Signals & Systems

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TUTORIAL NO.2

Q.1. Convolve $x[n] = \left(\frac{1}{3}\right)^n u[n]$ with $h[n] = \left(\frac{1}{2}\right)^n u[n]$

using convolution sum formula.

Soln. Convolution sum formula,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k].h[n-k]$$

$$x[k] = \left(\frac{1}{3}\right)^n .u[k] \quad , \quad h[n-k] = \left(\frac{1}{2}\right)^{n-k} u[n-k]$$

$$y[n] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k .u[k] . \left(\frac{1}{2}\right)^n 2^k \; u[n-k]$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{\infty} \left(\frac{2}{3}\right)^k .u[k] . u[n-k]$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\frac{2}{3}\right)^k$$

$$y[n] = \left(\frac{1}{2}\right)^n . \left[\frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}\right] \qquad \left(\sum_{k=0}^{n} a^n = \frac{1-a^{n+1}}{1-a}\right)$$

$$y[n] = \left(\frac{1}{2}\right)^n . 3 \left[1-\left(\frac{2}{3}\right)^{n+1}\right]$$

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Q.2. Perform convulation of $x_1(t)$ and $x_2(t)$ using convolution theorem and sketch resultant waveform, where

$x_1(t) = u(t) - u(t-1)$

$x_2(t) = u(t) - u(t-2)$

Soln.

$x_1(t) = u(t) - u(t-1)$

using laplace transform,

$X_1(S) = L\{x_1(t)\} = L\{u(t) - u(t-1)\}$

$X_1(S) = \frac{1}{S} - \frac{e^{-S}}{S}$

$x_2(t) = u(t) - u(t-2)$

$X_2(S) = L\{x_2(t)\} = L\{u(t)\} - L\{u(t-2)\}$

$= \frac{1}{S} - \frac{e^{-2S}}{S}$

Convolution Theorem of Laplace Transform,

$L\{x_1(t) * x_2(t)\} = X_1(S) \cdot X_2(S)$

$= \left(\frac{1}{S} - \frac{e^{-S}}{S}\right) \cdot \left(\frac{1}{S} - \frac{e^{-2S}}{S}\right)$

$= \frac{1}{S^2} - \frac{e^{-2S}}{S^2} - \frac{e^{-S}}{S^2} + \frac{e^{-3S}}{S^2}$

$x_3(t) = x_1(t) * x_2(t) = L^{-1}\left\{\frac{1}{S^2} - \frac{e^{-2S}}{S^2} - \frac{e^{-S}}{S^2} + \frac{e^{-3S}}{S^2}\right\}$

using $\left(L\{\{t-a\}.u(t-a)\} = \frac{e^{-aS}}{S^2}\right)$

$x_3(t) = t\,u(t) - (t-1).u(t-1) - (t-2)u(t-2) + (t-3).u(t-3)$

when $t = 0$ to 1

$u(t) = 1, \quad u(t-1) = 0, \quad u(t-2) = 0, \quad u(t-3) = 0$

$\therefore x_3(t) = t \times 1 - 0 - 0 + 0$

$\underline{x_3(t) = t}$

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when $1 \leq t \leq 2$,

$u(t) = 1$, $u(t-1) = 1$, $u(t-2) = 0$, $u(t-3) = 0$

$x_3(t) = 1$

when ~~t>3~~ $2 \leq t \leq 3$

$u(t) = 1$, $u(t-1) = 1$, $u(t-2) = 1$, $u(t-3) = 0$

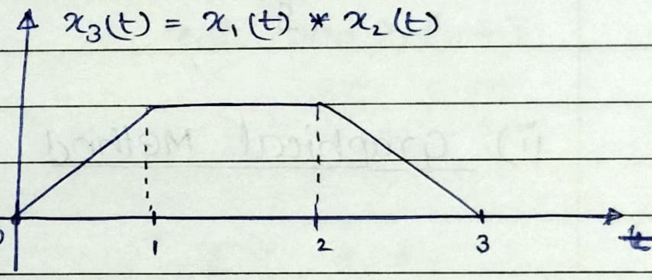
$x_3(t) = 3 - t$

when $t > 3$,

$u(t) = 1$, $u(t-1) = 1$, $u(t-2) = 1$, $u(t-3) = 1$

$\therefore x_3(t) = t - (t-1) - (t-2) + t - 3 = 0$

$\therefore$ The resultant waveform is,



Q.3 Perform convolution of $x_1(t) = e^{-3t} . u(t)$ and $x_2(t) = t . u(t)$ using mathematical method and also by graphical method.

Soln. (i) Mathematical method.

$x_1(t) = e^{-3t} . u(t) \Rightarrow x_1(t) = e^{-3t}$; $t \geq 0$

$x_2(t) = t . u(t) \Rightarrow x_2(t) = t$; $t \geq 0$

$x_3(t) = x_1(t) * x_2(t)$

$$= \int_{\lambda=\lambda_1}^{\lambda_2} x_1(\lambda) . x_2(t-\lambda) \, d\lambda$$

As $x_1(t)$ and $x_2(t)$ are causal limits of integration is 0 to 1.

$$= \int_0^t e^{-3\lambda}. (t-\lambda). \, d\lambda$$

$$= \int_0^t t. e^{-3\lambda}. d\lambda - \int_0^t \lambda . e^{-3\lambda}. d\lambda$$

$$= \left. \frac{t.e^{-3\lambda}}{-3} \right]_0^t - \left. \frac{\lambda.e^{-3\lambda}}{3} - \frac{e^{-3\lambda}}{9} \right]_0^t$$

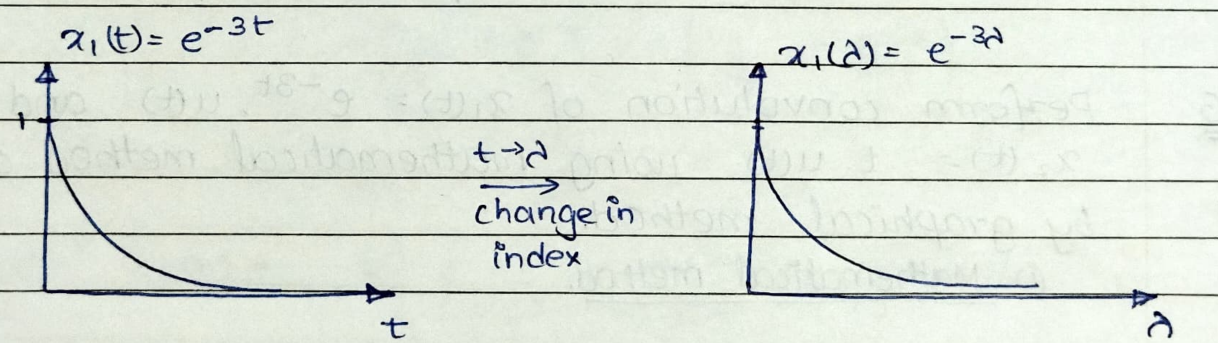
$$= \frac{-t.e^{-3t}}{3} + \frac{t}{3} + \frac{t.e^{-3t}}{3} - \frac{e^{-3t}}{9} - \frac{1}{9}$$

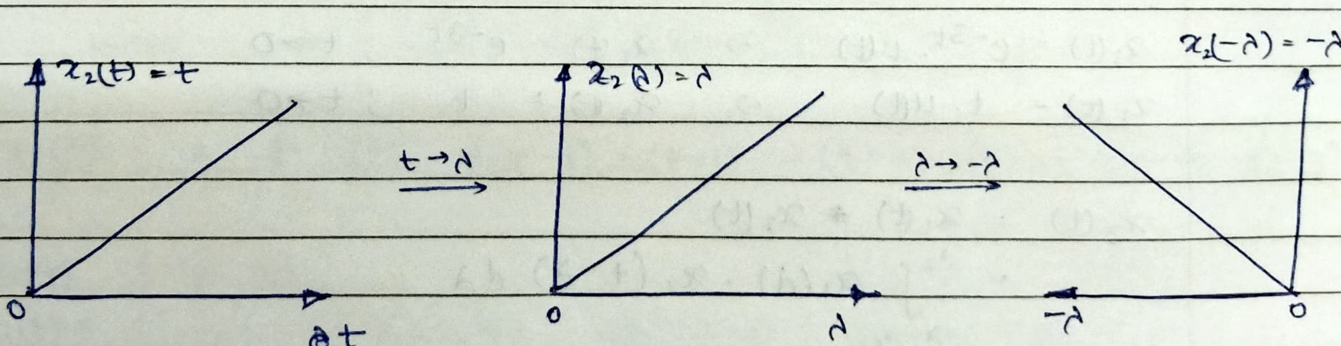
$$x_3(t) = \frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9} \quad ; \; t \geq 0$$

$$x_3(t) = \frac{1}{9} \left(e^{-3t} + 3t - 1\right). u(t) \; ;$$

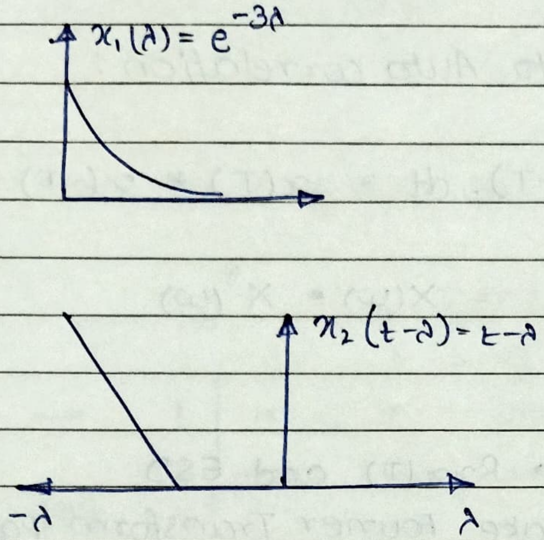
ii). <u>Graphical Method</u>

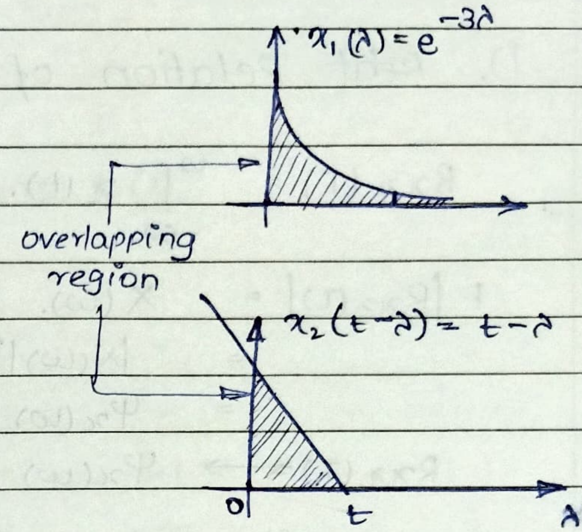
$$x_3(t) = \int_{\lambda=-\infty}^{+\infty} x_1(\lambda).\, x_2(t-\lambda).\, d\lambda$$





Shifting $x_2(-\lambda)$ by t units,





$x_1(\lambda)$ and $x_2(t-\lambda)$ when $t<0$

$x_1(\lambda)$ and $x_2(t-\lambda)$ when $t>0$

Overlapping from 0 to t
when time shift $t>0$,

$$\therefore\; x_3(\lambda) = \int_0^t x_1(\lambda).\, x_2(t-\lambda) = \int_0^t e^{-3\lambda}.\; t-\lambda$$

$$= \int_0^t t.e^{-3\lambda} - \int_0^t \lambda.e^{-3\lambda} = \left.\frac{t.e^{-3\lambda}}{3}\right]_0^t - \left.\frac{\lambda.e^{-3\lambda}}{-3} - \frac{e^{-3\lambda}}{9}\right]_0^t$$

$$= \frac{-t.e^{-3t}}{3} + \frac{t}{3} + \frac{t.e^{-3t}}{3} + \frac{e^{-3t}}{9} - \frac{1}{9}$$

$$x_3(\lambda) = \frac{t}{3} + \frac{e^{-3t}}{9} - \frac{1}{9} \;;\; t \geq 0$$

$$x_3(\lambda) = \frac{1}{9}\left(e^{-3t} + 3t - 1\right).\, u(t) \;;\; \text{for all } t.$$

Q.4 Write notes on the relation of ESD, PSD with auto-correlatio

Soln.

i). ~~Relat~~ Relation of ESD to Auto correlation :

$$R_{xx}(T) = \int_{-\infty}^{\infty} x(t).\, x(t-T).\, dt = x(T) * x(-T)$$

$$F\,[R_{xx}(T)] = X(\omega).\ X(-\omega) = X(\omega) \cdot X^*(\omega)$$

$$= |X(\omega)|^2$$

$$= \Psi_x(\omega)$$

$R_{xx}(T) \longleftrightarrow \Psi_x(\omega)$ $\longrightarrow$ $R_{xx}(T)$ and ESD make Fourier Transform Pair

ii). Relation of PSD to Autocorrelation :

$$R_{xx}(T) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t).\, x(t-T).\, dt$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} x_T(t).\, x(t-T).\, dt$$

$$R_{xx}(T) = \lim_{T\to\infty} \frac{1}{T} \left[x_T(T) * x_T(-T) \right]$$

$$F(R_{xx}(T)) = \lim_{T\to\infty} \frac{1}{T} \cdot X_T(\omega).\, X_T(-\omega) = \lim_{T\to\infty} \frac{1}{T}\, X_T(\omega).\, X_T^*(\omega)$$

$$= \lim_{T\to\infty} \frac{1}{T} \cdot |X_T(\omega)|^2$$

$$= G_x(\omega)$$

$R_{xx}(T) \longleftrightarrow G_x(\omega)$ $\longrightarrow$ $R_{xx}(T)$ and PSD $G_x(\omega)$ makes a Fourier Transform pair.

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Q.5 Compute the convolution $y(n) = x(n) * h(n)$ using the tabulation method. Where $x[n] = \{\underset{\uparrow}{1}, 1, 0, 1, 1\}$ and $h[n] = \{1, -2, \underset{\uparrow}{-3}, 4\}$

Soln.

We form a matrix of $x(n)$ and $h(n)$.

$h(n)$ \ $x[n]$	1	1	0	1	1
→ 1	1×1	1×1	0×1	1×1	1×1
−2	−2×1	−2×1	−2×0	−2×1	−2×1
−3	−3×1	−3×1	−3×0	−3×1	−3×1
4	4×1	4×1	4×0	4×1	4×1

	1	1	0	1	1
+1	1	1	0	1	1
−2	−2	−2	0	−2	−2
−3	−3	−3	0	−3	−3
4	4	4	0	4	4

Adding each elements from each diagonal.

The sum $= y[n]$

$\therefore y[n] = \{1, -1, -5, 2, 3, -5, 1, 4\}$

Now, length of $y[n]$ = length of $x[n]$ + length of $h[n] - 1$

$= 5 + 4 - 1$

$\therefore$ Length of $y[n] = 8$

Also, $\Sigma x[n] \cdot \Sigma h[n] = \Sigma y[n]$

$4 \cdot 0 = 0$

$\Sigma y[n] = 1 - 1 - 5 + 2 + 3 - 5 + 1 + 4 = 0$

Hence, the result is correct.

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Q.6 Determine cross correlation of sequence $x[n] = \{1, 1, 2, 2\}$ and $y[n] = \{1, 3, 1\}$

$r_{xy}(m)$ be the cross correlation of $x[n]$ and $y[n]$

$$r_{xy} = \sum_{-\infty}^{+\infty} x[n] \cdot y[n-m]$$

$x[n] = \{1, 1, 2, 2\}$, $y[n] = \{1, 3, 1\}$

4 samples, starts from $n=1$

3 samples, starts from $n=-1$

$\therefore$ Total samples in $y[n] = 4+3-1 = 6$

$\therefore$ $y[n]$ starts at $n=-1$, ends at $n=4$

$y(n) = \{1, 3, 1\}$

	1	3	1
1	1	3	1
1	1	3	1
2	2	6	2
2	2	6	2

$r_{xy} = \{1, 4, 6, 9, 8, 2\}$ (By adding diagonal elements).

Q.7 Consider two LTI systems connected in series. Their impulse responses are $h_1[n]$ and $h_2[n]$ respectively. Find the output of the system if $x[n]$ is the input being applied to one of the systems.

$$x[n] = \{\underset{\uparrow}{1}, 2\}, \quad h_1[n] = \{1, 0, \underset{\uparrow}{-1}\}, \quad h_2[n] = \{2, 1, \underset{\uparrow}{-1}\}$$

Soln. As $h_1[n]$ and $h_2[n]$ connected in series, their equivalent impulse responses

$$h[n] = h_1[n] * h_2[n]$$

By matrix form,

$$\begin{array}{c|ccc} & 1 & 0 & \overset{\downarrow}{-1} \\ \hline \rightarrow 2 & 2 & 0 & -2 \\ 1 & 1 & 0 & -2 \\ -1 & -1 & 0 & 1 \end{array}$$

$$\therefore h[n] = \{2, 1, \underset{\uparrow}{-3}, -1, 1\}$$

Final output of system $y[n] = x[n] * h[n]$

$$\begin{array}{c|ccccc} & 2 & 1 & -3 & -1 & 1 \\ \hline 1 & 2 & 1 & -3 & -1 & 1 \\ 2 & 4 & 2 & -6 & -2 & 2 \end{array}$$

$$y[n] = \{2, 5, \underset{\uparrow}{-1}, -7, -1, 2\}$$