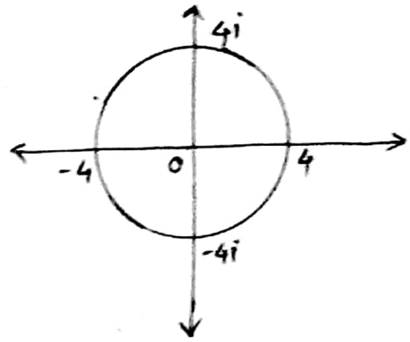
Ex ⑧ Using residue theorem evaluate $\oint_C \frac{e^z}{(z^2+\pi^2)^2}\,dz$

where C is $|z| = 4$

Solution: let $f(z) = \frac{e^z}{(z^2+\pi^2)^2}$

Given: $C: \quad |z| = 4$ which is equation of circle with centre origin and radius 4



Note that $(z^2+\pi^2)^2 = 0$

$\Rightarrow (z^2+\pi^2)(z^2+\pi^2) = 0$

$\Rightarrow z = \pi i, \pi i, -\pi i, -\pi i$ all lies inside C

Therefore, $z = \pi i$ is pole of $f(z)$ of order 2

and $z = -\pi i$ is pole of $f(z)$ of order 2

Now, Residue of $f(z)$ at $z = z_0 = \pi i$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$= \frac{1}{1!} \lim_{z \to \pi i} \frac{d}{dz} \left[(z-\pi i)^2 \frac{e^z}{(z^2+\pi^2)^2} \right]$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left[(z-\pi i)^2 \frac{e^z}{(z-\pi i)^2 (z+\pi i)^2} \right]$$

$$= \lim_{z \to \pi i} \frac{d}{dz} \left[\frac{e^z}{(z+\pi i)^2} \right]$$

$$= \lim_{z \to \pi i} \left[\frac{(z+\pi i)^2 \cdot e^z - e^z\, 2(z+\pi i)}{(z+\pi i)^4} \right]$$ ($\because$ By division Rule of derivative)

$$= \lim_{z \to \pi i} \left[\frac{e^z (z+\pi i - 2)}{(z+\pi i)^3} \right]$$

$$= \frac{e^{\pi i}(\pi i+\pi i-2)}{(\pi i+\pi i)^3}$$

$$= \frac{e^{\pi i}\cdot 2i(\pi+i)}{(2\pi i)^3}$$

$$= \frac{e^{\pi i}\, 2i(\pi+i)}{-8\pi^3 i}$$

$$= \frac{\pi+i}{4\pi^3} \qquad (\because\ e^{\pi i} = \cos\pi + i\sin\pi = -1)$$

Residue of $f(z)$ at $z = z_0 = -\pi i$

$$= \frac{1}{(2-1)!} \lim_{z\to -\pi i} \frac{d}{dz}\left[(z+\pi i)^2 \frac{e^z}{(z-\pi i)^2(z+\pi i)^2}\right]$$

$$= \lim_{z\to -\pi i} \frac{d}{dz}\left[\frac{e^z}{(z-\pi i)^2}\right]$$

$$= \lim_{z\to -\pi i}\left[\frac{(z-\pi i)^2 e^z - e^z\cdot 2(z-\pi i)}{(z-\pi i)^4}\right]$$

$$= \lim_{z\to -\pi i} \frac{e^z(z-\pi i-2)}{(z-\pi i)^3}$$

$$= \frac{e^{-\pi i}(-\pi i-\pi i-2)}{(-\pi i-\pi i)^3} = \frac{-e^{-\pi i}\, 2i(\pi-i)}{-8\pi^3 i^3}$$

$$= \frac{-e^{-\pi i}(\pi-i)}{4\pi^3} = \frac{\pi-i}{4\pi^3} \qquad (\because\ e^{-\pi i} = \cos\pi - i\sin\pi = -1)$$

$\therefore$ By Residue Theorem,

$$\oint_C f(z)\,dz = 2\pi i\,[\text{sum of residues}]$$

$$= 2\pi i\left[\frac{\pi+i}{4\pi^3} + \frac{\pi-i}{4\pi^3}\right] = 2\pi i\left[\frac{2\pi}{4\pi^3}\right]$$

$$\therefore \int \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{i}{\pi}$$

Ex. (4) Using Residue theorem evaluate

$$\int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz \quad \text{where } C \text{ is } |z|=1$$

Solution: let $f(z) = e^{-\frac{1}{z}} \cdot \sin\left(\frac{1}{z}\right)$

Clearly, $z=0$ is singular point of $f(z)$ which lies inside C

Now, $f(z) = e^{-\frac{1}{z}} \cdot \sin\left(\frac{1}{z}\right)$

$$\Rightarrow \quad f(z) = \left[1 - \frac{1}{z} + \frac{1}{2!\,z^2} - \cdots\right]\left[\frac{1}{z} - \frac{1}{3!\,z^3} + \cdots\right]$$

$$\left(\because e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \right.$$
$$\left. \& \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \right)$$

$$\Rightarrow e^{-\frac{1}{z}} \cdot \sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{z^2} + \frac{1}{3z^3} - \cdots$$

which is Laurent's series expansion around $z=0$

$\therefore$ Residue of $f(z)$ at $z=0$ = coefficient of $\frac{1}{z}$

$= 1$

$\therefore$ By Cauchy Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\,[\text{Residues at } z=0]$$

$$\Rightarrow \int_C f(z)\,dz = 2\pi i\,(1)$$

$$\Rightarrow \int_C e^{-\frac{1}{z}} \sin\left(\frac{1}{z}\right) dz = 2\pi i$$

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Homework :

Ex. 1 Using cauchy residue theorem evaluate $\oint_C \frac{z^3+3}{z^2-1}\,dz$ where C is the circle $|z+1|=1$

Ans: $-4\pi i$

Ex 2 Evaluate $\int_C \frac{dz}{z^3(z+4)}$ where C is $|z|=2$

Ans: $\frac{\pi i}{32}$