdot \frac{2+1}{2^2} = 2\pi i \cdot \frac{3}{4} = \frac{3\pi i}{2}.$$

(c) $|z - 1 - 2i| = 2$ is a circle with centre at (1, 2) and radius 2.

$\therefore$ The points $z_0 = 0$ and $z = 2$ both lie **outside** the circle.

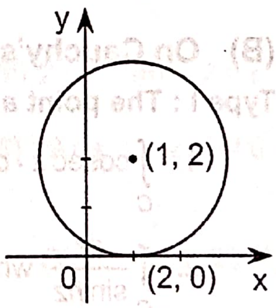


Fig. 2.48

$$\therefore \quad f(z) = \frac{z+1}{z^3 - 2z^2} \text{ is analytic in } C.$$

$\therefore$ By Cauchy's Theorem $\int_C f(z) dz = 0$.

Example 5 : Evaluate $\int_C \frac{1}{(z^3 - 1)^2} dz$ where C is the circle $|z - 1| = 1$. **(M.U. 1998)**

Sol. : The circle $|z - 1| = 1$ has centre at (1, 0) and radius 1.

$$\therefore \quad z^3 - 1 = 0 \qquad \therefore \quad (z-1)(z^2 + z + 1) = 0$$

$$\therefore \quad z = 1 \text{ or } z = \frac{-1 \pm \sqrt{3}\, i}{2}$$

The point $z = 1$ *i.e.* (1, 0) lies **inside** the circle and the points $z = \dfrac{-1 \pm \sqrt{3}\,i}{2}$ *i.e.* $\left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$ and $\left(-\dfrac{1}{2}, \dfrac{-\sqrt{3}}{2}\right)$ lie in the second and third quadrant and hence, **outside** the circle. Further note that the roots are **repeated twice.**

$\therefore$ We take $f(z) = \dfrac{1}{(z^2+z+1)^2}$ which is analytic in C.

Then by the corollary of Cauchy's formula,

$$\int_C \frac{f(z)}{(z-z_0)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0) \text{ and } z_0 = 1.$$

$$\therefore\ f'(z) = -\frac{2}{(z^2+z+1)^3}\cdot(2z+1)$$

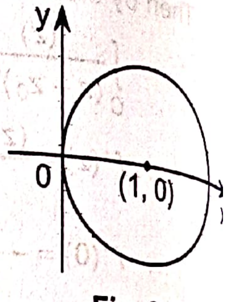


Fig. 2.49

$$\therefore\ f'(z_0) = f'(1) = -\frac{2}{(1+1+1)^3}(2\cdot 1+1) = -\frac{6}{27} = -\frac{2}{9}.$$

$$\therefore\ \int_C \frac{1}{(z^3+1)^2}\,dz = \frac{2\pi i}{1!}\left(-\frac{2}{9}\right) = -\frac{4\pi i}{9}.$$

EXERCISE - II

(A) On Cauchy's Theorem

1. Evaluate $\int_C (x^2 - y^2 + 2i\,xy)\,dz$, where C is the circle $|z| = 2$. [Ans. : $I = 0$

2. $\int_C \cot z \cdot dz$ where C is $\left|z + \dfrac{1}{2}\right| = \dfrac{1}{3}$. (M.U. 2004) [Ans. : $I = 0$

(B) On Cauchy's Formula

Type I : The point *a* lies inside *C*

1. $\int_C \operatorname{cosec} z \cdot dz$ where C is $|z| = 1$. (M.U. 2003) [Ans. : $2\pi i$

2. $\int_C \dfrac{dz}{\sin hz}$ where C is the circle $x^2 + y^2 = 16$. (M.U. 2001, 03) [Ans. : 2π

3. $\int_C \dfrac{\sin 3z}{z + (\pi/2)}\,dz$ where C is the circle $|z| = 5$. (M.U. 2003) [Ans. : 2π

4. $\int_C \dfrac{dz}{z-2}$, where C is the circle (i) $|z-2| = 1$, (ii) $|z-1| = 1/2$. (M.U. 1991, 93)

[Ans. : (i) $2\pi i$, (ii)

5. $\int_C \dfrac{e^{2z}}{z-1}\,dz$, where C is the circle, find (i) $|z| = 2$, (ii) $|z| = 1/2$. (M.U. 1998)

[Ans. : (i) $2\pi i e^2$, (ii)

6. $\int_C \frac{\sin^6 z}{(z-\pi/6)^n} dz$ where C is the circle $|z| = 1$ for $n = 1, n = 3$. **(M.U. 2004, 18)**

[Ans. : (i) $\frac{\pi i}{32}$, **(ii)** $\frac{21\pi i}{16}$. **]**

7. $\int_C \frac{dz}{z^4 e^z}$, where C is the circle $|z| = 1$. **[Ans. :** $-\frac{\pi i}{3}$ **]**

8. $\int_C \frac{ze^{2z}}{(z-1)^3} dz$, where C is $|z+i| = 2$. **(M.U. 1995) [Ans. :** $8\pi i e^2$ **]**

Type II : Both points lie inside C

1. $\int_C \frac{\cos(\pi z^2)}{(z^2 - 3z + 2)} dz$, where C is the circle $|z| = 3$. **[Ans. :** $4\pi i$ **]**

2. $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where C is the circle $|z| = 3$. **(M.U. 1995, 2003, 06, 15)**

[Ans. : $4\pi i$ **]**

3. If $f(z) = z^3 + iz^2 - 4z - 4i$, evaluate $\int_C \frac{f'(z)}{f(z)} dz$ where C is a simple closed curve enclosing zeros of $f(z)$. **(M.U. 2004) [Ans. :** $6\pi i$ **]**

Type III : One point lies inside and the other lies outside C

1. $\int_C \frac{z+6}{z^2-4} dz$, where (i) C is the circle $|z| = 1$, (ii) C is the circle $|z-2| = 1$, (iii) C is the circle $|z+2| = 1$. **(M.U. 1994,2001) [Ans. : (i)** 0, **(ii)** $4\pi i$, **(iii)** $-2\pi i$ **]**

2. $\int_C \frac{z^2+1}{z^2-1} dz$, where the contour C is a circle with radius 1 and centre at (i) $z = i$, (ii) $z = 1$, (iii) $z = -1$. **[Ans. : (i)** 0, **(ii)** $2\pi i$, **(iii)** $-2\pi i$ **]**

3. $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$, where C is $|z-i| = 2$. **(M.U. 1998, 2005) [Ans. :** $-2\pi i/9$ **]**

4. $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{z^2 + 3z + 2} dz$, where C is (i) $|z| = \frac{1}{2}$, (ii) $|z| = \frac{3}{2}$. **(M.U. 1998, 2011)**

[Ans. : (i) 0, **(ii)** $-2\pi i$ **]**

5. $\int_C \frac{z \cos \pi z}{z^2 - z - 2} dz$, where C is $|z-i| = 2$. **(M.U. 1995) [Ans. :** $\frac{-2\pi i}{3}$ **]**

6. $\int_C \frac{(z-1)(z-2)}{(z-3)(z-4)} dz$, where C is (i) $|z| = 3{\cdot}5$, (ii) $|z| = 4{\cdot}5$. **(M.U. 1996, 2006)**

[Ans. : (i) $-4\pi i$, **(ii)** $8\pi i$ **]**

7. $\int_C \frac{z+1}{z^3 - 2z} dz$, where C is the circle $|z| = 1$. **(M.U. 1992, 2002) [Ans. :** $-3\pi i/2$ **]**

8. $\int_C \frac{z^2+4}{(z-2)(z+3i)}dz$, where C is (i) $|z+1|=2$, (ii) $|z-2|=2$.

(M.U. 1991, 94, 95, 2003, 06) [**Ans. : (i)** 0, **(ii)** $16\pi i/(2+3i)$]

9. $\int_C \frac{dz}{z^2+9}$, when (i) the point $3i$ lies inside C and the point $-3i$ lies outside C. (ii) the point $3i$ lies outside C and the point $-3i$ lies inside C. (iii) both the points $3i$ and $-3i$ lie outside C.

(M.U. 1991, 92, 97) [Ans. : (i) $\pi/3$, **(ii)** $-\pi/3$, **(iii)** 0]

10. $\int_C \frac{\cos \pi z}{z^2-1}dz$, where C is the rectangle whose vertices are (i) $2 \pm i$, $-2 \pm i$, (ii) $-i$, $2-i$, $2+i$, i.

(M.U. 1998) [Ans. : (i) 0, **(ii)** $-\pi i$]

11. $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}dz$ where C is $|z|=1$.

(M.U. 2003, 04) [Ans. : $-\frac{16\pi i}{9}$]

12. $\int_C \frac{z\,dz}{(z-1)(z-3)}$, where C is the circle (i) $|z|=3$, (ii) $|z|=1{\cdot}5$. **[Ans. : (i)** $2\pi i$, **(ii)** $-\pi i$]

13. $\int_C \frac{e^z}{(z-1)(z-4)}dz$, where C is the circle $|z|=2$.

[Ans. : $\frac{-2\pi ie}{3}$]

14. $\int_C \frac{dz}{z^3(z+4)}$, where C is the circle $|z|=2$.

(M.U. 2000) [Ans. : $\frac{2\pi i}{27}$]

15. $\int_C \frac{z^2+2z+3}{z^2+3z-4}dz$, where C is the ellipse $4x^2+9y^2=36$.

[Ans. : $\frac{12\pi i}{5}$]

16. $\int_C \frac{z+1}{z^3-2z^2}dz$, where C is (a) the circle $|z|=1$, (b) the circle $|z-2-i|=2$, (c) the circle $|z-1-2i|=2$.

(M.U. 2002) [Ans. : (a) $-\frac{3\pi i}{2}$, **(b)** $\frac{3\pi i}{2}$, **(c)** 0]

Type IV : To find the value of the function $\Phi(\alpha)$

1. If $\Phi(\alpha) = \int_C \frac{ze^z}{z-\alpha}dz$, where C is $|z-2i|=3$, find the values of (i) $\Phi(1)$, (ii) $\Phi'(2)$, (iii) $\Phi(3)$.

(M.U. 1996, 2014) [Ans. : (i) $2\pi i e$, **(ii)** $6\pi i e^2$, **(iii)** 0]

2. If $\Phi(\alpha) = \int_C \frac{4z^2+z+5}{z-\alpha}dz$, where C is the contour of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$, find the values of (i) $\Phi(3{\cdot}5)$, (ii) $\Phi(i)$, (iii) $\Phi'(-1)$, (iv) $\Phi''(-i)$.

[Ans. : (i) 0, **(ii)** $2\pi(i-1)$, **(iii)** $-14\pi i$, **(iv)** $16\pi i$]

3. If $f(\xi) = \int_C \frac{4z^2+z+4}{z-\xi}dz$, where C is the ellipse $4x^2+9y^2=36$, find the values of (i) $f(4)$, (ii) $f(1)$, (iii) $f(i)$, (iv) $f'(-1)$, (v) $f''(-i)$.

(M.U. 1997, 2006)

[Ans. : (i) 0, **(ii)** $18\pi i$, **(iii)** -2π, **(iv)** $-14\pi i$, **(v)** $16\pi i$]

4. If $f(\xi) = \int_C \frac{3z^2 + 7z + 1}{z - \xi} dz$, where C is a circle $|z| = 2$, find the values of (i) $f(-3)$, (ii) $f(i)$, (iii) $f'(1-i)$, (iv) $f''(1-i)$. **(M.U. 1993, 99, 2006, 15)**

[Ans. : (i) 0, **(ii)** $-2\pi(2i+7)$, **(iii)** $2\pi(6+13i)$, **(iv)** $12\pi i$ **]**

5. Verify Cauchy's Theorem for $\oint_C f(z)\,dz = 3z^2$ where C is the circle $|z| = 2$. **(M.U. 2004)**

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12. Taylor's and Laurent's Series : Introduction

Now, we shall study how to expand an analytic function as a power series. We shall also learn in this chapter some important concepts in complex analysis *viz.* zeros, poles and residues and applications of residues to find the integrals in the next chapter.

13. Complex Power Series

A series of the type $(a_1 + ib_1) + (a_2 + ib_2) + \ldots + (a_n + ib_n) + \ldots$ where $a_1, a_2, \ldots a_n, \ldots$ and $b_1, b_2, \ldots b_n, \ldots$ are real numbers is called a series of **complex numbers.** It can be written as

$$\sum_{n=1}^{\infty} (a_n + ib_n)$$

The above series is convergent if both the series Σa_n and Σb_n are convergent.

A series of the type $c_0 + c_1(z-a) + c_2(z-a)^2 + \ldots + c_n(z-a)^n + \ldots$ is called **a power series** in powers of $(z-a)$. It can be written as

$$\sum_{n=1}^{\infty} c_n(z-a)^n$$

where z is a complex number. The constants $c_0, c_1, c_2, \ldots$ are called the **coefficients** and the constant a is called **the centre** of the series.

It can be proved that there exists a real number R such that the power series $\Sigma c_n(z-a)^n$ is convergent for $|z-a| < R$, is divergent for $|z-a| > R$ and may or may not be convergent for $|z-a| = R$. This means that there is a circle with centre a and radius R, such that the power series $\Sigma c_n(z-a)^n$ is convergent at all points inside the circle $|z-a| = R$, is divergent at all points outside the circle and may and may not be convergent at points on the circle. For this reason the circle $|z-a| = R$ is called **the circle of convergence** for the power series $\Sigma C_n(z-a)^n$ and R is called the **radius of convergence.**

Power series are important in complex analysis because every power series is analytic and every analytic function can be represented as power series. Such series are known as **Taylor's Series.**

Analytic functions can also be represented by another type of series containing positive as well as negative powers of $(z-a)$. Such series are called **Laurent's Series**. They are useful for evaluating integrals real and complex.

We shall accept the validity of the expansions of functions as Taylor's Series and Laurent's Series without proof.

Brook Taylor (1685 - 1731)

He was an English mathematician best known for Tayl Theorem and Taylor's Series. Initially he had interest in law and doctorate in law in 1714. But he had also keen interes mathematics. His publication **'Methodus Incrementorum Dir et Inversa'** is considered as the beginning of new branch mathematics called **"Calculus of finite differences"**. The famo Taylor's theorem remained unrecognised untill 1712 when Lagra realised its powers. He was elected to Royal Society in the sa year. From 1715 he took interest in the studies of religion a philosoply.

Pierre Alphonse Laurent (1813 - 1854)

He studied in Ecole polytechnic and got his degree with the highest rank in his class. He was then immediately given the rank of second lieutenant in the engineering corps. After few years in Algeria he returned to France. He spent six years in the project of enlargement of the port of Le Harve. In the midst of these technical operations he submitted a "Memoire Sur le calcul des variations" to the Academy des Sciences. In 1843 he discovered the well-known theorem for the expansion of a function in the form of series now known as **Laurent Series**. Though he had keen interest in Mathematics, by profession he was a military engineer.

To find Radius of Convergence

We accept the following two formulae for finding the radius of convergence of power se $\Sigma\, C_n (z-a)^n$.

(i) D'Alembert's Ratio Formula : $R = \lim\limits_{n\to\infty}\left|\dfrac{C_n}{C_{n+1}}\right|$

(ii) *N*-th Root Formula : $R = \lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{C_n}}$

Example 1 : Find the radius of convergence of each of the following power series.

(i) $\Sigma\dfrac{z^n}{3^n+1}$ (ii) $\Sigma\dfrac{(n!)^2}{(2n)!}z^n$ (iii) $\Sigma\dfrac{n+1}{(n+2)(n+3)}z^n$

(iv) $\Sigma\left(1+\dfrac{1}{n}\right)^{n^2}\cdot z^n$ (v) $\Sigma\, a^n\cdot z^n \ (a>0)$

Sol. : (i) We have $C_n = \dfrac{1}{3^n+1}$ $\qquad\therefore\quad C_{n+1} = \dfrac{1}{3^{n+1}+1}$

$$\therefore\quad \left|\frac{C_n}{C_{n+1}}\right| = \frac{1}{3^n+1}\cdot\frac{3^{n+1}+1}{1} = \frac{3+(1/3^n)}{1+(1/3^n)}$$

$$\lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\frac{3+(1/3^n)}{1+(1/3^n)} = 3$$

$\therefore$ Radius of convergence = 3.

(ii) We have $C_n = \dfrac{(n!)^2}{(2n)!}$. $\therefore$ $C_{n+1} = \dfrac{[(n+1)!]^2}{[2(n+1)!]}$

$$\therefore \left|\frac{C_n}{C_{n+1}}\right| = \frac{n!\cdot n!}{(2n)!}\cdot\frac{(2n+2)!}{(n+1)!(n+1)!} = \frac{n!n!(2n+2)(2n+1)(2n)!}{(2n)!(n+1)n!(n+1)n!}$$

$$= \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$$

$$\therefore \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)} = \lim_{n\to\infty}\frac{[2+(2/n)]\,[2+(1/n)]}{[1+(1/n)]\,[1+(1/n)]}$$

$$= 2\cdot 2 = 4$$

$\therefore$ Radius of convergence = 4.

(iii) Here $C_n = \dfrac{n+1}{(n+2)(n+3)}$. $\therefore$ $C_{n+1} = \dfrac{n+2}{(n+3)(n+4)}$

$$\therefore \left|\frac{C_n}{C_{n+1}}\right| = \frac{n+1}{(n+2)(n+3)}\cdot\frac{(n+3)(n+4)}{(n+2)} = \frac{(n+1)(n+4)}{(n+2)(n+2)}$$

$$\therefore \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)(n+4)}{(n+2)(n+2)}$$

$$= \lim_{n\to\infty}\frac{[1+(1/n)]\,[1+(4/n)]}{[1+(2/n)]\,[1+(2/n)]} = 1$$

$\therefore$ Radius of convergence = 1.

(iv) Here $C_n = \left(1+\dfrac{1}{n}\right)^{n^2}$ $\therefore$ $\sqrt[n]{C_n} = \left[\left(1+\dfrac{1}{n}\right)^{n^2}\right]^{1/n}$

$$\therefore \lim_{n\to\infty}\sqrt[n]{C_n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$\therefore$ Radius of convergence $= \lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{C_n}} = \dfrac{1}{e}$.

(v) Here $C_n = a^n$.

$$\therefore \lim_{n\to\infty}\sqrt[n]{C_n} = \lim_{n\to\infty}\sqrt[n]{a^n}\ \lim_{n\to\infty}(a^n)^{1/n}$$

$$= \lim_{n\to\infty} a = a.$$

$\therefore$ Radius of convergence $= \lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{C_n}} = \dfrac{1}{a}$.

EXERCISE - III

Find the radius of convergence of the following series.

(i) $\Sigma \frac{1}{n^p} z^n$ (ii) $\Sigma \left(1-\frac{1}{n}\right)^{n^2} z^n$ (iii) $\Sigma \frac{n+3}{3^n} z^n$ (iv) $\Sigma \frac{z^n}{2^n+1}$

[**Ans. :** (i) 1, (ii) $1/e$, (iii) $1/3$, (

14. Expansion of A Complex Function $f(z)$ as Taylor's Series

If $f(z)$ is analytic inside a circle C with centre at z_0, then for all z inside C, $f(z)$ c expanded as

$$\boxed{f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z-z_0)^2}{2!} f''(z_0) + \ldots\ldots \infty} \quad \ldots\ldots\ldots$$

The series is convergent at every point inside C and is known as Taylor's Series.

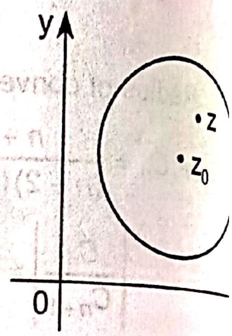


Fig. 2.50

Cor. 1 : If we put $z = z_0 + h$, then

$$f(z_0 + h) = f(z_0) + h f'(z_0) + \frac{h^2}{2!} f''(z_0) + \ldots\ldots \infty$$

Cor. 2 : If we put $z_0 = 0$, then

$$f(z) = f(0) + z f'(0) + \frac{z^2}{2!} f''(0) + \ldots\ldots \infty$$

This is known as **Maclaurin's Series.**

For every point z_0 we can always find a circle C such that Taylor's series expansion of $f($ possible. The largest circle with centre z_0 such that $f(z)$ is analytic inside it is called **the circl convergence.** Its radius is called the **radius of convergence.** Clearly **the radius of converge is the distance between z_0 and the nearest singularity of $f(z)$.**

From (A), we see that Taylor's series consists of positive integral powers of $(z - z_0$

15. Laurent's Series Expansion

If C_1 and C_2 are two concentric circles of radii r_1 and r_2 with centre at z_0 and if $f(z)$ is ana on C_1 and C_2 and in the annular region R between the two circles, then for any point z in R,

$$\boxed{f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}} \quad \ldots\ldots\ldots$$

where $\boxed{a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(\omega)}{(\omega - z_0)^{n+1}} d\omega}$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (1)

and $\boxed{b_n = \frac{1}{2\pi i} \oint_{C} \frac{f(\omega)}{(\omega - z_0)^{-n+1}} d\omega}$ $\ldots\ldots\ldots\ldots\ldots\ldots\ldots$ (2)

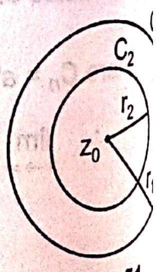


Fig. 2.51

We accept this without proof.

From (B), we see that Laurent's series consists of positive as well as negative integ powers of $(z - z_0)$.

It is clear that if $f(z)$ is analytic inside C_2, then Laurent's series reduces to Taylor's series. In this case the coefficients b_n of negative powers are zero.

The Taylor's expansion and Laurent's expansion are **unique** in the given regions and are usually found by using Binomial Series as illustrated below.

The part $\sum_{n=0}^{\infty} a_n(z-a)^n$ consisting of positive integral powers of $(z-a)$ is called the **analytic part** or **regular part** and the part $\sum_{n=0}^{\infty} b_n(z-a)^{-n}$ consisting of negative integral powers of $(z-a)$ is called **principal part of Laurent's Series.**

Notes ....

1. The Maclaurin's Series of some elementary functions are given below for ready reference.

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\ldots \qquad |z| < \infty$

2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\ldots \qquad |z| < \infty$

3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots\ldots \qquad |z| < \infty$

4. $\sin hz = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\ldots \qquad |z| < \infty$

5. $\cos hz = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots\ldots \qquad |z| < \infty$

6. $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \ldots\ldots \qquad |z| < 1$

2. We often need the following expansions.

$(1+z)^{-1} = 1 - z + z^2 - z^3 + z^4 - \ldots\ldots$ where $|z| < 1$

$(1-z)^{-1} = 1 + z + z^2 + z^3 + z^4 + \ldots\ldots$ where $|z| < 1$

$(1+z)^{-2} = 1 - 2z + 3z^2 - 4z^3 + \ldots\ldots$ where $|z| < 1$

$(1-z)^{-2} = 1 + 2z + 3z^2 + 4z^3 + \ldots\ldots$ where $|z| < 1$

(A) Taylor's Series Expansion

Example 1 : Expand $f(z) = \sin z$ as a Taylor's series around $z = \pi/4$. **(M.U. 1993)**

Sol. : We first note that $f(z) = \sin z$ is analytic everywhere. By Taylor's series,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \ldots\ldots$$

Now $\quad f(z) = \sin z \qquad \therefore f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z \qquad \therefore f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

$$f''(z) = -\sin z \qquad \therefore\ f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -\cos z \qquad \therefore\ f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$$

$$\therefore\ \sin z = \frac{1}{\sqrt{2}}\left[1 + \frac{[z-(\pi/4)]}{1!} - \frac{[z-(\pi/4)]^2}{2!} - \frac{[z-(\pi/4)]^3}{3!} + \ldots\ldots\infty\right]$$

Example 2 : Expand $\cos z$ as Taylor's series at $z = \pi/2$. **(M.U. 1995)**

Sol. : By Taylor's series,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots\ldots$$

Here $f(z) = \cos z$ and $a = \pi/2 \quad \therefore\ f(\pi/2) = \cos(\pi/2) = 0$

$\therefore\ f'(z) = -\sin z \quad \therefore\ f'(\pi/2) = -\sin(\pi/2) = -1$

$f''(z) = -\cos z \quad \therefore\ f''(\pi/2) = -\cos(\pi/2) = 0$

$f'''(z) = \sin z \quad \therefore\ f'''(\pi/2) = \sin(\pi/2) = 1$

$$\therefore\ f(z) = 0 + [z-(\pi/2)](-1) + \frac{[z-(\pi/2)]^3}{3!}(1) + \frac{[z-(\pi/2)]^5}{5!}(-1) + \frac{[z-(\pi/2)]^7}{7!}(1) + \ldots\ldots$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\cdot\frac{[z-(\pi/2)]^{2n+1}}{(2n+1)!}$$

Aliter : Put $z = u + (\pi/2)$

$$\therefore\ \cos z = \cos[u + (\pi/2)] = \cos u\cos(\pi/2) - \sin u\sin(\pi/2) = -\sin u$$

$$= -\left[u - \frac{u^3}{3!} + \frac{u^5}{5!} - \frac{u^7}{7!} + \ldots\ldots\right] = -u + \frac{u^3}{3!} - \frac{u^5}{5!} + \frac{u^7}{7!} + \ldots\ldots$$

$$= -[z-(\pi/2)] + \frac{[z-(\pi/2)]^3}{3!} - \frac{[z-(\pi/2)]^5}{5!} + \ldots\ldots$$

$$= \sum_{n=0}^{\infty}(-1)^{n+1}\frac{[z-(\pi/2)]^{2n+1}}{(2n+1)!}$$

Example 3 : Show that $\dfrac{1}{z^2} = 1 + \sum_{n=1}^{\infty}\dfrac{(n+1)!}{n!}\cdot(z+1)^n$, where $|z+1| < 1$. **(M.U. 1991)**

Sol. : By Taylor's series,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!}f''(a) + \ldots\ldots$$

Here $f(z) = \dfrac{1}{z^2}$ and $a = -1 \quad \therefore\ f(-1) = 1$

$\therefore\ f'(z) = -\dfrac{2}{z^3} \quad \therefore\ f'(-1) = 2!\ ; \qquad f''(z) = \dfrac{2\cdot 3}{z^4} \quad \therefore\ f''(-1) = 3!$

$f'''(z) = -\dfrac{2\cdot 3\cdot 4}{z^5} \quad \therefore\ f'''(-1) = 4!$ and so on.

$$\therefore\ \frac{1}{z^2}=1+(z+1)\cdot 2!+\frac{(z+1)^2}{2!}\cdot 3!+\frac{(z+1)^3}{3!}\cdot 4!+\ldots..$$

$$=1+\sum_{n=1}^{\infty}\frac{(n+1)!}{n!}\cdot(z+1)^n$$

Aliter : $$\frac{1}{z^2}=\frac{1}{[1-(z+1)]^2}=[1-(z+1)]^{-2} \text{ where } |z+1|<1$$

$$=1+2(z+1)+3(z+1)^2+4(z+1)^3+\ldots..$$

$$=1+\sum_{n=1}^{\infty}\frac{(n+1)!}{n!}(z+1)^n$$

Example 4 : Show that for every finite value of z, $e^z=e+e\,\Sigma\dfrac{(z-1)^n}{n!}$. **(M.U. 1993, 94)**

Sol. : By Taylor's series

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\ldots..$$

Here $f(z)=e^z$ and the r.h.s. suggests that $a=1$.

Also $f'(z)=f''(z)=f'''(z)=\ldots..=f^n(z)=e^z$

$$\therefore\ f'(1)=f''(1)=f'''(1)=\ldots..=f^n(1)=e^1$$

$$\therefore\ e^z=e+(z-1)\cdot e+\frac{(z-1)^2}{2!}\cdot e+\frac{(z-1)^3}{3!}\cdot e+\ldots..$$

$$=e+e\,\Sigma\frac{(z-1)^n}{n!}.$$

Example 5 : Obtain Taylor's expansion of $f(z)=\dfrac{1-z}{z^2}$ in powers of $(z-1)$.

Sol. : $f(z)$ is not analytic at $z=0$. However if we consider the region $0<|z-1|<1$, the function is analytic in the region.

Now $$f(z)=\frac{1-z}{z^2}=\frac{1}{z^2}-\frac{1}{z}\quad\therefore\ f(1)=0$$

$$\therefore\ f^n(z)=(-1)^n\cdot\frac{(n+1)!}{z^{n+2}}-(-1)^n\cdot\frac{n!}{z^{n+1}}$$

$$\therefore\ f^n(1)=(-1)^n\cdot(n+1)!-(-1)^n\cdot n!$$

$$=(-1)^n\cdot n!\,[(n+1)-1]=(-1)^n\cdot n!\,n$$

By Taylor's series,

$$f(z)=f(a)+(z-a)f'(a)+\frac{(z-a)^2}{2!}f''(a)+\frac{(z-a)^3}{3!}f'''(a)+\ldots..$$

$$\therefore\ f(z)=\sum_{n=1}^{\infty}(z-1)^n(-1)^n\cdot\frac{n!\,n}{n!}=\sum_{n=1}^{\infty}(-1)^n\cdot n\cdot(z-1)^n$$

Alternatively we can obtain the above expansion by using Binomial theorem as follows.

$$f(z)=-\frac{(z-1)}{(z-1+1)^2}$$

Now put $z - 1 = u$

$$\therefore \quad f(z) = -\frac{u}{(u+1)^2} = -u[1+u]^{-2}$$

$$= -u\left[1 + (-2)u + \frac{(-2)(-3)}{2!}u^2 + \frac{(-2)(-3)(-4)}{3!}u^3 + ...\right]$$

$$= -u\left[1 - 2u + 3u^2 - 4u^3 + ...\right]$$

$$= -u + 2u^2 + 3u^3 - 4u^4 + ...$$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot u^n = \sum_{n=1}^{\infty} (-1)^n \cdot n \cdot (z-1)^n.$$

Example 6 : Obtain Taylor's expansion of $f(z) = \dfrac{z+2}{(z-1)(z-4)}$ at $z = 2$.

Sol. : Let us put $z - 2 = u$.

$$\therefore \quad f(z) = \frac{u+4}{(u+1)(u-2)} = -\frac{1}{u+1} + \frac{2}{u-2}$$

$$= -\frac{1}{(1+u)} - \frac{1}{2[1-(u/2)]} = -(1+u)^{-1} - \frac{1}{2}\left(1 - \frac{u}{2}\right)^{-1}$$

$$= -(1 - u + u^2 - u^3 + ...) - \frac{1}{2}\left(1 + \frac{u}{2} + \frac{u^2}{2^2} + \frac{u^3}{2^3} + ...\right)$$

$$= -\Sigma(-1)^n \cdot u^n - \frac{1}{2}\Sigma\frac{1}{2^n} \cdot u^n = -\Sigma\left[(-1)^n + \frac{1}{2^{n+1}}\right]u^n$$

$$= -\Sigma\left[(-1)^n + \frac{1}{2^{n+1}}\right](z-2)^n$$

The region of convergence is $|z-2| < 1$ and the singularities $z = 1$, $z = 4$ lie outside the regio of convergence.

Example 7 : Expand the function $f(z) = \dfrac{\sin z}{z - \pi}$ about $z = \pi$. **(M.U. 2003, 0**

Sol. : $f(z)$ is not analytic at $z = \pi$. Hence, we put $z - \pi = u$ *i.e.* $z = \pi + u$.

$$\therefore \quad f(z) = \frac{\sin(\pi + u)}{u} = -\frac{\sin u}{u} = -\frac{1}{u}\left[u - \frac{u^3}{3!} + \frac{u^4}{5!} - .....\right]$$

$$= -1 + \frac{u^2}{3!} - \frac{u^4}{5!} + ... = -1 + \frac{(z-\pi)^2}{3!} - \frac{(z-\pi)^4}{5!} + ...$$

Example 8 : Expand e^{3z} about $z = 3i$.

Sol. : $$e^{3z} = e^{3(z-3i)+9i} = e^{9i} \cdot e^{3(z-3i)}$$

$$= e^{9i}\left[1 + \frac{3(z-3i)}{1!} + \frac{3^2(z-3i)^2}{2!} + \frac{3^3(z-3i)^3}{3!} + ...\right]$$

Example 9 : Obtain Taylor's series for $f(z) = \dfrac{2z^3+1}{z(z+1)}$ about $z = i$.

Sol. : By actual division and partial fractions.

$$f(z) = 2z - 2 + \frac{1}{z} + \frac{1}{z+1}$$

Since, we want Taylor's series about $z = i$, we have to express $f(z)$ in positive powers of $(z - i)$.

$$\therefore \quad f(z) = 2(z-i) + (2i-2) + \frac{1}{(z-i)+i} + \frac{1}{(z-i)+(1+i)}$$

$$= 2(z-i) + 2(i-1) + \frac{1}{i\left[1+\dfrac{(z-i)}{i}\right]} + \frac{1}{(1+i)\left[1+\dfrac{(z-i)}{(1+i)}\right]}$$

$$= 2(z-i) + 2(i-1) + \frac{1}{i}\left[1+\frac{(z-i)}{i}\right]^{-1} + \frac{1}{(1+i)}\left[1+\frac{(z-i)}{(1+i)}\right]^{-1}$$

$$= 2(z-i) + 2(i-1) + \frac{1}{i}\left[1-\frac{(z-i)}{i}+\frac{(z-i)^2}{i^2}-\ldots..\right]$$

$$+ \frac{1}{(1+i)}\left[1-\frac{(z-i)}{(1+i)}+\frac{(z-i)^2}{(1+i)^2}-\ldots..\right]$$

$$= 2(z-i) + 2(i-1) + (-1)^n \sum\left(\frac{1}{i^{n+1}} + \frac{1}{(1+i)^{n+1}}\right)(z-i)^n$$

(B) Laurent's Series Expansion

Example 1 : Find the Laurent's series for $f(z) = z^3 e^{1/z}$ about $z = 0$. **(M.U. 2006)**

Sol. : $f(z)$ is not analytic at $z = 0$. Hence, for $|z| > 0$.

$$f(z) = z^3\left[1+\frac{1}{z}+\frac{1}{2!}\cdot\frac{1}{z^2}+\frac{1}{3!}\cdot\frac{1}{z^3}+\ldots..\right]$$

$$= z^3 + z^2 + \frac{1}{2!}\cdot z + \frac{1}{3!} + \frac{1}{4!}\cdot\frac{1}{z} + \frac{1}{5!}\cdot\frac{1}{z^2} + \ldots..$$

Example 2 : Find Laurent's series for $f(z) = \dfrac{e^{3z}}{(z-1)^3}$ about $z = 1$. **(M.U. 2004, 05)**

Sol. : $f(z)$ is not analytic at $z = 1$. Hence, for $|z-1| > 0$.

$$f(z) = \frac{e^{3(z-1)}\cdot e^3}{(z-1)^3} = \frac{e^3}{(z-1)^3}\left[e^{3(z-1)}\right]$$

$$= \frac{e^3}{(z-1)^3}\left[1+3(z-1)+\frac{3^2(z-1)^2}{2!}+\frac{3^3(z-1)^3}{3!}+\frac{3^4(z-1)^4}{4!}+\ldots..\right]$$

$$= e^3\left[\frac{1}{(z-1)^3}+\frac{3}{(z-1)^2}+\frac{3^2}{2!(z-1)}+\frac{3^3}{3!}+\frac{3^4}{4!}\cdot\frac{1}{(z-1)}+\ldots..\right]$$

Example 3 : Find Laurent's series for $f(z) = (z-3)\sin\left(\frac{1}{z+2}\right)$ about $z = -2$. (M.U. 200

Sol. : $f(z)$ is not analytic at $z = -2$. Hence, put $z+2 = u$.

$$\therefore \quad f(z) = (u-5)\sin\frac{1}{u}$$

$$\text{But } \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots..$$

$$\therefore \quad f(z) = (u-5)\left[\frac{1}{u} - \frac{1}{3!}\cdot\frac{1}{u^3} + \frac{1}{5!}\cdot\frac{1}{u^5} - \ldots..\right]$$

$$= \left(1 - \frac{5}{u}\right) - \frac{1}{3!}\cdot\frac{1}{u^2} + \frac{5}{3!}\cdot\frac{1}{u^3} + \ldots..$$

$$= 1 - \frac{5}{(z+2)} - \frac{1}{3!}\cdot\frac{1}{(z+2)^2} + \frac{5}{3!}\cdot\frac{1}{(z+2)^3} + \ldots..$$

Example 4 : Expand the unction $f(z) = \frac{1}{z^2 \sin hz}$ about $z = 0$. (M.U. 20

Sol. : We have

$$f(z) = \frac{1}{z^2 \sin hz} = \frac{1}{z^2\left[z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots..\right]} = \frac{1}{z^3\left[1 + \frac{z^2}{6} + \frac{z^4}{120} + \ldots..\right]}$$

$$= \frac{1}{z^3}\left[1 + \left(\frac{z^2}{6} + \frac{z^4}{120} + \ldots..\right)\right]^{-1}$$

$$= \frac{1}{z^3}\left[1 - \left(\frac{z^2}{6} + \frac{z^4}{120} + \ldots..\right) + \left(\frac{z^2}{6} + \frac{z^4}{120} + \ldots..\right)^2 + \ldots.\right]$$

$$= \frac{1}{z^3}\left[1 - \frac{z^2}{6} - \frac{z^4}{120} + \frac{z^4}{36} + \ldots..\right] = \frac{1}{z^3} - \frac{1}{6z} + \frac{7}{360}z + \ldots...$$

Example 5 : Find all possible Laurent's expansions of the function

$$f(z) = \frac{2 - z^2}{z(1-z)(2-z)}$$

about $z = 0$ indicating the region of convergence in each case. (M.U. 20

Sol. : Let $f(z) = \frac{a}{z} + \frac{b}{1-z} + \frac{c}{2-z}$

$$\therefore \quad 2 - z^2 = a(1-z)(2-z) + bz(2-z) + cz(1-z)$$

When $z = 0$, $\quad 2 = 2a \quad \therefore \; a = 1$

When $z = 1$, $\quad 1 = b \quad \therefore \; b = 1$

When $z = 2$, $\quad -2 = -2c \quad \therefore \; c = 1$

$$\therefore \quad \frac{2 - z^2}{z(1-z)(2-z)} = \frac{1}{z} + \frac{1}{1-z} + \frac{1}{2-z}$$

Clearly $f(z)$ is not analytic at $z = 0$, $z = 1$ and $z = 2$.

Hence, we consider the regions (i) $0 < |z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$.

In these regions $f(z)$ is analytic and can be expanded by Laurent's series.

Hence, we consider the following three cases.

Note ....

We note that the series $\dfrac{1}{1-z} = (1-z)^{-1} = 1 + z + z^2 + \ldots\ldots$ .... (A)

is convergent only if $|z| < 1$. Hence, in $\dfrac{1}{a-z}$, we take a common if $a > z$ and write

$$\frac{1}{a-z} = \frac{1}{a[1-(z/a)]} = \frac{1}{a}\cdot\left(1-\frac{z}{a}\right)^{-1}$$

In this case now $\left|\dfrac{z}{a}\right| < 1$ and the expansion (A) can be used.

In the same way in $\dfrac{1}{a-z}$, we take z common if $z > a$ and write

$$\frac{1}{a-z} = \frac{1}{z[(a/z)-1]} = -\frac{1}{z[1-(a/z)]} = -\frac{1}{z}\left(1-\frac{a}{z}\right)^{-1}$$

In this case now $\left|\dfrac{a}{z}\right| < 1$ and the expansion (A) can be used.

Case (i) : $f(z) = \dfrac{1}{z} + (1-z)^{-1} + \dfrac{1}{2}\left(1-\dfrac{z}{2}\right)^{-1}$

$$= \frac{1}{z} + (1 + z + z^2 + z^3 + \ldots\ldots) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \ldots\ldots\right)$$

$$= \frac{1}{z} + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2^2}\right)z + \left(1 + \frac{1}{2^3}\right)z^2 + \left(1 + \frac{1}{2^4}\right)z^3 + \ldots\ldots$$

$$= \frac{1}{z} + \left(1 + \frac{1}{2^{n+1}}\right)z^n$$

The series is convergent when $|z| < 1$. But this includes the point $z = 0$, which is a singularity of $f(z)$. Hence, the region of convergence of the series is $0 < |z| < 1$.

This is the unit circle without the centre $z = 0$.

Case (ii) : $f(z) = \dfrac{1}{z} - \dfrac{1}{z[1-(1/z)]} + \dfrac{1}{2[1-(z/2)]}$

$$= \frac{1}{z} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\ldots\right) + \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \ldots\right)$$

$$= \frac{1}{z} - \Sigma\frac{1}{z^{n+1}} + \Sigma\frac{z}{2^{n+1}}$$

..U. 1990, 2011)

The series is convergent if $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$ *i.e.* $1 < |z|$ and $|z| < 2$ *i.e.* $1 < |z| < 2$ and this excludes $z = 0$. Hence, the region of convergence of the series is $1 < |z| < 2$.

This is the annular region between two circles of radii $r = 1$ and $r = 2$ with centre at the origin.

Case (iii) : $f(z) = \frac{1}{z} - \frac{1}{z[1-(1/z)]} - \frac{1}{z[1-(2/z)]}$

$$= \frac{1}{z} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1 - \frac{2}{z}\right)^{-1}$$

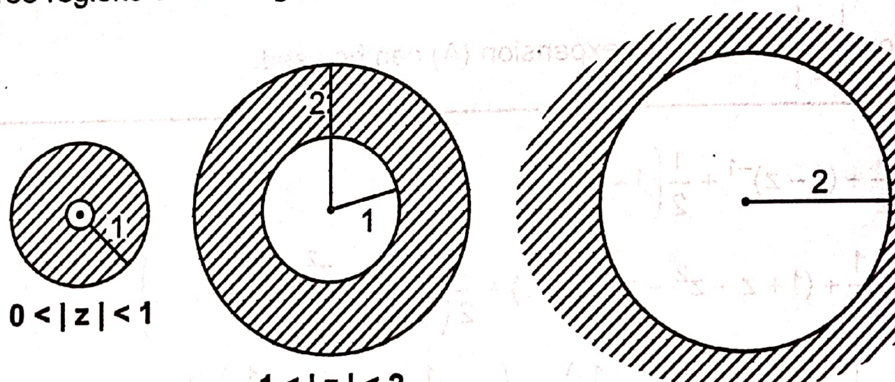
$$= \frac{1}{z} - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right) - \frac{1}{z}\left(1 + \frac{2}{z} + \frac{2^2}{z^2} + \frac{2^3}{z^3} + \ldots\right)$$

$$= \frac{1}{z} - \Sigma\frac{1}{z^{n+1}} - \Sigma\frac{2^n}{z^{n+1}}.$$

The series is convergent if $\left|\frac{1}{z}\right| < 1$ and $\left|\frac{2}{z}\right| < 1$ *i.e.* $1 < |z|$ and $2 < |z|$ *i.e.* if $|z| > 2$. Hence the region of convergence of the series is $2 < |z| < \infty$.

This is the exterior of the circle $|z| = 2$.

The three regions of convergence are shown below.



$0 < |z| < 1$ $\quad$ $1 < |z| < 2$ $\quad$ $2 < |z| < \infty$

Fig. 2.52

Example 6 : Expand $f(z) = \frac{1}{z(z+1)(z-2)}$ (i) within the unit circle about the origin, (ii) within the annulus region between the concentric circles about the origin having radii 1 and 2 respectively (iii) in the exterior of the circle with centre at the origin and radius 2. **(M.U. 1998, 2000, 01, 17)**

S.Sol. : Let $f(z) = \frac{1}{z(z+1)(z-2)} = \frac{a}{z} + \frac{b}{z+1} + \frac{c}{z-2}$

$\therefore \quad 1 = a(z+1)(z-2) + bz(z-2) + cz(z+1)$

When $z = 0$, $\quad 1 = -2a \quad \therefore \ a = -1/2$

When $z = 1$, $\quad 1 = 3b \quad \therefore \ b = 1/3$

When $z = 2$, $\quad 1 = 6c \quad \therefore \ c = 1/6.$

$\therefore \quad \frac{2-z^2}{z(1-z)(2-z)} = \frac{1}{z-1} + \frac{1}{6(z-2)}$

***Case (i)* :** When $0 < |z| < 1$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)} \text{ as}$$

$$f(z) = -\frac{1}{2z} + \frac{1}{3(1+z)} - \frac{1}{12[1-(z/2)]}$$

When $|z| < 1$, clearly $|z| < 2$

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3}(1+z)^{-1} - \frac{1}{12}[1-(z/2)]^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3}[1 - z + z^2 - z^3 + z^4 - .....] - \frac{1}{12}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\right]$$

***Case (ii)* :** When $1 < |z| < 2$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)} \text{ as}$$

$$f(z) = -\frac{1}{2z} + \frac{1}{3z[1+(1/z)]} - \frac{1}{12[1-(z/2)]}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left(1 + \frac{1}{z}\right)^{-1} - \frac{1}{12}\left(1 - \frac{z}{2}\right)^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + ...\right] - \frac{1}{12}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\right]$$

***Case (iii)* :** When $|z| > 2$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{3(z+1)} + \frac{1}{6(z-2)} \text{ as}$$

$$f(z) = -\frac{1}{2z} + \frac{1}{3z[1+(1/z)]} + \frac{1}{6z[1-(2/z)]}$$

When $|z| > 2$, clearly $|z| > 1$.

$$\therefore \quad f(z) = -\frac{1}{2z} + \frac{1}{3z}\left[1 + \left(\frac{1}{z}\right)\right]^{-1} + \frac{1}{6z}\left[1 - \left(\frac{2}{z}\right)\right]^{-1}$$

$$= -\frac{1}{2z} + \frac{1}{3z}\left[1 - \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + ...\right] + \frac{1}{6z}\left[1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ...\right]$$

The regions of convergence are shown on the previous page.

Example 7 : Find the Laurent's series for

$$f(z) = \frac{4z+3}{z(z-3)(z+2)} \text{ valid for } 2 < |z| < 3.$$

(M.U. 1990, 2011)

Sol. : Let $f(z) = \dfrac{a}{z} + \dfrac{b}{z-3} + \dfrac{c}{z+2}$

$$\therefore \quad 4z + 3 = a(z-3)(z+2) + bz(z+2) + cz(z-3)$$

When $z = 0$, $3 = -6a$ $\therefore a = -1/2$

When $z = -2$, $-5 = 10c$ $\therefore c = -1/2$

When $z = 3$, $15 = 15b$ $\therefore b = 1$

$$\therefore \frac{4z+3}{z(z-3)(z+2)} = -\frac{1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)}$$

When $2 < |z| < 3$, we write

$$f(z) = -\frac{1}{2z} + \frac{1}{z-3} - \frac{1}{2(z+2)} \text{ as}$$

$$f(z) = -\frac{1}{2z} - \frac{1}{3[1-(z/3)]} - \frac{1}{2z[1+(2/z)]}$$

$$= -\frac{1}{2z} - \frac{1}{3}\left[1-\left(\frac{z}{3}\right)\right]^{-1} - \frac{1}{2z}\left[1+\left(\frac{2}{z}\right)\right]^{-1}$$

$$= -\frac{1}{2z} - \frac{1}{3}\left[1+\left(\frac{z}{3}\right)+\left(\frac{z}{3}\right)^2+\left(\frac{z}{3}\right)^3+.....\right] - \frac{1}{2z}\left[1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\left(\frac{2}{z}\right)^3+.....\right]$$

Example 8 : Find Laurent's series which represents the function $f(z) = \dfrac{2}{(z-1)(z-2)}$ wh...

(i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$. **(M.U. 1990, 2003, 09, 10,...**

Sol. : Let $\dfrac{2}{(z-1)(z-2)} = \dfrac{a}{z-1} + \dfrac{b}{z-2}$ $\therefore 2 = a(z-2) + b(z-1)$

When $z = 1$, $2 = -a$ $\therefore a = -2$

When $z = 2$, $2 = b$

$$\therefore \frac{2}{(z-1)(z-2)} = \frac{-2}{z-1} + \frac{2}{z-2}$$

Case (i) : When $|z| < 1$, clearly $|z| < 2$

$$\therefore f(z) = \frac{2}{1-z} - \frac{2}{2[1-(z/2)]} = 2[1-z]^{-1} - [1-(z/2)]^{-1}$$

$$\therefore f(z) = 2\left[1+z+z^2+z^3+.....\right] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+.....\right]$$

Case (ii) : When $1 < |z| < 2$, we write

$$\frac{2}{(z-1)(z-2)} = -\frac{2}{(z-1)} + \frac{2}{(z-2)} \text{ as}$$

$$= -\frac{2}{z[1-(1/z)]} - \frac{2}{2[1-(z/2)]}$$

$$= -\frac{2}{z}[1-(1/z)]^{-1} - [1-(z/2)]^{-1}$$

$$= -\frac{2}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+.....\right] - \left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+.....\right]$$

Case (iii) : When $|z| > 2$, $\dfrac{|z|}{2} > 1$ *i.e.* $\dfrac{2}{|z|} < 1$.

Also when $|z| > 2$, $|z| > 1$ $\therefore$ $\dfrac{1}{|z|} < 1$. We write

$$f(z) = -\frac{2}{z}\cdot\frac{1}{[1-(1/z)]} + \frac{2}{z}\cdot\frac{1}{[1-(2/z)]}$$

$$= -\frac{2}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{2}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -\frac{2}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....\right) + \frac{2}{z}\left(\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+.....\right)$$

$$= -2\left(\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+.....\right) + 4\left(\frac{1}{z^2}+\frac{2}{z^3}+\frac{4}{z^4}+.....\right)$$

Example 9 : Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the regions

(i) $1 < |z-1| < 1$, (ii) $1 < |z-3| < 2$, (iii) $|z| < 1$. **(M.U. 2002, 03)**

Sol. : Let $f(z) = \dfrac{a}{z-1} + \dfrac{b}{z-2}$ $\quad\therefore\quad \dfrac{1}{(z-1)(z-2)} = \dfrac{a(z-2)+b(z-1)}{(z-1)(z-2)}$

$\therefore\ 1 = a(z-2) + b(z-1)$

Putting $z = 1$, we get $1 = a(1-2)$ $\quad\therefore\ a = -1$.

Putting $z = 2$, we get $1 = b(2-1)$ $\quad\therefore\ b = 1$.

$$\therefore\ f(z) = -\frac{1}{z-1} + \frac{1}{z-2}$$

Case (i) : When $1 < |z-1| < 2$, we write

$$f(z) = -\frac{1}{z-1} + \frac{1}{(z-1)-1} = -\frac{1}{z-1} - \frac{1}{1-(z-1)} = -\frac{1}{z-1} - [1-(z-1)]^{-1}$$

$$= -\frac{1}{z-1} - \left[1+(z-1)+(z-1)^2+(z-1)^3+.....\right]$$

Case (ii) : When $1 < |z-3| < 2$, we write

$$f(z) = -\frac{1}{(z-3)+2} + \frac{1}{(z-3)+1}$$

$$= -\frac{1}{2}\cdot\frac{1}{1+[(z-3)/2]} + \frac{1}{(z-3)}\cdot\frac{1}{1+[1/(z-3)]}$$

$$= -\frac{1}{2}\left[1+\left(\frac{z-3}{2}\right)\right]^{-1} + \frac{1}{(z-3)}\left[1+\frac{1}{(z-3)}\right]^{-1}$$

$$= -\frac{1}{2}\left[1-\left(\frac{z-3}{2}\right)+\left(\frac{z-3}{2}\right)^2-\left(\frac{z-3}{2}\right)^3+.....\right]$$

$$+\frac{1}{(z-3)}\left[1-\frac{1}{z-3}+\frac{1}{(z-3)^2}-\frac{1}{(z-3)^3}+.....\right]$$

Case (iii) : When $|z| < 1$, clearly $|z| < 2$. Hence, we write

$$f(z) = -\frac{1}{z-1} + \frac{1}{z-2} = \frac{1}{1-z} - \frac{1}{2-z}$$

$$= \frac{1}{1-z} - \frac{1}{2} \cdot \frac{1}{[1-(z/2)]} = [1-z]^{-1} - \frac{1}{2}\left[1 - \frac{z}{2}\right]^{-1}$$

$$= 1 + z + \frac{z^2}{2!} + \ldots\ldots - \frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{2!.4} + \ldots\ldots\right]$$

Example 10 : Obtain Taylor's or Laurent's series for the function

$$f(z) = \frac{1}{(1+z^2)(z+2)} \text{ for (i) } 1 < |z| < 2 \text{ and (ii) } |z| > 2.$$

(M.U. 2003, 1

Sol. : Let $f(z) = \frac{a}{z+2} + \frac{bz+c}{z^2+1}$

$$\therefore \quad \frac{1}{(z^2+1)(z+2)} = \frac{a(z^2+1) + (bz+c)(z+2)}{(z^2+1)(z+2)}$$

$\therefore \quad 1 = (a+b)z^2 + (2b+c)z + (a+2c)$

Equating the coefficients of like powers of z, on both sides

$a + b = 0, \ 2b + c = 0 \ \therefore \ a + 2c = 1.$

$\therefore \ 2c - b = 1$ *i.e.* $4c - 2b = 2$ and $2b + c = 0.$

$\therefore \ 5c = 2$ *i.e.* $c = 2/5, \ \therefore \ a = 1 - 2c = 1 - (4/5) = 1/5$

$\therefore \ b = -a = -1/5$

$$\therefore \quad f(z) = \frac{1}{5} \cdot \frac{1}{z+2} - \frac{z-2}{5(z^2+1)} = \frac{1}{5}\left[\frac{1}{z+2} - \frac{z-2}{z^2+1}\right]$$

Case (i) : When $1 < |z| < 2$, $|z| < 2$ *i.e.* $\frac{|z|}{2} < 1$ and $|z^2| > 1$ *i.e.* $\frac{1}{|z^2|} < 1.$

Hence, we write,

$$f(z) = \frac{1}{5} \cdot \frac{1}{z+2} - \frac{1}{5} \cdot \frac{z-2}{z^2+1}$$

$$= \frac{1}{5} \cdot \frac{1}{2} \cdot \frac{1}{1+(z/2)} - \frac{z-2}{5} \cdot \frac{1}{z^2} \cdot \frac{1}{1+(1/z^2)}$$

$$= \frac{1}{10}\left(1 + \frac{z}{2}\right)^{-1} - \left(\frac{z-2}{5}\right) \cdot \left(1 + \frac{1}{z^2}\right)^{-1}$$

$$= \frac{1}{10}\left[1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \ldots\ldots\right] - \left(\frac{z-2}{5}\right)\left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \ldots\ldots\right]$$

Case (ii) : When $|z| > 2$, $\frac{|z|}{2} > 1$ *i.e.* $\frac{2}{|z|} < 1.$

Also when $|z| > 2$, $|z| > 1$ *i.e.* $|z^2| > 1 \ \therefore \ \frac{1}{|z^2|} < 1.$

Hence, we write,

$$f(z) = \frac{1}{5}\cdot\frac{1}{z}\cdot\frac{1}{1+(2/z)} - \frac{(z-2)}{5}\cdot\frac{1}{z^2}\cdot\frac{1}{1+(1/z^2)}$$

$$= \frac{1}{5z}\left[1+\frac{2}{z}\right]^{-1} - \frac{(z-2)}{5\cdot z^2}\cdot\left[1+\frac{1}{z^2}\right]^{-1}$$

$$= \frac{1}{5z}\left[1-\frac{2}{z}+\frac{4}{z^2}-\frac{8}{z^3}+.....\right] - \frac{(z-2)}{5\cdot z^2}\left[1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+.....\right]$$

Example 11 : Obtain Taylor's and Laurent's expansions of $f(z) = \dfrac{z-1}{z^2-2z-3}$ indicating regions of convergence. **(M.U. 1996, 2004, 10, 14, 16, 18)**

Sol : Let $f(z) = \dfrac{z-1}{(z+1)(z-3)} = \dfrac{a}{z+1} + \dfrac{b}{z-3}$ $\quad\therefore\ z-1 = a\,(z-3) + b\,(z+1)$

Putting $z = -1$, $\quad -2 = -4a \quad \therefore\ a = 1/2$

Putting $z = 3$, $\quad 2 = b\cdot 4 \quad \therefore\ b = 1/2$

$$\therefore\ \frac{z-1}{(z+1)(z-3)} = \frac{1/2}{z+1} + \frac{1/2}{z-3}$$

Hence, $f(z)$ is not analytic at $z = -1$ and $z = 3$.

$\therefore$ $f(z)$ is analytic in (i) $|z| < 1$, (ii) $1 < |z| < 3$, (iii) $|z| > 3$.

Case (i) : When $|z| < 1$, we also get $|z| < 3$.

$$\therefore\ f(z) = \frac{1/2}{z+1} + \frac{1/2}{z-3} = \frac{1}{2}\cdot\frac{1}{1+z} + \frac{1}{2}\cdot\frac{1}{(-3)}\cdot\frac{1}{1-(z/3)}$$

$$= \frac{1}{2}\cdot[1+z]^{-1} - \frac{1}{6}\left(1-\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2}\left[1-z+z^2-z^3+.....\right] - \frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+.....\right]$$

$$= \frac{1}{3} - \frac{5}{9}z + \frac{13}{27}z^2 + ...$$

This is the required Taylor's series.

Case (ii) : When $1 < |z| < 3$, we get $\left|\dfrac{1}{z}\right| < 1$ and $\left|\dfrac{z}{3}\right| < 1$.

$$\therefore\ f(z) = \frac{1}{2}\cdot\frac{1}{1+z} + \frac{1}{2}\cdot\frac{1}{z-3} = \frac{1}{2\cdot z}\cdot\frac{1}{1+(1/z)} + \frac{1}{2}\cdot\frac{1}{(-3)}\cdot\frac{1}{1-(z/3)}$$

$$= \frac{1}{2z}\cdot\left[1+\frac{1}{z}\right]^{-1} - \frac{1}{6}\left[1-\frac{z}{3}\right]^{-1}$$

$$= \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+.....\right] - \frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+.....\right]$$

$$= \frac{1}{2}\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+.....\right] - \frac{1}{6}\left[1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+.....\right]$$

This is the required Laurent's series.

Case (iii) : When $|z| > 3$, clearly, $|z| > 1$.

$$\therefore \quad \frac{|z|}{3} > 1 \text{ and } \frac{|z|}{1} > 1 \quad \therefore \quad \frac{3}{|z|} < 1 \text{ and } \frac{1}{|z|} < 1$$

$$\therefore \quad f(z) = \frac{1}{2} \cdot \frac{1}{z+1} + \frac{1}{2} \cdot \frac{1}{z-3} = \frac{1}{2z} \cdot \frac{1}{1+(1/z)} + \frac{1}{2 \cdot z} \cdot \frac{1}{1-(3/z)}$$

$$= \frac{1}{2z} \cdot \left[1 + \frac{1}{z}\right]^{-1} + \frac{1}{2z} \cdot \left[1 - \frac{3}{z}\right]^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \ldots\ldots\right] + \frac{1}{2z}\left[1 + \frac{3}{z} + \frac{9}{z^2} + \frac{27}{z^3} + \ldots\ldots\right]$$

$$= \frac{1}{2z}\left[2 + \frac{2}{z} + \frac{10}{z^2} + \frac{26}{z^3} + \ldots\ldots\right] = \frac{1}{z} + \frac{1}{z^2} + \frac{5}{z^3} + \frac{13}{z^4} + \ldots\ldots$$

This is the required Laurent's series.

The regions of convergence are as below.

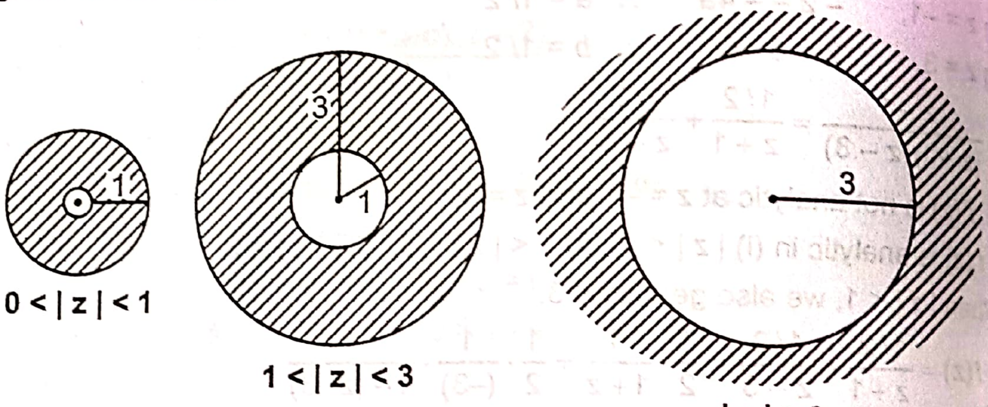


Fig. 2.53

Example 12 : Expand $f(z) = \dfrac{1}{z^2(z-1)(z+2)}$ about $z = 0$ for

(i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$. **(M.U. 1990, 91, 97, 201**

Sol. : Let $f(z) = \dfrac{a}{z} + \dfrac{b}{z^2} + \dfrac{c}{z-1} + \dfrac{d}{z+2}$

$$\therefore \quad 1 = az(z-1)(z+2) + b(z-1)(z+2) + cz^2(z+2) + dz^2(z-1)$$

When $z = 0$, $\quad 1 = -2b \quad \therefore \; b = -1/2$

When $z = 1$, $\quad 1 = 3c \quad \therefore \; c = 1/3$

When $z = -2$, $\quad 1 = -12c \quad \therefore \; d = -1/12$

Equating powers of z^3,

$$0 = a + c + d \quad \therefore \; a = -\frac{1}{3} + \frac{1}{12} = -\frac{3}{12} = -\frac{1}{4}$$

$$\therefore \quad f(z) = -\frac{1}{4} \cdot \frac{1}{z} - \frac{1}{2} \cdot \frac{1}{z^2} + \frac{1}{3} \cdot \frac{1}{z-1} - \frac{1}{12} \cdot \frac{1}{z+2}$$

Case (i) : When $0 < |z| < 1$,

$$\frac{1}{z-1} = -\frac{1}{1-z} = -(1-z)^{-1} = -(1 + z + z^2 + z^3 + \ldots\ldots)$$

$$\frac{1}{z+2}=\frac{1}{2[1+(z/2)]}=\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}=\frac{1}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+.....\right)$$

Hence, from (1), we get,

$$f(z)=-\frac{1}{4}\cdot\frac{1}{z}-\frac{1}{2}\cdot\frac{1}{z^2}-\frac{1}{3}(1+z+z^2+z^3+.....)-\frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+.....\right)$$

Case (ii) : When $1<|z|<2$

$$\frac{1}{z-1}=\frac{1}{z[1-(1/z)]}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....\right)$$

$$\frac{1}{z+2}=\frac{1}{2[1+(z/2)]}=\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}=\frac{1}{2}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+.....\right)$$

Hence, from (1), we get

$$f(z)=-\frac{1}{4}\cdot\frac{1}{z}-\frac{1}{2}\cdot\frac{1}{z^2}+\frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....\right)-\frac{1}{24}\left(1-\frac{z}{2}+\frac{z^2}{2^2}-\frac{z^3}{2^3}+.....\right)$$

Case (iii) : When $|z|>2$

When $|z|>2$ clearly $|z|>1$, and we get

$$\frac{1}{z-1}=\frac{1}{z[1-(1/z)]}=\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}=\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....\right)$$

And $$\frac{1}{z+2}=\frac{1}{z[1+(2/z)]}=\frac{1}{z}\left(1+\frac{2}{z}\right)^{-1}=\frac{1}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+.....\right)$$

Hence, from (1), we get

$$f(z)=-\frac{1}{4}\cdot\frac{1}{z}-\frac{1}{2}\cdot\frac{1}{z^2}+\frac{1}{3z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....\right)-\frac{1}{12z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+.....\right)$$

(See figures given on page 2-54.)

Example 13 : Expand $\dfrac{z^2-1}{z^2+5z+6}$ around $z=0$. **(M.U. 1991, 99, 2001)**

Sol. : Since the degree of the numerator is equal to the degree of the denominator we **first divide the numerator by the denominator.**

$$\therefore\quad f(z)=\frac{z^2-1}{z^2+5z+6}=1-\frac{5z+7}{z^2+5z+6}$$

Let $\dfrac{-5z-7}{z^2+5z+6}=\dfrac{a}{z+3}+\dfrac{b}{z+2}$ $\quad\therefore\ -5z-7=a(z+2)+b(z+3)$

When $z=-2$, $\quad b=3$; $\quad$ When $z=-3$, $\quad a=-8$.

$$\therefore\quad f(z)=\frac{z^2-1}{z^2+5z+6}=1-\frac{8}{z+3}+\frac{3}{z+2}\qquad\text{.................... (1)}$$

Case (i) : When $|z|<2$, we write

$$f(z)=1-\frac{8}{3[1+(z/3)]}+\frac{3}{2[1+(z/2)]}$$

When $|z| < 2$, clearly $|z| < 3$

$$\therefore \quad f(z) = 1 - \frac{8}{3}[1 + (z/3)]^{-1} + \frac{3}{2}[1 + (z/2)]^{-1}$$

$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \ldots\right] + \frac{3}{2}\left[1 - \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \ldots\right]$$

Case (ii) : When $2 < |z| < 3$, we write

$$f(z) = 1 - \frac{8}{3[1 + (z/3)]} + \frac{3}{z[1 + (2/z)]}$$

$$= 1 - \frac{8}{3}\left(1 + \frac{z}{3}\right)^{-1} + \frac{3}{z}\left(1 + \frac{2}{z}\right)^{-1}$$

$$= 1 - \frac{8}{3}\left[1 - \left(\frac{z}{3}\right) + \left(\frac{z}{3}\right)^2 - \ldots\right] + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \ldots\right]$$

Case (iii) : When $|z| > 3$, we write

$$f(z) = 1 - \frac{8}{z[1 + (3/z)]} + \frac{3}{z[1 + (2/z)]}$$

When $|z| > 3$, clearly $|z| > 2$

$$\therefore \quad f(z) = 1 - \frac{8}{z}\left(1 + \frac{3}{z}\right)^{-1} + \frac{3}{2}\left(1 + \frac{2}{z}\right)^{-1}$$

$$= 1 - \frac{8}{z}\left[1 - \left(\frac{3}{z}\right) + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \ldots\right] + \frac{3}{z}\left[1 - \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \ldots\right]$$

Example 14 : Expand $f(z) = \dfrac{z^2 - 1}{z^2 + 5z + 6}$ around $z = 1$. **(M.U. 1993, 2016)**

Sol. : As in the previous example dividing numerator by denominator

$$f(z) = 1 - \frac{8}{z + 3} + \frac{3}{z + 2}$$

Since, we want expansion around $z = 1$, we have to obtain Laurent's series in powers of $(z-1)$.

$$\therefore \quad \frac{z^2 - 1}{z^2 + 5z + 6} = 1 - \frac{8}{(z - 1) + 4} + \frac{3}{(z - 1) + 3}$$

There now arise three cases.

Case (i) : When $|z - 1| < 3$, we write

$$f(z) = 1 - \frac{8}{(z - 1) + 4} + \frac{3}{(z - 1) + 3} \quad \text{as}$$

$$\therefore \quad f(z) = 1 - \frac{8}{4[1 + (z - 1)/4]} + \frac{3}{3[1 + (z - 1)/3]}$$

(When $|z - 1| < 3$ clearly $|z - 1| < 4$.)

$$\therefore \quad f(z) = 1 - \frac{8}{4}\left[1 + \left(\frac{z - 1}{4}\right)\right]^{-1} + \frac{3}{3}\left[1 + \left(\frac{z - 1}{3}\right)\right]^{-1}$$

$$\therefore \quad f(z) = 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \ldots\ldots\right]$$
$$+ \left[1 - \left(\frac{z-1}{3}\right) + \left(\frac{z-1}{3}\right)^2 - \left(\frac{z-1}{3}\right)^3 + \ldots\ldots\right]$$

Case (ii) : When $3 < |z-1| < 4$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \quad \text{as}$$
$$= 1 - \frac{8}{4[1+(z-1)/4]} + \frac{3}{(z-1)[1+3/(z-1)]}$$
$$= 1 - 2\left[1 + \left(\frac{z-1}{4}\right)\right]^{-1} + \frac{3}{(z-1)}\left[1 + \left(\frac{3}{z-1}\right)\right]^{-1}$$
$$= 1 - 2\left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \ldots\ldots\right]$$
$$+ \frac{3}{(z-1)}\left[1 - \left(\frac{3}{z-1}\right) + \left(\frac{3}{z-1}\right)^2 - \left(\frac{3}{z-1}\right)^3 + \ldots\ldots\right]$$

Case (iii) : When $|z-1| > 4$, we write

$$f(z) = 1 - \frac{8}{(z-1)+4} + \frac{3}{(z-1)+3} \quad \text{as}$$
$$= 1 - \frac{8}{(z-1)[1+4/(z-1)]} + \frac{3}{(z-1)[1+3/(z-1)]}$$

When $|z-1| > 4$, clearly $|z-1| > 3$

$$\therefore \quad f(z) = 1 - \frac{8}{z-1}\left[1 + \left(\frac{4}{z-1}\right)\right]^{-1} + \frac{3}{z-1}\left[1 + \left(\frac{3}{z-1}\right)\right]^{-1}$$
$$= 1 - \frac{1}{z-1}\left[1 - \left(\frac{4}{z-1}\right) + \left(\frac{4}{z-1}\right)^2 - \ldots\ldots\right] + \frac{3}{z-1}\left[1 - \left(\frac{3}{z-1}\right) + \left(\frac{3}{z-1}\right)^2 + \ldots\ldots\right]$$

The regions of convergence of the three series are shown below.

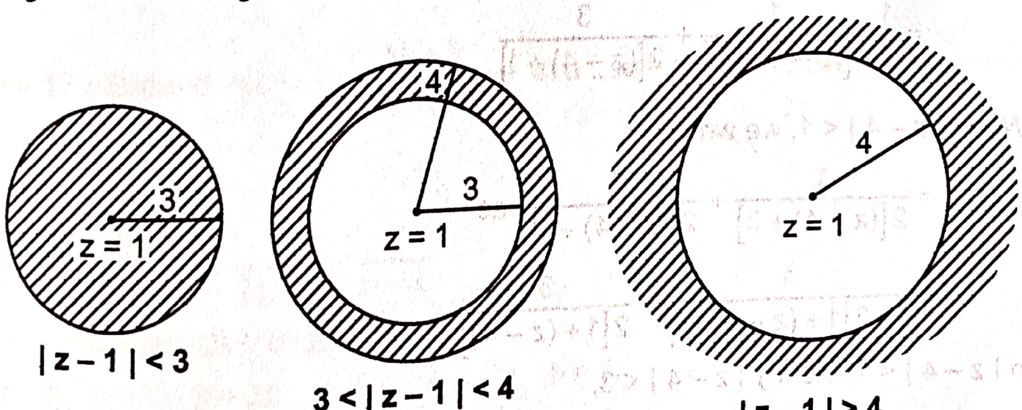


Fig. 2.54

Example 15 : Obtain two distinct Laurent's series expansions of

$$f(z) = \frac{1}{z^2(2-z)}.$$

(M.U. 1997, 200

Sol. : $f(z)$ is not analytic at $z = 0$ and $z = 2$. However, if we consider the regions $0 < |z| < 2$ a $|z| > 2$, the function is analytic in the regions.

Case (i) : When $0 < |z| < 2$. We write

$$f(z) = \frac{1}{z^2(2-z)} = \frac{1}{2z^2[1-(z/2)]} = \frac{1}{2z^2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{2z^2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+.....\right) = \frac{1}{2z^2}+\frac{1}{4z}+\frac{1}{8}+\frac{z}{16}+.....$$

Case (ii) : When $|z| < 2$. We write

$$f(z) = -\frac{1}{z^2(z-2)} = -\frac{1}{z^3[1-(2/z)]}$$

$$= -\frac{1}{z^3}\left(1-\frac{2}{z}\right)^{-1} = -\frac{1}{z^3}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+.......\right)$$

$$= -\frac{1}{z^3}-\frac{2}{z^4}-\frac{4}{z^5}-\frac{8}{z^6}-.....$$

These are the two required expansions of $f(z)$.

Example 16 : Obtain Taylor's and Laurent's series for $f(z) = \dfrac{2z-3}{z^2-4z-3}$ in powers of (z- indicating the regions of convergence.

(M.U. 1996, 2004, 13,

Sol. : Let $f(z) = \dfrac{2z-3}{z^2-4z-3} = \dfrac{a}{(z-1)} + \dfrac{b}{(z-3)}$

$\therefore\ 2z-3 = a(z-3) + b(z-1)$

When $z = 1$, $\quad -1 = a(-2) \qquad \therefore\ a = 1/2$

When $z = 3$, $\quad 3 = 2b \qquad \therefore\ b = 3/2$

$$\therefore\ f(z) = \frac{1}{2}\cdot\frac{1}{(z-1)} + \frac{3}{2}\cdot\frac{1}{(z-3)}$$

$$= \frac{1}{2}\cdot\frac{1}{[(z-4)+3]} + \frac{3}{2[(z-4)+1]}$$

Case (i) : When $|z-4| < 1$, we write

$$f(z) = \frac{1}{2[(z-4)+3]} + \frac{3}{2[(z-4)+1]} \text{ as}$$

$$= \frac{1}{2\cdot 3[1+(z-4)/3]} + \frac{3}{2[1+(z-4)]}$$

When $|z-4| < 1$, clearly $|z-4| < 3$.

$$\therefore\ f(z) = \frac{1}{6}\left[1+\left(\frac{z-4}{3}\right)\right]^{-1} + \frac{3}{2}[1+(z-4)]^{-1}$$

$$\therefore \quad f(z) = \frac{1}{6}\left[1-\left(\frac{z-4}{3}\right)+\left(\frac{z-4}{3}\right)^2-\left(\frac{z-4}{3}\right)^3+.....\right]$$

$$+\frac{3}{2}\left[1-(z-4)+(z-4)^2-(z-4)^3+.....\right]$$

This is the required Taylor's series Expansion.

Case (ii) : When $|z-4|>3$, we write

(When $|z-4|>3$, clearly $|z-4|>1$)

$$f(z) = \frac{1}{2[(z-4)+3]}+\frac{3}{2[(z-4)+1]} \quad \text{as}$$

$$= \frac{1}{2(z-4)[1+3/(z-4)]}+\frac{3}{2(z-4)[1+1/(z-4)]}$$

$$= \frac{1}{2(z-4)}\left[1+\left(\frac{3}{z-4}\right)\right]^{-1}+\frac{3}{2(z-4)}\left[1+\left(\frac{1}{z-4}\right)\right]^{-1}$$

$$= \frac{1}{2(z-4)}\left[1-\left(\frac{3}{z-4}\right)+\left(\frac{3}{z-4}\right)^2-.....\right]$$

$$+\frac{3}{2(z-4)}\left[1-\left(\frac{1}{z-4}\right)+\left(\frac{1}{z-4}\right)^2-.....\right]$$

This is the required Laurent's series expansion.

The regions of convergence are shown below.

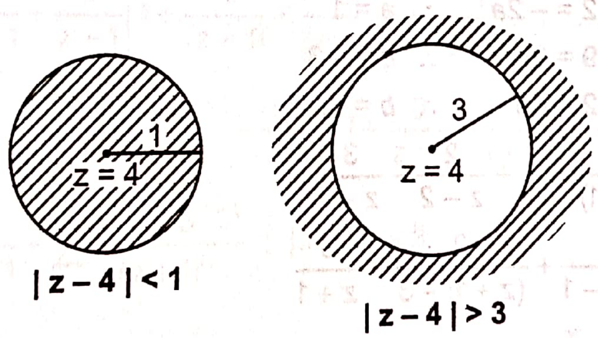


Fig. 2.55

Example 17 : Expand $f(z)=\dfrac{3z-3}{(2z-1)(z-2)}$ in a Laurent's series about $z=1$ convergent in $\frac{1}{2}<|z-1|<1$. **(M.U. 2004)**

Sol. : Let $\dfrac{3z-3}{(2z-1)(z-2)}=\dfrac{a}{2z-1}+\dfrac{b}{z-2}$

$\therefore \quad 3z-3 = a(z-2)+b(2z-1)$

When $z=2$, $\quad 3=3b \quad \therefore\ b=1.$

When $z=1/2$, $\quad -3/2=-(3/2)\cdot a \quad \therefore\ a=1.$

$$\therefore \quad f(z)=\frac{1}{2z-1}+\frac{1}{z-2}$$

When $\dfrac{1}{2} < |z-1| < 1$, we get

$$f(z) = \frac{1}{2(z-1)+1} + \frac{1}{(z-1)-1}$$

$$= \frac{1}{2(z-1)\left[1+\dfrac{1}{2(z-1)}\right]} - \frac{1}{1-(z-1)}$$

$$= \frac{1}{2(z-1)}\left[1+\frac{1}{2(z-1)}\right]^{-1} - [1-(z-1)]^{-1}$$

[Since $\dfrac{1}{2} < |z-1| < 1 \quad \therefore\ 1 < 2|z-1| < 2 \quad \therefore\ 1 > \dfrac{1}{2|z-1|} > \dfrac{1}{2} \quad \therefore\ \dfrac{1}{2|z-1|} < 1.$]

$$\therefore\ f(z) = \frac{1}{2(z-1)}\left[1 - \frac{1}{2(z-1)} + \frac{1}{4(z-1)^2} - \frac{1}{8(z-1)^3} + \ldots\right]$$

$$-\left[1+(z-1)+(z-1)^2+(z-1)^3+\ldots\right]$$

Example 18 : Find all possible Laurent's expansions of the function

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} \text{ about } z = -1.$$

(M.U. 1999, 2003, 09, 14, 16)

Sol. : Let $\dfrac{7z-2}{z(z-2)(z+1)} = \dfrac{a}{z} + \dfrac{b}{z-2} + \dfrac{c}{z+1}$

$\therefore\ 7z-2 = a(z-2)(z+1) + bz(z+1) + cz(z-2)$

When $z = 0$, $\quad -2 = -2a \quad \therefore\ a = 1$

When $z = -1$, $\quad -9 = 3c \quad \therefore\ c = -3$

When $z = 2$, $\quad 12 = 6b \quad \therefore\ b = 2.$

$$\therefore\ \frac{7z-2}{z(z-2)(z+1)} = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$\therefore\ f(z) = \frac{1}{(z+1)-1} + \frac{2}{(z+1)-3} - \frac{3}{z+1}$$

Case (i) : When $|z+1| < 1$, we write

$$f(z) = -\frac{3}{z+1} - \frac{1}{1-(z+1)} - \frac{2}{3-(z+1)}$$

When $|z+1| < 1$, clearly $|z+1| < 3$.

$$\therefore\ f(z) = -\frac{3}{z+1} - [1-(z+1)]^{-1} - \frac{2}{3}\left[1-\left(\frac{z+1}{3}\right)\right]^{-1}$$

$$= -\frac{3}{z+1} - \left[1+(z+1)+(z+1)^2+(z+1)^3+\ldots\right]$$

$$-\frac{2}{3}\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\frac{(z+1)^3}{3^3}+\ldots\right]$$

$$= -\frac{3}{z+1} - \sum\left(1+\frac{2}{3^{n+1}}\right)(z+1)^n, \quad 0 < |z+1| < 1$$

***Case (ii)* :** When $1 < |z+1| < 3$

$$f(z) = -\frac{3}{z+1} + \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} - \frac{2}{3\left[1-\frac{(z+1)}{3}\right]}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} - \frac{2}{3}\left(1-\frac{z+1}{3}\right)^{-1}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left[1+\frac{1}{(z+1)^1}+\frac{1}{(z+1)^2}+\ldots..\right]$$

$$-\frac{2}{3}\left[1+\frac{(z+1)}{3}+\frac{(z+1)^2}{3^2}+\frac{(z+1)^3}{3^3}+\ldots..\right]$$

$$= -\frac{3}{z+1} + \sum\frac{1}{(z+1)^{n+1}} - 2\sum\frac{(z+1)^n}{3^{n+1}}\ ;\ 1<|z+1|<3$$

***Case (iii)* :** When $|z+1| > 3$, we write

$$f(z) = -\frac{3}{z+1} + \frac{1}{(z+1)\left(1-\frac{1}{z+1}\right)} + \frac{2}{(z+1)\left(1-\frac{3}{z+1}\right)}$$

When $|z+1| > 3$, $|z+1| > 1$.

$$\therefore\ f(z) = -\frac{3}{z+1} + \frac{1}{z+1}\left(1-\frac{1}{z+1}\right)^{-1} + \frac{2}{z+1}\left(1-\frac{3}{z+1}\right)^{-1}$$

$$= -\frac{3}{z+1} + \frac{1}{z+1}\left[1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\frac{1}{(z+1)^3}+\ldots..\right]$$

$$+\frac{2}{z+1}\left[1+\frac{3}{z+1}+\frac{3^2}{(z+1)^2}+\frac{3^3}{(z+1)^3}+\ldots..\right]$$

$$= -\frac{3}{z+1} + \sum\frac{1+2\cdot 3^{n-1}}{(z+1)^n}\ ;\ |z+1|>3.$$

Example 19 : Find all possible Laurent's expansions of $\frac{z}{(z-1)(z-2)}$ about $z = -2$.

(M.U. 2016)

Sol. : Let $\frac{z}{(z-1)(z-2)} = \frac{a}{z-1} + \frac{b}{z-2}$ $\quad\therefore\ z = a(z-2) + b(z-1)$

Putting $z = 1$, $\quad 1 = -a \quad \therefore\ a = -1$

Putting $z = 2$, $\quad 2 = b \quad \therefore\ b = 2$.

$$\therefore\ f(z) = -\frac{1}{z-1} + \frac{2}{z-2} = -\frac{1}{(z+2)-3} + \frac{2}{(z+2)-4}$$

***Case (i)* :** When $[z-(-2)] < 3$, $|z+2| < 3$.

When $|z+2| < 3$, clearly $|z+2| < 4$.

We write,

$$\therefore \quad f(z) = -\frac{1}{(z+2)-3} + \frac{2}{(z+2)-4}$$

$$= \frac{1}{3} \cdot \frac{1}{\left[1 - \frac{z+2}{3}\right]} - \frac{2}{4\left[1 - \frac{z+2}{4}\right]} = \frac{1}{3}\left[1 - \frac{z+2}{3}\right]^{-1} - \frac{1}{2}\left[1 - \frac{z+2}{4}\right]^{-1}$$

$$= \frac{1}{3}\left[1 + \left(\frac{z+2}{3}\right) + \left(\frac{z+2}{3}\right)^2 + \ldots\right] - \frac{1}{2}\left[1 + \left(\frac{z+2}{4}\right) + \left(\frac{z+2}{4}\right)^2 + \ldots\right]$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{3^{n+1}} - \frac{1}{2 \cdot 4^n}\right)(z+2)^n$$

***Case (ii)* : When** $3 < |z+2| < 4$, we write,

$$f(z) = -\frac{1}{(z+2)-3} + \frac{2}{(z+2)-4}$$

$$= -\frac{1}{(z+2)\left[1 - \frac{3}{z+2}\right]} - \frac{2}{4\left[1 - \frac{z+2}{4}\right]}$$

$$= -\frac{1}{(z+2)}\left[1 - \frac{3}{z+2}\right]^{-1} - \frac{1}{2}\left[1 - \frac{z+2}{4}\right]^{-1}$$

$$= -\frac{1}{(z+2)}\left[1 + \left(\frac{3}{z+2}\right) + \left(\frac{3}{z+2}\right)^2 + \ldots\right] - \frac{1}{2}\left[1 + \left(\frac{z+2}{4}\right) + \left(\frac{z+2}{4}\right)^2 + \ldots\right]$$

$$= -\sum_{n=0}^{\infty} \frac{3^n}{(z+2)^{n+1}} - \frac{1}{2}\sum\left(\frac{z+2}{4}\right)^n$$

***Case (iii)* :** When $|z+2| > 4$, clearly $|z+2| > 3$, we write,

$$f(z) = -\frac{1}{(z+2)\left[1 - \frac{3}{z+2}\right]} + \frac{2}{(z+2)\left[1 - \frac{4}{z+2}\right]}$$

$$= -\frac{1}{(z+2)}\left(1 - \frac{3}{z+2}\right)^{-1} + \frac{2}{(z+2)}\left(1 - \frac{4}{z+2}\right)^{-1}$$

$$= -\frac{1}{(z+2)}\left[1 + \left(\frac{3}{z+2}\right) + \left(\frac{3}{z+2}\right)^2 + \ldots\right] + \frac{2}{(z+2)}\left[1 + \left(\frac{4}{z+2}\right) + \left(\frac{4}{z+2}\right)^2 + \ldots\right]$$

$$= \sum(2 \cdot 4^n - 3^n) \cdot \frac{1}{(z+2)^{n+1}}$$

EXERCISE - IV

1. Expand the following in Taylor's series

   (i) cos z about $z = \pi/4$. **(M.U. 2003)**

   (ii) cos z about $z = 0$.

   (iii) sin z about $z = 0$.

   (iv) e^z about $z = 0$.

   (v) cos z about $z = \pi/2$.

   (vi) e^{2z} about $z = 2i$.

(vii) $\cos z$ about $z = \pi/3$. **(M.U. 2003)** (viii) e^z about $z = -i$.

(ix) e^z about $z = 1$. (x) $\log\left(\dfrac{1+z}{1-z}\right)$ about $z = 0$.

[**Ans. :** (i) $\dfrac{1}{\sqrt{2}}\left[1-\left(z-\dfrac{\pi}{4}\right)+\dfrac{1}{2!}\left(z-\dfrac{\pi}{4}\right)-.....\right]$ (ii) $1-\dfrac{z^2}{2!}+\dfrac{z^4}{4!}-.....$

(iii) $z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-.....$ (iv) $1+z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\dfrac{z^4}{4!}+.....$

(v) $\dfrac{[z+(\pi/2)]}{1!}-\dfrac{[z+(\pi/2)]^3}{3!}+\dfrac{[z+(\pi/2)]^5}{5!}-\dfrac{[z+(\pi/2)]^7}{7!}+.....$

(vi) $e^{4i}\left[1+\dfrac{2(z-2i)}{1!}-\dfrac{2^2(z-2i)^2}{2!}+\dfrac{2^3(z-2i)^3}{3!}+.....\right]$

(vii) $\dfrac{1}{2}\left[1-\sqrt{3}\left(z-\dfrac{\pi}{3}\right)-\dfrac{1}{2!}\left(z-\dfrac{\pi}{3}\right)^2+\dfrac{\sqrt{3}}{3!}\left(z-\dfrac{\pi}{3}\right)^3+.....\right]$

(viii) $e^{-i}\left[1+\dfrac{(z+i)}{1}+\dfrac{(z+i)^2}{2!}+.....\right]$ (ix) $e^{-1}\left[1-\dfrac{(z-1)}{1!}+\dfrac{(z-1)^2}{2!}-\dfrac{(z-1)^3}{3!}+.....\right]$

(x) $2\left(z+\dfrac{z^3}{3}+\dfrac{z^5}{5}+.....\right)$; $|z|<1$]

2. Find the radius of convergence in each of the following cases if the given function is expanded as a Taylor's series about the indicated point without actually expanding.

(i) $\dfrac{z+1}{z-1}$ about $z = 0$. (ii) $\dfrac{z}{z^2+9}$ about $z = 0$.

(iii) $\dfrac{z+2}{(z-3)(z-4)}$ about $z = 2$. (iv) $\sec \pi z$ about $z = 1$.

(**Hint :** The radius of convergence is the distance between the centre of Taylor's series and the nearest singularity of the function.)

[**Ans. :** (i) $|z|<1$, (ii) $|z|<3$, (iii) $|z-2|<1$, (iv) $|z-1|<1/2$]

3. Find the Taylor's expansion of $f(z)=\dfrac{1}{z(1-z)}$ about $z = -1$, indicating the region of convergence.

[**Ans. :** $f(z)=\sum\left(\dfrac{1}{2^{n+1}}-1\right)(z+1)^n$, $|z+1|<1$]

4. Find the Taylor's expansion of $f(z)=\dfrac{z}{(z+1)(z+2)}$ about $z = i$, indicating the region of convergence.

[**Ans. :** $f(z)=\sum(-1)^n\left\{\dfrac{2}{(2+i)^{n+1}}-\dfrac{1}{(1+i)^{n+1}}\right\}(z-i)^n$; $|z-i|<1$]

5. Obtain Taylor's expansion of $f(z)=\dfrac{z-1}{z+1}$ indicating the region of convergence.

(M.U. 1998) [**Ans. :** $f(z)=1-2(1+z+z^2+z^3+.....)$; $|z|<1$]

6. Find Taylor's expansion of $f(z)=\dfrac{1}{z^2+4}$ about $z=-i$. (M.U. 200

[Ans. : $\dfrac{1}{3}+\dfrac{5}{18}(z+i)\cdot i+\dfrac{7}{27}(z+i)^2-\dfrac{20}{81}(z+i)^3 i+....$

7. Find the Taylor's series expansion of $f(z)=\dfrac{1}{(z-1)(z-3)}$ about the point $z=4$. Find region of convergence. (M.U. 2000,

[Ans. : $\dfrac{1}{2}\left[1-(z-4)+(z-4)^2-(z-4)^3+.....\right]-\dfrac{1}{6}\left[1-\left(\dfrac{z-4}{3}\right)+\left(\dfrac{z-4}{3}\right)^2-...\right.$

8. Obtain Laurent's series for $f(z)=\dfrac{1}{z(z+2)(z+1)}$ about $z=-2$. (M.U. 20

[Ans. : $\dfrac{1}{2u}-\dfrac{1}{4}\left(1+\dfrac{u}{2}+\dfrac{u^2}{4}+.....\right)+(1+u+u^2+u^3+.....)$ where, $u=z+2$

9. Obtain the expansion of $f(z)=\dfrac{z+1}{(z-3)(z-4)}$ about $z=2$. (M.U. 20

[Ans. : $4\left[1+u+u^2+u^3+.....\right]-\dfrac{5}{2}\left[1+\dfrac{u}{2}+\dfrac{u^2}{4}+\dfrac{u^3}{8}+.....\right]$ where $u=z-2$

10. Obtain Laurent's series expansion of $f(z)=\dfrac{1}{z^2+4z+3}$ when (i) $1<|z|<3$, (ii) $|z|>$ (M.U. 20

[Ans. : (i) $\dfrac{1}{2}\left[\dfrac{1}{z}-\dfrac{1}{z^2}+\dfrac{1}{z^3}-.....\right]-\dfrac{1}{3}\left[1-\dfrac{z}{3}+\dfrac{z^2}{9}-\dfrac{z^3}{27}+.....\right]$

(ii) $\dfrac{1}{z^2}-\dfrac{4}{z^3}+\dfrac{13}{z^4}-....$]

11. Expand $f(z)=\dfrac{1}{z^3-3z^2+2z}$ as Laurent's series about $z=0$ for

(i) $|z|<1$, (ii) $1<|z|<2$, (iii) $|z|>2$. (M.U. 1997, 98, 20

[Ans. : (i) $\dfrac{1}{2z}+\dfrac{3}{4}+\dfrac{7}{8}z+\dfrac{15}{16}z^2+.....$

(ii) $-\dfrac{1}{4}-\dfrac{1}{8}\cdot z-\dfrac{1}{16}\cdot z^2-.....-\dfrac{1}{z^2}-\dfrac{1}{z^3}-\dfrac{1}{z^4}-.....$

(iii) $\dfrac{1}{z^3}+\dfrac{3}{z^4}+\dfrac{7}{z^5}+\dfrac{15}{z^6}+.....$]

12. Obtain Taylor's and Laurent's expansions of $f(z)=\dfrac{z-1}{z^2-2z-3}$ indicating regions convergence. (M.U. 199

[Ans. : (i) $\dfrac{1}{2}\left[\dfrac{2}{3}-\dfrac{10}{9}z+\dfrac{29}{27}z^2+.....\right];\ |z|<1$

(ii) $\frac{1}{2}\left[-\frac{1}{3}-\frac{z}{9}-\frac{z^2}{27}\ldots\ldots-\frac{1}{z^2}+\frac{1}{z^3}-\ldots\ldots\right]$; $1<|z|<3$

(iii) $\frac{1}{z}+\frac{1}{z^2}+\frac{5}{z^3}+\ldots\ldots, |z|>3.$]

13. Obtain the Laurent's series valid in the indicated region.

(i) $\frac{1}{z^2(z-2)}$; $0<|z|<2$, **(M.U. 1997)**

(ii) $\frac{1+2z}{z+z^2}$; $0<|z|<1$

(iii) $\frac{1}{z-z^2}$; $1<|z+1|<2$

(iv) $\frac{7z-2}{z(z-2)(z+1)}$; $1<|z+1|<3$

(v) $\frac{1}{z^2-z}$; $0<|z-1|<1$

(vi) $\frac{1}{z^3(1-z)}$; $|z|>1$

(vii) $\frac{1}{z(4-z)}$; $0<|z|<4$

(viii) $\frac{z}{(z+1)(z+2)}$; $0<|z+2|<1$

(ix) $\frac{1}{1-z^2}$; $0<|z-1|<2$

(x) $\frac{z-1}{z^2}$; $|z-1|>1$ **(M.U. 2003)**

(xi) $\frac{(z-2)(z+2)}{(z+1)(z+4)}$ (a) $1<|z|<4$, (b) $|z|>4$. **(M.U. 2004, 05)**

[**Ans. :** (i) $-\frac{1}{2z^2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\ldots\ldots\right)$ (ii) $\frac{1}{z}(1+z+z^2+z^3-\ldots\ldots)$

(iii) $\frac{1}{z+1}\left(1+\frac{1}{(z+1)}+\frac{1}{(z+1)^2}+\ldots\ldots\right)+\frac{1}{2}\left[1+\left(\frac{z+1}{2}\right)+\left(\frac{z+1}{2}\right)^2+\ldots\ldots\right]$

(iv) $-\frac{2}{(z+1)}+\sum_{n=2}^{\infty}\frac{1}{(z+1)^n}-\frac{2}{3}\Sigma\left(\frac{z+1}{3}\right)^n$

(v) $\frac{1}{z-1}\left[1-(z-1)+(z-1)^2-(z-1)^3+\ldots\ldots\right]$

(vi) $-\frac{1}{z^4}\left[1+\frac{1}{z}+\frac{1}{z^2}+\ldots\ldots\right]$ (vii) $\sum_{n=0}^{\infty}\frac{z^{n-1}}{4^{n+1}}$

(viii) $\frac{2}{z+2}+1+(z+2)+(z+2)^2+(z+2)^3+\ldots\ldots$

(ix) $\Sigma\,\frac{(-1)^{n+1}}{2^{n+1}}(z-1)^{n-1}$

(x) $(z-1)^{-1}-2(z-1)^{-2}+3(z-1)^{-3}-4(z-1)^{-4}+\ldots\ldots$

(xi) (a) $1-\left[\frac{1}{z}-\frac{1}{z^2}+\ldots\right]-\left[1-\frac{z}{4}+\left(\frac{z}{4}\right)^2-\ldots\right]$

(b) $1-\left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}+\ldots\right]-4\left[\frac{1}{z}-\frac{4}{z^2}+\frac{16}{z^2}+\ldots\right]$

14. Find all possible Laurent's expansions of the following functions indicating the region of convergence.

(i) $\dfrac{4-3z}{z(1-z)(2-z)}$ about $z = 0$.

(ii) $\dfrac{z^2-1}{z^2+5z+6}$ about $z = 0$.

(iii) $\dfrac{z}{(z-1)(z-2)}$ about $z = -2$.

(iv) $\dfrac{z^3-6z-1}{(z-1)(z-3)(z+2)}$ about $z = 3$.

(v) $\dfrac{1}{z^2(1-z)}$ about $z = 0$.

(vi) $\dfrac{z^2-1}{(z+2)(z+3)}$ about $z = 0$.

[**Ans. : (i)** $f(z) = \dfrac{2}{z} + \sum_{n=0}^{\infty}\left(1+\dfrac{1}{2^{n+1}}\right)z^n$; $|z| < 1$

$f(z) = \dfrac{2}{z} - \sum_{n=0}^{\infty}\dfrac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\dfrac{z^n}{2^{n+1}}$; $1 < |z| < 2$

$f(z) = \dfrac{2}{z} - \sum_{n=0}^{\infty}(1-2^n)\cdot\dfrac{1}{z^{n+1}}$; $2 < |z| < \infty$

(ii) $f(z) = 1 + \sum(-1)^n\left\{\dfrac{3}{2^{n+1}} - \dfrac{8}{3^{n+1}}\right\}z^n$; $|z| < 2$

$f(z) = 1 + 3\sum(-1)^n\cdot\dfrac{2^n}{z^{n+1}} - 8\sum(-1)^n\cdot\dfrac{z^n}{3^{n+1}}$; $2 < |z| < 3$

$f(z) = 1 + \sum(-1)\cdot\left\{3\cdot 2^n - 8\cdot 3^n\right\}\dfrac{1}{z^{n+1}}$; $|z| > 3$

(iii) $f(z) = \sum\left(-\dfrac{1}{2\cdot 4^n} + \dfrac{1}{3^{n+1}}\right)(z+2)^n$; $|z+2| < 3$

$f(z) = -\dfrac{1}{2}\sum\dfrac{(z+2)^n}{4^n} - \sum\dfrac{3^n}{(z+2)^{n+1}}$; $3 < |z+2| < 4$

$f(z) = \sum(2\cdot 4^n - 3^n)\dfrac{1}{(z+2)^{n+1}}$; $|z+2| > 4$

(iv) $f(z) = 1 + \dfrac{4/5}{z-3} + \sum(-1)^n\cdot\left\{\dfrac{1}{2^{n+1}} + \dfrac{1}{5^{n+2}}\right\}(z-3)^n$; $0 < |z-3| < 2$

$f(z) = 1 + \dfrac{4/5}{z-3} + \sum(-1)^n\cdot\left\{\dfrac{2^n}{(z-3)^{n+1}} + \dfrac{1}{5^{n+2}}\right\}(z-3)^n$; $2 < |z-3| < 5$

$f(z) = 1 + \dfrac{4/5}{3} + \sum(-1)^n\cdot\left\{2^n + 5^{n-1}\right\}\cdot\dfrac{1}{(z-3)^n}$; $|z-3| > 5$

(v) $f(z) = \sum z^{n+2}$; $0 < |z| < 1$; $f(z) = -\sum\dfrac{1}{z^{n+3}}$; $|z| > 1$

(vi) $f(z) = 1 + \frac{3}{2}\sum(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum(-1)^n\left(\frac{z}{3}\right)^3$; $|z| < 2$

$f(z) = 1 + \frac{3}{z}\sum(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{z}\sum(-1)^n\left(\frac{3}{z}\right)^n$; $|z| > 3$

$f(z) = 1 + \frac{3}{2}\sum(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum(-1)^n\left(\frac{z}{3}\right)^n$; $2 < |z| < 3$]

16. Residues : Introduction

Now we shall learn some new concepts *viz.* a zero of an analytic function, poles, residues etc. We shall then study a very important theorem *viz.* residue theorem. We shall then see how to use this theorem to find some integrals.

17. Zero of an Analytic Function

If a function which is analytic in a region R is equal to zero at a point $z = z_0$ in R then z_0 is called **a zero** of $f(z)$ in R.

If $f(z_0) = 0$ but $f'(z_0) \neq 0$ then z_0 is called a **simple zero** or **a zero of first order.**

If $f(z_0) = 0$ and also $f'(z_0) = f''(z_0) = \ldots\ldots = f^{n-1}(z_0) = 0$ but $f^n(z_0) \neq 0$ then z_0 is called **a zero of order *n*.**

Since $f(z)$ is analytic at $z = z_0$ there exists a neighbourhood of z_0 in which $f(z)$ can be expanded as a Taylor's series.

$$f(z) = f(z_0) + (z - z_0) f'(z_0) + \frac{(z - z_0)^2}{2!} f''(z_0) + \ldots\ldots + \frac{(z - z_0)^n}{n!} f^n(z_0) + \ldots\ldots \quad (|z - z_0| < r)$$

(a) If $z = z_0$ is a simple zero then $f(z_0) = 0$ and $f'(z_0) \neq 0$.

(b) If $z = z_0$ is a zero of order n then, since

$f'(z_0) = f''(z_0) = \ldots\ldots = f^{n-1}(z_0) = 0$, we get from the above series

$$f(z) = \frac{(z - z_0)^n}{n!} f^n(z_0) + \ldots\ldots$$

Example 1 : Find the zeros of $f(z) = (z - 1)\, e^z$.

Sol. : Clearly $f(z) = 0$ when $z = 1$ and $f'(z) = (z - 1)\, e^z + e^z = e$ at $z = 1$. Thus, $f'(z) \neq 0$. Hence, $f(z)$ has simple zero at $z = 1$.

Example 2 : Find the zero of $f(z) = \sin z$.

Sol. : Clearly, $\sin z = 0$ when $z = 0, \pm\pi, \pm 2\pi, \ldots\ldots$ and $f'(z) = \cos z$ is not equal to zero for these values.

Hence, $f(z)$ has simple zeros at $z = 0, \pm\pi, \pm 2\pi, \ldots\ldots$

Example 3 : Find the zeros of $f(z) = z^2 \sin z$.

Sol. : $f(z) = z^2 \sin z = 0$ when $z = 0$, and $z = \pm\pi, \pm 2\pi, \ldots\ldots$

Now, $f(z) = z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\ldots\right) = z^3 - \frac{z^5}{3!} + \frac{z^7}{5!} - \ldots\ldots$

Now, $f'(z) = 3z^2 - \frac{5z^4}{3!} + \ldots,$ $f''(z) = 6z - \frac{20z^3}{3!} + \ldots,$

$f'''(z) = 6 - \frac{60z^2}{3!} + \ldots$

Clearly for $z = 0$, $f'(z)$, $f''(z)$ are zero but $f'''(z) = 6 \neq 0$.

$\therefore$ $z = 0$ is a zero order 3.

But $f'(z) \neq 0$ for $z = \pm \pi, \pm 2\pi, \ldots$

Hence, $z = \pm \pi, \pm 2\pi, \ldots$ each is a simple zero.

Example 4 : Find the zeros of $f(z) = \left(\frac{z-1}{z^2+2}\right)^3$.

Sol. : $f(z) = 0$ $\therefore$ $(z-1)^3 = 0$

$\therefore$ $z = 1$ is a zero of order 3.

EXERCISE - V

Find the zeros of the following functions.

1. $\cos z$ **2.** $\frac{(z+1)^3}{z^2+3}$ **3.** $(z-1)^2 (z-2)^3$ **4.** $\frac{1}{(z-2)^2}$

[**Ans. : (1)** $z = \pm \pi/2, \pm 3\pi/2, \ldots$ are simple zeros. **(2)** $z = -1$ is a zero of order 3. **(3)** $z =$ is a zero of order 2 and $z = 2$ is a zero of order 3. **(4)** a zero of order 2 at infinity.]

2. Find the zeros of the following functions and determine their order.

(i) $z \tan z$, **(ii)** $(z^2 - 1)(z^2 + 3z + 2)$, **(iii)** $z^3 \sin z$, **(iv)** $(z-1)^3 (z+2)$.

[**Ans. : (i)** $z = 0$ is a zero of order 2 and $z = \pm \pi, \pm 2\pi, \pm 3\pi \ldots$ each is a zero of order 1 **(ii)** $z = 1, 2$ is a zero of order 1, $z = -1$ is a zero of order 2. **(iii)** $z = 0$ is a zero of order 4 and $z = \pm\pi$, $\pm 2\pi, \pm 3\pi \ldots$ each is a zero of order 1. **(iv)** $z = 1$ is a zero of order 3 and $z = -2$ is a zero of order 1.]

18. Singular Points

Definition : If a function $f(z)$ is analytic at every point in the neighbourhood of a point z_0 except at z_0 itself then $z = z_0$ is called **a singular point** or **a singularity** of $f(z)$.

For example, if $f(z) = \frac{z^2}{z-2}$, $z = 2$ is a singularity of $f(z)$.

If $f(z) = \frac{z}{z(z+1)}$, $z = 0$ and $z = -1$ are two singularities of $f(z)$.

There are different types of singularities. They are :-

(a) Isolated Singularity

If the singular point z_0 of $f(z)$ is such that there is no other singular point in the neighbourhood of z_0 then such a singularity is called an **isolated singularity** ; the point z_0 is called the **isolated singular point**. In other words z_0 is an isolated singularity or an isolated singular point, if we can find a circle $|z - z_0| = \delta$ with centre at z_0 and radius δ, such that there is no singular point of $f(z)$ within the circle other than z_0.

If we cannot find any such δ *i.e.* we cannot find a circle $|z - z_0| = \delta$ which does not contain a singularity other than z_0, z_0 is called **non-isolated singularity.**

(i) Isolated Singularity : There is no other singular point inside the neighbourhood of z_0 other than z_0 which **is** a singular point.

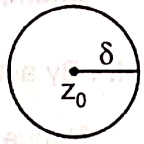


Fig. 2.56

For example, if $f(z) = \dfrac{z^2}{z-2}$, then $z = 2$ is an isolated singularity.

If $f(z) = \dfrac{z-2}{z(z+1)(z-1)}$, then $z = 0$, $z = 1$, $z = -1$ are the finite isolated singularities of $f(z)$.

If $f(z) = \dfrac{1}{z(z^2+1)(z^2-1)}$ then $z = 0, i, -i, +1, -1$ are the finite isolated singularities of $f(z)$.

If $f(z) = \dfrac{1}{\sin \pi z}$, then $z = 0, \pm 1, \pm 2, \ldots, \pm n, \ldots$ are infinite singularities of $f(z)$.

Note ....

If a function $f(z)$ has only **a finite** number of singularities then they are necessarily **isolated.**

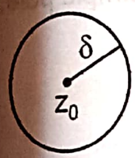


Fig. 2.57

If z_0 is **not** a singularity and the circle $|z - z_0| = \delta$ **does not contain** any singular point of $f(z)$, z_0 is called a **regular point** or **ordinary point** of $f(z)$.

(ii) Ordinary Point or **Regular Point :** No singular point inside the neigh-bourhood and z_0 is **not** a singular point.

(iii) Non-isolated Singularity : Every neighbourhood of z_0 contains **atleast one more singular point z_1** other than z_0.

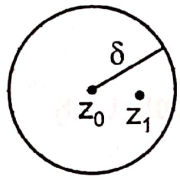


Fig. 2.58

1. Pole : If $z = z_0$ is an isolated singularity of $f(z)$ then we can find a region $0 < |z - z_0| < \delta$ in which $f(z)$ is analytic. In such a region $f(z)$ can be expanded by a Laurent's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n} \qquad \ldots\ldots\ldots\ldots (1)$$

The part $\Sigma a_n (z - z_0)^n$ of the above expansion is called the **analytic part, regular part** and the part $\Sigma b_n (z - z_0)^{-n}$ is called the **principal part** of $f(z)$ in the neighbourhood of z_0.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots + \frac{b_n}{(z - z_0)^n} + \ldots \qquad \ldots\ldots\ldots\ldots (2)$$

(i) If in the above series the coefficients, $b_{n+1} = b_{n+2} = \ldots = 0$

i.e. if $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \dfrac{b_1}{z - z_0} + \dfrac{b_2}{(z - z_0)^2} + \ldots + \dfrac{b_n}{(z - z_0)^n}$ then $z = z_0$ is called a **pole of order n.**

(ii) A pole of order 1 is called a **simple pole.** If $z = z_0$ is a simple pole then $f(z)$ can be written as

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0}.$$

2. Isolated Essential Singularity : In the above series (i) if we have infinite terms of negative powers of $(z - z_0)$ *i.e.* if the principal part of $f(z)$ does not terminate as in (i) *i.e.* if $b_1, b_2, \ldots, b_n, \ldots, \infty$ are **not** zero then $z = z_0$ is called an **isolated essential singularity.**

We note here that the behaviour of a function at an essential singularity is very complicated.

Example 1 : Show that $f(z) = \dfrac{z^2 - 3z + 4}{z - 3}$ has a simple pole at $z = 3$.

Sol. : By actual division $f(z) = z + \dfrac{4}{z-3} = 3 + (z-3) + \dfrac{4}{z-3}$

Hence, $f(z)$ has a simple pole at $z = 3$.

Example 2 : Show that $f(z) = \dfrac{1}{z(z-1)}$ has a simple pole at $z = 0$ and at $z = 1$.

Sol. : We can expand $f(z)$ around the singularities $z = 0$ and $z = 1$ as Laurent's series follows.

$$f(z) = -\frac{1}{z(1-z)} = -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}(1 + z + z^2 + z^3 + z^4 + \ldots\ldots \infty)$$

$$= -(1 + z + z^2 + \ldots\ldots) - \frac{1}{z} \text{ in } 0 < |z| < 1$$

It has only one term in the principal part, $b_1 \neq 0$, $b_2 = b_3 = \ldots\ldots = 0$. Hence, $z = 0$ is a simple pole.

Again $f(z) = \dfrac{1}{(z-1)[1 + (z-1)]} = \dfrac{1}{z-1}[1 + (z-1)]^{-1}$

$$= \frac{1}{z-1}\left[1 - (z-1) + (z-1)^2 - (z-1)^3 + \ldots\ldots\right]$$

$$= -1 + (z-1) - (z-1)^2 + \ldots\ldots + \frac{1}{(z-1)} \qquad 0 < |z-1| < 1$$

As before it has only one term in the principal part, $b_1 \neq 0$, $b_2 = b_3 = \ldots = 0$.

Hence, $z = 1$, is a simple pole.

Example 3 : Show that $f(z) = \dfrac{1}{(z-1)^2 (z-2)^3}$ has a pole of order 2 at $z = 1$ and a pole order 3 at $z = 2$.

Sol. : We can expand $f(z)$ around the singularities $z = 1$, and $z = 2$ as Laurent's series follows.

$$f(z) = \frac{1}{(z-1)^2 (z-1-1)^3} = -\frac{1}{(z-1)^2 [1 - (z-1)]^3}$$

$$= -\frac{1}{(z-1)^2}[1 - (z-1)]^{-3}$$

$$= -\frac{1}{(z-1)^2}\left[1 + 3(z-1) + 6(z-1)^2 + 10(z-1)^3 + \ldots\right]$$

$$= -6 - 10(z-1) - \ldots\ldots\ldots - \frac{3}{z-1} - \frac{1}{(z-1)^2}$$

It has two terms in the principal part, $b_1 \neq 0$, $b_2 \neq 0$, $b_3 = b_4 = \ldots\ldots = 0$.

Hence, $z = 1$ is a pole order 2.

Again $f(z) = \dfrac{1}{(z-1)^2(z-2)^3} = \dfrac{1}{(z-2+1)^2(z-2)^3}$

$$= \frac{1}{(z-2)^3[1+(z-2)]^2} = \frac{1}{(z-2)^3}[1+(z-2)]^{-2}$$

$$= \frac{1}{(z-2)^3}\left[1-2(z-2)+3(z-2)^2-4(z-2)^3+5(z-2)^4-\ldots\right]$$

$$= -4+5(z-2)+\ldots+\ldots+\frac{3}{z-2}-\frac{2}{(z-2)^2}+\frac{1}{(z-2)^3}$$

It has 3 terms in the principal part, $b_1 \neq 0,\ b_2 \neq 0,\ b_3 \neq 0,\ b_4 = b_5 = \ldots = 0$.

Hence, $z = 2$ is a pole of order 3.

Example 4 : Find the order of the pole of $f(z) = \dfrac{\sin hz}{z^7}$.

Sol. : We have $f(z) = \dfrac{1}{z^7}\left(z+\dfrac{z^3}{3!}+\dfrac{z^5}{5!}+\dfrac{z^7}{7!}+\dfrac{z^9}{9!}+\ldots\right)$

$$= \frac{1}{z^6}+\frac{1}{3!\,z^4}+\frac{1}{5!\,z^2}+\frac{1}{7!}+\frac{1}{9!}z^2+\ldots$$

$\therefore$ $f(z)$ has a pole at $z = 0$ of order 6.

Example 5 : Find the poles of cosec $h\ 2z$ within $|z| = 4$. **(M.U. 2004)**

Sol. : The poles of $\operatorname{cosec} h\, 2z = \dfrac{1}{\sin h\, 2z}$ are given by $\sin h\ 2z = 0$.

$$\therefore \frac{e^{2z}-e^{-2z}}{2} = 0 \quad \therefore e^{2z} = e^{-2z} \quad \therefore e^{4z} = 1$$

$$\therefore e^{4z} = e^{2\pi i} = e^{2n\pi i} \quad \therefore 4z = \pm 2n\pi i \quad \therefore z = \pm\frac{n\pi}{2}i$$

The poles within $|z| = 4$ are $\pm\dfrac{\pi}{2}i$ and $\pm\pi i$.

(For $n = 3, 4, \ldots;\ |z| > 4$.)

Example 6 : Find the singular points of $f(z) = \dfrac{1}{z^4+1}$. **(M.U. 2001)**

Sol. : The singular points of $f(z)$ are given by

$$z^4+1 = 0 \quad \therefore z^4 = -1, \quad \therefore z = (-1)^{1/4}$$

$$z = (\cos\pi + i\sin\pi)^{1/4}$$

$$= \cos\frac{(2n+1)\pi}{4} + i\sin\frac{(2n+1)\pi}{4}, \quad n = 0, 1, 2, 3.$$

Example 7 : Show that $f(z) = e^{1/z}$ has an isolated essential singularity at $z = 0$.

Sol. : We have $e^{1/z} = 1+\dfrac{1}{z}+\dfrac{1}{z^2}+\ldots+\dfrac{1}{z^n}+\ldots\infty$

The principal part of $f(z)$ has in infinite number of terms. Hence, $f(z)$ has an isolated **essential singularity** at $z = 0$.

Example 8 : Show that $f(z) = (z+2)\sin\left(\frac{1}{z-1}\right)$ has isolated essential singularity at $z = 1$.

Sol. : We have $f(z) = (z-1+3)\left[\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - \ldots\ldots\right]$

$$= (z-1)\left[\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - \ldots\ldots\right]$$

$$+ 3\left[\frac{1}{z-1} - \frac{1}{3!(z-1)^3} + \frac{1}{5!(z-1)^5} - \ldots\ldots\right]$$

$$= 1 - \frac{1}{3!(z-1)^2} + \frac{1}{5!(z-1)^4} - \ldots\ldots + \frac{3}{(z-1)} - \frac{3}{3!(z-1)^3} + \ldots\ldots$$

The principal part has infinite number of terms. Hence, $f(z)$ has **essential** isolated singularity at $z = 1$.

Limit point of sequence : Loosely speaking if a small neighbourhood of a point 'a' contains a large number of elements of a sequence, then it is called the **limit point** (or a **cluster point**) of the sequence.

Example (i) : The sequence $\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots \frac{1}{n} \ldots\ldots\right\}$ has limit point zero because in a small neighbourhood of zero, there lie a large number of members of the elements of the sequence.

(ii) The sequence $\left\{1, \frac{1}{2}, 1, \frac{1}{3}, 1, \frac{1}{4}, \ldots\ldots\right\}$ has two limit points 1 and 0.

(iii) The sequence $(-1)^n\left(1+\frac{1}{n}\right)$ has two limit points -1 and 1.

(b) Non-isolated Essential Singularity

If we have a sequence of poles of $f(z)$, $z_1, z_2, \ldots, z_n, \ldots$ such that z_0 is the limit point of these poles then z_0 is called **non-isolated essential** singularity.

Example 1 : If $f(z) = \tan\left(\frac{1}{z}\right)$ then $f(z)$ has singularities at $\cos\frac{1}{z} = 0$ *i.e.* at $\frac{1}{z} = \frac{n\pi}{2}$, where $n = \pm 1, \pm 3, \pm 5, \ldots\ldots$ *i.e.* when $z = \frac{2}{n\pi}$, $n = \pm 1, \pm 3, \pm 5, \ldots\ldots$ The limit point of the sequence of the poles $\frac{2}{\pm\pi}, \frac{2}{\pm 3\pi}, \frac{2}{\pm 5\pi}, \ldots\ldots$ is $z = 0$. Hence, $z = 0$ is non-isolated essential singularity.

Example 2 : State the nature of singularity of $f(z) = \left[\sin\left(\frac{1}{z}\right)\right]^{-1}$. **(M.U. 2001)**

Sol. : We have $f(z) = \frac{1}{\sin(1/z)}$

$$\sin\left(\frac{1}{z}\right) = 0 \text{ when } \frac{1}{z} = n\pi \text{ where, } n = 0, \pm 1, \pm 2, \pm 3, \ldots\ldots$$

The limit point of the sequence of the poles

$$\left\{z=\frac{1}{n\pi}\right\} \; i.e. \; \left\{\frac{1}{\pm\pi}, \frac{1}{\pm 2\pi}, \frac{1}{\pm 3\pi}, \ldots\right\} \text{ is } z = 0.$$

Hence, $z = 0$ is non-isolated essential singularity.

Example 3 : State the nature of the singularity of $f(z)=\left[\sin\frac{\pi}{z}\right]^{-1}$. **(M.U. 2000)**

Sol. : We have $f(z)=\dfrac{1}{\sin(\pi/z)}$

$$\sin\left(\frac{\pi}{z}\right) = 0 \text{ when } \frac{\pi}{z} = n\pi \text{ where, } n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

The limit point of the sequence of the pole

$$\left\{z=\frac{1}{n}\right\} \; i.e. \; \left\{\frac{1}{\pm\pi}, \frac{1}{\pm 2\pi}, \frac{1}{\pm 3\pi}, \ldots\right\} \text{ is } z = 0$$

Hence, $z = 0$ is non-isolated essential singularity.

Example 4 : Determine the nature of singularities of

(i) $\dfrac{z-2}{z^2}\sin\left(\dfrac{1}{z-1}\right)$ **(M.U. 2005)** (ii) $\dfrac{z^2+4}{e^z}$ **(M.U. 1998)**

Sol. : (i) Clearly $z = 0$ is a pole of order 2.

$$\text{Now, } \frac{z-2}{z^2}\sin\left(\frac{1}{z-1}\right) = \left(\frac{z-1-1}{z^2}\right)\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} + \ldots\right]$$

$$= \frac{(z-1)}{z^2}\cdot\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} + \ldots\right] - \frac{1}{z^2}\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} - \ldots\right]$$

$$= \frac{1}{z^2}\left[1 - \frac{1}{6(z-1)^2} + \ldots\right] - \frac{1}{z^2}\left[\frac{1}{(z-1)} - \frac{1}{6(z-1)^3} + \ldots\right]$$

$\therefore$ $z = 1$ is an essential singularity.

(ii) $f(z) = \dfrac{z^2+4}{e^z}$ is analytic everywhere. $z = -\infty$ is its singularity.

(c) Removable Singularity

If $z = z_0$ is a singularity of $f(z)$ such that $\lim_{z\to z_0} f(z)$ exists then $z = z_0$ is called a **removable singularity.**

Alternatively : If the expansion of $f(z)$ about $z = z_0$ does not contain negative powers of $(z - z_0)$ *i.e.* $b_n = 0$ for all n then $z = z_0$ is called a **removable singularity** of $f(z)$. (See Ex. 1 below)

Note ....

From the above discussion it is clear that

(i) if $\lim_{z\to z_0} f(z)$ exists then $z = z_0$ is a removable singularity.

This type of singularity can be removed by suitably defining $f(z)$ at z_0.

(ii) $\lim_{z \to z_0} f(z) = \infty$ if z is a pole.

Example 1 : Show that $f(z) = \dfrac{\sin z}{z}$ has a removable singularity at $z = 0$. (M.U. 200

Sol. : Now $f(z) = \dfrac{\sin z}{z}$ is not defined at $z = 0$ but $\lim_{z\to 0} \dfrac{\sin z}{z} = 1$. Hence, $z = 0$ is a removable singularity of $f(z)$.

Laurent's expansion of $f(z)$ is

$$f(z) = \frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots\right] = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots$$

It does not contain negative powers of z.

The singularity is removed if we ***define*** $f(z) = \dfrac{\sin z}{z} = 0$ at $z = 0$.

Example 2 : Show that $f(z) = \dfrac{1-\cos z}{z}$ has a removable singularity at $z = 0$.

Sol. : Now $f(z) = \dfrac{1-\cos z}{z}$ is not defined at $z = 0$, but

$$\lim_{z\to 0} \frac{1-\cos z}{z} = \lim_{z\to 0} \frac{2\sin^2(z/2)}{z} = \lim_{z\to 0} \frac{2\sin^2(z/2)}{(z/2)^2} \cdot \frac{z}{2} = 0$$

Hence, $z = 0$ is a removable singularity.

Laurent's expansion of $f(z)$ is

$$f(z) = \frac{1}{z}\left[1 - \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right)\right] = \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - \ldots$$

It does not contain negative powers of z.

The singularity is removed if we ***define*** $f(z) = \dfrac{1-\cos z}{z} = 0$ at $z = 0$.

(d) Singularity At ∞

The singularity of $f(z)$ at $z = \infty$ is the same as the singularity of $f(1/\omega)$ at at $\omega = 0$.

Example : $f(z) = z^4$ has a pole of order 4 at $z = \infty$, since $f(\omega) = \dfrac{1}{\omega^4}$ has a pole of order 4 at $\omega = 0$.

Notes ....

1. A function which is analytic everywhere in the finite z-plane is called an **entire function** or **integral function**. An entire function can be expressed as a Taylor's series where radius of convergence is ∞ and conversely a power series whose radius of convergence is ∞ is **entire** or **integral function**.

   For example, functions e^z, $\sin z$, $\cos hz$ are entire functions.

2. A function which is analytic everywhere in the finite z-plane except at a finite number of poles is called a **mermorphic function.**

For example, $f(z) = \dfrac{1}{z(z-1)^2}$, $f(z) = \dfrac{3z}{(z-1)^2(z-2)^3}$ are mermorphic functions, since they have finite number of poles.

Example 3 : Determine the nature of singularities of the following functions.

(i) $\dfrac{e^z}{(z-1)^4}$, **(ii)** $(z+1)\cdot\sin\left(\dfrac{1}{z+2}\right)$, **(iii)** $\dfrac{1}{z^2(e^z-1)}$, **(iv)** $e^{1/z}$,

(v) $\dfrac{\cot\pi z}{(z-a)^3}$, **(vi)** $\sin\left(\dfrac{1}{z-1}\right)$, **(vii)** $\sec\left(\dfrac{1}{z}\right)$, **(viii)** $\left[\sin\dfrac{1}{z}\right]^{-1}$ **(M.U. 2001)**

Sol. : (I) $z = 1$ is a pole of order 4.

(II) We have

$$f(z) = (z+2-1)\cdot\sin\left(\frac{1}{z+2}\right)$$

$$= [(z+2)-1]\cdot\left[\frac{1}{z+2} - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \ldots..\right]$$

$$= (z+2)\left[\frac{1}{z+2} - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \ldots..\right]$$

$$-1\left[\frac{1}{z+2} - \frac{1}{3!(z+2)^3} + \frac{1}{5!(z+2)^5} - \ldots..\right]$$

$$= 1 - \frac{1}{(z+2)} - \frac{1}{3!(z+2)^2} + \frac{1}{3!(z+2)^3} - \ldots..$$

Since the principal part of $f(z)$ contains infinite number of terms in $(z+2)$, $z = -2$ is an isolated essential singularity.

(III) We have $f(z) = \dfrac{1}{z^2(e^z-1)}$

The singularities are given by $z^2(e^z - 1) = 0 \quad \therefore \; z = 0$

or $\quad e^z = 1 = e^{2n\pi i}, \; n = 0, \pm 1, \pm 2, \ldots..$

Hence, $z = 0$ is a pole of order **three.** The other singularities $\pm 2\pi, \pm 4\pi, \ldots..$ are simple poles.

(IV) $f(z) = e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{z^2\cdot 2!} + \dfrac{1}{z^3\cdot 3!} + \ldots..$

Since, the principle part contains an infinite number of terms, $z = 0$ is an isolated essential singularity.

(V) Singularities of $f(z) = \dfrac{\cot\pi z}{(z-a)^3}$ *i.e.* of $f(z) = \dfrac{\cos\pi z}{(z-a)^3\sin\pi z}$ are given by $z - a = 0$ and $\sin\pi z = 0$.

$\therefore \; z = a$ is a pole of order three.

Further, $\sin n\pi = 0, \; n = 0, \pm 1, \pm 2, \ldots..$ *i.e.* $z = n$.

Hence, $z = 0, \pm 1, \pm 2, \ldots..$ are simple poles.

(vi) $f(z) = 0$ when $\dfrac{1}{z-1} = n\pi$ i.e. $z - 1 = \dfrac{1}{n\pi}$ i.e. $z = 1 + \dfrac{1}{n\pi}$ when $n = 0, \pm 1, \pm 2, \ldots$

The limit point of the sequence of poles is 1. Hence, $z = 1$ is non-isolated essential singu...

(vii) $f(z) = \sec\dfrac{1}{z} = \dfrac{1}{\cos(1/z)} = 0$ when $\dfrac{1}{z} = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}, \ldots$

i.e. $z = \pm\dfrac{2}{\pi}, \pm\dfrac{2}{3\pi}, \pm\dfrac{2}{5\pi}, \ldots$

The limit point of the sequence of poles is $z = 0$. Hence, $z = 0$ is non-isolated essential singu...

(viii) $f(z) = \dfrac{1}{\sin(1/z)}$ is singular when $\sin\dfrac{1}{z} = 0$ i.e. when $\dfrac{1}{z} = 0$ or $n\pi$.

$\therefore$ The singularities are $z = 0, \pm\dfrac{1}{\pi}, \pm\dfrac{1}{2\pi}, \pm\dfrac{1}{3\pi}, \ldots$

These are all isolated singularities.

EXERCISE - VI

1. Determine the nature of singularities, if any, for the following functions.

(i) $f(z) = \dfrac{z-2}{4}\sin\left(\dfrac{1}{z-1}\right)$ **(M.U. 1997)** (ii) $f(z) = \dfrac{z^2+1}{e^z}$ **(M.U. 1997)** (iii) $f(z) = z^2 e^{1/z}$

(iv) $f(z) = \dfrac{\sin 3z}{z}$ (v) $f(z) = (z+1)\,e^{1/(z-3)}$ (vi) $\dfrac{\sinh(z-z_0)}{z-z_0}$

(vii) e^{-1/z^2} **(M.U. 2005)**

[**Ans. : (i)** $z = 1$ is an essential singularity. **(ii)** $z = 0$ is an ordinary point. **(iii)** $z = 0$ is essential singularity. **(iv)** $z = 0$ is a removable singularity. **(v)** $z = 3$ is an essential singular... **(vi)** $z = z_0$ is a removable singularity. **(vii)** No singularity.]

2. Determine the nature of singularities of the following functions.

(i) $\dfrac{\tan z}{z}$, (ii) $z^3 \cdot e^{1/(z-1)}$, (iii) $\dfrac{z^2+1}{(z-1)^2(z+1)}$, (iv) $z^2 e^{-z}$,

(v) $\dfrac{2-e^z}{z^3}$, (vi) $\dfrac{z}{\cos z}$, (vii) $\dfrac{z}{\sin z}$.

[**Ans. : (i)** Removable singularity at $z = 0$. **(ii)** $z = 1$ is an essential singularity. **(iii)** $z = +1$ is isolated singularity of order 2 and $z = -1$ is an isolated singularity of order 1. **(iv)** $z = 0$ is an ordin... point. **(v)** $z = 0$ is a pole of order 3. **(vi)** $z = (2n+1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$ each is a simple p... **(vii)** $z = n\pi$, $n = \pm 1, \pm z, \ldots$ each is a simple pole.]

3. Expand each of the following functions in Laurent's series about $z = 0$ and identify ... singularity.

(i) $z^2 e^{-z^2}$, (ii) $\dfrac{1-\cos 2z}{z}$, (iii) $z^{-1}e^{-z}$, (iv) $z^{-3}e^z$,

(v) $(z+1)\sin\dfrac{1}{z}$, (vi) $\dfrac{1-e^z}{z}$, (vii) $\dfrac{1-e^{2z}}{z^3}$.

[**Ans. :** (i) $z^2e^{-z^2} = z^2\left[1 - z^2 + \frac{z^4}{2!} - \frac{z^6}{3!} + \ldots\right] = z^2 - 1 + \frac{z^6}{2!} - \frac{z^8}{3!} + \ldots$

$\therefore$ $z = 0$ is an ordinary point.

(ii) $\frac{1-\cos 2z}{z} = \frac{1}{z}\left[1 - \left(1 - \frac{4z^2}{2!} + \frac{16z^4}{4!} - \ldots\right)\right]$

$= 2z - \frac{2}{3}z^3 + \ldots\ ; \quad \lim_{z\to 0}\frac{1-\cos 2z}{z} = 0$

$\therefore$ $z = 0$ is a removable singularity.

(iii) $z^{-1}e^{-z} = \frac{1}{z}\left(1 - z + \frac{z^2}{2!} - \frac{z^3}{3!} + \frac{z^4}{4!} - \ldots\right) = \frac{1}{z} - 1 + \frac{z}{2!} - \frac{z^2}{3!} + \ldots$

$\therefore$ $z = 0$ is a pole of order 1 *i.e.* a simple pole.

(iv) $z^{-3}e^{z} = \frac{1}{z^3}\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots\right) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z\cdot 2!} + \frac{1}{3!} + \frac{z}{4!} + \ldots$

$\therefore$ $z = 0$ is a pole of order 3.

(v) $(z+1)\sin\frac{1}{z} = (z+1)\left[1 - \frac{1}{z^3\cdot 3!} + \frac{1}{z^5\cdot 5!} - \ldots\right]$

$= z - \frac{1}{z^2\cdot 3!} + \frac{1}{z^4\cdot 5!} - \ldots + 1 - \frac{1}{z^3\cdot 3!} + \ldots$

$\therefore$ $z = 0$ is an essential singularity.

(vi) $\frac{1-e^z}{z} = \frac{1}{z}\left[1 - \left(1 + z + \frac{z^2}{2!} + \ldots\right)\right]$

$= -1 - \frac{z}{2!} - \frac{z^2}{3!} - \ldots\ ; \quad \lim_{z\to 0}\frac{1-e^z}{z} = -1.$

$\therefore$ $z = 0$ is a removable singularity.

(vii) $\frac{1-e^{2z}}{z^3} = \frac{1}{z^3}\left[1 - \left(1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \ldots\right)\right] = \frac{2}{z^2} + \frac{2}{z} + \frac{4}{3} + \frac{2}{3}z + \ldots$

$\therefore$ $z = 0$ is a pole of order 2.]

4. Expand each of the following functions in Laurent's series about $z = 0$ and identify the singularity

(i) $z\,e^{1/z^2}$ **(ii)** $\frac{\sin^2 z}{z}$ **(iii)** $\frac{1}{z(4-z)}$ **(M.U. 2004)**

[**Ans. :** (i) $ze^{1/z^2} = z + \frac{1}{z} + \frac{1}{2!\,z^3} + \ldots$ $\quad\therefore$ $z = 0$ is an essential singularity.

(ii) $\frac{\sin^2 z}{z} = z - \frac{1}{3}\cdot z^3 + \frac{2}{45}z^5 - \ldots$ $\quad\therefore$ $z = 0$ is an ordinary point.

(iii) $\frac{1}{z(4-z)} = \frac{1}{4z(z-(z/4))} = \frac{1}{4z}\left(1 - \frac{z}{4}\right)^{-1}$

$$\frac{1}{z(4-z)}=\frac{1}{4z}\left(1+\frac{z}{4z}+\frac{z^2}{64}+\frac{z^3}{256}+\ldots\ldots\right)=\frac{1}{4z}+\frac{1}{16}+\frac{z}{64}+\ldots\ldots$$

$\therefore \quad z=0$ is a simple pole.]

19. Residues

If $z = z_0$ is an isolated singularity then the constant b_1 *i.e.* **the coefficient of** $\frac{1}{z-z_0}$ Laurent's expansion of $f(z)$ at $z = z_0$ is called the **residue** of $f(z)$ at $z = z_0$.

$\therefore$ Residue of $f(z)$ (at $z = z_0$) $= b_1 =$ coefficient of $\frac{1}{z-z_0}$

$$= \frac{1}{2\pi i}\oint_C f(z)\,dz$$

Or $\quad \oint_C f(z)\,dz = 2\pi i$ (Residue at $z = z_0$)

Calculation of Residues at Poles

(i) If $z = z_0$ is a simple pole of $f(z)$ then

$$\boxed{\textbf{Residue of } f(z) \textbf{ (at } z = z_0) = \lim_{z\to z_0}(z-z_0)f(z)}$$

Proof : Since $z = z_0$ is a simple pole, the Laurent's series of

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n+\frac{b_1}{(z-z_0)} \text{ becomes}$$

$$\therefore \quad (z-z_0)f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^{n+1}+b_1$$

$$\therefore \quad \lim_{n\to z_0}(z-z_0)f(z)=b_1=\text{Residue of } f(z) \text{ (at } z=z_0)$$

(ii) If $z = z_0$ is a simple pole of $f(z) = \frac{P(z)}{Q(z)}$ then

$$\boxed{\textbf{Residue of } f(z) \textbf{ (at } z = z_0) = \lim_{z\to z_0}\frac{P(z)}{Q'(z)}}$$

Proof : Since $z = z_0$ is a simple pole by the above result,

$$\text{Residue (at } z=a_0) = \lim_{z\to z_0}\left[(z-z_0)\frac{P(z)}{Q(z)}\right]$$

Since at $z = z_0$, $Q(z) = 0$, the limit in the r.h.s. is of the form $\frac{0}{0}$. Hence, by Hospital's

$$\text{Residue (at } z=a_0) = \lim_{z\to z_0}\left[\frac{(z-z_0)P'(z)+P(z)}{Q'(z)}\right]=\lim_{z\to z_0}\frac{P(z)}{Q'(z)}$$

(iii) If $z = z_0$ is a pole of order m then

$$\boxed{\textbf{Residue of } f(z) \textbf{ (at } z = z_0) = \frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]}$$

Proof : Since $z = z_0$ is a pole of order m, the Laurent's series of $f(z)$ becomes

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \ldots.. + \frac{b_m}{(z - z_0)^m}$$

$$(z - z_0)^m f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{m+n} + b_1 (z - z_0)^{m-1} + b_2 (z - z_0)^{m-2} + \ldots.. + b_m$$

Differentiating this $(m - 1)$ times and taking the limit as $z \to z_0$, we get

$$\frac{1}{m - 1!} \lim_{z \to z_0} \frac{d^{m-1}}{d z^{m-1}} (z - z_0)^m f(z) = b_1 = \text{Residue of } f(z) \text{ (at } z = z_0).$$

Note ....

1. To find the residues and to evaluate an integral using residues you are advised to draw a rough sketch of the contour or find the distance of centre z_0 of the given circle from the pole z_1 by using

$$| z_1 - z_0 | = | (x_1 - x_0) + i (y_1 - y_0) | = [(x_1 - x_0)^2 + (y_1 - y_0)^2]$$

and also use L' Hospitals rule wherever necessary.

2. Residue at $z = z_0$ can be obtained either by limit method (Ex. 1) or by expansion method [Ex. 2 (ii), Ex. 3].

Type I : To find the residues by taking limits

Example 1 : Determine the pole of the function $f(z) = \dfrac{z^2}{(z-1)^2 (z+2)}$ and also find the residue at each pole. **(M.U. 1998, 2003, 05, 11)**

Sol. : $\because$ $(z - 1)^2 (z + 2) = 0$ gives $z = -2$, 1 and 1. Hence, $f(z)$ has a simple pole at $z = -2$ and a pole of order 2 at $z = 1$.

(i) Residue of $f(z)$ (at $z = -2$) $= \lim\limits_{z \to -2} (z + 2) f(z) = \lim\limits_{z \to -2} \dfrac{z^2}{(z-1)^2} = \dfrac{4}{9}$

(ii) Residue of $f(z)$ (at $z = 1$) $= \lim\limits_{z \to 1} \dfrac{d}{dz} \left[(z-1)^2 f(z) \right] = \lim\limits_{z \to 1} \dfrac{d}{dz} \left[\dfrac{z^2}{z+2} \right]$

$$= \lim_{z \to 1} \frac{(z+2) 2z - z^2}{(z+2)^2} = \lim_{z \to 1} \frac{z^2 + 4z}{(z+2)^2} = \frac{5}{9}.$$

Example 2 : Determine the nature of poles of the following functions and find the residue at each pole.

(i) $\dfrac{z e^z}{(z-a)^3}$ **(M.U. 1998)** **(ii)** $\dfrac{1 - e^{2z}}{z^3}$ **(M.U. 1996, 2005)**

Sol. : (i) $z = a$ is a pole of order 3.

$\therefore$ Residue of $f(z)$ (at $z = a$) $= \lim\limits_{z \to a} \dfrac{1}{2!} \dfrac{d^2}{dz^2} \left[(z-a)^3 f(z) \right]$

$$= \lim_{z \to a} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-a)^3 \frac{z e^z}{(z-a)^3} \right]$$

$$\therefore \text{ Residue of } f(z) \text{ (at } z = a) = \frac{1}{2!} \lim_{z \to a} \frac{d^2}{dz^2}(ze^z) = \frac{1}{2!} \lim_{z \to a} \frac{d}{dz}(ze^z + e^z)$$

$$= \frac{1}{2!} \lim_{z \to a} \left[ze^z + e^z + e^z\right] = \frac{1}{2}(ae^a + 2e^a) = \frac{1}{2}(a+2)e^a$$

(ii) $z = 0$ is a pole of order 3.

$$\therefore \text{ Residue of } f(z) \text{ (at } z = 0) = \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[z^3 f(z)\right]$$

$$= \lim_{z \to 0} \frac{1}{2!} \frac{d^2}{dz^2}\left[\frac{z^3 \cdot (1 - e^{2z})}{z^3}\right]$$

$$= \frac{1}{2!} \lim_{z \to 0} \frac{d^2}{dz^2}(1 - e^{2z}) = \frac{1}{2!} \lim_{z \to 0} \frac{d}{dz}(-2e^{2z})$$

$$= \frac{1}{2!} \lim_{z \to 0} (-4e^{2z}) = \frac{1}{2} \cdot (-4) = -2$$

Alternatively

$$f(z) = \frac{1 - e^{2z}}{z^3} = \frac{1 - \left\{1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \ldots\right\}}{z^3} = -\frac{2}{z^2} - \frac{2}{z} - \frac{4}{3} - \ldots$$

$$\therefore \text{ Residue (at } z = 0) = b_1 = \text{coefficient of } \frac{1}{z} = -2.$$

Example 3 : Find the residue of $\frac{1 - e^{2z}}{z^4}$ at its pole. **(M.U. 1992,**

Sol. : Clearly $z = 0$ is a pole of order 4.

$$\therefore \text{ Residue of } f(z) \text{ (at } z = 0) = \lim_{z \to 0} \frac{1}{3!} \frac{d^3}{dz^3}\left[z^4 f(z)\right]$$

$$= \lim_{z \to 0} \frac{1}{3!} \frac{d^3}{dz^3}\left[\frac{z^4 \cdot (1 - e^{2z})}{z^4}\right] = \frac{1}{3!} \lim_{z \to 0} \frac{d^3}{dz^3}(1 - e^{2z})$$

$$= \frac{1}{3!} \lim_{z \to 0} \frac{d^2}{dz^2}(-2e^{2z}) = \frac{1}{3!} \lim_{z \to 0} \frac{d}{dz}(-4e^{2z})$$

$$= \frac{1}{3!} \lim_{z \to 0} (-8e^{2z}) = -\frac{8}{3!} = -\frac{4}{3}.$$

Alternatively

$$f(z) = \frac{1 - \left[1 + 2z + \frac{4z^2}{2!} + \frac{8z^3}{3!} + \frac{16z^4}{4!} + \frac{32z^5}{5!} + \ldots\right]}{z^4}$$

$$= -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3} \cdot \frac{1}{z} - \frac{2}{3} - \frac{4}{15}z - \ldots$$

$$\therefore \text{ Residue (at } z = 0) = b_1 = \text{coefficient of } \frac{1}{z} = -\frac{4}{3}.$$

Example 4 : Determine the nature of poles and find the residue thereat for $f(z) = z^2 \sec \pi z$. **(M.U. 1996)**

Sol. : We have $f(z) = \dfrac{z^2}{\cos \pi z}$

The singularities are given by $\cos \pi z = 0$.

$$\therefore \quad \pi z = (2n+1)\frac{\pi}{2} \qquad \therefore \quad z = \frac{(2n+1)}{2}, \quad n = 0, \pm 1, \pm 2, \ldots\ldots$$

and each is a simple pole.

$$\therefore \quad \text{Residue}\left[\text{at } z = \frac{(2n+1)}{2}\right] = \lim_{z \to (2n+1)/2} \frac{[z-(2n+1)/2]z^2}{\cos \pi z}$$

$$= \lim_{z \to (2n+1)/2} \left[\frac{2n+1}{2}\right]^2 \cdot \frac{z-(2n+1)/2}{\cos \pi z}$$

$$= \left(\frac{2n+1}{2}\right)^2 \cdot \lim_{z \to (2n+1)/2} \cdot \frac{1}{-\sin \pi z \cdot \pi} \qquad \text{[By L'Hospital's Rule]}$$

$$= \left(\frac{2n+1}{2}\right)^2 \left[-\frac{1}{\pi \sin[(2n+1)\pi/2]}\right]$$

$$= (-1)^{n+1}\left(\frac{2n+1}{2}\right)^2 \cdot \frac{1}{\pi}$$

Example 5 : Find the residues of $f(z) = \dfrac{\sin \pi z}{(z-1)^2 (z-2)}$ at its poles. **(M.U. 2003)**

Sol.: Clearly $z = 1$ is a pole of order 2 and $z = 2$ is a simple pole.

$$\therefore \quad \text{Residue (at } z = 1) = \lim_{z \to 1} \frac{1}{1!}\frac{d}{dz}\left[(z-1)^2 \cdot \frac{\sin \pi z}{(z-1)^2 (z-2)}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz}\left[\frac{\sin \pi z}{z-2}\right] = \lim_{z \to 1} \frac{(z-2)(\cos \pi z) \cdot \pi - \sin \pi z}{(z-2)^2}$$

$$= \frac{(1-2)(-1)\pi - 0}{(1-2)^2} = \pi$$

$$\therefore \quad \text{Residue (at } z = 2) = \lim_{z \to 2} (z-2) \cdot \frac{\sin \pi z}{(z-1)^2 (z-2)} = \lim_{z \to 2} \frac{\sin \pi z}{(z-1)^2} = 0$$

Example 6 : Find the residues of $\dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$ at its poles. **(M.U. 1998, 2000, 15)**

Sol. : $f(z)$ has a simple pole at $z = 1$ and a pole of order 2 at $z = 2$.

$$\therefore \quad \text{Residue (at } z = 1) = \lim_{z \to 1} (z-1)\left\{\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}\right\}$$

$$= \lim_{z \to 1} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)^2} = \frac{\sin \pi + \cos \pi}{(1-2)^2} = -1$$

$$\text{Residue (at } z = 2) = \lim_{z\to 2} \frac{1}{1!}\frac{d}{dz}\left[\frac{(z-2)^2(\sin \pi z^2 + \cos \pi z^2)}{(z-1)(z-2)^2}\right]$$

$$= \lim_{z\to 2} \frac{d}{dz}\left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1}\right]$$

$$= \lim_{z\to 2} \left[(z-1)\left\{(\cos \pi z^2)(2\pi z) - (\sin \pi z^2)(2\pi z)\right\} - (\sin \pi z^2 + \cos \pi z^2)\right] / (z-1)^2$$

$$= \frac{(2-1)[4\pi - 0 - 0 - 1]}{(2-1)^2} = 4\pi - 1.$$

Type II : To find the residues using Laurent's Series

Example 1 : Find the residues at each pole of the following by Laurent's Expansion.

(i) $z^2 e^{1/z}$ **(M.U. 2000)** (ii) $e^{-1/(z-1)^2}$ **(M.U. 2000)**

Sol. : (i) Since $f(z) = z^2 e^{1/z} = z^2\left[1 + \frac{1}{z} + \frac{1}{z^2 \cdot 2!} + \frac{1}{z^3 \cdot 3!} + ...\right]$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{6z} + \frac{1}{4!\,z^2} + \frac{1}{5!\,z^3} + ...$$

$z = 0$ is an essential singularity.

$\therefore$ Residue (at $z = 0$) $= b_1$ = coefficient of $\frac{1}{z} = \frac{1}{6}$.

(ii) $f(z) = e^{-1/(z-1)^2} = 1 - \frac{1}{(z-1)^2} + \frac{1}{(z-1)^4 \cdot 2!} - .....$

$z = 1$ is an essential singularity.

$\therefore$ Residue (at $z = 1$) $= b_1$ = coefficient of $\frac{1}{z-1} = 0$.

Example 2 : Find the residue of $f(z) = \frac{1}{z - \sin z}$ at its singularity, using Laurent's expansion. **(M.U.**

Sol. : We have

$$f(z) = \frac{1}{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - ......\right]} = \frac{1}{\frac{z^3}{3!} - \frac{z^5}{5!} + ...} = \frac{1}{\frac{z^3}{3!}\left[1 - 3!\frac{z^2}{5!} + ...\right]}$$

Hence, $z = 0$ is a pole of order 3.

Now, $f(z) = \frac{1}{z^3/3!}\left[1 - \frac{1}{20}z^2 + ...\right]^{-1} = \frac{6}{z^3}\left[1 + \frac{z^2}{20} + ...\right] = \frac{6}{23} + \frac{3}{10}\cdot\frac{1}{z} + ...$

$\therefore$ Residue (at $z = 0$) $= b_1$ = coefficient of $\frac{1}{z} = \frac{3}{10}$.

Example 3 : Find the residue of $f(z)=\dfrac{z}{\cos z-\cos hz}$ at its singularity, using Laurent's series expansion.

Sol. : We have

$$f(z)=\frac{z}{\left[1-\frac{z^2}{2!}+\frac{z^4}{4!}-.....\right]-\left[1+\frac{z^2}{2!}+\frac{z^4}{4!}+.....\right]}$$

$$=\frac{z}{-z^2-\frac{2}{6!}z^6-.....}=-\frac{1}{z+\frac{2}{6!}z^6+.....}=-\frac{1}{z\left[1+\frac{2}{6!}z^5+.....\right]}$$

Hence, $z=0$ is a simple pole.

Now, $f(z)=-\dfrac{1}{z}\left[1+\dfrac{2}{6!}z^5+.....\right]^{-1}=-\dfrac{1}{z}\left[1+\dfrac{2}{6!}z^5+.....\right]=-\dfrac{1}{z}+\dfrac{2}{6!}z^4-.....$

$\therefore$ Residue $(z=0)=b_1=$ coefficient of $=\dfrac{1}{z}=-1.$

Example 4 : Find the residue $f(z)=\dfrac{1-z}{1-\cos z}$ at its singularity, using Laurent's series expansion.

Sol. : We have

$$f(z)=\frac{1-z}{1-\left[1-\frac{z^2}{2!}+\frac{z^4}{4!}-.....\right]}=\frac{(1-z)}{\frac{z^2}{21}\left[1-\frac{2}{4!}z^2+\frac{2}{6!}z^4-.....\right]}$$

$$=2\left(\frac{1-z}{z^2}\right)\left[1-\frac{2}{4!}z^2+\frac{2}{6!}z^4-.....\right]^{-1}$$

$$=2\left(\frac{1}{z^2}-\frac{1}{z}\right)\left[1+\left(\frac{2}{4!}z^2-.....\right)+\left(\frac{2z^2}{4!}-.....\right)^2+.....\right]$$

$$\therefore\quad f(z)=\frac{2}{z^2}-\frac{2}{z}+2\cdot\frac{2}{4!}-2\cdot\frac{2}{4!}z+.....$$

$\therefore$ Residue (at $z=0$) $=b_1=$ coefficient of $\dfrac{1}{z}=-2.$

Example 5 : Find the residues of at $f(z)=\dfrac{z}{(z-1)(z+2)^2}$ at its isolated singularities using Laurent's series expansion. **(M.U. 2003)**

Sol. : Now $\dfrac{z}{(z-1)(z+2)^2}=\dfrac{1/9}{z-1}-\dfrac{1/9}{z+2}+\dfrac{2/3}{(z+2)^2}$

Clearly $z=1$ and $z=-2$ are isolated singularities.

To find the residue at $z=1$, we have to expand $f(z)$ in powers of $(z-1)$, in $0<|z-1|<r$ and find the coefficient of $\dfrac{1}{z-1}$ in it.

$$f(z) = \frac{1/9}{(z-1)} - \frac{1/9}{3+(z-1)} + \frac{2/3}{[3+(z-1)]^2}$$

$$= \frac{1}{9} \cdot \frac{1}{(z-1)} - \frac{1}{9 \cdot 3}\left[1+\left(\frac{z-1}{3}\right)\right]^{-1} + \frac{2}{3 \cdot 9}\left[1+\left(\frac{z-1}{3}\right)\right]^{-2}$$

$$= \frac{1}{9} \cdot \frac{1}{(z-1)} - \frac{1}{27}\left[1-\frac{(z-1)}{3}+\frac{(z-1)^2}{3^2}-\frac{(z-1)^3}{3^3}+\ldots\right]$$

$$+ \frac{2}{27}\left[1-\frac{2(z-1)}{3}+\frac{2\cdot 3}{2!}\cdot\frac{(z-1)^2}{3^2}-\frac{2\cdot 3\cdot 4}{3!}\cdot\frac{(z-1)^3}{3^3}\ldots\right]$$

$$= \frac{1}{9} \cdot \frac{1}{(z-1)} - \frac{1}{27}\Sigma(-1)^n\frac{(z-1)^n}{3^n} + \frac{2}{27}\Sigma(-1)^n(n+1)\frac{(z-1)^n}{3^n}$$

The expansion is valid in $\left|\frac{z-1}{3}\right| < 1$ *i.e.* $0 < |z-1| < 3$.

$\therefore$ Residue (at $z = 1$) $= b_1 =$ coefficient of $\frac{1}{z-1} = \frac{1}{9}$.

To find the residue at $z = -2$, we have to expand $f(z)$ in powers of $(z+2)$ in $0 < |z+2| < r$ and find the coefficient of $\frac{1}{z+2}$.

$$f(z) = \frac{1}{9[(z+2)-3]} - \frac{1}{9(z+2)} + \frac{2}{3} \cdot \frac{1}{(z+2)^2}$$

$$= -\frac{1}{27}\left[1-\left(\frac{z+2}{3}\right)\right]^{-1} - \frac{1}{9(z+2)} + \frac{2}{3} \cdot \frac{1}{(z+2)^2}$$

$$= -\frac{1}{27}\left[1-\frac{(z+2)}{3}+\frac{(z+2)^2}{3^2}-\ldots\right] - \frac{1}{9(z+2)} + \frac{2}{z(z+2)^2}$$

The expansion is valid in $\left|\frac{z+2}{3}\right| < 1$ *i.e.* $0 < |z+2| < 3$.

$\therefore$ Residue (at $z = -2$) $= b_1 =$ coefficient of $\frac{1}{z+2} = -\frac{1}{9}$.

Type III : To find the sum of the residues

Example 1 : Prove that the sum of the residues of $f(z) = \frac{e^z}{z^2 - a^2}$ is $\frac{\sin ha}{a}$.

Sol. : Clearly the poles of $f(z)$ are given by $z^2 - a^2 = 0$.

$\therefore$ $(z-a)(z+a) = 0$ $\therefore$ $z = a, -a$.

$$r_1 = \text{Residue (at } z = a) = \lim_{z\to a}(z-a)\cdot\frac{e^z}{(z^2-a^2)} = \lim_{z\to a}\frac{e^z}{z+a} = \frac{e^a}{2a}.$$

$$r_2 = \text{Residue (at } z = -a) = \lim_{z \to -a} (z+a) \cdot \frac{e^a}{z^2 - a^2} = \lim_{z \to -a} \frac{e^z}{z-a} = \frac{e^{-a}}{-2a}$$

$$\therefore \text{ Sum of the residues} = r_1 + r_2 = \frac{e^a}{2a} - \frac{e^{-a}}{2a} = \frac{1}{a}\left[\frac{e^a - e^{-a}}{2}\right] = \frac{\sin ha}{a}.$$

Example 2 : Find the sum of the residues at singular points of

$$f(z) = \frac{z}{(z-1)^2(z^2-1)}.$$ **(M.U. 2001, 2016)**

Sol. : We have $f(z) = \dfrac{z}{(z-1)^2(z-1)(z+1)} = \dfrac{z}{(z-1)^3(z+1)}$

$\therefore$ $z = 1$ is a pole of order 3 and $z = -1$ is a simple pole.

$$\therefore \text{ Residue (at } z = 1) = \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2}\left[(z-1)^3 \cdot \frac{z}{(z-1)^3(z+1)}\right]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2}\left[\frac{z}{z+1}\right] = \frac{1}{2} \lim_{z \to 1} \frac{d}{dz}\left[\frac{(z+1)-(z)(1)}{(z+1)^2}\right]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz}\left[\frac{1}{(z+1)^2}\right]$$

$$= \frac{1}{2} \lim_{z \to 1}\left[-\frac{2}{(z+1)^3}\right] = \frac{1}{2}\left[-\frac{2}{2^3}\right] = -\frac{1}{8}$$

$$\text{Residue (at } z = -1) = \lim_{z \to -1}\left[(z+1) \cdot \frac{z}{(z+1)(z-1)^3}\right]$$

$$= \lim_{z \to -1}\left[\frac{z}{(z-1)^3}\right] = \frac{-1}{(-1-1)^3} = \frac{1}{8}$$

$$\therefore \text{ Sum of the residues} = -\frac{1}{8} + \frac{1}{8} = 0.$$

Example 3 : Find the sum of the residues at singular points of

$$f(z) = \frac{z}{az^2 + bz + c}$$ **(M.U. 2001)**

Sol. : $f(z)$ is singular at $z = \alpha, \beta$ where α, β are the roots of $az^2 + bz + c = 0$.

$\therefore$ Further $z = \alpha$, $z = \beta$ are simple poles.

Also it should be noted that $az^2 + bz + c = a(z-\alpha)(z-\beta)$.

$$\therefore \text{ Residue (at } z = \alpha) = \lim_{z \to \alpha} (z-\alpha) \cdot \frac{z}{a(z-\alpha)(z-\beta)}$$

$$= \lim_{z \to \alpha} \frac{z}{a(z-\beta)} = \frac{\alpha}{a(\alpha-\beta)}$$

$$\text{Residue (at } z = \beta) = \lim_{z \to \beta} (z-\beta) \cdot \frac{z}{a(z-\alpha)(z-\beta)}$$

$$= \lim_{z \to \beta} \frac{z}{a(z-\alpha)} = \frac{\beta}{a(\beta-\alpha)}$$

$$\therefore \text{ Sum of residues } = \frac{\alpha}{a(\alpha-\beta)} + \frac{\beta}{a(\beta-\alpha)}$$

$$= \frac{1}{a}\left[\frac{\alpha}{\alpha-\beta} - \frac{\beta}{\alpha-\beta}\right] = \frac{1}{a}\left(\frac{\alpha-\beta}{\alpha-\beta}\right) = \frac{1}{a}.$$

EXERCISE - VII

1. Determine the poles of the following and find the residue at each pole.

1. $\frac{3z+1}{z(z-2)}$, 2. $\frac{z+3}{z(z-1)(z+2)}$, 3. $\frac{z+2}{z^2(z-1)}$, 4. $\frac{1-e^{2z}}{z^4}$,

5. $\frac{e^{2z}}{z^2+\pi^2}$, 6. $\frac{z^2-z}{(z+1)^2(z^2+4)}$, **(M.U. 2006)** 7. $\frac{1}{(z^2+1)^3}$, 8. $\frac{z}{\cos z}$,

9. $\frac{1}{(z^2+a^2)^2}$, 10. $\frac{e^z}{(z-1)^3}$, **(M.U. 2004)** 11. $\frac{z^3}{(z-1)(z-2)(z-3)}$,

12. $\frac{z+2}{(z-2)(z+1)^2}$, 13. $\frac{1}{(z^2+1)^2}$, 14. $\frac{1}{z^3+z^5}$ **(M.U. 2005)**

15. e^{1/z^2} 16. $\frac{\sin^2 z}{z^3}$ **(M.U. 2002)**

[**Ans. :** **(1)** $z=0, z=2; -1/2, 7/2$. **(2)** $z=0, 1, -2; -3/2, 4/3, 1/6$. **(3)** $z=0, z=1; -3, 3$.
(4) $z=0; -4/3$. **(5)** $z=\pi i, z=-\pi i; 1/2\pi i, -1/2\pi i$.
(6) $z=-1, z=\pm 2i; -11/25, (11 \pm 2i)/50$. **(7)** $z=i, z=-i; 3/16, 3/16$.
(8) $z=(2n+1)\pi/2; (-1)^{n+1}(2n+1)\pi/2$ where n is any integer.
(9) $z=ai, -ai, 1/(4a^3 i), -1/(4a^3 i)$. **(10)** $z=1; e/2$.
(11) $z=1, z=2, z=3; 1/2, -8, 27/2$. **(12)** $z=2, -1; 4/9, -4/9$.
(13) $z=+i, -i, -i/4, i/4$. **(14)** $(1/2, -1/2, -1)$,
(15) $(0, 0)$, **(16)** $(0, 1/4)$]

2. Find the sum of the residues at singular points of the following functions.

(i) $\frac{z}{2z^2+3z+1}$ (ii) $\frac{z}{3z^2+2z+1}$ (iii) $\frac{z-4}{z(z-1)(z-2)}$

(iv) $\frac{z}{(z-1)(z-2)^2}$ (v) $\frac{z}{z^3+1}$ **(M.U. 2002)**

[**Ans. :** **(i)** 1/2, **(ii)** 1/3, **(iii)** 0, **(iv)** 0, **(v)** 0]

3. Find the residue of the following.

(i) $\frac{e^z}{\sin z}$ at $z=0$, (ii) $\frac{1+e^z}{z\cos z+\sin z}$ at $z=0$, (iii) $\frac{e^z}{z^2+a^2}$ at $z=ai$.

[**Ans. :** **(i)** 1, **(ii)** 1, use Hospital's rule for (i) and (ii). **(iii)** $e^{ai}/2ai$]

4. Find the residues of the following functions at their singularities, using Laurent's series expansion.

(i) $\dfrac{z}{\cos hz - \cos z}$ (ii) $\dfrac{1+z}{1-\cos z}$ (iii) $z^3 \cdot e^{1/z}$ **(M.U. 2006)**

(iv) $\dfrac{1}{z - \sin hz}$ (v) $\dfrac{z^2}{\sin hz - \sin z}$ (vi) $\text{cosec}^2 z$ **(M.U. 2004)**

[Ans. : (i) 1, **(ii)** 2, **(iii)** 1/24, **(iv)** 3/10, **(v)** 3, **(vi)** 0]

5. Determine the nature of poles and find the residue at each pole of

$$\frac{z^2+1}{(z^2-1)(z^2+4)}$$

(M.U. 1997) [Ans. : $z = 1, -1, 2i, -2i;\ 1, -1, -\dfrac{3i}{20}, \dfrac{3i}{20}$.]

6. Prove that the sum of the residues of

(i) $f(z) = \dfrac{e^z}{z^2+a^2}$ is $\dfrac{\sin a}{a}$, (ii) $f(z) = \dfrac{e^{-z}}{z+a^2}$ is $-\dfrac{\sin a}{a}$,

(iii) $f(z) = \dfrac{e^{-z}}{z^2-a^2}$ is $-\dfrac{\sin ha}{a}$.

7. Find the residue of $f(z) = \dfrac{\sin z}{z \cos z}$ at its pole inside the circle $|z| = 2$. **(M.U. 2005)**

[Ans. : 0]

8. Find the residue of $f(z) = z^2 \sin\dfrac{1}{z}$ at $z = 0$. **(M.U. 2000) [Ans. :** $-\dfrac{1}{6}$]

20. Cauchy's Residue Theorem

If $f(z)$ is analytic inside and on a simple closed curve C, except at a finite number of isolated singular points $z_1, z_2 \ldots\ldots z_n$ inside C then

$$\boxed{\oint_C f(z)\,dz = 2\pi i\ [\text{sum of residues at } z_1, z_2 \ldots\ldots z_n]}$$

(M.U. 2001)

Note ....

When integration of $f(z)$ is carried out along a simple closed curve C, it is denoted by putting a small circle on the integration sign $\int$ like this $\oint$. But when integration is carried out along a specified circle or an ellipse or a rectangle which obviously is a simple closed curve we may denote the integral by $\int$ or by $\oint$.

Proof : We draw small circles $C_1, C_2, \ldots\ldots C_n$ with centres at $z_1, z_2 \ldots\ldots z_n$ which lie wholly inside C and which do not intersect mutually. Since, $f(z)$ is now analytic in multiply connected region bounded by C, $C_1, C_2 \ldots\ldots C_n$, we have by Cauchy's Integral Theorem

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \ldots\ldots + \oint_{C_n} f(z)\,dz$$

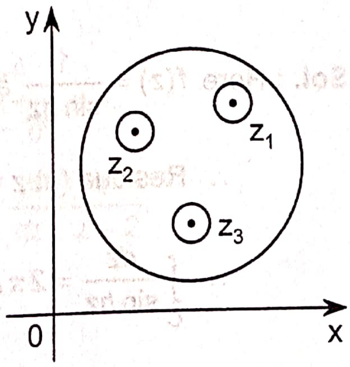


Fig. 2.59

$$\oint_C f(z)\,dz = 2\pi i\,[\text{Res. at } z_1] + 2\pi i\,[\text{Res. at } z_2] + \ldots + 2\pi i\,[\text{Res. at } z_n]$$

$$= 2\pi i\ [\text{Sum of residues at } z_1, z_2, \ldots\ z_n]$$

Note ....

Cauchy's integral formula for $f(z_0)$ that $\oint_C \frac{f(z)}{z-z_0}dz = 2\pi i\, f(z_0)$ and for $f^n(z)$ that $\oint_C \frac{f(z)}{(z-z_0)^n}dz = \frac{2\pi i}{(n-1)!} f^{n-1}(z_0)$ can be deduced from Cauchy's residue theorem.

Proof : Let $f(z)$ be an analytic function on and inside a closed curve C of a simple connected region R and let z_0 be any point within C and consider $\oint_C \frac{f(z)}{z-z_0}dz$.

Now $\frac{f(z)}{z-z_0}$ has a simple pole at $z = z_0$.

Further residue (at $z = z_0$) $= \lim_{z \to z_0} (z - z_0) \cdot \frac{f(z)}{(z-z_0)} = \lim_{z \to z_0} f(z) = f(z_0)$

$\therefore$ By Cauchy's residue theorem

$$\int_C \frac{f(z)}{z-z_0}dz = 2\pi i\,(\text{Residue}) = 2\pi i\, f(z_0)$$

No consider $\int_C \frac{f(z)}{(z-z_0)^n}dz$.

It has a pole of order n at $z = z_0$.

And residue (at $z = z_0$) $= \frac{1}{(n-1)!} \lim_{z \to 0} \frac{d^{n-1}}{dz^{n-1}} (z-z_0)^n \cdot \frac{f(z)}{(z-z_0)^n}$

$$= \frac{1}{(n-1)!} \cdot f^{n-1}(z_0)$$

$\therefore$ By Cauchy's residue theorem

$$\int_C \frac{f(z)}{(z-z_0)^n}dz = 2\pi i\,(\text{Residue}) = 2\pi i \frac{1}{(n-1)!} f^{n-1}(z_0).$$

Example 1 : Evaluate $\int_C \frac{dz}{\sin hz}$ where C is $x^2 + y^2 = 16$. **(M.U. 2003)**

Sol. : Here $f(z) = \frac{1}{\sin hz}$ and C is $|z| = 4$. $\therefore$ $z = 0$ is a simple pole.

$\therefore$ Residue (at $z = 0$) $= \lim_{z \to 0} z \cdot \frac{1}{\sin hz} = \lim_{z \to 0} \frac{z}{\sin hz} = 1.$

$$\therefore \int_C \frac{dz}{\sin hz} = 2\pi i\,(1) = 2\pi i.$$

Example 2 : Evaluate $\int_C \frac{dz}{\cos z}$ where C is $|z| = 2$. **(M.U. 2000)**

Sol. : Singularities of $f(z) = \frac{1}{\cos z}$ are given $\cos z = 0$. $\therefore\ z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\ldots$

Now, the poles lying within $|z| = 2$ are $\pm\pi/2$ which are simple poles.

$$\therefore \text{ Residue (at } z = \pi/2) = \lim_{z \to \pi/2} \left(z - \frac{\pi}{2}\right) \cdot \frac{1}{\cos z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to \pi/2} \frac{1}{-\sin z} = -1 \qquad \text{[By Hospital's rule]}$$

Similarly, residue (at $z = -\pi/2$) $= -(-1) = 1$.

$$\therefore \int_C \frac{dz}{\sin z} = 2\pi i \text{ (Sum of the residues)} = 2\pi(-1+1) = 0.$$

Example 3 : Evaluate $\int_C \frac{dz}{z \sin z}$ where C is $x^2 + y^2 = 1$.

Sol. : Poles are given by $z \sin z = 0$ *i.e.* $z = 0$ and $\sin z = 0$.

$\therefore\ z = 0$ and $z = n\pi$, $n = 0, \pm 1, \pm 2$.

Thus, $z = 0$ is a pole of order 2 in C.

$$\therefore \text{ Residue } f(z) \text{ (at } z = 0) = \lim_{z \to 0} \frac{d}{dz}\left[z^2 \cdot \frac{1}{z \sin z}\right] = \lim_{z \to 0} \frac{d}{dz}\left(\frac{z}{\sin z}\right)$$

$$= \lim_{z \to 0} \frac{\sin z - z \cos z}{\sin^2 z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z \to 0} \frac{\cos z - \cos z + z \sin z}{2 \sin z \cos z} \qquad \text{[By Hospital's Rule]}$$

$$= \lim_{z \to 0} \frac{z}{2 \cos z} = 0$$

$$\therefore \int_C \frac{dz}{z \sin z} = 0.$$

Example 4 : Evaluate $\int_C \frac{z^2}{(z-1)^2(z-2)} dz$ where C is the circle $|z| = 2{\cdot}5$.
(M.U. 1993, 2001, 03, 05)

Sol. : $f(z)$ has a simple pole at $z = 2$ and a pole of order 2 at $z = 1$.

Both poles lie inside C.

$$\therefore \text{ Residue (at } z = 2) = \lim_{z \to 2} (z-2) \cdot \frac{z^2}{(z-1)^2(z-2)} = \lim_{z \to 2} \frac{z^2}{(z-1)^2} = 4.$$

$$\text{Residue (at } z = 1) = \lim_{z \to 1} \frac{d}{dz}(z-1)^2 \cdot \frac{z^2}{(z-1)^2(z-2)} = \lim_{z \to 1} \frac{d}{dz} \cdot \frac{z^2}{z-2}$$

$$= \lim_{z \to 1} \frac{(z-2)2z - z^2}{(z-2)^2} = -3$$

$$\therefore \int_C f(z)\, dz = 2\pi i \text{ (Sum of residues)} = 2\pi i(4-3) = 2\pi i$$

Note ....

This example and also the other examples which follow can also be solved by using Cauchy's Integral Theorem.

Example 5 : Using Cauchy's residue theorem evaluate $\oint_C \dfrac{z^2+3}{z^2-1}\,dz$ where C is the circle (i) $|z-1|=1$, (ii) $|z+1|=1$. **(M.U. 2016)**

Sol. : The poles are given by $z^2-1=0$.

$\therefore\ (z-1)(z+1)=0 \qquad \therefore\ z=1,-1$ are simple poles.

(i) Now $|z-1|=1$ is a circle with centre at $(1, 0)$ and radius 1.

$\therefore\ z=1$ lies inside C and $z=-1$ lies outside C.

$$\therefore\ \text{Residue at } (z=1) = \lim_{z\to 1}(z-1)\cdot\frac{z^2+3}{(z-1)(z+1)} = \lim_{z\to 1}\frac{z^2+3}{z+1} = \frac{4}{2} = 2$$

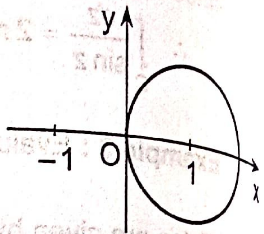


Fig. 2.60

$$\therefore\ \oint_C \frac{z^2+3}{z^2-1}\,dz = 2\pi i(2) = 4\pi i$$

(ii) $|z+1|=1$ is a circle with centre at $(-1, 0)$ and radius 1.

$\therefore\ z=-1$ lies inside C and $z=1$ lies outside C.

$$\therefore\ \text{Residue at } (z=-1) = \lim_{z\to -1}(z+1)\cdot\frac{z^2+3}{(z+1)(z-1)} = \lim_{z\to -1}\frac{z^2+3}{z-1} = \frac{4}{-2} = -2$$

$$\therefore\ \oint_C \frac{z^2+3}{z^2-1}\,dz = 2\pi i(-2) = -4\pi i$$

Example 6 : Evaluate $\int_C \dfrac{z^2}{(z-1)^2(z+1)}\,dz$ where C is (i) $|z|=1/2$, (ii) $|z|=2$.

Sol. : $f(z)$ has a simple pole at $z=-1$ and a pole of order two at $z=1$.

(i) If C is $|z|=1/2$, both poles lie outside C and hence by Cauchy's Theorem

$$\int_C f(z)\,dz = \int_C \frac{z^2}{(z-1)^2(z+1)}\,dz = 0$$

(ii) If C is $|z|=2$, both poles lie inside C.

$$\text{Now, Residue (at } z=-1) = \lim_{z\to -1}(z+1)f(z) = \lim_{z\to -1}\frac{z^2}{(z-1)^2} = \frac{1}{4}.$$

$$\text{Residue (at } z=1) = \lim_{z\to 1}\frac{d}{dz}\left[(z-1)^2 f(z)\right] = \lim_{z\to 1}\frac{d}{dz}\left[\frac{z^2}{z+1}\right]$$

$$= \lim_{z\to 1}\frac{(z+1)2z-z^2}{(z+1)^2} = \lim_{z\to 1}\frac{z^2+2z}{z+1} = \frac{3}{2}.$$

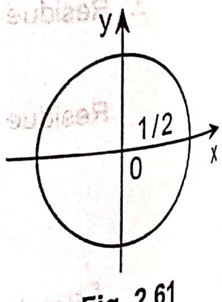


Fig. 2.61

$$\therefore \int_C f(z)\,dz = 2\pi i\,(\text{sum of residues}) = 2\pi i\left[\frac{1}{4}+\frac{3}{2}\right] = \frac{7\pi i}{2}.$$

Example 7 : Evaluate $\oint_C \frac{e^z}{\cos \pi z}\,dz$ where C is the circle $|z| = 1$. **(M.U. 1998, 2002)**

Sol. : $f(z)$ has simple poles when $\cos \pi z = 0$ *i.e.* when $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$..... Of these $z = \pm\frac{1}{2}$ lie within the circle $|z| = 1$.

$$\therefore \text{ Residue }\left(\text{at } z = \frac{1}{2}\right) = \lim_{z\to 1/2} \frac{[z-(1/2)]e^z}{\cos \pi z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z\to 1/2} \frac{[z-(1/2)]\cdot e^z + e^z}{-\pi\cdot \sin \pi z} \qquad [\text{L' Hospital}]$$

$$= -\frac{e^{1/2}}{\pi}$$

Similarly, $\quad \text{Residue}\left(\text{at } z = -\frac{1}{2}\right) = \frac{e^{-1/2}}{\pi}$

$$\therefore \oint_C \frac{e^z}{\cos \pi z}\,dz = 2\pi i\left[-\frac{e^{1/2}}{\pi}+\frac{e^{-1/2}}{\pi}\right] = -4i\left(\frac{e^{1/2}-e^{-1/2}}{2}\right) = -4i\sinh\left(\frac{1}{2}\right)$$

Example 8 : Evaluate $\int \frac{z\sec z}{(1-z^2)}\,dz$ where C is the **(i)** ellipse $4x^2+\frac{9}{4}y^2 = 9$.
(ii) circle $|z| = 2$.

Sol. : (i) $f(z) = \frac{z}{(1-z^2)\cos z}$. It has simple poles at $z = \pm 1$ and $z = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}$...... The ellipse $\frac{x^2}{9/4}+\frac{y^2}{4} = 1$ intersects the x-axis at $x = \pm\frac{3}{2}$ and the y-axis at $y = \pm 2$. Hence, the only poles which lie within the ellipse are ± 1.

$$\therefore \text{ Residue (at } z = 1) = \lim_{z\to 1} (z-1)\frac{z}{(1-z^2)\cos z}$$

$$= \lim_{z\to 1} -\frac{z}{(1+z)\cos z} = -\frac{1}{2\cos(1)}$$

Similarly, $\quad \text{Residue (at } z = -1) = -\frac{1}{2\cos(1)}$

$$\therefore \oint_C \frac{z\sec z}{(1-z^2)}\,dz = 2\pi i\left[-\frac{1}{2\cos(1)}-\frac{1}{2\cos(1)}\right] = -\frac{2\pi i}{\cos(1)}.$$

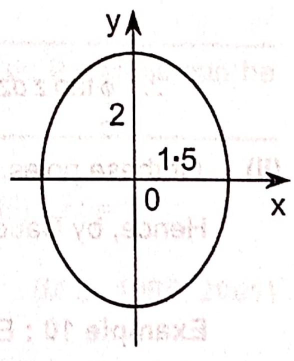


Fig. 2.62

(ii) There are four poles within the circle $|z| = 2$. They are $\pm 1, \pm\frac{\pi}{2}$.

As before the residues at $z = +1$ and $z = -1$ are each $-\frac{1}{2\cos 1}$.

$$\text{Residue}\left(\text{at } z=\frac{\pi}{2}\right)=\lim_{z\to\pi/2}\left(z-\frac{\pi}{2}\right)\cdot\frac{z}{(1-z^2)\cos z}$$

$$=\lim_{t\to 0} t\cdot\frac{(\pi/2+t)}{[1-(\pi/2+t)^2]}\cdot\frac{1}{(-\sin t)} \quad \text{where } z-\frac{\pi}{2}=t$$

$$=-\frac{\pi/2}{1-(\pi/2)^2}=\frac{2}{\pi(\pi^2-2^2)} \qquad \left[\because \lim_{t\to 0}\left(\frac{t}{\sin t}\right)=1\right]$$

$$\text{Similarly, Residue}\left(\text{at } z=-\frac{\pi}{2}\right)=\frac{2}{\pi(\pi^2-2^2)}$$

$$\therefore \int_C \frac{z\sec z}{(1-z)^2}dz = 2\pi i \text{ (Sum of the residues)}$$

$$=2\pi i\left[-\frac{1}{\cos 1}+\frac{4}{\pi(\pi^2-2^2)}\right]=\frac{8i}{\pi^2-2^2}-\frac{2\pi i}{\cos(1)}.$$

Example 9 : Evaluate $\int_C \tan z\, dz$ where C (i) is the circle $|z|=2$, (ii) is the circle $|z|=1$.

(M.U. 1995, 98, 2009)

Sol. : The poles of $\tan z=\dfrac{\sin z}{\cos z}$ are given by $\cos z=0$ *i.e.* $z=\pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}......$

(i) Of these $z=+\pi/2$ and $z=-\pi/2$ lie inside the circle

$$\therefore \text{Residue}\left(\text{at } z=\frac{\pi}{2}\right)=\lim_{z\to\pi/2}\frac{[z-(\pi/2)]\cdot\sin z}{\cos z}$$

$$=\lim_{z\to\pi/2}\frac{[z-(\pi/2)]\cos z+\sin z}{-\sin z}=-1 \qquad \text{[By L' Hospital's Rule]}$$

$$\text{Residue}\left(\text{at } z=-\frac{\pi}{2}\right)=\lim_{z\to-\pi/2}\frac{[z+(\pi/2)]\cdot\sin z}{\cos z}$$

$$=\lim_{z\to-\pi/2}\frac{[z+(\pi/2)]\cos z+\sin z}{-\sin z} \qquad \text{[By L' Hospital's Rule]}$$

$$=-1$$

$$\therefore \oint_C \tan z\,dz=2\pi i(-1-1)=-4\pi i.$$

(ii) Of these poles, none lies within the circle $|z|=1$.

Hence, by Cauchy's Integral Theorem $\oint_C \tan z\,dz=0$.

Example 10 : Evaluate $\oint_C \dfrac{z-1}{z^2+2z+5}dz$ where C is the circle

(i) $|z|=1$, (ii) $|z+1+i|=2$, (iii) $|z+1-i|=2$. **(M.U. 2010)**

Sol. : $(z^2+2z+1)+4=0 \qquad \therefore\ z+1+2i=0,\ z+1-2i=0.$

$\therefore\ z=-1-2i,\ z=-1+2i$ are simple poles.

(i) Since both pole lie outside the circle $|z| = 1$.

By Cauchy's Integral Theorem

$$\oint_C \frac{z-1}{z^2+2z+5}\,dz = 0 \text{ where } C \text{ is } |z| = 1.$$

(ii) The centre C of the circle $|z+1+i| = 2$ is $-1-i$ and radius is 2. If A is $z = -1-2i$ then

$$CA = |(-1-2i)-(-1-i)| = |i| = 1 < 2$$

Hence, A lies inside C the circle.

$$\therefore \text{ Residue (at } z = -1-2i) = \lim_{z\to(-1-2i)} \frac{z-1}{z+1-2i} = \frac{-1-2i-1}{-1-2i+1-2i} = \frac{-2-2i}{-4i} = \frac{1+i}{2i}$$

$$\therefore \oint_C \frac{z-1}{z^2+2z+5}\,dz = 2\pi i\left(\frac{1+i}{2i}\right) = \pi(1+i)$$

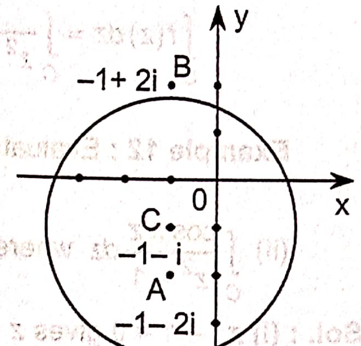


Fig. 2.63

(iii) The centre C of the circle $|z+1-i| = 2$ is $-1+i$ and radius is 2. If B is $z = -1+2i$ then

$$CB = |-1+2i+1-i| = |i| = 1 < 2$$

Hence, B lies inside the circle.

$$\therefore \text{ Residue (at } z = -1+2i) = \lim_{z\to(-1+2i)} \frac{z-1}{z+1+2i} = \frac{-1+2i-1}{-1+2i+1+2i} = \frac{-2+2i}{4i} = \frac{-1+i}{2i}$$

$$\therefore \oint_C \frac{z-1}{z^2+2z+5}\,dz = 2\pi i\left(\frac{-1+i}{2i}\right) = \pi(-1+i)$$

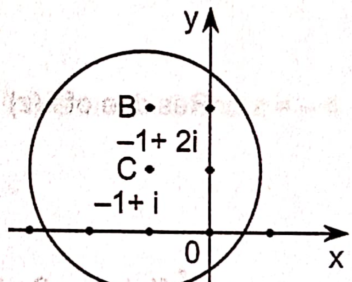


Fig. 2.64

Note ....

Whether a point at which the function is not analytic lies inside or outside the circle can be determined either (i) by calculating the distances or (ii) by drawing a sketch.

Example 11 : Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where C is (i) $|z+1-i| = 2$, (ii) $|z| = 1$.

(M.U. 1995, 2001)

Sol. : $z^2+2z+5 = 0 \quad \therefore \quad z = \frac{-2\pm\sqrt{4-20}}{2}$

$\therefore$ $z = -1 \pm 2i$. Both are simple poles.

(i) The centre C of the circle $|z+1-i| = 2$ is $-1+i$ and radius is 2. Hence, as in Example 9, the point $z = -1+2i$ lies inside the circle

$$\therefore \text{ Residue (at } z = -1+2i) = \lim_{z\to(-1+2i)} \frac{z+4}{z+1+2i}$$

$$\therefore \quad \text{Residue (at } z = -1 + 2i) = \frac{-1+2i+4}{-1+2i+1+2i} = \frac{3+2i}{4i}$$

$$\therefore \quad \oint_C \frac{z+4}{z^2+2z+5}\, dz = 2\pi i \left(\frac{3+2i}{4i}\right) = \frac{\pi(3+2i)}{2}$$

(ii) If C is $|z| = 1$ both the poles lie outside the circle and the function is analytic every where inside C. Hence by Cauchy's Integral Theorem.

$$\int_C f(z)\, dz = \int_C \frac{z-4}{z^2+2z+5}\, dz = 0.$$

Example 12 : Evaluate (i) $\int_C \frac{3z^2+z}{z^2-1}\, dz$ where C is the circle $|z| = 2$.

(ii) $\int_C \frac{\cos \pi z}{z^2-1}\, dz$ where C is the rectangle whose vertices are $2 \pm i, -2 \pm i$. **(M.U. 1998)**

Sol. : (i) $z^2 - 1 = 0$ gives $z = +1, -1$. Hence, both the poles lie within the circle $|z| = 2$.

$$\text{Residue of } f(z) \text{ (at } z = 1) = \lim_{z \to 1} (z-1) \cdot \frac{3z^2+z}{(z-1)(z+1)}$$

$$= \lim_{z \to 1} \frac{3z^2+z}{z+1} = \frac{4}{2} = 2$$

$$\text{Residue of } f(z) \text{ (at } z = -1) = \lim_{z \to -1} (z+1) \cdot \frac{3z^2+z}{(z-1)(z+1)}$$

$$= \lim_{z \to -1} \frac{3z^2+z}{z-1} = \frac{2}{-2} = -1.$$

$$\therefore \quad \int_C f(z)\, dz = 2\pi i \text{ (Sum of the residues)} = 2\pi i (2-1) = 2\pi i$$

(ii) As before the poles are $z = \pm 1$. Both the poles lie within the rectangle.

$$\text{Residue of } f(z) \text{ (at } z = 1) = \lim_{z \to 1} (z-1) \cdot \frac{\cos \pi z}{(z-1)(z+1)}$$

$$= \lim_{z \to 1} \frac{\cos \pi z}{z+1} = \frac{-1}{2}$$

$$\text{Residue of } f(z) \text{ at } (z = -1) = \lim_{z \to -1} (z+1) \cdot \frac{\cos \pi z}{(z+1)(z-1)}$$

$$= \lim_{z \to -1} \frac{\cos \pi z}{z-1} = \frac{-1}{-2} = \frac{1}{2}$$

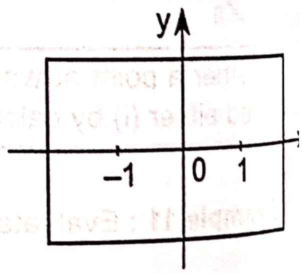


Fig. 2.65

$$\therefore \quad \int_C f(z)\, dz = 2\pi i \text{ (Sum of the residues)} = 2\pi \left(-\frac{1}{2} + \frac{1}{2}\right) = 0.$$

Example 13 : Evaluate $\int_C \frac{2z-1}{z(2z+1)(z+2)}\, dz$ using residue theorem, where C is the circle $|z| = 1$. **(M.U. 2010)**

Sol. : The poles of $f(z)$ are given by $z(2z+1)(z+2) = 0$.

$\therefore$ The poles are $z = 0$, $z = -1/2$, $z = -2$. Of these poles $z = 0$, $z = -1/2$ lie inside C.

$$\text{Residue (at } z = 0) = \lim_{z \to 0} z \cdot \frac{2z-1}{z(2z+1)(z+2)} = \lim_{z \to 0} \frac{2z-1}{(2z+1)(z+2)}$$

$$= \frac{-1}{1 \cdot 2} = -\frac{1}{2}$$

$$\text{Residue (at } z = -1/2) = \lim_{z \to (-1/2)} \left(z + \frac{1}{2}\right) \cdot \frac{2z-1}{z(2z+1)(z+2)}$$

$$= \lim_{z \to (-1/2)} \left(z + \frac{1}{2}\right) \cdot \frac{2z-1}{z \cdot 2[z+(1/2)](z+2)}$$

$$= \frac{-2}{(-1)(2)(3/2)} = \frac{2}{3}.$$

$$\therefore \int_C f(z)\, dz = 2\pi i \text{ (Sum of the residues)}$$

$$= 2\pi i\left(-\frac{1}{2} + \frac{2}{3}\right) = 2\pi i\left(\frac{1}{6}\right) = \frac{\pi i}{3}$$

Example 14 : Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is $|z| = 2$. **(M.U. 1994, 95)**

Sol. : The poles are given by $z^3(z+4) = 0$.

$\therefore$ $z = 0$ is a pole of order 3 and $z = -4$ is a simple pole.

$|z| = 2$ is a circle with centre at the origin and radius 2. Hence, $z = 0$ lies inside C and $z = -4$ lies outside.

$$\text{Residue (at } z = 0) = \lim_{z \to 0} \frac{1}{2!} \cdot \frac{d^2}{dz^2}\left[z^3 \cdot \frac{1}{z^3(z+4)}\right]$$

$$= \lim_{z \to 0} \frac{1}{2} \cdot \frac{d^2}{dz^2}\left(\frac{1}{z+4}\right) = \lim_{z \to 0} \frac{1}{2} \cdot \frac{d}{dz}\left(-\frac{1}{z+4}\right)$$

$$= \lim_{z \to 0} \frac{1}{2} \cdot \frac{2}{(z+4)^3} = \frac{1}{64}$$

$$\therefore \int_C \frac{dz}{z^3(z+4)} = 2\pi i\left(\frac{1}{64}\right) = \frac{\pi i}{32}$$

Example 15 : Evaluate the following using residue theorem.

(i) $\int_C \frac{dz}{4z^2+1}$ where C is $|z| = 1$. **(M.U. 1993)**

(ii) $\int_C \frac{(z+4)^2}{z^4+5z^3+6z^2}\, dz$ where C is $|z| = 1$. **(M.U. 1993, 2003)**

(iii) $\int_C \operatorname{cosec} z\, dz$ where C is $|z| = 1$. **(M.U. 1992, 2003)**

Sol. : (i) We have $f(z) = \frac{1}{4z^2+1}$ $\quad\therefore\quad f(z) = \frac{1}{(2z+i)(2z-i)}$

$\therefore$ The poles of $f(z)$ are $z = -i/2, i/2$

$\therefore$ Residue of $f(z)$ (at $z = -i/2$) $= \lim\limits_{z \to -i/2} [z + (i/2)] \cdot \dfrac{1}{(2z+i)(2z-i)}$

$$= \lim_{z \to -i/2} \frac{(2z+i)}{2} \cdot \frac{1}{(2z+i)(2z-i)}$$

$$= \frac{1}{2 \cdot (-i-i)} = -\frac{1}{4i}$$

$\therefore$ Residue of $f(z)$ (at $z = i/2$) $= \lim\limits_{z \to i/2} [z - (i/2)] \cdot \dfrac{1}{(2z+i)(2z-i)}$

$$= \lim_{z \to i/2} \frac{(2z-i)}{2} \cdot \frac{1}{(2z+i)(2z-i)}$$

$$= \frac{1}{(i+i) \cdot 2} = \frac{1}{4i}$$

$\therefore$ $\oint\limits_C f(z)\,dz = 2\pi i$ (Sum of the residues) $= 2\pi i \left[-\dfrac{1}{4i} + \dfrac{1}{4i}\right] = 0$

(ii) The poles of $f(z)$ are given by $z^2(z^2 + 5z + 6) = 0$

i.e. $z^2(z+2)(z+3) = 0$

$\therefore$ The poles are $z = 0, z = -2, z = -3$.

The last two poles lie outside the circle $|z| = 1$ and $z = 0$ is a pole of order 2.

$\therefore$ Residue of $f(z)$ (at $z = 0$) $= \lim\limits_{z \to 0} \dfrac{d}{dz} z^2 \cdot f(z)$

$$= \lim_{z \to 0} \frac{d}{dz} z^2 \cdot \frac{(z+4)^2}{z^2(z+2)(z+3)}$$

$$= \lim_{z \to 0} \frac{d}{dz} \cdot \frac{(z+4)^2}{z^2 + 5z + 6}$$

$$= \lim_{z \to 0} \frac{(z^2+5z+6)2 \cdot (z+4) - (z+4)^2(2z+5)}{(z^2+5z+6)^2}$$

$$= \frac{6 \cdot 2 \cdot 4 - 16 \cdot 5}{36} = -\frac{32}{36} = -\frac{8}{9}$$

$\therefore$ $\oint f(z)\,dz = 2\pi i$ (Sum of the residues) $= 2\pi i \left(-\dfrac{8}{9}\right) = -\dfrac{16\pi i}{9}$

(iii) $\operatorname{cosec} z = \dfrac{1}{\sin z}$

$\sin z = 0$ when $z = 0, \pm\pi, \pm 2\pi, \ldots$

Of these poles $z = 0$ lies within the circle $|z| = 1$.

$\therefore$ $z = 0$ is a simple pole.

$\therefore$ Residue of $f(z)$ (at $z = 0$) $= \lim\limits_{z \to 0} z f(z)$

$$= \lim_{z \to 0} z \cdot \frac{1}{\sin z}$$

$\therefore$ $\oint\limits_C f(z)\,dz = 2\pi i$ (Sum of the residues) $= 2\pi i(1) = 2\pi i$.

Example 16 : Using residue theorem evaluate $\oint_C \dfrac{e^z}{(z^2+\pi^2)^2}\,dz$ where C is $|z| = 4$.

(M.U. 2009, 16, 18)

Sol. : The poles of $f(z)$ are given by

$$(z^2+\pi^2) = 0 \qquad \therefore \ (z-\pi i)(z+\pi i) = 0$$

$\therefore$ $z = \pi i, -\pi i$ are the poles of order 2.

$$\text{Residue (at } z = \pi i) = \frac{1}{1!}\lim_{z\to\pi i}\frac{d}{dz}\left[(z-\pi i)^2\cdot\frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right]$$

$$= \lim_{z\to\pi i}\frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right] = \lim_{z\to\pi i}\left[\frac{(z+\pi i)^2\cdot e^z - e^z\cdot(z+\pi i)\cdot 2}{(z+\pi i)^4}\right]$$

$$= \lim_{z\to\pi i}\frac{e^z(z+\pi i-2)}{(z+\pi i)^3} = e^{\pi i}\cdot\frac{2\cdot(\pi i-1)}{(2\pi i)^3}$$

$$= e^{\pi i}\cdot\frac{2i(\pi+i)}{-8\pi^3 i} = -\frac{e^{\pi i}(\pi+i)}{4\pi^3}$$

$$= \frac{\pi+i}{4\pi^3} \qquad \left[\because e^{\pi i} = \cos\pi + i\sin\pi = -1\right]$$

$$\text{Residue (at } z = -\pi i) = \frac{1}{1!}\lim_{z\to-\pi i}\frac{d}{dz}\left[(z+\pi i)^2\cdot\frac{e^z}{(z+\pi i)^2(z-\pi i)^2}\right]$$

$$= \lim_{z\to-\pi i}\frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right] = \lim_{z\to-\pi i}\left[\frac{(z-\pi i)^3 e^z - e^z(z-\pi i)^2}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to-\pi i}\left[\frac{e^z(z-\pi i-2)}{(z-\pi i)^3}\right] = e^{-\pi i}\,\frac{(-2\pi i-2)}{(-2\pi i)^3} = e^{-\pi i}\,\frac{(-2i)(\pi-i)}{8\pi^3 i}$$

$$= e^{-\pi i}\cdot\frac{(\pi-i)}{4\pi^3} = \frac{\pi-i}{4\pi^3}$$

$$\therefore \oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i\,(\text{Sum of the residues})$$

$$= 2\pi i\left[\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right] = \frac{4\pi^2 i}{4\pi^3} = \frac{i}{\pi}.$$

Example 17 : Using the residue theorem evaluate

$$\oint_C \frac{\cos\pi z^2 + \sin\pi z^2}{z-z^2}\,dz \quad \text{where } C \text{ is } |z-2| = 4.$$

(M.U. 2003)

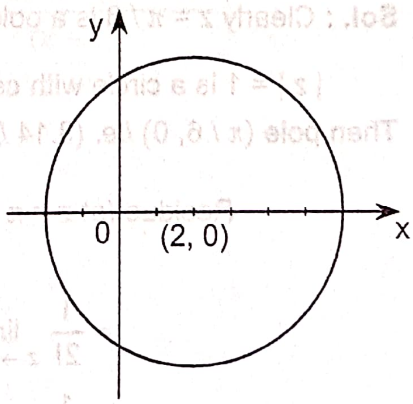


Fig. 2.66

Sol. : The poles are given by $z - z^2 = 0$ *i.e.* $z(1-z) = 0$

$\therefore$ $z = 0, 1$.

$|z-2| = 4$ is a circle with center at (2, 0) and radius 4.

Hence, both the poles lie inside C.

$$\text{Residue (at } z = 0) = \lim_{z \to 0} (z-0) \cdot \frac{\cos \pi z^2 + \sin \pi z^2}{(z-0)(1-z)}$$

$$= \lim_{z \to 0} \frac{\cos \pi z^2 + \sin \pi z^2}{1-z} = 1$$

$$\text{Residue (at } z = 1) = \lim_{z \to 1} (z-1) \cdot \frac{\cos \pi z^2 + \sin \pi z^2}{z(1-z)}$$

$$= \lim_{z \to 1} \frac{\cos \pi z^2 + \sin \pi z^2}{(-1)z} = 1$$

$$\therefore \oint_C f(z)\, dz = 2\pi i \text{ (Sum of the residues)} = 2\pi i (1+1) = 4\pi i.$$

Example 18 : Using residue theorem evaluate $\oint_C \frac{e^{2z}}{(z-\pi i)^3} dz$ where C is $|z-2i|=2.$

(M.U. 2

Sol. : Clearly $z = \pi i$ is a pole of order 3.

$|z-2i| = 4$ is a circle with centre at (0, 2) and radius 4.
Hence, the pole $(0, \pi)$ lies inside C.

$$\text{Residue (at } z = \pi i) = \lim_{z \to \pi i} \frac{1}{2!} \frac{d^2}{dz^2} \left[(z-\pi i)^3 \cdot \frac{e^{2z}}{(z-\pi i)^3} \right]$$

$$= \lim_{z \to \pi i} \frac{1}{2!} \frac{d^2}{dz^2} (e^{2z})$$

$$= \lim_{z \to \pi i} \frac{1}{2} \cdot \frac{d}{dz} (2 e^{2z})$$

$$= \lim_{z \to \pi i} \frac{1}{2} \cdot 4 e^{2z} = 2 e^{2\pi i}.$$

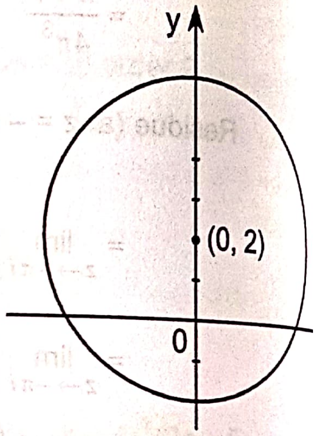


Fig. 2.67

$$\therefore \oint_C f(z)\, dz = 2\pi i (2 e^{2\pi i}) = 4\pi i (\cos 2\pi + i \sin 2\pi) = 4\pi i.$$

Example 19 : By using residue theorem evaluate $\oint_C \frac{\sin^6 z}{(z-\pi/6)^3} dz$ where C is $|z|=1.$

(M.U. 2000

Sol. : Clearly $z = \pi/6$ is a pole of order 3.

$|z| = 1$ is a circle with center at (0, 0) and radius 1.
Then pole $(\pi/6, 0)$ *i.e.* (3.14 / 6, 0) lies inside C.

$$\text{Residue (at } z = \pi/6) = \frac{1}{2!} \lim_{z \to \pi/6} \frac{d^2}{dz^2} \left[(z-\pi/6)^3 \cdot \frac{\sin^6 z}{(z-\pi/6)^3} \right]$$

$$= \frac{1}{2!} \lim_{z \to \pi/6} \frac{d^2}{dz^2} (\sin^6 z) = \frac{1}{2!} \lim_{z \to \pi/6} \frac{d}{dz} (6 \sin^5 z \cdot \cos z)$$

$$= \frac{1}{2!} \lim_{z \to \pi/6} \left[6 \cdot 5 \sin^4 z \cos^2 z - 6 \sin^5 z \sin z \right]$$

$$\text{Residue (at } z=\pi/6) = \frac{1}{2}\left[30\cdot\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2 - 6\left(\frac{1}{2}\right)^6\right]$$

$$= \frac{1}{2}\left[30\cdot\frac{1}{16}\cdot\frac{3}{4} - 6\cdot\frac{1}{64}\right] = \frac{1}{2}\cdot\frac{84}{64} = \frac{21}{32}$$

$$\therefore \oint_C f(z)\,dz = 2\pi i\left(\frac{21}{32}\right) = \frac{21}{16}\pi i.$$

Example 20 : Using residue theorem evaluate $\oint_C e^{-1/z}\sin\left(\frac{1}{z}\right)dz$ where C is $|z| = 1$.

(M.U. 1997)

Sol. : We expand $f(z)$ as Laurent's series.

$$e^{-1/z}\cdot\sin\left(\frac{1}{z}\right) = \left[1 - \frac{1}{z} + \frac{1}{2z^2} - \ldots\right]\left[\frac{1}{z} - \frac{1}{6z^3} + \ldots\right]$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{3z^3} - \ldots$$

$z = 0$ is an isolated essential singularity and

Residue (at $z = 0$) $= b_1 =$ coefficient of $1/z = 1$

$$\therefore \int_C e^{-1/z}\sin\left(\frac{1}{z}\right)dz = 2\pi i\,(\text{Residue}) = 2\pi i\,(1) = 2\pi i$$

Example 21 : Evaluate $\int_C \frac{z-1}{(z+1)^2(z-2)}dz$ where C encloses both poles of $f(z)$.

(M.U. 2006, 19)

Sol. : Clearly $z = -1$ is a pole of order 2 and $z = 2$ is a simple pole.

$$\therefore \text{Residue (at } z=2) = \lim_{z\to 2}\left[\frac{(z-2)\cdot(z-1)}{(z+1)^2(z-2)}\right] = \lim_{z\to 2}\frac{z-1}{(z+1)^2} = \frac{1}{9}.$$

$$\text{Residue (at } z=-1) = \lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\cdot\frac{z-1}{(z+1)^2(z-2)}\right]$$

$$= \lim_{z\to -1}\frac{d}{dz}\left(\frac{z-1}{z-2}\right) = \lim_{z\to -1}\left[\frac{(z-2)\cdot 1 - (z-1)\cdot 1}{(z-2)^2}\right]$$

$$= \lim_{z\to -1}\left[-\frac{1}{(z-2)^2}\right] = \lim_{z\to -1} -\frac{1}{9}.$$

$$\therefore \int_C f(z)\,dz = 2\pi i\ (\text{Sum of the residues})$$

$$= 2\pi i\left(\frac{1}{9} - \frac{1}{9}\right) = 0.$$

Example 22 : If $f(z) = \dfrac{\Phi(z)}{\Psi(z)}$, where $\Phi(z)$ and $\Psi(z)$ are complex polynomials of degree 2 has (i) pole of order 2 at $z = 1$. (ii) residue at $z = 2$ is -1. (iii) $f(0) = f(-1) = 0$, find $f(z)$.

Sol. : Since $z = 1$ is a single pole of order 2, we assume the required function $f(z)$ as

$$f(z) = \frac{az^2 + bz + c}{(z-1)^2}$$

Since $f(0) = 0$, $c = 0$.

Since $f(-1) = 0$, $a - b = 0 \quad \therefore\ a = b$.

$$\therefore\ f(z) = \frac{az^2 + az}{(z-1)^2}$$

$$\text{Now, residue (at } z = 1) = \frac{1}{1!} \lim_{z \to 1} \frac{d}{dz}\left[(z-1)^2 \cdot \frac{az^2 + az}{(z-1)^2}\right]$$

$$= \lim_{z \to 1} \frac{d}{dz}(az^2 + az)$$

$$= \lim_{z \to 1} 2az + a = 2a + a = 3a$$

But this is $-1 \quad \therefore\ 3a = -1 \quad \therefore\ a = -1/3$

$\therefore\ b = a = -1/3$ and $c = 0$

$$\therefore\ f(z) = -\frac{1}{3} \cdot \frac{z^2 + z}{(z-1)^2}$$

Example 23 : Construct a function $f(z) = \dfrac{\Phi(z)}{\Psi(z)}$ which is analytic except at four distinct points z_1, z_2, z_3 and z_4 where it has (i) a simple pole at z_1, (ii) simple zeroes at z_2, z_3, z_4, (iii) simple pole at $z = 0$, (iv) $\lim\limits_{|z| \to \infty} \dfrac{f(z)}{z} = 2$. **(M.U. 199**

Sol. : **(i)** Since $z = z_1$ is a simple pole $\Psi(z)$ must contain the factor $z - z_1$.

(ii) Since z_2, z_3, z_4 are simple zeros, $\Phi(z)$ must contain the factor $(z - z_2)(z - z_3)(z - z_4)$.

(iii) Since $f(z)$ has a simple pole at zero, $\Psi(z)$ must contain a factor z.

(iv) Since $\lim\limits_{|z| \to \infty} \dfrac{f(z)}{z} = 2$ we introduce a constant k (to be determined) in $f(z)$ and consid

$$f(z) = k\frac{(z - z_2)(z - z_3)(z - z_4)}{z(z - z_1)}$$

Since $\lim\limits_{|z| \to \infty} \dfrac{f(z)}{z} = 2$, we have $\lim\limits_{|z| \to \infty} \dfrac{k(z - z_2)(z - z_3)(z - z_4)}{z^2(z - z_1)} = 2$.

$$\therefore\ \lim_{|z| \to \infty} \frac{k(1 - z_2/z)(1 - z_3/z)(1 - z_4/z)}{(1 - z_1/z)} = 2 \quad \therefore\ k = 2$$

$$\therefore\ f(z) = \frac{2(z - z_2)(z - z_3)(z - z_4)}{z(z - z_1)}$$

Example 24 : If $f(z) = \dfrac{\Phi(z)}{\Psi(z)}$, where $\Phi(z)$ and $\Psi(z)$ are complex polynomials of degree 3 has

(i) poles of order 1 and 2 at $z = 2$, $z = 1$ respectively.

(ii) residues at 2 and 1 equal to 3 and 1 respectively.

(iii) $f(0) = 3/2$, $f(-1) = 1$, find $f(z)$.

Sol. : Since $z = 1$ and $z = 2$ are the poles of order 2 and 1, we assume the required function $f(z)$ as

$$f(z) = \frac{az^3 + bz^2 + cz + d}{(z-2)(z-1)^2}$$

$$\text{Now, Residue (at } z = 2) = \lim_{z\to 2} \frac{(z-2)(az^3 + bz^2 + cz + d)}{(z-2)(z-1)^2}$$

$$= \lim_{z\to 2} \frac{az^3 + bz^2 + cz + d}{(z-1)^2}$$

$$3 = 8a + 4b + 2c + d$$

$$\text{Residue (at } z = 1) = \lim_{z\to 1} \frac{d}{dz}(z-1)^2 \cdot \frac{(az^3 + bz^2 + cz + d)}{(z-1)^2(z-2)}$$

$$1 = \lim_{z\to 1} \frac{d}{dz}\left\{\frac{az^3 + bz^2 + cz + d}{z-2}\right\}$$

$$= \lim_{z\to 1} \frac{(z-2)(3az^2 + 2bz + c) - (az^3 + bz^2 + cz + d)}{(z-2)^2}$$

$$1 = -4a - 3b - 2c - d$$

$$f(0) = \frac{d}{-2} = \frac{3}{2} \qquad \therefore\ d = -3$$

$$f(-1) = \frac{-a + b - c + d}{-12} = 1 \quad \therefore\ \frac{a - b + c - d}{12} = 1$$

$\therefore$ $8a + 4b + 2c = 6$ and $4a + 3b + 2c = 2$ and $a - b + c = 9$.

Solving these simultaneous equations, we get $a = 2$, $b = -4$, $c = 3$.

$$\therefore\ f(z) = \frac{2z^3 - 4z^2 + 3z - 3}{(z-2)(z-1)^2}.$$

EXERCISE - VIII

Using Cauchy's residue theorem evaluate the following.

1. $\oint_C e^{1/z^2} dz$ where C is the circle $|z| = 1$. [**Ans. :** 0]

2. $\oint_C \dfrac{\sin z}{z^6} dz$ where C is the circle $|z| = 1$. [**Ans. :** $\dfrac{\pi i}{60}$]

3. $\oint_C z\, e^{1/z} dz$ where C is the circle $|z| = 1$. [**Ans. :** πi]

4. $\oint_C z^4 e^{1/z} dz$ where C is $|z| = 1$. (M.U. 2000) [Ans. : $\frac{\pi i}{60}$

(**Hint :** In Ex. 2, 3, 4, expand $\sin z$, $e^{1/z}$ to find the residue *i.e.* b_1.)

5. $\oint_C \frac{z}{(z-1)^2(z+1)} dz$ where C is the circle (i) $|z| = \frac{5}{4}$, (ii) $|z| = \frac{3}{2}$. [Ans. : (i) 0, (ii) 0

6. $\oint_C \frac{z-2}{z^2-z} dz$ where C is the ellipse $4x^2 + 9y^2 = 36$. [Ans. : $2\pi i$

7. $\oint_C \frac{e^z}{(z+1)^2} dz$ where C is the circle $(x-1)^2 + y^2 = 3^2$. [Ans. : $\frac{2\pi i}{e}$

8. $\oint_C \frac{z}{\cos z}$ where C is $|z + (\pi/2)| = \pi/2$. [Ans. : $-\pi^2 i$

9. $\oint_C \frac{1}{(z-a)^m}$ where C is the circle $|z - a| = a$ and m is any positive integer.

[Ans. : If $m = 1$, $I = 2\pi i$; if $m \neq 1$, $I = 0$

10. $\oint_C \frac{z}{(z+1)^2(z-2)} dz$ where C is the circle $|z - i| = 2$. [Ans. : $-\frac{4\pi i}{9}$

11. $\oint_C \frac{dz}{(z^2+1)^2}$ where C is the circle $|z - i| = 1$. [Ans. : $\frac{\pi}{2}$

12. $\oint_C \frac{1-2z}{z(z-1)(z-2)} dz$ where C is the circle $|z| = 1{\cdot}5$. [Ans. : $3\pi i$

13. $\oint_C \frac{dz}{4z^2+1}$ where C is the circle $|z| = 1$. (M.U. 1993) [Ans. : 0

14. $\oint_C \frac{(z+4)^2}{z^4+5z^3+6z^2} dz$ where C is the circle $|z| = 1$. (M.U. 1993) [Ans. : $\frac{-16\pi i}{9}$

15. $\oint_C \frac{3z^2+2z-4}{z^3-4z} dz$ where C is the circle $|z - i| = 3$. (M.U. 1995) [Ans. : $6\pi i$

16. $\oint_C \frac{4z^2+1}{(2z-3)(z+1)^2}$ where C is the circle $|z| = 4$. (M.U. 2003) [Ans. : $4\pi i$

17. $\oint_C \frac{z^2+4}{(z-2)(z+3i)} dz$ where C is (i) $|z + 1| = 2$, (ii) $|z - 2| = 2$. (M.U. 2006

[Ans. : (i) 0, (ii) $16\pi i/(2+3i)$

18. $\oint_C \frac{4z-1}{z^2-3z-4}$ where C is the ellipse $x^2 + 4y^2 = 4$. (M.U. 2003) [Ans. : $2\pi i$

19. $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{z^2+3z+2}$ where C is (i) $|z| = 0{\cdot}5$, (ii) $|z| = 1{\cdot}5$. (M.U. 1998, 2003

[Ans. : (i) 0, (ii) $-2\pi i$

20. $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz$ where C is (i) $|z| = 5$. (M.U. 1998) [Ans. : $2\pi i$]

21. $\oint_C \frac{z+3}{2z^2+3z-2}\,dz$ where C is the circle with centre $(0, 1)$ and radius 2. (M.U. 1998)
(Hint : C is $|z-i| = 2$) [Ans. : $7\pi i/5$]

22. $\oint_C \operatorname{cosec} z\,dz$ where C is the circle $|z| = 1$. (M.U. 1992) [Ans. : $2\pi i$]

23. $\oint_C \frac{z-1}{(z+1)^2(z-2)}\,dz$ where C is the circle $|z-i| = 2$. (M.U. 1994, 98, 2005) [Ans. : $-2\pi i/9$]

24. $\oint_C \frac{ze^{2z}}{(z-1)^3}\,dz$ where C is $|z+i| = 2$. (M.U. 1996) [Ans. : $8\pi i e^2$]

25. $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\,dz$ where C is the circle $|z| = 3$. (M.U. 1991, 95) [Ans. : $4\pi i$]

26. $\oint_C \frac{dz}{z^2+1}$ where C is the circle $x^2 + y^2 = 4$. [Ans. : 0]

27. $\oint_C \frac{12z-7}{(z-1)^2(2z+3)}\,dz$ where C is the circle (i) $|z| = 1/2$, (ii) $|z| = 2$, (iii) $|z+i| = \sqrt{3}$.
(M.U. 2009) [Ans. : (i) 0, (ii) 0, (iii) $4\pi i$]

28. $\oint_C \frac{e^z}{\cos \pi z}\,dz$ where C is the circle $|z| = 1$. (M.U. 1998) [Ans. : $-4\pi \sin h\left(\frac{1}{2}\right)$]

29. $\oint_C \frac{2z-1}{z(z+1)(z-3)}\,dz$ where C is the circle $x^2 + y^2 = 4$. [Ans. : $-5\pi i/6$]

30. $\oint_C \frac{z-3}{z^2+2z+5}\,dz$ where C is the circle (i) $x^2 + y^2 = 1$, (ii) $|z+1+i| = 2$, (iii) $|z+1-i| = 2$.
(M.U. 1993) [Ans. : (i) 0, (ii) $\pi(2+i)$, (iii) $\pi(-2+i)$]

31. $\oint_C \frac{z^2+3}{z^2-1}$ where C is the circle (i) $|z-1| = 1$, (ii) $|z+1| = 1$. (M.U. 2016)
[Ans. : (i) $4\pi i$, (ii) $-4\pi i$]

32. $\oint_C \frac{15z+9}{z^3-9z}\,dz$ where C is $|z-1| = 3$. (M.U. 2004) [Ans. : $4\pi i$]

21. Applications of Residues

Cauchy's residue theorem can be used to evaluate certain types of definite integrals of real variables by using suitable contours - a circle or a semi-circle and its diameter.

Integral of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

If f is a rational function of $\sin\theta$, $\cos\theta$ then we put $z = e^{i\theta}$. Then, we get,

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2+1}{2z}$$

$$\sin\theta = \frac{e^{i\theta} - e^{i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2-1}{2iz}$$

$$dz = i e^{i\theta}\, d\theta = i z\, d\theta \qquad \therefore d\theta = \frac{dz}{iz}.$$

Now, as θ changes from 0 to 2π it is clear that $z = e^{i\theta} = \cos\theta + i\sin\theta$ moves around t circle $|z| = 1$.

The new integral z can be evaluated by using Cauchy's residue theorem. **Because this integ is equal to $2\pi i$ (sum of the residues) at poles in $|z| = 1$.**

Example 1 : Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 3\sin\theta}$. **(M.U. 2004, 09, 15, 1**

Sol. : Let $e^{i\theta} = z$ $\therefore e^{i\theta} \cdot i\, d\theta = dz$; $d\theta = \frac{dz}{iz}$ and $\sin\theta = \frac{z^2-1}{2iz}$

$$I = \int_C \frac{1}{5 + 3\left(\frac{z^2-1}{2iz}\right)} \cdot \frac{dz}{iz} = \int_C \frac{2}{3z^2 + 10iz - 3}\, dz$$

$$= \int_C \frac{2}{(3z+i)(z+3i)}\, dz \text{ where } C \text{ is the circle } |z| = 1.$$

Now, the poles of $f(z)$ are given by $(3z + i)(z + 3i) = 0$ $\therefore$ $z = -(i/3)$ and $z = -3i$ are simp poles. But $z = -(i/3)$ lies inside and $z = -3i$ lies outside the circle $|z| = 1$.

$$\text{Residue}\left(\text{at } z = -\frac{i}{3}\right) = \lim_{z \to -\frac{i}{3}} [z + (i/3)] \cdot \frac{2}{(3z+i)(z+3i)}$$

$$= \lim_{z \to -\frac{i}{3}} \frac{2}{3(z+3i)} = \frac{1}{4i}$$

$$\therefore \quad I = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}.$$

Example 2 : Evaluate $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$, $a > b$. **(M.U. 200**

Sol. : Let $z = e^{i\theta}$ $\therefore dz = i e^{i\theta}\, d\theta$

$$\therefore dz = i \cdot z \cdot d\theta \qquad \therefore d\theta = \frac{dz}{iz} \qquad \therefore \sin\theta = \frac{z - (1/z)}{2i}$$

$$\therefore \; I = \int_C \frac{1}{a + b\left[\dfrac{z-(1/z)}{2i}\right]} \cdot \frac{dz}{iz} = \int_C \frac{1}{a + b\left[\dfrac{z^2-1}{2iz}\right]} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{2}{bz^2 + 2aiz - b} dz$$

The roots of $bz^2 + 2aiz - b = 0$ are

$$z = \frac{-2ai \pm \sqrt{-4a^2 + 4b^2}}{2b} = -\frac{-ai \pm \sqrt{a^2 - b^2} \cdot i}{b} \quad [\because a > b]$$

$$= \left[\frac{-a \pm \sqrt{a^2 - b^2}}{b}\right] i$$

Let $\alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b} \cdot i, \; \beta = \dfrac{-a - \sqrt{a^2 - b^2}}{b} \cdot i$

Now, $\alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b} \cdot i \cdot \dfrac{-a - \sqrt{a^2 - b^2}}{-a - \sqrt{a^2 - b^2}}$

$$= \frac{a^2 - (a^2 - b^2)}{-b(a + \sqrt{a^2 - b^2})} \cdot i = \frac{-b}{a + \sqrt{a^2 - b^2}} \cdot i$$

$$\therefore \; |\alpha| = \left|\frac{-b \cdot i}{a + \sqrt{a^2 - b^2}}\right| = \frac{b}{a + \sqrt{a^2 - b^2}}$$

Since $a > b$, $|\alpha| < 1$. $\therefore$ α lies inside $|z| = 1$.

By the same reasing $\beta > 1$ and hence, β lies outside C.

$$\therefore \; \text{Residue of } f(z) \text{ (at } z = \alpha) = \lim_{z \to \alpha} (z - \alpha) \cdot \frac{2}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z \to \alpha} \frac{2}{z - \beta} = \frac{2}{\alpha - \beta}$$

Now, $\alpha - \beta = \dfrac{-a + \sqrt{a^2 - b^2}}{b} \cdot i - \dfrac{-a - \sqrt{a^2 - b^2}}{b} \cdot i = \dfrac{2\sqrt{a^2 - b^2}}{b} \cdot i$

$$\therefore \; \text{Residue of } f(z) \text{ (at } z = \alpha) = \frac{2b}{2\sqrt{a^2 - b^2} \cdot i} = \frac{b}{\sqrt{a^2 - b^2}} \cdot i$$

$$\therefore \; I = 2\pi i \,(\text{Residue}) = 2\pi i \cdot \frac{1}{\sqrt{a^2 - b^2} \cdot i} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

Example 3 : Using residues evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{(2 + \cos\theta)^2}$. **(M.U. 1997, 2004, 15)**

Sol. : Let $e^{i\theta} = z$, $e^{i\theta} \cdot i \, d\theta = dz$ $\quad \therefore \; d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \dfrac{z + z^{-1}}{2}$.

$$\therefore \; I = \int_C \frac{1}{\left(2 + \frac{z^2+1}{2z}\right)^2} \cdot \frac{dz}{iz} = \frac{4}{i} \cdot \int_C \frac{z\,dz}{(z^2+4z+1)^2}$$ where C is the circle $|z| = 1$.

Now the poles of $f(z)$ are given by $z^2 + 4z + 1 = 0$.

$$\therefore \; z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

Let $\alpha = -2 - \sqrt{3}$ and $\beta = -2 + \sqrt{3}$.

Both are the poles of order 2. But the pole α lies outside the circle and the pole β lies inside t circle. ($\because \sqrt{3} = 1{\cdot}73$)

Since, $f(z) = \dfrac{z}{(z^2+4z+1)^2} = \dfrac{z}{(z-\alpha)^2(z-\beta)^2}$

$$\therefore \text{ Residue (at } z = \beta) = \lim_{z \to \beta} \frac{1}{1!} \frac{d}{dz}\left[(z-\beta)^2 \cdot \frac{z}{(z-\alpha)^2(z-\beta)^2}\right]$$

$$= \lim_{z \to \beta} \frac{d}{dz}\left[\frac{z}{(z-\alpha)^2}\right] = \lim_{z \to \beta}\left[\frac{(z-\alpha)^2 \cdot 1 - z \cdot 2(z-\alpha)}{(z-\alpha)^4}\right]$$

$$= \lim_{z \to \beta} \frac{-z-\alpha}{(z-\alpha)^3} = -\frac{\beta+\alpha}{(\beta-\alpha)^3}$$

But $\beta + \alpha = -4$ and $\beta - \alpha = 2\sqrt{3}$

$$\therefore \text{ Residue (at } z = \beta) = \frac{4}{24\sqrt{3}} = \frac{1}{6\sqrt{3}}$$

$$\therefore \; I = 2\pi i \cdot \frac{4}{i} \cdot \frac{1}{6\sqrt{3}} = \frac{4\pi}{3\sqrt{3}}$$

Example 4 : From the integral $\displaystyle\int_C \frac{dz}{z+2}$ where C denotes the circle $|z| = 1$, deduce that

$$\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0.$$ **(M.U. 199**

Sol. : Let $z = e^{i\theta} = \cos\theta + i\sin\theta \;\; \therefore dz = (-\sin\theta + i\cos\theta)\,d\theta$.

$$\therefore \int_C \frac{dz}{z+2} = \int_0^{2\pi} \frac{(-\sin\theta + i\cos\theta)}{(\cos\theta + i\sin\theta) + 2}\, d\theta$$

$$= \int_0^{2\pi} \frac{(-\sin\theta + i\cos\theta)}{[(\cos\theta+2) + i\sin\theta]} \cdot \frac{[(\cos\theta+2) - i\sin\theta]}{[(\cos\theta+2) - i\sin\theta]}\, d\theta$$

$$= \int_0^{2\pi} \frac{(-\sin\theta\cos\theta + i\cos^2\theta - 2\sin\theta + 2i\cos\theta + i\sin^2\theta + \sin\theta\cos\theta)}{\cos^2\theta + 4\cos\theta + 4 + \sin^2\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{-2\sin\theta + i(1 + 2\cos\theta)}{5 + 4\cos\theta}\, d\theta$$

Now, by Cauchy's Integral Theorem $\int_C \frac{dz}{z+2} = 0$ because $f(z) = \frac{1}{z+2}$ is analytic inside and on the circle $|z| = 1$.

$$\therefore \int_0^{2\pi} \frac{-2\sin\theta + i(1+2\cos\theta)}{5+4\cos\theta}\, d\theta = 0$$

Equating to zero (real and) imaginary parts,

$$\therefore \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0$$

Now, $\int_0^a f(x)\,dx = 2\int_0^{\pi/2} f(x)$ if $f(a-x) = f(x)$.

And since $\cos(2\pi - \theta) = \cos\theta$, we get $\int_0^{\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\, d\theta = 0.$

Example 5 : Evaluate $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$ where $a > b > 0$. **(M.U. 1996, 2006, 19)**

Sol. : Let $z = e^{i\theta}$ $\therefore\ dz = i e^{i\theta} \cdot d\theta = i z\, d\theta$ $\therefore\ d\theta = \frac{dz}{iz}$; $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore\ I = \int_C \frac{1}{a + b\cdot\frac{(z^2+1)}{2z}} \cdot \frac{dz}{iz} = \int_C \frac{2\,dz}{(bz^2+2az+b)\,i}$$ where C is the circle $|z| = 1$.

Now, the poles of $f(z)$ are given by $z = \frac{-2a \pm \sqrt{4a^2-4b^2}}{2b}$ *i.e.* say $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$ which are simple poles. Since $a > b > 0$, α lies inside and β lies outside the circle $|z| = 1$.

$$\therefore \text{ Residue of } f(z) \text{ (at } z=\alpha) = \lim_{z\to\alpha} (z-\alpha)\cdot\frac{2}{b(z-\alpha)(z-\beta)\,i} = \frac{2}{bi(\alpha-\beta)}$$

But $\alpha - \beta = \frac{2\sqrt{a^2-b^2}}{b}$

$$\therefore \text{ Residue of } f(z) \text{ (at } z=\alpha) = \frac{2}{bi\cdot\frac{2\sqrt{a^2-b^2}}{b}} = \frac{1}{i\sqrt{a^2-b^2}}$$

$$\therefore\ I = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Example 6 : Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta.$ **(M.U. 1994, 2003, 05, 09, 14,**

Sol. : Consider $\int_0^{2\pi} \frac{e^{2i\theta}}{5 + 4\cos\theta} d\theta.$

Now, put $z = e^{i\theta}$ $\therefore dz = i e^{i\theta} \cdot d\theta$ $\therefore dz = i z d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + (1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5 + 4\cos\theta} d\theta = \int_C \frac{z^2}{5 + 4\left(\frac{z + (1/z)}{2}\right)} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{z^2}{i(2z^2 + 5z + 2)} dz \quad \text{where } C \text{ is the circle } |z| = 1.$$

Now, the poles are given by $2z^2 + 5z + 2 = 0$ $\therefore (2z + 1)(z + 2) = 0$

$\therefore z = -1/2$ and $z = -2.$

The pole $z = -1/2$ lies inside the unit circle and the pole $z = -2$ lies outside.

Now, Residue of $f(z)$ (at $z = -1/2$) $= \lim_{z \to -\frac{1}{2}} \left(z + \frac{1}{2}\right) \cdot \frac{z^2}{2[z + (1/2)](z + 2)i}$

$$= \frac{(-1/2)^2}{2[-(1/2) + 2]i} = \frac{1}{12i}$$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{5 + 4\cos\theta} d\theta = 2\pi i \left(\frac{1}{12i}\right) = \frac{\pi}{6}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta = \text{Real part of } \int_0^{2\pi} \frac{e^{2i\theta}}{5 + 4\cos\theta} d\theta = \frac{\pi}{6}.$$

Example 7 : Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta.$ **(M.U. 1995, 96, 2013, 14**

Sol. : Consider $\int_0^{2\pi} \frac{e^{3i\theta}}{5 - 4\cos\theta} d\theta.$

Now, put $z = e^{i\theta}$ $\therefore dz = i e^{i\theta} \cdot d\theta$ $\therefore dz = i z d\theta$ $\therefore d\theta = \frac{dz}{iz}$

And $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + (1/z)}{2} = \frac{z^2 + 1}{2z}$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5 - 4\cos\theta} d\theta = \int_C \frac{z^3}{5 - 4\left(\frac{z^2 + 1}{2z}\right)} \cdot \frac{dz}{iz}$$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta = \int_C \frac{z^3}{-2z^2+5z-2}\cdot\frac{dz}{i} = -\int_C \frac{z^3 dz}{i(2z^2-5z+2)}$$

where C is the circle $|z| = 1$.

Now, the poles are given by $2z^2 - 5z + 2 = 0 \quad \therefore (2z-1)(z-2) = 0$

$\therefore \; z = 1/2$ and $z = 2$.

The pole $z = 1/2$ lies inside the unit circle and $z = 2$ lies outside it.

Now, Residue of $f(z)$ (at $z = 1/2$) $= \lim\limits_{z\to\frac{1}{2}} \left(z-\frac{1}{2}\right)\cdot\frac{(-1)\cdot z^3}{i\cdot 2[z-(1/2)](z-2)}$

$$= \lim_{z\to\frac{1}{2}} \frac{(-1)\cdot(1/2)^3}{i\cdot 2[(1/2)-2]} = \frac{1}{24i}$$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta = 2\pi i\left(\frac{1}{24i}\right) = \frac{\pi}{12}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\,d\theta = \text{Real part of} \int_0^{2\pi} \frac{e^{3i\theta}}{5-4\cos\theta}\,d\theta = \frac{\pi}{12}.$$

Example 8 : Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta}$ **(M.U. 2013)**

Sol : Put $z = e^{i\theta} \quad \therefore dz = i e^{i\theta}\cdot d\theta \quad \therefore dz = iz\,d\theta \quad \therefore d\theta = \dfrac{dz}{iz}$

And $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+(1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{d\theta}{25-16\cos^2\theta} = \int_C \frac{1}{25-16\left[\dfrac{z+(1/z)}{2}\right]^2}\cdot\frac{dz}{iz}$$

$$= \int_C \frac{1}{25-\left[4z^2+8+(4/z^2)\right]}\cdot\frac{dz}{iz} = -\frac{1}{i}\int_C \frac{z}{4z^4-17z^2+4}\,dz$$

where C is the circle $|z| = 1$.

Now, the poles of $4z^4 - 17z^2 + 4 = 0$ *i.e.* of $(4z^2-1)(z^2-4) = 0$ are $z = 1/2, -1/2, 2, -2$.

The poles $z = 1/2, -1/2$ lie inside the unit circle and $z = 2, -2$ lie outside it.

Residue of $f(z)$ (at $z = 1/2$) $= \lim\limits_{z\to 1/2} \left(z-\dfrac{1}{2}\right)\cdot\dfrac{-z}{i(2z-1)(2z+1)(z^2-4)}$

$$= \lim_{z\to 1/2} \left(\frac{2z-1}{2}\right)\cdot\frac{-z}{i(2z-1)(2z+1)(z^2-4)}$$

$$= \lim_{z\to 1/2} \frac{1}{2}\cdot\frac{-z}{i(2z+1)(z^2-4)}$$

$$= \lim_{z\to 1/2} \frac{1}{2}\cdot\frac{-1/2}{i(2)[(1/4)-4]} = \frac{-1/2}{-15i} = \frac{1}{30i}$$

$$\text{Residue of } f(z) \text{ (at } z = -1/2) = \lim_{z \to -1/2} \left(\frac{2z+1}{2}\right) \cdot \frac{-z}{i\,(2z-1)(2z+1)(z^2-4)}$$

$$= \lim_{z \to -1/2} \frac{1}{2} \cdot \frac{-z}{i\,(2z-1)(z^2-4)}$$

$$= \frac{1}{2} \cdot \frac{1/2}{i\,(-2)[(1/4)-4]} = \frac{1/2}{15\,i} = \frac{1}{30\,i}$$

$$\therefore \oint_C f(z)\,dz = 2\pi i \text{ (Sum of the residues)} = 2\pi i \left(\frac{2}{30\,i}\right) = \frac{2\pi}{15}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{25 - 16\cos^2\theta} = \frac{2\pi}{15}.$$

Example 9 : Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\,d\theta$ **(M.U. 2003, 04, 1**

Sol. : Consider $\displaystyle\int_0^{2\pi} \frac{e^{3i\theta}}{5 + 4\cos\theta}\,d\theta$

Now, put $z = e^{i\theta}$ $\quad\therefore\ dz = i\,e^{i\theta} \cdot d\theta \quad \therefore\ dz = i\,z\,d\theta \quad \therefore\ d\theta = \dfrac{dz}{i\,z}$

And $\quad \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2} = \dfrac{z + (1/z)}{2}$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5 + 4\cos\theta}\,d\theta = \int_C \frac{z^3}{5 + 4\left[\dfrac{z + (1/z)}{2}\right]} \cdot \frac{dz}{i\,z}$$

$$= \int_C \frac{z^3}{i\,(2z^2 + 5z + 2)}\,dz \qquad \text{where, } C \text{ is the cirlce } |z| = 1.$$

Now, the roots of $2z^2 + 5z + 2 = 0$ *i.e.* of $(2z + 1)(z + 2) = 0$ are $z = -1/2$ and $z = -2$.

$\therefore$ The pole $z = -1/2$ lies inside the unit circle and $z = -2$ lies outside it.

$$\text{Now, residue of } f(z) \text{ (at } z = -1/2) = \lim_{z \to -1/2} \left(z + \frac{1}{2}\right) \cdot \frac{3}{i\,(2z+1)(z+2)}$$

$$= \lim_{z \to -1/2} \left(\frac{2z+1}{2}\right) \cdot \frac{z^3}{i\,(2z+1)(z+2)}$$

$$= \lim_{z \to -1/2} \frac{1}{2} \cdot \frac{z^3}{i\,(z+2)} = \frac{1}{2} \cdot \frac{(-1/2)^3}{i\,[(-1/2)+2]}$$

$$= -\frac{1}{8 \cdot 3i} = -\frac{1}{24i}$$

$$\therefore \int_0^{2\pi} \frac{e^{3i\theta}}{5 + 4\cos\theta}\,d\theta = 2\pi i \left(-\frac{1}{24\,i}\right) = -\frac{\pi}{12}$$

$$\therefore \int_0^{2\pi} \frac{\cos 3\theta}{5 + 4\cos\theta}\,d\theta = \text{Real part of} \int_0^{2\pi} \frac{e^{3i\theta}}{5 + 4\cos\theta}\,d\theta = -\frac{\pi}{12}.$$

Example 10 : Evaluate $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta}$. (M.U. 2001, 03, 10, 16, 17)

Sol. : Let $z = e^{i\theta}$ $\therefore$ $dz = i e^{i\theta} \cdot d\theta$ $\therefore$ $dz = i z\, d\theta$

$$\therefore \quad d\theta = \frac{dz}{iz} \quad \text{and} \quad \cos\theta = \frac{(z^2+1)}{2z}$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \int_C \frac{1}{3+2\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz}$$ where, C is the unit circle $|z| = 1$.

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{i}\int_C \frac{1}{z^2+3z+1} dz$$

The roots of $z^2 + 3z + 1 = 0$ are $z = \frac{-3 \pm \sqrt{9-4}}{2} = \frac{-3 \pm \sqrt{5}}{2}$.

Let $\alpha = \frac{-3+\sqrt{5}}{2}$ and $\beta = \frac{-3-\sqrt{5}}{2}$

Clearly α lies within the unit circle and β lies outside it.

$$\therefore \quad \text{Residue of } f(z) \text{ (at } z = \alpha) = \lim_{z\to\alpha} (z-\alpha) \cdot \frac{1}{(z-\alpha)(z-\beta)} \cdot \frac{1}{i}$$

$$= \lim_{z\to\alpha} \frac{1}{z-\beta} \cdot \frac{1}{i} = \frac{1}{(\alpha-\beta)\cdot i}$$

Now, $\alpha - \beta = \frac{-3+\sqrt{5}}{2} - \frac{-3-\sqrt{5}}{2} = \frac{2\sqrt{5}}{2} = \sqrt{5}$

$$\therefore \quad \text{Residue of } f(z) \text{ (at } z = \alpha) = \frac{1}{\sqrt{5}\cdot i}.$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = 2\pi i \,(\text{Residue}) = 2\pi i \cdot \frac{1}{\sqrt{5}\, i} = \frac{2\pi}{\sqrt{5}}$$

$$\therefore \quad \int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta} = \frac{1}{2} \cdot \frac{2\pi}{\sqrt{5}} = \frac{\pi}{\sqrt{5}}$$

$$\left[\int_0^{2\pi} \frac{dx}{3+\cos 2x} = \int_0^{\pi} \frac{dx}{3+2\cos x} + \int_{\pi}^{2\pi} \frac{dx}{3+2\cos x}. \right.$$

In the second integral put $x = 2\pi - t$ $\therefore$ $dx = -dt$

When $x = \pi$, $t = \pi$, when $x = 2\pi$, $t = 0$.

$$\therefore \quad \int_{\pi}^{2\pi} \frac{dx}{3+2\cos x} = \int_{\pi}^{0} \frac{-dt}{3+2\cos t} = \int_0^{\pi} \frac{dt}{3+2\cos t} = \int_0^{\pi} \frac{dx}{3+2\cos x}$$

$$\left. \therefore \quad \int_0^{2\pi} \frac{dx}{3+2\cos x} = 2\int_0^{\pi} \frac{dx}{3+2\cos x} \right]$$

Example 11 : Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} d\theta$ where $-1 < a < 1$.

(M.U. 1993, 95, 99, 200..)

Sol. : Let $z = e^{i\theta}$ $\therefore$ $dz = i e^{i\theta} \cdot d\theta = i z d\theta$ and $\cos\theta = \frac{z^2+1}{2z}$.

Now consider

$$\int_0^{2\pi} \frac{e^{2i\theta}}{1 - 2a\cos\theta + a^2} d\theta = \int_C \frac{z^2}{1 - 2a\left(\frac{z^2+1}{2z}\right) + a^2} \cdot \frac{dz}{iz}$$

$$= \int_C \frac{z^2}{a^2 z - az^2 - a + z} \cdot \frac{dz}{iz} = \frac{1}{i} \int_C \frac{z^2}{(az-1)(a-z)} dz$$

where C is the unit circle $|z| = 1$.

Now, $f(z)$ has simple poles at $z = 1/a$ and $z = a$. But as $-1 < a < 1$, the pole $z = a$ lies within the unit circle and $z = 1/a$ lies outside it.

$$\text{Residue of } f(z) \text{ (at } z = a) = \lim_{z \to a} (z-a) \cdot \frac{z^2}{i(az-1)(a-z)}$$

$$= \lim_{z \to a} \frac{-z^2}{i(az-1)} = \frac{-a^2}{i(a^2-1)}$$

$$\therefore \int_0^{2\pi} \frac{e^{2i\theta}}{1 - 2a\cos\theta + a^2} d\theta = \frac{1}{i} \cdot 2\pi i \cdot \left(\frac{-a^2}{a^2-1}\right) = \frac{2\pi a^2}{1-a^2}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1-a^2}$$

Equating real parts,

$$\int_0^{2\pi} \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1-a^2}.$$

Example 12 : Show that $\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta} d\theta = \frac{2\pi}{b^2}\left(a - \sqrt{a^2 - b^2}\right)$. $(0 < b < a)$

Sol. : Let $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{z^2+1}{2z}$, $\sin\theta = \frac{z^2-1}{2iz}$

$$\therefore \ I = \int_C \frac{[(z^2-1)/2iz]^2}{a + b\left(\frac{z^2+1}{2z}\right)} \cdot \frac{dz}{iz} = -\frac{1}{2i} \int_C \frac{(z^2-1)^2}{z^2(bz^2 + 2az + b)} dz$$

where C is the circle $|z| = 1$.

Now, the poles of $f(z)$ are given by $z = 0$, which is a pole of order 2 and $z = \frac{-2a \pm \sqrt{4a^2 - 4b^2}}{2b}$

i.e. say $\alpha = \dfrac{-a + \sqrt{a^2 - b^2}}{b}$ and $\beta = \dfrac{-a - \sqrt{a^2 - b^2}}{b}$ which are simple poles. Since $a > b > 0$, α lies inside and β lies outside the circle $|z| = 1$.

Residue of $f(z)$ (at $z = 0$)

$$= \lim_{z \to 0} \frac{d}{dz}\left[z^2 \cdot \frac{(z^2-1)^2}{z^2(bz^2 + 2az + b)}\right] = \lim_{z \to 0} \frac{d}{dz}\left[\frac{(z^2-1)^2}{bz^2 + 2az + b}\right]$$

$$= \lim_{z \to 0} \frac{(bz^2 + 2az + b) \cdot 2(z^2-1) \cdot 2z - (z^2-1)^2 (2bz + 2a)}{(bz^2 + 2az + b)^2}$$

$$= -\frac{2a}{b^2}$$

Note that $bz^2 + 2az + b = b(z - \alpha)(z - \beta)$ (since the roots of $ax^2 + bx + c = 0$ are obtained after dividing by a, the function must be multiplied by a). Also note that in this case $\alpha\beta = 1$.

$\therefore\ \beta = 1/\alpha.$

Now, Residue of $f(z)$ (at $z = \alpha$) $= \lim\limits_{z \to \alpha} (z - \alpha) \dfrac{(z^2-1)^2}{z^2 b(z-\alpha)(z-\beta)}$

$$= \lim_{z \to \alpha} \frac{(z^2-1)^2}{z^2 b(z-\beta)} = \frac{(\alpha^2-1)^2}{b\alpha^2(\alpha-\beta)} = \frac{[\alpha - (1/\alpha)]^2}{b(\alpha-\beta)}$$

$$= \frac{(\alpha-\beta)^2}{b(\alpha-\beta)} = \frac{1}{b}(\alpha-\beta) = \frac{1}{b} \cdot \frac{2\sqrt{a^2-b^2}}{b} \qquad \left[\because\ \beta = \frac{1}{\alpha}\right]$$

$$\therefore\ I = 2\pi i\left(-\frac{1}{2i}\right)\left[-\frac{2a}{b^2} + \frac{2\sqrt{a^2-b^2}}{b^2}\right] = \frac{2\pi}{b^2}\left[a - \sqrt{a^2-b^2}\right]$$

Alternatively $I = \displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1 - \cos 2\theta}{a + b\cos\theta}\, d\theta$

$$= \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1 - e^{2i\theta}}{a + b\cos\theta}\, d\theta$$

Then putting as before $z = e^{i\theta}$

$$I = \text{Real part of } \frac{1}{2}\int_0^{2\pi} \frac{1 - z^2}{bz^2 + 2az + b} \frac{2\,dz}{i}$$

Then proceed as above.

Example 13 : Evaluate $\displaystyle\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta.$

Sol. : Putting $a = 5$, $b = -4$, we get from the above result

$$\int_0^{2\pi} \frac{\sin^2\theta}{5 - 4\cos\theta}\, d\theta = \frac{2\pi}{16}\left[5 - \sqrt{25 - 16}\right] = \frac{\pi}{4}$$

Or proceed independently by putting $z = e^{i\theta}$.

Example 14 : Evaluate $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta,\ a > 1.$

Sol. : Let $z = e^{i\theta}$ $\therefore\ d\theta = \frac{dz}{iz},\ \cos\theta = \frac{z^2+1}{2z}$

Consider $\int_0^{2\pi} \frac{a\cos\theta}{a+\cos\theta}\, d\theta = \int_C \frac{a(z^2+1)/2z}{a+(z^2+1)/2z} \cdot \frac{dz}{iz} = \int_C \frac{a(z^2+1)}{z(z^2+2az+1)} \cdot \frac{dz}{i}$

$\therefore$ The roots of $z(z^2+2az+1) = 0$ are $z = 0,\ z = \frac{-2a \pm \sqrt{4a^2-4}}{2} = -a \pm \sqrt{a^2-1}.$

$\therefore$ The pole $z = 0,\ z = \alpha = -a + \sqrt{a^2-1}$ lie inside the unit circle and $z = \beta = -a - \sqrt{a^2-1}$ lies outside it because $a > 1$.

$\therefore$ Residue of $f(z)$ (at $z = 0$) $= \lim_{z \to 0} z \cdot \frac{a(z^2+1)}{z(z^2+2az+1)i} = \frac{a}{i}$

$\therefore$ Residue of $f(z)$ (at $z = \alpha$) $= \lim_{z \to \alpha} (z-\alpha) \cdot \frac{a(z^2+1)}{z(z-\alpha)(z-\beta)} \cdot \frac{1}{i}$

$$= \lim_{z \to \alpha} \frac{a(z^2+1)}{z(z-\beta)} \cdot \frac{1}{i} = \frac{a(\alpha^2+1)}{\alpha(\alpha-\beta)} \cdot \frac{1}{i}$$

Now, $\frac{a(\alpha^2+1)}{\alpha(\alpha-\beta)i} = \frac{a[\alpha+(1/\alpha)]}{(\alpha-\beta)} \cdot \frac{1}{i}$

Now, $\alpha + \frac{1}{\alpha} = \left(-a+\sqrt{a^2-1}\right) + \frac{1}{-a+\sqrt{a^2-1}}$

$$= \frac{\left(-a+\sqrt{a^2-1}\right)^2 + 1}{-a+\sqrt{a^2-1}} = \frac{a^2+a^2-1-2a\sqrt{a^2-1}+1}{-a+\sqrt{a^2-1}}$$

$$= \frac{2a^2-2a\sqrt{a^2-1}}{-a+\sqrt{a^2-1}} = \frac{2a\left(a-\sqrt{a^2-1}\right)}{-a+\sqrt{a^2-1}} = -2a$$

And $\alpha - \beta = \left(-a+\sqrt{a^2-1}\right) - \left(-a+\sqrt{a^2-1}\right) = 2\sqrt{a^2-1}$

Hence, from (1), (2), (3) and (4)

$\therefore$ Residue of $f(z)$ (at $z = \alpha$) $= a \cdot (-2a) \frac{1}{2\sqrt{a^2-1}} \cdot \frac{1}{i} = \frac{-a^2}{\sqrt{a^2-1} \cdot i}$

$\therefore$ $\int_C \frac{a(z^2+1)}{z(z^2+2az+1)} dz = 2\pi i$ (Sum of the residues)

$$= 2\pi i \left[\frac{a}{i} - \frac{a^2}{\sqrt{a^2-1} \cdot i}\right] = 2\pi a \left[1 - \frac{a}{\sqrt{a^2-1}}\right]$$

Now, $\int_{-\pi}^{\pi} \frac{a\cos\theta}{a+\cos\theta}\,d\theta = 2\int_0^{\pi} \frac{a\cos\theta}{a+\cos\theta}\,d\theta = 2\cdot\frac{1}{2}\int_0^{2\pi} \frac{a\cos\theta}{a+\cos\theta}\,d\theta$

$$= \int_0^{2\pi} \frac{a\cos\theta}{a+\cos\theta}\,d\theta = 2\pi a\left[1-\frac{a}{\sqrt{a^2-1}}\right]$$

Example 15 : By using Cauchy's residue theorem evaluate $\int_0^{2\pi} \frac{\cos^2\theta}{5+4\cos\theta}\,d\theta$. **(M.U. 2016)**

Sol. : Let $z = e^{i\theta}$ $\quad\therefore\ dz = e^{i\theta}\cdot i\,d\theta \quad \therefore\ dz = iz\,d\theta \quad \therefore\ d\theta = \frac{dz}{iz}$.

Now, $\cos\theta = \frac{z^2+1}{2z}$

$$\therefore\ \frac{\cos^2\theta}{5+4\cos\theta} = \left(\frac{z^2+1}{2z}\right)^2\cdot\frac{1}{5+4\left(\frac{z^2+1}{2z}\right)}$$

$$= \frac{z^4+2z^2+1}{4z^2}\cdot\frac{z}{2z^2+5z+2} = \frac{z^4+2z^2+1}{4z(2z^2+5z+2)}$$

$$\therefore\ I = \int_C \frac{z^4+2z^2+1}{4z(2z^2+5z+2)}\cdot\frac{dz}{iz} = \frac{1}{4i}\int_C \frac{z^4+4z^2+1}{z^2(2z^2+5z+2)}\,dz$$

$$= \frac{1}{4i}\int_C \frac{z^4+4z^2+1}{z^2(2z+1)(z+2)}\,dz \text{ where } C \text{ is the circle } |z| = 1.$$

Now, the poles are given by $z^2 = 0$, $2z+1 = 0$ and $z+2 = 0$.

$\therefore$ The poles are $z = 0$, $z = -1/2$, $z = 2$.

Of these poles $z = 0$ and $z = -1/2$ lie within the unit circle and $z = 2$ lies outside it.

Further, $z = -1/2$ is a simple pole and $z = 0$ is a pole of order 2.

Now, residue at $\left(z = -\frac{1}{2}\right) = \lim_{z\to(-1/2)}\left(z+\frac{1}{2}\right)\cdot\frac{z^4+2z^2+1}{z^2\cdot 2[z+(1/2)]\cdot(z+2)}$

$$= \lim_{z\to(-1/2)} \frac{z^4+2z^2+1}{z^2\cdot 2(z+2)} = \frac{(1/16)+2(1/4)+1}{(1/4)\cdot 2[(-1/2)+2]}$$

$$= \frac{25}{16}\cdot\frac{4}{3} = \frac{25}{12}$$

Residue at $(z = 0)$ $= \lim_{z\to 0} \frac{d}{dz}\left[\frac{z^4+2z^2+1}{2z^2+5z+2}\right]$

$$= \lim_{z\to 0} \frac{(2z^2+5z+2)(4z^3+4z)-(z^4+2z^2+1)(4z+5)}{(2z^2+5z+2)^2}$$

$$= \frac{0-5}{4} = -\frac{5}{4}$$

$\therefore$ Sum of the residues $= \dfrac{25}{12} - \dfrac{5}{4} = \dfrac{10}{12} = \dfrac{5}{6}$

$\therefore \quad I = 2\pi i \cdot \dfrac{1}{4i} \cdot \dfrac{5}{6} = \dfrac{5\pi}{12}.$

Example 16 : Evaluate $\oint_C \dfrac{e^{kz}}{z} dz$ where C is $|z| = 1$. Hence, deduce that

$$\int_0^{\pi} e^{k \sin\theta} \cos(k \sin\theta)\, d\theta = \pi$$

(M.U. 2003)

Sol. : Let $f(z) \dfrac{e^{kz}}{z}$. Clearly $z = 0$ is a simple pole.

Residue (at $z = 0$) $= \lim_{z \to 0} z \cdot \dfrac{e^{kz}}{z} = \lim_{z \to 0} e^{kz} = e^0 = 1.$

$\therefore \quad \int_C \dfrac{e^{kz}}{z} dz = 2\pi i(1) = 2\pi i$

Now, put $z = e^{i\theta}$ and proceed as in Ex. 1 page 2-37.

Example 17 : If $f(z) = z^3 + iz^2 - 4z - 4i$, evaluate $\oint_C \dfrac{f'(z)}{f(z)} dz$ where C encloses zeros of f

(M.U. 2

Sol. : $\because f(z) = z^3 + iz^2 - 4z - i \quad \therefore f'(z) = 3z^2 + 2iz - 4$

$$\therefore \quad \frac{f'(z)}{f(z)} = \frac{3z^2 + 2iz - 4}{z^2(z+i) - 4(z+i)} = \frac{3z^2 + 2iz - 4}{(z^2 - 4)(z+i)}$$

Now, zeros of $f(z)$ are given by $(z^2 - 4)(z + i) = 0$.

$\therefore \quad z = 2, -2, -i.$

Since, C encloses all zeros of $f(z)$ we calculate residues at $z = 2, -2, -i$ which are si poles.

$$\text{Residue (at } z = 2) = \lim_{z \to 2} (z-2) \cdot \frac{3z^2 + 2iz - 4}{(z-2)(z+2)(z+i)}$$

$$= \lim_{z \to 2} \frac{3z^2 + 2iz - 4}{(z+2)(z+i)} = \frac{12 + 4i - 4}{4(2+i)} = \frac{8 + 4i}{8 + 4i} = 1$$

$$\text{Residue (at } z = -2) = \lim_{z \to -2} (z+2) \cdot \frac{3z^2 + 2iz - 4}{(z-2)(z+2)(z+i)}$$

$$= \lim_{z \to -2} \frac{3z^2 + 2iz - 4}{(z-2)(z+i)} = \frac{12 - 4i - 4}{-4(-2+i)} = \frac{8 - 4i}{8 - 4i} = 1$$

$$\text{Residue (at } z = -i) = \lim_{z \to -i} (z+i) \cdot \frac{3z^2 + 2iz - 4}{(z+i)(z^2 - 4)}$$

$$\text{Residue (at } z = -i) = \lim_{z \to -i} \frac{3z^2 + 2iz - 4}{z^2 - 4}$$

$$= \frac{-3+2-4}{-5} = \frac{-5}{-5} = 1$$

$$\therefore \oint_C \frac{f'(z)}{f(z)} dz = 2\pi i \text{ (Sum of the residues)} = 2\pi i (3) = 6\pi i.$$

EXERCISE - IX

Using the residue theorem evaluate

1. $\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$ (M.U. 2004)

2. $\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$ (M.U. 2005)

3. $\int_0^{2\pi} \frac{d\theta}{5 - 4\sin\theta}$

4. $\int_0^{2\pi} \frac{d\theta}{17 - 8\cos\theta}$

5. $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$ (M.U. 2001, 03, 10, 11)

6. $\int_0^{\pi/2} \frac{d\theta}{97 - 72\cos\theta}$

7. $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta}$ $(a > b > 0)$

8. $\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$

9. $\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}$ $a^2 < 1$

10. $\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}$ $|a| < 1$

11. $\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$, $|a| < 1$

12. $\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}$ $(a > b > 0)$

13. $\int_0^{2\pi} \frac{d\theta}{(5 - 4\cos\theta)^2}$

14. $\int_0^{2\pi} \frac{d\theta}{1 - 2a\sin\theta + a^2}$ $0 < a < 1$ (M.U. 1998, 2006)

15. $\int_0^{2\pi} \frac{d\theta}{5 + 4\cos\theta}$ (M.U. 1993, 2002, 04)

16. $\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}$ $|a| < 1$ (M.U. 2000)

17. $\int_0^{2\pi} \frac{d\theta}{13 + 5\cos\theta}$ (M.U. 2002)

18. $\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$ (M.U. 2015)

[Ans. : (1) $\frac{\pi}{6}$, (2) $\frac{2\pi}{\sqrt{3}}$, (3) $\frac{2\pi}{3}$, (4) $\frac{2\pi}{15}$, (5) $\frac{\pi}{2}$, (6) Factors are $(4z - 9)(9z - 4)$; $\frac{2\pi}{65}$,

(7) $\frac{2\pi}{\sqrt{a^2 - b^2}}$, (8) $\frac{2\pi}{\sqrt{a^2 - b^2}}$, (9) $\frac{2\pi}{\sqrt{1 - a^2}}$, (10) $\frac{2\pi}{\sqrt{1 - a^2}}$,

(11) $\frac{2\pi}{(1 - a^2)}$, (12) $\frac{2\pi a}{(a^2 - b^2)^{3/2}}$, (13) $\frac{10\pi}{27}$, (14) $\frac{2\pi}{(1 - a^2)}$,

(15) $\frac{2\pi}{3}$, (16) $\frac{2\pi}{\sqrt{1 - a^2}}$, (17) $\frac{\pi}{6}$, (18) $\frac{\pi}{6}$.]

EXERCISE - X

Theory : Complex Integration

1. Define line integral of $f(z)$ where $f(z)$ is a function of complex variable z.
2. Explain how the line integral $\int_C f(z)\,dz$ is evaluated.
3. Explain why the line integral $\int_C f(z)\,dz$ depends upon the path. When is it independent of the path ?

Theory : Cauchy's Theorem

1. Define a simple curve and also simply and multiply connected region with suitable diagrams. (M.U. 200...)
2. State Cauchy's Integral Theorem.
3. State and prove Cauchy's Theorem. (M.U. 200...)
4. State and prove Cauchy's integral formula. (M.U. 1999, 2003, 0...)
5. State Morera's Theorem.
6. State Cauchy's extended integral formula for a doubly connected region.
7. State Cauchy's integral formula for the n-th derivative of an analytic function.
8. If $f(z)$ is continuous on a closed curve C of length l, where $|f(z)| < M$, then prove that
$$\left|\int_C f(z)\,dz\right| \le M\,l.$$
9. Find the following integrals.
   (i) $\int_C \cot z\,dz$, where C is $|z| = 1$.
   (ii) $\int_C \tan hz\,dz$, where C is $|z| = 3$. (M.U. 200...)
   (iii) $\int_C \frac{(z^2+2z+1)}{z-4}\,dz$, where C is $|z| = 1$.
   (iv) $\int_C (z-a)^n\,dz$, $n \ne -1$ where C is $|z-a| = r$. (M.U. 200...)

   [**Ans. :** Each is zero by Cauchy's Theorem.

10. State true or false with proper justification.
   (i) $\oint_C z^3 dz = \oint_C z^{-3} dz$, where C is $|z-3| = 4$.
   (ii) If $\oint_C f(z)\,dz = 0$ for a any closed curve C then $f(z)$ is an analytic function. (M.U. 199...)
   (iii) $\oint_C z^3\,dz = \oint_C \frac{1}{z^3}\,dz$ where C is $|z-2i| = 1$. (M.U. 200...)
   (iv) $\int_C \frac{dz}{z} = 2\pi i$ where C is the circle $|z| = r$. (M.U. 200...)

(v) $\int_C (z^2 + 2z)\,dz = 0$ where C is a triangle whose vertices are (0, 0), (0, 1), (1, 1).

(M.U. 2001)

(vi) If $f(z) = (x^2 - 2x - y^2) + i\,(2xy - 2y)$, then $\int_A^B f(z)\,dz$ is the same along

(i) $y = x^2$ and (ii) $y = x$ where A is (0, 0) and B is (1, 1). **(M.U. 2002)**

(vii) $\int_C \dfrac{dz}{(z-a)^4} = 0$ where C is $|z - a| = a$. **(M.U. 2002)**

(viii) $\int_0^{1+i} [(x^2 - y^2 + 2x) + 2iy\,(x+1)]\,dz$ is same along $y = x^2$ and $x = y^2$. **(M.U. 2002)**

(ix) If $\Phi(a) = \oint_C \dfrac{z^2 + 2}{z - a}\,dz$ where C is $|z - 1 + i| = 2$, then $\Phi'(1) = 2\pi i$. **(M.U. 2002)**

(x) $\oint_C \dfrac{dz}{z - 4} = 0$ where C is $|z - 4| = 2$. **(M.U. 2002)**

(xi) $\int_C (x^2 - y^2 + 2i\,xy)\,dz = 0$ where C is $|z| = 2$. **(M.U. 2004)**

[**Ans. : (i)** z^3 is analytic in and on C.

$\therefore \oint z^3 dz = 0.$ But z^{-3} is not analytic in C. Hence, $\oint z^{-3} dz \neq 0.$ Hence, false.

(ii) If $\oint_C f(z)\,dz = 0$ for any closed curve C, then $f(z)$ is analytic.

Morera's Theorem. Hence, True.

(iii) True. z^3 and $1/z^3$ are both analytic on C. **(iv)** True. **(v)** True.

(vi) True $f(z)$ is analytic. **(vii)** False. a is inside C.

(viii) True. **(ix)** False. **(x)** False. **(xi)** True.

Theory : Taylor's and Laurent's Series

1. Define radius and circle of convergence of a power series.
2. Define Taylor's series.
3. Define Laurent's series.
4. Define a Laurent's series and state its analytic part and the principal part.
5. State whether the following statements are true or false.
   (i) A function analytic at z_0 may have two Taylor series expansions centred at z_0.
   (ii) A function $f(z)$ may have two Laurent's series centre at z_0 depending upon the annulus of convergence. **(M.U. 1999)** [**Ans. :** Both are false.]

Theory : Residues

1. State and prove Cauchy's Integral Theorem. **(M.U. 1993, 96, 99)**
2. State and prove Cauchy's Integral Formula. **(M.U. 1995, 96, 98)**

3. If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then prove that

$$\oint_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a).$$

(M.U. 1997)

4. Prove that $f'''(a) = \frac{3!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^4} dz$, if C is a closed curve enclosing $z = a$ and $f(z)$ is analytic inside and on C. Hence, evaluate $\oint_C \frac{e^{iz}}{z^4} dz$, where C is the circle $|z| = 2$.

(M.U. 1997) [Ans. : $\frac{2\pi}{3!} e^{ia}$]

5. Define : (i) Essential Singularity, (ii) Residue of a function, (iii) Pole of order m.

(M.U. 1994, 97, 2003)

6. State Cauchy's residue theorem. (M.U. 1994)
7. Define (i) Pole, (ii) Isolated Singular Point. (M.U. 1995, 97, 2003)
8. State and prove Cauchy's residue theorem. (M.U. 1995, 97, 2003)
9. "Cauchy's integral formula can be deduced from Cauchy's residue theorem". Explain.

(M.U. 1999)

10. If $f(z)$ has a pole of order m at $z = a$, then prove that

$$\text{Residue (at } z = a) = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-a)^m \cdot f(z) \right]$$

11. Define : (i) Singular Point, (ii) Essential Singularity,
(iii) Removable Singularity, (iv) Residue of a function. (M.U. 2006)

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