Syllabus...

Mumbai University
Revised syllabus (Rev-2016) from Academic Year 2017-18

Signals and Systems

Course Code	Course Name	Teaching Scheme			Credits Assigned			
		Theory	Practical	Tutorial	Theory	Practical	Tutorial	Total
ECC404	Signals and Systems	04	–	2@	04	–	01	05

Subject Code	Subject Name	Examination Scheme							
		Theory Marks				Term Work	Practical and Oral	Oral	Total
		Internal assessment							
		Test 1	Test 2	Avg. of Test 1 and Test 2	End Sem Exam				
ECC404	Signals and Systems	20	20	20	80	25	–	–	125

@2 hour to be taken as tutorial classwise.

Course Prerequisite :

- Applied Maths – III
- Circuit Theory and Networks

Course Objectives :

1. To introduce students the concept and theory of signals and systems needed in electronics and telecommunication engineering fields.
2. To introduce students to the basic idea of signals and system analysis and its characterization in time and frequency domain.

Course Outcome :

After successful completion of the course student will be able to

1. Understand about various types of signals and systems, classify them, analyze them and perform various operations on them.
2. Understand use of transforms in analysis of signals and system in continuous and discrete time domain.
3. Observe the effect of various properties and operations of signals and systems.
4. Evaluate the time and frequency response of Continuous and Discrete time systems which are useful to understand the behavior of electronic circuits and communication systems.

Signals and Systems : (Semester IV – Electronics & Telecomm. Engineering (MU))

Module 1 : Introduction to Signals and Systems

1.1 **Introduction to signals :** Definition. Sampling theorem, Sampling of continuous time signals, Elementary signals, Exponential, Sine, Step, Impulse, Ramp, Rectangular, Triangular, Signum, Sinc, Operations on signals.

1.2 **Classification of signals :** Continuous and discrete time, Deterministic and non deterministic, Periodic and aperiodic, Symmetric (even) and asymmetric (odd), Energy and power, Causal and anti-causal signals. Case study of different signals from communication and biomedical field.

1.3 **Introduction to systems :** Definition, classification of systems : Static and dynamic, Time variant and time invariant, Linear and nonlinear, Causal and noncausal, Stable and unstable systems, Communication and control system as examples. **(Refer chapters 1 and 2)**

Module 2 : Time Domain Analysis of Continuous Time and Discrete Time Systems

2.1 Representation of systems using differential/difference equation. Impulse, step and exponential response, system stability.

2.2 Use of convolution integral and convolution sum for analysis of LTI systems, properties of convolution integral/sum. Impulse response of interconnected systems.

2.3 **Correlation and spectral density :** Auto correlation, Cross correlation, Analogy between correlation and convolution, Energy spectral density, Power spectral density, Relation between ESD, PSD with auto correlation. **(Refer chapters 3 and 4)**

Module 3 : Frequency Domain Analysis of Continuous and Discrete Signals

3.1 **Review of Fourier Series :** Trigonometric and exponential Fourier series representation of signals, Gibbs phenomenon, Discrete time Fourier series, Properties, Analogy between Continuous Time Fourier Series (CTFS) and Discrete Time Fourier Series (DTFS).

3.2 **Fourier Transform (FT) :** Fourier transform and inverse Fourier transform on periodic and non periodic signals, Limitations of CT/DT Fourier transform and need for Laplace/Z-transform.

3.3 **Overview of Laplace Transform :** Need of Laplace Transform, review of unilateral and bilateral Laplace Transform, properties, inverse of Laplace Transform, concept of Region of Convergence (ROC) poles and zeros, relation between continuous time Fourier Transform and Laplace Transform. **(Refer chapters 5, 6 and 7)**

Module 4 : Z-Transform

4.1 Need of Z-Transform, definition of unilateral and bilateral Z-Transform. Z-Transform of finite and infinite duration sequences, properties, Inverse Z-Transform, relation between discrete time Fourier Transform and Z-Transform, Z-Transform of standard signals, ROC for ZT, plotting poles and zeros of transfer function.

4.2 Analysis of discrete time LTI systems using Z-Transform : Transfer Function, causality and stability of systems, frequency response (impulse and step), relation between Laplace Transform and Z-Transform. **(Refer chapter 8)**

Module 5 : State Space Analysis and Realization Structures

5.1 **State variable analysis :** Introduction to the notion of 'state', systematic procedure for determining state equations, solution of state equations using Laplace transform, definition of exp (A) where A is a matrix, time domain solution of state equations.

5.2 Systems with finite duration and infinite duration, impulse response, recursive and non-recursive discrete time system, realization structures : direct form-I, direct form-II, Transpose, cascade, and parallel forms. **(Refer chapters 9 and 10)**

Module 6 : Applications of Signals and Systems

6.1 Signal processing Applications : Speech and Audio Processing, Multimedia (image and video) processing, Underwater acoustic signal processing Biological signal analysis.

6.2 Communication and Control System Application : Modulation (Analog and Digital) processing, Feedback/Feedforward Control system. **(Refer chapter 11)**

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Module 1

Chapter 1 : Introduction to Signals **1-1 to 1-60**

Syllabus : Introduction to signals : Definition. Sampling theorem, Sampling of continuous time signals, Elementary signals, Exponential, Sine, Step, Impulse, Ramp, Rectangular, Triangular, Signum, Sinc, Operations on signals.

Classification of signals : Continuous and discrete time, Deterministic and non deterministic, Periodic and aperiodic, Symmetric (even) and asymmetric (odd), Energy and power, Causal and anti-causal signals. Case study of different signals from communication and biomedical field.

1.1 Definition of Signal 1-1
1.2 Low Pass and Band Pass Signals 1-1
1.3 Sampling Process 1-2
1.4 Sampling Theorem for Low Pass Signals in Time Domain 1-2
1.4.1 Proof of Sampling Theorem 1-2
1.4.2 Aliasing or Foldover Error 1-6
1.4.3 Nyquist Rate and Nyquist Interval 1-7
1.4.4 Effect of Nonideal Filter 1-7
1.4.5 Examples on Sampling Theorem for Low Pass Signals 1-7
1.4.6 Sampling Theorem for Bandpass Signals 1-8
1.4.6.1 Quadrature Sampling of Bandpass Signals 1-8
1.5 Classification of Signals 1-11
1.5.1 Continuous and Discrete Time Signals 1-11
1.5.2 Continuous Valued or Discrete Valued Signals 1-12
1.5.3 Periodic and Non-periodic Signals 1-12
1.5.4 Deterministic and Random Signals 1-16
1.5.5 Symmetrical (Even) or Antisymmetrical (Odd) Signals 1-16
1.5.6 Decomposing a CT Signal into Even and Odd Parts 1-17
1.5.7 Even and Odd Components for a D.T. Signal x (n) 1-17
1.5.8 Energy and Power Signals 1-18
1.6 Energy and Power of the Signals 1-18
1.6.1 Time Average (dc Value) of a Signal 1-18
1.6.2 Power 1-18
1.6.3 Energy 1-19
1.6.4 Comparison of Energy and Power Signals 1-19
1.6.5 Average Power of a DT Signal 1-19
1.6.6 Energy of a DT Signal 1-19
1.6.7 Power of the Energy Signals 1-19
1.6.8 Energy of a Power Signal 1-20
1.6.9 Multichannel and Multidimensional Signals 1-24
1.7 Elementary or Basic Signals used for Testing 1-24
1.7.1 Reasons for using the Standard Signals 1-24
1.7.2 DC Signal 1-24
1.7.3 Sinusoidal Signals 1-25
1.7.4 Unit Step Signal 1-25
1.7.5 Signum Function 1-25
1.7.6 Rectangular Signals 1-26
1.7.7 Delta or Unit Impulse Function [δ (t)] 1-26
1.7.7.1 Important Properties of a Delta Function 1-27
1.7.8 Unit Sample Signal δ (n) 1-27
1.7.9 Unit Ramp Signal 1-28
1.7.10 C.T. Complex Exponential Signals 1-28
1.7.11 Discrete Time Exponential Signals 1-29
1.7.12 Sinc Function 1-30
1.7.13 Unit Triangle Function 1-31
1.8 Operations on Signals 1-31
1.8.1 Operations Performed on Dependent Variables 1-32
1.8.1.1 Amplitude Scaling of a Signal 1-32
1.8.1.2 Addition of Signals 1-32
1.8.1.3 Subtraction of Signals 1-33
1.8.1.4 Multiplication of Two Signals 1-34
1.8.1.5 Even and Odd Parts 1-34
1.8.1.6 Differentiation 1-35
1.8.1.7 Integration 1-35
1.8.1.8 Accumulation for D.T. Signal 1-35
1.8.2 Operations Performed on Independent Variable 1-35
1.8.2.1 Time Shifting 1-35
1.8.2.2 Time Scaling 1-37
1.8.2.3 Time Reversal (Time Inversion) or Folding 1-40
1.9 Relationship between Unit Step and Unit Impulse 1-43
1.9.1 Precedence Rule for Time Shifting and Time Scaling 1-43
1.10 Additional Solved Examples 1-45
1.11 University Questions and Answers 1-59
• **Review Questions** 1-59

Module 1

Chapter 2 : Introduction to Systems **2-1 to 2-20**

Syllabus : Introduction to systems : Definition, classification of systems : Static and dynamic, Time variant and time invariant, Linear and nonlinear, Causal and noncausal, Stable and unstable systems, Communication and control system as examples.

2.1 System 2-1
2.1.1 Types of Systems 2-1
2.1.2 Examples of Practical Systems 2-1
2.1.3 Classification of C.T. Systems 2-1

2.1.4 Classification of D.T. Systems 2-2
2.2 Linear and Non linear Systems 2-2
2.2.1 Linear and Nonlinear C.T. Systems 2-2
2.2.2 Linearity of D.T. System 2-2
2.3 Time Variant or Time Invariant Systems 2-5
2.3.1 Time Invariant and Time Variant C.T. Systems 2-5
2.3.2 Shift Invariant and Shift Variant D.T. Systems 2-5
2.4 Causal or Noncausal C.T. Systems 2-8
2.4.1 Causal and Noncausal C.T. Systems 2-8
2.4.2 Causal DT Systems 2-8
2.4.3 Anticausal or Non-causal DT Systems 2-8
2.5 Stable or Unstable Systems 2-9
2.5.1 Stable and Unstable C.T. Systems 2-9
2.5.2 Stable and Unstable D.T. Systems 2-9
2.6 Static or Dynamic D.T. Systems (Dynamicity Property) 2-10
2.6.1 Static DT Systems 2-10
2.6.2 Dynamic DT Systems 2-10
2.6.3 A Static or Dynamic C.T. System 2-10
2.7 Invertibility 2-11
2.7.1 Comparisons 2-12
2.8 Types (Examples) of Systems 2-12
2.8.1 Communication System 2-13
2.8.2 Control System 2-14
2.9 Solved Examples 2-14
2.10 Examples for Practice 2-19
2.11 Solved University Examples 2-20
• **Review Questions** 2-19

Module 2

Chapter 3 : Linear Time Invarient (LTI) Systems

3-1 to 3-51

Syllabus : Representation of systems using differential / difference equation. Impulse, step and exponential response, system stability.

Use of convolution integral and convolution sum for analysis of LTI systems, properties of convolution integral/sum. Impulse response of interconnected systems.

3.1 Representation of Systems Using Differential or Difference Equations 3-1
3.1.1 CT System Representation 3-1
3.1.1.1 RC Circuit Described by Difference Equation 3-2
3.1.1.2 RLC Circuit Described by Difference Equation 3-2
3.1.1.3 Impulse Response of LTI System 3-3
3.1.2 DT System Representation 3-4
3.1.2.1 LTI Systems Characterized by Constant Coefficient Difference Equations 3-6
3.1.2.2 Solution of Linear Constant Coefficient Difference Equation 3-6
3.2 Convolution Integral 3-10
3.2.1 Representation of C.T. Signal in terms of Impulses 3-10
3.2.2 Properties of Convolution Integral 3-10
3.3 Computation of Convolution Integral using Graphical Method 3-11
3.3.1 Unit step to Unit Step 3-11
3.3.2 Unit Step to Exponential 3-11
3.3.3 Exponential to Exponential 3-15
3.3.4 Exponential to Rectangular 3-15
3.3.5 Unit Step to Rectangular 3-17
3.3.6 Rectangular to Rectangular 3-17
3.4 Impulse Response of DT-LTI System 3-20
3.4.1 Representation of Discrete Time Signal as Weighted Impulses 3-21
3.5 Linear Convolution (Convolution Sum) 3-23
3.6 Methods of Convolution 3-23
3.6.1 Graphical Method 3-23
3.6.2 Calculation using Basic Equation of Convolution 3-33
3.6.3 Tabulation Method of Linear Convolution 3-35
3.6.4 Multiplication Method of Linear Convolution 3-37
3.7 Properties of Linear Convolution 3-38
3.8 Impulse Response of Interconnected Systems 3-40
3.9 System Properties in Terms of Impulse Response 3-44
3.9.1 Causal or Non-causal 3-44
3.9.2 Memory Less (Static) and With Memory (Dynamic) 3-45
3.9.3 Stable Systems 3-45
3.9.3.1 Proof of Stability Criteria For LTI (or LSI) Systems in Terms of Unit Impulse Response 3-46
3.9.3.2 Stability for CT Systems 3-46
3.9.3.3 Solved Examples on Stability 3-47
3.9.4 Invertible Systems and Deconvolution 3-47
3.9.4.1 Solved Examples on Invertible Systems 3-47
3.9.5 Step Response in terms of Impulse Response 3-48
3.9.5.1 Solved Examples on System Response 3-48
3.10 Exponential Response 3-51
• **Review Questions** 3-51

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Module 2

Chapter 4 : Correlation and Spectral Density 4-1 to 4-14

Syllabus : Correlation and spectral density : Auto correlation, Cross correlation, Analogy between correlation and convolution, Energy spectral density, Power spectral density, Relation between ESD, PSD with auto correlation.

4.1 Introduction 4-1
4.1.1 Correlation 4-1
4.2 Spectral Density Functions 4-1
4.2.1 Energy Spectral Density (ESD) 4-1
4.3 Power Spectral Density (PSD) 4-1
4.4 Correlation of CT Energy Signals 4-2
4.4.1 Auto-Correlation Function for the CT Energy Signals 4-2
4.4.2 Analogy between Autocorrelation and Convolution 4-2
4.4.3 Properties of Auto-Correlation Function of CT Energy Signals 4-2
4.4.4 Auto-Correlation Function of CT Power Signals 4-2
4.4.5 Properties of Auto-Correlation Function of a CT Periodic Signal 4-3
4.5 Cross Correlation of CT Signals 4-6
4.5.1 Cross-Correlation of CT Energy Signals 4-6
4.5.2 Analogy between Cross-Correlation and Convolution 4-6
4.5.3 Properties of Cross-Correlation of the CT Energy Signals 4-6
4.5.4 Cross-Correlation of CT Power Signals 4-7
4.5.5 Properties of Cross-Correlation of CT Periodic Signals 4-7
4.6 Correlation of Discrete Time Signals 4-8
4.6.1 Autocorrelation of D.T. Signals 4-9
4.6.2 Autocorrelation of D.T. Energy Signals 4-9
4.6.3 Analogy between Correlation and Convolution 4-9
4.6.4 Properties of Autocorrelation Function of D.T. Energy Signals 4-9
4.6.5 Autocorrelation of D.T. Power Signals 4-9
4.6.6 Properties of Autocorrelation Function of a D.T. Power Signal 4-9
4.6.7 Cross Correlation of D.T. Signals 4-9
4.6.8 Cross Correlation of D.T. Energy Signals 4-9
4.6.9 Properties of Cross Correlation of D.T. Energy Signals 4-10
4.6.10 Cross Correlation of D.T. Power Signals 4-10
4.7 Autocorrelation of Sum of Signals 4-13
4.8 Additional Solved Examples 4-14
• **Review Questions** 4-14

Module 3

Chapter 5 : Review of Fourier Series 5-1 to 5-34

Syllabus : Trignometric and exponential Fourier series representation of signals, Gibbs phenomenon, Discrete time Fourier series, Properties, Analogy between Continuous Time Fourier Series (CTFS) and Discrete Time Fourier Series (DTFS).

5.1 Frequency Domain Representation 5-1
5.1.1 How to Plot Line Spectrum ? 5-2
5.1.2 Double Sided Line Spectrum 5-2
5.2 CT Fourier Series 5-3
5.2.1 Trigonometric or Quadrature Fourier Series 5-3
5.2.2 Polar Fourier Series 5-4
5.2.3 Exponential Fourier Series [or Complex Exponential Fourier Series] 5-4
5.2.4 Dirichlet Conditions for the Existence of Fourier Series 5-5
5.2.4.1 Knowledge of Waveform Symmetry used to Calculate Fourier Coefficient 5-5
5.2.5 Examples on Fourier Series 5-6
5.3 Properties of C.T. Fourier Series 5-19
5.3.1 Linearity 5-19
5.3.2 Time Shifting 5-20
5.3.3 Time Reversal 5-20
5.3.4 Time Scaling 5-20
5.3.5 Frequency Shifting 5-21
5.3.6 Differentiation 5-21
5.4 Applications of Fourier Series 5-21
5.5 Gibb's Phenomenon 5-21
5.6 Fourier Series for Discrete Time Periodic Signals (DTFS) 5-25
5.6.1 Periodicity Condition 5-25
5.6.2 Properties of DTFS 5-25
5.7 Properties of DTFS 5-25
5.7.1 Linearity 5-25
5.7.2 Time Shifting 5-25
5.7.3 Frequency Shift 5-26
5.7.4 Convolution 5-26
5.7.5 Modulation or Multiplication 5-26
5.7.6 Parseval's Theorem 5-27
5.7.7 Duality 5-27
5.7.8 Symmetry 5-27
5.7.9 Summary of Properties of DTFS 5-28
5.8 Solved Examples on DTFS 5-28
5.9 Convergence of DTFS 5-30
5.9.1 Analogy between CTFS and DTFS 5-31
5.10 Additional Solved Examples 5-31
• **Review Questions** 5-34

Module 3

Chapter 6 : Fourier Transform 6-1 to 6-39

Syllabus : Fourier transform and inverse Fourier transform on periodic and non periodic signals, Limitations of CT/DT Fourier transform and need for Laplace/Z-transform.

6.1 C.T. Fourier Transform 6-1
6.1.1 Necessity of Fourier Transform 6-1
6.1.2 Definition of Fourier Transform 6-1
6.1.3 Definition of Inverse Fourier Transform 6-1
6.1.4 Conditions for the Existence of Fourier Transform 6-2
6.1.5 Amplitude and Phase Spectrums 6-2
6.2 Properties of Fourier Transform 6-2
6.2.1 Property 1 : Linearity or Superposition 6-2
6.2.2 Property 2 : Time Scaling 6-2
6.2.3 Property 3 : Duality or Symmetry Property 6-7
6.2.4 Property 4 : Time Shifting 6-8
6.2.5 Property 5 : Area under x(t) 6-9
6.2.6 Property 6 : Area under X (f) 6-9
6.2.7 Property 7 : Frequency Shifting 6-10
6.2.8 Property 8 : Differentiation in Time Domain 6-10
6.2.9 Property 9 : Integration in Time Domain 6-11
6.2.10 Property 10 : Multiplication in Time Domain (Multiplication Theorem) 6-11
6.2.11 Property 11 : Convolution in the Time Domain (Convolution Theorem) 6-11
6.2.12 Conjugate Functions 6-12
6.2.13 Parseval's Power Theorem 6-13
6.2.14 Solved University Examples on Properties of F.T. 6-13
6.3 Solved Examples on C.T. Fourier Transform 6-15
6.4 Fourier Transform of a Periodic Signal 6-21
6.5 Application of Fourier Series and Fourier Transform to the System Analysis 6-26
6.5.1 Frequency Response of an LTI System 6-26
6.5.2 Amplitude and Phase Response of a System 6-26
6.5.3 Differential Equations 6-27
6.6 Merits of Fourier Transform 6-29
6.7 Limitations of Fourier Transform and Need of LT and ZT 6-29
6.8 Introduction to Discrete Time Fourier Transform 6-29
6.9 Analogy between CTFT and DTFT 6-29
6.10 Fourier Transform of Discrete Time Aperiodic Signals (DTFT) 6-29
6.10.1 Periodic Nature of X (ω) 6-30
6.11 Discrete Fourier Transform of Standard Signals 6-30
6.11.1 Summary of Properties of DTFT 6-34
6.12 Additional Solved Examples 6-35
6.13 University Questions and Answers 6-38
• **Review Questions** 6-37

Module 3

Chapter 7 : Laplace Transform 7-1 to 7-39

Syllabus : Overview of Laplace transform : Need of Laplace Transform, review of unilateral and bilateral Laplace Transform, properties, inverse of Laplace Transform, concept of Region of Convergence (ROC) poles and zeros, relation between continuous time Fourier Transform and Laplace Transform.

7.1 Introduction 7-1
7.1.1 Need of Laplace Transform (Application of Laplace Transform) 7-1
7.1.2 What is the 's' Domain ? 7-1
7.1.3 Definition of Laplace Transform 7-1
7.2 Sufficient Conditions for Existance of Laplace Transform 7-2
7.2.1 Region of Convergence 7-2
7.2.2 Pole - Zero Concept 7-2
7.3 Laplace Transform of Elementary Signals 7-3
7.3.1 Delta Function 7-3
7.3.2 Unit step 7-3
7.3.3 Unit Ramp 7-4
7.3.4 Positive Sided Growing Exponential Pulse 7-4
7.3.5 Positive Sided Decaying Exponential Signal 7-5
7.3.6 Negative Sided (Left Handed) Exponential Signal 7-5
7.3.7 Laplace Transform of Two Sided Exponential Signal 7-6
7.3.8 Laplace Transform of Sine Wave 7-6
7.3.9 Laplace Transform of Cosine Wave 7-7
7.3.10 Laplace Transform of Damped Sine Wave 7-7
7.3.11 Laplace Transform of Damped Cosine Wave 7-8
7.3.12 x (t) = sinh at 7-8
7.3.13 x (t) = cosh at 7-9
7.3.14 x (t) = t^n 7-9

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7.4 Standard Laplace Transform Pairs 7-10
7.5 Properties of Laplace Transform 7-10
7.5.1 Linearity 7-10
7.5.1.1 Solved Examples on Linearity Property 7-10
7.5.2 Time Shifting (Shifting in Time Domain) 7-12
7.5.3 Shifting in s - Domain 7-12
7.5.3.1 Solved Examples on Shifting Property 7-12
7.5.4 Time Scaling 7-13
7.5.5 Time Differentiation (Differentiation in Time Domain) 7-14
7.5.6 Differentiation in s-Domain 7-14
7.5.6.1 Solved Examples on Scaling and Differentiation Property 7-14
7.5.7 Convolution Property 7-15
7.5.8 Time Integration (Integration in Time Domain) 7-15
7.5.9 Integration in s-domain (Division by t) 7-15
7.5.10 Laplace Transform of Periodic Signal 7-16
7.5.11 Conjugation 7-16
7.5.12 Initial Value Theorem 7-16
7.5.13 Final Value Theorem 7-16
7.5.14 Summary of Laplace Transform Properties 7-16
7.5.14.1 Solved Examples on Properties of Laplace Transform 7-17
7.6 Laplace Transform used for Waveform Synthesis 7-20
7.7 One Sided (Unilateral) Laplace Transform 7-26
7.7.1 Properties of U.T. (Unilateral Laplace Transform) 7-26
7.7.2 Initial Value Theorem 7-26
7.7.3 Final Value Theorem 7-26
7.8 Inverse Laplace Transform 7-27
7.8.1 Necessity of Inverse Laplace Transform 7-28
7.8.2 Definition of Inverse Laplace Transform 7-28
7.8.3 Calculation of Inverse Laplace using Duality 7-28
7.8.3.1 Solved Examples on Inverse Laplace using Duality 7-28
7.8.4 Partial Fraction Expansion Method (P.F.E.) 7-29
7.8.4.1 Solved Examples on P.F.E. 7-29
7.9 Relationship between Laplace Transform and C.T.F.T. 7-37
7.10 Examples for Practice 7-38
• **Review Questions** 7-38

Module 4

Chapter 8 : Z-Transform 8-1 to 8-89

Syllabus : Need of Z-Transform, definition of unilateral and bilateral Z-Transform. Z-Transform of finite and infinite duration sequences, properties, Inverse Z-Transform, relation between discrete time Fourier Transform and Z-Transform, Z-Transform of standard signals, ROC for ZT, plotting poles and zeros of transfer function. Analysis of discrete time LTI systems using Z-Transform : Transfer Function, causality and stability of systems, frequency response (impulse and step), relation between Laplace Transform and Z-Transform.

8.1 Need of Z-transform 8-1
8.1.1 Advantages of Z-transform 8-1
8.2 Definition of Uniliteral and Bilateral Z-transform 8-1
8.2.1 Region of Convergence (ROC) (Meaning and Significance) 8-2
8.2.1.1 Solved Examples using Definition of Z 8-2
8.2.2 Z-transform of Standard Sequences 8-5
8.2.3 Properties of ROC 8-11
8.3 Properties of Z-transform 8-11
8.3.1 Linearity 8-11
8.3.1.1 Solved Examples on Linearity Property 8-11
8.3.2 Time Shifting 8-16
8.3.2.1 Solved Examples on Time Shifting Property 8-16
8.3.3 Scaling in the Z-domain (Multiplication by Exponential Sequence) 8-17
8.3.3.1 Solved Examples on Scaling Property 8-18
8.3.4 Time Reversal Property 8-20
8.3.4.1 Solved Examples on Time Reversal Property 8-21
8.3.5 Differentiation in Z domain 8-23
8.3.5.1 Solved Examples of Differential Property 8-23
8.3.6 Convolution of Two Sequences 8-25
8.3.7 Initial Value Theorem 8-29
8.3.8 Final Value Theorem 8-29
8.4 Summary of the Properties of Z- Transform 8-31

8.5 Inverse Z Transform (IZT) 8-32
8.5.1 Power Series Expansion Method 8-32
8.5.1.1 Solved Examples on IZT using Power Series Expansion 8-32
8.5.2 Partial Fraction Expansion (P.F.E.) Method 8-36
8.5.2.1 Solved Examples using P.F.E. 8-36
8.5.2.2 Numericals on Calculations of Convolution using Z-transform 8-46
8.5.2.3 Calculation of Inverse Z using Convolution Theorem 8-49
8.5.3 IZT using Cauchy's Residue Theorem 8-51
8.5.3.1 Solved Examples using Residue Method 8-51
8.6 Relationship between Fourier Transform and Z-Transform 8-52
8.7 Plotting Poles and Zeros of Transfer Function : 8-53
8.7.1 Pole Zero Plot 8-53
8.7.2 Behaviours of Causal DT Signals 8-54
8.7.3 Solved Examples on Pole Zero Plot 8-56
8.8 System Function H (Z) 8-58
8.9 Solved Examples on Transfer Function H (Z) 8-58
8.10 Analysis of Causality and Stability of DT System 8-63
8.11 Solved Examples on Stability and Causality 8-64
8.12 Unilateral (One-Sided) Z-transform 8-76
8.12.1 Shifting Property of Unilateral Z-Transform 8-76
8.12.2 Solution of Difference Equation using Unilateral Z Transform (Impulse and Step Response) 8-77
8.13 Numericals Based on Calculation of Frequency Response 8-83
8.14 Frequency Response from Pole-Zero Plot using Simple Geometric Construction 8-85
8.15 Relationship of Z-transform with Laplace Transform (Mapping between S-plane and Z-plane) 8-89
• **Review Questions** 8-89

Module 5

Chapter 9 : State Space Analysis 9-1 to 9-14

Syllabus : State variable analysis : Introduction to the notion of 'state', systematic procedure for determining state equations, solution of state equations using Laplace transform, definition of exp (A) where A is a matrix, time domain solution of state equations.

9.1 Introduction to Notation of State 9-1
9.1.1 Definition of State and State Variables, State Vectors and State Space 9-1
9.2 Systematic Procedure for Determining State Equations 9-2
9.2.1 State Model of Linear Systems 9-3
9.2.2 State Diagram Representation 9-4
9.2.3 Non-uniqueness of the State Variable 9-5
9.2.4 Different Representation of State Model 9-5
9.2.5 Forming State Models by Physical Variables 9-5
9.2.6 Solution of LTI State Equations 9-8
9.2.7 Solution of Homogeneous State Equation 9-8
9.3 Definition of exp (A), where 'A' is Matrix 9-9
9.3.1 Properties of State Transition Matrix 9-9
9.4 Time Domain Solution of State Equations 9-12
9.5 Solution of State Equations using Laplace Transform 9-13
• **Review Questions** 9-14

Module 5

Chapter 10 : Realization Structure 10-1 to 10-30

Syllabus : Systems with finite duration and infinite duration, impulse response, recursive and non-recursive discrete time system, realization structures : direct form-I, direct form-II, Transpose, cascade, and parallel forms.

10.1 Systems with Finite Duration and Infinite Duration, Impulse Response, Recursive and Nonrecursive Systems 10-1
10.2 Realization Structures for FIR Systems 10-1
10.2.1 Direct Form Structure 10-1
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Module 6

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Communication and Control System Application : Modulation (Analog and Digital) process, Feedback / Feedforward Control system.

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❑❑❑

CHAPTER 1

Module 1

Introduction to Signals

Syllabus :

Introduction to signals : Definition. Sampling theorem, Sampling of continuous time signals, Elementary signals, Exponential, Sine, Step, Impulse, Ramp, Rectangular, Triangular, Signum, Sinc, Operations on signals.

Classification of signals : Continuous and discrete time, Deterministic and non deterministic, Periodic and aperiodic, Symmetric (even) and asymmetric (odd), Energy and power, Causal and anti-causal signals. Case study of different signals from communication and biomedical field.

1.1 Definition of Signal :

- In a communication system, the word 'signal' is used very commonly. Therefore we must know its exact meaning.
- Mathematically, signal is described as a function of one or more independent variables.
- Basically it is a physical quantity. It varies with some dependent or independent variables.
- So the term signal is defined as "A physical quantity which contains some information and which is function of one or more independent variables."
- The signals can be one-dimensional or multidimensional.

One dimensional signals :

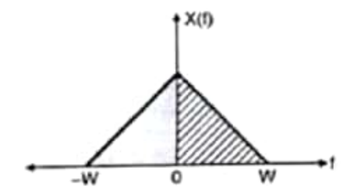
- When a function depends only on a single variable, the signal is said to be one dimensional.
- Example of one dimensional signal is speech signal whose amplitude varies with time.

Multidimensional signals :

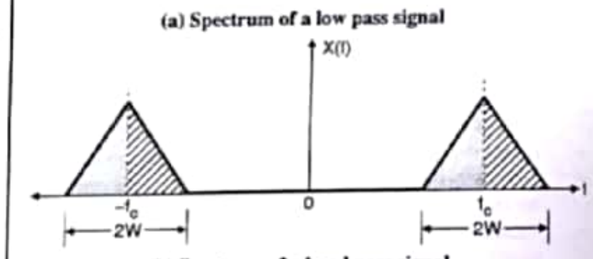
- When a function depends on two or more variables, the signal is said to be multidimensional.
- The example of a multidimensional signal is an image because it is a two dimensional signal with horizontal and vertical co-ordinates.

1.2 Low Pass and Band Pass Signals :

- A low pass signal is defined as the signal which has a frequency spectrum extending right from 0 Hz to W Hz, if a single sided spectrum is plotted and it extends from – W to + W, if a double sided spectrum is plotted as shown in Fig. 1.2.1(a).
- A video signal which has a frequency spectrum from 0 to 5 MHz is an example of low pass signal.



(a) Spectrum of a low pass signal



(b) Spectrum of a band pass signal

(D-406) **Fig. 1.2.1**

- But the nature of the spectrum of a bandpass signal is completely different. It is as shown in Fig. 1.2.1(b). The bandwidth is 2W Hz but it is centered about a frequency $\pm f_c$ rather than zero. We assume that $f_c > W$, therefore $(f_c - W) > 0$.
- Thus bandpass signals are the signals having their spectrum extending from f_1 to f_2 with $f_2 > f_1$ and both f_1 and f_2 being of nonzero value. For example, the voice signal which has a spectrum extending from 20 Hz to 3.4 kHz is a bandpass signal.

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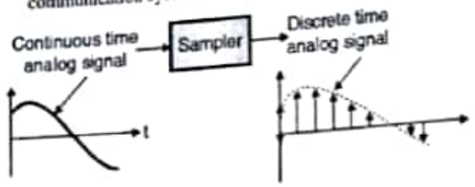
1.3 Sampling Process :

- In the pulse modulation and digital modulation systems, the signal to be transmitted must be in the discrete time form.
- If the message signal is coming from a digital source (e.g. a digital computer) then it is in the proper for a digital communication system to be processed.



(L-156) **Fig. 1.3.1 : Sampling process**

- But this is not always the case. The message signal can be analog in nature (e.g. speech or video signal).
- In such a case it has to be first converted into a discrete time signal. We use the "sampling process" to do this.
- Thus using the sampling process we convert a continuous time signal into a discrete time signal.
- For the sampling process to be of practical utility it is necessary to choose the sampling rate properly. The sampling process should satisfy the following requirements :
  1. Sampled signal should represent the original signal faithfully.
  2. We should be able to reconstruct the original signal from its sampled version.
- Fig. 1.3.1 summarizes the sampling process.
- Thus sampling is the process of converting a continuous analog signal to a discrete analog signal and the sampled signal is the discrete time representation of the original analog signal.

1.4 Sampling Theorem for Low Pass Signals in Time Domain :

- In order to represent the original message signal "faithfully" (without loss of information), it is necessary to take as many samples of the original signal as possible.
- Higher the number of samples, closer is the representation.
- The number of samples depends on the "sampling rate" and the maximum frequency of the signal to be sampled.
- Sampling theorem was introduced to the communication theory in 1949 by Shannon. Therefore this theorem is also called as **"Shannon's sampling theorem"**.
- The statement of sampling theorem in time domain, for the bandlimited signals of finite energy is as follows :

Statement :

1. If a finite energy signal x (t) contains no frequencies higher than "W" Hz (i.e. it is a band limited signal) then it is completely determined by specifying its values at the instants of time which are spaced (1/2W) seconds apart.
2. If a finite energy signal x (t) contains no frequency components higher than "W" Hz then it may be completely recovered from its samples which are spaced (1/2W) seconds apart.

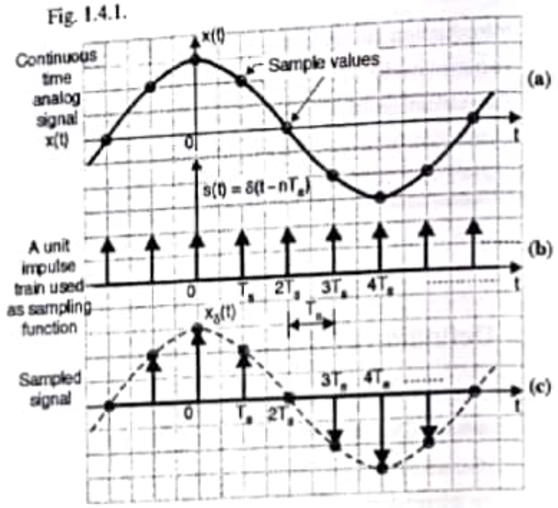
> Combined statement of sampling theorem : A continuous time signal x (t) can be completely represented in its sampled form and recovered back from the sampled form if the sampling frequency $f_s \geq 2W$ where "W" is the maximum frequency of the continuous time signal x (t).

1.4.1 Proof of Sampling Theorem :

Let us now prove the sampling theorem in time domain. The assumptions made for this proof are as follows :

Assumptions :

- Let x (t) be a continuous time analog signal as shown in Fig. 1.4.1.



(D-408) **Fig. 1.4.1 : Sampling of a continuous time signal x (t)**

- Let x (t) be a signal with finite energy and infinite duration.
- Let x (t) be a strictly bandlimited signal. That means it does not contain any frequency components above "W" Hz.
- Let s (t) be the sampling function as shown in Fig. 1.4.1. It is a train of unit impulses, spaced by a period of T_s seconds. This sampling function samples the original signal at a rate of "f_s" samples per second. Therefore "T_s" represents the sampling period such that,

$$T_s = \frac{1}{f_s} = \text{Sampling period} \qquad ...(1.4.1)$$

and $$f_s = \frac{1}{T_s} = \text{Sampling rate.}$$

Procedure to be followed :

We are going to follow the steps given below to prove the sampling theorem :

Step 1 :	Represent the sampling function s (t) mathematically.
Step 2 :	Represent the sampled signal x_δ (t) mathematically.
Step 3 :	Obtain the Fourier transform of the sampled signal.
Step 4 :	Prove that the sampled signal x_δ (t) completely represents x (t).
Step 5 :	Represent x (t) as summation of sinc functions (interpolation).
Step 6 :	Graphical representation of the interpolation process.
Step 7 :	Actual recovery of x (t) using an ideal low pass filter.

Part 1 : Sampling theorem :

Spectrum of the sampled signal :

Step 1 : Represent the sampling function s(t) mathematically :

- Fig. 1.4.1 shows the sampling function s (t) which is a train of unit impulses.
- The spacing between the adjacent unit impulses is T_s seconds, therefore the frequency of the sampling function is equal to the sampling frequency f_s.
- The sampled signal is denoted by x_δ (t) and it is as shown in Fig. 1.4.1.
- The sample function s (t) can be represented mathematically as follows :

$$s(t) = ...\delta(t + 2T_s) + \delta(t + T_s) + \delta(t) + \delta(t - T_s) + \delta(t - 2T_s) + ...$$

$$\therefore\ s(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \qquad ...(1.4.2)$$

Step 2 : Represent the sampled signal x_δ (t) mathematically :

- Fig. 1.4.1 shows the sampled signal x_δ (t) graphically. It is present only at the sampling instants i.e. T_s, 2 T_s etc. and its instantaneous amplitude is equal to the amplitude of original signal x (t) at the sampling instants.
- This is shown by the encircled points in Fig. 1.4.1. Let us represent the instantaneous amplitude of x (t) at the various sampling points t = n T_s as x (n T_s). This is the amplitude of the encircled points of Fig. 1.4.1.
- Looking at the sampled signal x_δ (t) we can say that the sampled signal is obtained by multiplying x (t) and s (t).

$$\therefore\ x_\delta(t) = x(t) \times s(t) = x(nT_s) \times s(t) \qquad ...(1.4.3)$$

- Substituting the expression for s (t) from Equation (1.4.2) we get the mathematical expression for the sampled signal x_δ (t) as,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s) \qquad ...(1.4.4)$$

Step 3 : Obtain the Fourier transform of the sampled signal :

- The fourier transform of a train of impulses (dirac delta function) is given by,

$$X(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

- Here we have the similar pulse train as sampling function s (t). Therefore the Fourier transform of the sampling function is given by,

$$S(f) = f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \qquad ...(1.4.5)$$

- Note that f_0 has been replaced by f_s in the above equation.
- The sampled signal in the time domain is represented as product of x (t) and s (t).

i.e. $$x_\delta(t) = x(t) \times s(t) \qquad ...(1.4.6)$$

- Taking the Fourier transform of both the sides we get,

i.e. $$X_\delta(f) = X(f) * S(f) \qquad ...(1.4.7)$$

- This is because the Fourier transform of the product of two signals in the time domain is the convolution of their Fourier transforms. Substituting the value of S(f) from Equation (1.4.5) we get,

$$X_\delta(f) = X(f) * \left[f_s \sum_{n=-\infty}^{\infty} \delta(f - nf_s) \right] \qquad ...(1.4.8)$$

where * denotes convolution. Interchanging the orders of convolution and summation results in :

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f) * \delta(f - nf_s) \qquad ...(1.4.9)$$

- From the properties of delta function, we find that the convolution of X (f) and δ (f − nf_s) is equal to X (f − nf_s). Hence the above equation can be simplified as follows :

F.T. of the sampled signal,

$$X_\delta(f) = f_s \sum_{n=-\infty}^{\infty} X(f - nf_s) \qquad ...(1.4.10)$$

where X (f) = Fourier transform of the original signal x (t).

Conclusion from Equation (1.4.10) :

1. The term X (f − nf_s) in Equation (1.4.10) represents the shifted version of the spectrum X (f) of the original signal x (t). Thus depending on the value of "n" (which extends from − ∞ to + ∞) we will get infinite number of original spectrums X (f) centered at frequencies 0, ± f_s, ± 2f_s, ± 3f_s, ± 4f_s ...etc. In other words,

$$X(f - nf_s) = X(f) \text{ at } f = 0, \pm f_s, \pm 2f_s, \pm 3f_s \qquad ...(1.4.11)$$

2. This concept will be clear if we open Equation (1.4.10) and write the terms separately as shown below.

Now open the summation sign in Equation (1.4.10) to get,

$$X_\delta(f) = ... + \underbrace{f_s X(f + 2f_s)}_{} + \underbrace{f_s X(f + f_s)}_{} + \underbrace{f_s X(f)}_{} + \underbrace{f_s X(f - f_s)}_{} + ...$$

X(f) shifted right by f_s

Spectrum X(f)

X(f) shifted left by f_s

X(f) shifted left by 2f_s

(D-409)

The spectrum X_δ (f) of the sampled signal is plotted as shown in Fig. 1.4.2.

Equation (1.4.10) can also be written as :

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} f_s X(f - nf_s) \qquad ...(1.4.12)$$

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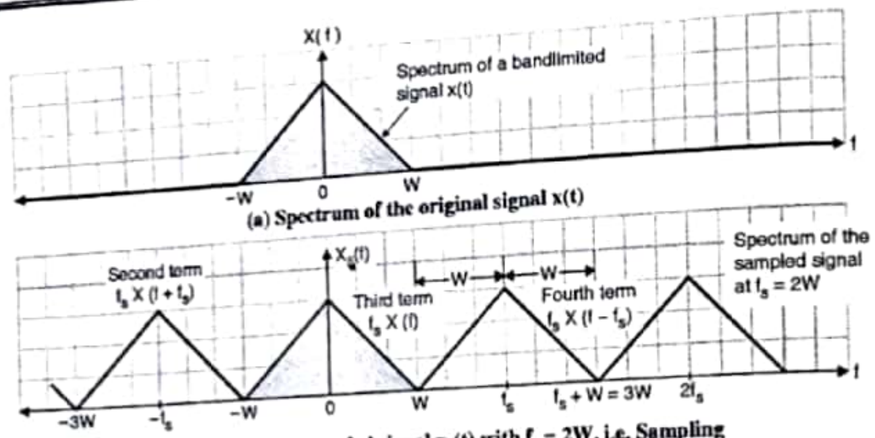


(a) Spectrum of the original signal x(t)

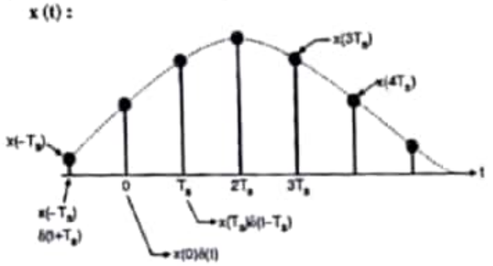
(b) Spectrum of the sampled signal $x_\delta(t)$ with $f_s = 2W$, i.e. Sampling is done exactly at Nyquist rate

(D-410) **Fig. 1.4.2**

Comment :

From Equation (1.4.12) we conclude that the process of uniform sampling of a signal in the time domain results in a periodic spectrum in the frequency domain with a period equal to the sampling rate f_s.

3. Prove that sampled signal x_δ (t) completely represents x (t) :



(D-411) **Fig. 1.4.3 : Sampled signal x_δ (t)**

– x_δ (t) can be represented in the summation form as follows (Refer Fig. 1.4.3).

$$x_\delta(t) = \ldots x(-T_s)\,\delta(t + T_s) + x(0)\,\delta(t) + x(T_s)\,\delta(t - T_s) + \ldots \quad \ldots(1.4.12(a))$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t - nT_s)$$

– We can obtain another useful expression for the fourier transform X_δ (f) by taking the fourier transform of both the sides of the equation stated above as,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\,e^{-j2\pi fT_s} \quad \ldots(1.4.13)$$

– This equation is the fourier transform of a discrete time signal x_δ (t). Therefore it is called as the discrete fourier transform (DFT). Compare it with the definition of fourier transform of a continuous time signal. i.e.

$$X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$$

– As the signal is discrete, the integration sign has been replaced by the summation sign and "t" has been replaced by "nT_s".

– Now consider Equation (1.4.12)

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f_s X(f - nf_s)$$

$$\therefore\ X(f) = \frac{1}{f_s}X_\delta(f) - \sum X(f - nf_s)$$

– But in the range $-W \le f \le W$ the second term of the above expression will not be present

$$\therefore\ X(f) = \frac{1}{f_s}X_\delta(f) \quad \ldots(1.4.14)$$

– Substitute $f_s = 2W$ and X_δ (f) from Equation (1.4.13) to get,

$$X(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j2\pi fT_s} \quad \ldots(1.4.15)$$

– This is the frequency spectrum of x (t) in terms of x (nT_s) i.e. the sampled signal.

Substitute $T_s = 1/2W$ to get,

$$X(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W)\cdot e^{-j2\pi f/2W}$$

$$\ldots -W \le f \le W \quad \ldots(1.4.16)$$

– This equation shows that the spectrum of x (t) is same as the spectrum of x_δ (t) in the frequency range – W to + W . Hence the sampled signal represents the original signal x (t) successfully.

Thus if the sample values x (n/2W) of the signal x (t) are specified for all time, then the Fourier transform X (f) of the original signal is uniquely determined by using the Equation (1.4.16). Because x (t) is related to X (f) by the inverse Fourier transform, it follows that the signal x (t) is itself uniquely determined by the sample values x (n/2W) for $-\infty \le n \le \infty$. Or in other words the sequence of samples {x (n / 2W)} contains all the information of x (t).

– Thus we have proved first part of the sampling theorem.

Part 2 of the sampling theorem :

4. Reconstruction of signal from samples :

– This is the second part of the sampling theorem. From Equation (1.4.16) we can obtain x (t) by taking the inverse fourier transform (IFT).

$$x(t) = \text{IFT}\{X(f)\}$$

$$= \text{IFT}\left\{\frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W).\,e^{-j\pi f n/W}\right\}$$

– Using the definition of inverse Fourier transform,

$$x(t) = \int_{-W}^{W}\frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W)\cdot e^{-j\pi f n/W}\,e^{j2\pi ft}\,df$$

– Interchanging the order of summation and integration we get,

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{1}{2W}\int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}\,df$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\cdot\frac{1}{2W}$$

$$\times\frac{1}{j2\pi\left[t-\frac{n}{2W}\right]}\cdot\left[e^{j2\pi f(t-n/2W)}\right]_{-W}^{W}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{1}{j4\pi W\left[t-\frac{n}{2W}\right]}\cdot$$

$$\left[e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n/2W)\cdot\left[\frac{e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}}{j4\pi W\left[t-\frac{n}{2W}\right]}\right]$$

– The term inside the square bracket is a "sine" function.

$$\therefore\ x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)} \quad \ldots(1.4.17)$$

– We can simplify the equation above by using the definition of the "sinc function". The sinc function is defined as :

$$\text{sinc}\,x = \frac{\sin(\pi x)}{\pi x} \quad \ldots(1.4.18)$$

– Therefore Equation (1.4.17) can be written as :

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\,\text{sinc}(2Wt - n) \quad \ldots(1.4.19)$$

– Equation (1.4.19) provides an interpolation formula for reconstructing the original signal x (t) from the sequence of sample values {x (n /2W)}. The "sinc" function plays the role of an interpolation function. Each sample x (n / 2W) is multiplied by a delayed version of the interpolation function i.e. sinc function. Then all these resulting waveforms are added to obtain x (t).

5. Graphical representation of the interpolation process :

Let us re-arrange Equation (1.4.19) as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2W\left(t - \frac{n}{2W}\right)$$

This is because $\frac{1}{2W} = T_s$

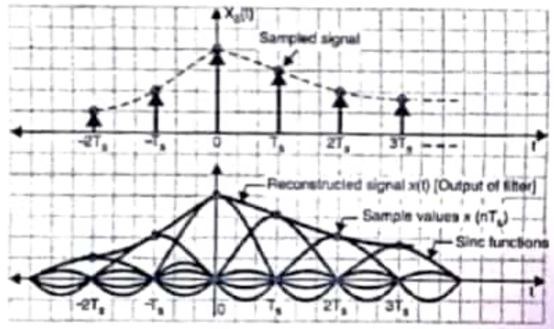
$$\therefore\ x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2W(t - nT_s) \quad \ldots(1.4.20)$$

Let us expand this equation to write,

$$x(t) = x(0)\,\text{sinc}\,2Wt + x(\pm T_s)\,\text{sinc}\,2W(t \pm T_s) + x(\pm 2T_s)\,\text{sinc}\,2W(t \pm 2T_s) + \ldots \quad \ldots(1.4.21)$$

(a) First term : x (0) sinc 2Wt :

– This will have a maximum amplitude at t = 0. The maximum amplitude is equal to the sample value x (0) at t = 0. This sinc function will pass through zeros at t = ± 1/2 W,± 1/4 W ...etc. This is as shown in Fig. 1.4.4.



(D-412) **Fig. 1.4.4 : Reconstruction of the original signal x (t) from its samples using the interpolation**

(b) Second term : x (± T_s) sinc 2W (t ± T_s) :

– This sinc function will have maximum amplitude at t = ± T_s. The maximum amplitude is equal to the sample value x (± T_s) at t = ± T_s respectively. Thus sinc 2W (t ± T_s) represents shifted sinc function i.e. "sinc 2Wt" by a period ± T_s. This is as shown in Fig. 1.4.4.

– Similarly the third term, x (± 2T_s) sinc 2W (t ± 2T_s) represents shifted sinc function "sinc 2Wt" by a period of ± 2T_s and so on. We can plot all these sinc functions along with the sampled signal x_δ (t) as shown in Fig. 1.4.4. Note that the peak amplitude of any sinc function is equal to the corresponding sample value x (nT_s).

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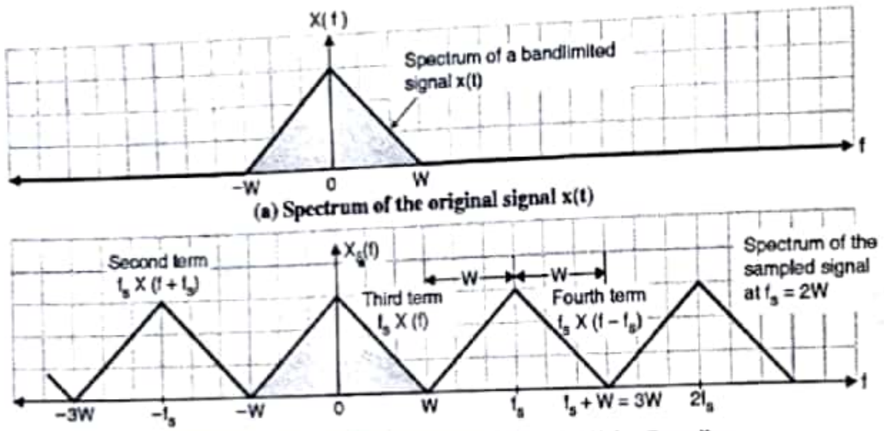


(a) Spectrum of the original signal x(t)

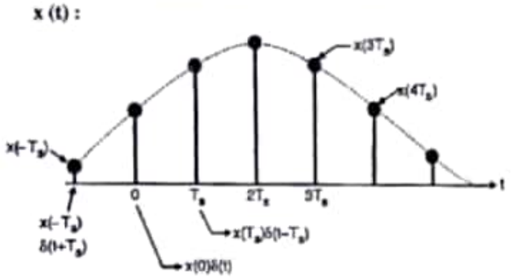
(b) Spectrum of the sampled signal $x_\delta(t)$ with $f_s = 2W$, i.e. Sampling is done exactly at Nyquist rate

(D-410) **Fig. 1.4.2**

Comment :

From Equation (1.4.12) we conclude that the process of uniform sampling of a signal in the time domain results in a periodic spectrum in the frequency domain with a period equal to the sampling rate f_s.

3. Prove that sampled signal x_δ (t) completely represents x (t) :



(D-411) **Fig. 1.4.3 : Sampled signal x_δ (t)**

- x_δ (t) can be represented in the summation form as follows (Refer Fig. 1.4.3).

$$x_\delta(t) = \ldots x(-T_s)\,\delta(t+T_s) + x(0)\,\delta(t) + x(T_s)\,\delta(t-T_s) + \ldots \quad \ldots(1.4.12(a))$$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t-nT_s)$$

- We can obtain another useful expression for the fourier transform X_δ (f) by taking the fourier transform of both the sides of the equation stated above as,

$$X_\delta(f) = \sum_{n=-\infty}^{\infty} x(nT_s)\,e^{-j2\pi fT_s} \quad \ldots(1.4.13)$$

- This equation is the fourier transform of a discrete time signal x_δ (t). Therefore it is called as the discrete fourier transform (DFT). Compare it with the definition of fourier transform of a continuous time signal. i.e.

$$X(f) = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt$$

- As the signal is discrete, the integration sign has been replaced by the summation sign and "t" has been replaced by "nT_s".
- Now consider Equation (1.4.12)

$$X_\delta(f) = f_s X(f) + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} f_s X(f-nf_s)$$

$$\therefore\ X(f) = \frac{1}{f_s}X_\delta(f) - \sum X(f-nf_s)$$

- But in the range $-W \le f \le W$ the second term of the above expression will not be present

$$\therefore\ X(f) = \frac{1}{f_s}X_\delta(f) \quad \ldots(1.4.14)$$

- Substitute $f_s = 2W$ and X_δ (f) from Equation (1.4.13) to get,

$$X(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} x(nT_s)\cdot e^{-j2\pi fT_s} \quad \ldots(1.4.15)$$

- This is the frequency spectrum of x (t) in terms of x (nT_s) i.e. the sampled signal.

Substitute $T_s = 1/2W$ to get,

$$X(f) = \frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W)\cdot e^{-j2\pi nf/2W}$$

$$\ldots -W \le f \le W \quad \ldots(1.4.16)$$

- This equation shows that the spectrum of x (t) is same as the spectrum of x_δ (t) in the frequency range – W to + W . Hence the sampled signal represents the original signal x (t) successfully.

Thus if the sample values x (n/2W) of the signal x (t) are specified for all time, then the Fourier transform X (f) of the original signal is uniquely determined by using the Equation (1.4.16). Because x (t) is related to X (f) by the inverse Fourier transform, it follows that the signal x (t) is itself uniquely determined by the sample values x (n/2W) for $-\infty \le n \le \infty$. Or in other words the sequence of samples {x (n / 2W)} contains all the information of x (t).

- Thus we have proved first part of the sampling theorem.

Part 2 of the sampling theorem :

4. Reconstruction of signal from samples :

- This is the second part of the sampling theorem. From Equation (1.4.16) we can obtain x (t) by taking the inverse fourier transform (IFT).

$$x(t) = \text{IFT}\{X(f)\}$$

$$= \text{IFT}\left\{\frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W).\,e^{-j\pi fn/W}\right\}$$

- Using the definition of inverse Fourier transform,

$$x(t) = \int_{-W}^{W}\frac{1}{2W}\sum_{n=-\infty}^{\infty} x(n/2W)\cdot e^{-j\pi fn/W}e^{j2\pi ft}\,df$$

- Interchanging the order of summation and integration we get,

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{1}{2W}\int_{-W}^{W} e^{j2\pi f\left(t-\frac{n}{2W}\right)}\,df$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\cdot\frac{1}{2W}\times\frac{1}{j2\pi\left[t-\frac{n}{2W}\right]}\cdot\left[e^{j2\pi f(t-n/2W)}\right]_{-W}^{W}$$

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{1}{j4\pi W\left[t-\frac{n}{2W}\right]}\cdot\left[e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n/2W)\cdot\left[\frac{e^{j2\pi W(t-n/2W)} - e^{-j2\pi W(t-n/2W)}}{j4\pi W\left[t-\frac{n}{2W}\right]}\right]$$

- The term inside the square bracket is a "sine" function.

$$\therefore\ x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\frac{\sin(2\pi Wt - n\pi)}{(2\pi Wt - n\pi)} \quad \ldots(1.4.17)$$

- We can simplify the equation above by using the definition of the "sinc function". The sinc function is defined as :

$$\text{sinc}\,x = \frac{\sin(\pi x)}{\pi x} \quad \ldots(1.4.18)$$

- Therefore Equation (1.4.17) can be written as :

$$x(t) = \sum_{n=-\infty}^{\infty} x(n/2W)\,\text{sinc}\,(2Wt - n) \quad \ldots(1.4.19)$$

- Equation (1.4.19) provides an interpolation formula for reconstructing the original signal x (t) from the sequence of sample values {x (n /2W)}. The "sinc" function plays the role of an interpolation function. Each sample x (n / 2W) is multiplied by a delayed version of the interpolation function i.e. sinc function. Then all these resulting waveforms are added to obtain x (t).

5. Graphical representation of the interpolation process :

Let us re-arrange Equation (1.4.19) as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2W\left(t-\frac{n}{2W}\right)$$

This is because $\frac{1}{2W} = T_s$

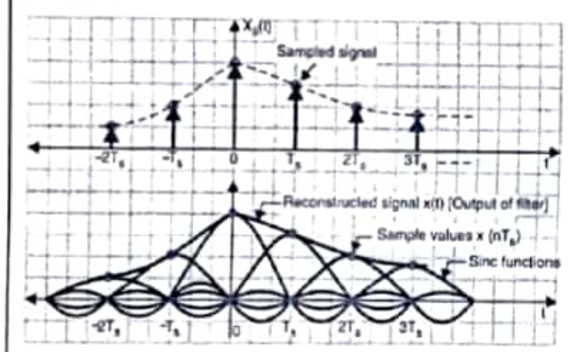
$$\therefore\ x(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\text{sinc}\,2W(t-nT_s) \quad \ldots(1.4.20)$$

Let us expand this equation to write,

$$x(t) = x(0)\,\text{sinc}\,2Wt + x(\pm T_s)\,\text{sinc}\,2W(t\pm T_s) + x(\pm 2T_s)\,\text{sinc}\,2W(t\pm 2T_s) + \ldots \quad \ldots(1.4.21)$$

(a) First term : x (0) sinc 2Wt :

- This will have a maximum amplitude at t = 0. The maximum amplitude is equal to the sample value x (0) at t = 0. This sinc function will pass through zeros at t = ± 1/2 W, ± 1/4 W ...etc. This is as shown in Fig. 1.4.4.



(D-412) **Fig. 1.4.4 : Reconstruction of the original signal x (t) from its samples using the interpolation**

(b) Second term : x (± T_s) sinc 2W (t ± T_s) :

- This sinc function will have maximum amplitude at t = ± T_s. The maximum amplitude is equal to the sample value x (± T_s) at t = ± T_s respectively. Thus sinc 2W (t ± T_s) represents shifted sinc function i.e. "sinc 2Wt" by a period ± T_s. This is as shown in Fig. 1.4.4.
- Similarly the third term, x (± 2T_s) sinc 2W (t ± 2T_s) represents shifted sinc function "sinc 2Wt" by a period of ± 2T_s and so on. We can plot all these sinc functions along with the sampled signal x_δ (t) as shown in Fig. 1.4.4. Note that the peak amplitude of any sinc function is equal to the corresponding sample value x (nT_s).

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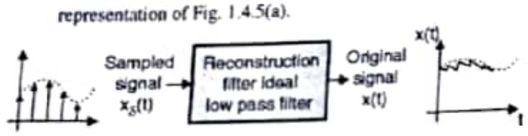
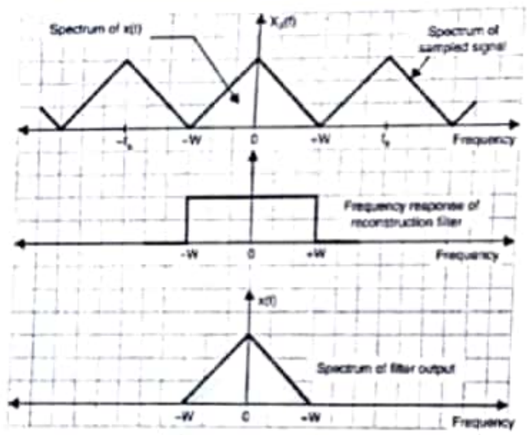
6. Actual reconstruction of the original signal by using a low pass filter :

- Thus the peaks of the sinc pulses represent the amplitudes of the samples.
- The signal x (t) expressed in Equation (1.4.19) is then passed through an ideal low pass filter to recover the original signal x (t). This low pass filter is therefore called as the reconstruction filter. This is shown in the graphical representation of Fig. 1.4.5(a).



(D-413) **Fig. 1.4.5(a) : Reconstruction filter**

- Assume that the cut-off frequency of the ideal low pass filter is adjusted precisely to W Hz. The frequency response of the reconstruction filter is shown in Fig. 1.4.5(b).

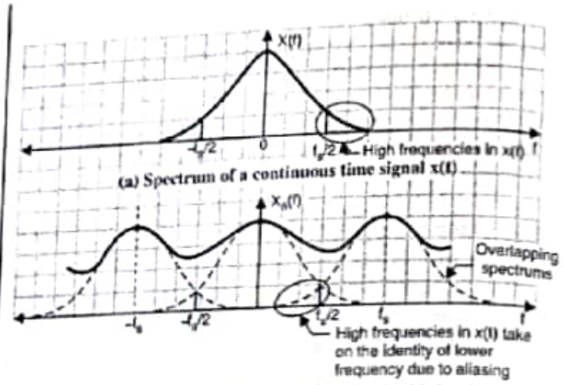


(D-414) **Fig. 1.4.5(b) : Operation of reconstruction**

- When the sampled signal x_δ (t) is applied at the input, this filter will allow only the shaded portion in the spectrum of x_δ (t) to pass through to the output and will block all other frequency components.
- Thus the frequency components only corresponding to x (t) will be passed through to the output and the original signal x (t) is recovered.

1.4.2 Aliasing or Foldover Error :

- If the signal x (t) is not strictly bandlimited and / or if the sampling frequency f_s is less than 2W, then an error called **aliasing** or **foldover error** is observed. The adjacent spectrums overlap if $f_s < 2W$. This is shown in Fig. 1.4.6(b).
- The signal x (t) is not strictly bandlimited. The spectrum of signal x (t) is shown in Fig. 1.4.6(b).
- The spectrum X_δ (f) of the discrete time signal x_δ (t) is shown in Fig. 1.4.6(b) which is nothing but the sum of X (f) and infinite number of frequency shifted replicas of it as explained earlier.



(b) Spectrum of the sampled version of x(t) with $f_s < 2W$

(D-415) **Fig. 1.4.6**

- Consider the two replicas of X (f) which are centered about the frequencies f_s and $-f_s$.
- If we use a reconstruction filter with its pass-band extending from $-f_s/2$ to $+f_s/2$ then its output will not be an undistorted version of the original signal x (t). Some distortion will be present in the filter output.
- The distortion occurs due to the overlapping of the adjacent spectrums as shown in Fig. 1.4.6(b). Due to this overlapping, it is seen that the portions of the frequency shifted replicas are "folded over" inside the desired spectrum.
- Due to this "fold over", high frequencies in X (f) are reflected into low frequencies in X_δ (f). This can be understood by comparing the shaded portions of the spectra shown in Fig. 1.4.6(a) and (b).
- **Aliasing** : This phenomenon of a high frequency in the spectrum of the original signal x (t), taking on the identity of lower frequency in the spectrum of the sampled signal x_δ (t) is called as aliasing or fold over error.

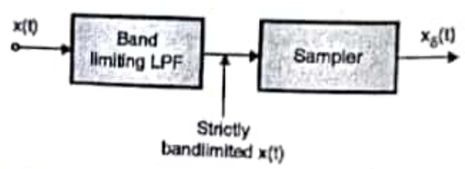
Effect of aliasing :

Due to aliasing some of the information contained in the original signal x (t) is lost in the process of sampling.

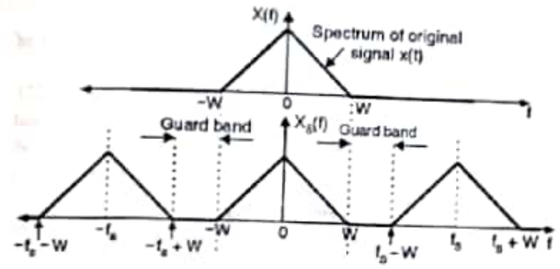
How to eliminate aliasing ?

Aliasing can be completely eliminated if we take the following action :

- Use a bandlimiting low pass filter and pass the signal x (t) through it before sampling as shown in Fig. 1.4.7(a).
- This filter has a cutoff frequency at $f_c = W$, therefore it will strictly bandlimit the signal x (t) before sampling takes place. This filter is also called as antialiasing filter or prealias filter.



(D-416) **Fig. 1.4.7(a) : Use of a bandlimiting filter to eliminate aliasing**



(D-417) **Fig 1.4.7(b) : Spectrum of a sampled signal for $f_s > 2W$**

- Increase the sampling frequency f_s to a great extent i.e. $f_s >>> 2W$. Due to this, even though x (t) is not strictly bandlimited, the spectrums will not overlap. A guard band is created between the adjacent spectrums as shown in Fig. 1.4.7(b).
- Thus aliasing can be prevented by :

1. Using an antialiasing or prealiasing filter and
2. Using the sampling frequency $f_s > 2W$.

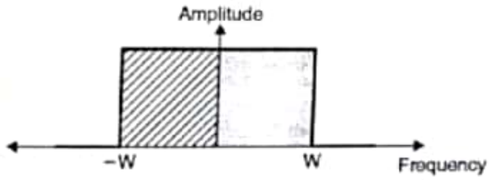
1.4.3 Nyquist Rate and Nyquist Interval :

- The minimum sampling rate of "2W" samples per second for a signal x (t) having maximum frequency of "W" Hz is called as "Nyquist rate". The reciprocal of Nyquist rate i.e. 1/2W is called as the Nyquist interval.

$$\text{Nyquist rate} = 2W \text{ Hz}$$
$$\text{Nyquist interval} = 1/2W \text{ seconds}$$

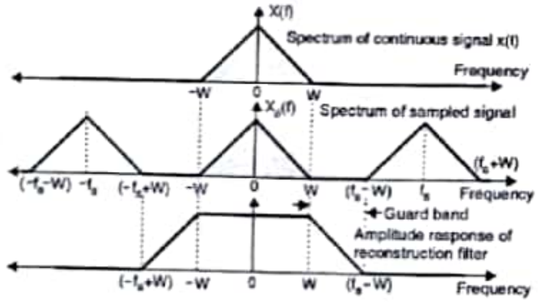
1.4.4 Effect of Nonideal Filter :

- As mentioned earlier the reconstruction filter is a low pass filter. It is expected to pass all the frequencies in the range of (– W to + W) Hz.
- This is because the original signal x (t) is bandlimited to "W" Hz.
- Therefore the frequency response of a reconstruction filter should be as shown in Fig. 1.4.8. This is the frequency response of an ideal low pass filter.



(D-418) **Fig. 1.4.8 : Frequency response of an ideal low pass filter used as a reconstruction filter**

- But it is not possible to practically realize an ideal low pass filter. Therefore a practical low pass filter with a frequency response as shown in Fig. 1.4.9 is used.
- It is possible to use the practical low pass filter without introducing any distortion due to the presence of the guard bands between the adjacent frequency spectrums as shown in Fig. 1.4.8. That is why it is necessary to have $f_s > 2W$.



(D-419) **Fig. 1.4.9 : Amplitude response of a practical reconstruction filter**

1.4.5 Examples on Sampling Theorem for Low Pass Signals :

Ex. 1.4.1 : Find the Nyquist rate and Nyquist interval for each of the following signals :

(a) x (t) = 5 cos 1000 πt cos 4000 πt.

(b) $x(t) = \frac{\sin 200\pi t}{\pi t}$.

Soln. :

(a) The given signal x (t) is in the form of product of cosine term. So let us use the following standard expression :

$$2\cos A \cos B = \cos(A+B) + \cos(A-B)$$

$$\therefore 5\cos 1000\pi t \cos 4000\pi t = 2.5\cos 5000\pi t + 2.5\cos 3000\pi t$$

$$\therefore x(t) = 2.5\cos 5000\pi t + 2.5\cos 3000\pi t \quad ...(1)$$

- From Equation (1) it is clear that the maximum frequency component present in the signal x (t) is of 2500 Hz. In other words x (t) is bandlimited to 2.5 kHz (W = 2.5 kHz).

$\therefore$ Nyquist rate = 2W = 2 × 2.5 kHz = 5 kHz **...Ans.**

$$\text{and Nyquist interval} = \frac{1}{2W} = \frac{1}{5\times10^3}$$
$$= 0.2 \text{ msec} \quad \textbf{...Ans.}$$

(b) $x(t) = \frac{\sin 200\pi t}{\pi t}$

- In order to calculate the Nyquist rate we need to calculate the maximum frequency component present in its spectrum. The spectrum of x (t) can be obtained by taking its fourier transform.
- Multiply numerator and denominator of x (t) by 200 to get,

$$x(t) = \frac{200\sin(200\pi t)}{(200\pi t)}$$

But $\frac{\sin \pi t}{\pi t} = \text{sinc } t \therefore \frac{\sin(200\pi t)}{(200\pi t)} = \text{sinc}(200t)$

$$\therefore x(t) = 200 \text{ sinc}(200t)$$

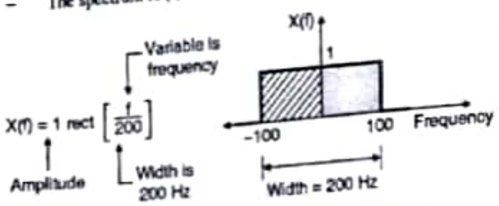
- We know that $A \text{ sinc } 2Wt \overset{F}{\leftrightarrow} \frac{A}{2W} \text{ rect}\left[\frac{f}{2W}\right]$

$$\therefore 200 \text{ sinc}(200t) \overset{F}{\leftrightarrow} \frac{200}{200} \text{ rect}\left[\frac{f}{200}\right]$$

$\therefore$ 200 sinc (200 t) $\overset{F}{\leftrightarrow}$ rect $\left[\frac{f}{200}\right]$

$\therefore$ X (f) = rect [f/200]

- The spectrum X (f) have been shown in Fig. P. 1.4.1.



(D-420) **Fig. P. 1.4.1 : Spectrum of the given signal**

- From Fig. P. 1.4.1 the maximum frequency in the frequency spectrum is 100 Hz.

Nyquist rate = 2 × 100 = 200 Hz ...Ans.

And Nyquist interval = 1/200 = 5 mS ...Ans.

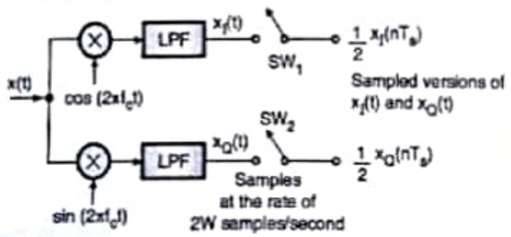
1.4.6 Sampling Theorem for Bandpass Signals :

The sampling theorem for the bandpass signals can be stated as follows :

A bandpass signal x (t), having a maximum bandwidth of 2W Hz can be completely represented in its sampled form and recovered back from the sampled form if it is sampled at a rate which is at least twice the maximum bandwidth. (i.e. $f_s \geq 4W$.)

1.4.6.1 Quadrature Sampling of Bandpass Signals :

- In this section, we consider a scheme called "quadrature sampling" for the uniform sampling of bandpass signals. This scheme is actually a natural extension of the sampling of low pass signals. The scheme is as follows :
- In this scheme, we do not sample the bandpass signal directly. Instead, before sampling we represent the bandpass signal x (t) in terms of its "in-phase" and "quadrature" components, x_I (t) and x_Q (t) respectively.
- The in-phase and quadrature components can be obtained by multiplying the bandpass signal x (t) by cos ($2\pi f_c$ t) and sin ($2\pi f_c$ t) respectively and then by suppressing the sum frequency components by means of low pass filters as shown in Fig. 1.4.10(a).



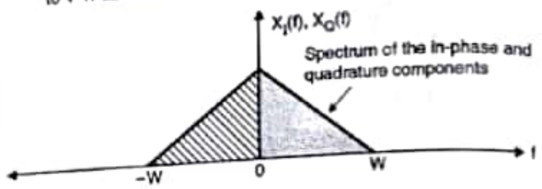
(D-423) **Fig. 1.4.10(a) : Generation of in-phase and quadrature samples from the bandpass signal x (t)**

- If x_I (t) = In-phase component and x_Q (t) = Quadrature component.

Then we can express the bandpass signal x (t) in terms of x_I (t) and x_Q (t) as follows :

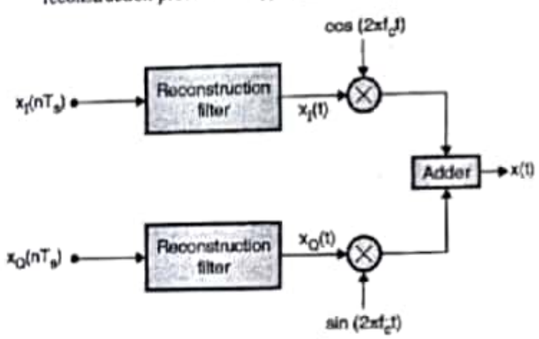
$$x(t) = x_I(t) \cos(2\pi f_c t) - x_Q(t) \sin(2\pi f_c t) \quad ...(1.4.22)$$

- Under the assumption of $f_c > W$, it is found that x_I (t) and x_Q (t) both are "low pass signals" extending from − W to + W as shown in Fig. 1.4.10(b).



(D-424) **Fig. 1.4.10(b) : Spectrum of the in-phase and quadrature components of x (t)**

- Then both the in-phase and quadrature components are separately sampled at a rate of 2W samples per second by the switches SW_1 and SW_2 as shown in Fig. 1.4.10(a) to obtain the sampled versions of x_I (t) and x_Q (t).
- In order to reconstruct the original bandpass signal from its quadrature sampled version, we first reconstruct the in-phase component x_I(t) and quadrature component x_Q (t) from their respective sampled versions $x_I(nT_s)$ and $x_Q(nT_s)$ by means of reconstruction filters. Then multiply x_I (t) and x_Q (t) by cos ($2\pi f_c$ t) and sin ($2\pi f_c$ t) respectively and add the result. The reconstruction process of x (t) is shown in Fig. 1.4.10(c).



(D-425) **Fig. 1.4.10(c) : Reconstruction of the bandpass signal x(t)**

Ex. 1.4.2 : The given signal is m (t) = 10 cos 2000 πt cos 8000 πt

(a) What is the minimum sampling rate based on the low pass uniform sampling theorem ?

(b) Repeat (a) based on the bandpass sampling theorem.

Soln. :

(a) m (t) = 10 cos 2000 πt cos 8000 πt

$\therefore$ m (t) = 5 cos 10000 πt + 5 cos 6000 πt ...(1)

Thus the highest frequency present in m (t) is W = 5000 Hz. Therefore as per the low pass sampling theorem, the minimum sampling rate is given by,

$f_{s\,min}$ = 2W = 2 × 5000 = 10 kHz ...Ans.

(b) Looking at Equation (1) it is clear that the given signal m (t) contains two frequency components which are,

f_1 = 3000 Hz and f_2 = 5000 Hz

The bandwidth of m (t) is,

B = $f_2 - f_1$ = 2000 Hz

When f_1 and f_2 are not harmonically related to the sampling frequency f_s, the bandpass sampling theorem stated in section 1.4.5 is stated in a more generalized form as follows :

If a bandpass signal x (t) has a bandwidth "B" and the highest frequency "f_M" then x (t) can be recovered from its sampled version if $f_s = \frac{2f_M}{k}$ where k is the largest integer not exceeding $\frac{f_M}{B}$.

All higher sampling rates are not necessarily usable unless they exceed $2f_M$.

Thus $f_M = f_2$ = 5 kHz

and B = 2 kHz

$\therefore$ k = $\frac{5}{2}$ = 2.5. Therefore the value of k is 2

$\therefore$ $f_s = \frac{2 \times 5 \text{ kHz}}{2}$ = 5 kHz ...Ans.

Ex. 1.4.3 : A bandpass signal has a spectral range that extends from 20 kHz to 82 kHz. Find the sampling frequency f_s.

Soln. : f_1 = 20 kHz and f_2 = 82 kHz.

$\therefore$ Bandwidth B = $f_2 - f_1$ = 82 − 20 = 62 kHz ...(1)

Let us assume that f_s = 2B = 2 × 62 = 124 kHz ...(2)

From Equations (1) and (2) we observe that neither f_1 nor f_2 is harmonically related to f_s. Hence we have to use the general bandpass sampling theorem stated in the preceding example.

$\therefore$ $k = \frac{f_M}{B} = \frac{82}{62} = 1.32 \rightarrow 1$

$\therefore$ $f_s = \frac{2f_M}{k} = \frac{2 \times 82}{1}$ = 164 kHz ...Ans.

Ex. 1.4.4 : Consider the analog signal x_a(t) = 5 sin 200πt

1. Determine the minimum required sampling rate to avoid sampling.
2. Suppose that the signal is sampled at the rate f_s = 100 Hz. What is the discrete time signal obtained after sampling ?
3. Suppose that the signal is sampled at the rate f_s = 300 Hz, what is the discrete time signal obtained after sampling. **May 10, 10 Marks**

Soln. :

Given : x_a (t) = 5 sin 200 πt

1. The corresponding standard equation is,

x(t) = A sin 2πft

$\therefore$ 2 πft = 200 πt

$\therefore$ f = 100 Hz

Thus minimum sampling frequency, f_s = 2 × 100 = 200 Hz.

$\therefore$ Minimum sampling rate = $\frac{1}{f_s} = \frac{1}{200}$ = 5 msec.

2. Now take f_s = 100 Hz.

The discrete time signal is obtained by replacing 't' by $\frac{n}{f_s}$

$\therefore$ $x(n) = 5 \sin\left(\frac{200\pi n}{100}\right) = 5 \sin(2\pi n) = 0$

It indicates that alising takes place.

3. Given f_s = 300 Hz

So replace 't' by $\frac{n}{300}$

$\therefore$ $x(n) = 5 \sin\left(\frac{200\pi n}{300}\right) = 5 \sin\left(\frac{2}{3}\pi n\right)$

It will not produce aliasing effect.

Ex. 1.4.5 : An analog signal,

x_a (t) = sin [480 π t] + 3 sin (720 π t) is sampled 600 times per second.

1. Determine the Nyquist sampling rate for x_a (t).
2. Determine the folding frequency.
3. What are the frequencies, in radians, in resulting discrete time signal x (n) ?
4. If x (n) is passed through an ideal D/A converter, what is the reconstructed signal Y_a (t) ? **Dec. 11, 10 Marks**

Soln. :

1. The given analog signal is,

x (t) = sin (480 πt) + 3 sin (720 πt) ...(1)

The discrete time signal x (n) is obtained by replacing t by nT.

$\therefore$ x (n) = sin (480 πnT) + 3 sin (720 πnT) ...(2)

Now T = Sampling time = $\frac{1}{\text{Sampling frequency}}$

Given sampling frequency = 600 samples/sec.

$\therefore$ T = $\frac{1}{600}$

Putting this value in Equation (2) we get,

$x(n) = \sin\left(\frac{480\pi n}{600}\right) + 3 \sin\left(\frac{720\pi n}{600}\right)$

$\therefore$ $x(n) = \sin\frac{4}{5}\pi n + 3 \sin\frac{6\pi n}{5}$

$\therefore$ $x(n) = \sin\frac{4}{5}\pi n + 3 \sin\left(2\pi - \frac{4\pi}{5}\right) n$ ...(3)

Now we have sin (2π − θ) = − sin θ

$\therefore$ $x(n) = \sin\frac{4}{5}\pi n - 3 \sin\frac{4}{5}\pi n$

$\therefore$ $x(n) = -2 \sin\frac{4}{5}\pi n$ ...(4)

We have standard equation of discrete time signal,

x (n) = A sin 2π fn = A sin ωn ...(5)

Comparing Equations (4) and (5) we get,

Frequency in D.T. signal = $\omega = \frac{4\pi}{5}$ radians ...Ans.

2. When x (n) is passed through DAC then the reconstructed signal y (t) is given by,

$$y(t) = x(n)\Big|_{n=\frac{t}{T}} \quad ...(6)$$

This is because we have,

$$x(n) = x(t)\Big|_{t=nT}$$

Since $t = nT \Rightarrow n = \frac{t}{T}$

Putting Equation (4) in Equation (6) we get,

$$y(t) = -2\sin\frac{4}{5}\pi n\Big|_{n=\frac{t}{T}}$$

$$\therefore\ y(t) = -2\sin\frac{4}{5}\pi\frac{t}{T}$$

But $T = \frac{1}{600}$ (given)

$$\therefore\ y(t) = -2\sin\left(\frac{4}{5}\pi t \times 600\right)$$

$$\therefore\ y(t) = -2\sin 480\pi t$$ ...Ans.

Ex. 1.4.6 : Consider an analog signal
$x(t) = 5\cos 50\pi t + 2\sin 200\pi t + 2\cos 1000\pi t$

1. Determine nyquist rate of sampling
2. If the signal x (t) is sampled at the rate of 200 Hz, what is the discrete time signal obtained after sampling ?

Dec. 12, 5 Marks

Soln. :

1. Given,

$$x(t) = 5\cos 50\pi t + 2\sin 200\pi t + 2\cos 1000\pi t \quad ...(1)$$

The given signal consists of following components

$2\pi f_1 t = 50\pi t$,

$2\pi f_2 t = 200\pi t$, $2\pi f_3 t = 1000\pi t$

$\therefore\ f_1 = 25$ Hz.

$f_2 = 100$ Hz, $f_3 = 500$ Hz

Thus maximum frequency $= f_{max} = W = 500$ Hz

$\therefore$ Nyquist rate $= 2W = 1000$ Hz

2. The discrete time signal is obtained by putting $t = \frac{n}{f_s} = \frac{n}{200}$ in Equation (1)

$$\therefore\ x(n) = 5\cos\left(50\pi\cdot\frac{n}{200}\right) + 2\sin\left(200\pi\cdot\frac{n}{200}\right) + 2\cos\left(1000\pi\cdot\frac{n}{200}\right)$$

$$\therefore\ x(n) = 5\cos\left(\frac{\pi}{4}n\right) + 2\sin(\pi n) + 2\cos(5\pi n)$$

But $\sin(\pi n) = 0$

$$\therefore\ x(n) = 5\cos\left(\frac{\pi}{4}n\right) + 2\cos(5\pi n)$$

This is the discrete signal obtained after sampling at $f_s = 200$ Hz

Ex. 1.4.7 : Consider the analog signal $x_a(t) = 3\cos 100\pi t$

1. Determine the minimum required sampling rate to avoid aliasing.
2. Suppose that the signal is sampled at the rate $f_s = 200$ Hz. What is the discrete time signal obtained after sampling ?
3. Suppose that the signal is sampled at the rate $f_s = 75$ Hz. What is the discrete time signal obtained after sampling.

(Similar to Ex. 1.4.4) **Dec. 09, 10 Marks**

Ex. 1.4.8 : The analog signal x(t) is given by
$x(t) = 2\cos(2000\pi t) + 3\sin(6000\pi t) + 8\cos(12000\pi t)$
Calculate :

1. Nyquist sampling rate.
2. If x(t) is sampled at the rate $f_s = 5$kHz. What is the discrete time signal obtained after sampling ?
3. What is the analog signal y(t) we can reconstruct from the samples if ideal interpolation is used ?

Dec. 10, 10 Marks

Soln. :

Given :

1. $x(t) = 2\cos(2000\pi t) + 3\sin(6000\pi t) + 8\cos(12000\pi t)$

$\therefore\ 2\pi f_1 = 2000\pi$, $2\pi f_2 = 6000\pi$, $2\pi f_3 = 12000\pi$

$\therefore f_1 = 1000$ Hz, $f_2 = 3000$ Hz, $f_3 = 6000$ Hz.

Thus maximum frequency $= W = 6000$ Hz.

$\therefore$ Nyquist rate of sampling $= 2W = 12000$ Hz.

2. Given $f_s = 5$ kHz $= 5000$ Hz.

Discrete signal is obtained by replacing t by $\frac{n}{f_s}$

$$\therefore\ x(n) = 2\cos\left(\frac{2000\pi n}{5000}\right) + 3\sin\left(\frac{6000\pi n}{5000}\right) + 8\cos\left(\frac{12000\pi n}{5000}\right)$$

$$\therefore\ x(n) = 2\cos\left(\frac{2}{5}\pi n\right) + 3\sin\left(\frac{6}{5}\pi n\right) + 8\cos\left(\frac{12}{5}\pi n\right)$$

$$\therefore\ x(n) = 2\cos\left(\frac{2}{5}\pi n\right) + 3\sin\left(\frac{6}{5}\pi n\right) + 8\cos\left(2\pi n + \frac{2}{5}\pi n\right)$$

Now $\cos(2\pi n + \theta) = \cos\theta$

$$\therefore\ x(n) = 2\cos\left(\frac{2}{5}\pi n\right) + 3\sin\left(\frac{6}{5}\pi n\right) + 8\cos\left(\frac{2}{5}\pi n\right)$$

$$\therefore\ x(n) = 10\cos\left(\frac{2}{5}\pi n\right) + 5\sin\left(\frac{6}{5}\pi n\right)$$

3. For ideal case $f_s = 12000$ Hz.

$$\therefore\ x(n) = 2\cos\left(\frac{2000\pi n}{12000}\right) + 3\sin\left(\frac{6000\pi n}{12000}\right) + 8\cos\left(\frac{12000\pi n}{12000}\right)$$

$$\therefore\ x(n) = 2\cos\left(\frac{\pi n}{6}\right) + 3\sin\left(\frac{\pi n}{2}\right) + 8\cos(\pi n).$$

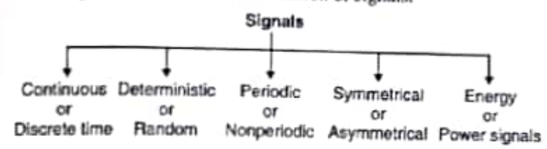
To obtain y(t) replace 'n' by $t f_s = 12000t$

$$\therefore\ y(t) = 2\cos\left(\frac{\pi}{6}12000t\right) + 3\sin\left(\frac{\pi}{2}12000t\right) + 8\cos(\pi\, 12000\, t)$$

$$\therefore\ y(t) = 2\cos(6000\pi t) + 3\sin(6000\pi t) + 8\cos(12000\pi t)$$

1.5 Classification of Signals :

Fig. 1.5.1 shows the classification of signals.



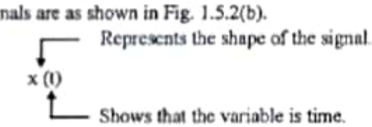
(G-1250) **Fig. 1.5.1 : Classification of signals**

- Out of these we will concentrate only on periodic or non periodic signals.

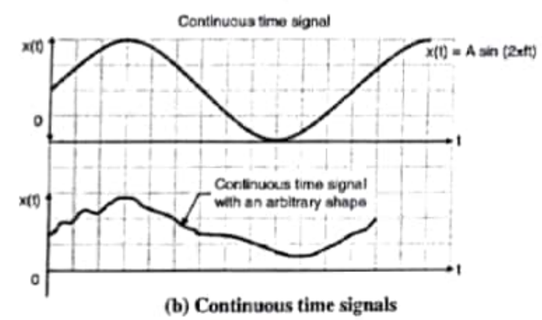
1.5.1 Continuous and Discrete Time Signals :

Continuous Time (CT) signal :

- A signal of continuous amplitude and time is known as a continuous time signal or an analog signal. This signal will have some finite "value" at every instant of time.
- The electrical signals derived in proportion with the physical quantities such as temperature, pressure, sound etc. are generally continuous time signals.
- The other examples of continuous signals are sine wave, cosine wave, triangular wave etc. Some of the continuous signals are as shown in Fig. 1.5.2(b).



(a)



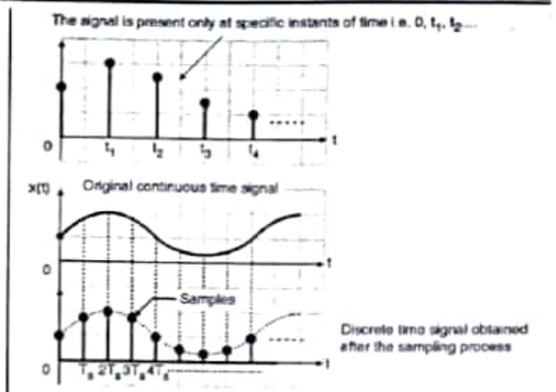
(b) Continuous time signals

(H-6) **Fig. 1.5.2**

- The continuous time signals are represented at x(t) where "x" represents the shape of the signal and "t" shows that the variable is time.

Discrete time signals :

- If the signal has finite values only at "discrete instants of time," then it is known as a discrete time signal. The discrete time signals have "values" only at certain instants of time.

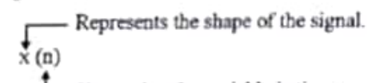


(H-7) **Fig. 1.5.2(c) : Discrete time signals**

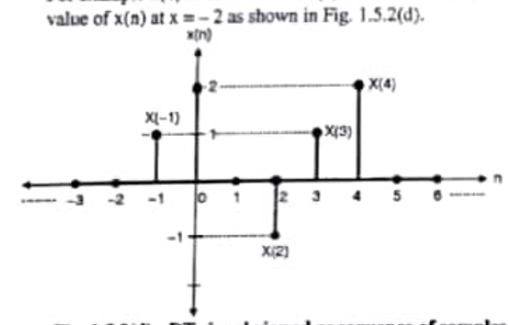
- If we take the blood pressure readings of a patient after every one hour and plot the graph then the resultant signal will be a discrete signal.
- One more way of obtaining a discrete signals is by "sampling" the analog signals as shown in Fig. 1.5.2(c). The dotted line joining the "sample values" is an imaginary line.
- The sampled version of the continuous time signal is represented by $x_s(t)$. The signals which are discrete in time and discrete in amplitude are called as "digital signals".
- The digital signals can be obtained from the continuous time analog signal by a process called "analog to digital conversion".

Sequence of samples :

- The discrete time (DT) signal can also be visualised as a sequence of samples taken at uniform intervals and denoted as x(n).
- Where "x" represents the shape of the signal and "n" is the variable (n is actually an integer). Such a sequence is shown in Fig. 1.5.2(d).



- For example x(1) is the value of x(n) at n = 1, or x(–2) is the value of x(n) at x = – 2 as shown in Fig. 1.5.2(d).



(H-9) **Fig. 1.5.2(d) : DT signal viewed as sequence of samples**

- Mathematically it is denoted as follows :

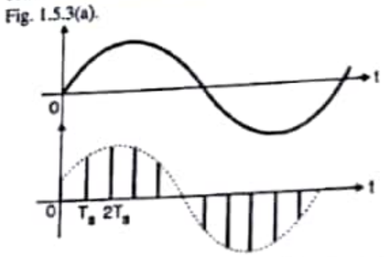
$$x(n) = \{\ldots\ldots 0, 0, 1, 2, 0, -1, 1, 2, 0, 0 \ldots\}$$
$$\uparrow$$

Where the arrow indicates the value of x(n) at n = 2.

1.5.2 Continuous Valued or Discrete Valued Signals :

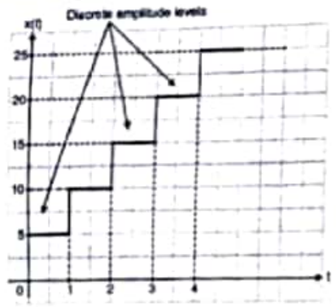
Continuous valued signal :

If the variation in the amplitude of signal is continuous then, it is called as continuous valued signal. Such signal may be continuous or discrete in nature. Such signals are as shown in Fig. 1.5.3(a).



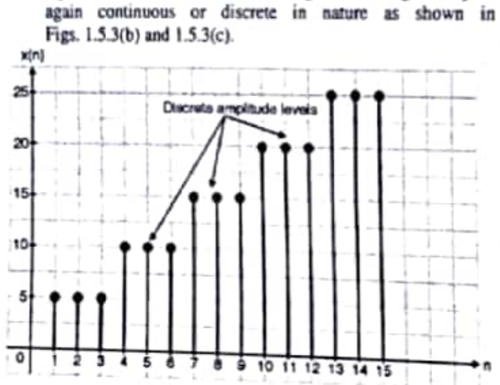
(H-10) **Fig. 1.5.3(a) : Continuous valued signals**

Discrete valued signal :



(H-11) **Fig. 1.5.3(b) : Discrete amplitude signal continuous in nature**

- If the variation in the amplitude of signal is not continuous; but the signal has certain discrete amplitude levels then such signal is called as discrete valued signal. Such signal may be again continuous or discrete in nature as shown in Figs. 1.5.3(b) and 1.5.3(c).



(H-12) **Fig. 1.5.3(c) : Discrete amplitude signal discrete in nature**

- As shown in Fig. 1.5.3(b), the signal is defined at all instants of time. So it is continuous signal. But it takes only certain discrete amplitude levels. The amplitude is not continuously changing with time. So it is discrete amplitude signal continuous in nature.
- As shown in Fig. 1.5.3(c), the signal is defined only at discrete intervals of time. So it is discrete signal. And this signal takes only certain discrete amplitude levels. So it is discrete amplitude signal discrete in nature.

Digital signals :

- A **digital signal** is defined as the signal which has only a finite number of distinct values.
- Digital signals are not continuous signal. They are discrete signals as shown in Fig. 1.5.3(d).
- **Binary signal :** If a digital signal has only two distinct values, i.e. 0 and 1 then it is called as a binary signal.

(H-13) **Fig. 1.5.3(d) : Binary signal (Digital signal)**

- **Octal signal :** A digital signal having eight distinct values is called as an octal signal.
- **Hexadecimal signal :** A digital signal having sixteen distinct values is called as the hexadecimal number.

Type of digital signal	Number of distinct values
Binary	2
Octal	8
Hexadecimal	16

1.5.3 Periodic and Non-periodic Signals :

MU : May 07

University Questions

Q. 1 Prove that for discrete time signal to be periodic, its frequency must be a rational number. What is the minimum and maximum range for discrete time sinusoidal frequency ? What is its unit ? **(May 07, 5 Marks)**

Periodic CT signal :

- A CT signal which repeats itself after a fixed time period is called as a periodic signal. The periodicity of a CT signal can be defined mathematically as follows :

$$x(t) = x(t + T_0) \quad \text{: Condition of periodicity} \qquad ...(1.5.1)$$

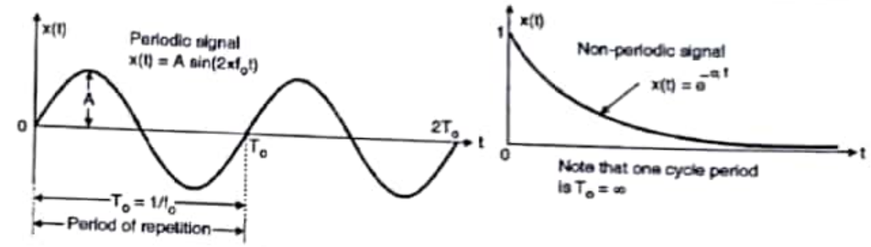
- where T_0 is called as the period of signal x(t), in other words, signal x(t) repeats itself after a period of T_0 sec.
- Examples of periodic signals are sine wave, cosine wave, square wave etc. Fig. 1.5.4 shows a sine wave which is periodic because it repeats itself after a period T_0.

Non-periodic CT signal :

- A CT signal which does not repeat itself after a fixed time period or does not repeat at all is called as a non-periodic or aperiodic signal.
- The non-periodic signals do not satisfy the condition of periodicity stated in Equation (1.5.1).

$$\therefore \quad \text{For a non-periodic signal } x(t) \neq x(t + T_0) \qquad ...(1.5.2)$$

- Sometimes it is said that an aperiodic signal has a period $T_0 = \infty$. Fig. 1.5.4 shows a decaying exponential signal.
- This exponential signal is non-periodic but it is deterministic because we can mathematically express it as $x(t) = e^{-\alpha t}$



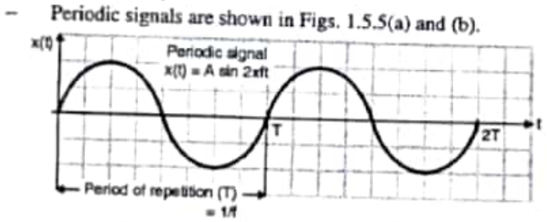
(H-14) **Fig. 1.5.4 : Periodic and non-periodic signals**

Periodic discrete time signal :

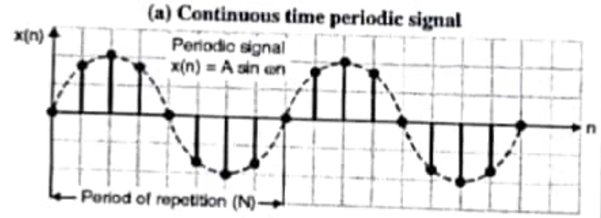
- For the discrete time signal, the condition of periodicity is,

$$x(n) = x(n + N) \qquad ...(1.5.3)$$

- Here number 'N' is the period of signal. The smallest value of N for which the condition of periodicity exists is called as fundamental period.
- Periodic signals are shown in Figs. 1.5.5(a) and (b).



(a) Continuous time periodic signal



(b) Discrete time periodic signal

(H-15) **Fig. 1.5.5**

Non-periodic DT signal :

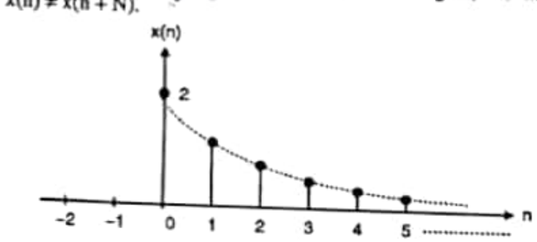
- A signal which does not repeat itself after a fixed time period or does not repeat at all is called as non-periodic or aperiodic signal. Thus mathematical expression for non-periodic signal is,

$$x(t) \neq x(t + T_0) \qquad ...(1.5.4)$$

and

$$x(n) \neq x(n + N) \qquad ...(1.5.5)$$

Fig. 1.5.5(c) shows a D.T. non-periodic signal, for which $x(n) \neq x(n + N)$.



(H-16) **Fig. 1.5.5(c) : Non-periodic D.T. signal**

Condition for periodicity of a discrete time signal :

A discrete time sinusoidal signal is periodic only if its frequency (f_0) is rational. That means frequency f_0 should be in the form of ratio of two integers.

Proof :

- For the discrete signal, the condition of periodicity is,

$$x(n + N) = x(n) \qquad ...(1.5.6)$$

- Let x(n) be the cosine wave. So it can be expressed as,

$$x(n) = A\cos(2\pi f_0 n + \theta) \qquad ...(1.5.7)$$

Here A = Amplitude

and θ = Phase shift

- Now the equation of x(n + N) can be obtained by replacing 'n' by 'n + N' in Equation (1.5.7).

$$\therefore \quad x(n + N) = A\cos[2\pi f_0 (n + N) + \theta] \qquad ...(1.5.8)$$

- According to condition of periodicity Equation (1.5.6) ; we can equate Equations (1.5.7) and (1.5.8).

$$\therefore \quad A\cos[2\pi f_0 (n + N) + \theta] = A\cos(2\pi f_0 n + \theta)$$

$$\therefore \quad A\cos(2\pi f_0 n + 2\pi f_0 N + \theta) = A\cos(2\pi f_0 n + \theta) \qquad ...(1.5.9)$$

- To satisfy this equation,

$$2\pi f_0 N = 2\pi k \qquad ...(1.5.10)$$

Where k is an integer

$$\therefore \quad f_0 = \frac{k}{N} \qquad \text{—Proved.} \qquad ...(1.5.11)$$

- Here k and N both are integers. Thus discrete time (DT) signal is periodic if its frequency f_0 is rational.

Periodicity condition for $x(n) = x_1(n) + x_2(n)$:

- Here input sequence x(n) is expressed as summation of two discrete time sequences. We can calculate the values of f_1 and f_2 corresponding to $x_1(n)$ and $x_2(n)$.
- Let $x_1(n)$ and $x_2(n)$ both be periodic discrete time signals (sequences).
- So according to condition of periodicity,

$$f_1 = \frac{k_1}{N_1} \qquad \text{and} \quad f_2 = \frac{k_2}{N_2}$$

- The resultant signal x(n) is periodic if $\frac{N_1}{N_2}$ is ratio of two integers. The period of x(n) will be least common multiple of N_1 and N_2.
- Similarly if continuous time signals is,

$$x(t) = x_1(t) + x_2(t)$$

- We can calculate the values of T_1 and T_2 corresponding to $x_1(t)$ and $x_2(t)$. Then the resultant x(t) is periodic if $\frac{T_1}{T_2}$ is ratio of two integers. The fundamental period of x(t) will be least common multiple of T_1 and T_2.

Solved examples :

Ex. 1.5.1 : Prove that the sinewave shown in Fig. 1.5.4 is a periodic signal.

Soln. :

The sinewave shown in the Fig. 1.5.4 can be mathematically represented as,

$$x(t) = A \sin \omega_o t \quad ...(1)$$

Now, let us test if it satisfies the condition for periodicity i.e. if,

$$x(t) = x(t + T_o) \quad ...(2)$$

So, let us find the expression for $x(t + T_o)$

$$x(t + T_o) = A \sin \omega_o (t + T_o) \quad ...(3)$$

$$= A \sin [\omega_o t + \omega_o T_o]$$

But $\omega_o = 2\pi f_o$ and $T_o = \frac{1}{f_o}$. Therefore $\omega_o T_o = 2\pi f_o \times \frac{1}{f_o} = 2\pi$.

Substitute this in Equation (3), to get,

$$x(t + T_o) = A \sin [\omega_o t + 2\pi]$$

$$= A [\sin(\omega_o t) \cos 2\pi + \cos(\omega_o t) \sin 2\pi] \quad ...(4)$$

$$\therefore \quad x(t + T_o) = A \sin \omega_o t = x(t)$$

Therefore the sinewave shown in Fig. 1.5.4 is a periodic signal.

Ex. 1.5.2 : Prove that the exponential signal shown in Fig. 1.5.4 is non-periodic.

Soln. : The exponential signal shown in Fig. 1.5.4 is expressed mathematically as,

$$x(t) = e^{-\alpha t}$$

Substitute $t = (t + T_o)$ to get,

$$x(t + T_o) = e^{-\alpha(t+T_o)} = e^{-\alpha t} e^{-\alpha T_o}$$

But $T_o = \infty$

$$\therefore \quad e^{-\alpha T_o} = e^{-\infty} = 0$$

$$\therefore \quad x(t + T_o) = e^{-\alpha t} \cdot 0 = 0$$

$$\therefore \quad x(t) \neq x(t + T_o)$$

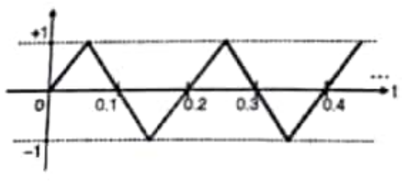
Hence the exponential signal shown in Fig. 1.5.4 is a non-periodic signal.

Ex. 1.5.3 : What is the fundamental frequency of the waveform shown in Fig. P. 1.5.3, in Hz and rad/sec ?

Soln. :

- One cycle corresponds to 0.2 sec. Hence $T_o = 0.2$ sec.

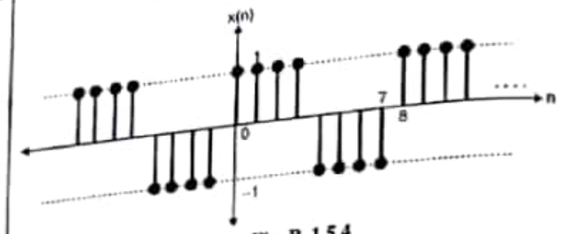
$\therefore$ Frequency $f_o = \frac{1}{T_o} = \frac{1}{0.2} = 5$ Hz ...Ans.



(B-17) **Fig. P. 1.5.3**

Frequency in rad/sec. $= \omega_o = 2\pi f_o = 2 \times 3.14 \times 5 = 31.4$ rad/s...Ans.

Ex. 1.5.4 : What is the fundamental frequency of the D.T. square wave shown in Fig. P. 1.5.4.



(B-18) **Fig. P. 1.5.4**

Soln. :

- The fundamental angular frequency or simply fundamental frequency of x (n) is given by,

$$\Omega = \frac{2\pi}{N}$$

When N = A positive integer indicating number of samples in one cycle.

- For the given signal N = 8.

$$\therefore \quad \Omega = \frac{2\pi}{8} = \frac{\pi}{4} \text{ radians}$$ ...Ans.

Ex. 1.5.5 : State whether the following signals x(t) is periodic or not, giving reasons. If it is periodic, find the corresponding period.

$$x(t) = 2 \cos 100\pi t + 5 \sin 50 t$$

Soln. :

1. The given signal is,

$$x(t) = 2 \cos 100\pi t + 5 \sin 50 t \quad ...(1)$$

$$\text{Let } x(t) = x_1(t) + x_2(t) \quad ...(2)$$

$$\text{Here, } x_1(t) = 2 \cos 10\pi t \quad ...(3)$$

$$x_2(t) = 5 \sin 50 t \quad ...(4)$$

The standard equation can be expressed as $x_1(t) = A \cos \omega_1 t$ ...(5)

Comparing Equations (3) and (5) we get, $\omega_1 = 100\pi$

Now, we have,

$$\omega_1 = 2\pi f_1 = \frac{2\pi}{T_1}$$

$$\therefore \frac{2\pi}{T_1} = 100\pi$$

$$\therefore T_1 = \frac{2\pi}{100\pi} = \frac{1}{50} \quad ...(6)$$

Similarly, standard equation $x_2(t)$ can be written as

$$x_2(t) = A \sin \omega_2 t \quad ...(7)$$

Comparing Equations (4) and (7) we get,

$$\omega_2 = 50$$

$$\therefore 2\pi f_2 = \frac{2\pi}{T_2} = 50, \quad \therefore T_2 = \frac{2\pi}{50} \quad ...(8)$$

Here, x(t) is expressed as summation of two signals. We know that the resultant signal x(t) is periodic if $\frac{T_1}{T_2}$ Is the ratio of two integers. From Equations (6) and (7) we get,

$$\frac{T_1}{T_2} = \frac{1}{50} \cdot \frac{50}{2\pi} = \frac{1}{2\pi}$$

It is not the ratio of two integers. Thus x(t) is **non periodic.**

Ex. 1.5.6 : Determine whether or not each of the following signal is periodic. If periodic, determine its fundamental period.

1. $x(t) = \left[\cos\left(2t - \frac{\pi}{3}\right)\right]^2$
2. $x(n) = \sin\left(\frac{6\pi}{7} n + 1\right)$

May 07, Dec. 07, 4 Marks

Soln. :

1. $$x(t) = \left[\cos\left(2t - \frac{\pi}{3}\right)\right]^2 = \cos^2\left(2t - \frac{\pi}{3}\right)$$

$$x(t) = \cos\left(2t - \frac{\pi}{3}\right) \cos\left(2t - \frac{\pi}{3}\right) \quad ...(1)$$

The standard equation can be expressed as,

$$x(n) = \cos \omega_1 n \cos \omega_2 n \quad ...(2)$$

Comparing Equations (1) and Equation (2),

$$\omega_1 = \left(2t - \frac{\pi}{3}\right), \omega_2 = \left(2t - \frac{\pi}{3}\right)$$

But $\omega = 2\pi f$

$$\therefore \quad 2\pi f_1 = \left(2t - \frac{\pi}{3}\right)$$

$$6\pi f_1 = (6t - \pi)$$

$$\therefore \quad f_1 = \left(\frac{6t - \pi}{6\pi}\right)$$

$$\therefore \quad f_1 = \left(\frac{6t}{6\pi} - \frac{\pi}{6\pi}\right) = \left(\frac{1}{\pi} - \frac{1}{6}\right) = f_2$$

Hence f_1 and f_2 are ratio of non- integer values, so it is **nonperiodic.**

2. $$x(n) = \sin\left(\frac{6\pi}{7} n + 1\right)$$

$$\sin(a + b) = \sin a \cos b + \sin b \cos a$$

$$\sin\left(\frac{6\pi}{7} n + 1\right) = \sin \frac{6\pi}{7} n \cos 1 + \sin 1 \cos \frac{6\pi}{7} n$$

$$\sin\left(\frac{6\pi}{7} n + 1\right) = \sin \frac{6\pi}{7} n \cdot 1 + 0 \cdot \cos \frac{6\pi}{7} n$$

$$\therefore \quad \left(\frac{6\pi}{7} n + 1\right) = \sin \frac{6\pi}{7} n \quad ...(3)$$

$$\text{Comparing with, } x(n) = \sin \omega_1 n \quad ...(4)$$

Comparing Equations (3) and (4),

$$\therefore \quad \omega_1 = \frac{6\pi}{7} \qquad \text{But } \omega = 2\pi f$$

$$2\pi f_1 = \frac{6\pi}{7} \qquad f_1 = \frac{6\pi}{7 \times 2\pi}$$

$$\therefore \quad f_1 = \frac{3}{7} \text{ cycles per samples} \quad ...(5)$$

Since frequency 'f_1' is expressed as the ratio of two integers, this sequence is **periodic.**

To determine fundamental period :

$$f_1 = \frac{k}{N} \quad ...(6)$$

Here 'N' indicates, the fundamental period.

Comparing Equations (5) and Equation (6),

Fundamental period = N = 7 samples

Ex. 1.5.7 : Find whether the following are periodic. If yes, find period also :

1. $\sum_{k=-\infty}^{+\infty} (-1)^k \delta(t - 2k)$
2. $\cos\left(\frac{\pi n}{5}\right) \sin\left(\frac{\pi n}{5}\right)$

May 03, 5 Marks, May 11, 2 Marks

Soln. :

1. $$x(t) = \sum_{k=-\infty}^{+\infty} (-1)^k \delta(t - 2k)$$

First we will draw the nature of signal by considering some range of k. Let k = – 2 to + 2

$$\therefore \quad x(t) = \sum_{k=-2}^{2} (-1)^k \delta(t - 2k)$$

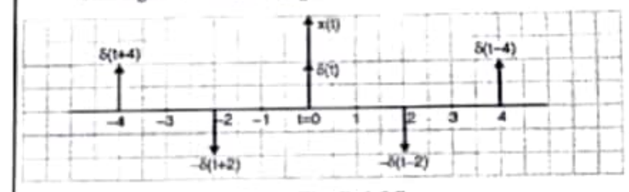
$$\therefore \quad x(t) = (-1)^{-2} \delta(t + 4) + (-1)^{-1} \delta(t + 2) + (-1)^0 \delta(t) + (-1)^1 \delta(t - 2) + (-1)^2 \delta(t - 4)$$

$$\therefore \quad x(t) = \delta(t + 4) - \delta(t + 2) + \delta(t) - \delta(t - 2) + \delta(t - 4)$$

The given signal is existing from $-\infty$ to $+\infty$,

$$\therefore x(t) = ...\delta(t + 4) - \delta(t + 2) + \delta(t) - \delta(t - 2) + \delta(t - 4) + ...$$

This signal is shown in Fig. P. 1.5.7.



(B-203) **Fig. P. 1.5.7**

Thus the signal is periodic and period = T = 2.

2. $$x(n) = \cos\left(\frac{\pi n}{5}\right) \sin\left(\frac{\pi n}{5}\right) \quad ...(1)$$

The standard equation can be written as,

$$x(n) = \cos 2\pi f_1 n \sin 2\pi f_2 n \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$2\pi f_1 = \frac{\pi}{5} \text{ and } \qquad 2\pi f_2 = \frac{\pi}{5}$$

$$\therefore \quad f_1 = \frac{1}{10} \text{ and } \qquad f_2 = \frac{1}{10}$$

Since f_1 and f_2 are ratio of two integers, the signal is periodic and period = N = 10

Ex. 1.5.8 : A discrete time periodic sequence is given by

$$x_p(n) = A \cos\left[\frac{n\pi}{2}\right]$$

1. Determine the period of the sequence
2. Sketch the sequence x(n) for the variable n for one period

Dec. 09, 5 Marks

Soln. :

1. Given $$x_p(n) = A \cos\left(\frac{n\pi}{2}\right) \quad ...(1)$$

We have standard equation,

$$x_p(n) = A \cos(2\pi f_o n) \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$2\pi f_o n = \frac{n\pi}{2}$$

$$\therefore \quad f_o = \frac{1}{4} = \frac{k}{N}$$

Thus period of signal is N = 4 samples.

2. We will part different values of n to plot x (n) as follows :

For $n = 0 \Rightarrow x_p(0) = A\cos(0) = A$

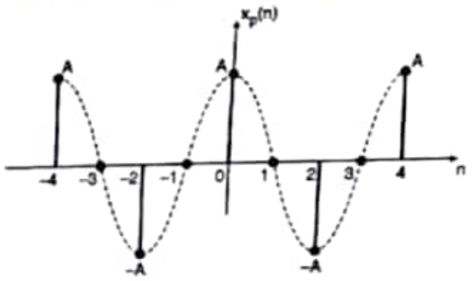
$n = \pm 1 \Rightarrow x_p(1) = A\cos\left(\pm\frac{\pi}{2}\right) = 0$

$n = \pm 2 \Rightarrow x_p(2) = A\cos(\pm\pi) = -A$

$n = \pm 3 \Rightarrow x_p(3) = A\cos\left(\pm\frac{3\pi}{2}\right) = 0$

$n = \pm 4 \Rightarrow x_p(4) = A\cos(\pm 2\pi) = A$

The sketch of signal is as shown in Fig. P. 1.5.8.



(H-1194) Fig. P. 1.5.8

Ex. 1.5.9 : Determine whether each of the following signal is periodic. Is so, find its fundamental period :

1. $x(t) = 3\cos\sqrt{2}\cdot t + 4\cos 5\pi t$
2. $x(t) = \frac{\cos 4\pi n}{12}$

Dec. 12. 4 Marks

Soln. :

1. Given : $x(t) = 3\cos\sqrt{2}\cdot t + 4\cos 5\pi t$ ...(1)

The corresponding standard equation is

$$x(t) = A\cos 2\pi f_1 t + B\cos 2\pi f_2 t \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$2\pi f_1 t = \sqrt{2}\, t \quad \text{and} \quad f_2 = \frac{5}{2}$$

$$\therefore \quad f_1 = \frac{2\pi}{\sqrt{2}} \quad \text{and} \quad T_2 = \frac{2}{5}$$

$$\therefore \quad \frac{T_1}{T_2} = \frac{2\pi\sqrt{2}}{2/5} \text{ and } \frac{2\pi}{\sqrt{2}}\cdot\frac{5}{2} = \frac{5\pi}{\sqrt{2}}$$

Since it is ratio of non-integer values ; the given signal is non-periodic.

2. Given : $x(n) = \frac{\cos 4\pi n}{12} = \frac{1}{12}\cos(4\pi n)$ ...(1)

The corresponding standard equation is,

$$x(n) = A\cos(2\pi f_0 n) \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$2\pi f_0 n = 4\pi n$$

$$\therefore \quad f_0 = \frac{4}{2} = \frac{K}{N}$$

Since it is ratio of two integers ; this signal is periodic with period, N = 2 samples.

1.5.4 Deterministic and Random Signals :

Deterministic signal :

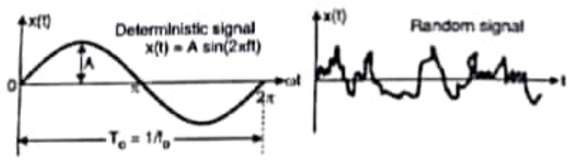
- A signal which can be described by a mathematical expression, look up table or some well defined rule is known as the deterministic signal.
- The examples of deterministic signals are sine wave, cosine wave, square wave etc. Fig. 1.5.6 shows a sine wave which is a deterministic signal because it can be represented mathematically as,

$$x(t) = A\sin(2\pi f t)$$

- Where A is the peak amplitude and f is the frequency of the signal.
- The deterministic signals such as sine, cosine etc. are periodic but some deterministic signals may not be periodic. Example of such signal is an exponential signal.

Random signals :

- A signal which cannot be described by any mathematical expression is called as a random signal. Due to this it is not possible to predict about the amplitude of such signals at a given instant of time.
- The example of random signal is "Noise" in the communication systems. The deterministic and random signals are as shown in Fig. 1.5.6.



(H-19) Fig. 1.5.6 : Deterministic and random signals

1.5.5 Symmetrical (Even) or Antisymmetrical (Odd) Signals : MU - Dec. 09

University Questions

Q. 1 Show that if $x_e(n)$ is an even signal and $x_o(n)$ is an odd signal, then $x_e(n)\cdot x_o(n)$ is an odd signal. **(Dec. 09, 20 Marks)**

Symmetrical signal (Continuous time) :

- A CT signal x (t) is said to be symmetrical or even if it satisfies the following condition,

Condition for symmetry : $x(t) = x(-t)$ ...(1.5.12)

Where, $x(t)$ = Value of the signal for positive "t".

and $x(-t)$ = Value of the signal for negative "t".

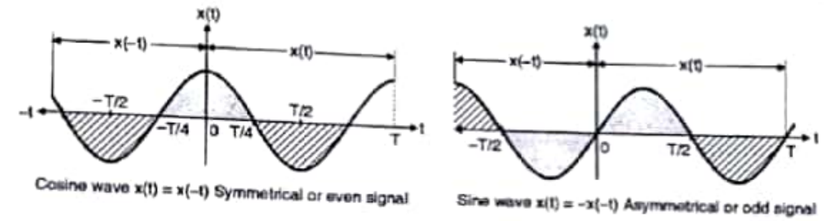
An example of symmetrical signal is a cosine wave shown in Fig. 1.5.7.

Antisymmetrical signal (Continuous time) :

A CT signal x (t) is said to be antisymmetric or odd if it satisfies the following condition,

Condition for antisymmetry : $x(t) = -x(-t)$ ...(1.5.13)

An example of odd signal is a sine wave shown in Fig. 1.5.7.



(H-20) Fig. 1.5.7 : Symmetrical and antisymmetrical signals

Even and odd discrete time signals :

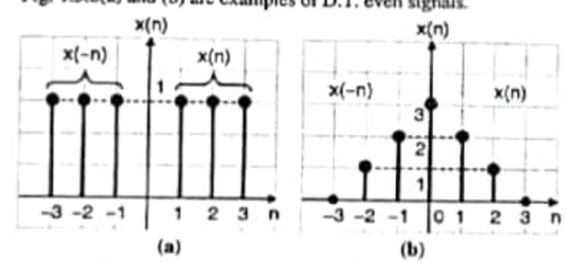
D.T. symmetric signals :

A discrete time real valued signal is said to be symmetric(even) if they satisfy the following condition,

Condition for symmetry : $x(n) = x(-n)$ ...For D.T. signal

Examples :

Fig. 1.5.8(a) and (b) are examples of D.T. even signals.



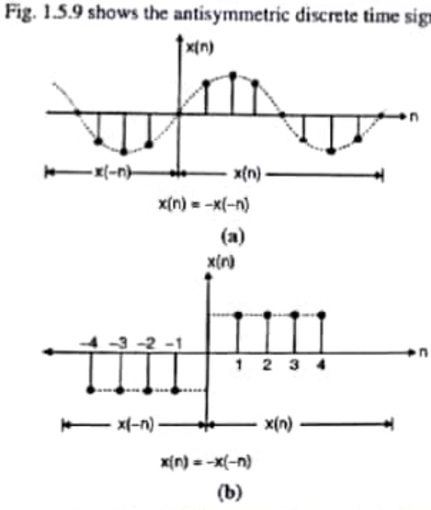
(H-21) Fig. 1.5.8 : D.T. symmetric (even) signals

Odd (Antisymmetric) D.T. signal :

- Similarly discrete time (D.T.) signal x (n) is said to be antisymmetric or odd if it satisfies the following condition,

Condition of antisymmetric : $x(n) = -x(-n)$ ... For D.T. signal

- Fig. 1.5.9 shows the antisymmetric discrete time signals.



(H-22) Fig. 1.5.9 : Antisymmetric (odd) D.T. signals

1.5.6 Decomposing a CT Signal Into Even and Odd Parts :

- Any continuous or discrete time signal can be expressed as the summation of even part and odd part.

$$\therefore \quad x(t) = x_e(t) + x_o(t) \quad ...(1.5.14)$$

Here $x_e(t)$ = Even component of signal x (t)

and $x_o(t)$ = Odd component of signal x (t)

Putting $t' = -t$ in Equation (1.5.14) we get,

$$x(-t) = x_e(-t) + x_o(-t) \quad ...(1.5.15)$$

- Let us now obtain the expressions for the even and odd part $x_e(t)$ and $x_o(t)$.

Expression for the even part $x_e(t)$:

- For the even signal we have,

$$x_e(t) = x_e(-t) \quad ...(1.5.16)$$

- And for odd signal we have,

$$x_o(-t) = -x_o(t) \quad ...(1.5.17)$$

- Putting Equations (1.5.16) and (1.5.17) in Equation (1.5.15) we get,

$$x(-t) = x_e(t) - x_o(t) \quad ...(1.5.18)$$

- Adding Equations (1.5.14) and (1.5.18),

$$x(t) + x(-t) = 2x_e(t)$$

$$\therefore \quad x_e(t) = \frac{1}{2}[x(t) + x(-t)] \quad ...(1.5.19)$$

Equation (1.5.19) gives even component of x (t).

Expression for the odd part $x_o(t)$:

Now subtracting Equation (1.5.18) from Equation (1.5.14) we get,

$$x(t) - x(-t) = 2x_o(t)$$

$$\therefore \quad x_o(t) = \frac{1}{2}[x(t) - x(-t)] \quad ...(1.5.20)$$

Equation (1.5.20) gives odd component of x(t).

1.5.7 Even and Odd Components for a D.T. Signal x (n) :

- The discrete time signal x (n) can be expressed in terms of its even and odd components as follows, by replacing "t" by "n".
- So expression for the even component $x_e(n)$ is given by,

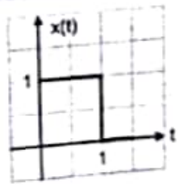
$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \quad ...(1.5.21)$$

- And equation of odd component is,

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)] \quad ...(1.5.22)$$

Solved examples :

Ex. 1.5.10 : Function x (t) is as shown in Fig. P. 1.5.10(a). Draw even and odd parts of x (t) **Dec. 04. 10 Marks**



(H-23) Fig. P. 1.5.10(a) : Given function x(t)

Soln. :

- The even part is given by,

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

- And the odd part is given by,

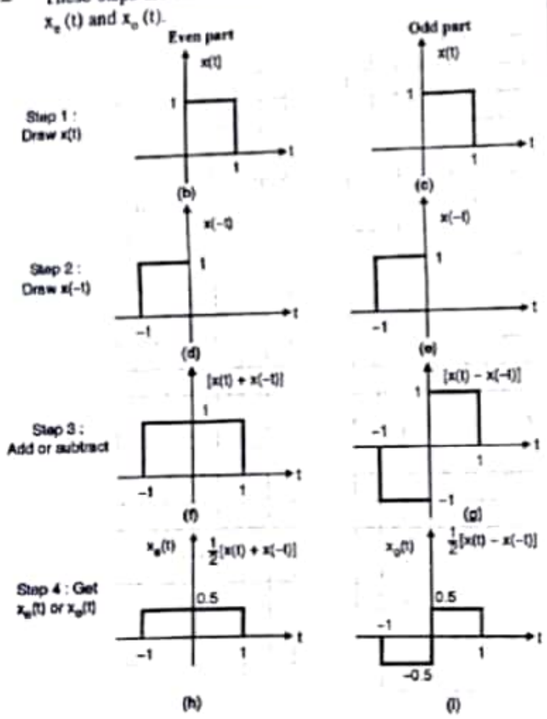
$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

where x (– t) represents the folded version of x (t). The folding operation is discussed later on.

Steps to be followed :

Step 1 : Draw the signal x (t).
Step 2 : Draw its folded version x (– t).
Step 3 : Add x (t) and x (– t) or subtract x (– t) from x (t).
Step 4 : Divide the addition or subtraction by 2 to get $x_e(t)$ and $x_o(t)$.

- These steps are followed in Fig. P. 1.5.10(b) to (i) to obtain $x_e(t)$ and $x_o(t)$.



(H-24) Fig. P. 1.5.10

- The above definitions of even and odd signals assume that the signals are real valued.
- If the signals are complex valued, then we have to talk in terms of conjugate symmetry.
- A complex valued signal is said to be having a conjugate symmetry if it satisfies the following condition,

$$x(-t) = x^*(t) \quad ...(1.5.23)$$

- Remember that if x (t) = a + jb then its complex conjugate is given by,

$$x^*(t) = (a - jb)$$

- So the complex valued signal x (t) is conjugate symmetric if its real part is even and its imaginary part is odd.

1.5.8 Energy and Power Signals :

Power signal :

- A signal is called as a power signal if its "average normalized power" is non-zero and finite.
- It has been observed that almost all the periodic signals are power signals.

Energy signals :

- A signal having a finite non-zero total normalized energy is called as an energy signal.
- It has been observed that almost all the non-periodic signals defined over a finite period, are energy signals.
- As these signals are defined over a finite period, they are called as time limited signals.

1.6 Energy and Power of the Signals :

1.6.1 Time Average (dc Value) of a Signal :

- The time average or dc value of a signal x (t) over all time is defined as,

Time average or dc value :

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)\, dt \quad ...(1.6.1)$$

- The integration of x (t) from – T / 2 to T / 2 gives us the area under x (t) between the time limits – T / 2 to T / 2.
- This area is then divided by T to obtain the average value of x (t). And as the limit of T → ∞ is taken, the average is calculated over the entire time range from – ∞ to + ∞.
- If the signal x (t) is periodic signal with period "T_o" then its average value is given by,

Time average of periodic signal :

$$\langle x(t) \rangle = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x(t)\, dt \quad ...(1.6.2)$$

1.6.2 Power :

MU - May 07

University Questions

Q. 1 State and prove the expression for normalized power and normalized energy of continuous time signal x (t). Periodic signals are always power signals. Justify. **(May 07, 9 Marks)**

Instantaneous power :

- If x (t) is a voltage across a resistor R, the instantaneous power is defined as,

$$\text{Instantaneous power} = \frac{x^2(t)}{R} \quad ...(1.6.3)$$

- However if x (t) is a current signal then the expression for the instantaneous power is given by,

$$\text{Instantaneous power} = x^2(t) \times R \quad ...(1.6.4)$$

Normalized power :

- Everytime we may not know whether x (t) is a voltage signal or a current signal. Hence in order to make the expression for power independent of the nature of x (t), we normalize it by substituting R = 1Ω in Equations (1.6.3) and (1.6.4).
- Therefore the normalized instantaneous power is given by,

$$\text{Normalized instantaneous power} = x^2(t) \quad ...(1.6.5)$$

Average normalized power :

- Next step is to obtain the average value of this normalized power. From the basic expression for time average defined by Equation (1.6.1) we can write that,

$$\text{Average normalized power} = P = \langle x^2(t) \rangle$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \quad ...(1.6.6)$$

- The above definition can be generalized for a complex signal x (t) as,

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt \quad ...(1.6.7)$$

- From Equations (1.6.6) and (1.6.7) it is observed that the signal power P is the time average (mean) of the signal amplitude squared that is the "mean squared" value of x (t).
- Therefore the square root of P is the root mean square (rms) value of x (t).

$$\therefore \; P = \text{Mean square value of } x(t)$$

$$\therefore \; \sqrt{P} = \text{rms value of } x(t) \quad ...(1.6.8)$$

- For a periodic signal with a period T_o, the Equations (1.6.6) and (1.6.7) get modified to,

$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x^2(t)\, dt \quad ...(1.6.9)$$

- For a complex periodic signal x (t) the average normalized power is given by,

$$P = \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} |x^2(t)|\, dt \quad ...(1.6.10)$$

Based on these definitions of average normalized power, we have defined the power signal in the preceding section.

1.6.3 Energy :

- The total normalized energy for a "real" signal x (t) is given by,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt \quad ...(1.6.11)$$

- However if the signal is complex then the expression for total normalized energy is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \quad ...(1.6.12)$$

- These equations indicate that the energy is equal to the area under $x^2(t)$ the curve over $(-\infty \le t \le \infty)$ hence it is always positive.

1.6.4 Comparison of Energy and Power Signals :

Table 1.6.1

Sr. No.	Power signals	Energy signals
1	The signal having finite non-zero power are called as power signals.	The signals having a finite non-zero energy are called as energy signals.
2	Almost all the periodic signals in practice are power signals.	Almost all the non-periodic signals are energy signals.
3	Power signals can exist over an infinite time. They are not time limited.	Energy signals exist over a short period of time. They are time limited.
4	Energy of a power signal is infinite.	Power of an energy signal is zero.

1.6.5 Average Power of a DT Signal :

For a discrete time signals, average power P is given by,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2(n) \quad ...(1.6.13)$$

1.6.6 Energy of a DT Signal :

The energy of a D.T. signal is denoted by E and it is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad ...(1.6.14)$$

Note : The nonperiodic signals are generally energy signals whereas the periodic signals are generally the power signals.

Signal	Condition
Energy signal	$0 < E < \infty$
Power signal	$0 < P < \infty$

1.6.7 Power of the Energy Signals :

Let x (t) be an energy signal i.e. x (t) has a finite non-zero energy. Let us calculate the power of x (t). By definition, stated in Equation (1.6.10) the power of x (t) is given by,

$$P = \lim_{T_o \to \infty} \frac{1}{T_o} \int_{-T_o/2}^{T_o/2} x^2(t)\, dt$$

$$= \lim_{T_e \to \infty} \frac{1}{T_e} \cdot \lim_{T_e \to \infty} \int_{-T_e/2}^{T_e/2} x^2(t)\, dt$$

$$= \lim_{T_e \to \infty} \frac{1}{T_e} \cdot \int_{-\infty}^{\infty} x^2(t)\, dt$$

$$= \lim_{T_e \to \infty} \frac{1}{T_e} \cdot E \quad \text{Using Equation (1.6.12)}$$

$$\therefore \quad P = 0 \times E = 0, \quad \text{As} \lim_{T_e \to \infty} \frac{1}{T_e} = 0$$

Thus the power of an energy signal is zero over a finite time.

1.6.8 Energy of a Power Signal :

Let x (t) be a power signal. The normalized energy of this signal is given by,

$$E = \int_{-\infty}^{\infty} x^2(t)\, dt$$

This expression can be written in a different way as follows :

$$E = \lim_{T_e \to \infty} \int_{-T_e/2}^{T_e/2} x^2(t)\, dt$$

$$= \lim_{T_e \to \infty} \left[T_e \cdot \frac{1}{T_e} \int_{-T_e/2}^{T_e/2} x^2(t)\, dt \right]$$

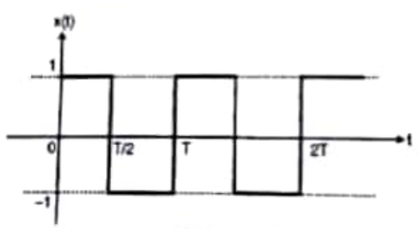
$$= \lim_{T_e \to \infty} T_e \cdot \lim_{T_e \to \infty} \frac{1}{T_e} \int_{-T_e/2}^{T_e/2} x^2(t)\, dt$$

$$= \lim_{T_e \to \infty} T_e \cdot [P]$$

$$\therefore \quad E = \infty \quad \text{As} \lim_{T_e \to \infty} T_e = \infty$$

Thus the energy of a power signal is infinite over a finite time.

Ex. 1.6.1 : What is the average power of the square wave shown in Fig. P. 1.6.1 ?



(H-34) Fig. P. 1.6.1

Soln. :

Given signal is periodic. So consider one cycle from 0 to T.

$$P = \frac{1}{T} \int_0^T x^2(t)\, dt$$

But, $x(t) = 1$ For $0 \le x(t) \le T/2$

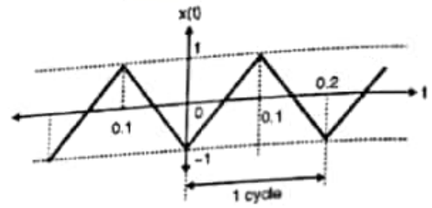
$T = -1$ For $T/2 \le x(t) \le T$

$$\therefore \quad P = \frac{1}{T} \left[\int_0^{T/2} (1)^2\, dt + \int_{T/2}^{T} (-1)^2\, dt \right]$$

$$= \frac{1}{T} \left[1\,(t)_0^{T/2} + 1\,(t)_{T/2}^{T} \right]$$

$$= \frac{1}{T} \left[\frac{T}{2} + \frac{T}{2} \right] = 1 \quad \text{...Ans.}$$

Ex. 1.6.2 : Calculate the average power of the triangular wave shown in Fig. P. 1.6.2.

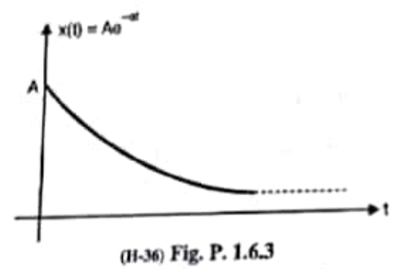


(H-35) Fig. P. 1.6.2

Ans. : P = 1/3 W

Ex. 1.6.3 : Consider the following signal :
$x(t) = A\, e^{-\alpha t}\, u(t), \alpha > 0.$
Is x (t) an energy signal or power signal ? As $\alpha \to 0$, what is the nature of signal x (t).
May 16, 4 Marks

Soln. : The given signal is as shown in Fig. P. 1.6.3. It is positive sided exponential signal; which is non-periodic, so it is an energy signal.



(H-36) Fig. P. 1.6.3

Now energy of continuous signal is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

$$\therefore \quad E = \int_{-\infty}^{\infty} |A\, e^{-\alpha t}\, u(t)|^2\, dt$$

Since the signal $A\, e^{-\alpha t}$ is multiplied by u (t); we will change the limits of integration. The new limits will be from t = 0 to t = ∞; sin t * u (t) exists only in this range and its magnitude is 1.

$$\therefore \quad E = \int_0^{\infty} |A\, e^{-\alpha t}|^2\, dt = \int_0^{\infty} A^2 e^{-2\alpha t}\, dt$$

$$\therefore \quad E = A^2 \left[\frac{e^{-2\alpha t}}{-2\alpha} \right]_0^{\infty} = -\frac{A^2}{2\alpha} [e^{-\infty} - e^0]$$

$$= -\frac{A^2}{2\alpha} \left[\frac{1}{e^{\infty}} - 1 \right]$$

$$\therefore \quad E = -\frac{A^2}{2\alpha} (-1)$$

$$\therefore \quad E = \frac{A^2}{2\alpha}$$

Hence 'A' and 'α' and constant terms. Thus energy is finite and non-zero.

Now the given signal is,

$$x(t) = A\, e^{-\alpha t}\, u(t), \alpha > 0.$$

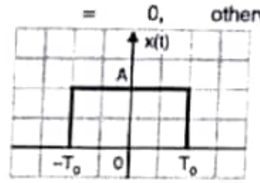
As $\alpha \to 0$; the equation of x (t) becomes,

$$x(t) = A\, u(t)$$

Thus as $\alpha \to 0$; x (t) tends to become unit step of magnitude 'A'.

Ex. 1.6.4 : Find if the following signal is energy or power signals. If yes determine its energy or power.

$$x(t) = A, \quad -T_0 \le t \le T_0$$
$$= 0, \quad \text{otherwise}$$



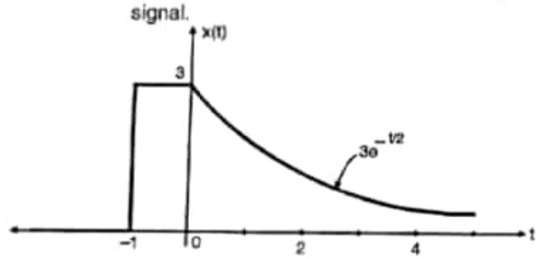
(H-37) Fig. P. 1.6.4

Ans. : Energy signal. $E = 2\, A^2\, T_e$

Ex. 1.6.5 : Find whether the signal sin ωt is energy or power signal.

Ans. : Power signal

Ex. 1.6.6 : Determine whether the signal shown in Fig. P. 1.6.6 is an energy signal or a power signal.



(H-39) Fig. P. 1.6.6

Soln. :

The signal shown in Fig. P. 1.6.6 tends to zero as $|t| \to \infty$. Therefore the average power content of this signal will be zero. Let us calculate the total normalized energy of this signal.

$$x(t) = 3 \qquad -1 \le t \le 0$$
$$= 3\, e^{-t/2} \qquad t > 0$$

$$\therefore \quad E = \int_{-\infty}^{\infty} x^2(t)\, dt = \int_{-1}^{0} x^2(t)\, dt + \int_0^{\infty} x^2(t)\, dt$$

$$= \int_{-1}^{0} (3)^2\, dt + \int_0^{\infty} (3\, e^{-t/2})^2\, dt$$

$$= 9\,[t]_{-1}^{0} + 9 \left[\frac{e^{-t}}{-1} \right]_0^{\infty} = 9\,[0 + 1] - 9\,[e^{-\infty} - e^0]$$

$$\therefore \quad E = 9 + 9 = 18 \quad \text{...Ans.}$$

Thus the total normalized energy of the signal is non-zero and finite. Hence this signal is an energy signal.

Ex. 1.6.7 : Explain whether the following signal is power signal or energy signal ?
cos ωt
May 05, 4 Marks

Soln. :

x (t) = cos ωt

The signal cos ωt does not tend to zero as (t) → ∞. Thus the energy of the signal will be ∞.

$$P = \frac{1}{T} \int_{-T/2}^{T/2} \cos^2 \omega t\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \frac{1 + \cos 2\omega t}{2}\, dt$$

$$= \frac{1}{2T} \int_{-T/2}^{T/2} 1\, dt + \int_{-T/2}^{T/2} \cos 2\omega t\, dt$$

$$= \frac{1}{2T} \left[[t]_{-T/2}^{T/2} + \frac{1}{2} [\sin 2\omega t]_{-T/2}^{T/2} \right]$$

$$= \frac{1}{2T} \left[\frac{T}{2} - \left(-\frac{T}{2} \right) + \frac{1}{2} \left[\sin 2\omega \frac{T}{2} - \sin 2\omega \frac{T}{2} \right] \right]$$

$$= \frac{1}{2T} \left[T + \frac{1}{2} [\sin \omega T - \sin \omega T] \right] = \frac{1}{2T} \times T = \frac{1}{2}$$

As the power is non-zero and finite the given signal is a power signal.

Ex. 1.6.8 : Determine whether the following signal is energy signals or power signals and evaluate their normalized energy and power.
$x(t) = \cos^2(\omega_0 t)$
Dec. 07, 4 Marks, May 11, 2 Marks

Soln. :

The signal $\cos^2(\omega_0 t)$ does not tend to zero as $t \to \infty$. Thus the energy of the signal ∞.

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt = \frac{1}{T} \int_{-T/2}^{T/2} [\cos^2 \omega_0 t]^2\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1 + \cos 2\omega_0 t}{2} \right)^2 dt$$

$$P = \frac{1}{4T} \int_{-T/2}^{T/2} (1 + \cos 2\omega_0 t)^2\, dt$$

$$P = \frac{1}{4T} \int_{-T/2}^{T/2} (1 + \cos^2 2\omega_0 t + 2 \cos 2\omega_0 t)\, dt$$

$$P = \frac{1}{4T} \left[\int_{-T/2}^{T/2} 1\, dt + \int_{-T/2}^{T/2} \cos^2 2\omega_0 t\, dt + 2 \int_{-T/2}^{T/2} \cos 2\omega_0 t\, dt \right]$$

$$P = \frac{1}{4T} \left[[t]_{T/2}^{T/2} + \int_{-T/2}^{T/2} \left(\frac{1 + \cos 4\omega_0 t}{2} \right) dt + 2\, [\sin 2\omega_0 t]_{T/2}^{T/2} \right]$$

$$P = \frac{1}{4T}\left[\left[\frac{T}{2}-\left(\frac{T}{2}\right)\right]+\frac{1}{2}\left[\int_{-T/2}^{T/2} 1\,dt + \int \cos 4\omega_0 t\,dt\right]\right.$$
$$\left. +2\left[\sin 2\omega_0\frac{T}{2} - \sin 2\omega_0\left(\frac{-T}{2}\right)\right]\right]$$

$$P = \frac{1}{4T}\left[T+\frac{1}{2}\left[[t]_{-T/2}^{T/2} + [\sin 4\omega_0 t]_{-T/2}^{T/2}\right]+2\left[\sin\omega_0 T + \sin\omega_0 T\right]\right]$$

$$P = \frac{1}{4T}\left[T+\frac{1}{2}\left[\left(\frac{T}{2}+\frac{T}{2}\right)+\left[\sin 4\omega_0\frac{T}{2}+\sin 4\omega_0\frac{T}{2}\right]\right]+4\sin\omega_0 T\right]$$

$$P = \frac{1}{4T}\left[T+\frac{T}{2}+\frac{1}{2}[\sin 2\omega_0 T + \sin 2\omega_0 T) + 4\sin\omega_0 T\right]$$

$$P = \frac{1}{4T}\left[T+\frac{T}{2}+\sin\omega_0 T + 4\sin\omega_0 T\right]$$

$$P = \frac{1}{4T}\left[T+\frac{T}{2}+5\sin\omega_0 T\right]$$

As the power is non-zero and finite, the given signal is a **power signal**.

Ex. 1.6.9 : Find the energy and power of the following signals :

1. $x(t) = A\cos(2\pi f_0 t + \theta)$
2. $x(t) = \text{rect}[t/T_0]$

Dec. 07. 4 Marks. May 11. Dec. 11. 2 Marks

Soln. :

1. $x(t) = A\cos(2\pi f_0 t + \theta)$

This is a periodic signal with period $T_0 = 2\pi/\omega_0$ or $1/f_0$. Therefore the power is given by,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} A^2\cos^2(2\pi f_0 t+\theta)\,dt$$
$$= \frac{1}{T}\int_{-T/2}^{T/2}\frac{A^2}{2}[1+\cos 2(2\pi f_0 t+\theta)\,dt$$
$$= \frac{A^2}{2T}\int_{-T/2}^{T/2} dt + \frac{A^2}{2T}\int_{-T/2}^{T/2}\cos(4\pi f_0 t + 2\theta)\,dt$$
$$= \frac{A^2}{2T}\times(T/2+T/2)+0$$
$$\therefore\ P = \frac{A^2}{2} \quad \text{...Ans.}$$

The second term reduces to zero because the integral appearing in this second term represents the area under a sinusoid over a very large time period i.e. $T \to \infty$.

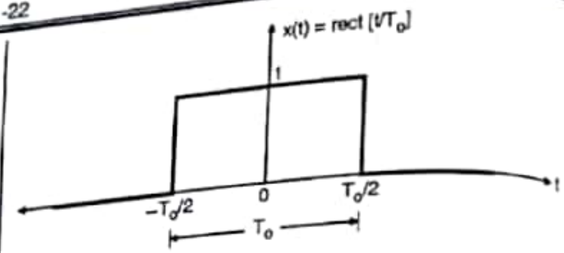
Conclusion : Thus the power of a sinusoid of peak amplitude A is $A^2/2$ irrespective of the value of its frequency f_0 or phase angle θ.

2. $x(t) = \text{rect}[t/T_0]$

We have defined a rectangular pulse of amplitude A and duration T which is centered about the y-axis as,

$$x(t) = A\,\text{rect}[t/T]$$
$$= A \quad \text{for } -T/2 \le t \le T/2$$
$$= 0 \quad \text{otherwise}$$

Comparing the given signal x (t) with this standard rectangular pulse we know that A = 1 and $T = T_0$. Therefore rect $[t/T_0]$ can be plotted graphically as shown in Fig. P. 1.6.9.



(H-41) **Fig. P. 1.6.9 : Rectangular pulse**

The given signal is $x(t) = \text{rect}[t/T_0]$

$$\therefore \text{ Normalized energy } E = \int_{-\infty}^{\infty} x^2(t)\,dt$$
$$= \int_{-T_0/2}^{T_0/2}(1)^2\,dt = [t]_{-T_0/2}^{T_0/2}$$
$$\therefore\ E = (T_0/2 + T_0/2) = T_0 \quad \text{...Ans.}$$

Ex. 1.6.10 : Determine whether the following signal is energy or power signal : $x(t) = 0.9\,e^{-3t}\,u(t)$.

Dec. 12. 3 Marks

Soln. :

Given signal is exponential. So it is non-periodic here from t = 0 to ∞. So, multiplication amplitude 1 and it is existing that exponential signal is from t = 0 to t = ∞.

Now energy of signal is given by,

$$E = \int_{-\infty}^{\infty}|x(t)|^2\,dt = \int_0^{\infty}[|0.9\,e^{-3t}|]^2\,dt$$
$$= (0.9)^2\int_0^{\infty} e^{-\sigma t}\,dt = (0.9)^2\left[\frac{e^{-\sigma t}}{-\sigma}\right]_0^{\infty}$$
$$= \frac{(0.9)^2}{(-6)}[e^{-\infty} - e^0] = \frac{(0.9)^2}{(-6)}[0-1]$$
$$\therefore\ E = 0.135\ J \quad \text{...Ans.}$$

Ex. 1.6.11 : $x(n) = (0.5)^n u(n)$. State whether it is an energy or power signal. Justify.

May 05. May 11. 2 Marks

Soln. :

The signal is discrete exponential, that is, it is non periodic. Energy of discrete signal is given by,

$$E = \sum_{n=-\infty}^{\infty}|x(n)|^2 \quad \text{... (1)}$$

The given signal is,

$$x(n) = (0.5)^n u(n) \quad \text{... (2)}$$

Since it is multiplied by unit step; this signal is present from n = 0 to n = ∞. Thus Equation (1) becomes,

$$E = \sum_{n=0}^{\infty}[(0.5)^n]^2 = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{2n}$$

$$E = \sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n \quad \text{...(3)}$$

We have the standard geometric series formula,

$$\sum_{n=0}^{\infty} A^n = 1 + A + A^2 + \ldots = \frac{1}{1-A}$$
$$\therefore\ E = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

That means this is finite energy.

Justification :

If $0 < E < \infty$ then the signal is energy signal. Since the calculated value of energy is finite; the given signal is **energy signal**.

Ex. 1.6.12 : Find signal power of $x(n) = u(n)$

Dec. 03. May 13. 5 Marks. Dec. 12. 2 Marks

Soln. :

The unit step sequence is power signal. We will prove it as follows :

Step 1 : Calculation of energy :

Energy of input sequence x (n) is given by,

$$E = \sum_{n=-\infty}^{\infty}|x(n)|^2 \quad \text{...(1)}$$

Here x (n) = Unit step, u (n). Unit step is present in the range n = 0 to n = ∞.

$$\therefore\ E = \sum_{n=0}^{\infty}|u(n)|^2 \quad \text{..(2)}$$

The value of unit step is 1 for energy value of n.

$$E = 1 + 1 + 1 + \ldots\ldots\infty$$

This is infinite value.

Thus unit step is not an energy signal.

Step 2 : Calculation of power :

The average power of input signal is given by,

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 \quad \text{...(3)}$$

But here x (n) = Unit step, u (n). Since the range of unit step is from n = 0 to n = ∞, we will change the limits of summation as follows :

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N}|u(n)|^2$$

We know that $|u(n)| = 1$ and $|u(n)|^2 = 1^2 = 1$. Thus summation can be expanded as :

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\underbrace{[1+1+1\ldots\ldots+1]}_{n=0\text{ to }N}$$

$$\text{Thus we can write, } P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N}(1)^n \quad \text{...(4)}$$

Now use standard equation of summation

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases}\dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{for } a \ne 1\\ N_2 - N_1 + 1 & \text{for } a = 1\end{cases} \quad \text{...(5)}$$

Comparing summation term of Equation (4) with Equation (5),

$$a = 1,\ N_1 = 0 \text{ and } N_2 = N.$$

Thus Equation (4) becomes,

$$P = \lim_{N\to\infty}\frac{1}{2N+1}[N_2 - N_1 + 1]$$
$$= \lim_{N\to\infty}\frac{1}{2N+1}[N - 0 + 1]$$
$$\therefore\ P = \lim_{N\to\infty}\frac{N+1}{2N+1}$$

Dividing numerator and denominator by N we get,

$$P = \lim_{N\to\infty}\frac{1+\frac{1}{N}}{2+\frac{1}{N}} \therefore\ P = \frac{1}{2}$$

Since the power is finite; unit step is power signal.

Ex. 1.6.13 : Determine whether following signals are energy signals or power signals ? Calculate their energy or power.

1. $x(t) = A\cos(2\pi f_0 t + \theta)$ **(Ex. 1.6.9))**
2. $x[n] = \left[\frac{1}{4}\right]^n u[n]$

(Similar to Ex. 1.6.11) Dec. 11. 4 Marks

Ex. 1.6.14 : Determine whether following signals are periodic or non periodic ? If periodic, find fundamental period.

$x(t) = 5\cos(4\pi t) + 3\sin(8\pi t)$

Dec. 11. 4 Marks

Soln. :

Given : $x(t) = 5\cos(4\pi t) + 3\sin(8\pi t)$ ...(1)

The corresponding standard equation is,

$$x(t) = A\cos 2\pi f_1 t + A\sin 2\pi f_2 t \quad \text{...(2)}$$

Comparing Equations (1) and (2) we get,

$$2\pi f_1 = 4\pi \quad \text{and} \quad 2\pi f_2 = 8\pi$$
$$\therefore\ f_1 = \frac{4\pi}{2\pi} = 2 \quad \text{and} \quad f_2 = \frac{8\pi}{2\pi} = 4$$
$$\therefore\ T_1 = \frac{1}{2} \quad \text{and} \quad T_2 = \frac{1}{4}$$
$$\text{Now } \frac{T_1}{T_2} = \frac{\frac{1}{2}}{\frac{1}{4}} = \frac{1}{2}\cdot\frac{4}{1} = \frac{2}{1}$$

Since $\frac{T_1}{T_2}$ is rational number; the resultant signal is periodic and the fundamental period is $\frac{1}{2}$ sec.

1.6.9 Multichannel and Multidimensional Signals :

Multichannel signals :

- As the name indicates, multichannel signals are generated by multiple sources or multiple sensors.
- The resultant signal is the vector sum of signals from all channels.

Example :

- A common example of multichannel signal is ECG waveform. To generate ECG waveform; different leads are connected to the body of a patient.
- Each lead is acting as individual channel. Since there are 'n' number of leads; the final ECG waveform is a result of multichannel signal. Mathematically final wave is expressed as,

$$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \text{ ; If three leads are used.}$$

Multidimensional signals :

- If a signal is a function of single independent variable, then such a signal is called as one-dimensional signal. On the other hand, if a signal is a function of many independent variables then it is called as multidimensional signal.
- A good example of multidimensional signal is the picture displayed on the TV screen. To locate a pixel (a point) on the TV screen two co-ordinates namely X and Y are required.
- Similarly this point is a function of time also. So to display a pixel, minimum three dimensions are required; namely x, y and t.
- Thus this is multidimensional signal. Mathematically it can be written as, P (x, y, t).

Comparison of multichannel and multidimensional signal :

Sr. No.	Multichannel signal	Multidimensional signal
1.	Such signals are generated by multiple sources or multiple sensors.	Such signals are function of many independent variables.
2.	Example : ECG signal. Mathematically it is represented by, $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$	Example : Picture displayed on TV screen. Mathematically it is represented by , P (x, y, t).

Causal and Anticausal signals :

- A continuous time signal, x (t) is causal, if it is existing only for positive values of time. That means, x (t) is causal if x (t) = 0 for t < 0. It is anticausal, if x (t) = 0 for t > 0.
- A discrete time signal, x (n) is causal if x (n) = 0 for n < 0. It is anticausal if x (n) = 0 for n > 0.
- Thus, if a signal is multiplied with unit step then it becomes causal signal.

1.7 Elementary or Basic Signals used for Testing :

- In signals and systems we need to use some standard or elementary signals.
- In this section we will show some important standard signal graphically and express them mathematically.
- Some of the standard continuous time and discrete time signals are :
  1. A dc signal
  2. Unit step signal
  3. Sinusoidal signal
  4. Signum function
  5. Triangular signals
  6. A rectangular pulse
  7. Delta or unit impulse function
  8. Exponential signals
  9. Sinc function.

  Let us understand them one-by-one.

1.7.1 Reasons for using the Standard Signals :

There are two basic reasons for using these standard signals for testing purpose :

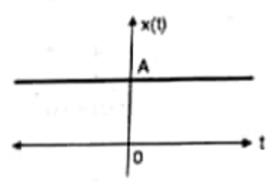
1. These elementary signals serve as basic building blocks for the construction of more complex signals.
2. These signals may be used to model many physical signals which occur in nature.

1.7.2 DC Signal :

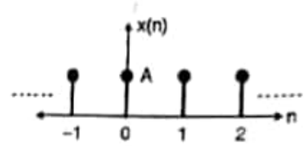
Continuous time DC signal :

- A CT dc signal is as shown in Fig. 1.7.1(a). As seen from the waveform the amplitude "A" of a direct current (dc) signal remains constant independent of time.
- The CT dc signal can be represented mathematically as follows :

  A CT dc signal : $x(t) = A \quad -\infty < t < \infty$ ...(1.7.1)



(H-43) (a) A CT dc signal



(H-44) (b) A discrete time dc signal

Fig. 1.7.1

Discrete time DC signal :

- Fig. 1.7.1(b) shows the discrete time dc signal. It is a sequence of samples each of amplitude A and extending from $-\infty < n < \infty$.
- This signal can be mathematically represented as follows :

  $x(n) = A \quad \ldots -\infty < n < \infty$

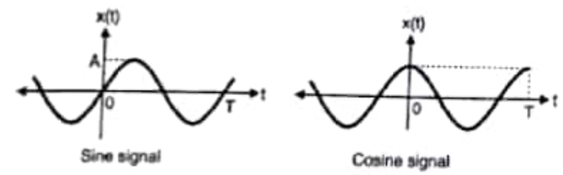
- Or it can be represented in the infinite sequence form as follows :

  $x(n) = \{\ldots\ldots A, A, A, A, A, \ldots\}$
  ↑

1.7.3 Sinusoidal Signals :

C.T. sinusoidal signals :

- The CT sinusoidal signals include sine and cosine signals. They are as shown in the Fig. 1.7.2.



(H-45) Fig. 1.7.2 : Sinusoidal signals

- Mathematically they can be represented as follows :

A sine signal : $x(t) = A \sin \omega t = A \sin (2 \pi f t)$ ...(1.7.2)

A cosine signal : $x(t) = A \cos \omega t = A \cos (2 \pi f t)$ ...(1.7.3)

Discrete time sinusoidal wave :

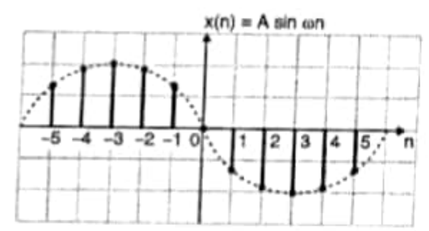
- A discrete time sinusoidal waveform is denoted by,

  $x(n) = A \sin \omega n = A \sin (2\pi f n)$

  Here A = Amplitude

  ω = Angular Frequency = $2 \pi f$
- This waveform is as shown in Fig. 1.7.3.
- Similarly a discrete time cosine wave is expressed as,

  $x(n) = A \cos \omega n = A \cos (2\pi f n)$

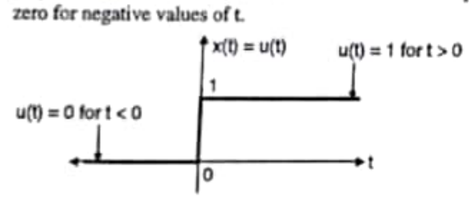


(H-46) Fig. 1.7.3 : DT sinusoidal waveform

1.7.4 Unit Step Signal :

C.T. unit step signal :

- The CT unit step signal is as shown in Fig. 1.7.4. It has a constant amplitude of unity(1) for the zero or positive values of time "t". Whereas its value is equal to zero for negative values of t.



(H-47) Fig. 1.7.4 : Unit step signal

- The CT unit step signal is mathematically represented as,

  Unit step signal : $u(t) = 1 \quad \text{For } t \geq 0$

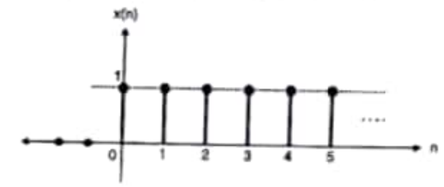
  $= 0 \quad \text{For } t < 0$ ...(1.7.4)

A D.T. unit step signal :

- A discrete time unit step signal is denoted by u(n). Its value is unity(1) for all non negative values of "n". That means its value is one for $n \geq 0$. While for other values of n, its value is zero.

  $\therefore \quad u(n) = \begin{cases} 1 & \text{For } n \geq 0 \\ 0 & \text{For } n < 0 \end{cases}$
- In the form of sequence it can be written as,

  $u(n) = \{1, 1, 1, 1, \ldots\}$
  ↑
- This signal is graphically represented in Fig. 1.7.5.



(H-48) Fig. 1.7.5 : A D.T. unit step signal

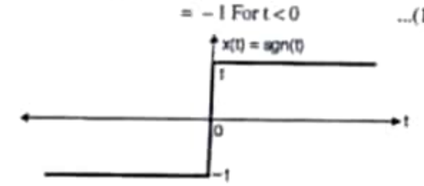
1.7.5 Signum Function :

CT signum function :

- The CT signum function is as shown in the Fig. 1.7.6.
- It is represented mathematically as follows :

  Signum function : $\text{sgn}(t) = 1 \quad \text{For } t > 0$

  $= -1 \quad \text{For } t < 0$ ...(1.7.5)

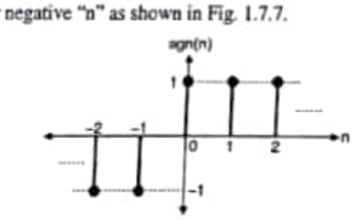


(H-49) Fig. 1.7.6 : Signum function

- The signum function is an "odd" or antisymmetric function.

Discrete time signum function :

- A D.T. signum function can be obtained by sampling the continuous time signum function at regular intervals i.e. at $n = 0, \pm 1, \pm 2 \ldots$
- It is a train of samples of values + 1 for positive "n" and – 1 for negative "n" as shown in Fig. 1.7.7.



(H-50) Fig. 1.7.7 : D.T. signum function

- The DT signum function is defined mathematically as follows :

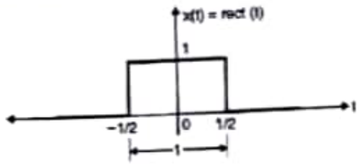
  $x(n) = \text{sgn}(n) = 1 \quad n \geq 0$

  $= -1 \quad n < 0$ ...(1.7.5(a))

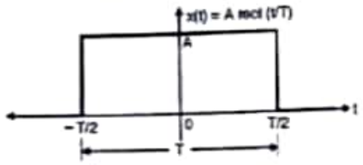
1.7.6 Rectangular Signals :

C.T. rectangular pulse of unit amplitude and unit duration :

- A rectangular pulse of unit amplitude and duration is as shown in Fig. 1.7.8(a).



(a) Rectangular pulse of unit duration



(b) Rectangular pulse of durationand unit amplitude T and amplitude A

(H-51) **Fig. 1.7.8**

- The unit CT rectangular pulse is centered about the y-axis i.e. about t = 0. It is represented mathematically as follows,

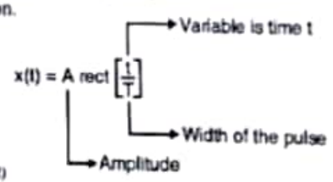
$$\text{rect}(t) = 1 \quad -1/2 \le t \le 1/2$$
$$= 0 \quad \text{Elsewhere} \qquad ...(1.7.6)$$

C.T. rectangular pulse of amplitude "A" and duration "T" :

- The other general type of CT rectangular pulse having an amplitude of "A" over a duration of "T" is as shown in Fig. 1.7.8(b).
- This pulse also is centered about t = 0. It is mathematically represented as,

$$A \text{ rect}\left[\frac{t}{T}\right] = A \quad -T/2 \le t \le T/2$$
$$= 0 \quad \text{Elsewhere} \qquad ...(1.7.7)$$

- In this expression, "t" shows that it is a function of time and "T" represents the width of the rectangular pulse, and A is the amplitude. Rectangular pulse is an even or symmetrical function.

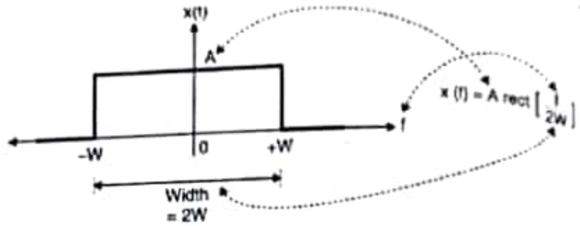


(H-52)

- Hence it is possible to represent the rectangular signal in frequency domain as follows :

$$x(f) = A \text{ rect}\left[\frac{f}{2W}\right]$$

and it is graphically represented as shown in Fig. 1.7.9.

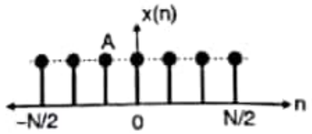


(H-53) **Fig. 1.7.9**

D.T. rectangular pulse :

- Fig. 1.7.10 shows the discrete time rectangular pulse.
- The D.T. rectangular pulse of Fig. 1.7.10 is represented mathematically as :

$$x(n) = A \text{ rect}\left[\frac{n}{N}\right]$$



(H-54) **Fig. 1.7.10 : A DT rectangular pulse**

where A is its amplitude and N is its width.

$$A \text{ rect}\left[\frac{n}{N}\right] = A \;.....\; -N/2 \le n \le N/2$$
$$= 0 \quad .....\; \text{elsewhere}$$

1.7.7 Delta or Unit Impulse Function [δ (t)] :

MU - May 06, May 07, Dec. 09

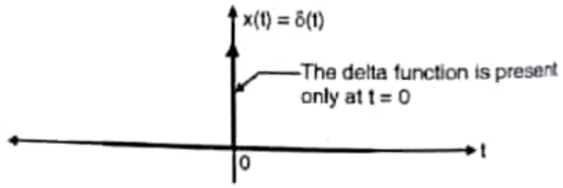
University Questions

Q. 1 What is a generalized delta function ? **(May 06, 2 Marks)**

Q. 2 Impulse function has uniform spectral density over the entire frequency interval. Justify **(May 07, 4 Marks)**

Q. 3 Evaluate the following convolutions :
1. $\delta(t) * \delta(t)$ 2. $x(t) * \delta(t - t_0)$ **(Dec. 09, 10 Marks)**

- The delta function δ (t) is an extremely important function which is used for the analysis of communication systems. The impulse response of a system is its response (output) to a delta function applied at the input.



(H-55) **Fig. 1.7.11 : Delta function**

- The delta function is as shown in Fig. 1.7.11. It is present only at t = 0, its width tends to 0 and its amplitude at t = 0 is infinitely large so that the area under the delta function is unity (i.e. 1). Due to unity area, it is called as a unit impulse function.
- The unit impulse or delta function is denoted by δ (t) and represented mathematically as follows :

Delta function : $\delta(t) = 0 \quad \text{For } t \ne 0 \qquad ...(1.7.8)$

$$\text{OR} \quad \delta(t) \ne 0 \quad \text{For } t = 0$$

The area under the unit impulse is given as,

$$\text{Area under unit impulse : } \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad ...(1.7.9)$$

1.7.7.1 Important Properties of a Delta Function :

Here let us understand two important properties of the delta function. They are :

1. Shifting property 2. Replication property.

- The shifting property of delta function states that,

Shifting property : $$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_m)\, dt = x(t_m) \qquad ...(1.7.10)$$

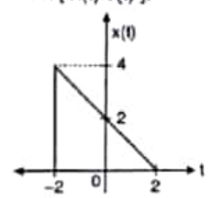
- where $\delta(t - t_m)$ represents the time shifted delta function. This delta function is present only at $t = t_m$. The RHS of Equation (1.7.10) represents the value of x (t) at $t = t_m$. The LHS of Equation (1.7.10) is the area under the product term $x(t) \cdot \delta(t - t_m)$.
- This result indicates that the area under the product of a function x (t) with a shifted impulse $\delta(t - t_m)$ is equal to the value of that function at the instant where the unit impulse is located i.e. $x(t_m)$.
- This property is also known as the **sampling property.**
- The other property of delta function is the replication property. It states that,

Replication property : $x(t) * \delta(t) = x(t) \qquad ...(1.7.11)$

- This property can be stated in words as : The convolution of any function x (t) with delta function yields the same function. The sign * in Equation (1.7.11) represents "convolution".

Ex. 1.7.1 : Write the properties of impulse signal δ(t) and evaluate the following using δ(t) properties for given x(t) shown in Fig. P. 1.7.1(a) ahead :

1. $\int_{-\infty}^{+\infty} x(t)\, \delta(t)\, dt$
2. $\int_{-\infty}^{+\infty} x(t-1)\, \delta(t-1)\, dt$
3. $\int_{-\infty}^{+\infty} x(t)\, \delta(4t)\, dt$
4. $x(t) \cdot \sin t \cdot \delta(t)$
5. $\sin[x(t)\, \delta(t)]$.



(H-734) **Fig. P. 1.7.1(a)**

Soln. :

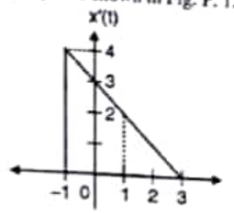
1. According to shifting property,

$$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_0)\, dt = x(t_0)$$

Let $t_0 = 0$

$$\therefore \int_{-\infty}^{\infty} x(t)\, \delta(t)\, dt = x(0) = 2.$$

2. Let $x'(t) = x(t-1)$. It is shown in Fig. P. 1.7.1(b).



(H-735) **Fig. P. 1.7.1(b)**

Now $$\int_{-\infty}^{\infty} x'(t)\, \delta(t-1)\, dt = x'(1) = 2$$

3. According to scaling property,

$$\delta(\alpha t) = \frac{1}{|\alpha|}\, \delta(t)$$

$$\therefore \delta(4t) = \frac{1}{4}\, \delta(t)$$

$$\therefore \int_{-\infty}^{\infty} x(t)\, \delta(4t)\, dt = \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{1}{4}\, \delta(t)\right] dt$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} x(t)\, \delta(t) \cdot dt$$

Let $t_0 = 0$

Using shifting property,

$$\int_{-\infty}^{\infty} x(t)\, \delta(4t)\, dt = \frac{1}{4}\, x(t_0) = \frac{1}{4} \times 2 = \frac{1}{2}$$

4. Consider the term $\sin t \cdot \delta(t)$

According to product property,

$$y(t) \cdot \delta(t) = y(0)\, \delta(t)$$
$$\therefore \sin t \cdot \delta(t) = \sin(0) \cdot \delta(t) = 0 \times \delta(t) = 0$$
$$\therefore x(t) \cdot \sin t \cdot \delta(t) = 0$$

5. According to product property,

$$x(t) \cdot \delta(t) = x(0) \cdot \delta(t) = 2$$
$$\therefore \sin[x(t) \cdot \delta(t)] = \sin(2) = 0.0348°$$

1.7.8 Unit Sample Signal δ (n) :

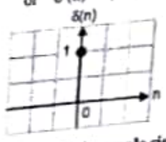
- The D.T. version of unit impulse signal is the unit sample signal.
- A discrete time unit impulse function is denoted by δ (n). Its amplitude is 1 at n = 0 and for all other values of n; its amplitude is zero.

$$\therefore \delta(n) = \begin{cases} 1 & \text{For } n = 0 \\ 0 & \text{For } n \ne 0 \end{cases}$$

- In the sequence form it can be represented as,

$$\delta(n) = \{\ldots, 0, 0, 0, \underset{\uparrow}{1}, 0, 0, 0\}$$

or $\delta(n) = \{1\}$



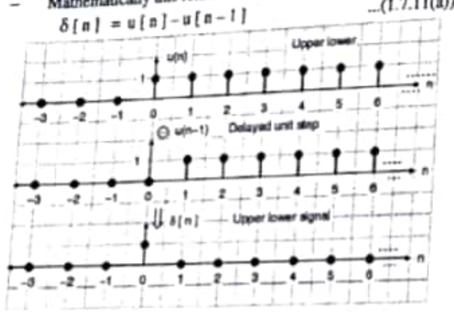
(H-56) **Fig. 1.7.12 : Unit sample signal δ(n)**

- The graphical representation of unit sample signal is as shown in Fig. 1.7.12.

Relation between D.T. unit impulse and D.T. unit step signals :

- Fig. 1.7.12(a) shows how to obtain a D.T. unit impulse signal $\delta[n]$ from a D.T. unit step signal $u[n]$.
- Mathematically this relation can be expressed as,

$$\delta[n] = u[n] - u[n-1] \quad \ldots(1.7.11(a))$$



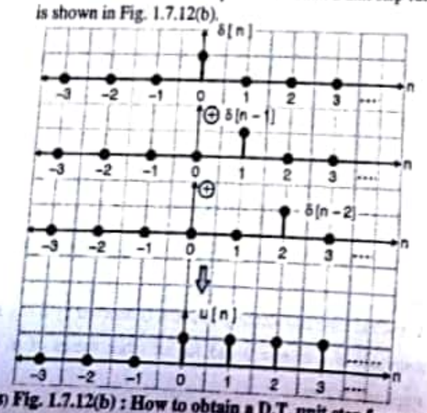
(H-57) **Fig. 1.7.12(a) : How to obtain a unit sample signal from a unit step signal**

- We can obtain a D.T. unit step signal u[n] by taking the sum of unit samples. This is mathematically expressed as,

$$u[n] = \ldots + \delta[n] + \delta[n-1] + \delta[n-2] + \ldots$$

$$= \sum_{k=0}^{\infty} \delta[n-k] \quad \ldots(1.7.11(b))$$

- The summation of unit impulses to obtain a unit step function is shown in Fig. 1.7.12(b).



(H-58) **Fig. 1.7.12(b) : How to obtain a D.T. unit step from unit impulses**

Sampling using a unit sample signal :

- The unit sample sequence can be used to sample the value of a signal.
- If signal x (n) is multiplied by a unit sample δ (n), then we get the value of x (n) at n = 0 as the product. That means,

$$x(n)\,\delta(n) = x(0) \quad \ldots(1.7.11(c))$$

- This happens because δ (n) = 1 only at n = 0. Similarly we can write that,

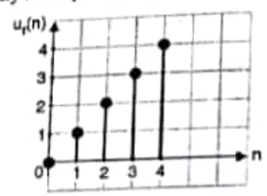
$$x(n)\,\delta(n - n_0) = x(n_0) \quad \ldots(1.7.11(d))$$

1.7.9 Unit Ramp Signal :

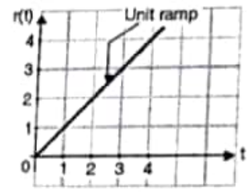
- A discrete time unit ramp signal is denoted by $u_r(n)$. Its value increases linearly with sample number n. Mathematically it is defined as,

$$u_r(n) = \begin{cases} n & \text{For } n \geq 0 \\ 0 & \text{For } n < 0 \end{cases}$$

- Graphically it is represented as shown in Fig. 1.7.13(a).



(a) Unit ramp signal u_r (n)



(b) A CT ramp signal r(t)

(H-59) **Fig. 1.7.13**

- A continuous time ramp signal is denoted by r(t). Mathematically it is expressed as,

$$r(t) = \begin{cases} t & \text{For } t \geq 0 \\ 0 & \text{For } t < 0 \end{cases}$$

It is as shown in Fig. 1.7.13(b).

1.7.10 C.T. Complex Exponential Signals :

- The continuous time (C.T.) complex exponential signal is of the following form,

$$x(t) = C e^{\alpha t} \quad \ldots(1.7.12)$$

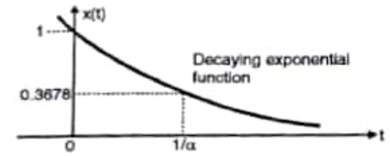
- Here C and α are in general complex numbers. Depending on the values of these parameters, the complex exponent can have several different characteristics.

Types of complex exponential signal :

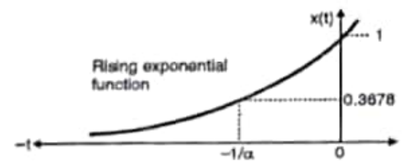
- Depending on the type of C and α, the complex exponential signal can be classified as follows :
  1. Real exponential signals
  2. Periodic complex exponential signals

1. CT real exponential signals :

- If C and α both are real, then the corresponding exponential signal is called as the real exponential signal.
- The exponential functions also are used extensively in the signal analysis. There are two types of exponential functions viz., rising and decaying exponential functions as shown in the Fig. 1.7.14.



(a) Decaying exponential function



(b) Rising exponential function

(H-60) **Fig. 1.7.14**

- They are mathematically represented as follows :
  1. Decaying exponential function :

$$x(t) = e^{-\alpha t} \quad \ldots(1.7.13)$$

  2. Rising exponential function :

$$x(t) = e^{\alpha t} \quad \ldots(1.7.14)$$

- Note that we have assumed C = 1 for both the equations stated above.

2. CT periodic complex exponential signals :

- This is the second important type of complex exponential signals. For this type, α is assumed to be purely imaginary.
- Such an exponential is mathematically expressed as,

$$x(t) = e^{j\omega_0 t} \quad \ldots(1.7.15)$$

- The most important property of this signal is that it is a periodic signal. Applying the condition of periodicity we can write that,

$$x(t) = e^{j\omega_0 t} = e^{j\omega_0 (t+T)} \quad \ldots(1.7.16)$$

$$e^{j\omega_0 t} = e^{j\omega_0 t} \cdot e^{j\omega_0 T}$$

- The above expression will be true if and only if,

$$e^{j\omega_0 T} = 1$$

- So the conclusion is that for $e^{j\omega_0 t}$ to be periodic, $e^{j\omega_0 T}$ has to be equal to 1.
- If $\omega_0 = 0$, then x (t) = 1 for any value of T. If $\omega_0 \neq 0$, then the fundamental period T_0 of x (t) that is the smallest positive value of T for which $e^{j\omega_0 T} = 1$ is

$$T_0 = \frac{2\pi}{|\omega_0|} \quad \ldots(1.7.17)$$

- Thus the signals $e^{j\omega_0 t}$ and $e^{-j\omega_0 t}$ will have the same fundamental period of T_0.

1.7.11 Discrete Time Exponential Signals :

- A discrete time exponential signal is expressed as,

$$x(n) = a^n \quad \ldots(1.7.18)$$

Here 'a' is some real constant.

- If 'a' is the complex number then x (n) is written as,

$$x(n) = r e^{j\theta} \quad \ldots(1.7.19)$$

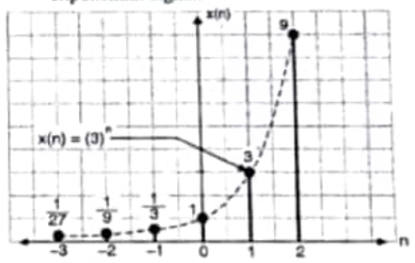
- Depending on the values of "a" we have four different cases,
  1. Case 1 : $a > 1$
  2. Case 2 : $0 < a < 1$
  3. Case 3 : $a < -1$
  4. Case 4 : $-1 < a < 0$

Case 1 : For a > 1 : Rising D.T. exponential

- Let a = 3. Thus we have,

$$x(n) = a^n = 3^n$$

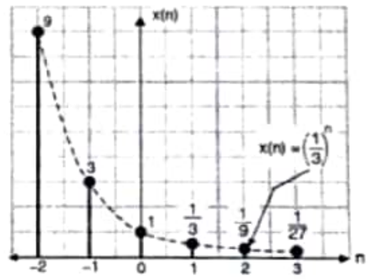
- Graphically such signal is represented as shown in Fig. 1.7.15(a).
- Since the signal is exponentially growing; it is called as rising exponential signal.



n	3^n
−3	1/27
−2	1/9
−1	1/3
0	1
1	3
2	9

(H-61) **Fig. 1.7.15(a) : Rising exponential signal (a > 1)**

Case 2 : For 0 < a < 1 : Decaying exponential signal



n	$(1/3)^n$
−3	27
−2	9
−1	3
0	1
1	1/3
2	1/9
3	1/27

(H-62) **Fig. 1.7.15(b) : Decaying exponential signal**

- In this case we will get decaying exponential sequence. Let $a = \frac{1}{3}$

$$\therefore \quad x(n) = a^n = \left(\frac{1}{3}\right)^n.$$

- Graphically such signal is represented as shown in Fig. 1.7.15(b).

Case 3 : For a < −1 :

- In this case we will get double sided rising exponential signal.

Let a = −3

$$\therefore \quad x(n) = (-3)^n$$

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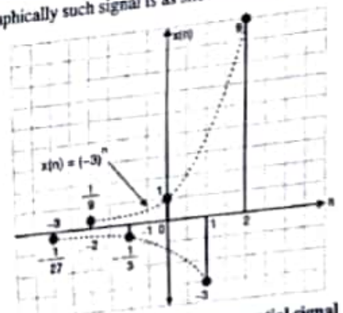
- Graphically such signal is represented as shown in Fig. 1.7.15(c).

Case 4 : For $-1 < a^r < 0$:

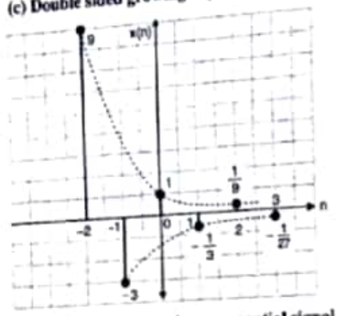
- In this case we will get double sided decaying exponential signal.

Let $a = -\frac{1}{3} \therefore x(n) = a^n = \left(-\frac{1}{3}\right)^n$

- Graphically such signal is as shown in Fig. 1.7.15(d).



(c) Double sided growing exponential signal



(d) Double sided decaying exponential signal

(H-63) Fig. 1.7.15

Relation between the complex exponential and sinusoidal signals :

- The C.T. complex exponential signal is given by,

$$x(t) = e^{j\omega_0 t} \quad ...(1.7.20)$$

- By Euler's relation, the complex exponential signal can be written in terms of sinusoidal signals as,

$$e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t \quad ...(1.7.21)$$

General complex exponential signal :

- The general complex exponential signal is given by,

$$x(t) = C e^{\alpha t}$$

- If we express C in the polar form and α in the rectangular form, then

$$C = |C| e^{j\theta}$$

$$\text{and} \quad \alpha = r + j\omega_0$$

- Substituting we get

$$C e^{\alpha t} = |C| e^{j\theta} e^{(r + j\omega_0)t}$$

$$= |C| e^{rt} e^{j(\omega_0 t + \theta)} \quad ...(1.7.22)$$

- Using the Euler's expression, we can expand this equation as follows :

$$C e^{\alpha t} = |C| e^{rt} \cos(\omega_0 t + \theta) + j |C| e^{rt} \sin(\omega_0 t + \theta) \quad ...(1.7.23)$$

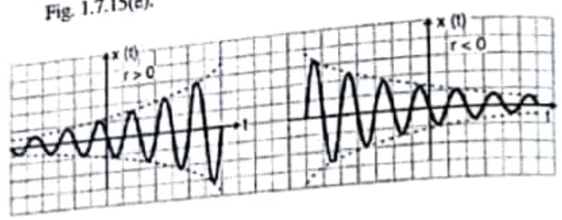
- If r = 0 then $e^{rt} = 1$

$$\therefore C e^{\alpha t} = |C| \cos(\omega_0 t + \theta) + j |C| \sin(\omega_0 t + \theta) \quad ...(1.7.24)$$

- Thus for r = 0, the real and imaginary parts of the complex exponential are sinusoidal.

For r > 0 : Growing exponential :

- If r > 0, then e^{rt} in Equation (1.7.23) will be a growing exponential signal.
- So in Equation (1.7.24), the cosine and sine terms are multiplied by a growing exponential signal.
- Hence we get a growing sinusoidal signal as shown in Fig. 1.7.15(e).



(e) Growing exponential signal (f) Decaying exponential signal

(H-64) Fig. 1.7.15

For r < 0 : Decaying exponential :

- If r is negative (r < 0), then e^{rt} in Equation (1.7.23) will be a decaying exponential.
- Hence each sinusoidal signal in Equation (1.7.24) is multiplied by a decaying exponential. So we get a decaying sinusoidal signal as shown in Fig. 1.7.15(f).

Damped sinusoids :

Sinusoidal signals, multiplied by decaying exponentials are called as damped sinusoids.

1.7.12 Sinc Function :

- The sinc function or sinc pulse is mathematically expressed as,

$$\text{Sinc function : } \operatorname{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)} \quad \text{... For } x \neq 0 \quad ...(1.7.25)$$

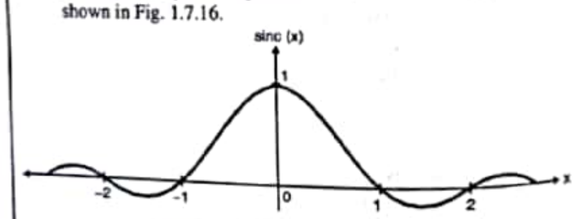
where x is the independent variable. The procedure of plotting the sinc function is explained in Ex. 1.7.1.

- It is proved in Ex. 1.7.1, that

$$\operatorname{sinc}(x) = 1 \quad \text{... At } x = 0$$

$$\text{and} \quad \operatorname{sinc}(x) = 0 \quad \text{... At } x = \pm 1, \pm 2, \pm 3 ....$$

- Hence the graphical representation of a sinc function is as shown in Fig. 1.7.16.



(H-65) Fig. 1.7.16 : sinc function

- Fig. 1.7.16 shows that sinc function has the shape of a sinewave.
- Its magnitude goes on decreasing as the value of |x| increases.
- sinc x → 0 when |x| → ∞.

Ex. 1.7.2 : Plot graphically the sinc function.

Soln. :

- The sinc function is mathematically represented as,

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{(\pi x)} \quad \text{... For } x \neq 0 \quad ...(1)$$

- Here x is an independent variable. We have to find the value of sinc function for different values of x including zero and negative values of x.

Substitute x = 0 in Equation (1)

$$\operatorname{sinc}(x) = \frac{\sin 0}{0}.$$

- This is an indefinite form. Therefore we must find out the value of sinc function at x = 0 separately.

To find sinc(x) at x = 0 :

- Express sin (πx) in the exponential form. Using Euler's theorem we can express sin (πx) as,

$$\sin(\pi x) = \frac{e^{j\pi x} - e^{-j\pi x}}{2j} \quad ...(2)$$

- To solve Equation (2), let us take help of the standard exponential series. It is given as,

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + ...$$

Substitute t = jπx to get,

$$\therefore e^{(j\pi x)} = 1 + j\pi x + \frac{(j\pi x)^2}{2!} + \frac{(j\pi x)^3}{3!} + \frac{(j\pi x)^4}{4!} + ... \quad ...(3)$$

$$\text{Similarly, } e^{(-j\pi x)} = 1 - j\pi x + \frac{(-j\pi x)^2}{2!} + \frac{(-j\pi x)^3}{3!} + \frac{(-j\pi x)^4}{4!} + ... \quad ...(4)$$

- Subtracting Equation (4) from Equation (3), we get,

$$e^{(j\pi x)} - e^{(-j\pi x)} = 2 j\pi x + \frac{2}{3!}(j\pi x)^3 + \frac{2}{5!}(j\pi x)^5 + ... \quad ...(5)$$

- Divide both the sides of Equation (5) by "2j" to get,

$$\sin(\pi x) = \frac{e^{(j\pi x)} - e^{(-j\pi x)}}{2j} = \pi x t + \frac{(j\pi x)^3}{j3!} + \frac{(j\pi x)^5}{j5!} + ... \quad ...(6)$$

- Now divide both the sides of Equation (6) by (πx) to get,

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x} = 1 + \frac{1}{3!}\frac{(j\pi x)^3}{j\pi x} + \frac{1}{5!}\frac{(j\pi x)^5}{j\pi x} + ...$$

$$= 1 + \frac{1}{3!}(j\pi x)^2 + \frac{1}{5!}(j\pi x)^4 + ...$$

$$= 1 + \frac{1}{3!} j^2 \pi^2 x^2 + \frac{1}{5!} j^4 \pi^4 x^4 + ...$$

$$\therefore \operatorname{sinc}(x) = 1 - \frac{\pi^2 x^2}{3!} + \frac{\pi^4 x^4}{5!} + ... \quad ...(7)$$

- Now substitute x = 0 in the Equation (7), to get,

$$\operatorname{sinc}(0) = 1 \quad ...(8)$$

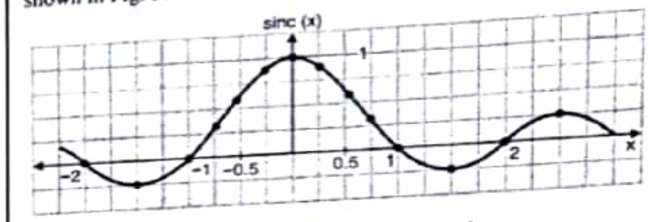
Values of sinc(x) at other values of x :

At other values of x, find out sinc(x) using the Equation (1).

Table P. 1.7.2

x	sinc (x)	x	sinc (x)
0.25	0.9	– 0.25	0.9
0.50	0.6366	– 0.5	0.6366
0.75	0.3	– 0.75	0.3
1.5	– 0.2122	– 1.5	– 0.2122
2.5	0.1273	– 2.5	– 0.1273

By plotting these values, we get the waveform for sinc(x) as shown in Fig. P. 1.7.2.



(H-66) Fig. P. 1.7.2 : A sinc function

Conclusion from the Fig. P. 1.7.2 :

The sinc function passes through zero at the multiple values of x i.e. at x = ± 1, ± 2, ± 3..... This is because sin (πx) = 0 for multiple values of x and the amplitude of the "sinc" function decreases with increase in the magnitude of x.

1.7.13 Unit Triangle Function :

A continuous and discrete time unit triangular functions are shown in Fig. 1.7.17 and 1.7.18 respectively.

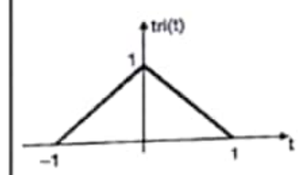
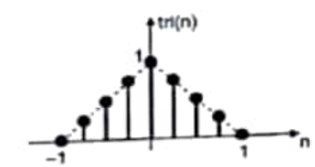


Fig. 1.7.17



(H-1226) Fig. 1.7.18

It is called as unit triangle function because its height and area both are one.

$$\text{tri}(t) = \begin{cases} 1 - |t| & , |t| < 1 \\ 0 & , |t| \geq 1 \end{cases}$$

Similarly D.T. triangular function is defined as,

$$\text{tri}(n) = \begin{cases} 1 - |n| & , |n| < 1 \\ 0 & , |n| \geq 1 \end{cases}$$

1.8 Operations on Signals :

- The independent variable for a C.T. signal is time "t" and for a D.T. signal is "n".
- The transformation of the signal is an important concept in signal and system analysis.
- Transformation of independent variable needs simple modifications of "t" and "n".
- Some of the important transformations of the independent variable are :
  1. Time shifting 2. Time scaling
  3. Folding or time reversal.
- In this section, we are going to discuss these three useful signal operations :

- However the discussion is valid for functions which have independent variables other than time (e.g. frequency) as well.
- Some useful signal operations are :
  1. Amplitude scaling
  2. Addition and subtraction
  3. Multiplication
  4. Time scaling
  5. Time shifting.

1.8.1 Operations Performed on Dependent Variables :

1.8.1.1 Amplitude Scaling of a Signal :

Amplitude scaling of a CT signal :

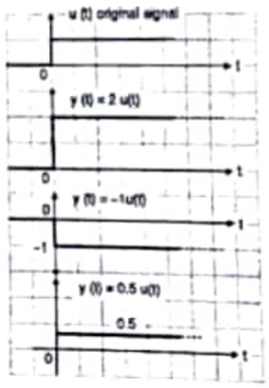
- Amplitude scaling means changing the amplitude of given continuous time signal.
- We will denote a continuous time signal by x (t). If it is multiplied by some constant 'A' then the resulting signal is,

$$y(t) = A\,x(t)$$

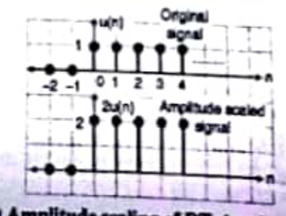
- y (t) is the amplitude scaled version of x (t) and A is a constant.
- Depending on the value of A we can get different versions of x (t). This is illustrated in Fig. 1.8.1(a).
- Note that the scaling takes place on the y-axis and not on the x-axis (i.e. time axis).

Example :

- Let x (t) = u (t) be the original signal. Then the time scaled signal y (t) = A u (t) for different values of A is shown in Fig. 1.8.1(a).



(H-83) (a) Amplitude scaling (H-84)



(b) Amplitude scaling of DT signal

(H-84) Fig. 1.8.1

- The same concept is applicable to the discrete time signals as well.

Amplitude scaling of a DT signal :

- Fig. 1.8.1(b) illustrates the amplitude scaling of a DT signal.
- The original signal is given by,

$$y(n) = u(n) = \{\ldots, 0, 0, \ldots, \underset{\uparrow}{1}, 1, 1, 1, \ldots\}$$

- The amplitude scaled signal is given by,

$$y(n) = 2u(n) = \{\ldots, 0, 0, \ldots, \underset{\uparrow}{2}, 2, 2, 2, \ldots\}$$

1.8.1.2 Addition of Signals :

Addition of CT signals :

- Let the two signals to be added by $x_1(t)$ and $x_2(t)$. Then their addition is given by,

$$y(t) = x_1(t) + x_2(t)$$

- Note that the addition is to be made on the basis of instant to instant addition of the two signals.

Example :

Let $x_1(t) = u(t)$ and $x_2(t) = r(t)$

- Suppose we want to obtain $y(t) = x_1(t) + x_2(t)$ that means, $y(t) = u(t) + r(t)$. This operation is performed as follows :

Here u (t) = Unit step = 1 for t = 0 to ∞.

and r (t) = Unit ramp = t for t = 0 to ∞.

- The addition is shown in the tabulated form in Table 1.8.1 and graphically in Fig. 1.8.2(a).

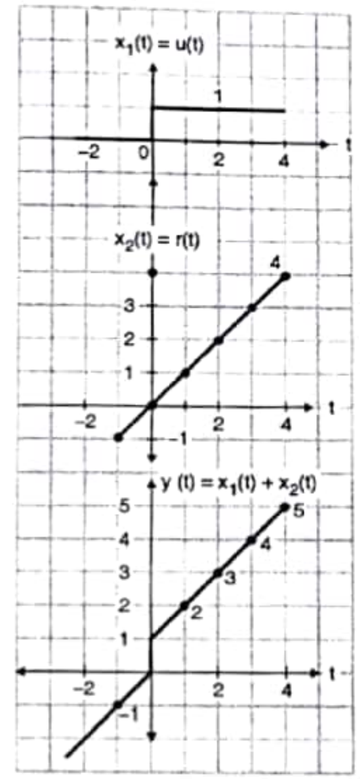
Table 1.8.1

t	–2	–1	0	1	2	3	4	5	.....
$x_1(t) = u(t)$	0	0	1	1	1	1	1	1	.....
$x_2(t) = r(t)$	–2	–1	0	1	2	3	4	5	.....
y (t)	–2	–1	1	2	3	4	5	6	.....

- The same principle is applicable to the addition of discrete time signals as well.

Addition of D.T. signals :

- The addition of two or more D.T. signals takes place on the sample by sample basis.
- That means for the two sequences $x_1(n)$ and $x_2(n)$ to be added, we add $x_1(0)$ and $x_2(0)$ to get the addition at n = 0. Same principle is used for addition at the other values of n.



(H-85) Fig. 1.8.2(a) : Addition of signals

- Let $x_1(n) = u(n)$ and $x_2(n) = u(n)$. Table 1.8.2 shows their addition in the tabular form while Fig. 1.8.2(b) shows the addition graphically.

Table 1.8.2

n	–2	–1	0	1	2	3	4	5	.....
$x_1(n)$	0	0	1	1	1	1	1	1	.....
$x_2(n)$	–2	–1	0	1	2	3	4	5	.....
$y(n) = x_1(n) + x_2(n)$	–2	–1	1	2	3	4	5	6	.....

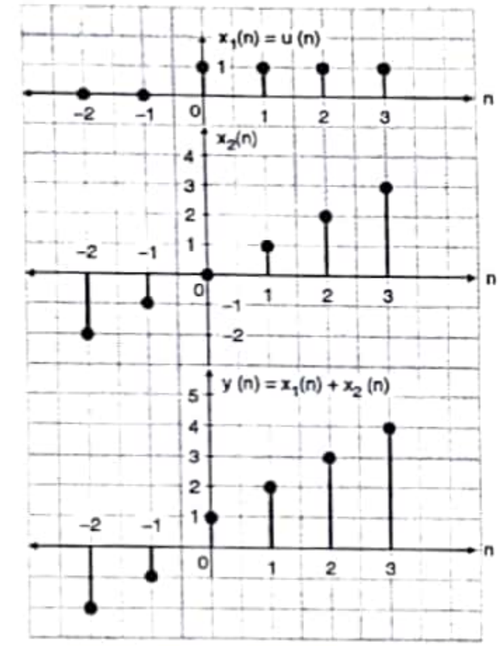
- This addition can be mathematically represented as follows :

$$x_1(n) = \{0, 0, \underset{\uparrow}{1}, 1, 1, 1, \ldots\}$$

$$x_2(n) = \{\ldots, -2, -1, \underset{\uparrow}{0}, 1, 2, 3, 4, \ldots\}$$

- And their addition is

$$y(n) = x_1(n) + x_2(n) = \{\ldots, -2, -1, \underset{\uparrow}{1}, 2, 3, 4, \ldots\}$$



(H-86) Fig. 1.8.2(b) : Addition of D.T. signals

Example :

A physical example of a device that adds two signals is an audio mixer which adds music and sound signal.

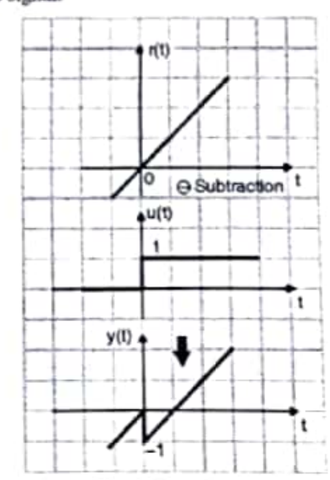
1.8.1.3 Subtraction of Signals :

Subtraction of C.T. signals :

- Similarly we can obtain the subtraction of two signals on the instantaneous basis.

$$y(t) = x_1(t) - x_2(t) = x_1(t) + [-1 \times x_2(t)]$$

- This shows that in order to carry out the subtraction, we have to use the principle of addition only.
- Fig. 1.8.3 shows the subtraction of a unit ramp signal and a unit step signal.



(H-87) Fig. 1.8.3 : Subtraction of CT signals

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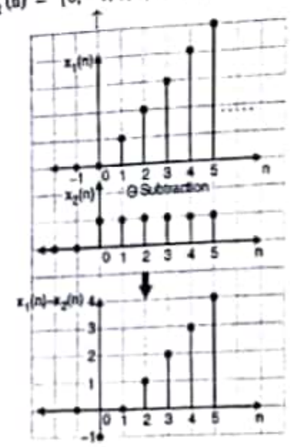
Subtraction of DT signals :

- The subtraction of DT signals takes place on the sample by sample basis. This is illustrated in Fig. 1.8.4.

$$x_1(n) = \{0, 00, 1, 2, 3, 4, 5 \ldots\}$$
$$\uparrow$$
$$\text{and } x_2(n) = \{0, 0, 1, 1, 1, 1 \ldots\}$$
$$\uparrow$$
$$\therefore x_1(n) - x_2(n) = \{0, -1, 0, 1, 2, 3, 4, \ldots\}$$
$$\uparrow$$



(H-88) Fig. 1.8.4 : Subtraction of D.T. signals

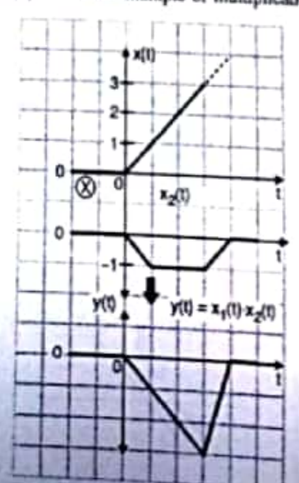
1.8.1.4 Multiplication of Two Signals :

Multiplication of CT Signals :

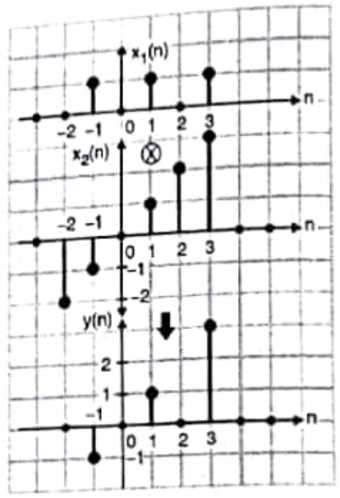
- If $x_1(t)$ and $x_2(t)$ are two continuous signals then the product of $x_1(t)$ and $x_2(t)$ is,

$$y(t) = x_1(t) \cdot x_2(t)$$

- In this case, the multiplication of amplitudes of two signals take place. For example, $y(t) = u(t) \cdot r(t)$.
- Since the amplitude of u (t) is 1; this multiplication will not change the ramp signal r (t). But if we perform, $y(t) = 2u(t) \cdot r(t)$ then it will make the slope of r (t) equal to 2.
- For each prescribed instant of time (t) the value of y (t) is given by the product of corresponding values of $x_1(t)$ and $x_2(t)$ at the same instant.
- Fig. 1.8.5(a) shows the example of multiplication of two CT signals.



(a) Multiplication of CT signals



(b) Multiplication of DT signals

(H-89) Fig. 1.8.5 : Multiplication of signals

Multiplication of D.T. Signal :

- Similarly for D.T. signals

$$y(n) = x_1(n) \cdot x_2(n)$$

- Fig. 1.8.5(b) shows the multiplication of two sequences. Note that the multiplication of DT signals is performed on the sample by sample basis.
- In Fig. 1.8.5(b) $x_1(n) = \{0, 1, 0, 1, 0, 1\}$
$$\uparrow$$
$$\text{and } x_2(n) = \{-2, -1, 0, 1, 2, 3\}$$
$$\uparrow$$

Then their product sequence is given by,

$$y(n) = \{0, -1, 0, 1, 0, 3\}$$
$$\uparrow$$

Real life example :

A real life example is AM radio signal, in which $x_1(t)$ consists of an audio signal plus dc component and $x_2(t)$ is a carrier wave (sine wave).

1.8.1.5 Even and Odd Parts :

For CT signal :

- Even part of signal x (t) is given by,

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

- and odd part of x (t) is given by,

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

- We have already studied these operations.

For a D.T. signal x (n) :

- The discrete time signal x (n) can be expressed in terms of its even and odd components as follows, by replacing "t" by "n".
- So expression for the even component $x_e(n)$ is given by,

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

- And equation of odd component is,

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$

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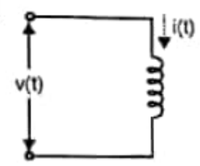
1.8.1.6 Differentiation :

- If x (t) is continuous time signal then its derivative is given by,

$$y(t) = \frac{d}{dt}x(t)$$

- For example, an inductor performs differentiation operation.
- Let the current passing through inductor be denoted by i (t) as shown in Fig. 1.8.6.
- Then the voltage across an inductor is,

$$v(t) = L\frac{d}{dt}i(t)$$



(H-90) Fig. 1.8.6 : Differentiation

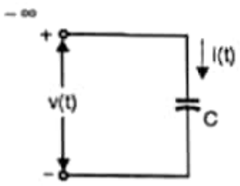
1.8.1.7 Integration :

- Similarly integration of x (t) with respect to time is given by,

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

- Here τ is an integration variable.
- For example, a capacitor performs integration operation. Let i (t) be the current passing through capacitor as shown in Fig. 1.8.7.
- Then the voltage developed across the capacitor is,

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)\, d\tau$$



(H-91) Fig. 1.8.7 : Integration

1.8.1.8 Accumulation for D.T. Signal :

The integration is basically a process of summation. This process is equivalent to accumulation or summation of sample values over the specified interval of time.

1.8.2 Operations Performed on Independent Variable :

- The independent variable in our discussion is time "t" or "n". So the operations discussed in this section are performed on "t", or "n".
- We will discuss the following operations :
  1. Time shifting
  2. Time scaling 3. Folding or time inversion.

1.8.2.1 Time Shifting :

- Let x (t) be the original signal.
- A signal x (t) is said to be "shifted in time" if we replace "t" by (t – T).
- Thus x (t – T) represents the time shifted version of x (t) and the amount of time shift is "T" second.
- If T is positive, then the shift is to right (delay) and if "T" is negative then the shift is to the left (advance).
- Thus x (t – 3) is x (t) delayed (right shifted) by 3 seconds and x (t + 2) is x (t) advanced (left shifted) by 2 seconds.
- Similarly let x (n) be the original D.T. signal. Then x (n – N) represents the shifted version of x (n), and N represents the amount of positional shift.
- If N is positive, then the signal shifts right or get delayed and if N is negative, then the signal is shifted left or it gets advanced.
- Thus x (n – 3) is x (n) delayed (right shifted) by 3 positions whereas x (n + 2) is x (n) advanced (left shifted) by 2 positions.

Time advance of CT signals :

- Let x (t) = r (t) is the original signal which is present only for t ≥ 0.

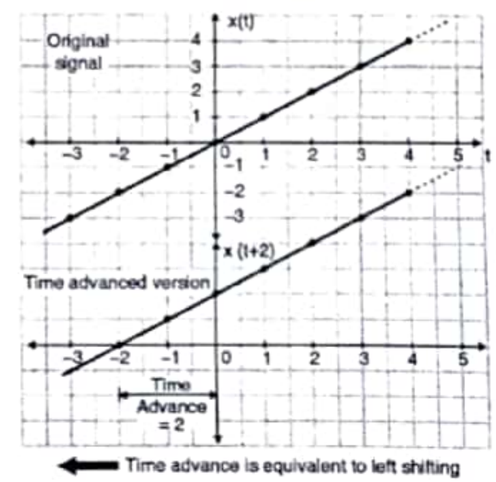
$$\therefore x(t) = r(t) = t \quad \ldots t \geq 0$$
$$= 0 \quad \ldots t < 0$$

- Then x (t + 2) represents the time advanced (left shifted) version of x (t).
- Table 1.8.3 shows the values of x (t) and x (t + 2) for different values of t and Fig. 1.8.8(a) shows the time advanced version.

Table 1.8.3 : Time advance

	t	–3	–2	–1	0	1	2	3	...
Original	x(t)	–3	–2	–1	0	1	2	3	...
Time advanced	x(t+2)	x(–3+2) = x(–1) = –1	x(–2+2) = x(0) = 0	1	2	3	4	5	...

Time advance by 2

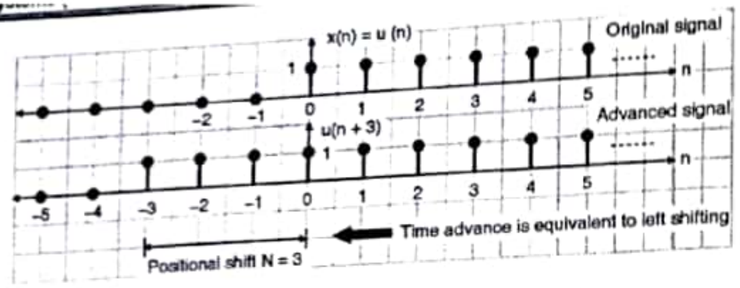


(H-92) Fig. 1.8.8(a) : Time advancing

Note : When time advanced the signal is shifted left on the time axis (x-axis). So left shifting and time advance are one and the same.

Time advance for a D.T. signal :

Let x (n) = u (n) be the original signal. Then x (n + 3) = u (n + 3) is the advanced version of u (n) and the left shift is of three positions as shown in Fig. 1.8.8(b).



(H-93) **Fig. 1.8.8(b) : Positional advance of D.T. signal**

Mathematical representation of shifting operation :

From Fig. 1.8.8(b) we can write the two sequences mathematically as follows :

Original signal : $x(n) = u(n) = \{ \ldots 0, 0, 0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1 \ldots\ldots \}$

Shifted signal : $u(n+3) = \{ \ldots 0, \underbrace{1, 1, 1, \underset{\uparrow}{1}}_{N=3}, 1, 1, 1, 1 \ldots\ldots \}$

Note that the arrow is shifted right by N = 3 positions

(H-94)

Note : The arrow indicating n = 0, has shifted three positions (N = 3) to the right as compared to the original sequence.

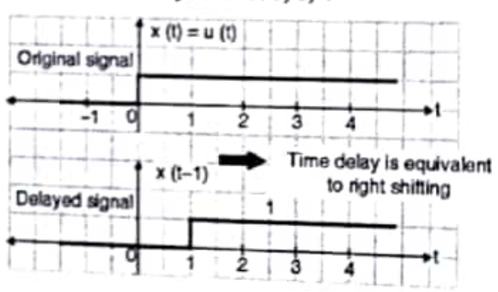
Time delay of CT signals :

- Let x (t) = u (t) be the original signal. Then x (t – 1) i.e. u (t – 1) represents the delayed version of x (t) with a delay equal to 1.
- Table 1.8.4 explains the concept of time delaying and Fig. 1.8.9(a) shows the waveforms.

Table 1.8.4 : Time delay

t	–1	0	1	2	3	4
x(t) = u(t)	0	1	1	1	1	1
x(t–1) = u(t–1)	u (–1–1) = u(–2) = 0	u (0–1) = u(–1) = 0	u (1–1) = u(0) = 1	u (2–1) = u(1) = 1	1	1

Time delay by 1



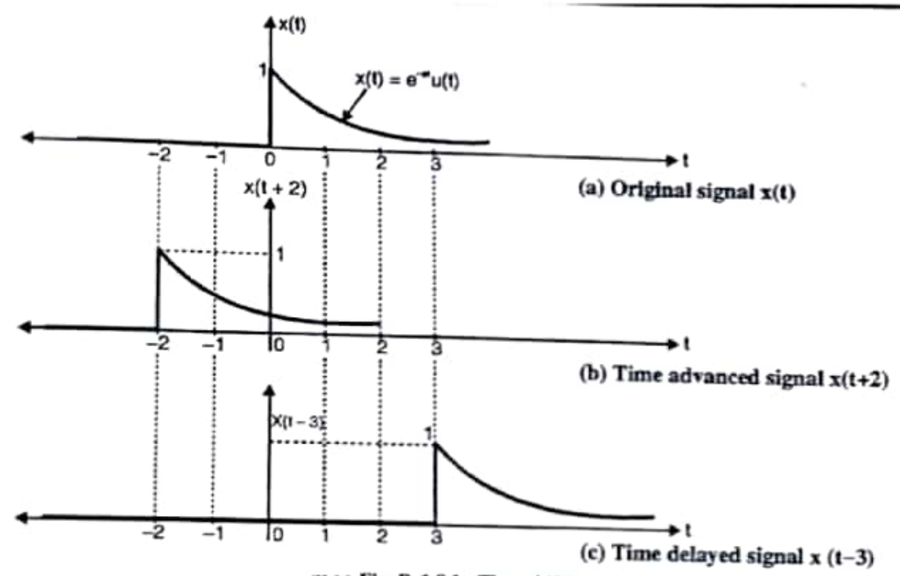
(H-95) **Fig. 1.8.9(a) : Time delay**

Note : When time delayed, the signal is shifted right on the x-axis.

Ex. 1.8.1 : If the given signal $x(t) = e^{-at} u(t)$, draw the signals x (t + 2) and x (t – 3).

Soln. :

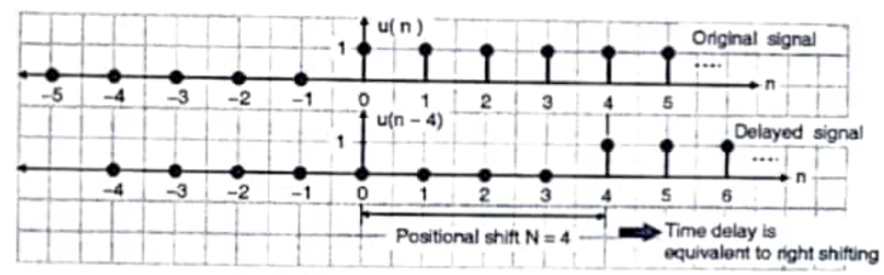
- x (t + 2) is a time advanced signal with a left shift of 2 places.
- x (t – 3) is a time delayed signal with a right shift of 3 places.
- These waveforms are shown in Fig. P. 1.8.1.



(H-96) **Fig. P. 1.8.1 : Time shifting**

Time delay for a D.T. signal :

- For the original D.T. signal x (n), the delayed signal is denoted by x (n – N). The delayed signal is always the right shifted.
- Consider x (n) = u (n) as the original signal. Then x (n – 4) represents its delayed version or right shifted version and the amount of positional shift is N = 4.
- The original and delayed signals are shown in Fig. 1.8.9(b).



(H-97) **Fig. 1.8.9(b) : Positional delay of a D.T. signal**

Mathematical representation :

From Fig. 1.8.9(b) we can express the two sequences mathematically as follows :

Original signal $x(n) = u(n) = \{ \ldots 0, 0, 0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1, \ldots \}$

Delayed signal $u(n-4) = \{ \ldots 0, 0, 0, 0, \underset{\uparrow}{1}, 1, 1, 1, 1 \ldots \}$

(H-97(a))

Note that the arrow is shifted left by N = 4 positions

Note : The arrow indicating n = 0 has shifted N = 4 positions to the left as compared to the original signal.

1.8.2.2 Time Scaling :

Time scaling of CT signals :

- If x (t) is the original signal, then x (at) is its time scaled version, where a is a constant.
- Depending on the value of "a" we can either have signal compression or signal expansion.

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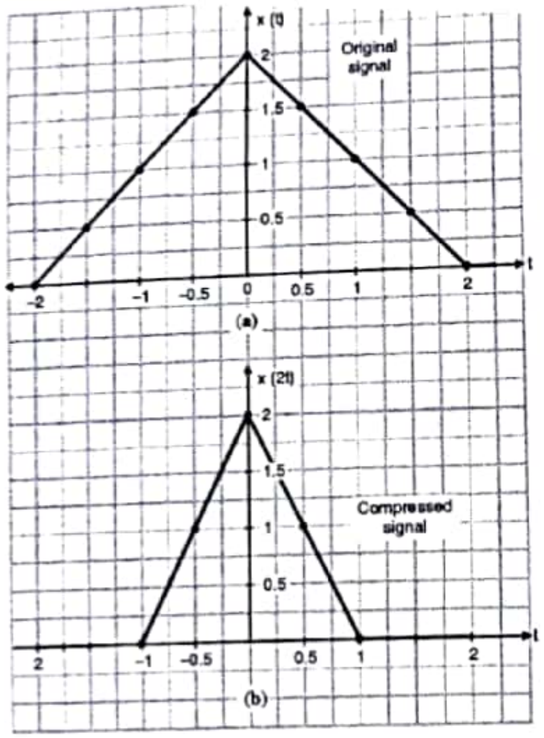


(H-98) **Fig. 1.8.10 : Signal compression**

Signal compression a > 1 :

- For the values of "a" greater than 1, compression of the original signal will take place.
- Let the signal x (t) be a triangular wave as shown in Fig. 1.8.10(a). Then for a = 2 the compressed version of x (t) i.e. x (2t) is shown in Fig. 1.8.10(b).
- The time scaling (signal compression) process is also illustrated in Table 1.8.5.

Table 1.8.5 : Signal compression

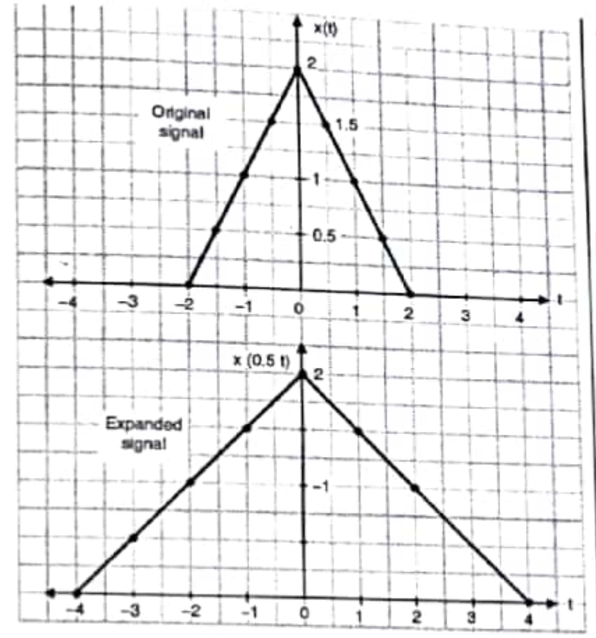
t	−2	−1.5	−1	0.5	0	0.5	1	1.5
x (t)	0	0.5	1	1.5	2	1.5	1	0.5
x(2t)	x (2x − 2) = x (− 4) = 0	x(− 3) = 0	x(− 2) = 0	x(− 1) = 1	x(0) = 2	x(1) = 1	x(2) = 0	x(3) = 0

Signal expansion :

- In x (at) if a is positive and less than 1 (0 < a < 1) then the time scaled signal is the expanded version of x (t).
- This process is illustrated in Table 1.8.6 and in Fig. 1.8.11.
- Note that the value of a is assumed to be equal to 0.5.

Table 1.8.6 : Signal expansion

t	−4	−3	−2	−1	0	1	2	3	4
x (t)	0	0	0	1	2	1	0	0	0
x(0.5 t)	0	0.5	1	1.5	2	1.5	1	0.5	0



(H-99) **Fig. 1.8.11 : Signal expansion**

Time scaling of DT signals :

- If x(n) is the original DT signal then x (an) is called as the time scaled version of x(n), where a is a constant.
- Depending on the value of "a", we can either have signal compression or signal expansion.

Signal compression (a > 1) :

- For a > 1, the signal compression will take place. Let the original signal be represented by,

$$x(n) = \{0, 0, 1, 1, 1, 1, 1, 1, 1, 0, 0, \ldots\}$$
$$\uparrow$$

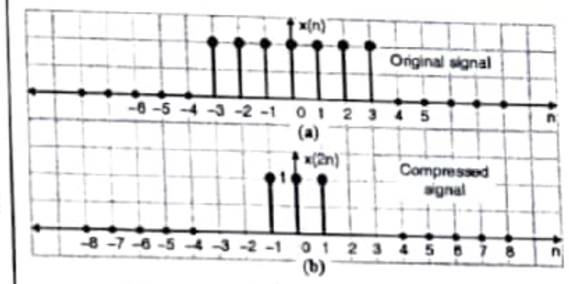
This signal is as shown in Fig. 1.8.12(a).

- For a = 2 the compressed version of x(n) i.e. x (2n) is as shown in Fig. 1.8.12(b). The time scaling process is illustrated in Table 1.8.7. The time scaled signal is given by,

$$x(2n) = \{0, 0, 0, 0, 1, 1, 1, 0, 0, 0, 0\}$$
$$\uparrow$$

Table 1.8.7 : Compression of a DT signal

n	−4	−3	−2	−1	0	1	2	3	4
x(n)	0	1	1	1	1	1	1	1	0
x (2n)	x (− 8) = 0	x (− 6) = 0	x (− 4) = 0	x (− 2) = 1	x (0) = 1	x (2) = 1	x (4) = 0	x (6) = 0	x (8) = 0



(H-100) **Fig. 1.8.12 : Compression of a DT signal**

Signal expansion (a < 1) :

- In x (an) if a is positive and less than 1 (0 < a < 1) then the time scaled signal x (an) is the expanded version of the original signal x(n).
- This process is illustrated in Fig. 1.8.13 and Table 1.8.8 with a = 0.5.

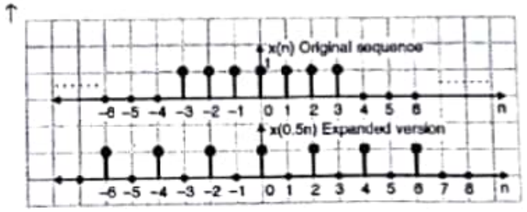
The original signal x (n) = {0, 0, 0, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, ...}
↑

Table 1.8.8 : Expansion of a DT signal

n	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6
x (n)	0	0	0	1	1	1	1	1	1	1	0	0	0
x (0.5 n)	x (− 3) = 1	x (− 2.5) = 0	x (− 2) = 1	x (− 1.5) = 0	x (− 1) = 1	x (0.5) = 0	x (0) = 1	x (0.5) = 0	x(1) = 1	0	1	0	1

and the expanded signal is

$$x(0.5(n)) = \{0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0 \ldots\}$$
$$\uparrow$$



(H-101) **Fig. 1.8.13 : Expansion of a DT signal**

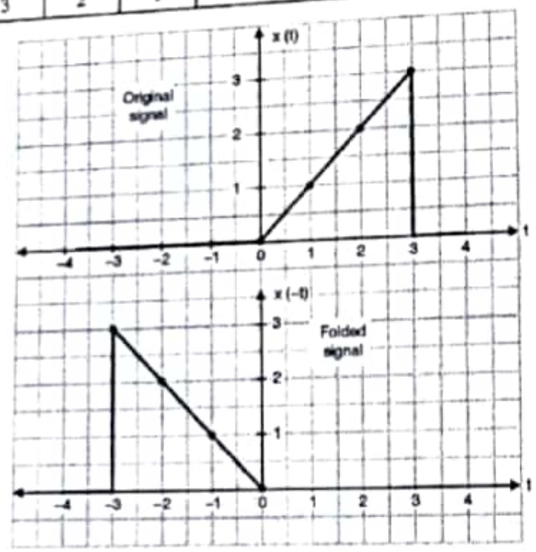
1.8.2.3 Time Reversal (Time Inversion) or Folding :

Folding of CT signals :

- Time inversion can be considered as a special case of time scaling with a = – 1. Thus x (– t) represents the time reversed version of the signal x (t).
- Time reversal is also called as "folding", because x (– t) is the folded version of x (t).
- The folding of x (t) will take place about the vertical axis. x (– t) is the mirror image of x (t) about the y axis as shown in Fig. 1.8.14(a).
- Let x (t) = r (t) for the duration 0 to 3 and let x (t) = 0 elsewhere. Table 1.8.9 explains the folding operations and Fig. 1.8.14 illustrates it graphically.

Table 1.8.9

t	– 4	– 3	– 2	– 1	0	1	2	3	4	.....
x (t)	0	0	0	0	0	1	2	3	0	.....
x(– t)	x[– (– 4)] = x(4) = 0	x(3) = 3	x(2) = 2	x(1) = 1	x(0) = 0	x(– 1) = 0	x(– 2) = 0	x(– 3) = 0	x(– 4) = 0	.....



(H-102) **Fig. 1.8.14(a) : Process of folding**

Folding of D.T. signal :

- If x (n) is the original sequence then x (– n) represents the folded version.
- We can obtain the folded signal simply by replacing "n" by "– n". The folded sequence is actually the mirror image of the original signal, with the mirror assumed to be placed at the y-axis.
- Let the original sequence be,

$$x(n) = \{\ldots, 0, 0, \underset{\uparrow}{1}, 2, 3, 4, 0, 0, \ldots\}$$

- Then the folded sequence is obtained as shown in Table 1.8.10 and it is graphically plotted as shown in Fig. 1.8.14(b).

Table 1.8.10 : Folding of a D.T. signal

n	– 5	– 4	– 3	– 2	– 1	0	1	2	3	4	5	6	...
x(n)	0	0	0	0	0	0	1	2	3	4	0	0	...
x(– n)	x(5) = 0	x(4) = 4	x(3) = 3	x(2) = 2	x(1) = 1	x(0) = 0	x(– 1) = 0	x(– 2) = 0	x(– 3) = 0	x(– 4) = 0	x(– 5) = 0	x(– 6) = 0	...

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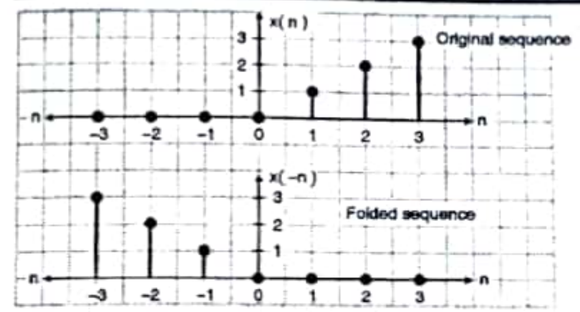
By AnAnT Sawal



(H-103) **Fig. 1.8.14(b) : Folding of a D.T. signal**

- The folded sequence of Fig. 1.8.14(b) is represented as follows :

$$x(-n) = \{0, 4, 3, 2, 1, 0, 0, \ldots\}$$

Folding and Delay :

This operation is a combination of folding and delaying operation. Now let us take a quick view at the modifications used for different operations.

1. **For drawing diagrams :**
   - To obtain delayed sequence, shift the original diagram towards right by 'k' samples.
   - To obtain advanced sequence, shift the original diagram towards left by 'k' samples.
   - To obtain the folded version, take the mirror image of the diagram at n = 0.
2. **For writing the sequence :**
   - The delayed version of x (n) is denoted by x (n – k)
     Negative sign always indicates delay operation of x (n).
   - The advanced version of x (n) is denoted by x (n + k)
     Positive sign always indicates advanced operation of x (n).
   - To obtain the folded version of x (n); replace n by – n

Now folding and delay operation means :

- First fold the sequence x (n); that means obtain x (– n)
- Then delay the folded sequence by k samples.

One major difference between x (n) and x (– n) is that x (– n) denotes the mirror image of x (n). We know that in case of mirror image; left and right sides are reversed.

Now if x (n) is original sequence then its delayed version is denoted by x (n – k). The folded sequence is denoted by x (– n). Since it is a mirror image the delayed version of folded signal is denoted by x (– n + k).

$$\therefore\ x(n) \xrightarrow{\text{Delay}} x(n-k)$$

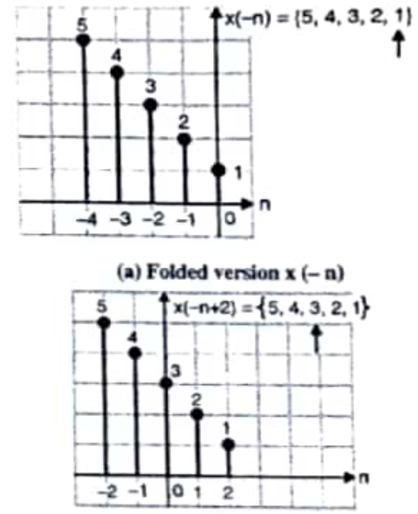
Folding ↓

$$x(-n) \xrightarrow{\text{Delay}} x[-(n-k)] = x(-n+k)$$

But stick to the basic concepts. Delay means shift the diagram towards right by k samples.

Consider the same original sequence $x(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 5\}$ as shown in Fig. 1.8.15(a).

The folded sequence x (– n) is shown in Fig. 1.8.15(a). Suppose we want to delay this folded sequence x (– n) by '2' samples then it will be denoted by x (– n + 2). This sequence is as shown in Fig. 1.8.15(b).



(a) Folded version x (– n)

(b) Delay of folded sequence x (– n + 2)

(H-104) **Fig. 1.8.15**

From Fig. 1.8.15(b), the sequence x (– n + 2) can be written as,

$$x(-n+2) = \{5, 4, 3, 2, \underset{\uparrow}{1}\}$$

Folding and Advance :

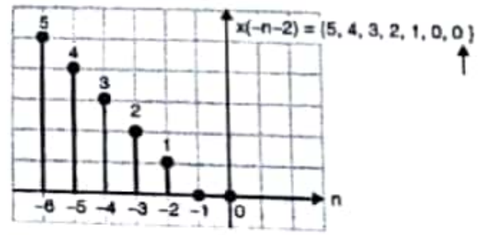
The advanced version of original sequence x (n) is denoted by x (n + k). The folded version of x (n) is denoted by x (– n). Since folding means mirror image of sequence; the advanced version of folded sequence is denoted by x (– n – k).

$$\therefore\ x(n) \xrightarrow{\text{Advance}} x(n+k)$$

Folding ↓

$$x(-n) \xrightarrow{\text{Advance}} x[-(n+k)] = x(-n-k)$$

Consider the same folded sequence x (– n) shown in Fig. 1.8.15(a). Suppose we want to advance this sequence by '2' samples, then the advanced version is denoted by x (– n – 2). Such a sequence is as shown in Fig. 1.8.16.



(H-105) **Fig. 1.8.16 : Advance of folded sequence, x (– n – 2)**

Remember the basic rule. Advancing the sequence means shifting the diagram towards left by 'k' samples. From Fig. 1.8.16, we can write the sequence x (– n – 2) as,

$$x(-n-2) = \{5, 4, 3, 2, 1, 0, \underset{\uparrow}{0}\}$$

Ex. 1.8.2 : Prove that folding and time delaying or advancing of a signal are not commutative operations.

Soln. : In this case we have to prove, ...(1)

Delaying (Folding) ≠ Folding (Delaying)

Let the input sequence be,

$$x(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$$

This sequence is represented in Fig. P. 1.8.2(a).

Now consider L.H.S.

Step 1 : First fold the sequence x (n). The folded sequence x (– n) is shown in Fig. P. 1.8.2(b).

Step 2 : Delay this sequence by 1 sample. This gives delaying of folding operation. This sequence is shown in Fig. P. 1.8.2(c).

Step 3 : Now consider the R.H.S.

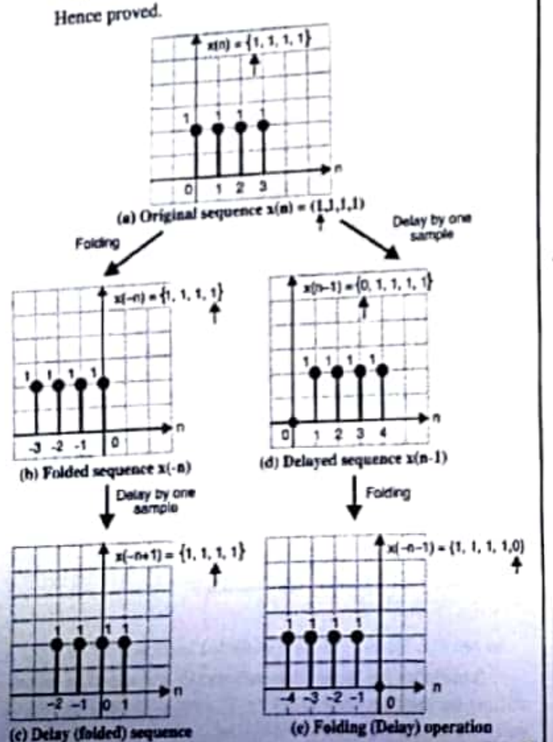
First delay the sequence x (n) by one sample. This sequence is as shown in Fig. P. 1.8.2(d).

Step 4 : Now take the folded version of signal x (n – 1). This gives folding of delaying operation.Such sequence is as shown in Fig. P. 1.8.2(e).

Compare Fig. P. 1.8.2(c) and Fig. P. 1.8.2(e). Since these two diagrams are not same so;

Delaying (Folding) ≠ Folding (Delaying)

Hence proved.



(H-1206) Fig. P. 1.8.2

Ex. 1.8.3 : Find and sketch the even and odd components of the following : **May 11, 4 Marks**

$$x(t) = \begin{cases} t & , 0 \le t \le 1 \\ 2-t & , 1 \le t \le 2 \end{cases}$$

Soln. :

Given : $x(t) = \begin{cases} t & , 0 \le t \le 1 \\ 2-t & , 1 \le t \le 2 \end{cases}$

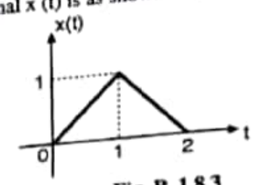
In the range $0 \le t \le 1$ it is a ramp wave. Now for the range $1 \le t \le 2$, we will put some values of t.

For $t = 1 \Rightarrow 2 - 1 = 1$

For $t = 1.5 \Rightarrow 2 - 1.5 = 0.5$

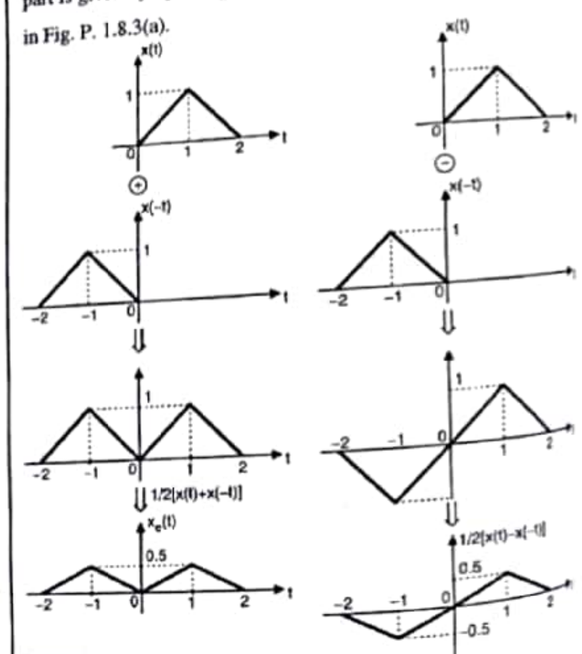
For $t = 2 \Rightarrow 2 - 2 = 0$

Thus the signal x (t) is as shown in Fig. P. 1.8.3.



(H-1211) Fig. P. 1.8.3

Now even part is given by $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$ and odd part is given by $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$. This calculation is shown in Fig. P. 1.8.3(a).



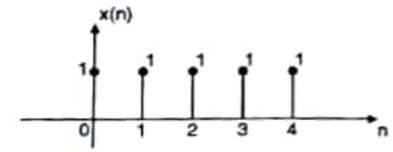
(H-1212) Fig. P. 1.8.3(a)

Ex. 1.8.4 : Let x[n] = u[n] – u [n – 5]. Find and sketch even and odd parts of x [n]. **Dec. 10, 5 Marks, May 14, 4 Marks**

Soln. :

Given : $x(n) = u(n) - u(n-5)$

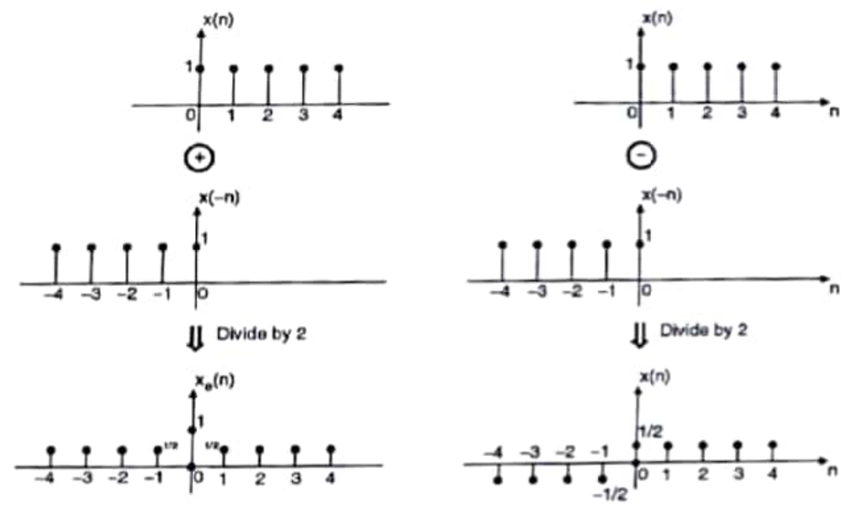
This signal is shown in Fig. P. 1.8.4.



(H-1205) Fig. P. 1.8.4

Now even part is $x_e(n) = \frac{x(n) + x(-n)}{2}$ and odd part is $x_o(n) = \frac{x(n) - x(-n)}{2}$

This calculation is shown in Fig. P. 1.8.4(a).

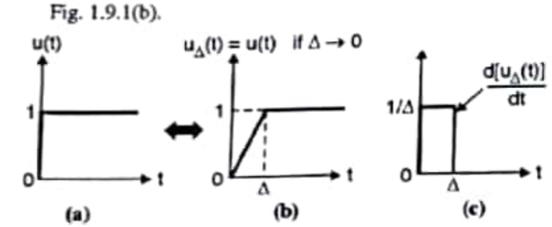


(H-1206) Fig. P. 1.8.4(a)

1.9 Relationship between Unit Step and Unit Impulse :

Unit step u (t) and unit impulse δ (t) of CT can be related as follows :

- Mathematically δ (t) is derivative of u (t) or
  u (t) is integral of δ (t)
- Unit step is not differentiable because there is discontinuity at t = 0 as shown in Fig. 1.9.1(a).
- Hence u (t) has limiting case of $u_\Delta(t)$ as shown in Fig. 1.9.1(b).



(H-157) Fig. 1.9.1

- We know that $u_\Delta(t)$ is continuous function and hence it is differentiable. Derivative of $u_\Delta(t)$ as shown in Fig. 1.9.1(c).
- As $\Delta \to 0$ Fig. 1.9.1(c) becomes unit impulse.

$$\therefore \lim_{\Delta \to 0} \frac{d[u_\Delta(t)]}{dt} = \delta t$$

$$\therefore \delta(t) = \frac{du(t)}{dt}$$ Or same can be represented as

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$

Importance of impulse :

- To convert CT-signal into DT-signal samples of CT signals are required to take at regular interval of time.
- For sampling of CT signal a train of unit impulses is required.

1.9.1 Precedence Rule for Time Shifting and Time Scaling :

- Let x (t) and y (t) be two continuous time signals with y (t) derived from x (t) through a combination of time shifting and time scaling operations.

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- Let the relation between x (t) and y (t) be given mathematically as follows :

$$y(t) = x(at - b) \quad ...(1.9.1)$$

where a and b are constants.

- Let the relation between x (t) and y (t) satisfy the following two conditions :

Condition - 1 $y(0) = x(-b)$

and

Condition - 2 $y(b/a) = x(0)$

- In order to obtain y (t) correctly from x (t), it is necessary to perform the time shifting and time scaling operations in the correct order.
- The proper order is decided on the basis of the fact that in the scaling operation "t" is replaced by "at" and in the time shifting operation "t" is replaced by (t – b) :
- So the time shifting operation is carried out on x (t) to get an intermediate signal y′ (t) as,

$$y'(t) = x(t - b) \quad ...(1.9.2)$$

- Then the time scaling operation is performed on y′ (t) by replacing t by "at". The result is, desired signal y (t).

$$\therefore \quad y(t) = y'(t)\big|_{t \to at}$$

$$\therefore \quad y(t) = x(at - b) \quad ...(1.9.3)$$

Conclusion :

The precedence rule states that for obtaining y (t) from x (t) the proper order needs to be followed. First time shifting and then the time scaling needs to be performed.

Diagrammatic representation of precedence rule :

The precedence rule can be represented diagrammatically as shown in Fig. 1.9.2.

(H-158) Fig. 1.9.2 : Precedence rule

What if the precedence rule is not followed ?

If the rule is not followed, then we do not obtain the desired signal y (t). This is demonstrated as follows :

- First perform the time scaling operation on x (t) to get y′ (t)

$$y'(t) = x(at)$$

- Then perform the time shifting operation on y′ (t) by replacing t by (t – b).

$$\therefore \quad y(t) = x[a(t - b)] = x[at - ab]$$

- We have not obtained the desired signal y (t) = x (at – ab).
- To understand the precedence rule, solve the following example :

Ex. 1.9.1 : If x (t) = rect(t/3) then obtain y (t) = x (2t – 3) first by following the precedence rule and then by violating the rule.

Soln. :

Step 1 : Draw x (t) :

x (t) is given by,

$$x(t) = \text{rect}\left[\frac{t}{3}\right]$$

So it is a rectangular signal of width 3 time units and an amplitude equal to 1. It is plotted as shown in Fig. P. 1.9.1(a).

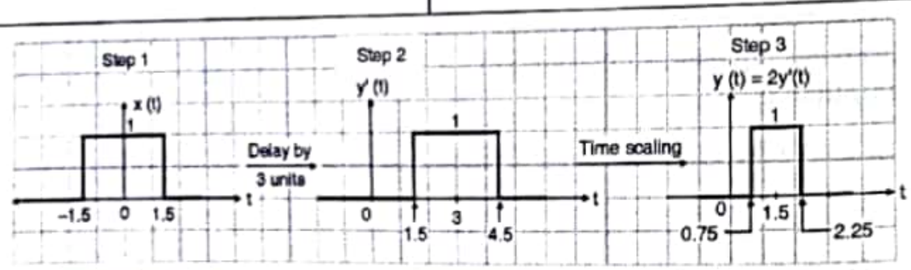
Part I : Solution by following the precedence rule :

Step 2 : Delay x (t) by 3 units :

As shown in Fig. P. 1.9.1(b), x (t) is delayed by 3 time units to obtain the intermediate signal y′ (t) = x (t – 3).

Step 3 : Apply time scaling to y′ (t) :

Now apply time scaling to y′ (t) with a = 2 to get y (t) = x (2t – 3) as shown in Fig. P. 1.9.1(c). Note that we have obtained the correct signal.



(a) Original signal (b) Delayed signal (c) Time scaled signal

(H-159) Fig. P. 1.9.1 : The proper order of operations

Part II : Solution by violating the precedence rule :

Step 4 : Draw the original signal :

Refer Fig. P. 1.9.1(d).

Step 5 : Time-scaling :

Refer Fig. P. 1.9.1(e). The time scaled signal is y′ (t) = x (2t).

Step 6 : Time-shifting :

Refer Fig. P. 1.9.1(f). The intermediate signal y′(t) has been delayed by 3 time units, to obtain y (t).

But y (t) = 2y′ (t) = x [2 (t – 3)] = x [2t – 6] which is not the desired signal.

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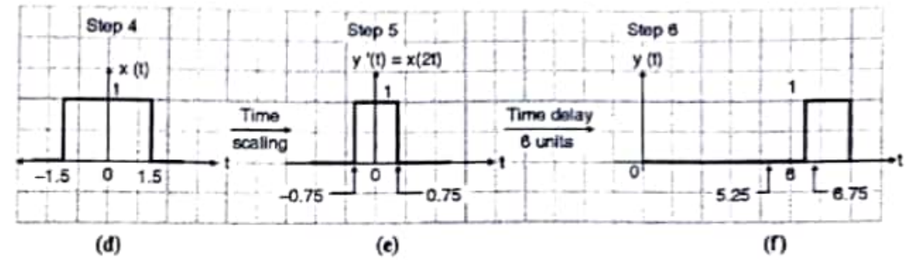


(d) (e) (f)

(H-160) Fig. P. 1.9.1 : Solution by violation of precedence rule

1.10 Additional Solved Examples :

Ex. 1.10.1 : State whether the following signals x(t) is periodic or not, giving reasons. If it is periodic, find the corresponding period.

x (t) = 2 cos 100 π t + 5 sin 50 t

May 13, 5 Marks

Soln. :

1. The given signal is,

$$x(t) = 2\cos 100\pi t + 5\sin 50t \quad ...(1)$$

Let $x(t) = x_1(t) + x_2(t)$ ...(2)

Here $x_1(t) = 2\cos 100\pi t$ ...(3)

and $x_2(t) = 5\sin 50t$ ...(4)

The standard equation can be expressed as,

$$x_1(t) = A\cos\omega_1 t \quad ...(5)$$

Comparing Equations (3) and (5) we get,

$$\omega_1 = 100\pi$$

Now we have,

$$\omega_1 = 2\pi f_1 = \frac{2\pi}{T_1}$$

$$\therefore \quad \frac{2\pi}{T_1} = 100\pi$$

$$\therefore \quad T_1 = \frac{2\pi}{100\pi} = \frac{1}{50} \quad ...(6)$$

Similarly standard equation for $x_2(t)$ can be written as,

$$x_2(t) = A\sin\omega_2 t \quad ...(7)$$

Comparing Equations (4) and (7) we get,

$$\omega_2 = 50$$

$$\therefore \quad 2\pi f_2 = \frac{2\pi}{T_2} = 50$$

$$\therefore \quad T_2 = \frac{2\pi}{50} \quad ...(8)$$

Here x (t) is expressed as summation of two signals. We know that the resultant signal x (t) is periodic if $\frac{T_1}{T_2}$ is the ratio of two integers. From Equations (6) and (7) we get,

$$\frac{T_1}{T_2} = \frac{1}{50}\cdot\frac{50}{2\pi} = \frac{1}{2\pi}$$

It is not the ratio of two integers. Thus x (t) is **non-periodic.**

Ex. 1.10.2 : Determine which of the following signals are periodic. If periodic determine the fundamental period :

1. 10 sin (12 π t) + 4 sin (18 π t)
2. 3 sin 4t
3. $3 + t^2$

Dec. 03, 5 Marks

Soln. : 1. Given signal is,

$$x(t) = 10\sin(12\pi t) + 4\sin(18\pi t) \quad ...(1)$$

Let $x(t) = x_1(t) + x_2(t)$

Here $x_1(t) = 10\sin(12\pi t)$ ...(2)

and $x_2(t) = 4\sin(18\pi t)$ ...(3)

The standard equation can be written as,

$$x(t) = A\sin\omega t \quad ...(4)$$

Comparing Equations (2) and (4) we get,

$$\omega_1 = 12\pi, \quad \therefore \quad 2\pi f_1 = 12\pi$$

$$\therefore \quad f_1 = \frac{12\pi}{2\pi} = \frac{6}{1}$$

$$\therefore \quad T_1 = \frac{1}{f_1} = \frac{1}{6} \quad ...(5)$$

Similarly comparing Equations (3) and (4) we get,

$$\omega_2 = 18\pi,$$

$$\therefore \quad 2\pi f_2 = 18\pi$$

$$\therefore \quad f_2 = \frac{18\pi}{2\pi} = 9$$

$$\therefore \quad T_2 = \frac{1}{f_2} = \frac{1}{9} \quad ...(6)$$

The resultant signal x (t) is periodic if $\frac{T_1}{T_2}$ is the ratio of two integers. From Equations (5) and (6) we get,

$$\frac{T_1}{T_2} = \frac{1/6}{1/9} = \frac{1}{6}\cdot\frac{9}{1} = \frac{9}{6}$$

It is the ratio of two integers. **Thus x (t) is periodic.** Now the fundamental period of x (t) is least common multiple of T_1 and T_2.

Thus fundamental period is $\frac{1}{3}$ sec. ...Ans.

2. Given $x(t) = 3\sin 4t$ ...(7)

We have the standard equation,

$$x(t) = A\sin\omega t \quad ...(8)$$

Comparing Equations (7) and (8) we get,

$$\omega = 4$$

$$\therefore \quad 2\pi f = 4$$

$$\therefore \quad f = \frac{4}{2\pi}$$

It is not the ratio of integer values. Thus **this signal is non-periodic in nature.** ...Ans.

3. Given $x(t) = 3 + t^2$ ...(9)

We know that a continuous time signal is periodic in nature if it satisfies the equation,

$$x(t) = x(t + T_0) \quad ...(10)$$

Where T_0 is the fundamental period of repetation.

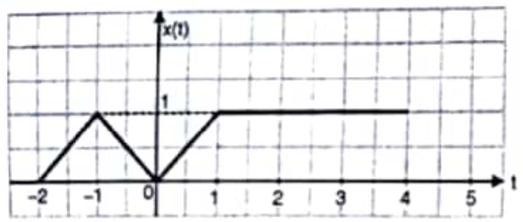
From Equation (9) we can write,

$$x(t + T_0) = 3 + (t + T_0)^2$$

$$\therefore \; x(t + T_0) = 3 + t^2 + 2t\,T_0 + T_0^2 \quad ...(11)$$

For any value of 'T_0' Equations (9) and (11) cannot be made equal. Thus **given signal is non-periodic.** ...Ans.

Ex. 1.10.3 : Find even and odd parts of signal shown in Fig. P. 1.10.3(a). **May 06, 5 Marks**



(H-28) **Fig. P. 1.10.3(a) : Given signal**

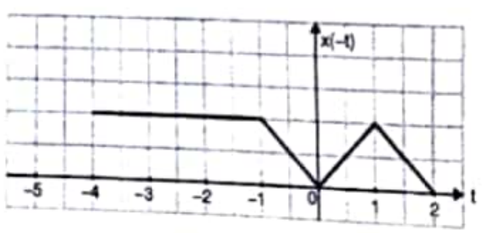
Soln. :

Step 1 : Find the even part :

Even part of signal is given by,

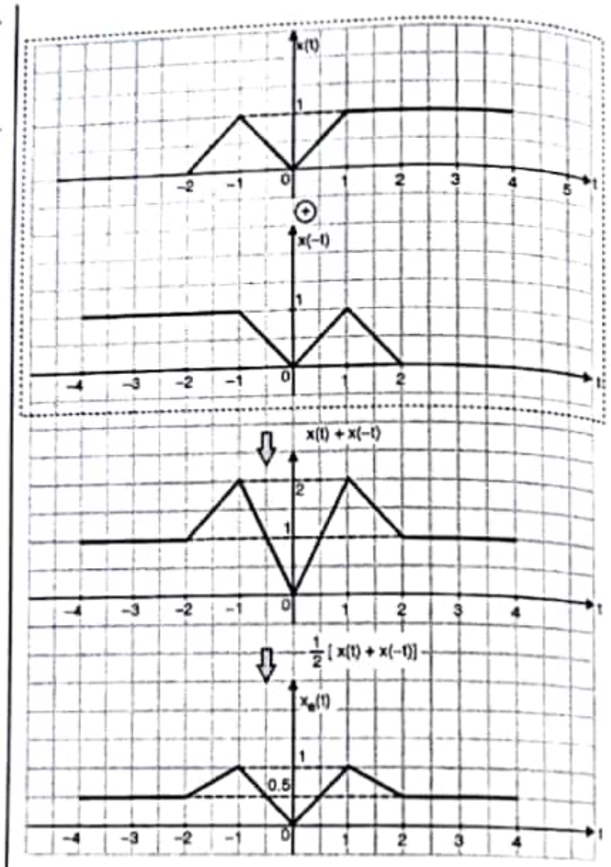
$$x_e(t) = \frac{1}{2}[x(t) + x(-t)] \quad ...(1)$$

Here $x(-t)$ represents the folded version of $x(t)$. It is shown in Fig. P. 1.10.3(b).



(H-29) **Fig. P. 1.10.3(b) : Folded version of given signal**

Equation (1) indicates that we have to add $x(t)$ and $x(-t)$ and then the magnitude of resultant signal is divided by 2. This operation is shown in Fig. P. 1.10.3(c).



(H-30) **Fig. P. 1.10.3(c)**

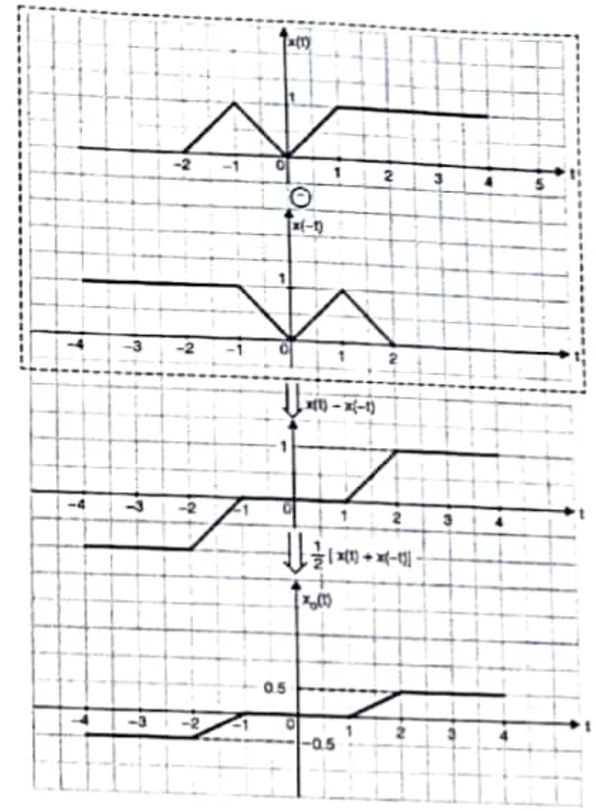
Step 2 : Find the odd part :

Odd part of signal $x(t)$ is given by,

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)] \quad ...(2)$$

Equation (2) indicates that, we have to substract folded signal from given signal $x(t)$ and then the magnitude of resultant signal is divided by 2.

This operation is shown in Fig. P. 1.10.3(d).



(H-31) **Fig. P. 1.10.3(d)**

Ex. 1.10.4 : Sketch $x(t) = \text{ramp}(t) - \text{ramp}(t-1) - \text{unit}(t-2)$. Find odd and even part of it. Is the signal energy or power ? **Dec. 05, 3 Marks, May 07, 4 Marks**

Soln. :

$$x(t) = r(t) - r(t-1) - u(t-2)$$

The sketch of $x(t)$ is shown in Fig. P. 1.10.4.

The even part of $x(t)$ is given by

$$x(t) = \frac{1}{2}[x(t) + x(-t)]$$

It is shown in Fig. P. 1.10.4(a).

The odd part of $x(t)$ is given by

$$x(t) = \frac{1}{2}[x(t) - x(-t)]$$

The energy of signal is given by,

$$E = \int_{-\infty}^{\infty} x^2(t)\,dt$$

$$= \int_0^1 t^2\,dt + \int_0^2 (-t)^2\,dt$$

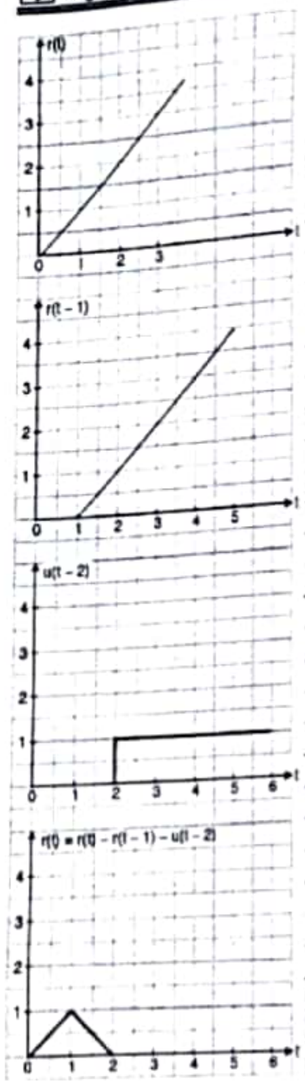
$$= \left[\frac{t^3}{3}\right]_0^1 + \left[\frac{(-t)^3}{-3}\right]_1^2$$

$$= \left[\frac{1}{3} - \frac{0}{3}\right] + \left[\frac{(-2)^3 - (-1)^3}{-3}\right]$$

$$= \frac{1}{3} - \frac{1}{3}(-8) + 1$$

$$= \frac{1}{3} + \frac{8}{3} = \frac{9}{3}$$

Since this is finite value, $x(t)$ is energy signal.



(H-827) Fig. P. 1.10.4(a)

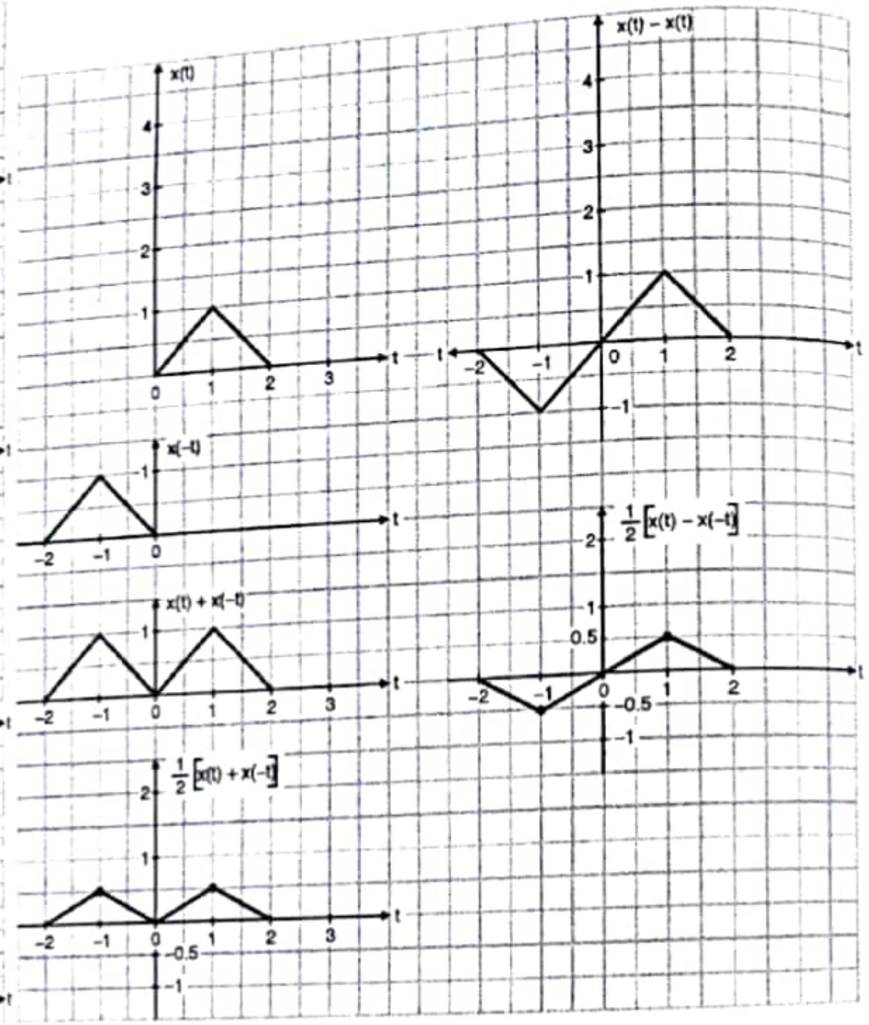


Fig. P. 1.10.4(b) : Even part of x (t)

(H-828) Fig. P. 1.10.4(c) : Odd part of x (t)

Ex. 1.10.5 : Determine which of the following signals are periodic :

1. $x_1(t) = \sin 15\pi t$
2. $x_2(t) = \sin 20\pi t$
3. $x_3(t) = x_1(t) + x_2(t)$ **Dec. 07, 4 Marks**

Soln. :

1. **$x_1(t) = \sin 15\pi t$:**

For satisfying condition of periodicity,

$$x_1(t) = x_1(t+T_0)$$
$$\therefore x_1(t+T_0) = \sin 15\pi(t+T_0)$$
$$\therefore x_1(t+T_0) = \sin 15\pi t + \sin 15\pi T_0$$
$$\therefore x_1(t) \neq x_1(t+T_0)$$

Therefore $x_2(t)$ signal is **nonperiodic.**

2. **$x_2(t) = \sin 20\pi t$:**

For satisfying condition of periodicity,

$$x_2(t) = x_2(t+T_0)$$
$$x_2(t+T_0) = \sin 20\pi(t+T_0)$$
$$x_2(t+T_0) = \sin 20\pi t + \sin 20\pi T_0$$
$$\therefore x_2(t) \neq x_2(t+T_0)$$

Therefore $x_2(t)$ signal is **nonperiodic.**

3. **$x_3(t) = x_1(t) + x_2(t)$:**

$$\therefore x_3(t) = \sin 15\pi t + \sin 20\pi t$$

The standard equation can be expressed as,

$$x_1(t) = \sin \omega_1 t$$
$$x_1(t) = \sin 15\pi t$$
$$\therefore \omega_1 = 15\pi$$
$$\therefore 2\pi f_1 = \frac{2\pi}{T_1} = 15\pi$$
$$\frac{2}{T_1} = 15$$
$$\therefore T_1 = \frac{2}{15} \quad ...(1)$$

$$x_2(t) = \sin \omega_2 t$$
$$x_2(t) = \sin 20\pi t$$
$$\therefore \omega_2 = 20\pi$$
$$\therefore 2\pi f_2 = \frac{2\pi}{T_2} = 20\pi$$
$$\frac{1}{T_2} = 10$$
$$T_2 = \frac{1}{10} \quad ...(2)$$

Here $x_3(t)$ is expressed as summation of two signals. Therefore resultant signal $x_3(t)$ is periodic if $\frac{T_1}{T_2}$ is the ratio of two integer.

From Equation (1) and Equation (2),

$$\frac{T_1}{T_2} = \frac{2}{15} \times \frac{10}{1} = \frac{20}{15} = \frac{4}{3}$$

Therefore it is the ratio of two integers.

Thus $x_3(t)$ signal is **periodic.** ...Ans.

Ex. 1.10.6 : Determine which of the following signals are periodic. If periodic determine the fundamental period.

1. $10 \sin(12\pi t) + 4 \sin(18\pi t)$
2. $3 \tan 4t$
3. $3 + t^2$ **Dec. 03, 5 Marks**

Soln. : For 1. and 3. refer Ex. 1.10.2.

2. Given $x(t) = 3 \tan 4t$ ...(1)

The standard equation,

$$x(t) = A \tan \omega t \quad ...(2)$$

Comparing Equations (1) and (2),

$$\omega = 4 \qquad \therefore 2\pi f = 4$$
$$\therefore f = \frac{4}{2\pi}$$

It is not the ratio of integer values. Thus **this signal is non-periodic in nature.**

Ex. 1.10.7 : Find signal power of

1. $x(n) = A$
2. $x(n) = u(n)$ **Dec. 03, 5 Marks**

Soln. :

1. $x(n) = A$:

For the discrete time signals, average power P is given by,

$$P = \lim_{n \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2(n)$$

Here $x(n) = A$

$$= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} A^2$$
$$= A^2$$

2. Please refer Ex. 1.6.12.

Ex. 1.10.8 : Explain whether the following signal is power signal or energy signal :

$$x(t) = 4 \sin\left(6\pi t + \frac{\pi}{6}\right)$$

May 06, 5 Marks

Soln. :

$$x(t) = 4 \sin\left(6\pi t + \frac{\pi}{6}\right)$$

Power is given by,

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} (4)^2 \sin^2\left(6\pi t + \frac{\pi}{6}\right)$$
$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{16}{2}\left[1 + \cos\left(6\pi t + \frac{\pi}{6} - 90\right)\right] dt$$
$$= \lim_{T \to \infty} \frac{16}{2T} \int_{-T/2}^{T/2} dt$$
$$+ \lim_{T \to \infty} \frac{16}{2T} \int_{-T/2}^{T/2} \cos\left(6\pi t - \frac{\pi}{3}\right) dt$$
$$= 8/T(T/2 + T/2) + 0$$
$$P = 8$$

Energy signal :

$$E = \int_{-\infty}^{\infty} 16 \sin^2\left(6\pi t + \frac{\pi}{6}\right) dt$$
$$= \frac{16}{2} \int_{-\infty}^{\infty} 1 + \cos\left(6\pi t + \frac{\pi}{6} - \frac{\pi}{2}\right) dt$$
$$= 8 \int_{-\infty}^{\infty} dt + 8 \int_{-\infty}^{\infty} \cos\left(6\pi - \frac{\pi}{3}\right) dt$$
$$\therefore E = \infty + 0 = \infty$$

Thus the energy of the signal is ∞. Hence the given signal is a power signal.

Ex. 1.10.9 : Determine whether the following sequences are periodic or not. If periodic determine the fundamental period : **May 06, 5 Marks**

1. $x_1[n] = \sin\left[\frac{6\pi n}{7}\right]$
2. $x_2[n] = \sin\left[\frac{n}{8}\right]$

Soln. :

1. $x_1[n] = \sin\left[\frac{6\pi n}{7}\right]$

The standard equation,

$$x_1[n] = A \sin \omega n \quad ...(1)$$

given $$x_1[n] = \sin\left[\frac{6\pi n}{7}\right] \quad ...(2)$$

Comparing Equations (1) and (2),

$$\omega = \frac{6\pi}{7}$$
$$\therefore 2\pi f = \frac{6\pi}{7}$$

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$$\therefore \quad f = \frac{3}{7} \text{ cycles per sample}$$

$$\therefore \quad T = \frac{1}{f} = \frac{7}{3}$$

$\therefore$ It is ratio of two integers; the given sequence is periodic.

The fundamental period $N = \frac{7}{3}$.

2. $x_2[n] = \sin\left[\frac{n}{8}\right]$

Given $\quad x_2[n] = \sin\left[\frac{n}{8}\right] \quad$ ...(3)

The standard equation is given by,

$$x_2[n] = A \sin \omega n \quad \text{...(4)}$$

Comparing Equations (3) and (4)

$$\omega = \frac{1}{8} \qquad \therefore \; 2\pi f = \frac{1}{8}$$

$$\therefore \quad f = \frac{1}{16\pi}$$

It is not ratio of two integers. The given sequence is non-periodic.

Ex. 1.10.10 : Explain whether the following signal are power signals or energy signals

1. $x(t) = A \cos 100 \pi t$
2. $x(n) = (0.2)^n u(n)$ **Dec. 06. 6 Marks**

Soln. :

1. $x(t) = A \cos 100 \pi t$

Power is given by,

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2 \cos^2 (100 \pi t)\, dt$$

$$= \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \frac{A^2}{2} [1 + \cos 2 (100 \pi t)]\, dt$$

$$= \lim_{T \to \infty} \frac{A^2}{2T} \int_{-T/2}^{T/2} dt$$

$$+ \lim_{T \to \infty} \frac{A^2}{2T} \int_{-T/2}^{T/2} \cos (200 \pi t)\, dt$$

$$P = \frac{A^2}{2}$$

Power of a sinusoid of peak amplitude A is $A^2/2$ irrespective of frequency 50 Hz.

Energy of the signal

$$E = \int_{-\infty}^{\infty} A^2 \cos^2 (100 \pi t)\, dt$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} [1 + \cos 2 (100 \pi t)]\, dt$$

$$= \frac{A^2}{2} \int_{-\infty}^{\infty} dt + \frac{A^2}{2} \int_{-\infty}^{\infty} \cos 2 (100 \pi t)\, dt$$

$$E = \infty + 0 = \infty$$

Energy of signal is ∞. Hence signal $x(t) = A \cos 100 \pi t$ is a power signal.

2. Similar to Ex. 1.6.1...

Ex. 1.10.11 : 1. $x(t) = 3 \cos 500 \pi t + 4 \cos 300 \pi t$. What is the Nyquist rate for the signal ? Is the above signal periodic ? If yes then find the period of the signal ?

2. Distinguish between Deterministic and Stochastic signals. Give an example of each. **Dec. 06. 7 Marks**

Soln. : Please refer sections 1.5.4 and 1.5.5.

1. $x(t) = 3 \cos 500 \pi t + 4 \cos 300 \pi t$

Let $\quad x(t) = x_1 + x_2(t)$

$$x_1(t) = 3 \cos 500 \pi t$$

$$x_2(t) = 4 \cos 300 \pi t$$

The standard equation can be expressed as,

$$x_1(t) = A \cos \omega_1 t$$

$$\omega_1 = 500 \pi$$

$$\frac{2\pi}{T_1} = 500 \pi$$

$$\frac{1}{T_1} = \frac{500 \pi}{2 \pi}$$

$$\frac{1}{T_1} = 250$$

$$\therefore T_1 = \frac{1}{250}$$

$$x_2(t) = A \cos \omega_2 t$$

$$\omega_2 = 300 \pi$$

$$\frac{2\pi}{T_2} = 300 \pi$$

$$T_2 = \frac{2\pi}{300 \pi} = \frac{1}{150}$$

x (t) is expressed as summation of two signals. Thus resultant signal x (t) is periodic if $\frac{T_1}{T_2}$ is ratio of two integers.

$$\frac{T_1}{T_2} = \frac{150}{250} = \frac{3}{5}$$

It is ratio of two integers. Hence x (t) is periodic. The fundamental period is 1/5 sec.

Nyquist rate : The minimum sampling rate of "2W" samples per second for a signal x (t) having maximum frequency of "W" Hz is known as "Nyquist rate".

Nyquist rate = 2W Hz.

Ex. 1.10.12 : Determine whether the following signals are energy signals or power signals and evaluate their normalized energy and power.

1. $x(t) = \text{rect}\left(\frac{t}{T_0}\right)$
2. $x(t) = \cos^2 (\omega_0 t)$

Dec. 07. 4 Marks, May 11. 2 Marks

Soln. :

1. **Please refer Ex. 1.6.9.**
2. $x(t) = \cos^2 (\omega_0 t)$

The signal $\cos^2 (\omega_0 t)$ does not tend to zero as $t \to \infty$. Thus the energy of the signal ∞.

$$P = \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt = \frac{1}{T} \int_{-T/2}^{T/2} [\cos^2 \omega_0 t]^2\, dt$$

$$= \frac{1}{T} \int_{-T/2}^{T/2} \left(\frac{1 + \cos 2 \omega_0 t}{2}\right)^2 dt$$

$$= \frac{1}{4T} \left[\int_{-T/2}^{T/2} 1\, dt + \int_{-T/2}^{T/2} (\cos 2 \omega_0 t)^2\, dt \right]$$

$$P = \frac{1}{4T} \int_{-T/2}^{T/2} (1 + \cos 2 \omega_0 t)^2\, dt$$

$$P = \frac{1}{4T} \int_{-T/2}^{T/2} (1 + \cos^2 2 \omega_0 t + 2 \cos 2 \omega_0 t)\, dt$$

$$P = \frac{1}{4T} \left[\int_{-T/2}^{T/2} 1\, dt + \int_{-T/2}^{T/2} \cos^2 2 \omega_0 t\, dt + 2 \int_{-T/2}^{T/2} \cos 2 \omega_0 t\, dt \right]$$

$$P = \frac{1}{4T} \left[[t]_{-T/2}^{T/2} + \int_{-T/2}^{T/2} \left(\frac{1 + \cos 4 \omega_0 t}{2}\right) dt + 2 [\sin 2 \omega_0 t]_{-T/2}^{T/2} \right]$$

$$P = \frac{1}{4T} \left[\left[\frac{T}{2} - \left(\frac{T}{2}\right)\right] + \frac{1}{2} \left[\int_{-T/2}^{T/2} 1\, dt + \int \cos 4 \omega_0 t\, dt \right] + 2 \left[\sin 2 \omega_0 \frac{T}{2} - \sin 2 \omega_0 \left(\frac{-T}{2}\right) \right] \right]$$

$$P = \frac{1}{4T} \left[T + \frac{1}{2} \left[[t]_{-T/2}^{T/2} + [\sin 4 \omega_0 t]_{-T/2}^{T/2} \right] + 2 [\sin \omega_0 T + \sin \omega_0 T] \right]$$

$$P = \frac{1}{4T} \left[T + \frac{1}{2} \left[\left(\frac{T}{2} + \frac{T}{2}\right) + \left[\sin 4 \omega_0 \frac{T}{2} + \sin 4 \omega_0 \frac{T}{2} \right] \right] + 4 \sin \omega_0 T \right]$$

$$P = \frac{1}{4T} \left[T + \frac{T}{2} + \frac{1}{2} (\sin 2 \omega_0 T + \sin 2 \omega_0 T) + 4 \sin \omega_0 T \right]$$

$$P = \frac{1}{4T} \left[T + \frac{T}{2} + \sin \omega_0 T + 4 \sin \omega_0 T \right]$$

$$P = \frac{1}{4T} \left[T + \frac{T}{2} + 5 \sin \omega_0 T \right]$$

As the power is non-zero and finite, the given signal is a **power signal.**

Ex. 1.10.13 : Determine power and energy for the following signals : **May 14. 4 Marks**

1. $x(t) = 3 \cos 5 \omega_0 t$
2. $x[n] = \left(\frac{1}{4}\right)^n u[n]$

Soln. :

1. **$x(t) = 3 \cos 5 \omega_0 t$:**

This is a periodic signal, so we will calculate its power.

$$P = \frac{1}{T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} |x(t)|^2\, dt$$

$$= \frac{1}{T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} (3 \cos 5 \omega_0 t)^2\, dt = \frac{9}{T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} \cos^2 5 \omega_0 t$$

$$= \frac{9}{T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} \frac{1 + \cos 2 (5 \omega_0 t)}{2}\, dt$$

$$= \frac{9}{2T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} 1 + \cos (10 \omega_0 t)\, dt$$

$$= \frac{9}{2T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} 1\,dt + \frac{9}{2T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} \cos(10\omega_0 t)\,dt$$

The second integration is zero.

$$\therefore\ P = \frac{9}{2T_o}\int_{-\frac{T_o}{2}}^{\frac{T_o}{2}} 1\,dt = \frac{9}{2T_o}[t]_{-T_o/2}^{T_o/2}$$

$$\therefore\ P = \frac{9}{2T_o}\left[\frac{T_o}{2}+\frac{T_o}{2}\right] = \frac{9}{2}\text{W}$$

2. $x[n] = \left(\frac{1}{4}\right)^n u[n]$:

It is exponential signal. So, it is non-periodic signal. Now, energy is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Given $x(n) = \left(\frac{1}{4}\right)^n u(n)$. Due to multiplication by u (n), the range of summation will be from n = 0 to ∞.

$$\therefore\ E = \sum_{n=0}^{\infty}\left[\left(\frac{1}{4}\right)^n\right]^2 = \sum_{n=0}^{\infty}\left[\left(\frac{1}{4}\right)^2\right]^n$$

$$= \sum_{n=0}^{\infty}\left(\frac{1}{16}\right)^n$$

$$\therefore\ E = \frac{1}{1-\frac{1}{16}} = \frac{16}{15}\text{J}$$

Ex. 1.10.14 : For a continuous time signal x (t) = 8 cos 200 πt. Find :

1. Minimum sampling rate.
2. If f_s = 400 Hz, what is the discrete time signal ?
3. If f_s = 150 Hz, what is the discrete time signal ?
4. Comment on result obtained in 2 and 3 with proper justification.

May 14. 10 Marks

Soln. :

1. Given $x(t) = 8\cos 200\pi t$ ...(1)

Consider standard equation,

$x(t) = A\cos 2\pi f t$ ...(2)

Comparing Equations (1) and (2) we get,

$2\pi f t = 200\pi t$ $\therefore f = 100$ Hz

Thus maximum frequency, W = f = 100 Hz

∴ Minimum sampling frequency = f_s = 2W = 200 Hz

∴ Minimum sampling rate $= \frac{1}{f_s} = 5$ msec.

2. Given, f_s = 400 Hz

The discrete signal is obtained by replacing t by $\frac{n}{f_s} = \frac{n}{400}$

$$\therefore\ x(n) = 8\cos\left(200\pi \times \frac{n}{400}\right) = 8\cos\left(\frac{\pi n}{2}\right) \quad ...(3)$$

3. Given f_s = 150 Hz

Thus, corresponding discrete signal is,

$$x(n) = 8\cos\left(200\pi \times \frac{n}{150}\right) = 8\cos\left[2\pi\left(\frac{n}{1.5}\right)\right]$$

$$= 8\cos\left[2\pi\left(1-\frac{1}{3}\right)n\right]$$

$$= 8\cos\left(2\pi n - \frac{2\pi}{3}n\right)$$

$$x(n) = 8\cos\left(\frac{2\pi}{3}n\right) \quad ...(4)$$

4. If we reconstruct the signal from Equation (3) then we get,

$x(t) = 8\cos\left(\frac{\pi}{2}\times t\cdot f_s\right) = 8\cos\left(\frac{\pi}{2}\cdot t\cdot 400\right) = 8\cos(200\pi t)$. Note that for reconstruction, we have replaced n by $t\cdot f_s$. Thus for f_s = 400 Hz, original signal is obtained. That means, there is no aliasing.

Similarly we will reconstruct the signal from Equation (4).

$$\therefore x(t) = 8\cos\left(\frac{2\pi}{3}\cdot t\cdot f_s\right) = 8\cos\left(\frac{2\pi}{3}\cdot t\cdot 150\right)$$

$$= 8\cos(100\pi t).$$

It indicates aliasing error. This is because, the sampling frequency (150 Hz) is less than the minimum required sampling frequency (200 Hz).

Ex. 1.10.15 : Determine the fundamental period of the following signals :

1. $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$
2. $x[n] = \cos^2\frac{\pi}{8}n$

Dec. 14. 4 Marks

Soln. :

1. $x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t$:

$$x(t) = \cos\frac{\pi}{3}t + \sin\frac{\pi}{4}t \quad ...(1)$$

The corresponding standard equation is,

$$x(t) = A\cos 2\pi f_1 t + B\sin 2\pi f_2 t \quad ...(2)$$

Comparing Equation (1) and Equation (2),

$2\pi f_1 t = \frac{\pi}{3}t$ and $2\pi f_2 t = \frac{\pi}{4}t$

$\therefore f_1 = \frac{1}{6}$ and $f_2 = \frac{1}{8}$

$\therefore T_1 = 6$ and $T_2 = 8$

Since $\frac{T_1}{T_2} = \frac{6}{8}$ is rational number ; the signal is periodic. The fundamental period is M.C.D. (Most Common Divider) of T_1 and T_2. It is 2 sec.

2. $x[n] = \cos^2\frac{\pi}{8}n$:

We have standard identity, $\cos^2\theta = \frac{1}{2}+\frac{1}{2}\cos 2\theta$

$$x(n) = \frac{1}{2}+\frac{1}{2}\cos\left(\frac{\pi}{4}n\right) \quad ...(3)$$

First term is constant. The standard equation corresponding to second term is, $A\cos 2\pi f_o n$

$$\therefore\ 2\pi f_o n = \frac{\pi}{4}n$$

$$\therefore\ f_o = \frac{1}{8} = \frac{k}{N}$$

Thus fundamental period is N = 8 samples.

Ex. 1.10.16 : Determine whether the signals are power or energy signals. Calculate energy / power accordingly :

1. $x(t) = 0.9\,e^{-3t}u(t)$ 2. $x[n] = u[n]$

Dec. 14. 4 Marks

Soln. :

1. $x(t) = 0.9\,e^{-3t}u(t)$:

It is non-periodic signal. So it is energy signal

$$\therefore\ E = \int_{-\infty}^{\infty} |x^2(t)|\,dt$$

$$= \int_0^{\infty}(0.9)^2\cdot e^{-6t}\,dt = (0.9)^2\left[\frac{e^{-6t}}{-6}\right]_0^{\infty}$$

$$= \frac{0.81}{-6}[e^{-\infty} - e^0]$$

$$\therefore\ E = 0.135\text{ J}$$

2. x[n] = u[n] :

The unit step sequence is power signal. We will prove it as follows :

Step 1 : Calculation of energy :

Energy of input sequence x (n) is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad ...(1)$$

Here x (n) = Unit step, u (n). Unit step is present in the range n = 0 to n = ∞.

$$\therefore\ E = \sum_{n=0}^{\infty} |u(n)|^2 \quad ...(2)$$

The value of unit step is 1 for energy value of n.

$$E = 1+1+1+\ldots\ldots\infty$$

This is infinite value.

Thus unit step is not an energy signal.

Step 2 : Calculation of power :

The average power of input signal is given by,

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 \quad ...(3)$$

But here x (n) = Unit step, u (n). Since the range of unit step is from n = 0 to n = ∞, we will change the limits of summation as follows :

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N}|u(n)|^2$$

We know that $|u(n)| = 1$ and $|u(n)|^2 = 1^2 = 1$. Thus summation can be expanded as :

$$P = \lim_{N\to\infty}\frac{1}{2N+1}\underbrace{[1+1+1\ldots\ldots+1]}_{n=0\text{ to }N}$$

Thus we can write, $$P = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=0}^{N}(1)^n \quad ...(4)$$

Now use standard equation of summation

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1}-a^{N_2+1}}{1-a} & \text{for } a \neq 1 \\ N_2 - N_1 + 1 & \text{for } a = 1 \end{cases} \quad ...(5)$$

Comparing summation term of Equation (4) with Equation (5),

a = 1, N_1 = 0 and N_2 = N.

Thus Equation (4) becomes,

$$P = \lim_{N\to\infty}\frac{1}{2N+1}[N_2 - N_1 + 1]$$

$$= \lim_{N\to\infty}\frac{1}{2N+1}[N - 0 + 1]$$

$$\therefore\ P = \lim_{N\to\infty}\frac{N+1}{2N+1}$$

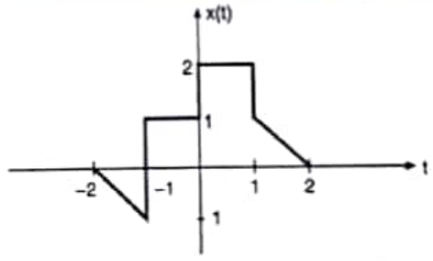
Dividing numerator and denominator by N we get,

$$P = \lim_{N\to\infty}\frac{1+\frac{1}{N}}{2+\frac{1}{N}} \quad \therefore P = \frac{1}{2}$$

Since the power is finite ; unit step is power signal.

Ex. 1.10.17 : For the signal x(t) depicted in the Fig. P. 1.10.17, sketch the signals :

Dec. 14. 10 Marks



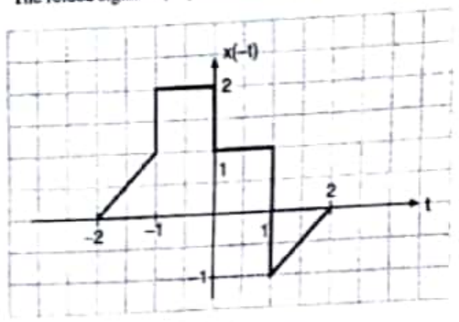
(H-1408) **Fig. P. 1.10.17**

1. x (– t) 2. x (t + 6)
3. x (3t) 4. x (t/2)

Soln. :

1. **x (– t) :**

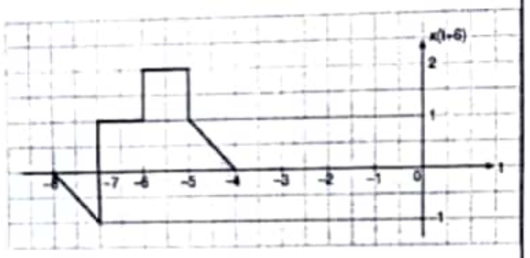
The folded signal x (– t) is shown in Fig. P. 1.10.17(a).



(H-1413) **Fig. P. 1.10.17(a)**

2. **x (t + 6) :**

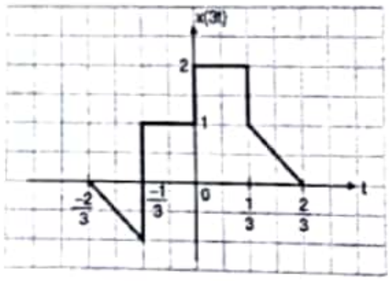
x (t + 6) is advance version of x (t) by 6 positions. It is shown in Fig. P. 1.10.17(b).



(H-1414) **Fig. P. 1.10.17(b)**

3. **x (3t) :**

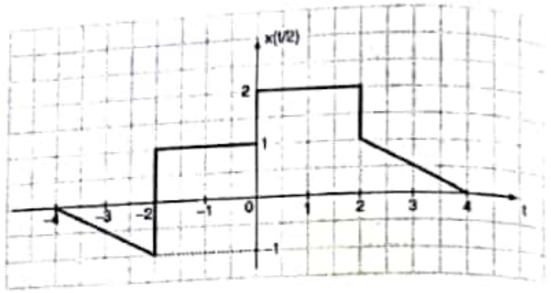
x (3t) represents compression of x (t) by 3 positions. It is shown in Fig. P. 1.10.17(c).



(H-1415) **Fig. P. 1.10.17(c)**

4. **x (t/2) :**

$x\left(\frac{t}{2}\right)$ represents expansion of x (t) by 2 positions. It is shown in Fig. P. 1.10.17(d).



(H-1416) **Fig. P. 1.10.17(d)**

Ex. 1.10.18 : Determine the fundamental period of the following signals :

1. $x(t) = 14 + 40 \cos(60 \pi t)$
2. $x[n] = \cos^2\left[\frac{\pi}{4} n\right]$

May 15, 4 Marks

Soln. :

1. **$x(t) = 14 + 40 \cos(60 \pi t)$:**

The first term is constant with zero frequency. Corresponding to second term, standard equation is,

$$x(t) = A \cos 2\pi f_o t \quad \therefore 2\pi f_o t = 60 \pi t$$

$$\therefore f_o = 30 \qquad \therefore T_o = \frac{1}{30} \text{ sec}$$

2. **$x[n] = \cos^2\left[\frac{\pi}{4} n\right]$:**

We have, $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$\therefore x(n) = \frac{1}{2} + \frac{1}{2}\cos\left[\frac{\pi}{2} n\right]$$

The first term is constant. Corresponding to second term the standard equation is,

$$x(n) = A \cos 2\pi f_o n$$

$$\therefore 2\pi f_o n = \frac{\pi}{2} n$$

$$\therefore f_o = \frac{1}{4} = \frac{K}{N}$$

Thus fundamental period, N = 4 samples

Ex. 1.10.19 : Find out even and odd component of the following two signals :

1. $x(t) = \cos^2 \frac{\pi t}{2}$
2. $x(t) = \begin{cases} t \ldots\ldots\ldots\ldots 0 \le t \le 1 \\ 2 - t \ldots\ldots\ldots 1 < t \le 2 \end{cases}$

May 15, 4 Marks

Soln. :

1. **$x(t) = \cos^2 \frac{\pi t}{2}$:**

$$\therefore x(t) = \frac{1}{2} + \frac{1}{2}\cos \pi t$$

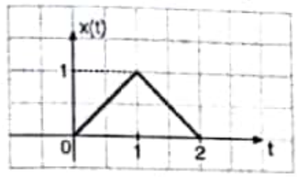
and $x(-t) = \frac{1}{2} + \frac{1}{2}\cos \pi(-t)$

$$= \frac{1}{2} + \frac{1}{2}\cos \pi t$$

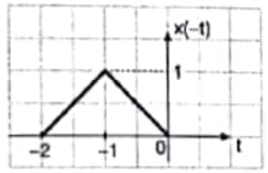
Thus given signal is even and odd part is zero.

2. **$x(t) = x(t) = \begin{cases} t \ldots\ldots\ldots\ldots 0 \le t \le 1 \\ 2 - t \ldots\ldots\ldots 1 < t \le 2 \end{cases}$:**

The sketch of signal x (t) is shown in Fig. P. 1.10.19(a).



(a)



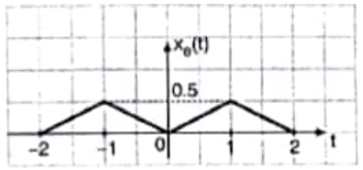
(b)

(H-1418) **Fig. P. 1.10.19**

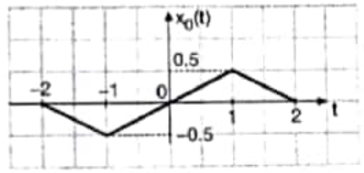
The signal x (– t) is shown in Fig. P. 1.10.19(b).

Even part is, $x_e(t) = \frac{1}{2}[x(t) + x(-t)]$

It is shown in Fig. P. 1.10.19(c).



(c)



(d)

(H-1419) **Fig. P. 1.10.19**

Odd part is, $x_o(t) = \frac{1}{2}[x(t) - x(-t)]$

It is shown in Fig. P. 1.10.19(d).

Ex. 1.10.20 : Determine whether the signals are power or energy signals. Calculate energy/power accordingly.

1. $x(t) = Ae^{-\alpha t}u(t) \ldots\ldots\ldots \alpha > 0$
2. $x[n] = u[n]$

May 15, 4 Marks

Soln. :

1. **$x(t) = Ae^{-\alpha t}u(t) \ldots\ldots\ldots \alpha > 0$:**

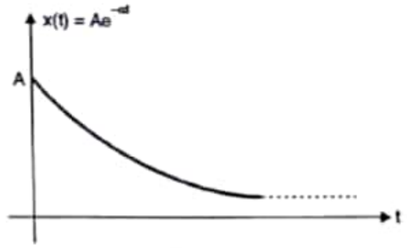
The given signal is as shown in Fig. P. 1.10.20. It is positive sided exponential signal; which is non-periodic, so it is an energy signal.

Now energy of continuous signal is given by,

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$\therefore E = \int_{-\infty}^{\infty} |A e^{-\alpha t} u(t)|^2 dt$$

Since the signal $A e^{-\alpha t}$ is multiplied by u (t); we will change the limits of integration. The new limits will be from t = 0 to t = ∞; sin t * u (t) exists only in this range and its magnitude is 1.



(H-36) **Fig. P. 1.10.20**

$$\therefore E = \int_0^{\infty} |A e^{-\alpha t}|^2 dt = \int_0^{\infty} A^2 e^{-2\alpha t} dt$$

$$\therefore E = A^2\left[\frac{e^{-2\alpha t}}{-2\alpha}\right]_0^{\infty} = -\frac{A^2}{2\alpha}[e^{-\infty} - e^0]$$

$$= -\frac{A^2}{2\alpha}\left[\frac{1}{e^{\infty}} - 1\right]$$

$$\therefore E = -\frac{A^2}{2\alpha}(-1) = \frac{A^2}{2\alpha}$$

Hence 'A' and 'α' and constant terms. Thus energy is finite and non-zero.

Now the given signal is,

$$x(t) = A e^{-\alpha t} u(t), \alpha > 0.$$

As $\alpha \to 0$; the equation of x (t) becomes,

$$x(t) = A u(t)$$

Thus as $\alpha \to 0$; x (t) tends to become unit step of magnitude 'A'.

2. **x [n] = u [n] :**

The unit step sequence is power signal. We will prove it as follows :

Step I : Calculation of energy :

Energy of input sequence x (n) is given by,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \quad \ldots(1)$$

Here x (n) = Unit step, u (n). Unit step is present in the range n = 0 to n = ∞.

$$\therefore E = \sum_{n=0}^{\infty} |u(n)|^2 \quad \ldots(2)$$

The value of unit step is 1 for energy value of n.

$$E = 1 + 1 + 1 + \ldots\ldots = \infty$$

This is infinite value.

Thus unit step is not an energy signal.

Step II : Calculation of power :

The average power of input signal is given by,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 \quad \ldots(3)$$

But here x (n) = Unit step, u (n). Since the range of unit step is from n = 0 to n = ∞, we will change the limits of summation as follows :

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} |u(n)|^2$$

We know that $|u(n)| = 1$ and $|u(n)|^2 = 1^2 = 1$. Thus summation can be expanded as :

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \underbrace{[1 + 1 + 1 \ldots\ldots + 1]}_{n = 0 \text{ to } N}$$

Thus we can write, $$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=0}^{N} (1)^n \quad \ldots(4)$$

Now use standard equation of summation

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{for } a \neq 1 \\ N_2 - N_1 + 1 & \text{for } a = 1 \end{cases} \quad \ldots(5)$$

Comparing summation term of Equation (4) with Equation (5),

$a = 1$, $N_1 = 0$ and $N_2 = N$.

Thus Equation (4) becomes,

$$P = \lim_{N \to \infty} \frac{1}{2N+1} [N_2 - N_1 + 1]$$

$$P = \lim_{N \to \infty} \frac{1}{2N+1} [N - 0 + 1]$$

$$\therefore \quad P = \lim_{N \to \infty} \frac{N+1}{2N+1}$$

Dividing numerator and denominator by N we get,

$$P = \lim_{N \to \infty} \frac{1 + \frac{1}{N}}{2 + \frac{1}{N}} \quad \therefore P = \frac{1}{2}$$

Since the power is finite; unit step is power signal.

Ex. 1.10.21 : A discrete time signal is given by $x[n] = \{1, 1, 1, 1, 2\}$.
↑

Sketch the following signals :

(a) $x[n]$

(b) $x[n-2]$

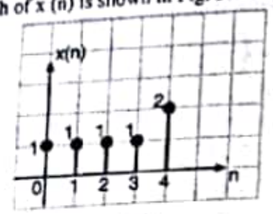
(c) $x[n] \cdot u[n-1]$

(d) $x[3-n]$ (e) $x[n-1] \cdot \delta[n-1]$

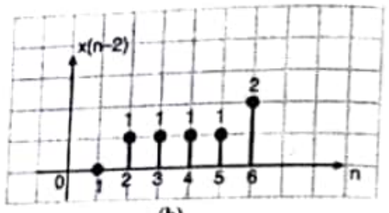
May 15. 10 Marks

Soln. :

(a) The sketch of x (n) is shown in Fig. P. 1.10.21(a).



(a)

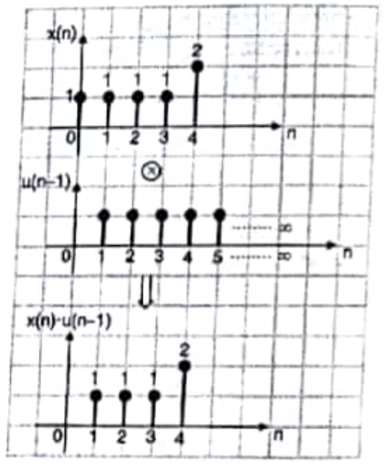


(b)

(H-1423) **Fig. P. 1.10.21**

(b) x (n − 2) is delay of x (n) by two positions. It is shown in Fig. P. 1.10.21(b).

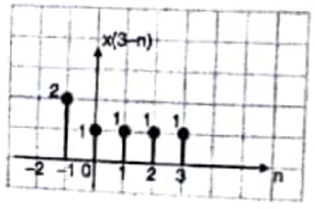
(c) x (n) · u (n − 1) is shown in Fig. P. 1.10.21(c).



(H-1424) **Fig. P. 1.10.21(c)**

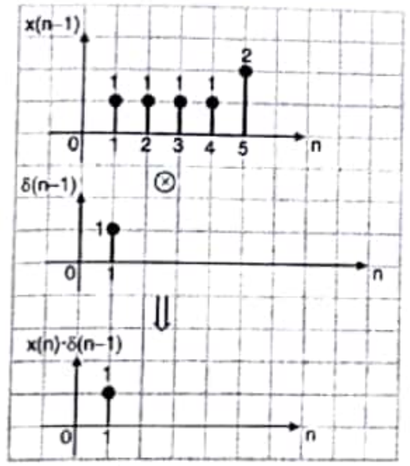
(d) x (3 − n) is delayed version of x (− n) by 3 positions. Here x (− n) = {2, 1, 1, 1, 1 }.
↑

The signal x (− n + 3) is shown in Fig. P. 1.10.21(d).



(H-1425) **Fig. P. 1.10.21(d)**

(e) x (n − 1) · δ (n − 1) is shown in Fig. P. 1.10.21(e).



(H-1426) **Fig. P. 1.10.21(e)**

Ex. 1.10.22 : Determine the fundamental period of the following signals :

1. $x(t) = 2\cos\frac{2\pi t}{3} + 3\cos\frac{2\pi t}{7}$

2. $x[n] = \cos^2\left[\frac{\pi}{4}n\right]$

Dec. 15. 5 Marks

Soln. :

1. Given : $x(t) = 2\cos\frac{2\pi t}{3} + 3\cos\frac{2\pi t}{7}$ ...(1)

The corresponding standard equation is,

$$x(t) = A\cos 2\pi f_1 t + A\cos 2\pi f_2 t \quad \ldots(2)$$

Comparing Equations (1) and (2),

$$2\pi f_1 t = \frac{2\pi t}{3} \text{ and } 2\pi f_2 t = \frac{2\pi t}{7}$$

$$\therefore f_1 = \frac{1}{3} \text{ and } f_2 = \frac{1}{7}$$

$$\therefore T_1 = 3 \text{ and } T_2 = 7$$

Since $\frac{T_1}{T_2}$ is rational number, it is periodic signal. The resultant period is MCD (Most Common Divider) of T_1 and T_2. So it is 1 sec.

2. Given : $x(n) = \cos^2\left(\frac{\pi}{4}n\right)$

Now $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$\therefore x(n) = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{\pi}{2}n\right) \quad \ldots(3)$$

The first term is constant. The standard equation corresponding to second term is $A\cos 2\pi f_0 n$.

$$\therefore 2\pi f_0 n = \frac{\pi}{2}n \quad \therefore f_0 = \frac{1}{4} = \frac{K}{N}$$

So, it is periodic signal with period N = 4 samples.

Ex. 1.10.23 : Find out even and odd component of the following signal. **Dec. 15. 5 Marks**

$$x(t) = \cos^2\left(\frac{\pi t}{2}\right)$$

Soln. :

Given : $x(t) = \cos^2\left(\frac{\pi t}{2}\right) = \frac{1}{2} + \frac{1}{2}\cos \pi t$

$$\therefore x(t) = \frac{1}{2} + \frac{1}{2}\cos \pi(-t) = \frac{1}{2} + \frac{1}{2}\cos \pi t$$

Now even part is, $x_e(t) = \frac{1}{2}[x(t) + x(-t)] = \frac{1}{2}[1 + \cos \pi t]$

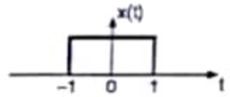
and odd part is, $x_o(t) = \frac{1}{2}[x(t) - x(-t)] = \frac{1}{2}[0] = 0$

Ex. 1.10.24 : A continuous time signal is shown in Fig. P. 1.10.24. Sketch the following transformed versions of the signal.

1. $x(t-3)$ 2. $-2x(t)$

3. $x(t-3) - 2x(t)$ 4. $\frac{dx(t)}{dt}$

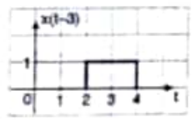
Dec. 15. 10 Marks



(H-1658) **Fig. P. 1.10.24**

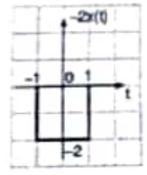
Soln. :

1. x (t − 3) is delay of x (t) by 3 positions. It is shown in Fig. P. 1.10.24(a).



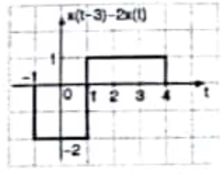
(H-1642) **Fig. P. 1.10.24(a)**

2. − 2x (t) means, multiply amplitude of x (t) by − 2. It is shown in Fig. P. 1.10.24(b).



(H-1643) **Fig. P. 1.10.24(b)**

3. x (t − 3) − 2x(t) is obtained by adding Figs. P. 1.10.24(a) and (b). It is shown in Fig. P. 1.10.24(c).



(H-1644) **Fig. P. 1.10.24(c)**

4. The given signal x (t) can be expressed in terms of unit step as,

$$x(t) = u(t+1) - u(t-1)$$

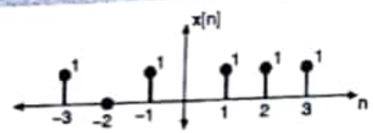
Now $\frac{d}{dt}u(t) = \delta(t)$

$$\therefore \frac{d}{dt}x(t) = \delta(t+1) - \delta(t-1)$$

Ex. 1.10.25 : A DT signal has been shown in Fig. P. 1.10.25. Sketch the following signals.

1. x [n – 4] 2. x [4 – n]
3. x [– 2n + 2] 4. x [n] u [3 – n]

May 16, 8 Marks

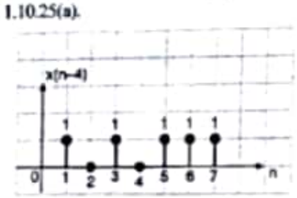


(H-1659) Fig. P. 1.10.25

Soln. :

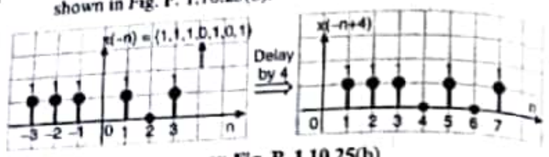
Given :

1. x (n – 4) indicates delay of x (n) by 4 positions. It is shown in Fig. P. 1.10.25(a).



(H-1651) Fig. P. 1.10.25(a)

2. x (4 – n) = x (– n + 4). It is delay of x (– n) by 4 positions as shown in Fig. P. 1.10.25(b).



(H-1652) Fig. P. 1.10.25(b)

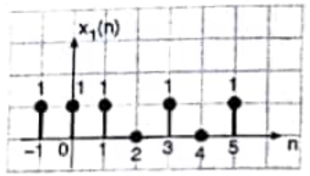
3. x (– 2n + 2)

According to precedence rule ; the sequence of operations will be as follows :

(a) Folding (b) Delay by 2 (c) Compression

Folding is shown in Fig. P. 1.10.25(b).

Delay by 2 of folding signal is shown in Fig. P. 1.10.25(c).



(H-1653) Fig. P. 1.10.25(c)

Here x (– 1) = 1, x (0) = 1, x (1) = 1, x (2) = 0, x (3) = 1, x (4) = 0, x (5) = 1

Now we will perform compression by 2. That means x_1 (2n).

Here 2n = – 1 $\quad \therefore \; n = -\frac{1}{2} = 0$

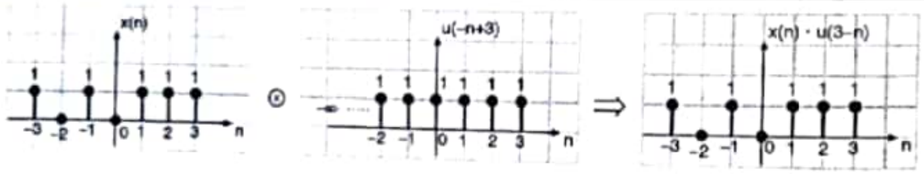
For n = 0 ⇒ x (0) = 1

n = 1 ⇒ x (2) = 0

n = 2 ⇒ x (4) = 0

The final sequence is, y (n) = {1}.

4. x (n) · u (3 – n) is shown in Fig. P. 1.10.25(d).



(H-1654) Fig. P. 1.10.25(d)

Review Questions

Q. 1 Define signal.

Q. 2 Give the classification of signals.

Q. 3 Define continuous time (CT) and discrete time (DT) signals.

Q. 4 Differentiate between CT and DT signals.

Q. 5 How is a CT signal represented mathematically ?

Q. 6 How do you obtain a DT signal from a CT signal ?

Q. 7 Define a digital signal.

Q. 8 Define periodic and nonperiodic signals.

Q. 9 State the condition of periodicity.

Q. 10 Compare periodic and nonperiodic signals.

Q. 11 Prove that $x(t) = A \cos \omega_0 t$ is a periodic signal.

Q. 12 Define deterministic and random signals.

Q. 13 Compare deterministic and random signals.

Q. 14 Define even and odd signals.

Q. 15 Compare energy and power signals.

Q. 16 Define signal energy.

Q. 17 Define signal power.

Q. 18 Define multichannel and multidimensional signals.

Q. 19 Define the following signals and write their mathematical expressions :

1. DC signal
2. Exponential signal
3. Rectangular pulse
4. Signum function.

Q. 20 State mathematical expression for unit step and rectangular signals.

Q. 21 State the important properties of delta function.

Q. 22 State the relation between DT unit impulse and DT unit step signals.

Q. 23 Define a unit ramp signal and draw it graphically.

Q. 24 State different types of complex exponential signal.

Q. 25 What is the relation between the complex exponential and sinusoidal signals.

Q. 26 Draw the x(t) = sinc (t) signal graphically.

Q. 27 Define the following :

1. Time shifting. 2. Time scaling

Q. 28 Define folding or time reversal.

Q. 29 What is the difference between amplitude scaling and time scaling ?

Q. 30 State the relationship between unit step and unit impulse signals.

Q. 31 State the precedence rule for time shifting and time scaling.

Q. 32 Define convolution and state its significance.

Q. 33 What is the practical application of convolution?

Q. 34 State convolution theorems.

Q. 35 Define a sinc signal and mathematically obtain its values at t = 0, ± 1, ± 2 .... Plot the sinc function.

Q. 36 Explain the transformation of independent variables.

1.11 University Questions and Answers :

Q. 1 Determine if the given sequence is periodic or not. If periodic, find out fundamental period

$$x[n] = \sin\left(\frac{6\pi}{7}n + 1\right) \qquad \textbf{(Dec. 16, 4 Marks)}$$

Ans. :

$$x(n) = \sin\left(\frac{6\pi}{7}n + 1\right)$$

$$\sin(a+b) = \sin a \cos b + \sin b \cos a$$

$$\sin\left(\frac{6\pi}{7}n + 1\right) = \sin\frac{6\pi}{7}n \cos 1 + \sin 1 \cos\frac{6\pi}{7}n$$

$$\sin\left(\frac{6\pi}{7}n + 1\right) = \sin\frac{6\pi}{7}n \cdot 1 + 0 \cdot \cos\frac{6\pi}{7}n$$

$$\therefore \left(\frac{6\pi}{7}n + 1\right) = \sin\frac{6\pi}{7}n \qquad \ldots(1)$$

Comparing with, $x(n) = \sin \omega_1 n \qquad \ldots(2)$

Comparing Equations (1) and (2),

$$\therefore \omega_1 = \frac{6\pi}{7}$$

But $\omega = 2\pi f$

$$2\pi f_1 = \frac{6\pi}{7}$$

$$f_1 = \frac{6\pi}{7 \times 2\pi}$$

$$\therefore f_1 = \frac{3}{7} \text{ cycles per samples} \qquad \ldots(3)$$

Since frequency 'f_1' is expressed as the ratio of two integers, this sequence is **periodic.**

To determine fundamental period :

$$f_1 = \frac{k}{N} \qquad \ldots(4)$$

Here 'N' indicates, the fundamental period.

Comparing Equations (3) and Equation (4),
Fundamental period = N = 7 samples

Q. 2 ACT signal has been shown below. Sketch the following signals **(Dec. 16, 10 Marks)**

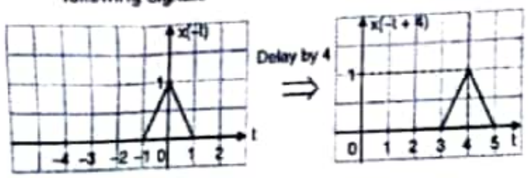


Fig. 1

(i) x (t – 4) (ii) x (4 – t) (iii) x (– 2t + 2) (iv) x (0.5 t)

Ans. :

(i) x (t – 4) is delay of x (t) by 4 positions. It is shown in Fig. 1(a).

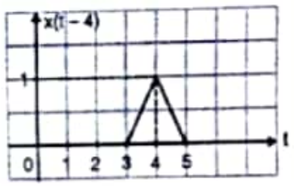


Fig. 1(a)

(ii) x (4t – t) = x (– t + 4). It is folding and then delay by 4 positions. It is shown in Fig. 1(b).

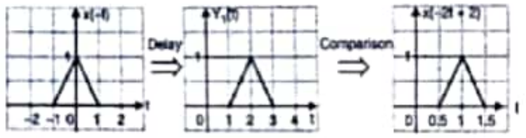


Fig. 1(b)

(iii) x (– 2t + 2). Using precedance rule; it is obtained by performing following operations, as shown in Fig. 1(c).

(a) Folding of x (t) (b) Delay by 2

(c) Compression by 2

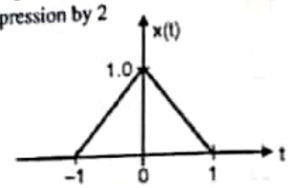


Fig. 1(c)

(iv) x (0.5t) = x (t/2). It is expansion of x (t) by 2 positions. It is shown in Fig. 1(d).

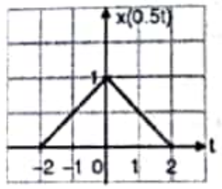


Fig. 1(d)

□□□

Introduction to Systems

Syllabus :

Introduction to systems : Definition, classification of systems : Static and dynamic, Time variant and time invariant, Linear and nonlinear, Causal and noncausal, Stable and unstable systems, Communication and control system as examples.

2.1 System :

Definition :

- A system is defined as an entity that operates on one or more signals to implement a function, thereby producing new signals.
- The block diagram representation of a system is shown in Fig. 2.1.1.
- The type of input and output signal depends on the type of system being used.

(H-220) **Fig. 2.1.1 : Block diagram representation of a signal**

- A system may be defined as a set of elements and functional blocks interconnected to produce an output y (t) in response to an input x (t).
- The input signal x (t) is also called as excitation of a system and y (t) is also called as response of the system.
- Some of the examples of continuous time (CT) systems around us are : amplifier, filters, and other electronic systems.
- The response y (t) of a system to the excitation x (t), in the time domain depends on the impulse response h (t) of the system. The relation between them is as follows :

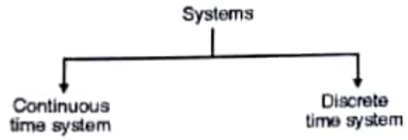
$$y(t) = x(t) * h(t) \quad ...(2.1.1)$$

where * represents the convolution of x (t) and h (t).

- In the frequency domain, the system response depends on the transfer function of the system.

2.1.1 Types of Systems :

Depending on the type of input signal (continuous time or discrete time) the systems are classified into two types as follows :



(H-221)

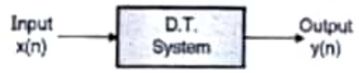
Continuous time (CT) system :

- A continuous time system is defined as the system which processes a continuous time (CT) signal and produces another continuous time (CT) signal at its output.
- The CT signal at the input of CT system is denoted by x(t) and the CT response of the system is denoted by y(t) as shown in Fig. 2.1.1.

Discrete time (DT) system :

- A Discrete Time (DT) system is defined as the system which processes a discrete time (DT) signal and produces another discrete time (DT) signal at its output.
- The DT signal at the input of DT system is denoted by x (n) and the DT response of the system is denoted by y (n).

Fig. 2.1.2 shows the block diagram of a D.T. system.



(H-222) **Fig. 2.1.2 : Block schematic of a D.T. system**

2.1.2 Examples of Practical Systems :

There are several types of practical systems. We can classify them into following categories :

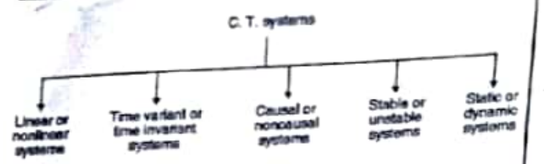
1. Communication system
2. Control system
3. Remote sensing
4. Biomedical signal processing
5. Auditory systems.

2.1.3 Classification of C.T. Systems :

The systems are classified as follows :

1. Linear and non-linear systems
2. Time variant and time invariant systems
3. Causal or non-causal systems
4. Stable and unstable systems.
5. Static or dynamic systems.

This classification is made based on various properties of systems.

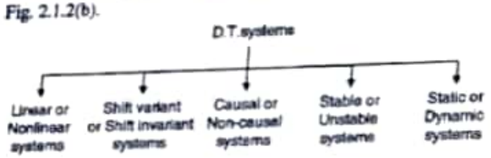


(H-223)

Fig. 2.1.2(a) : Classification of C.T. systems

2.1.4 Classification of D.T. Systems :

Similarly the D.T. systems can be classified as shown in Fig. 2.1.2(b).



(H-224)

Fig. 2.1.2(b) : Classification of D.T. systems

2.2 Linear and Non linear Systems :

2.2.1 Linear and Nonlinear C.T. Systems :

- A C.T. system is said to be "linear" if it satisfies the superposition theorem. Let the two C.T. systems be defined as follows :

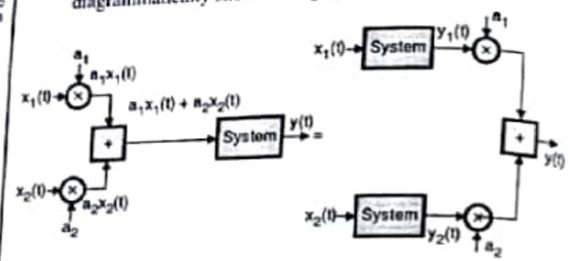
$$y_1(t) = f[x_1(t)]$$
$$\text{and} \quad y_2(t) = f[x_2(t)]$$

- Here $y_1(t)$ and $y_2(t)$ are responses and $x_1(t)$ and $x_2(t)$ are the excitations. Then a C.T. system will be called as a linear system if it satisfies the following expression :

$$f[a_1 x_1(t) + a_2 x_2(t)] = a_1 y_1(t) + a_2 y_2(t) \quad ...(2.2.1)$$

where a_1 and a_2 are constants.

- If you see the LHS of Equation (2.2.1), it shows that the inputs $x_1(t)$ and $x_2(t)$ are multiplied with a_1 and a_2, then the product terms are added and finally the addition is passed through the system. This is as shown in Fig. 2.2.1.
- Whereas the RHS of Equation (2.2.1) shows that $x_1(t)$ and $x_2(t)$ are passed through the individual systems to produce $y_1(t)$ and $y_2(t)$. The individual system responses are then multiplied by a_1 and a_2 and finally the product terms are added. This is as shown in Fig. 2.2.1.
- Equation (2.2.1) shows that even after following completely different sequences of operations we get the same output.
- This is called as linearity or superposition. This property is diagrammatically shown in Fig. 2.2.1.



(H-225) **Fig. 2.2.1 : Principle of linearity**

- A C.T. system is said to be **nonlinear** if it does not satisfy the superposition theorem stated in Equation (2.2.1). Communication channels and filters are examples of linear systems.

2.2.2 Linearity of D.T. System :

- We can define the linearity of D.T. system in the similar way as that of a C.T. system.
- A D.T. system is said to be linear if it satisfies the superposition theorem. Let the two D.T. systems be defined as follows :

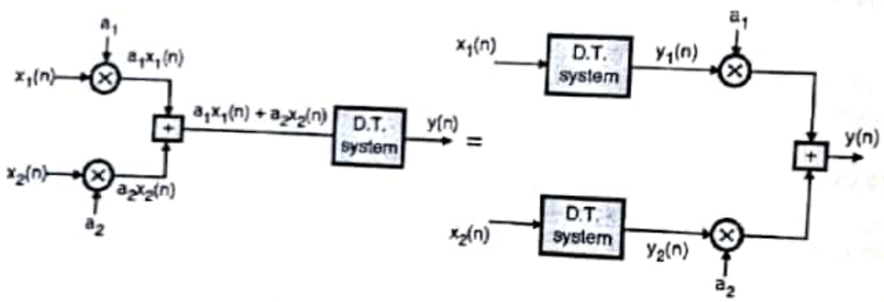
$$y_1(n) = f[x_1(n)] \text{ and } y_2(n) = f[x_2(n)]$$

- Then a D.T. system is called as a linear system if it satisfies the following expression :

$$f[a_1 x_1(n) + a_2 x_2(n)] = a_1 y_1(n) + a_2 y_2(n) \quad ...(2.2.2)$$

where a_1 and a_2 are constants.

- The procedure to be followed to check the linearity of a D.T. system is same as that followed for the C.T. system.
- This condition can be represented diagrammatically as shown in Fig. 2.2.2.
- A D.T system is said to be **non-linear** if it does not satisfy the superposition theorem stated earlier.



(H-226) **Fig. 2.2.2 : Principle of linearity**

Ex. 2.2.1 : Check whether the following system is linear or not $y(t) = e^{x(t)}$ **May 06, 5 Marks**

Soln. : We can write the given equation as,

$$y(t) = e^{x(t)}$$

Step 1 : If input $x(t)$ is zero then, output $y(t) = e^0 = 1$. Thus system is initially non-relaxed.

Step 2 : Here the function of system is to take exponential of $x(t)$. Now let there are two inputs, $x_1(t)$ and $x_2(t)$

$$\therefore \quad x_1(t) \xrightarrow{T} y_1(t) = e^{x_1(t)}$$
$$x_2(t) \xrightarrow{T} y_2(t) = e^{x_2(t)}$$

Add these output to get $y'(t)$

$$\therefore \quad y'(t) = e^{x_1(t)} + e^{x_2(t)} \quad ...(1)$$

Step 3 : First add input $x_1(t)$ and $x_2(t)$ and then apply it to the system. We know that the system produces exponential of input.

$$\therefore \quad [x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = e^{[x_1(t) + x_2(t)]}$$
$$\therefore \quad y''(t) = e^{[x_1(t) + x_2(t)]}$$
$$\therefore \quad y''(t) = e^{x_1(t)} \cdot e^{x_2(t)} \quad ...(2)$$

Step 4 : Compare Equations (1) and (2) :

Since $y'(t) \neq y''(t)$; the system is non-linear.

> **Note :** If applied input is zero and the system produces zero output then the system is said to be initially relaxed; otherwise it is initially non-relaxed. Step 1 is used just to check, whether the system is initially relaxed or not. But for checking linearity, other steps must be performed.

Ex. 2.2.2 : $y(n) = \cos x(n)$. Check the linearity of the system.

Soln. :

Step 1 : If input $x(n)$ is zero then,

$$y(n) = \cos(0) \quad \therefore \quad y(n) = 1$$

Thus system is initially non-relaxed

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these two inputs separately to the system. Observe that the function of system is to take 'cos' of input signal.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = \cos x_1(n)$$
$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = \cos x_2(n)$$

Now add two outputs to get $y'(n)$

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$
$$\therefore \quad y'(n) = \cos x_1(n) + \cos x_2(n) \quad ...(1)$$

Step 3 : Add two inputs and pass it through the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} \cos[x_1(n) + x_2(n)]$$
$$\therefore \quad y''(n) = \cos[x_1(n) + x_2(n)] \quad ...(2)$$

Use trignometric identity,

$$\cos[A + B] = \cos A \cos B + \sin A \sin B$$

Thus Equation (2) becomes,

$$y''(n) = \cos x_1(n) \cos x_2(n) + \sin x_1(n) \sin x_2(n) \quad ...(3)$$

Step 4 : Compare Equations (1) and (2) :

Since $y'(n) \neq y''(n)$; the system is non-linear. **...Ans.**

Ex. 2.2.3 : $y(n) = |x(n)|$. Is the system linear or non-linear ?

Soln. :

Step 1 : Apply zero input to the system that means $x(n) = 0$.

$$\therefore \quad y(n) = |x(n)| = 0$$

Thus system is initially relaxed.

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$ and apply it to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = |x_1(n)|$$
$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = |x_2(n)|$$

Add these two outputs to get $y'(n)$,

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$
$$\therefore \quad y'(n) = |x_1(n)| + |x_2(n)| \quad ...(1)$$

Step 3 : Add two inputs and apply it to the system. Observe that the function of system is to take magnitude of input signal.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} |x_1(n) + x_2(n)|$$
$$\therefore \quad y''(n) = |x_1(n) + x_2(n)| \quad ...(2)$$

Step 4 : Compare Equations (1) and (2) :

Since $y'(n) \neq y''(n)$; the system is non-linear. **...Ans.**

Ex. 2.2.4 : Explain the following system with respect to linearity property :

$$y(n) = x(n) + n\,x(n+1)$$

Soln. :

Step 1 : Apply input $x(n) = 0$ to the system. Thus system is initially relaxed.

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply two inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(n) + n\,x_1(n+1)$$
$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(n) + n\,x_2(n+1)$$

Add these two outputs to get $y'(n)$,

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$
$$\therefore \quad y'(n) = x_1(n) + n\,x_1(n+1) + x_2(n) + n\,x_2(n+1)$$
$$\therefore \quad y'(n) = x_1(n) + x_2(n) + n[x_1(n+1) + x_2(n+1)] \quad ...(1)$$

Step 3 : Combine two inputs and apply it to the system :

$$\therefore [x_1(n) + x_2(n)] \xrightarrow{T} [x_1(n) + x_2(n)] + n[x_1(n+1) + x_2(n+1)]$$
$$\therefore \quad y''(n) = [x_1(n) + x_2(n)] + n[x_1(n+1) + x_2(n+1)] \quad ...(2)$$

Step 4 : Compare Equations (1) and (2) :

Since $y'(n) = y''(n)$; the system is linear. **...Ans.**

Ex. 2.2.5 : Explain the following system with respect to linearity property :

$$y(n) = x(2n).$$

Soln. :

Step 1 : When input is zero, output $y(n)$ is also zero. Thus system is initially relaxed.

Step 2 : Consider two separate inputs $x_1(n)$ and $x_2(n)$. Apply it to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(2n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(2n)$$

Add $y_1(n)$ and $y_2(n)$ to get $y'(n)$,

$$\therefore \quad y'(n) = x_1(2n) + x_2(2n) \qquad ...(1)$$

Step 3 : Combine two inputs and apply it to the system :

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} x_1(2n) + x_2(2n)$$

$$\therefore \quad y''(n) = x_1(2n) + x_2(2n) \qquad ...(2)$$

Step 4 : Compare Equations (1) and (2) :

Since $y'(n) = y''(n)$; the system is linear. **...Ans.**

Ex. 2.2.6 : Find whether the following system is linear :

$$y(n) = x(-n+2)$$

Soln. :

Step 1 : Apply zero input to the system. It produces zero output. Thus system is initially relaxed.

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these inputs separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = x_1(-n+2)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = x_2(-n+2)$$

Add these two outputs to get $y'(n)$,

$$\therefore \quad y'(n) = y_1(n) + y_2(n)$$

$$\therefore \quad y''(n) = x_1(-n+2) + x_2(-n+2) \qquad ...(1)$$

Step 3 : Add two inputs and apply it to the system.

$$\therefore \quad x_1(n) + x_2(n) \xrightarrow{T} x_1(-n+2) + x_2(-n+2)$$

$$\therefore \quad y''(n) = x_1(-n+2) + x_2(-n+2) \qquad ...(2)$$

Step 4 : Since $y'(n) = y''(n)$; the system is linear. **Ans.**

Ex. 2.2.7 : $y(n) = n\,x^2(n)$ check the linearity :

Soln. :

Step 1 : When input $x(n)$ is zero; output is zero. Thus system is initially relaxed.

Step 2 : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply each input separately to the system.

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = n\,x_1^2(n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = n\,x_2^2(n)$$

Add these outputs to get $y'(n)$,

$$\therefore \quad y'(n) = n\left[x_1^2(n) + x_2^2(n)\right] \qquad ...(1)$$

Step 3 : Combine two inputs and apply it to the system.

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} n\,[x_1(n) + x_2(n)]^2$$

$$\therefore \quad y''(n) = n\,[x_1(n) + x_2(n)]^2 \qquad ...(2)$$

Step 4 : Compare Equations (1) and (2),

Since $y'(n) \neq y''(n)$; the system is non-linear. **...Ans.**

Ex. 2.2.8 : Check the following system for linearity :

$$y(t) = x(t) + x(t-100)$$

Soln. :

Step 1 : Apply zero input, $y(t) = 0$.

Step 2 : Apply each input separately.

$$\therefore \quad x_1(t) \xrightarrow{T} x_1(t) + x_1(t-100)$$

$$\therefore \quad x_2(t) \xrightarrow{T} x_2(t) + x_2(t-100)$$

$$\therefore \quad y'(t) = x_1(t) + x_1(t-100) + x_2(t) + x_2(t-100)$$

Step 3 : Apply both inputs combinely.

$$\therefore [x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = x_1(t) + x_1(t-100) + x_2(t) + x_2(t-100)$$

Step 4 : Since $y'(t) = y''(t)$; It is linear system.

Ex. 2.2.9 : Find whether the following systems are linear or non-linear systems :

1. $y(t) = t\,x(t)$ 2. $y(n) = n^2 x(n)$

Dec. 06, 6 Marks

Soln. :

1. $y(t) = t\,x(t)$:

Step 1 : If input is zero then output is zero. So system is initially relaxed

Step 2 : Let there be two inputs $x_1(t)$ and $x_2(t)$

$$\therefore \quad x_1(t) \xrightarrow{T} y_1(t) = t\,x_1(t)$$

$$x_2(t) \xrightarrow{T} y_2(t) = t\,x_2(t)$$

Adding these outputs to get $y'(t)$

$$\therefore \quad y'(t) = t\left[x_1(t) + x_2(t)\right] \qquad ...(1)$$

Step 3 :

$$[x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = t\left[x_1(t) + x_2(t)\right] \qquad ...(2)$$

Step 4 : Since $y'(t) = y''(t)$; system is linear

2. $y(n) = n^2 x(n)$:

Step 1 : If input is zero then output is zero. So, system is initially relaxed

Step 2 : Let there be two inputs $x_1(n)$ and $x_2(n)$

$$\therefore \quad x_1(n) \xrightarrow{T} y_1(n) = n^2 x_1(n)$$

$$\text{and} \quad x_2(n) \xrightarrow{T} y_2(n) = n^2 x_2(n)$$

Adding these outputs to get $y'(n)$

$$\therefore \quad y'(n) = n^2\left[x_1(n) + x_2(n)\right]$$

Step 3 : Combining two inputs and applying it to the system :

$$\therefore \quad [x_1(n) + x_2(n)] \xrightarrow{T} n^2 x_1(n) + n^2 x_2(n)$$

$$\therefore \quad y''(n) = n^2\left[x_1(n) + x_2(n)\right]$$

Step 4 : Since $y'(n) = y''(n)$; the system is linear

Example for Practice :

Ex. 2.2.10 : Check whether the following discrete time system is linear or not :

$$y(n) = x(n)\cos\omega_0 n.$$

Ans. : Linear

Ex. 2.2.11 : Determine whether the following system is linear or not :

$$y(n) = \frac{1}{3}\left[x(n) + x(n-1) + x(n-2)\right]$$

Ans. : Linear

Ex. 2.2.12 : $y(n) = n\,x(n)$ check for the linearity.

Ans. : Linear

Ex. 2.2.13 : $y(n) = x(n - n_0)$, n_0 is constant. Check for the linearity.

Ans. : Linear

Ex. 2.2.14 : $y(n) = x(n) \cdot n\,(n - n_0)$ check for the linearity.

Ans. : Linear

Ex. 2.2.15 : Examine the following system for linearity,

$$y(n) = x(-n)$$

Ans. : Linear

2.3 Time Variant or Time Invariant Systems :

2.3.1 Time Invariant and Time Variant C.T. Systems :

- A C.T. system is said to be a **time invariant** system if a time shift in the input signal results in a corresponding time shift in the output. If this condition is not satisfied, then the system is called time variant system.
- Let the relation between input and output of a C.T. system be defined as,

$$y(t) = f[x(t)]$$

- Now for a time invariant system if the input signal $x(t)$ is delayed by "t_d" sec. then the output also will be delayed by the same time i.e.

$$f[x(t - t_d)] = y(t - t_d) \qquad ...(2.3.1)$$

- A C.T. system is said to be a **time variant system** if it does not satisfy the Equation (2.3.1). Most of the practical systems which we come across are linear time invariant systems **(LTI)**.
- Examples of the C.T. linear time invariant systems are amplifiers, filters, etc.
- This property can be checked by following the procedure given below.

Procedure to test linearity of a C.T. system :

Step 1 : Apply $x(t)$ to a delay unit to obtain $x(t - t_d)$ i.e. the delayed version of $x(t)$.

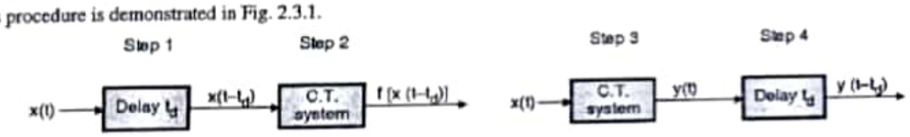
Step 2 : Pass the delayed signal $x(t - t_d)$ through the C.T. system to produce $f[x(t - t_d)]$.

Step 3 : Now pass the signal $x(t)$ through the C.T. system to produce $y(t)$.

Step 4 : Pass $y(t)$ through the delay unit to obtain delayed version of $y(t)$ i.e. $y(t - t_d)$.

Step 5 : Compare the results of step 2 and 4 to decide about the linearity of the C.T. system.

- This procedure is demonstrated in Fig. 2.3.1.



(H-227) **Fig. 2.3.1 : Principle of time invariance**

2.3.2 Shift Invariant and Shift Variant D.T. Systems :

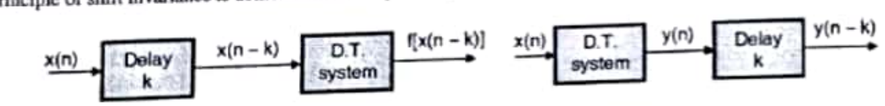
- A D.T. system is said to be a **shift invariant** system if a shift in the input signal results in corresponding shift in the output. Otherwise the system is **shift variant**.
- Let the relation between input and output of a D.T. system be given by,

$$y(n) = f[x(n)]$$

- Now if the input signal $x(n)$ is delayed by "k", then the output signal $y(n)$ will also get delayed by "k" if the D.T. system is a **shift invariant** system.
- This is expressed mathematically as follows :

$$f[x(n-k)] = y(n-k) \qquad ...(2.3.2)$$

- A D.T. system is said to be a **shift variant** system if it does not satisfy the above expression.
- The principle of shift invariance is demonstrated in Fig. 2.3.2.



(H-228) **Fig. 2.3.2 : Principle of shift invariance**

Theorem : A system is shift invariant if and only if

$$x(n) \xrightarrow{T} y(n) \quad \text{implies that} \quad x(n-k) \xrightarrow{T} y(n-k)$$

How to determine whether D.T. system is shift invariant or not ?

To check whether the D.T. system is shift invariant or shift variant follow the steps given below :

Step 1 : Delay the input x (n) by 'k' units to get x (n – k). Denote the corresponding output by y (n, k).

That means $x(n-k) \xrightarrow{T} y(n-k)$

Step 2 : In the given equation of system y (n) replace 'n' by (n – k) for every "n". This output is y (n – k).

Step 3 : If y (n, k) = y (n – k) then system is shift invariant and if y (n, k) ≠ y (n – k) then system is shift variant.

Ex. 2.3.1 : Show the following system is time variant.

$y(t) = t\,x(t)$ **May 03, 5 Marks**

Soln. :

Given, $y(t) = t\,x(t)$

Step 1 : Delay input by k units and denote this output by y (t, k).

$\therefore\ y(t, k) = t\,x(t-k)$

Step 2 : Replace t by t – k throughout the equation.

$\therefore\ y(t-k) = (t-k)\,x(t-k)$

Step 3 : Since y (t, k) ≠ y (t – k); the system is time variant.

Ex. 2.3.2 : Determine whether the following system is time invariant or not :

$$\frac{d}{dt}y(t) + t\,y(t) = x(t)$$

Soln. : The given equation contain a term ty (t). That means output changes with time. So this system is time variant.

Ex. 2.3.3 : Classify the following system for linearity and time invariance :

$$y''(t) + 3y'(t) = 2x'(t) + x(t)$$

Soln. : $y''(t) + 3y'(t) = 2x'(t) + x(t)$ ... Given

(A) Linearity :

Consider there are two inputs $x_1(t)$ and $x_2(t)$. Then if we apply individual input to the system; it satisfies following equations.

$$y_1''(t) + 3y_1'(t) = 2x_1'(t) + x_1(t) \quad ...(1)$$

$$\text{and } y_2''(t) + 3y_2'(t) = 2x_2'(t) + x_2(t) \quad ...(2)$$

The linear combinations of Equations (1) and (2) is,

$$y_1''(t) + y_2''(t) + 3y_1'(t) + 3y_2'(t) = 2x_1'(t) + 2x_2'(t) + x_1(t) + x_2(t)$$

$$\therefore \frac{d^2}{dt^2}y_1(t) + \frac{d^2}{dt^2}y_2(t) + 3\frac{d}{dt}y_1(t) + 3\frac{d}{dt}y_2(t) = 2\frac{d}{dt}x_1(t) + 2\frac{d}{dt}x_2(t) + x_1(t) + x_2(t)$$

$$\therefore \frac{d^2}{dt^2}[y_1(t) + y_2(t)] + 3\frac{d}{dt}[y_1(t) + y_2(t)] = 2\frac{d}{dt}[x_1(t) + x_2(t)] + x_1(t) + x_2(t) \quad ...(3)$$

This equation can be put in the form of original equation. Thus it is linear system.

(B) Time invariance :

This equation does not contain any term like t y (t) or t x (t). That means this equation does not vary with time. So it is time invariant system.

Ex. 2.3.4 : Determine whether the following system is time variant or not :

$y(t) = x(t) - 3u(t)$ **Dec. 03, 1 Mark**

Soln. :

$$y(t) = x(t) - 3u(t)$$

Step 1 : Delay input by k samples and denote output by y (t, k)

$\therefore\ y(t, k) = x(t-k) - 3u(t)$ ...(1)

Step 2 : Replace 't' by t – k throughout the equation.

$\therefore\ y(t-k) = x(t-k) - 3u(t-k)$ ...(2)

Step 3 : Since y (t, k) ≠ y (t – k); the system is time variant.

Ex. 2.3.5 : Determine whether the following system is shift invariant or not :

$T[x(n)] = e^{x(n)}$

Soln. : We know that T [x(n)] means x(n) is passed through the system to produce output. Thus given equation can be written as,

$\therefore\ y(n) = e^{x(n)}$ that means $x(n) \xrightarrow{T} y(n) = e^{x(n)}$

Step 1 : Delay input by k samples and denote this output by y (n, k)

$\therefore\ y(n, k) = e^{x(n-k)}$ ...(1)

Step 2 : Replace 'n' by 'n –k' throughout the given equation,

$\therefore\ y(n-k) = e^{x(n-k)}$ ...(2)

Step 3 : Compare Equations (1) and (2),

Since y (n, k) = y (n – k); the system is shift invariant ...Ans.

Ex. 2.3.6 : Determine whether the following system is shift invariant or not :

$$y(n) = \cos x(n)$$

Soln. :

Step 1 : Delay the input by 'k' units and denote this output by y (n, k)

$\therefore\ y(n, k) = \cos x(n-k)$ ...(1)

Step 2 : Replace 'n' by 'n – k' throughout the equation.

$\therefore\ y(n-k) = \cos x(n-k)$ ...(2)

Step 3 : Compare Equations (1) and (2)

Since y (n, k) = y (n – k); the system is shift invariant. ...Ans.

Ex. 2.3.7 : Determine whether the following system is shift invariant or not :

$$y(n) = x(n) + nx(n-1)$$

Soln. :

Step 1 : Delay input by 'k' units and denote this output by y (n, k)

$\therefore\ y(n, k) = x(n-k) + nx(n-k-1)$ ...(1)

Note that only input x (n) is delayed. That means in the given equation wherever x (n) is present, replace it by x (n – k). You should not replace 'n' present in the second term by n – k. That means in this step do not write the equation as,

$$y(n, k) = x(n-k) + (n-k)\,x(n-k-1)$$

This is not valid since we are delaying input, x (n) and not only n.

Step 2 : Replace 'n' by 'n – k' throughout the equation.

$\therefore\ y(n-k) = x(n-k) + (n-k)\,x(n-k-1)$ ...(2)

Step 3 : Compare Equations (1) and (2)

Since y (n, k) ≠ y (n – k); the system is shift variant. ...Ans.

Ex. 2.3.8 : Determine whether the following system is shift invariant or not :

$y(n) = x(2n)$ **May 03, 2 Marks**

Soln. :

Step 1 : Delay input by k units and denote this output by y (n, k)

$y(n, k) = x[2(n-k)]$ ...(1)

Step 2 : Replace 'n' by 'n – k' throughout the equation

$\therefore\ y(n-k) = x[2(n-k)]$ ...(2)

Step 3 : Compare Equations (1) and (2)

Since y (n, k) = y (n – k); the system is shift invariant. ...Ans.

Ex. 2.3.9 : Determine whether the following systems are time invariant or not

1. $y(t) = t\,x(t)$
2. $y(n) = x(2n)$
3. $3y(n) = 2x(n) - y(n-1)$

May 03, 5 Marks

Soln. : 1. Refer Ex. 2.3.1.

2. Refer Ex. 2.3.8.

3. $3y(n) = 2x(n) - y(n-1)$

$$y(n) = \frac{1}{3}[2x(n) - y(n-1)]$$

Step 1 : **Delay input by 'k' units and denote output by y (n, k) :**

$$y(n, k) + \frac{1}{3}y(n-1) = \frac{1}{3}2x(n-k)$$

Step 2 : **Replace n by (n – k) throughout the equation :**

$$y(n-k) + \frac{1}{3}y(n-k-1) = \frac{1}{3}\cdot 2x(n-k)$$

$$\because\ y(n, k) \neq y(n-k)$$

The given system is **time variant.**

Ex. 2.3.10 : Determine whether the following system is shift invariant or not :

$$y(n) = y(n-4) + x(n-4)$$

Soln. : Note that the term y (n – 4) denotes the delayed output. It also indicates that the feedback is taken from the output y (n) and it is delayed by 4 units.

Step 1 : Delay input by k units and denote this output by y (n, k)

$\therefore\ y(n, k) = y(n-4) + x(n-k-4)$ ...(1)

Step 2 : Replace 'n' by 'n – k' throughout the equations

$\therefore\ y(n-k) = y(n-k-4) + x(n-k-4)$ ...(2)

Step 3 : Compare Equations (1) and (2)

Since y (n, k) ≠ y (n – k); the system is shift variant. ...Ans.

Ex. 2.3.11 : Determine whether the following system is shift invariant or not :

$$y(n) = x(-n)$$

Soln. :

Step 1 : Delay input by 'k' units and denote this output by y(n, k)

$\therefore\ y(n, k) = x(-n-k)$ ...(1)

Note that only 'n' should not be replaced by n – k. This is because we want to delay the whole input x (n) by 'k' units. Thus x (– n) should be delayed by k samples.

Step 2 : Replace 'n' by 'n – k' throughout the equation

$\therefore\ y(n-k) = x[-(n-k)] = x(-n+k)$ ...(2)

Step 3 : Compare Equations (1) and (2).

Since y (n, k) ≠ y (n – k); the system shift variant system. ...Ans.

Ex. 2.3.12 : Find whether the following systems are time variant or time invariant

1. $y(t) = e^x(t)$ 2. $y(n) = x(n^2)$

Dec. 03, 3 Marks, Dec. 06, 6 Marks

Soln. :

Step 1 : **Delay input by t_0 and obtain the output :**

1. $\therefore\ y(t, t_0) = e^x(t - t_0)$

Step 2 : Replacing t by $t - t_0$ throughout the given equation

$$y(t - t_0) = e^x(t - t_0)$$

Step 3 : Since $y(t, t_0) = y(t - t_0)$; the system is shift (time) invariant

2. Given : $y(n) = x(n^2)$

Step 1 : Delay input by 'k' samples and output is denoted by y (n, k) = x (n^2 – k).

Step 2 : **Replacing n by n – k throughout the equation :**

$$\therefore\ y(n-k) = x(n-k)^2$$

Step 3 : Since y (n, k) ≠ y (n – k); the system is time variant.

Ex. 2.3.13 : Determine whether the following systems are time invariant or not.

1. $y(t) = x(t) - 3u(t)$
2. $y(n) = x(2n+2) + x(n)$

Dec. 03, 3 Marks

Soln. :

1. Refer Ex. 2.3.4

2. $y(n) = x(2n+2) + x(n)$

Step 1 : **Delay input by k units and denote this output by y (n, k) :**

$$y(n, k) = x(2n+2-k) + x(n-k)$$

Step 2 : **Replace n by (n – k) throughout the equation :**

$$y(n-k) = x(2n+2-k) + x(n-k)$$

Step 3 : Since y, (n – k) = y (n – k) the system is time invariant.

Examples for Practice :

Ex. 2.3.14 : Show the following system is shift variant :

$$y(n) = n\,x^2(n)$$

Ans. : Shift variant

Ex. 2.3.15 : Find out if the following systems are shift invariant or not :

1. $y(n) = x(n - n_0)$, n_0 is constant
2. $y(n) = ax(n) + b$, a and b are constants
3. $y(n) = x(n)\,n\,(n - n_0)$, n_0 is constant.

Ans. : 1. Shift invariant 2. Shift invariant 3. Shift variant

Ex. 2.3.16 : Determine whether the following system is shift invariant or not :

$$y(n) = x(n) \cos(\omega_0 n)$$

Ans. : Shift variant

Ex. 2.3.17 : Determine whether the following system is time invariant or not :

$$y(n) = x(n) - x(n-1)$$

Ans. : Time invariant

Ex. 2.3.18 : Determine whether the following system is time invariant or not :

$$T[x(n)] = ax(n) + 6$$

Ans. : Time invariant

Ex. 2.3.19 : Determine whether the following system is time invariant or not :

$$y(n) = |x(n)|$$

Ans. : Time invariant

Ex. 2.3.20 : Determine whether the following systems are time invariant or not :

1. $y(n) = g(n)\, x(n)$
2. $y(n) = x(n) + 3u(n+1)$

Ans. : 1. Time variant 2. Time variant

Ex. 2.3.21 : Determine whether the following system is time invariant or not :

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$$

Ans. : Time invariant

2.4 Causal or Noncausal C.T. Systems :

2.4.1 Causal and Noncausal C.T. Systems :

- A C.T. system is said to be "causal" if it produces a response y (t) only after the application of excitation x (t).
- That means for a causal C.T. system the response does not begin before the application of input x (t).
- The other way of defining a causal system is as follows : A system is "causal" if its output depends only on the present and past values of input and does not depend on its future values.
- If the input is applied at $t = t_m$ then the output at $t = t_m$ i.e. $y(t_m)$ will be dependent only on the values of x (t) for $t \le t_m$.

Condition for causality : $y(t_m) = f[x(t); t \le t_m]$ ...(2.4.1)

- Causal C.T. systems are physically realizable systems. The **noncausal systems** do not satisfy Equation (2.4.1). It is not possible to physically realize noncausal systems.

Condition for causality in terms of impulse response h (t) :

- The relation between y (t) and x (t) is given by,

$$y(t) = x(t) * h(t)$$

- where * represents convolution and h (t) is the impulse response of the system. The condition for causality in terms of the impulse response is as follows :

Condition for causality : $h(t) = 0$ for $t < 0$ ...(2.4.2)

- This condition states that a C.T. linear time invariant (LTI) system is "causal" if its impulse response h (t) has a zero value for all the negative values of time.

2.4.2 Causal DT Systems :

Definition :

A DT system is said to be causal system if output at any instant of time depends only on present and past inputs. But the output does not depend on future inputs.

The condition for causality of a D.T. system in terms of its impulse response is as follows :

Condition for causality $h(n) = 0$ for $n < 0$ ...(2.4.2(a))

Examples :

The output of system depends on present and past inputs that means output, y (n) is a function of x (n), x (n – 1), x (n – 2) ... etc. some examples of causal systems are :

1. $y(n) = x(n) + x(n-1)$
2. $y(n) = 3x(n)$
3. $y(n) = x(n) + 4x(n-1)$

Significance :

Since causal system does not include future input samples; such system is practically realizable. That means such system can be implemented practically. Generally all real time systems are causal systems; because in real time applications only present and past samples are present.

Since future samples are not present; **causal system is memoryless system.**

2.4.3 Anticausal or Non-causal DT Systems :

Definition :

A DT system is said to be anticausal system if its output depends not only on present and past inputs but also on future inputs.

Examples :

For a noncausal system, output y (n) is function of x (n), x (n – 1), x (n – 2) .... etc. as well x (n + 1), x (n + 2) ... etc. some examples of non-causal systems are :

1. $y(n) = x(n) + x(n+1)$
2. $y(n) = B\,x(n+2)$
3. $y(n) = x(n) + nx(n+1)$

Significance :

A non-causal system is practically not realizable. That means in practical cases it is not possible to implement a non-causal system.

But if the signals are stored in the memory and at a later time they are used by a system as the time advanced or future signal. Then in such cases it is possible to implement a non-causal system.

Ex. 2.4.1 : Determine if the systems described by following equations are causal or non-causal :

1. $y(n) = \cos x(n)$

2. $y(n) = |x(n)|$
3. $y(n) = x(n) + nx(n-1)$
4. $y(n) = x(n) + nx(n+1)$
5. $y(n) = x(2n)$
6. $y(n) = x(-n+2)$

Soln. :

1. **$y(n) = \cos x(n)$:**

This is a causal system because the function of system is to obtain cosine value of present input.

2. **$y(n) = |x(n)|$:**

This is a causal system because output depends on present input.

3. **$y(n) = x(n) + nx(n-1)$:**

This is a causal system because output depends on present and past input but not on the future input.

4. **$y(n) = x(n) + nx(n+1)$:**

Here output depends on future input i.e. x (n + 1). So this is non-causal system.

5. **$y(n) = x(2n)$:**

This is non-causal system because output expects future input. This can be verified by putting different values of n.

At $n = 0 \Rightarrow y(0) = x(0)$ Here present output excepts present input only.

At $n = 1 \Rightarrow y(1) = x(2)$ Here present output i.e. at n = 1, expects future value of input i.e. x (2).

6. **$y(n) = x(-n+2)$:**

This is non-causal system. This is because at n = – 1 we get,

$y(-1) = x[-(-1)+2] = x[1+2] = x(3)$

Thus present output at n = – 1, expects future input i.e. x (3).

Ex. 2.4.2 : Determine if the system described by the following equation is causal or non causal.

$$y(t) = e^{x(t)}$$

Soln. : The given equation is,

$$y(t) = e^{x(t)}$$

This is causal system since output depends on present input, x (t).

Ex. 2.4.3 : Check the following system for causality :

$$y(t) = \frac{1}{c}\int_{-\infty}^{\infty} x(t)\,dt$$

Soln. :

Given $$y(t) = \frac{1}{c}\int_{-\infty}^{\infty} x(t)\,dt$$

- Here limits of integration are from $-\infty$ to $+\infty$ and output y (t) depends on integration of input. That means output at any instant depends on future inputs also.
- Thus the system is non-causal.

Ex. 2.4.4 : Determine if the systems described by following equations are causal or non-causal :

1. $y(n) = x(n) + x(n-2)$
2. $T[x(n)] = ax(n) + 6$

Soln. :

1. **$y(n) = x(n) + x(n-2)$:**

It is a causal system because output depends only on present input i.e. x (n) and past input i.e. x (n – 2).

2. **$T[x(n)] = ax(n) + 6$:**

The given equation is,

$$y(n) = ax(n) + 6$$

This is causal system because output depends on present input x (n). There is no future input signal.

2.5 Stable or Unstable Systems :

2.5.1 Stable and Unstable C.T. Systems :

Before defining the stable and unstable systems, let us first define the "bounded" and "unbounded" signals.

Bounded signal

- A signal x(t) is called as a bounded signal if its magnitude is always finite i.e. If

$$|x(t)| \le M < \infty \quad ...(2.5.1)$$

- Then x (t) is a bounded signal. Here M is a real positive number. A signal which does not satisfy the condition stated in Equation (2.5.1) is called as an unbounded signal.

Definition of stability of a C.T. system :

- A C.T. system is said to be BIBO (Bounded Input Bounded Output) stable if it produces a bounded output for every bounded signal applied at its input. The system which does not satisfy this condition is an unstable system. The same definition is applicable to the discrete time (D.T.) system as well.
- That means for a BIBO (Bounded Input Bounded Output) stable system, the output signal y (t) satisfies the condition,

$|y(t)| \le M_y < \infty$ for all t

- Whenever the input signal satisfies the condition

$|x(t)| \le M_x < \infty$ for all t

M_x and M_y represent some finite positive numbers.

Condition for stability in terms of impulse response h (t) :

- The condition for stability in terms of the impulse response h (t) of a system is given by,

For an LTI system to be stable : $$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty \quad ...(2.5.2)$$

- The C.T. system which does not satisfy this equation is an unstable system.

2.5.2 Stable and Unstable D.T. Systems :

An initially relaxed system is BIBO stable if and only if every bounded input produces a bounded output. A relaxed system is that system which produces a zero output when its input is zero.

Mathematical representation :

- Let us consider some finite number M_x whose value is less than infinity. That means $M_x < \infty$, so it is a finite value. Then if input is bounded, we can write,

$$|x(n)| \le M_x < \infty \quad ...(2.5.3)$$

- Similarly consider some finite number M_y whose value is less than infinity. That means $M_y < \infty$, so it is a finite value. Then if output is bounded, we can write,

$$|y(n)| \le M_y < \infty \quad ...(2.5.4)$$

Definition of unstable system :

An initially relaxed system is said to be unstable if bounded input produces unbounded (infinite) output.

Significance :

- Unstable system shows erratic and extreme behaviour.
- When unstable system is practically implemented then it causes overflow.

Ex. 2.5.1 : Determine whether the following system is stable or not.

$y(t) = e^{x(t)}$

Soln. : $y(t) = e^{x(t)}$

The given equation is,

$$y(t) = e^{x(t)}$$

We have to check the stability of a system by applying bounded input. That means the value of x (t) should be finite (bounded). The value of 'e' is 2.718. So for bounded input, the output y (t) will be bounded. Thus this is stable system.

Ex. 2.5.2 : Determine whether the following discrete-time systems are stable or not :

1. $T[x(n)] = ax(n) + 6$
2. $y(n) = \cos[x(n)]$
3. $y(n) = x(-n)$
4. $y(n) = x(2n)$

Soln. :

1. **$T[x(n)] = ax(n) + 6$:**

The given equation is,

$$y(n) = ax(n) + 6$$

Here 'a' is some arbitrary constant. So as long as x (n) is bounded the output y (n) is also bounded. Thus it is stable system.

2. **$y(n) = \cos[x(n)]$:**

For every bounded value of x (n), its cosine value is always bounded (finite). So given system is stable.

3. **$y(n) = x(-n)$:**

Here x (– n) means folding of input sequence. That means taking mirror image of x (n). But in case of folding operation; the amplitude of signal is not changed. So if input is bounded the output y (n) = x (– n) will be bounded. Thus it is stable system.

4. **$y(n) = x(2n)$:**

If input x (n) is bounded then output y (n) is also bounded. So it is a stable system.

2.6 Static or Dynamic D.T. Systems (Dynamicity Property) :

2.6.1 Static DT Systems :

Definition : It is a DT system in which output at any instant of time depends on input sample at the same time.

Example :

1. $y(n) = 5x(n)$

Here 5 is constant which multiplies input x (n). But output at n^{th} instant that means y (n) depends on input at the same (n^{th}) time instant x (n). So this is static system.

2. $y(n) = x^2(n) + 5x(n) + 10$

Here also output at n^{th} instant, y (n) depends on the input at n^{th} instant. So this is static system.

Significance :

Observe the input-output relations of static system. Output does not depend on delayed [x (n – k)] or advanced [x (n + k)] input signals. It depends only on present (n^{th}) input signal. If output depends on delayed input signals then such signals should be stored in memory to calculate the output at n^{th} instant. This is not required in static systems. Thus for static systems, memory is not required. So **static systems are memoryless systems.**

2.6.2 Dynamic DT Systems :

Definition :

It is a system in which output at any instant of time depends on input sample at the same time as well as at other instants of time. Here other times means, other than the present time instant. It may be past time or future time. Note that if x (n) represents input signal at present instant then,

(a) x (n – k); that means delayed input signal is called as **past signal.**

(b) x (n + k); that means advanced input signal is called as **future signal.**

Thus in dynamic systems the output depends on present input as well as past or future inputs.

Examples :

1. $y(n) = x(n) + 5x(n-1)$

Here output at n^{th} instant depends on input at n^{th} instant, x (n) as well as $(n-1)^{th}$ instant x (n – 1) is previous (past) sample. So the system is dynamic.

2. $y(n) = 3x(n+2) + x(n)$

Here x (n + 2) indicates advanced version of input sample that means it is future sample; so this is dynamic system.

Significance :

Observe input-output relations of dynamic system. Since output depends on past or future input sample; we need a memory to store such samples. Thus **dynamic system has a memory.**

2.6.3 A Static or Dynamic C.T. System :

A C.T. system is static or memoryless if its output depends upon the present input only.

Example :

Voltage drop across a resistor.

It is given by, $v(t) = i(t) \cdot R$

Here the voltage drop depends on the value of current at that instant of time only so it is static system.

On the other hand a C.T. system is called as dynamic if output depends on present as well as past values.

Example :

A current flowing through inductor i (t) is related to applied voltage, v (t) as,

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau$$

Here the limits of integration are from – ∞ to + t. Thus the current at time 't' depends on all past values of voltage extending right from – ∞ upto that instant "t".

Ex. 2.6.1 : Check whether the following D. T. systems are static or dynamic :

1. $y(n) = n\,x(n) + b\,x^2(n)$
2. $y(n) = x(n^2)$
3. $y(n) = x(n)\cos\omega_0 n$
4. $y(n) = \sum_{k=0}^{2} x(n-k)$
5. $y(n) = x(-n)$

Soln. :

1. **$y(n) = nx(n) + b\,x^2(n)$:**

Note that $x^2(n)$ means, square the amplitude of input signal. So it is not related to time shifting. (It is not $x(n^2)$). Here 'b' is some constant. Since the equation does not contain any advanced or delayed input term; it is static system.

2. **$y(n) = x(n^2)$:**

It is dynamic system, because at n^{th} instant, system expects future inputs (n^2). For example,

At $n = 1$, $y(1) = x(1^2) = x(1)$

At $n = 2$, $y(2) = x(2^2) = x(4)$

Thus at n = 2, system requires future input, x (4). So it is dynamic system.

3. **$y(n) = x(n)\cos\omega_0 n$:**

It is static system because output depends on present input only.

4. **$y(n) = \sum_{k=0}^{2} x(n-k)$:**

First we will expand this summation.

$\therefore\ y(n) = x(n) + x(n-1) + x(n-2)$

Since output contains delayed input terms; it is dynamic system.

5. **$y(n) = x(-n)$:**

We know that x (– n) means folded input signal. Now we will put different values of n

At $n = 0$, $y(0) = x(-0) = x(0)$

At $n = 1$, $y(1) = x(-1)$

At $n = 2$, $y(2) = x(-2)$

Here x (– 1), x (– 2) ... etc. indicates past input sample. So the system is dynamic.

Ex. 2.6.2 : Check whether the following D.T. systems are static or dynamic :

1. $y(n) = n\,x(n)$,
2. $y(n) = 3x(n) + 5$,
3. $y(n) = 3|x(n)|$
4. $y(n) = x(n) - 3x(n-2)$
5. $y(n) = \sum_{k=0}^{2} x(n-k)$

Soln. :

1. **$y(n) = n\,x(n)$:**

It is static system because output depends on present input only.

2. **$y(n) = 3x(n) + 5$:**

It is static system because output, y (n) depends only on present input x (n).

3. **$y(n) = 3|x(n)|$:**

It is static system because output depends on present input only.

4. **$y(n) = x(n) - 3x(n-2)$:**

It is dynamic system because output requires delayed input.

5. **$y(n) = \sum_{k=0}^{2} x(n-k)$:**

First we will expand this summation.

$\therefore\ y(n) = x(n) + x(n-1) + x(n-2)$

Since output contains delayed input terms; it is dynamic system.

Ex. 2.6.3 : Determine whether the following C.T. systems are static or dynamic :

1. $y(t) = 3x(t) + 10$
2. $y(t) = x(t)\cos 50\pi t$
3. $\frac{dy(t)}{dt} + 10\,y(t) = x(t)$
4. $y(t) = e^{x(t)}$

Soln. :

1. **Given $y(t) = 3x(t) + 10$:**

Output depends on present input only. So it is static system.

2. **$y(t) = x(t)\cos 50\pi t$:**

Output at any instant depends on input at that instant. So it is static system.

3. **$\frac{dy(t)}{dt} + 10\,y(t) = x(t)$:**

This system involves differentiation with time. So it is dynamic system.

4. **$y(t) = e^{x(t)}$:**

Here output depends on present input only. So it is static system.

2.7 Invertibility :

- A system is said to be **invertible** if the input of the system can be recovered from the system output.
- A set of operations will be needed to be carried out in order to recover the input from output. This may be viewed as connecting a second system in cascade with the given system, such that the output signal of the second system is equal to the input signal of the given system. i.e. x(t) as shown in Fig. 2.7.1(a).

(H-229) **Fig. 2.7.1(a) : Concept of invertibility**

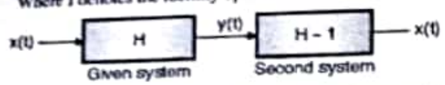
- In order to represent the invertibility, mathematically assume that operator H represents a C.T. given C.T. system with input signal x(t) and produces the output signal y(t).
- Let y(t) be applied to a second C.T. system represented by the operator H^{-1} as shown in Fig. 2.7.1(b). Then the output signal of the second system is given by,

$$\text{Output of second system} = H^{-1}\{y(t)\} = H^{-1}\{H\,x(t)\} = H^{-1}H\{x(t)\}$$

- If we want the output of second system to equal to the original input signal x(t), then it is necessary that

$$H^{-1}H = I \qquad \ldots(2.7.1)$$

Where I denotes the identity operator.



(H-230) **Fig. 2.7.1(b) : Invertibility expressed mathematically**

- The operator H^{-1} is called as the **inverse operator** and the second system is called as the **inverse system**.
- However it is important to understand that H^{-1} is not the reciprocal of H. It is inverse of H.
- There has to be a one-to-one mapping between the input and output signals if a system is be invertible.
- The meaning of one to one mapping is that an invertible system produces distinct outputs for distinct inputs.
- All the discussion done for the C.T. invertible system is applicable to the D.T. systems as well.

Application of invertibility :

- The property of invertibility is very important in designing the **communication systems**. When a signal travels over a communication channel, it gets distorted due to physical characteristics of the channel.
- An **equalizer** can be used to compensate for this distortion and recover back the original signal. The equalizer is added on the receiver side as shown in Fig. 2.7.2.
- An equalizer is an inverse system of the communication channel.

(H-231) **Fig. 2.7.2 : Equalizer is an inverse system of the communication channel**

Ex. 2.7.1 : For an inductor the input output relation is as follows :

$$y(t) = \frac{1}{L}\int_{-\infty}^{t} x(t)\,dt$$

Find the operation that represents its inverse system.

Soln. :

$$H = \frac{1}{L}\int_{-\infty}^{t} dt$$

$$\therefore\ H^{-1} = L \times \frac{d}{dt} \qquad \ldots\text{Ans.}$$

This is the operator of the inverse system so that

$$H^{-1}H = I$$

2.7.1 Comparisons :

Causal system and non-causal system :

Sr. No.	Causal system	Non-causal system
1.	A system is said to be causal system if output at any instant of time depends only on present and past inputs. But the output does not depend on future input.	A system is said to be non-causal system if its output depends not only on present and past inputs but also on future inputs.
2.	Causal system is practically realizable.	Non-causal system is practically not realizable.
3.	Condition for causality, h (t) = 0 for t < 0. i.e. h (t) is zero for all negative values of t only.	For non-causal system h (t) ≠ 0 for t < 0. i.e. h (t) exist for negative values of t also.
4.	Example : All real time systems.	Examples : Population growth, Weather broadcasting etc.

Time variant system and time invariant system :

Sr. No.	Time variant system	Time invariant system
1.	A system is time variant if its input-output characteristics changes with time.	A system is time invariant system if its input-output characteristic does not changes with time.
2.	If $x(n) \xrightarrow{T} y(n)$ then for time variant system, $x(n-k) \xrightarrow{T} y(n-k)$	If $x(n) \xrightarrow{T} y(n)$ then for TVI system, $x(n-k) \xrightarrow{T} y(n-k)$
3.	e.g. : y (n) = x (– n).	e.g. : y (n) = x (2 n).

Static system and dynamic system :

Sr. No.	Static system	Dynamic system
1.	It is a system in which output at any instant of time depends on input sample at the same time.	It is a system in which output at any instant of time depends on input sample at the same time as well as at other inputs.
2.	Output does not depend on delayed [x (n – k)] or advanced [x (n + k)] inputs.	Output depends on present [x (n)], delayed [x (n – k)] and advanced [x (n + k)] inputs.
3.	These are memoryless systems.	These system has a memory.
4.	e.g. : y (n) = 5x (n)	e.g. : y (n) = x (n) + 5x (n – 1)

2.8 Types (Examples) of Systems :

Signals and systems have several applications. Some of the important types of systems are as follows :

1. Communication system.

2. Control system.
3. Remote sensing system.
4. Biomedical signal processing.
5. Auditory system.

2.8.1 Communication System :

- There are three basic blocks of any communication system :
  1. Transmitter 2. Channel 3. Receiver.
- Fig. 2.8.1 shows the block diagram of a basic communication system.

(H-2) **Fig. 2.8.1 : A communication system**

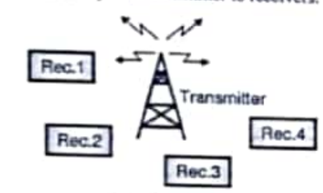
- The transmitter and receiver are located far away from each other. The channel is a physical medium that is used for connecting them.
- We can view each block as a separate system, with their own signals associated with them.
- The transmitter converts the message signal at its input into a signal which is suitable for transmission. The message signal can be a speech signal, video signal or computer data.
- The channel can be a wired channel such as a coaxial cable, optical fiber, satellite channel, or it can be a wireless channel such as a mobile radio channel, depending on the area of application.
- When the transmitted signal from the transmitter travels over the channel to receiver, it gets distorted due to the physical characteristics of the channel. The noise and other interferences get added to the transmitted signal.
- The received signal is a corrupted version of the transmitted signal. The receiver work on it to reconstruct an estimate of the original message signal.
- The role of the receiver is thus exactly opposite to that of the transmitter.
- The functions performed by the transmitter and receiver are dependant on what type of communication systems is being considered.
- A communication system can be one of the following two types :
  1. Analog system. 2. Digital system.
- The **analog communication systems** are relatively simple. The transmitter in such systems has a modulator whereas the receiver has a demodulator.
- **Modulation** is a process which converts the message signal into a form that is compatible with the transmission characteristics of the channel.
- The types of **analog modulation** are AM (amplitude modulation), FM (frequency modulation), or PM (phase modulation).
- We have to use an appropriate demodulator at the receiver in order to recover the original signal back.
- The **digital communication systems** are not as simple as the analog systems. The message signal at the transmitter input can be either an analog signal such as speech/video or a digital signal such as computer data.
- If the message is an analog signal then the transmitter performs the following operations to convert it to digital form :
  1. Sampling 2. Quantization
  3. Encoding.
- Note that the sampling and encoding operations can be reversed at the receiver but the quantization cannot be reversed. Quantization is a process of approximating the sampled signal and introduces errors due to approximation.
- For the digital message signal the sampling, quantization and encoding are not required to be used. We can use the message signal as it is.
- The transmitter may have to perform some additional operations such as data compression and channel encoding.
- The coded signal at the output of the encoder is used for modulating a high frequency carrier wave and the modulated signal is transmitted over the channel.
- The receiver performs all these operations in the reverse manner in order to recover the original message signal.

Modes of communication :

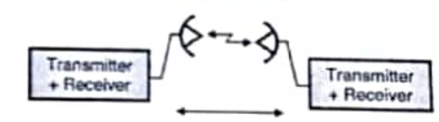
- There are two basic modes of communication :
  1. Broadcasting 2. Point to point communication.

1. Broadcasting :

Here a transmitter transmits the information in many directions and different receivers receive it. As shown in Fig. 2.8.2(a), in broadcasting the flow of information is **unidirectional** i.e. only from transmitter to receivers.



(a) Broadcasting



(b) Point to point communication

(H-3) **Fig. 2.8.2 : Modes of communication**

2. Point to point communication :

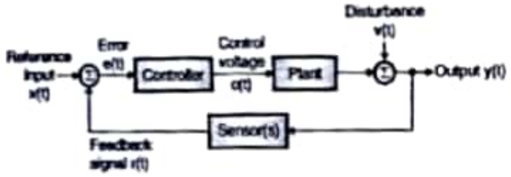
- In this type of communication, the communication takes place between a single transmitter and a single receiver as shown in Fig. 2.8.2(b).
- Both the sides of the point to point link have a transmitter and a receiver and the flow of information is **bi-directional, i.e. in both the directions.**
- The examples of broadcast communication are radio transmission or TV transmission. The examples of point to point communication system are the telephone system, and the deep space communication taking place between an earth station and space craft.

Problems :

- Every communication system has to deal with the presence of **channel noise**, which degrades the quality of the received signal severely.
- It is possible to overcome this problem by using large antennas and error controlling techniques.

2.8.2 Control System :

- Control of a physical system is an important application of signals and systems. Some of the examples in which control is applied are as follows :
  1. Machine tools
  2. Mass-transit vehicles
  3. Power plants
  4. Robots.
  5. Auto pilots
  6. Paper mills
  7. Nuclear reactors
  8. Automobile engines
- The object that is to be controlled is called as the **plant**. The reasons for using control systems are :
  1. To attain a satisfactory response.
  2. To improve the performance.
- We may describe the response to be a satisfactory response if plant's output follows or tracks a specified reference value. The plant output should be maintained close to this reference value. This process is called as regulation.
- A control system is called as a **robust control system** if it has a good regulation.
- We can attain these desirable properties by using the feedback control system as shown in Fig. 2.8.3. This system is also called as closed-loop control system.



(M-4) Fig. 2.8.3 : A feedback control system

- The plant is described by mathematical model which produces the output y(t) in response to plant input c(t) and external disturbance v(t).
- The sensors are included in the feedback path and they convert the plant output y(t) into the feedback signal r(t) which is generally an electrical signal.
- The feedback signal is compared with the reference signal x(t) and an error signal e(t) which is equal to the difference between x(t) and r(t) is produced.

$$\therefore \quad e(t) = x(t) - r(t)$$

- The error signal is then applied to the controller which generates a control voltage c(t). The control voltage c(t) then controls the plant.
- The control system of Fig. 2.8.3 has a single input and single output and it is known as a single input/single output (SISO) system.
- A control system with multiple inputs and outputs is known as a Multiple Input/Multiple Output (MIMO) system.
- The controller is in the form of a digital computer or microprocessor. Then the system is known as a digital control system.
- The digital control system is becoming more and more popular because of the following advantages :
  1. Flexibility 2. High accuracy.
- The digital control system uses the operations such as sampling, quantization and coding, stated earlier.

2.9 Solved Examples :

Ex. 2.9.1 : Determine whether the system described by : $y(t) = e^{tx(t)}$ is

1. Memoryless
2. Time invariant
3. Linear
4. Causal
5. Invertible
6. Stable

Give proper justification.

Soln. :

1. **Memoryless :** This is memoryless system because past samples are not present.
2. **Time invariant :**

Step 1 : Delay input by 'k' units and denote output by y (t, k).

$$\therefore \quad y(t, k) = e^{tx(t-k)}$$

Step 2 : Replace 't' by (t – k) throughout the equation.

$$\therefore \quad y(t-k) = e^{(t-k)\cdot x(t-k)}$$

Step 3 : Since $y(t, k) \neq y(t-k)$ the system is time variant.

3. **Linear :**

Step 1 : Apply zero input then output is,

$$y(t) = e^0 = 1$$

Step 2 : Consider two inputs $x_1(t)$ and $x_2(t)$.

$$x_1(t) \xrightarrow{T} y_1(t) = e^{tx_1(t)}$$

$$x_2(t) \xrightarrow{T} y_2(t) = e^{tx_2(t)}$$

$$\therefore \quad y'(t) = y_1(t) + y_2(t) = e^{tx_1(t)} + e^{tx_2(t)}$$

Step 3 :

$$[x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = e^{t[x_1(t) + x_2(t)]}$$

$$\therefore \quad y''(t) = e^{tx_1(t)} \cdot e^{tx_2(t)}$$

Step 4 : Since $y'(t) \neq y''(t)$; the system is nonlinear.

4. **Causal :** The system does not contain any future term; so it is causal system.
5. **Invariable :** The function of system is to multiply input by 't' and then take exponential value of input. This is invertible system; because invertible system is the system in which we can recover the input.
6. **Stable :** As long as an input is bounded, the output of system is always bounded. Thus it is stable system.

Ex. 2.9.2 : Determine whether the following systems are :

1. Linear
2. Memoryless
3. Stable
4. Causal
5. Time invariant
   (a) $y(t) = x(0.5\,t)$
   (b) $y(n) = \log_{10}(|x[n]|)$

Soln. :

(a) **$y(t) = x(0.5\,t)$:**

1. Let $x(t) = x_1(t) + x_2(t)$

$$x_1(t) \xrightarrow{T} x_1(0.5\,t)$$

$$x_2(t) \xrightarrow{T} x_2(0.5\,t)$$

$$\therefore \quad y'(t) = x_1(0.5\,t) + x_2(0.5\,t)$$

Now,

$$[x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = x_1(0.5\,t) + x_2(0.5\,t)$$

$$y'(t) = y''(t)$$

Thus it is linear system.

2. Put $t = 2 \Rightarrow y(2) = x(1)$:

That means at t = 2 it needs past input so it requires memory.

3. For any bounded input it produces bounded output; so it is stable system.
4. It is causal system.
5. $y(t, k) = x[0.5(t-k)]$
   $y(t-k) = x[0.5(t-k)]$
   so it is time invariant.

(b) **$y(n) = \log_{10}(|x(n)|)$:**

1. Let $x(n) = x_1(n) + x_2(t)$

$$x_1(n) \xrightarrow{T} \log_{10}(|x_1(n)|)$$

$$x_2(n) \xrightarrow{T} \log_{10}(|x_2(n)|)$$

$$y'(n) = \log(|x_1(n)|) + \log(|x_2(n)|)$$

Now, $[x_1(n) + x_2(n)] \xrightarrow{T} \log_{10}|[x_1(n) + x_2(n)]| = y''(n)$

Since $y'(n) \neq y''(n)$; the system is non-linear.

2. It is memoryless system; because output at any instant depends on input at that instant.
3. Output depends on present input only; so system is causal.
4. The system is stable.
5. $y(n, k) = \log_{10}|x(n-k)|$

and $y(n-k) = \log|x(n-k)|$

Since $y(n, k) = y(n-k)$; it is time invariant system.

Ex. 2.9.3 : Check whether the following systems are :

1. Static/dynamic 2. Causal/non-causal
3. Stable/unstable
4. Time invariant/time variant
   1. $y(t) = x(t + 10) + x^2(t)$
   2. $y[n] = \frac{1}{3}\{x[n] + x[n-1] + x[n-2]\}$.

Soln. :

1. **$y(t) = x(t + 10) + x^2(t)$:**
   1. It contains future input term ; so it is dynamic system.
   2. It has future input, so it is non-causal.
   3. For any finite input, output is always finite ; so it is stable system.

Step 1 : Delay input by k positions :

$$y(t, k) = x(t-k+10) + x^2(t-k)$$

Step 2 : Replace 't' by t-k in the given equation :

$$\therefore \quad y(t-k) = x(t-k+10) + x^2(t-k)$$

Step 3 : Since $y(t, k) = y(t-k)$; it is time in variant system.

2. **$y(n) = \frac{1}{3}\{x(n) + x(n-1) + x(n-2)\}$:**
   1. It contains past input terms ; so it is dynamic system.
   2. It does not contain any future input term ; so it is causal system.
   3. For finite input, the output is always finite. So it is stable system.

Step 1 : Delay input by k positions :

$$y(n, k) = \frac{1}{3}\{x(n-k) + x(n-k-1) + x(n-k-2)\}$$

Step 2 : Replace n by n-k in the given equation :

$$y(n-k) = \frac{1}{3}\{x(n-k) + x(n-k-1) + x(n-k-2)\}$$

Step 3 : Since $y(n, k) = y(n-k)$; it is time invariant system.

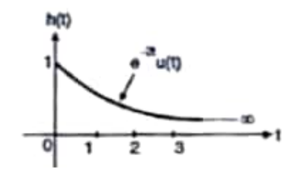
Ex. 2.9.4 : For each of the following impulse responses determine whether corresponding system is memoryless, causal and stable, justify answer. $h(t) = e^{-2t} \cdot u(t)$.

Soln. :

1. Given $h(t) = e^{-2t}u(t)$

This is a positive sided, decaying exponential signal as shown in Fig. P. 2.9.4.

It contains present and past samples, so it is not memoryless system.



(M-May) Fig. P. 2.9.4

2. The condition of causality is,

$$h(t) = 0 \text{ for } t < 0$$

This signal is only at positive side as shown in Fig. P. 2.9.4. So it is causal system.

3. According to the condition of stability,

$$S = \int_{-\infty}^{\infty} |h(t)| \cdot dt < \infty$$

Given $h(t) = e^{-2t}u(t)$

$$S = \int_0^{\infty} |e^{-2t}| dt = \left[\frac{e^{-2t}}{2}\right]_0^{\infty} = \left(-\frac{1}{2}\right)[e^{-\infty} - e^0]$$

$$\therefore \quad S = \frac{1}{2} < \infty$$

So it is a stable system.

Ex. 2.9.5 : Classify the following systems for linearity and time invariance

1. $y(n) = x(n) - x(n-1) + x(n+1)$
2. $y''(t) + 3y'(t) = 2x'(t) + x(t)$

May 04. 4 Marks

Soln. :

1. $y(n) = x(n) - x(n-1) + x(n+1)$

(A) Linearity :

Step 1 : **Apply zero input to the system. Then output $y(n) = 0$:**

Step 2 : **Apply individual input to the system :**

$$x_1(n) \xrightarrow{T} x_1(n) - x_1(n-1) + x_1(n+1)$$

$$\text{and } x_2(n) \xrightarrow{T} x_2(n) - x_2(n-1) + x_2(n+1)$$

$$\therefore y'(n) = x_1(n) - x_1(n-1) + x_1(n+1) + x_2(n) - x_2(n-1) + x_2(n+1)$$

Step 3 : Combine both inputs and apply to the system

$$[x_1(n) + x_2(n)] \xrightarrow{T} [x_1(n) + x_2(n)] - [x_1(n-1) + x_2(n-1)] + [x_1(n+1) + x_2(n+1)]$$

$$\therefore y''(n) = x_1(n) - x_1(n-1) + x_1(n+1) + x_2(n) - x_2(n-1) + x_2(n+1)$$

Step 4 : Since $y'(n) = y''(n)$; this system is linear

(B) Time invarience :

Step 1 : **Delay input by 'k' samples and denote output by $y(n, k)$:**

$$\therefore y(n, k) = x(n-k) - x(n-k-1) + x(n-k+1)$$

Step 2 : Replace 'n' by n – k throughout the equation :

$$\therefore y(n-k) = x(n-k) - x(n-k-1) + x(n-k+1)$$

Step 3 : Since $y(n, k) = y(n-k)$; the system is time invariant

2. Please refer Ex. 2.3.3.

Ex. 2.9.6 : Classify the systems as static/dynamic, linear/non-linear, Time varient/invarient, causal/non-causal.

1. $y(t) = x(2t)$.
2. $y(n) = x(n-1) + x(n)$.

Dec. 05, 6 Marks

Soln. :

1. **$y(t) = x(2t)$:**

1. **Static/dynamic :**

Here y (t) depends on present input so it is static system.

2. **Linear/non-linear :**

$$x_1(t) \to y_1(t) = x_1(2t)$$
$$x_2(t) \to y_2(t) = x_2(2t)$$
$$y'(t) = x_1(2t) + x_2(2t)$$
$$[x_1(t) + x_2(t)] \to x_1(2t) + x_2(2t)$$
$$\therefore y''(t) = x_1(2t) + x_2(2t)$$

Observing these equations

$$y'(t) = y''(t)$$

The system is linear.

3. **Time varient/invarient :**

$$y(t) = x(2t)$$
$$y(t, k) = x[2(t-k)]$$
$$y(t-k) = x[2(t-k)]$$

Since $y(t, k) = y(t-k)$ the system is time varient.

4. **Causal/non-causal :**

Since output depends on present input only the system is causal.

2. **$y(n) = x(n-1) + x(n)$:**

1. **Static/dynamic :**

Output y (n) depends on previous input so it is dynamic system.

2. **Linear/non-linear :**

Applying zero input to system

$$y(n) = x(-1) + 0$$
$$x_1(n) \to y_1(n) = x_1(n-1) + x_1(n)$$
$$x_2 \to y_2(n) = x_2(n-1) + x_2(n)$$
$$y'(n) = y_1(n) + y_2(n)$$
$$y'(n) = x_1(n-1) + x_1(n) + x_2(n-1) + x_2(n)$$
$$[x_1(n) + x_2(n)] \to [x_1(n-1) + x_1(n) + x_2(n-1) + x_2(n)]$$
$$y''(n) = x_1(n-1) + x_1(n) + x_2(n-1) + x_2(n)$$

As $y'(n) = y''(n)$

System is linear.

3. **Shift varient/invarient :**

$$y(n, n_0) = x(n-k-1) + x(n-k)$$
$$y(n-n_0) = x(n-k-1) + x(n-k)$$

As $y(n, n_0) = y(n-n_0)$ the system is time invarient.

4. **Causal/non-causal :**

$$y(n) = x(n-1) + x(n)$$

Since output depends on previous input system is non-causal.

Ex. 2.9.7 : For each of the following input-output relationships, determine whether the corresponding system is linear, time invariant or both.

1. $y(t) = t^2 x(t-1)$
2. $y(n) = x(n+1) - x(n-1)$

May 07, 4 Marks

Soln. :

1. **$y(t) = t^2 x(t-1)$:**

(A) Linearity :

Step 1 : Applying zero input to the system. Then output $y(t) = 0$.

Step 2 :

Considering there are two inputs $x_1(t-1)$ and $x_2(t-1)$. Then if applying individual input to the system; it satisfies following equations,

$$y_1(t) = t^2 x_1(t-1) \quad ...(1)$$

$$y_2(t) = t^2 x_2(t-1) \quad ...(2)$$

Step 3 : The linear combinations of Equations (1) and (2) is,

$$y'(t) = y_1(t) + y_2(t) = t^2 x_1(t-1) + t^2 x_2(t-1)$$
$$y'(t) = t^2 [x_1(t-1) + x_2(t-1)] \quad ...(3)$$
$$y''(t) = t^2 [x_1(t-1) + x_2(t-1)] \quad ...(4)$$

Step 4 : Comparing Equation (3) and Equation (4),

$y'(t) = y''(t)$; the system is **linear.**

(B) Time Invariance :

Given : $y(t) = t^2 x(t-1)$

Step 1 : Delaying input k units and denoting this output by $y(t, k)$.

$$y(t, k) = t^2 x(t-k-1)$$

Step 2 : Replacing t by (t – k) throughout the equation,

$$y(t-k) = (t-k)^2 x(t-k-1)$$

Step 3 : Since $y(t, k) \neq y(t-k)$; the system is time variant

2. **$y(n) = x(n+1) - x(n-1)$:**

(A) Linearity :

Step 1 : Applying zero input to the system. Then output $y(n) = 0$

Step 2 : Applying individual input to the system,

$$x_1(n) \xrightarrow{T} x_1(n+1) - x_1(n-1)$$

$$\text{and } x_2(n) \xrightarrow{T} x_2(n+1) - x_2(n-1)$$

$$\therefore y'(n) = x_1(n+1) - x_1(n-1) + x_2(n+1) - x_2(n-1)$$

Step 3 : Combining both inputs and applying to the system,

$$[x_1(n) + x_2(n)] \xrightarrow{T} [x_1(n+1) + x_2(n+1)] - [x_1(n-1) + x_2(n-1)]$$

$$\therefore y''(n) = x_1(n+1) - x_1(n-1) + x_2(n+1) - x_2(n-1)$$

Step 4 : Since $y'(n) = y''(n)$; the system is linear.

(B) Time Invariance :

Given : $y(n) = x(n+1) - x(n-1)$

Step 1 : **Delaying input by 'k' samples and denoting output by $y(n, k)$:**

$$\therefore y(n, k) = x(n-k+1) - x(n-k-1)$$

Step 2 : **Replacing 'n' by (n – k) throughout the equation :**

$$\therefore y(n-k) = x(n-k+1) - x(n-k-1)$$

Step 3 : Since $y(n, k) = y(n-k)$; the system is time invariant.

Ex. 2.9.8 : Check whether following systems are linear or nonlinear, time-invariant or time variant, causal or non causal.

1. $y(t) = x(t).\cos 100\pi t$
2. $y(n) = x(n) + n \cdot x[n+1]$

Dec. 11, 4 Marks

Soln. :

1. **$y(t) = x(t).\cos 100\pi t$:**

1. **Static or dynamic :**

In the given equation output depends to present input only. **So it is static system.**

2. **Linear or non-linear :**

Step 1 : **Apply zero input then output $y(t) = 0$.**

So system is initially relaxed.

Step 2 : **Apply each input separately to the system.**

$$\therefore x_1(t) \xrightarrow{T} x_1(t)\cos\pi t$$

$$\text{and } x_2(t) \xrightarrow{T} x_2(t)\cos\pi t$$

$$\therefore y'(t) = x_1(t)\cos\pi t + x_2(t)\cos\pi t$$

Step 3 : **Combine both inputs and apply it to the system.**

$$\therefore [x_1(t) + x_2(t)] \xrightarrow{T} x_1(t)\cos\pi t + x_2(t)\cos\pi t = y''(t)$$

Step 4 : Since $y'(t) = y''(t)$; it is linear system.

3. **Shift varient or shift invarient :**

Step 1 : **Delay input by 't0' units and denote output by $y(t, k)$:**

$$\therefore y(t, t_0) = x(t-t_0)\cos 100\pi t$$

Step 2 : **Replace t by t – t0 throughout the system.**

$$\therefore y(t-t_0) = x(t-t_0)\cos 100\pi(t-k)$$

Step 3 : **Since $y(t, t_0) \neq y(t-t_0)$; the system is shift varient.**

4. Here output depends on present input; thus **the system is causal system.**

5. Here equation of output contains cosine function. The maximum value of cosine function is 1. So if we apply bounded input then output is always bounded. **Thus it is stable system.**

So this system is static, linear, time varient, causal and stable.

2. **$y(n) = x(n) + n \cdot x[n+1]$:**

Linearity :

Step 1 : When input $x(n) = 0$, $y(n) = 0$

So the system is initially relaxed.

Step 2 : Let $x(n) = x_1(n) + x_2(n)$

$$x_1(n) \xrightarrow{T} x_1(n) + n\,x_1(n+1)$$

$$x_2(n) \xrightarrow{T} x_2(n) + 2\,x_2(n+1)$$

$$\therefore y'(n) = x_1(n) + n\,x_1(n+1) + x_2(n) + n\,x_2(n+1)$$
$$= x_1(n) + x_2(n) + n[x_1(n+1) + x_2(n+1)]$$

Step 3 : $[x_1(n) + x_2(n)] \xrightarrow{T} y''(n) = x_1(n) + x_2(n) + n[x_1(n+1) + x_2(n+1)]$:

Step 4 : **Since $y'(n) = y''(n)$; the system is linear :**

Time variance :

Step 1 : **Delay input by K positions :**

$$\therefore y(n, K) = x_1(n-K) + n\,x_2(n-K)$$

Step 2 : **Replace n by n – K :**

$$\therefore y(n-K) = x_1(n-K) + (n-K)\,x_2(n-K)$$

Step 3 : **Since $y(n, K) \neq y(n-K)$:**

The system is time invariance.

Causality : Since the equation contains advanced term; it is non-causal system.

Ex. 2.9.9 : For the following system, determine whether it is :

1. Memoryless, 2. Causal,
3. Linear, 4. Time-invariant

$y(n) = x[n^2]$.

Dec. 14, 4 Marks

Soln. :

1. **Memoryless :** Since equation contains future input term ; it is memoryless system.
2. **Causal :** Due to future input, it is non-causal system.
3. **Linear :**

Step 1 : Apply zero input then $y(n) = 0$

So, system is initially relaxed.

Step II : Let, $x(n) = x_1(n) + x_2(n)$

$$x_1(n) \xrightarrow{T} x_1(n^2)$$

$$x_2(n) \xrightarrow{T} x_2(n^2)$$

$$\therefore\ y'(n) = x_1(n^2) + x_2(n^2)$$

Step III : $[x_1(n) + x_2(n)] \xrightarrow{T} y''(n) = [x_1(n^2) + x_2(n^2)]$

Step IV : Since $y'(n) = y''(n)$; system is linear.

4. **Time-invariant :**

Step I : $y(n, k) = x[(n-k^2)]$

Step II : $y(n-k) = x[(n-k^2)]$

Step III : Since $y(n, k) = y(n-k)$; it is time invariant system.

Ex. 2.9.10 : For the given system, determine whether it is :
1. Memoryless 2. Causal
3. Linear 4. Time-invariant

$y[n] = x[-n]$ **May 15. 4 Marks**

Soln. :

1. **Memoryless :**

$y(n) = x(-n)$

We know that $x(-n)$ means folded input signal. Now we will put different values of n

At $n = 0, y(0) = x(-0) = x(0)$

At $n = 1, y(1) = x(-1)$

At $n = 2, y(2) = x(-2)$

Here $x(-1), x(-2)$ ...etc. indicates past input sample. So the system is dynamic.

2. **Causal :**

$y(n) = x(-n)$

For $n = -1$ we get, $y(-1) = x(2)$

That means it needs future input. So it is non-causal system.

3. **Linear :**

Step I : Apply zero input to the system; it produces zero output. Now we will check the remaining conditions.

Step II : Consider two inputs $x_1(n)$ and $x_2(n)$. Apply these inputs separately to the system.

$$\therefore\ x_1(n) \xrightarrow{T} y_1(n) = x_1(-n)$$

$$\text{and}\ \ x_2(n) \xrightarrow{T} y_2(n) = x_2(-n)$$

Add two outputs $y_1(n)$ and $y_2(n)$ to get $y'(n)$.

$$\therefore\ y'(n) = x_1(-n) + x_2(-n) \quad ...(1)$$

Step III : Combine two inputs and apply it to the system.

$$\therefore\ [x_1(n) + x_2(n)] \xrightarrow{T} x_1(-n) + x_2(-n)$$

$$\therefore\ y''(n) = x_1(-n) + x_2(n) \quad ...(2)$$

Step IV : From Equations (1) and (2) we observe that $y'(n) = y''(n)$;

The given system is linear.

4. **Time-invariant :**

Step I : Delay input by 'k' units and denote this output by $y(n, k)$

$$\therefore\ y(n, k) = x(-n-k) \quad ...(3)$$

Note that only 'n' should not be replaced by $n - k$. This is because we want to delay the whole input $x(n)$ by 'k' units. Thus $x(-n)$ should be delayed by k samples.

Step II : Replace 'n' by '$n - k$' throughout the equation

$$\therefore\ y(n-k) = x[-(n-k)] = x(-n+k) \quad ...(4)$$

Step III : Compare Equations (3) and (4)

Since $y(n, k) \neq y(n-k)$; the system Time Varient (TV) system.

Ex. 2.9.11 : For the given system, determine whether it is : 1. Memory less, 2. Causal, 3. Time-invariant $y[n] = nx[n]$. **Dec. 15. 5 Marks**

Soln. :

Given : $y(n) = n\,x(n)$

1. It does not contain any past input. So, it is memoryless.
2. It contains only present input. So, it is causal.
3.

Step 1 : Delay input by K positions.

$$\therefore\ y(n, K) = n\,x(n-K) \quad ...(1)$$

Step 2 : Replace every n by $n - K$ in the given equation.

$$\therefore\ y(n-K) = (n-K)\,x(n-K) \quad ...(2)$$

Step 3 : Since $y(n, K) \neq y(n-K)$; It is time variant.

Ex. 2.9.12 : Determine if the following system is memoryless, causal, linear, time invariant. $y(t) = x^2(t - t_0) + 2$ **May 16. 4 Marks**

Soln. :

Given : $y(t) = x^2(t - t_0) + 2$

1. It contains delayed term, so it is not memoryless system.
2. It does not contain future term ; so it is causal system.
3.

Step 1 : Apply zero input

$$\therefore\ y(t) = 2$$

So, system is initially non-relaxed.

Step 2 : Let $x(t) = x_1(t) + x_2(t)$

$$x_1(t) \xrightarrow{T} x_1^2(t - t_0) + 2$$

$$x_2(t) \xrightarrow{T} x_2^2(t - t_0) + 2$$

$$\therefore\ y'(t) = x_1^2(t - t_0) + x_2^2(t - t_0) + 4$$

Step 3 : $[x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = [x_1(t - t_0) + x_2(t - t_0)]^2 + 2$

Step 4 : Since, $y'(t) \neq y''(t)$; it is non-linear system.

Review Questions

Q. 1 Define system and state its types.

Q. 2 Define the CT and DT systems.

Q. 3 Classify CT and DT systems.

Q. 4 Define linear and nonlinear systems.

Q. 5 What do you understand by time variant and time invariant systems ? Give examples.

Q. 6 Define causal and noncausal systems.

Q. 7 What is the condition for causality in terms of impulse response of the system ?

Q. 8 Define the stable and unsatable systems.

Q. 9 What is a bounded signal ? What is BIBO stability ?

Q. 10 Define static and dynamic systems and state their examples.

Q. 11 Compare :

1. time variant / invariant systems
2. stable / unstable systems.

Q. 12 Explain the concept of invertibility.

2.10 Examples for Practice :

Ex. 2.10.1 : Check whether the following discrete time system is linear or not :

$$y(n) = x(n) \cos \omega_0 n.$$

Ans. : Linear

Ex. 2.10.2 : Determine whether the following system is linear or not :

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$$

Ans. : Linear

Ex. 2.10.3 : $y(n) = n\,x(n)$ check for the linearity.

Ans. : Linear

Ex. 2.10.4 : $y(n) = x(n - n_0)$, n_0 is constant. Check for the linearity.

Ans. : Linear

Ex. 2.10.5 : $y(n) = x(n) \cdot n(n - n_0)$ check for the linearity.

Ans. : Linear

Ex. 2.10.6 : Examine the following system for linearity,

$$y(n) = x(-n)$$

Ans. : Linear

Ex. 2.10.7 : Show the following system is shift variant :

$$y(n) = n\,x^2(n)$$

Ans. : Shift variant

Ex. 2.10.8 : Find out if the following systems are shift invariant or not :

1. $y(n) = x(n - n_0)$, n_0 is constant
2. $y(n) = ax(n) + b$, a and b are constants
3. $y(n) = x(n)\,n(n - n_0)$, n_0 is constant.

Ans. : 1. Shift invariant 2. Shift invariant 3. Shift variant

Ex. 2.10.9 : Determine whether the following system is shift invariant or not :

$$y(n) = x(n) \cos(\omega_0 n)$$

Ans. : Shift variant

Ex. 2.10.10 : Determine whether the following system is time invariant or not :

$$T[x(n)] = ax(n) + 6$$

Ans. : Time invariant

Ex. 2.10.11 : Determine whether the following system is time invariant or not :

$$y(n) = |x(n)|$$

Ans. : Time invariant

Ex. 2.10.12 : Determine whether the following systems are time invariant or not :

1. $y(n) = g(n)\,x(n)$
2. $y(n) = x(n) + 3u(n+1)$

Ans. : 1. Time variant 2. Time variant

Ex. 2.10.13 : Determine whether the following system is time invariant or not :

$$y(n) = \frac{1}{3}[x(n) + x(n-1) + x(n-2)]$$

Ans. : Time invariant

Ex. 2.10.14 : Determine if the following systems described by :

1. $y(t) = \sin[x(t+2)]$
2. $y[n] = x[2-n]$

are memoryless, causal, linear, time invariant, stable.

Ans. :

1. **Static, non-causal, TIV, non linear, stable**
2. **Dynamic, non-causal, T.V., linear, stable**

Ex. 2.10.15 : 1. Show that a system with excitation x (t) and response y (t) described by

$y(t) = x(t/2)$ is linear, time variant, non-causal.

Ex. 2.10.16 : Show that the moving average system described by input-output relation :

$$y(n) = \frac{1}{3}[x(n+1) + x(n) + x(n-1)]$$ is BIBO stable.

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Ex. 2.10.17 : Check whether the following CT systems are time invariant :

1. $y(t) = \sin x(t)$
2. $y[n] = x[-n]$
3. $y(t) = x(t) \cos 200 \pi t$
4. $y[n] = x[n] - x[n-1]$

Ans. : 1. T.V. 2. T.V. 3. T.V. 4. TIV

2.11 Solved University Examples :

Ex. 2.11.1 : Determine if the following is memoryless, causal, linear, time invariant $y(t) = t\,x(t)$ **Dec. 16. 4 Marks**

Soln. :

Given, $y(t) = t\,x(t)$

(i) It does not contain any past input sample. So, it is memoryless system.

(ii) It contains only present input; so it is causal system.

(iii) Linearity :

Step I : If input $x(t) = 0$ then $y(t) = 0$

So, the system is initially relaxed.

Step II : Let $x(t) = x_1(t) + x_2(t)$

$$x_1(t) \xrightarrow{T} t\,x_1(t)$$

$$x_2(t) \xrightarrow{T} t\,x_2(t)$$

$$\therefore\ y'(t) = t\,x_1(t) + t\,x_2(t)$$

Step III : $[x_1(t) + x_2(t)] \xrightarrow{T} y''(t) = t\,[x_1(t) + x_2(t)]$

$$\therefore\ y''(t) = t\,x_1(t) + t\,x_2(t)$$

Step IV : Since $y'(t) = y''(t)$; it is linear system

(iv) Time variance :

Step I : Delay input by k positions,

$$\therefore\ y(t, k) = t\,x(t-k)$$

Step II : Replace n by n – k in the system equation,

$$\therefore\ y(t-k) = (t-k)\,x(t-k)$$

Step III : $y(t, k) \neq y(t-k)$; so it is time variant system.

□□□

CHAPTER 3 — Module 2

Linear Time Invarient (LTI) Systems

Syllabus :

Representation of systems using differential/difference equation. Impulse, step and exponential response, system stability.

Use of convolution integral and convolution sum for analysis of LTI systems, properties of convolution integral/sum. Impulse response of interconnected systems.

3.1 Representation of Systems Using Differential or Difference Equations :

We will study representation of CT and DT system using differential equations separately.

3.1.1 CT System Representation :

The differential equations are used to represent a continuous time system. The constant coefficient difference equation for LTI system is given by,

$$\sum_{k=0}^{N} a_k \cdot \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} \cdot x(t) \qquad ...(3.1.1)$$

Here a_k and b_k are coefficients of difference equation.

Zero input response :

This is also called as natural response of a system. It is the response of a system when input is not applied. That means it is response of a system due to initial conditions only.

Thus putting input $x(t) = 0$ in Equation (3.1.1) we get,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = 0 \qquad ...(3.1.2)$$

Equation (3.1.2) is called as homogeneous difference equation.

The zero input response of a system is given by,

$$y_{zi}(t) = \sum_{i=1}^{N} C_i e^{r_i t} \qquad ...(3.1.3)$$

Here r_i represents roots of characteristic equation of a system. The characteristic equation is having the form,

$$\sum_{k=0}^{N} a_k r^k = 0 \qquad ...(3.1.4)$$

Here a_k represents the coefficients of equation.

Zero state response :

It is also called as a forced response. It is the response of a system due to input signal only. That means by assuming zero initial conditions. In this case input signal forces the system to produce the output so it is called as forced response.

The zero state response is denoted by $y_{zs}(t)$. It consists of following components :

1. Natural response of a system 2. A particular solution, $y_p(t)$

A particular solution is assumed to be having the same form as that of input. That means if input is u(t) then its particular solution is in the form $y_p(t) = k \cdot u(t)$ where 'k' is a constant.

Solved Examples :

Ex. 3.1.1 : Find the zero input response, zero state response and total response of the system. $2y''(t) + 3y'(t) + y(t) = u(t)$ with initial conditions,

$y(0) = -1$ and $y'(0) = 1$

Soln. : The given difference equation is,

$$2y''(t) + 3y'(t) + y(t) = u(t)$$

$$\therefore\ 2\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + y(t) = u(t) \qquad ...(1)$$

Now the characteristic equation is given by,

$$\sum_{k=0}^{N} a_k r^k = 0 \qquad ...(2)$$

Here order of given equation is N = 2.

Thus from Equations (1) and (2) we can write the characteristic equation as,

$$2r^2 + 3r + 1 = 0 \qquad ...(3)$$

Roots of Equation (3) are,

$$\therefore\ r_1 = -\frac{1}{2} \text{ and } r_2 = -1$$

Now zero input response is given by,

$$y_{zi}(t) = \sum_{i=1}^{N} C_i e^{r_i t}$$

$$\therefore \quad y_{zi}(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

$$\therefore \quad Y_{zi}(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t} \qquad ...(4)$$

Calculation of particular solution :

Given input $x(t) = u(t)$. Thus particular solution will be in the form,

$$y_p(t) = k\,u(t) \qquad ...(5)$$

We have to calculate the value of k. Thus putting $y(t) = y_p(t) = k\,u(t)$ and $x(t) = u(t)$ in Equation (1) we get,

$$2\frac{d^2}{dt^2}[k\,u(t)] + 3\frac{d}{dt}[k\,u(t)] + k\,u(t) = u(t)$$

But $u(t) = 1$

$$\therefore \quad 2\frac{d^2}{dt^2}k + 3\frac{d}{dt}k + k = 1$$

$$\therefore \quad k = 1 \qquad ...(6)$$

Putting this value in Equation (5) we get,

$$y_p(t) = u(t) \qquad ...(7)$$

Calculation of total response :

The total response is given by,

$$y(t) = y_{zi}(t) + y_p(t)$$

$$\therefore y(t) = C_1 e^{-\frac{1}{2}t} + C_2 e^{-t} + u(t) \qquad ...(8)$$

We will use initial conditions to calculate C_1 and C_2.

Thus putting $y(0) = -1$ (for $t = 0$) we get,

$$-1 = C_1 + C_2 + 1$$

$$\therefore \quad C_1 + C_2 = 2 \qquad ...(9)$$

Differentiating Equation (8) with respect to 't' we get,

$$\frac{dy(t)}{dt} = -\frac{1}{2}C_1 e^{-\frac{1}{2}t} - 1\,C_2 e^{-t}$$

Putting $t = 0$ we get,

$$\frac{d}{dt}y(0) = -\frac{1}{2}C_1 e^{0} - 1\,C_2 e^{0}$$

But we have initial condition

$$y'(0) = \frac{d}{dt}y(0) = 1$$

$$\therefore \quad 1 = -\frac{1}{2}C_1 - C_2$$

$$\therefore \quad +\frac{1}{2}C_1 + C_2 = -1 \qquad ...(10)$$

From Equation (9) we get,

$$C_1 = 2 - C_2 \qquad ...(11)$$

Putting this value in Equation (10),

$$\frac{1}{2}(2 - C_2) + C_2 = -1$$

$$\therefore \quad 1 - \frac{1}{2}C_2 + C_2 = -1$$

$$\therefore \quad \frac{1}{2}C_2 = -2, \qquad \therefore \quad C_2 = -4$$

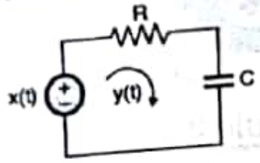
$$\text{and} \quad C_1 - 4 = 2, \qquad \therefore \quad C_1 = 6$$

Putting these values in Equation (8) we get,

$$y(t) = 6e^{-\frac{1}{2}t} - 4e^{-t} + u(t)$$

3.1.1.1 RC Circuit Described by Difference Equation :

Consider RC circuit shown in Fig. 3.1.1.



(H-265) Fig. 3.1.1

Here x (t) is input voltage applied to RC circuit. The current flowing through the loop is denoted by y (t).

Applying kVL we can write,

$$R\,y(t) + \frac{1}{c}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t) \qquad ...(3.1.5)$$

Taking differentiation of both sides we get,

$$R\frac{dy(t)}{dt} + \frac{1}{c}y(t) = \frac{dx(t)}{dt} \qquad ...(3.1.6)$$

This is the difference equation of RC circuit.

Here, N = 1, which is order of system.

Now the generalized difference equation is given by,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k}y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k}x(t)$$

In this case $N = M = 1$

$$\therefore \quad \sum_{k=0}^{1} a_k \frac{d^k}{dt^k}y(t) = \sum_{k=0}^{1} b_k \frac{d^k}{dt^k}x(t)$$

Expanding the summation terms we get,

$$a_0\,y(t) + a_1\frac{d}{dt}y(t) = b_0 + b_1\frac{d}{dt}x(t) \qquad ...(3.1.7)$$

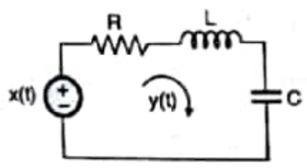
Comparing Equations (3.1.6) and (3.1.7),

$$a_0 = \frac{1}{c},\ a_1 = R,\ b_0 = 0 \text{ and } b_1 = 1.$$

These differential equations can be solved to obtain relationship between input and output.

3.1.1.2 RLC Circuit Described by Difference Equation :

Consider RLC circuit shown in Fig. 3.1.2.



(H-266) Fig. 3.1.2

Here x (t) is input signal applied to RLC circuit. The current generated in the loop is denoted by y (t).

Applying KVL to the closed loop we get,

$$R\,y(t) + L\frac{dy(t)}{dt} + \frac{1}{c}\int_{-\infty}^{t} y(\tau)\,d\tau = x(t) \qquad ...(3.1.8)$$

Differentiating both side we get,

$$R\frac{dy(t)}{dt} + L\frac{d^2y(t)}{dt^2} + \frac{1}{c}y(t) = \frac{dx(t)}{dt}$$

$$\therefore \quad L\frac{d^2y(t)}{dt^2} + R\frac{dy(t)}{dt} + \frac{1}{c}y(t) = \frac{d}{dt}x(t) \qquad ...(3.1.9)$$

This is differential equation for RLC circuit having order 2.

Thus putting N = M = 2 in the generalized difference equation we get,

$$\sum_{k=0}^{2} a_k \frac{d^k}{dt^k}y(t) = \sum_{k=0}^{2} b_k \frac{d^k}{dt^k}x(t)$$

Expanding the summation terms we get,

$$a_0\,y(t) + a_1\frac{d}{dt}y(t) + a_2\frac{d^2}{dt^2}y(t)$$

$$= b_0\,x(t) + b_1\frac{d}{dt}x(t) + b_2\frac{d^2}{dt^2}x(t) \qquad ...(3.1.10)$$

Comparing Equations (3.1.9) and (3.1.10) we get,

$$a_0 = \frac{1}{c},\ a_1 = R,\ a_2 = L$$

$$\text{and} \quad b_0 = 0,\ b_1 = 1,\ b_2 = 0.$$

We can solve the difference equation to obtain relationship between input and output.

3.1.1.3 Impulse Response of LTI System :

Impulse response is the response of a relaxed LTI system to unit impulse δ(t). It is shown in Fig. 3.1.3.

(H-267) Fig. 3.1.3

LTI system is said to be initially relaxed system if zero input produced zero output. In case of certain system; if we apply zero input then output is not zero. Some value of output is obtained. Such systems are called as non-relaxed systems.

An impulse response is useful to evaluate the zero-state response of LTI system. Consider a first order system described by the difference equation.

$$y'(t) + a\,y(t) = x(t) \qquad ...(3.1.11)$$

Its characteristic equation is,

$$s + a = 0 \qquad ...(3.1.12)$$

We are assuming initially relaxed system. Thus $x(t) = 0$ and here order of system is 1. Thus power of s is 1.

This characteristic equation produces a natural response.

$$Y_{zi}(t) = k\,e^{-at} \quad \text{for } t \geq 0 \qquad ...(3.1.13)$$

Here k is constant.

To find impulse response directly; we have to solve the difference equation.

$y'(t) + a\,y(t) = 0$, subject to zero initial conditions.

But an impulse response produces a sudden change in the state. For impulse response of first order system, note that h (0) = 1. This is because δ (t) = 1 for t = 0. Thus finding impulse response of $y'(t) + a\,y(t) = x(t)$ is then equivalent to solving homogeneous equation $y'(t) + a\,y(t) = 0$, with $y(0) = 1$.

Similarly impulse response of second order system $y''(t) + a_1 y'(t) + a_2 y(t) = x(t)$ can be calculated by solving the homogeneous equation,

$y''(t) + a_1 y'(t) + a_2 y(t) = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

Solved Examples :

Ex. 3.1.2 : Find impulse response of system described by the equation $2y'(t) + 3y(t) = x(t)$

Soln. : Given equation is the first order difference equation.

Its characteristics equation is,

$$2s + 3 = 0 \Rightarrow s + \frac{3}{2} = 0$$

$$\therefore \quad s = -\frac{3}{2} \text{ and } a = \frac{3}{2}$$

It represents the root of given system.

Now natural response for first order system is,

$$Y_{zi}(t) = h(t) = k\,e^{-at} \qquad ...(1)$$

$$\therefore \quad h(t) = k\,e^{-3/2t} \qquad ...(2)$$

But we know that h (0) = 1

Putting t = 0 in Equation (2) we get,

$$h(0) = k\,e^{0} \qquad \therefore \quad k = 1$$

Putting this value in Equation (1),

$$h(t) = e^{-3/2t} \quad \text{for } t \geq 0$$

$$\therefore \quad h(t) = e^{-\frac{3}{2}t}\,u(t)$$

This is the impulse response of system.

Ex. 3.1.3 : Find impulse response of a system described by :

$$y''(t) + 5y'(t) + 6y(t) = x(t)$$

Soln. :

Given, $y''(t) + 5y'(t) + 6y(t) = x(t)$

This is the second order difference equation.

Its characteristic equation is given by,

$$s^2 + 5s + 6 = 0 \qquad ...(1)$$

Thus roots are at,

$$s_1 = -2 \text{ and } s_2 = -3$$

$$\therefore \quad a_1 = 2 \text{ and } a_2 = 3$$

Now natural response is given by,

$$h(t) = k_1 e^{-a_1 t} + k_2 e^{-a_2 t}$$

$$\therefore \quad h(t) = k_1 e^{-2t} + k_2 e^{-3t} \qquad ...(2)$$

For second order system we have initial conditions,

$$h(0) = 0 \quad \text{and} \quad h'(0) = 1$$

Putting t = 0 in Equation (2) we get,

$$h(0) = k_1 + k_2$$

$$\therefore \quad k_1 + k_2 = 0 \qquad ...(3)$$

Differentiating Equation (2) with respect to t we get,

$$h'(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t} \qquad ...(4)$$

Putting t = 0 we get,

$$h'(t) = -2k_1 - 3k_2 \quad ...(5)$$

$$\therefore \; -2k_1 - 3k_2 = 1$$

We will solve Equations (4) and (5) to obtain values of k_1 and k_2.

From Equation (3) we get,

$$k_1 = -k_2$$

Putting this value in Equation (5),

$$+2k_2 - 3k_2 = 1 \qquad \therefore \quad k_2 = -1$$

$$\text{and} \quad k_1 = 1$$

Putting these values in Equation (2) we get,

$$h(t) = e^{-2t} - e^{-3t} \quad \text{for } t \geq 0$$

$$\therefore \; h(t) = e^{-2t} u(t) - e^{-3t} u(t)$$

This is impulse response of given system.

3.1.2 DT System Representation :

Note : We will discuss linear convolution later in this chapter.

The output of discrete time system can be calculated using equation of linear convolution.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad ...(3.1.14)$$

Here h (n) is the impulse response of system. By the use of h(n); we can calculate output y (n) for any input x (n) using Equation (3.1.14).

Now in this section we will discuss the difference equation which are useful to implement (realize) the discrete time system. These difference equations are especially used for designing the digital filter.

We discussed that impulse response h (n) is used to calculate output of discrete time system. Now if the system is causal then impulse response h (n) is zero for negative values of n. So for causal LTI system equation of convolution becomes,

$$y(n) = \sum_{k=0}^{\infty} x(k)\, h(n-k) \quad ...(3.1.15)$$

Equation (3.1.15) can also be written as,

$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \quad ...(3.1.16)$$

Since output depends on impulse response h (n); the number of samples, present in h (n) plays an important role to determine output y (n). The number of samples present in h (n) is called as length of impulse response. Depending upon the length of impulse response; there are two types of discrete time systems :

1. Finite impulse response (FIR) systems.
2. Infinite impulse response (IIR) systems.

1. Finite impulse response systems (FIR) :

As the name indicates; impulse response is having a finite length. That means h (n) contains some finite number of samples. Let the range of h (n) is from – M to + M. Thus equation of convolution for FIR system becomes,

$$y(n) = \sum_{k=-M}^{M} h(k)\, x(n-k) \quad \text{(Non-causal)} \quad ...(3.1.17)$$

Note that in Equation (3.1.17), negative terms are present, so this equation is for the non-causal system. Now for the causal FIR system we will consider only positive samples of h (n). Thus range of h (n) becomes '0' to M instead of – M to M. Equation (3.1.17) becomes,

$$y(n) = \sum_{k=0}^{M} h(k)\, x(n-k) \quad ...(3.1.18)$$

Note that the summation is taken from 0 to M. So there will be total M + 1 terms in h (n) or h (k). These terms are :

h (0), h (1), h (2) ....h (M).... Number of terms in h (n).

Now expanding Equation (3.1.18) we get,

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots + h(M)\,x(n-M) \quad ...(3.1.19)$$

In Equation (3.4.19) x (n) is present input and x (n – 1), x (n – 2) ... etc. are past inputs. Every input is multiplied by corresponding value of h (n). Thus Equation (3.1.19) indicates that output y (n) is weighted sum of inputs.

Note that in this case the system has to view input samples only from x (n) to x (n – M) and not all input samples. These input samples are most recent input samples. Thus this system has to store x (n – 1), x (n – 2) ... x (n – M), past samples in the memory. Since these are finite number of samples; **FIR has limited or finite memory requirement.**

2. Infinite impulse response (IIR) systems :

As the name indicates, impulse response h (n) contain '∞' samples. That means length of h (n) is infinite. Now the equation of convolution for non-causal IIR system becomes.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad \text{...(Non-causal)} \quad ...(3.1.20)$$

For the causal system, limits of summation will be from 0 to ∞.

$$\therefore \; y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k) \quad \text{... (Causal)} \quad ...(3.1.21)$$

Here the range of h (n) = h (k) is as follows :

h (0), h (1), h (2) ...., h(∞). So this is infinite impulse response system.

Now expanding Equation (3.1.21) we get,

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots + h(\infty)\,x(n-\infty) \quad ...(3.1.22)$$

In this case IIR system has to store all past samples from x (n – 1) to x (n – ∞). Thus **IIR system needs infinite memory.** This is practically not possible.

Note : We discussed that FIR system needs limited memory. So it is possible to design FIR systems using convolution equation. But IIR systems cannot be designed using convolution equation. This is because of the infinite memory requirement. So to implement such IIR systems; difference equations are used.

Now depending upon length of h (n) we have classified the systems as FIR and IIR. Similarly depending upon output of system; the discrete time systems are classified as follows :

1. Non-recursive systems
2. Recursive systems.

Why this type of classification is required ?

Equation of convolution indicates that for causal systems; the output depends on present and past input as well as it depends on the impulse response, h (n). But this is not the case always. There are some systems whose present output depends on present and past inputs as well as past (previous) output. For such systems we should take into account, the previous output to calculate present output. This type of classification gives the idea about the requirement of previous output to calculate present output.

1. Non-recursive systems :

Non-recursive systems do not require any past output sample to calculate the present output. Consider the equation of FIR causal system (Equation (3.1.18)). This equation is,

$$y(n) = \sum_{k=0}^{M} h(k)\, x(n-k)$$

To calculate present output we need present and past inputs only as shown by Equation (3.1.19). We will rewrite Equation (3.1.19) as follows :

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + \ldots + h(m)\,x(n-M)$$

This equation does not contain any previous output term like y (n – 1), y (n – 2) ... etc. So this is the equation of non-recursive system. Thus we can conclude that **causal FIR system is non-recursive system.** In other words we can say that non-recursive implementation of FIR causal system is possible.

Now for non-causal FIR system; the limits of summation will be from – M to M. Thus this system contain present, past and future inputs. But it does not contain future output. So this is also non-recursive system.

In case of non-recursive systems; past output terms are not required. Past (previous) output terms are obtained by taking feedback from output to the input. Thus **non-recursive system do not require any feedback.**

Is it possible to implement non-recursive IIR system ?

We know that,

Non-recursive ⇔ No feedback or no past output term present

and IIR ⇔ Infinite impulse response.

The equation of convolution for causal IIR system is,

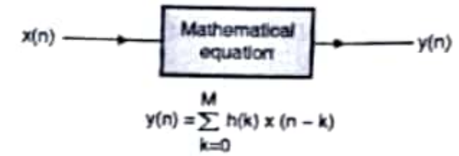
$$y(n) = \sum_{k=0}^{\infty} h(k)\, x(n-k)$$

Expanding the summation we get,

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots + h(\infty)\,x(n-\infty)$$

We have discussed earlier that such systems require to store all past inputs. That means it is required to store infinite input terms in the memory. So infinite memory is required. This is not possible practically. Thus non-recursive implementation of causal IIR system is not possible.

The representation of non-recursive system is shown in Fig. 3.1.4(a).



(H-1091) **Fig. 3.1.4(a) : Non-recursive system**

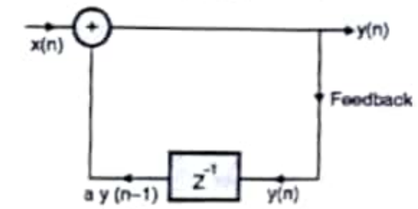
2. Recursive systems :

A discrete time system in which output y (n) depends on present input, past inputs as well as previous outputs; is called as recursive system.

Let us consider one example. Consider a system represented by block schematic as shown in Fig. 3.1.4(b).

First we will write the equation for system.

1. As shown in Fig. 3.1.4(b); a connection is drawn from output y (n) to the input side. This is a feedback.
2. This feedback signal, y (n) is passed through an unit delay (z^{-1}) block. So the output of delay block is y (n – 1). This is past output compared to y (n).
3. The signal y (n – 1) is multiplied by constant multiplier 'a' to obtain ay (n – 1).
4. An adder is used to add input x (n) and signal ay (n – 1).
5. Now the addition of x (n) and a y (n – 1) produces the output.



(H-1092) **Fig. 3.1.4(b) : Example of recursive system**

Thus the output of system is expressed as,

$$y(n) = x(n) + ay(n-1) \quad ...(3.1.23)$$

At instant n = 0 we get,

$$y(0) = x(0) + ay(-1)$$

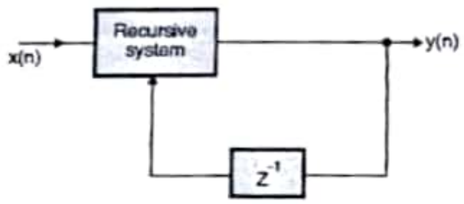
Here y (0) ⇒ present output

x (0) ⇒ present input

and y (– 1) ⇒ past output

Similarly at n = 1 we get,

$$y(1) = x(1) + ay(0)$$



(H-1093) **Fig. 3.1.4(c) : Representation of recursive system**

Thus y (0) is past output.

So at any instant present output depends on past output. Hence it is recursive system. The representation of recursive system is shown in Fig. 3.1.4(c).

3.1.2.1 LTI Systems Characterized by Constant Coefficient Difference Equations :

In the previous section, we discussed that, the behaviour of system depends on its impulse response. In this subsection we will discuss how LTI systems are characterized by input-output relations; called as difference equation with constant coefficients.

Consider a recursive system shown in Fig. 3.1.4(b). The equation of output is,

$$y(n) = x(n) + ay(n-1) \quad ...(3.1.24)$$

This is first order difference equation. Here 'a' is a constant coefficient. Now output y (n) is calculated by putting different values of n in Equation (3.1.24).

$$\text{For } n = 0 \Rightarrow y(0) = x(0) + ay(-1) \quad ...(1)$$

$$\text{For } n = 1 \Rightarrow y(1) = x(1) + ay(0) \quad ...(2)$$

Putting value of y (0) from Equation (1) we get,

$$y(1) = x(1) + a[x(0) + ay(-1)]$$

$$\therefore \quad y(1) = x(1) + ax(0) + a^2 y(-1)$$

Rearranging the terms,

$$y(1) = a^2 y(-1) + ax(0) + x(1) \quad ...(3)$$

$$\text{For } n = 2 \Rightarrow y(2) = x(2) + ay(1) \quad ...(4)$$

Putting value of y (1) from Equation (3) we get,

$$y(2) = x(2) + a[a^2 y(-1) + ax(0) + x(1)]$$

$$y(2) = x(2) + a^3 y(-1) + a^2 x(0) + ax(1)$$

Rearranging the terms,

$$y(2) = a^3 y(-1) + a^2 x(0) + ax(1) + x(2) \quad ...(5)$$

Observe Equations (3) and (5) carefully. Now we can directly write the equation for value 'n'.

$$\therefore \quad y(n) = a^{n+1} y(-1) + a^n x(0) + a^{n-1} x(1) + ... + ax(n-1) + x(n) \quad ...(3.1.25)$$

In terms of summation Equation (3.1.25) can be written as,

$$y(n) = a^{n+1} y(-1) + \sum_{k=0}^{n} a^k x(n-k) \quad ...(3.1.26)$$

Note that we have obtained output y (n) by putting only positive values of n. That means for $n \geq 0$. So these are the equations of causal system. Every equation contains the term y (– 1). It is assumed that, we are starting the operation of system at n = 0. Then the term y (– 1) is called as **initial condition of a system.**

Important terms related to Equation (3.1.26) :

We know that Equation (3.1.26) is representing the behaviour of causal system. It consists of two parts as follows :

1. The first term, y (– 1) is called initial condition of the system.
2. The second term, is the response of system to the input x (n).

Now depending upon initial conditions, two types of responses are obtained.

1. Zero state response or forced response :

If initial condition, y (– 1) of the system is zero then the response of the system to the input signal is called as **zero state response.** It is denoted by $y_{zs}(n)$. In this case, whatever output we are getting, that is because of input x (n) only. And since the system is forced to the input; it is also called as **forced response.** Since y (– 1) = 0; the system is said to be relaxed system.

Thus putting initial condition, y (– 1) = 0 in Equation (3.1.26) we get,

$$y(n) = y_{zs}(n) = \sum_{k=0}^{n} a^k x(n-k) \quad ... \text{for } n \geq 0 \quad ...(3.1.27)$$

2. Zero input response or natural response :

As the name indicates; input is not applied to the system and initial condition y (– n) ≠ 0. Thus, this kind of output without applying input x (n) is called as **zero input response** or **natural response.** Since y (– n) ≠ 0, this is called as non-relaxed initial condition. This output is denoted by y_{zi}.

Putting x (n) = 0 in Equation (3.1.26) we get,

$$y(n) = y_{zi}(n) = a^{n+1} y(-1) \quad ...(3.1.28)$$

This equation indicates that output is not dependent on input x (n) but it depends on initial condition, y (– 1).

3. Total response :

The total response of system is obtained by adding zero state response and zero input response.

$$\therefore \text{ Total response, } y(n) = y_{zi}(n) + y_{zs}(n) \quad ...(3.1.29)$$

Equation (3.1.24) is basically a linear constant coefficient difference equation. The general form of this equation can be expressed as,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad ...(3.1.30)$$

Here a_k and b_k are constant coefficients. And 'N' is the order of difference equation. That means 'N' represents the order of system.

Stability, linearity and time invariance property of recursive systems :

We will discuss these properties of recursive systems described by constant coefficient difference equations.

1. **Stability :** A recursive system is stable when we get a bounded output by applying bounded input.
2. **Linearity :** A recursive system described by Equation (3.1.30) is said to be linear if it satisfies following conditions :
   1. The total response of system is addition of zero input response and zero state response.
      $$\therefore \quad y(n) = y_{zs}(n) + y_{zi}(n)$$
   2. Zero state response should be linear that means principle of superposition should be applicable.
   3. Zero input response should be linear that means principle of superposition should be applicable.
3. **Time invariance :** As shown by Equation (3.1.30), the output y (n) contains constant coefficients a_k and b_k. These coefficients are not affected by the change in time (shifting). So the recursive system is time invariant.

3.1.2.2 Solution of Linear Constant Coefficient Difference Equation :

In this subsection we will discuss how to obtain exploit solution of a given linear constant coefficient difference equation. Basically there are two methods to obtain this solution.

These methods are as follows :

1. Direct method 2. Indirect method

Indirect method is based on z transformation. Presently we will discuss direct method for obtaining the solution.

In the direct method; it is assumed that the solution consists of two parts :

1. Homogeneous or complementary solution. It is denoted by $y_h(n)$.
2. Particular solution. It is denoted by $y_p(n)$.

Thus the total solution is expressed as,

$$y(n) = y_h(n) + y_p(n) \quad ...(3.1.31)$$

1. Homogeneous solution of difference equation :

Recall general form of difference equation (Equation (3.1.30)),

$$\text{It is, } y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad ...(3.1.32)$$

In the first term the limits of summation are from k = 1 to N. We can change these limits from k = 0 to N as follows. Take the limits from k = 0 to N then,

$$\sum_{k=0}^{N} a_k y(n-k) = a_0 y(n) + \sum_{k=1}^{N} a_k y(n-k)$$

$$\therefore \quad -\sum_{k=1}^{N} a_k y(n-k) = a_0 y(n) - \sum_{k=0}^{N} a_k y(n-k) \quad ...(3.1.33)$$

Putting this value in Equation (3.1.32) we get,

$$y(n) = a_0 y(n) - \sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Assume $a_0 = 1$

$$\therefore \quad y(n) = y(n) - \sum_{k=0}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$\therefore \quad \sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad ...(3.1.34)$$

We will obtain homogeneous solution by assuming input x (n) = 0.

Since $x(n) = 0 \Rightarrow x(n-k) = 0$. Thus Equation (3.1.34) becomes,

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \quad ...(3.1.35)$$

We will assume that the solution is in the form of exponential.

Thus let, homogeneous solution,

$$[y_h(n)] = \lambda^n \quad ...(3.1.36)$$

Replace n by n – k $\quad \therefore \quad y_h(n-k) = \lambda^{n-k}$

Putting this value in Equation (3.1.35) we get,

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \quad ...(3.1.37)$$

Expanding the summation we get,

$$a_0 \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + ... + a_N \lambda^{n-N} = 0$$

We have assumed that $a_0 = 1$

$$\therefore \quad \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + ... + a_N \lambda^{n-N} = 0$$

Taking λ^{n-N} common we get,

$$\lambda^{n-N}[\lambda^n \cdot \lambda^{-n+N} + a_1 \lambda^{n-1} \cdot \lambda^{-n+N} + a_2 \lambda^{n-2} \cdot \lambda^{-n+N} + ... + a_N] = 0$$

$$\therefore \quad \lambda^{n-N}[\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + ... + a_N] = 0 \quad ...(3.1.38)$$

The term inside the bracket is called as characteristic polynomial of the system. This polynomial has 'N' roots denoted by $\lambda_1, \lambda_2 ... \lambda_N$. These roots can be real or complex valued.

Case I : Roots are distinct :

When the roots are distinct then we can write the solution as,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n + ... + c_N \lambda_N^n \quad ...(3.1.39)$$

Here $c_1, c_2 .... c_N$ are constants.

Case II : Multiple roots :

In case of multiple roots; the solution can be written as,

$$y_h(n) = c_1 \lambda_1^n + c_2 n \lambda_2^n + c_3 n^2 \lambda_3^n + ... + c_N n^N \lambda_N$$

Solved Examples :

Ex. 3.1.4 : Determine the homogeneous solution of the system described by :

$$y(n) - 3y(n-1) - 4y(n-2) = x(n).$$

Soln. :

Step 1 : Assume input x (n) = 0

$$\therefore \quad y(n) - 3y(n-1) - 4y(n-2) = 0 \quad ...(1)$$

Step 2 : Assume the homogeneous solution is in the form

$$y_h(n) = \lambda^n \quad ...(2)$$

Thus we get,

$$y(n-1) = \lambda^{n-1} \text{ and } y(n-2) = \lambda^{n-2}$$

Step 3 : Putting these values in Equation (1) we get,

$$\lambda^n - 3\lambda^{n-1} - 4\lambda^{n-2} = 0 \quad ...(3)$$

Taking λ^{n-2} common we get,

$$\lambda^{n-2}[\lambda^n \cdot \lambda^{-n+2} - 3\lambda^{n-1} \cdot \lambda^{-n+2} - 4] = 0$$

$$\therefore \quad \lambda^{n-2}[\lambda^2 - 3\lambda - 4] = 0$$

$$\therefore \quad \lambda^2 - 3\lambda - 4 = 0 \quad ...(4)$$

Roots of Equation (4) are,

$$\therefore \quad \lambda_1 = -1 \text{ and } \lambda_2 = 4$$

Step 4 : The homogeneous solution is given by,

$$y_h(n) = c_1 \lambda_1^n + c_2 \lambda_2^n$$

$$\therefore \quad y_h(n) = c_1(-1)^n + c_2(4)^n \quad ...(5)$$

Now we will obtain values of c_1 and c_2.

Step 5 : From Equation (5) we get,

$$\text{For } n = 0 \Rightarrow y(0) = c_1 + c_2 \quad ...(6)$$

$$\text{and for } n = 1 \Rightarrow y(1) = -c_1 + 4c_2 \quad ...(7)$$

Step 6 : Now consider original equation,

$$y(n) - 3y(n-1) - 4y(n-2) = 0$$

$$\therefore \quad y(n) = 3y(n-1) + 4y(n-2) \quad ...(8)$$

$$\therefore \quad \text{For } n = 0 \Rightarrow y(0) = 3y(-1) + 4y(-2) \quad ...(9)$$

$$\text{For } n = 1 \Rightarrow y(1) = 3y(0) + 4y(-1) \quad ...(10)$$

Putting Equation (9) in Equation (10) we get,

$$y(1) = 3[3y(-1) + 4y(-2)] + 4y(-1)$$

$$\therefore \quad y(1) = 9y(-1) + 12y(-2) + 4y(-1)$$

$$\therefore \quad y(1) = 13y(-1) + 12y(-2) \quad ...(11)$$

Step 7 : Equate Equations in step (5) and (6). That means equating Equations (6) and (9) we get,

$$c_1 + c_2 = 3y(-1) + 4y(-2) \quad ...(12)$$

And equating Equations (7) and (11) we get,

$$-c_1 + 4c_2 = 13y(-1) + 12y(-1) \quad ..(13)$$

Now we will solve Equations (12) and (13) to obtain c_1 and c_2.

Adding Equations (12) and (13) we get,

$$5c_2 = 16y(-1) + 16y(-2)$$

$$\therefore \quad c_2 = \frac{16}{5}y(-1) + \frac{16}{5}y(-2) \quad ...(14)$$

Putting this value in Equation (12) we get,

$$c_1 + \frac{16}{5}y(-1) + \frac{16}{5}y(-2) = 3y(-1) + 4y(-2)$$

$$\therefore \quad c_1 = 3y(-1) - \frac{16}{5}y(-1) + 4y(-2) - \frac{16}{5}y(-2)$$

$$\therefore \quad c_1 = -\frac{1}{5}y(-1) + \frac{4}{5}y(-2) \quad ...(15)$$

Step 8 : Putting these values in Equation (5) we get,

$$y_h(n) = \left[-\frac{1}{5}y(-1) + \frac{4}{5}y(-2)\right](-1)^n + \left[\frac{16}{5}y(-1) + \frac{16}{5}y(-2)\right](4)^n \quad ..(16)$$

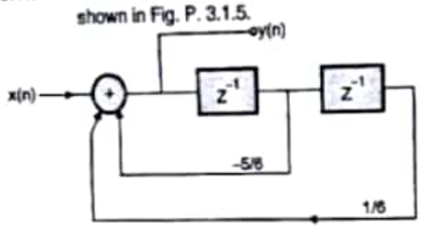
If we assume initial conditions zero that means,

$y(-1) = y(-2) = 0$ then Equation (16) becomes,

$$y_h(n) = (-1)^n + (4)^n \quad ...(17)$$

Note : To obtain this solution we have assumed input x (n) = 0. So this output is also called as zero input response.

Ex. 3.1.5 : Find the impulse response of the system shown in Fig. P. 3.1.5.



(II-271) **Fig. P. 3.1.5 : Given discrete time system**

Soln. :

Impulse response means, the response (output) of system when input $x(n) = \delta(n)$. Using homogeneous solution we can find the impulse response, assuming that the system is initially relaxed.

First we will write the difference equation from given Fig. P. 3.1.5. At the output of first delay we get $y(n-1)$ and it is multiplied by $\frac{-5}{6}$. While at the output of second delay we get $y(n-2)$ and it is multiplied by $\frac{1}{6}$. These two outputs are added with input signal x (n).

$$\therefore \quad y(n) = x(n) - \frac{5}{6}y(n-1) + \frac{1}{6}y(n-2) \quad ...(1)$$

Now we will find out the homogeneous solution.

Step 1 : Assume input $x(n) = 0$. Thus Equation (1) becomes,

$$y(n) + \frac{5}{6}y(n-1) - \frac{1}{6}y(n-2) = 0 \quad ...(2)$$

Step 2 : Assume that the homogeneous solution is in the form

$$y_h(n) = \lambda^n \quad ...(3)$$

Thus we get, $y(n-1) = \lambda^{n-1}$ and $y(n-2) = \lambda^{n-2}$

Step 3 : Putting these values in Equation (2) we get,

$$\lambda^n + \frac{5}{6}\lambda^{n-1} - \frac{1}{6}\lambda^{n-2} = 0$$

Taking λ^{n-2} common we get,

$$\lambda^{n-2}\left[\lambda^n \cdot \lambda^{-n+2} + \frac{5}{6}\lambda^{n-1} \cdot \lambda^{-n+2} - \frac{1}{6}\right] = 0$$

$$\therefore \quad \lambda^{n-2}\left[\lambda^2 + \frac{5}{6}\lambda - \frac{1}{6}\right] = 0$$

$$\therefore \quad \lambda^2 + \frac{5}{6}\lambda - \frac{1}{6} = 0 \quad ...(4)$$

Now the roots of Equation (4) are,

$$\therefore \quad \lambda_1 = \frac{1}{6} \text{ and } \lambda_2 = -1$$

Step 4 : The homogeneous solution is given by,

$$y_h(n) = c_1\lambda_1^n + c_2\lambda_2^n$$

$$\therefore \quad y_h(n) = c_1\left(\frac{1}{6}\right)^n + c_2(-1)^n$$

We will denote $y_h(n)$ by h (n)

$$\therefore \quad h(n) = c_1\left(\frac{1}{6}\right)^n + c_2(-1)^n \quad ...(5)$$

Now we will obtain values of c_1 and c_2.

Step 5 : We will apply initial conditions. In this case $x(n) = \delta(n)$ and $y(n) = h(n)$. Putting these values in Equation (1) we get,

$$h(n) = \delta(n) - \frac{5}{6}h(n-1) + \frac{1}{6}h(n-2)$$

For $n = 0 \Rightarrow h(0) = \delta(0) - \frac{5}{6}h(-1) + \frac{1}{6}h(-2)$

Here $\delta(0) = 1$ and since initial conditions are zero thus $h(-1) = 0$ and $h(-2) = 0$

$$\therefore \quad h(0) = 1 - \frac{5}{6} \times 0 + \frac{1}{6} \times 0$$

$$\therefore \quad h(0) = 1$$

For $n = 1 \Rightarrow h(1) = \delta(1) - \frac{5}{6}h(0) + \frac{1}{6}h(-1)$

Here $\delta(1) = 0$, $h(-1) = 0$ and $h(0) = 1$

$$\therefore \quad h(1) = 0 - \frac{5}{6} \times 1 + \frac{1}{6} \times 0$$

$$\therefore \quad h(1) = -\frac{5}{6}$$

Step 6 : Putting n = 0 and n = 1 in Equation (5) we get,

$$\text{For } n = 0 \Rightarrow c_1\left(\frac{1}{6}\right)^0 + c_2(-1)^0 = h(0)$$

$$\therefore \quad c_1 + c_2 = 1 \quad ...(6)$$

$$\text{For } n = 1 \Rightarrow c_1\left(\frac{1}{6}\right)^1 + c_2(-1)^1 = h(1)$$

$$\therefore \quad c_1 \cdot \frac{1}{6} - c_2 = -\frac{5}{6}$$

$$\therefore \quad \frac{1}{6}c_1 - c_2 = -\frac{5}{6} \quad ...(7)$$

We will obtain values of c_1 and c_2

From Equation (6) $c_2 = 1 - c_1$

Putting this value in Equation (7) we get,

$$\frac{1}{6}c_1 - 1 + c_1 = -\frac{5}{6}$$

$$\therefore \quad +\frac{7}{6}c_1 = -\frac{5}{6} + 1 = \frac{1}{6} \qquad \therefore \quad 7c_1 = 1$$

$$\therefore \quad c_1 = \frac{1}{7}$$

$$\text{and} \quad c_2 = 1 - c_1, \qquad \therefore \quad c_2 = 1 - \frac{1}{7} = \frac{6}{7}$$

$$\therefore \quad c_2 = \frac{6}{7}$$

Putting c_1 and c_2 in Equation (5) we get,

$$\therefore \quad h(n) = \frac{1}{7}\left(\frac{1}{6}\right)^n + \frac{6}{7}(-1)^n$$

This is impulse response of given system.

2. Particular solution of difference equation :

The particular solution is denoted by $y_p(n)$. It is calculated by assuming that the output has same from at that of input. For example if input $x(n) = u(n)$ then the output is assumed to be

$y_p(n) = k\,u(n)$; where 'k' is arbitrary constant.

Important rules to assume the solution :

1. If any term in a particular solution (y_p) is also present in homogeneous solution then multiply 'y_p' by n.
2. If such term appears 'r' times in a homogeneous solution then multiply 'y_p' by n^r.

Some commonly used input signals and the particular solution that we have to assume is shown in Table 3.1.1.

Table 3.1.1

Sr. No.	Input x (n)	Particular solution
1.	Constant term (A)	k
2.	u (n)	k u (n)
3.	$A \cdot c^n$	$k c^n$
4.	$A^n \cdot n^c$	$k_0 n^c + k_1 n^{M-1} + \ldots + k_c$
5.	$A \cos \omega_0 n$	$k_1 \cos \omega_0 n + k_2 \sin \omega_0 n$

A particular solution is basically used to calculate the zero state response (y_{zs}) of the system. The following steps are used to calculate the forced response of system :

Step 1 : Find the nature of zero input response from the roots of characteristics polynomials. This is $y_h(n)$ or y_{zi}.

Step 2 : Assume the particular solution $y_p(n)$

Step 3 : Determine the coefficients in the zero input response.

Then the forced response $y_{zs} = y_h(n) + y_p(n)$.

Solved Examples :

Ex. 3.1.6 : Find out the particular solution for the following differential equation :

$y(n) + 3y(n-1) = x(n)$

Assume $x(n) = u(n)$

Soln. : Here input $x(n) = u(n)$ = unit step

Now corresponding to this input, we will assume the particular solution as,

$$y_p(n) = k\,u(n) \quad ..(1)$$

Here k is some constant.

Now the given equation is,

$$y(n) + 3y(n-1) = x(n) \quad ...(2)$$

In Equation (1) replace n by n – 1,

$$y(n-1) = k\,u(n-1) \quad ...(3)$$

Putting Equations (1) and (3) in Equation (2) we get,

$$k\,u(n) + 3ku(n-1) = u(n) \quad ..(4)$$

Now we will calculate the value of k. We know that input is unit step, u (n). Its value is 1 for $n \geq 0$. Putting n = 1 in Equation (4) we get,

$$ku(1) + 3ku(0) = u(1)$$

Remember that while putting value of 'n' any term should not vanish.

$$\text{But} \quad u(0) = u(1) = 1$$

$$\therefore \quad k + 3k = 1$$

$$\therefore \quad 4k = 1 \Rightarrow k = \frac{1}{4}$$

Putting this value in Equation (1) we get,

$$y_p(n) = \frac{1}{4}u(n)$$

Ex. 3.1.7 : Determine the particular solution of the difference equation :

$y(n) = 3y(n-1) - 4y(n-2) + x(n)$

When the forcing function $x(n) = 2^n$, $n \geq 0$ and zero everywhere.

Soln. : Refer the table; for input $x(n) = 3^n$ we will assume the particular solution,

$$y_p(n) = k\,2^n \quad ...(1)$$

Now the given input $x(n) = 2^n$ is present only for positive value of 'n' ($n \geq 0$). Thus we can write,

$$x(n) = 2^n u(n) \quad ...(2)$$

Accordingly Equation (1) is modified as,

$$y_p(n) = k\,2^n u(n) \quad ..(3)$$

From Equation (3) we can write,

$$y(n-1) = k\,2^{n-1}u(n-1) \text{ and } y(n-2) = k\,2^{n-2}u(n-2)$$

Putting these values in the given equation of y (n) we get,

$$k\,2^n u(n) = 3k\,2^{n-1}u(n-1) - 4k\,2^{n-2}u(n-2) + 2^n u(n) \quad ...(4)$$

Now we will calculate value of k. Given input is present for $n \geq 0$. But while putting value of 'n' any term in equation should not vanish. For example if we will put n = 1 then the second term of R.H.S. becomes,

$$-4k\,2^{n-2}u(n-2) \Rightarrow -4k(2)^{-1}u(-1)$$

But $u(-1) = 0$; so this value of 'n' is not allowed. Thus we will put n = 2 in Equation (4) so that any term will not vanish.

$$\therefore \quad k\,2^2 u(2) = 3k(2)^1 u(1) - 4k(2)^0 u(0) + 2^2 u(2)$$

$$\text{Here} \quad u(0) = u(1) = u(2) = 1$$

$$\therefore \quad 4k = 6k - 4k + 4$$

$$\therefore \quad 2k = 4 \Rightarrow k = 2$$

Putting this value in Equation (3) we get,

$$y_p(n) = 2.2^n u(n)$$

Note : If total solution is asked to calculate, then first obtain $y_h(n)$ and $y_p(n)$ separately. The total solution is $y(n) = y_h(n) + y_p(n)$.

3.2 Convolution Integral :

Convolution is a mathematical operation which is used as a tool by communication engineers for system analysis, probability theory and transform calculations. Convolution can be performed in time as well as frequency domain.

The convolution of two functions x(t) and y (t) is defined as,

$$\text{Convolution : } x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau)\, d\tau \qquad ...(3.2.1)$$

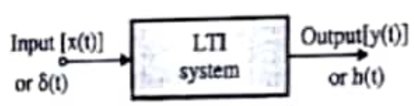
Equation (3.2.1) is called as convolution integral. Note that the independent variable is "t" which is same as the independent of the functions x (t) and y (t) which are being convolved. The integration is always performed with respect to a dummy variable such as τ and t is treated as a constant so far as the integration is concerned. The process of convolution involves following operations of y (τ) while the signal x (τ) remains unchanged :

1. Folding or time reversal to obtain y (– τ).
2. Time shifting the folded signal y (– τ) to obtain y (t – τ).
3. Multiplication of x (τ) and y (t – τ).
4. Integration of the product term x (τ) · y (t – τ).

Proof : Consider an LTI continuous time system with

h (t) = impulse response of the system
x (t) = input signal
y (t) = output signal

Writing the impulse response of the system in progressive form, i.e. Impulse response h (t) of an LTI system is defined to be response of system to unit impulse δ (t) [δ (t) → unit impulse at t = 0]



(H-273) **Fig. 3.2.1**

By **time invarient property** of LTI system, response to impulse applied at any time "nT" [δ (t – τ) is h (t – τ)].

It indicates that if unit impulse is delayed by 'τ' then impulse response is also delayed by the same amount.

According to the definition of linear convolution,

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(t-\tau)\, d\tau$$

In this case we can write,

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Here h (t – τ) is response of LTI system to input δ (t).

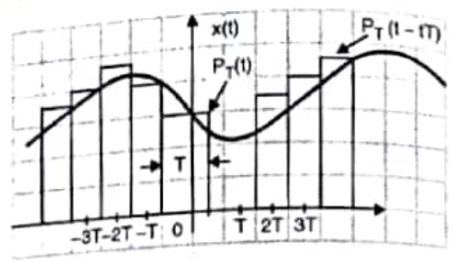
$$\therefore \quad Y(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

3.2.1 Representation of C.T. Signal in terms of Impulses :

Consider a signal x (t) which will be first approximated as a set of pulse of duration T as shown in Fig. 3.2.2. The function can be then expressed as,

$$x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, P_T(t-tT)$$

Where x (tT) = Is weight of signal, (t is an integer)



(H-274) **Fig. 3.2.2**

At every value of time; weight of signal is multiplied by pulse of duration T.

Note that, this approximation is valid even for small value of T.

The above equation can be written as,

$$\therefore \quad x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, T\, [P_T(t-tT)/T]$$

$$\text{But} \quad \left[\lim_{T \to 0} P_T(t-tT)/T\right] \to \delta(t-tT)$$

∴ The signal x (t) can be expressed as :

$$\therefore \quad x(t) = \sum_{t=-\infty}^{\infty} x(tT)\, \delta(t-tT) \qquad ...(3.2.2)$$

Where, T = Pulse width; f_s = Sampling frequency = 1/T

T is very small and f_s is high compared to the frequencies present in the signal.

3.2.2 Properties of Convolution Integral :

Some of the important properties of convolution integral are given below.

(a) Commutative property :

If $x_1(t)$ and $x_2(t)$ are continuous time signals then,

$$x_1(t) * x_2(t) = x_2(t) * x_1(t) \qquad ...(3.2.3)$$

(b) Distributive property :

$$x_1(t) * [x_2(t) + x_3(t)] = x_1(t) * x_2(t) + x_1(t) * x_3(t) \qquad ...(3.2.4)$$

Where $x_3(t)$ is another signal.

(c) Associative property :

This property states that,

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t) \qquad ...(3.2.5)$$

3.3 Computation of Convolution Integral using Graphical Method :

In this section, we will perform convolution integral of two signals with the help of some examples.

3.3.1 Unit step to Unit Step :

Solved Examples :

Ex. 3.3.1 : Perform the following convolution operation in time domain :

$x_1(t) = x_2(t) = u(t)$

Soln. : Let x (t) = u (t) and h (t) = u (t)

According to the definition of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) \qquad ...(1)$$

Given x (t) = h (t) = u (t)

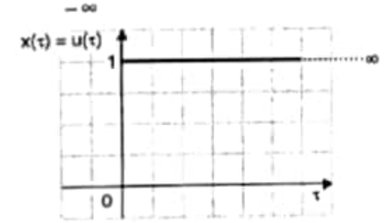
∴ x (τ) = h (τ) = u (τ).

It is unit step as shown in Fig. P. 3.3.1(a).

We will perform the convolution for different ranges.

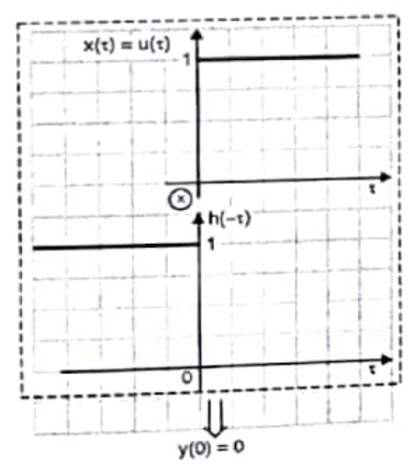
Consider t = 0 : Thus putting t = 0 in Equation (1) we get,

$$y(0) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau) \qquad ...(2)$$



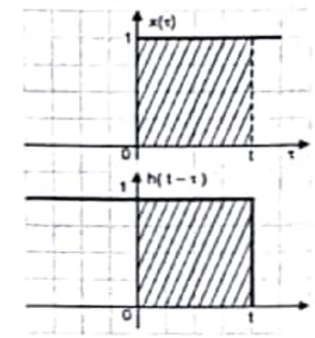
(H-275) **Fig. P. 3.3.1(a)**

This operation is shown in Fig. P. 3.3.1(b).



(b)

Fig. P. 3.3.1(Contd...)



(c)

(H-276) **Fig. P. 3.3.1**

There is no overlapping. ∴ y (0) = 0 for t = 0.

Note : The overlapping should be for some range. Here overlapping is only at t = 0.

Consider t > 0 :

As per Equation (1) we have the term h (t – τ). It is h (– τ + t). It indicates folding and delay operation. It is obtained by shifting h (– τ) towards right. This operation is shown in Fig. P. 3.3.1(c).

There is overlapping in the range '0' to 't'. And the value of x (τ) and h (t – τ) is 1.

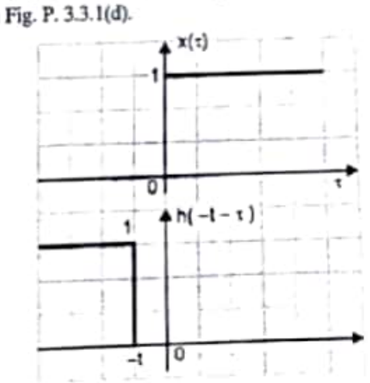
Thus, Equation (1) becomes,

$$y(t) = \int_0^t 1\, d\tau = [\tau]_0^t$$

$$\therefore y(t) = t \quad \text{for } t > 0.$$

Consider t < 0 :

That means value of 't' is negative. Thus second term in Equation (1) becomes h (– t – τ). It indicates folding and advanced operation. It is obtained by shifting h (– τ) towards left by 't' as shown in Fig. P. 3.3.1(d).



(H-278) **Fig. P. 3.3.1(d)**

There is no overlapping. Thus result of integration is zero.

Thus total convolution is, y (t) = t. for t > 0.

3.3.2 Unit Step to Exponential :

Ex. 3.3.2 : Perform the convolution operation between two functions in time domain.

$x_1(t) = u(t)$ and $x_2(t) = e^{-t}$ $t \geq 0$.

Soln. : We will denote the given functions using normal notations.

Let $x_1(t) = h(t) = u(t)$

and $x_2(t) = x(t) = e^{-t} u(t)$.

We will obtain the result of convolution using graphical method.

We have the equation of convolution.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad ...(1)$$

We have $x(t) = e^{-t} u(t)$

$\therefore\ x(\tau) = e^{-\tau} u(\tau)$ and $h(t) = u(t)$

$\therefore\ h(\tau) = u(\tau)$

The graph of $x(\tau)$ and $h(\tau)$ is shown in Figs. P. 3.3.2(a) and (b) respectively

First we will calculate $y(t)$ at $t = 0$. Thus putting $t = 0$ in Equation (1) we get,

$$y(0) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau)\, d\tau \qquad ...(2)$$

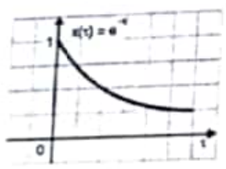
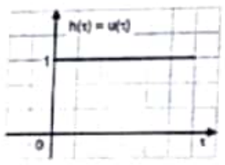
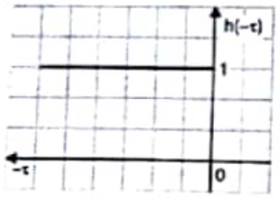


Fig. P. 3.3.2(a)



(H-279) Fig. P. 3.3.2(b)

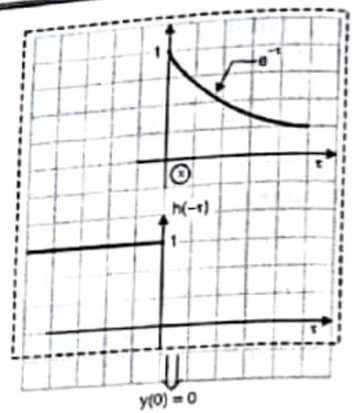
This equation indicates that we have to perform multiplication of $x(\tau)$ and $h(-\tau)$. Here $h(-\tau)$ is folded version of $h(\tau)$. It is shown in Fig. P. 3.3.2(c).



(H-280) Fig. P. 3.3.2(c)

The result integration of multiplication of $x(\tau)$ and $h(-\tau)$ is shown in Fig. P. 3.3.2(d). Since two waveforms do not overlap; this result is zero.

$$\therefore\ y(0) = 0 \qquad ...(3)$$



(H-281) Fig. P. 3.3.2(d)

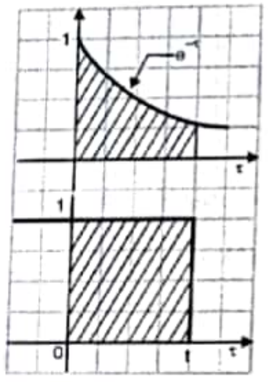
Now we will consider other possible ranges of 't'. First we will consider $t > 0$. Then Equation (1) becomes.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad ...(4)$$

Note that 't' is positive since it is greater than '0'.

Consider the second term that is $h(t-\tau)$. It can be written as $h(-\tau+t)$

This indicates folded and delay operation. That mean we have to shift the signal $h(-\tau)$ towards right by 't'. This operation is shown in Fig. P. 3.3.2(e).



(H-282) Fig. P. 3.3.2(e)

- Here the value (magnitude) of $h(t-\tau)$ is 1. There is overlapping of two signals in the range '0' to 't'. As per Equation (4) we will get the result of integration, only in the region where two signals overlap. Thus we will make the limits of integration as '0' to 't'.

$$\therefore\ y(t) = \int_0^t x(\tau) \cdot 1\, d\tau$$

$$\therefore\ y(t) = \int_0^t e^{-\tau} d\tau = \left[-1\, e^{-\tau}\right]_0^t = -\left[e^{-t} - e^{0}\right]$$

$$\therefore\ y(t) = 1 - e^{-t} \quad \text{... for } t > 0 \qquad ...(5)$$

Now consider the range $t < 0$. That means 't' is negative. Thus Equation (1) becomes,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(-t-\tau)\, d\tau.$$

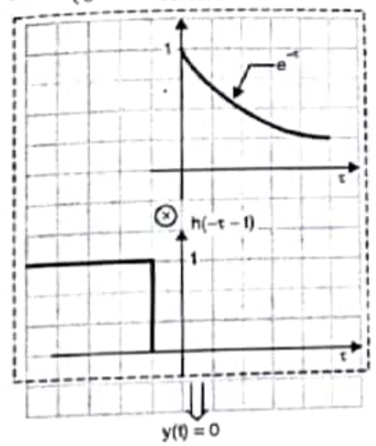
Consider the second term $h(-t-\tau)$. It can be written as $h(-\tau-t)$. It indicates folding and avance operation. It is obtained by shifting $h(-\tau)$ towards left. This operation is shown in Fig. P. 3.3.2(f).

There is no overlapping of signals. Thus result of convolution is zero.

$$\therefore\ y(t) = 0 \quad \text{for } t < 0.$$

Thus output of convolution is,

$$y(t) = \begin{cases} 1 - e^{-t} & \text{for } t > 0 \\ 0 & \text{for } t \le 0 \end{cases}$$



(H-283) Fig. P. 3.3.2(f)

Ex. 3.3.3 : The input and impulse responses of continuous time system are given below. Find out output of the continuous time systems using appropriate method.

$x(t) = u(t)$, $h(t) = e^{-2t} u(t)$

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Soln. :

Given : $x(t) = u(t)$ and $h(t) = e^{-2t} u(t)$

Taking Laplace transform,

$$X(s) = \frac{1}{s} \text{ and } H(s) = \frac{1}{s+2}$$

Now convolution becomes multiplication in Laplace domain.

$$\text{Let } Y(s) = X(s) \cdot H(s)$$

$$\therefore\ Y(s) = \frac{1}{s(s+2)}$$

$$\therefore\ Y(s) = \frac{A_1}{s} + \frac{A_2}{s+2}$$

$$\text{Here } A_1 = \frac{1}{s+2}\bigg|_{s=0} \quad \therefore\ A_1 = \frac{1}{2}$$

$$\text{and } A_2 = \frac{1}{s}\bigg|_{s=-2} \quad \therefore\ A_2 = -\frac{1}{2}$$

$$\therefore\ Y(s) = \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{1}{s+2}$$

Taking inverse Laplace transform.

Ex. 3.3.4 : A CT signal $x(t)$ is applied to the input of a CT LTI systems with unit impulse response $h(t)$. Find out $y(t)$ using convolution integral.

$x(t) = e^{-at} u(t) \quad a > 0$

$h(t) = u(t)$

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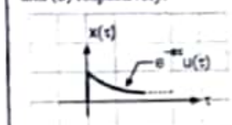
Soln. :

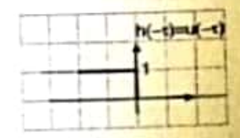
Given : $x(t) = e^{-at} u(t)$ and $h(t) = u(t)$

$\therefore\ x(\tau) = e^{-a\tau} u(\tau)$ and $h(\tau) = u(\tau)$ $\therefore\ h(-\tau) = u(-t)$

The signals $x(\tau)$ and $h(-\tau)$ are shown in Figs. P. 3.3.3(a) and (b) respectively.



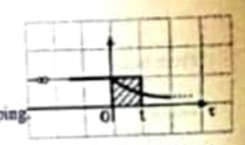
(H-1648) Fig. P. 3.3.4(a)



(H-1649) Fig. P. 3.3.4(b)

According to definition of convolution integral,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(-\tau+t)\, d\tau$$



(H-1650) Fig. P. 3.3.4(c)

Case 1 : If $t < 0$:

There will not be any overlapping.

$\therefore\ y(t) = 0$

Case 2 : If $t > 0$:

This condition is shown in Fig. P. 3.3.3(c).

Here overlapping is from 0 to t.

$$\therefore\ y(t) = \int_0^t e^{-a\tau} \cdot 1\, d\tau = \left[\frac{e^{-a\tau}}{-a}\right]_0^t$$

$$= -\frac{1}{a}\left[e^{-at} - e^{0}\right] = -\frac{1}{a}\left[e^{-at} - e^{0}\right]$$

$$= \frac{1}{a}\left[1 - e^{-at}\right]$$

Ex. 3.3.5 : Perform the following convolution operation of two functions in time domain.

$x_1(t) = e^{-4t} u(t)$, $x_2(t) = u(t-4)$

Soln. :

Let $x(t) = x_1(t) = e^{-4t} u(t)$,

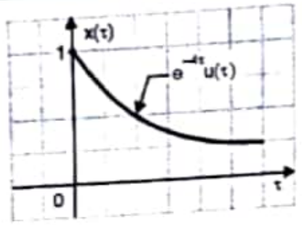
$\therefore\ x(\tau) = e^{-4\tau} u(\tau)$

This function is shown in Fig. P. 3.3.5(a).

Here $x_2(t) = u(t-4)$

Let $h(t) = u(t-4)$

$\therefore\ h(\tau) = u(\tau-4)$



(II-284) **Fig. P. 3.3.5(a)**

It indicates unit step delayed by '4'. It is shown in Fig. P. 3.3.5(b).

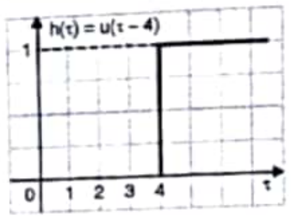
We have equation of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad ...(1)$$

Given $h(\tau) = u(\tau - 4) = 1$ ...(2)

for $\tau \geq 4$ as shown in Fig. P. 3.3.5(b).

But in Equation (1) we want the term $h(t-\tau)$.



(II-285) **Fig. P. 3.3.5(b)**

From Equation (2) we have,

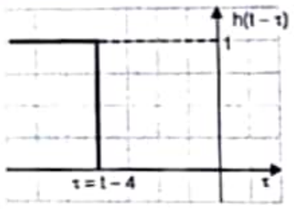
$h(\tau) = 1$ for $\tau \geq 4$

Replacing τ by $t-\tau$ we get,

$h(t-\tau) = 1$ for $t-\tau \geq 4$

$\therefore$ $h(t-\tau) = 1$ for $\tau \leq t-4$

This waveform is shown in Fig. P. 3.3.5(c).



(II-286) **Fig. P. 3.3.5(c)**

In Equation (1) we have the term $h(t-\tau)$. It is same as $h(-\tau + t)$. Here $h(-\tau)$ is folded unit step. Thus $h(-\tau + t)$ indicates that $h(-\tau)$ is shifted towards right by 't'. For any arbitrary value of 't' ; the two waveform are as shown in Fig. P. 3.3.5(d).

As we go on increasing the value of 't' then the waveform $h(t-\tau)$ will shift towards right. But always there is overlap in the range $\tau = 0$ to $\tau = t-4$.

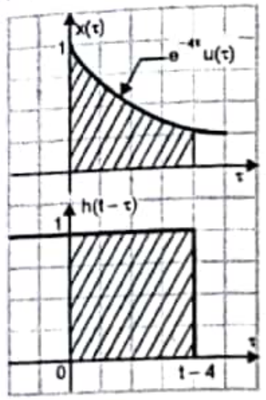
Thus we can write,

$$y(t) = \int_{0}^{t-4} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{0}^{t-4} e^{-4\tau} \cdot 1\, d\tau = -\left[\frac{e^{-4\tau}}{4}\right]_{0}^{t-4}$$

$$= -\frac{1}{4}\left[\frac{e^{-4\tau}}{4}\right]_{0}^{t-4} = -\frac{1}{4}\left[e^{-4(t-4)} - e^{0}\right]$$

$$\therefore\ y(t) = \frac{1}{4} - \frac{1}{4} e^{-4(t-4)}$$



(II-287) **Fig. P. 3.3.5(d)**

Ex. 3.3.6 : Evaluate continuous time (CT) convolution integral of $y(t) = e^{-2t} u(t) * u(t+2)$.

Soln. :

Let $x(t) = e^{-2t} u(t)$

$\therefore$ $x(\tau) = e^{-2\tau} u(\tau)$

This function is shown in Fig. P. 3.3.6(a).

and let $h(t) = u(t+2)$

$\therefore$ $h(\tau) = u(\tau + 2)$

It indicates unit step advanced by 2 units. It is shown in Fig. P. 3.3.6(b). We have equation of convolution.

$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau \qquad ...(1)$$

Here $h(\tau) = u(\tau + 2) = 1$ for $\tau \geq -2$ as shown in Fig. P. 3.3.6(b).

But in Equation (1) we want the term $h(t-\tau)$

We have,

$h(\tau) = 1$ for $\tau \geq -2$ ...(2)

Replacing τ by $t-\tau$ we get,

$h(t-\tau) = 1$ for $t-\tau \geq -2$

$\therefore$ $h(t-\tau) = 1$ for $\tau \leq t+2$ ...(3)

This waveform is shown in Fig. P. 3.3.6(c).

In Equation (1) the term $h(t-\tau)$ indicates that $h(-\tau)$ is shifted towards right by 't' positions. For any arbitrary value of 't' the waveform will overlap as shown in Fig. P. 3.3.6(d). As we go on increasing value of 't' the waveform will shift towards right. But always there will be overlap in the range $\tau = 0$ to $t+2$.

Thus we can write,

$$y(t) = \int_{0}^{t+2} x(\tau)\, h(t-\tau)\, d\tau$$

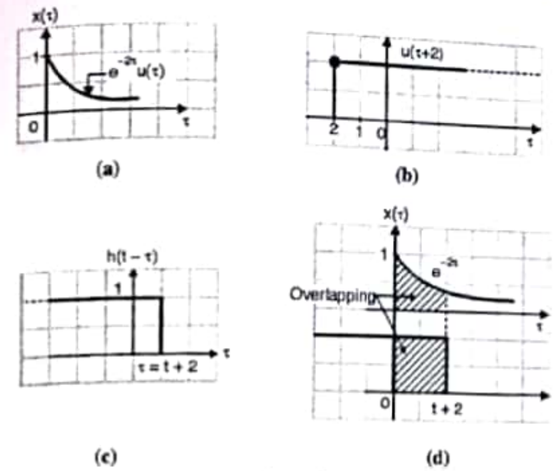
$$\therefore\ y(t) = \int_{0}^{t+2} e^{-2\tau} \cdot 1\, d\tau = -\left[\frac{e^{-2\tau}}{2}\right]_{0}^{t+2}$$

$$\therefore\ y(t) = -\frac{1}{2}\left[e^{-2\tau}\right]_{0}^{t+2} = -\frac{1}{2}\left[e^{-2t-4} - e^{0}\right]$$

$$\therefore\ y(t) = -\frac{1}{2}\left[e^{-2t} \cdot e^{-4} - e^{0}\right]$$

$$\therefore\ y(t) = \frac{1}{2} - \frac{1}{2} e^{-4} \cdot e^{-2t}$$

...Ans.



(II-288) **Fig. P. 3.3.6**

3.3.3 Exponential to Exponential :

Ex. 3.3.7 : For an LTI system with unit impulse response $h(t) = e^{-2t} u(t)$, determine output to the input $x(t) = e^{-t} u(t)$.

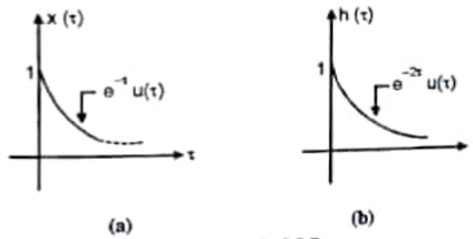
Soln. :

We have, $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

Given : $x(t) = e^{-t} u(t)$ $\therefore$ $x(\tau) = e^{-\tau} u(\tau)$

and $h(t) = e^{-2t} u(t)$ $\therefore$ $h(\tau) = e^{-2\tau} u(\tau)$

These signals are as shown in Fig. P. 3.3.7(a) and Fig. P. 3.3.7(b) respectively.

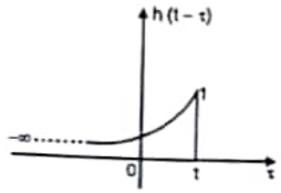


(II-289) **Fig. P. 3.3.7**

Now $h(t-\tau) = e^{-2(t-\tau)} u(t-\tau) = e^{-2(-\tau+t)} u(-\tau+t)$

The term $u(-\tau+t)$ represents folded unit step $u(-\tau)$ and it is existing between $\tau = -\infty$ to $\tau = t$

It is shown in Fig. P. 3.3.7(c)



(II-290) **Fig. P. 3.3.7(c)**

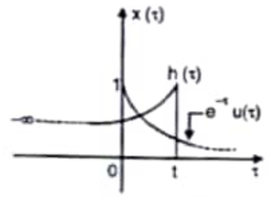
Case I : When t < 0

For $t < 0$; Fig. P. 3.3.7(a) and Fig. P. 3.3.7(c) will not overlap

$\therefore$ $y(t) = 0$

Case II : When t > 0

This condition is shown in Fig. P. 3.3.7(d)



(II-291) **Fig. P. 3.3.7(d)**

Here overlapping is in the range 0 to 't'.

$$\therefore\ y(t) = \int_{0}^{t} e^{-\tau} u(\tau) \cdot e^{-2(t-\tau+0)} u(t-\tau) \cdot d\tau$$

Amplitude of $u(\tau)$ and $u(t-\tau)$ are 1.

$$\therefore\ y(t) = \int_{0}^{t} e^{-\tau} e^{-2(t-\tau+0)} d\tau$$

$$\therefore\ y(t) = \int_{0}^{t} e^{-\tau} \cdot e^{+2\tau} \cdot e^{-2t} d\tau$$

$$\therefore\ y(t) = e^{-2t} \int_{0}^{t} e^{\tau} d\tau = e^{-2t}\left[e^{\tau}\right]_{0}^{t}$$

$$\therefore\ y(t) = e^{-2t}\left[e^{t} - 1\right]$$

3.3.4 Exponential to Rectangular :

Ex. 3.3.8 : Impulse response of an LTI system is given by :

$$h(t) = \begin{cases} 3e^{-0.5t} & ;\ t \geq 0 \\ 0 & ;\ \text{Otherwise} \end{cases}$$

Find the system output due to the input :

$$x(t) = \begin{cases} 1 & ;\ 0 \leq t \leq 2 \\ 0 & ;\ \text{Otherwise} \end{cases}$$

Also sketch the output.

Soln. : Output of LTI system is given as,

$$\therefore\ y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

OR $y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$

Given, $x(t) = 1$ for $0 \le t \le 2$

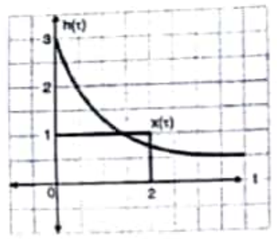
and $h(t) = 3e^{-0.5t}$ for $t \ge 0$

Replacing 't' by 'τ',

$x(\tau) = 1$ for $0 \le t \le 2$

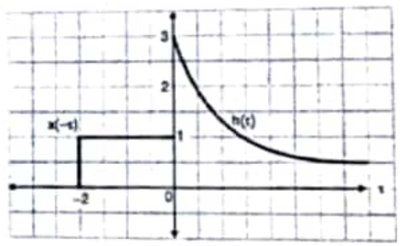
and $h(\tau) = 3e^{-0.5\tau}$ for $\tau \ge 0$

$x(t-\tau) = 1$ for $0 \le t-\tau \le 2$ i.e. $t-2 \le \tau$

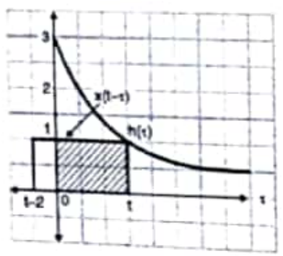


(H-432) Fig. P. 3.3.8(a)

(a) Signal $x(-\tau)$ and $h(\tau)$:



(H-433) (b)



(H-434) (c)

Fig. P. 3.3.8

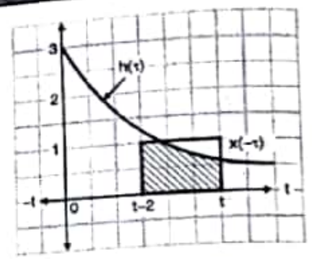
(b) $h(\tau)$ remains unchanged but $x(\tau)$ is folded to obtain $x(-\tau)$

(c) $h(\tau)$ remains unchanged, but $x(-\tau)$ is shifted i.e. $x(t-\tau)$.

(d) $x(-\tau)$ keeps shifting.

For $t < 0$:

As seen from Fig. P. 3.3.8(c), the function do not overlap for $t < 0$ hence their product, hence their product will be zero and the integration also will be zero.



(H-435) Fig. P. 3.3.8(d)

$\therefore\ x(t) * h(t) = 0$ for $t < 0$

For $0 < \tau < t$,

As seen from Fig. P. 3.3.8(d), the function overlap for $0 < \tau < t$ so,

$$\therefore x(t) * h(t) = \int_0^t 3e^{-0.5\tau} \cdot 1 \cdot d\tau = 3\left[\frac{e^{-0.5\tau}}{-0.5}\right]_0^t$$

$$= -3 \times 2\,[e^{-0.5t} - e^{0}]$$

$$= 6\,[1 - e^{-0.5t}]$$

For $t > 2$:

From Fig. P. 3.3.8(b), the function overlap for $(t-2) < \tau < t$,

$$\therefore\ x(t) * h(t) = \int_{t-2}^{t} 3e^{-0.5\tau} \cdot d\tau = 3\left[\frac{e^{-0.5\tau}}{-0.5}\right]_{t-2}^{t}$$

$$= -3 \times 2\,[e^{-0.5t} - e^{-0.5(t-2)}]$$

$$= -6\,[e^{-0.5t} - e^{-0.5t} \cdot e^{1}]$$

$$= -6e^{-0.5t}\,[1 - 2.716]$$

$$\therefore\ y(t) = x(t) * h(t) = 10.296\, e^{-0.5t}$$

Ex. 3.3.9 : Impulse response of an LTI system is given by :

$$h(t) = \begin{cases} e^{-2t} & ;\ t \ge 0 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Find the system output due to the input :

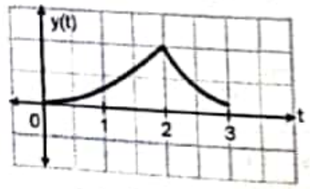
$$x(t) = \begin{cases} A & ;\ 0 \le t \le 2 \\ 0 & ;\ \text{otherwise} \end{cases}$$

Also sketch the output.

Soln. : Similar to the previous example.

$$\textbf{Output}: y(t) = x(t) * h(t) = \begin{cases} 0 & \text{for } t < 0 \\ \frac{A}{2}[1 - e^{-2t}] & \text{for } 0 < t < 2 \\ 26.799\, e^{-2t} & \text{for } t > 2 \end{cases}$$

The sketch of y (t) is shown in Fig. P. 3.3.9



(H-425) Fig. P. 3.3.9

3.3.5 Unit Step to Rectangular :

Ex. 3.3.10 : Obtain the convolution of :

$x(t) = u(t)$ and $h(t) = 1$ for $-1 \le t \le 1$.

Soln. :

Given : $x(t) = u(t)$ $\therefore\ x(\tau) = u(\tau)$

and $h(t) = 1$ for $-1 \le t \le 1$

$\therefore\ h(\tau) = 1$ for $-1 \le \tau \le 1$

According to the definition of convolution.

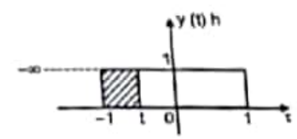
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Here $x(t-\tau) = u(t-\tau) = u(-\tau + t)$

These two signals are shown in Fig. P. 3.3.10(a) and Fig. P. 3.3.10(b) respectively.

(a) (b)

(H-292)



(H-293) (c) Fig. P. 3.3.10

Case - I : When $-1 < t < 0$:

This condition is shown in Fig. P. 3.3.10(c)

Here overlapping is in the range -1 to t.

$$\therefore\ y(t) = \int_{-1}^{t} u(\tau) \cdot 1\, d\tau = \int_{-1}^{t} d\tau = [\tau]_{-1}^{t}$$

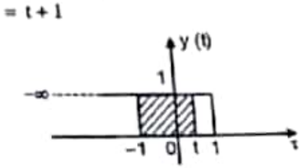
$$\therefore\ y(t) = t + 1$$

Case - II : When $0 < t < 1$:

It is shown in Fig. P. 3.3.10(d).

Here again overlapping is in the same range -1 to t.

$\therefore\ y(t) = t + 1$



(H-294) Fig. P. 3.3.10(d)

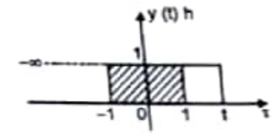
Case - III : When $t > 1$:

It is shown in Fig. P. 3.3.10(e).

For any value of t which is greater than 1 there is always overlapping from -1 to $+1$.

$$\therefore\ y(t) = \int_{-1}^{1} 1 \cdot d\tau = [\tau]_{-1}^{1}$$

$$\therefore\ y(t) = 2$$



(H-295) Fig. P. 3.3.10(e)

3.3.6 Rectangular to Rectangular :

Ex. 3.3.11 : Obtain convolution integral of :

$x(t) = 1$ for $-1 \le t \le 1$

$h(t) = 1$ for $0 \le t \le 2$

Soln. : We can write,

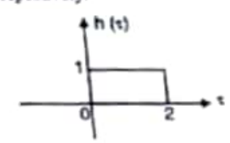
$x(\tau) = 1$ for $-1 \le \tau \le 1$

and $h(\tau) = 1$ for $0 \le \tau \le 2$

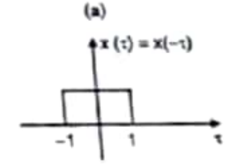
According to equation of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

The signals $h(\tau)$ and $x(-\tau)$ are shown in Fig. P. 3.3.11(a) and Fig. P. 3.3.11(b) respectively.



(a)



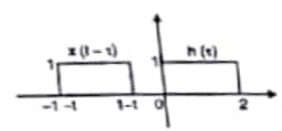
(b)

(H-296) Fig. P. 3.3.11

Case - I : When $t < -1$

This condition is shown in Fig. P. 3.3.11(c).

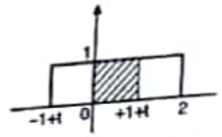
There is no overlapping so convolution is zero.



(H-297) Fig. P. 3.3.11(c)

Case - II : When $0 < t < 2$

This condition is shown in Fig. P. 3.3.11(d).



(H-298) Fig. P. 3.3.11(d)

The overlapping is in the range 0 to t + 1

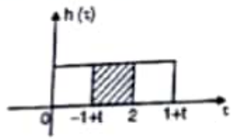
$$y(t) = \int_0^{t+1} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore\ y(t) = \int_0^{t+1} 1 \cdot 1 \cdot d\tau = [\tau]_0^{t+1}$$

$$\therefore\ y(t) = t+1$$

Case - II : When t > 2

This condition is shown in Fig. P. 3.3.11(e).



(H-299) Fig. P. 3.3.11(e)

Overlapping is in the range – 1 + t to 2.

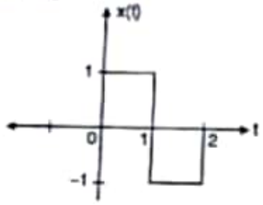
$$\therefore\ y(t) = \int_{-1+t}^{2} 1 \cdot 1 \cdot d\tau = [\tau]_{-1+t}^{2} = 2+1-t = 3-t$$

Ex. 3.3.12 : Evaluate the convolution integral x(t) * x(2 – t) where x(t) is shown in Fig. P. 3.3.12(a).

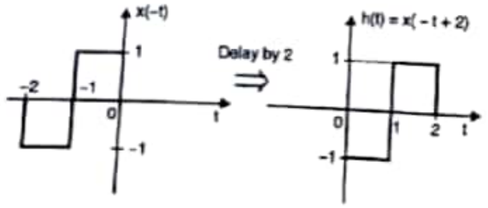
Soln. :

Let h (t) = x (2 – t) = x (– t + 2)

It is shown in Fig. P. 3.3.12(b).



(H-746) Fig. P. 3.3.12(a)

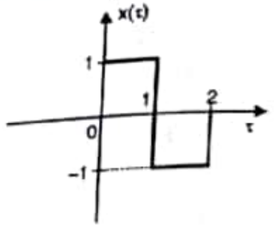


(H-747) Fig. P. 3.3.12(b)

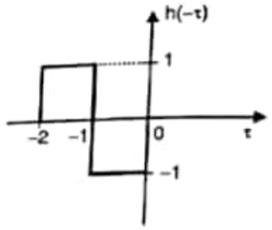
According to the definition of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

The signals x (τ) and h (– τ) are shown in Fig. P. 3.3.12(c) and P. 3.3.12(d) respectively.



(c)



(d)

(H-748) Fig. P. 3.3.12

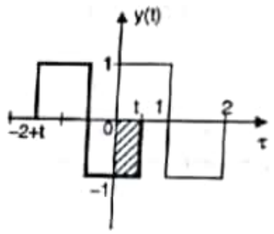
We will consider following cases for convolution.

Case 1 : When t < 0

Here h (– τ) will shift towards left. There will not be any overlapping. Thus y (t) = 0.

Case 2 : When 0 < t < 1

This condition is shown in Fig. P. 3.3.12(e).



(H-749) Fig. P. 3.3.12(e)

Here overlapping is from 0 to t

$$\therefore\ y(t) = \int_0^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore\ y(t) = \int_0^{t} 1 \cdot (-1) \cdot d\tau = -t$$

Case 3 : When 1 < t < 2

This condition is shown in Fig. P. 3.3.12(f).

Here overlapping is from 0 to t

We will perform integration for different periods as follows :

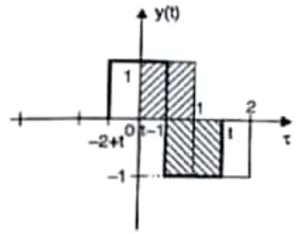
$$y(t) = \int_0^{t-1} x(\tau)\, h(t-\tau)\, d\tau + \int_{t-1}^{1} x(\tau)\, h(t-\tau)\, d\tau + \int_1^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$\therefore\ y(t) = \int_0^{t-1} 1 \times (1)\, d\tau + \int_{t-1}^{1} 1 \times (-1)\, d\tau + \int_1^{t} (-1) \times (-1)\, d\tau$$

$$\therefore\ y(t) = -(t-1) - [1-t+1] + [t-1]$$

$$\therefore\ y(t) = -t+1-2+t+t-1$$

$$\therefore\ t(t) = t-2$$



(H-750) Fig. P. 3.3.12(f)

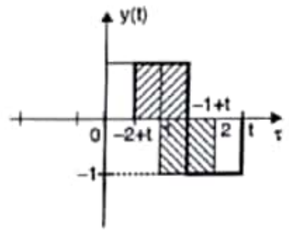
Case 4 : When 2 < t < 3

This condition is shown in Fig. P. 3.3.12(g). Here overlapping is from – 2 + t to 2.

$$\therefore\ y(t) = \int_{-2+t}^{1} 1 \times (1)\, d\tau + \int_{1}^{-1+t} (-1) \times (1)\, d\tau + \int_{-1+t}^{2} (-1) \times (-1)\, d\tau$$

$$\therefore\ y(t) = [1+2-t] - [-1+t-1] + [2+1-t]$$

$$\therefore\ y(t) = 3-t+2-t+3-t = 8-3t$$



(H-751) Fig. P. 3.3.12(g)

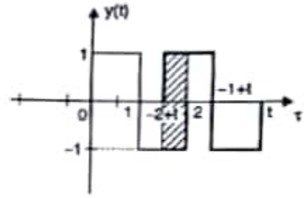
Case 5 : When 3 < t < 4

This condition is shown in Fig. P. 3.3.12(h).

Here overlapping is in the range – 2 + t to 2.

$$\therefore\ y(t) = \int_{-2+t}^{2} (-1) \times (1)\, d\tau = -[2+2-t]$$

$$\therefore\ y(t) = t-4$$



(H-752) Fig. P. 3.3.12(h)

Case 6 : When t > 4

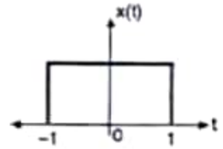
There will not be any overlapping

$$\therefore\ y(t) = 0$$

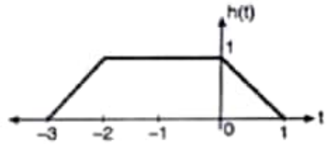
The result of convolution is as follows.

$$y(t) = \begin{cases} 0 & \text{For } t<0 \\ -t & \text{For } 0<t<1 \\ t-2 & \text{For } 1<t<2 \\ 8-3t & \text{For } 2<t<3 \\ t-4 & \text{For } 3<t<4 \\ 0 & \text{For } t>4 \end{cases}$$

Ex. 3.3.13 : Evaluate the convolution integral for input x(t) and impulse response h(t) shown in Fig. P. 3.3.13(a) and Fig. P. 3.3.13(b).



(H-801) Fig. P. 3.3.13(a)



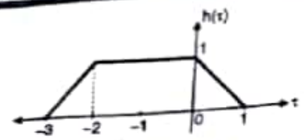
(H-802) Fig. P. 3.3.13(b)

Soln. : According to the equation of convolution integral.

$$u(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \qquad \text{....(1)}$$

The signals x(– τ) and h(τ) are as shown in Fig. P. 3.3.13(c) and P. 3.3.13(d) respectively.

Fig. P. 3.3.13(c)



(H-803)Fig. P. 3.3.13(d)

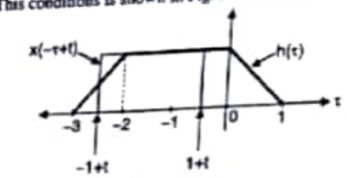
Case 1 : When t < – 4 :

This is the case of no overlapping. Because signal x(– τ) will be shifted towards, left by amount t < – 4.

$$\therefore \quad y(t) = 0$$

Case 2 : – 2 < t < 0 :

This conditions is shown in Fig. P. 3.3.13(c).



(H-804)Fig. P. 3.3.13(c)

Here overlapping is from – 1 + t to 1 + t.

The equation of h(τ) for this range is,

$$h(\tau) = \begin{cases} \tau + 3 & \text{for} \quad (-1+t) < \tau < -2 \\ 1 & \text{for} \quad -2 < \tau < 1+t \end{cases}$$

$$\therefore \quad y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-1+t}^{-2} (\tau+3)\cdot 1\, d\tau + \int_{-2}^{1+t} 1 \cdot 1 \cdot d\tau$$

$$= \left[\frac{\tau^2}{2} + 3\tau\right]_{-1+t}^{-2} + [\tau]_{-2}^{1+t}$$

$$= \left[\left(\frac{4}{2}+6\right) - \left(\frac{(-1+t)^2}{2} + 3(-1+t)\right)\right] + [1+t+2]$$

$$= 8 - \left[\frac{(-1+t)^2}{2} - 3 + 3t\right] + 3 + t$$

$$\therefore \quad y(t) = 8 - \frac{(-1+t)^2}{2} + 3 - 3t + 3 + t$$

$$= 14 - 2t - \frac{(-1+t)^2}{2}$$

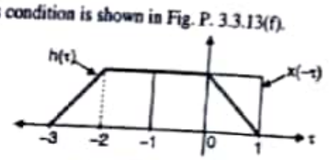
Case 3 : t = – 1 :

When t = – 1 then x(– τ + t) = x(– τ – 1) and this is the case of full overlapping. So range of integration is from – 2 to 0.

$$\therefore \quad y(t) = \int_{-2}^{0} 1 \cdot 1 \cdot d\tau = [\tau]_{-2}^{0} = 2$$

Case 4 : When t = 0 :

This condition is shown in Fig. P. 3.3.13(f).



(H-805)Fig. P. 3.3.13(f)

Here overlapping is from – 1 to + 1.

Now equation of h(τ) is,

$$h(\tau) = \begin{cases} 1 & \text{for} \quad -1 < t < 0 \\ -\tau + 1 & \text{for} \quad 0 < t < 1 \end{cases}$$

$$\therefore \quad y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot x(-\tau + t)\, d\tau$$

$$= \int_{-1}^{0} 1 \cdot 1 \cdot d\tau + \int_{0}^{1} (-\tau + 1) \cdot 1\, d\tau$$

$$= [\tau]_{-1}^{0} - \left[\frac{\tau^2}{2} - \tau\right]_{0}^{1} = 1 - \left[\left(\frac{1}{2}+1\right) - (0)\right] = \frac{3}{2}$$

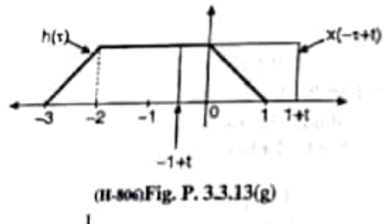
Case 5 : When 1 < t < 0 :

This condition is shown in Fig. P. 3.3.13(g).

Here overlapping is from – 1 + t to 1

$$\therefore \quad y(t) = \int_{-1+t}^{0} 1 \cdot 1\, d\tau + \int_{0}^{1} 1 \cdot (-\tau + 1)\, d\tau$$

$$= [\tau]_{-1+t}^{0} + \left[\frac{-\tau^2}{2} + \tau\right]_{0}^{1} = t - 1 + \frac{1}{2}$$



(H-806)Fig. P. 3.3.13(g)

$$\therefore \quad y(t) = t - \frac{1}{2}$$

3.4 Impulse Response of DT-LTI System :

May 15, Dec. 15, May 16

University Questions

Q. 1 Perform convolution of

(i) 2u(t) with u(t)

(ii) e^{-2t} u (t) with e^{-5t} u(t)

(iii) tu(t) with e^{-5t} u(t) **(May 15, 10 Marks)**

Q. 2 The input and impulse responses of continuous time system are given below. Find out output of the continuous time systems using appropriate method.

x (t) = u (t) h (t) = e^{-2t} u (t) **(Dec. 15, 10 Marks)**

Q. 3 A CT signal x (t) is applied to the input of a CT LTI systems with unit impulse response h (t). Find out y (t) using convolution integral. **(May 16, 10 Marks)**

x (t) = e^{-at} u (t) a > 0

h (t) = u (t)

It is also called as linear convolution and it is applicable for discrete time systems.

3.4.1 Representation of Discrete Time Signal as Weighted Impulses :

In this sub-section we will show that LTI system can be completely characterized by its impulse response. We will be using following notations related to discrete time systems.

x (n) ⇒ Input sequence

h (n) ⇒ Impulse response of system

y (n) ⇒ Output sequence

Consider input sequence,

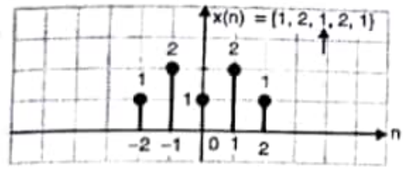
$$x(n) = \{1, \underset{\uparrow}{2}, 1, 2, 1\} \qquad ...(3.4.1)$$

This sequence is represented as shown in Fig. 3.4.1(a).

The different values of x (n) are as follows :

x (– 2) = 1, x (– 1) = 2,

x (0) = 1, x (1) = 2, x (2) = 1

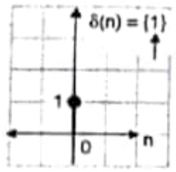


(H-300)Fig. 3.4.1(a) : Input sequence x (n)

Now consider unit impulse δ (n). It is expressed as,

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{elsewhere} \end{cases} \qquad ...(3.4.2)$$

That means its value is 1 at n = 0 and its value is zero for all other values of 'n'. Such unit impulse is as shown in Fig. 3.4.1(b).

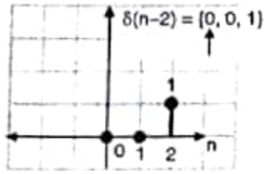


(H-301)Fig. 3.4.1(b) : Unit impulse, δ (n)

If we want to delay this unit impulse by '2' samples then it is denoted by δ (n – 2)

$$\text{And} \quad \delta(n-2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad ...(3.4.3)$$

This sequence is shown in Fig. 3.4.1(c).

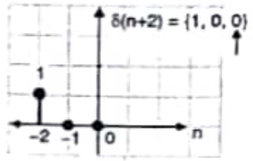


(H-302)Fig. 3.4.1(c) : Impulse sequence δ (n – 2)

Similarly if we want to advance this unit impulse by '2' samples then it is denoted by δ (n + 2).

$$\therefore \quad \delta(n+2) = \begin{cases} 1 & \text{for } n = 2 \\ 0 & \text{otherwise} \end{cases} \qquad ...(3.4.4)$$

This sequence is shown in Fig. 3.4.1(d).



(H-303)Fig. 3.4.1(d) : Impulse sequence δ (n + 2)

In general, if δ (n) is delayed by 'k' samples then we can write,

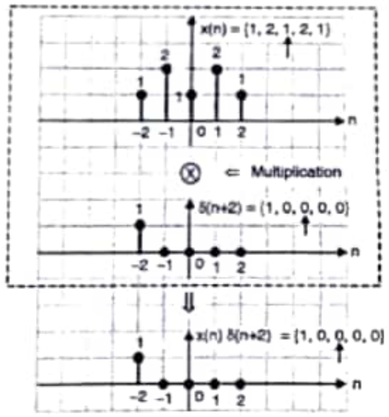
$$\therefore \quad \delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{otherwise} \end{cases} \qquad ...(3.4.5)$$

Decomposition of input sequence x (n) :

The given input sequence is,

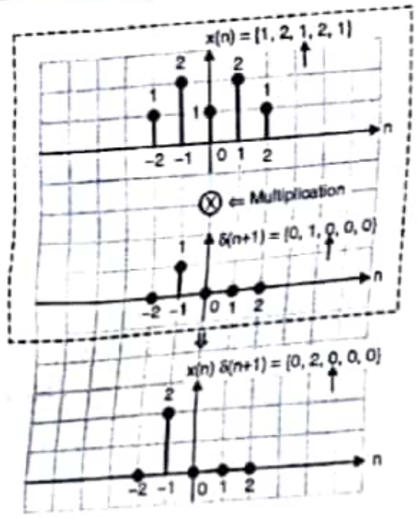
$$x(n) = \{1, \underset{\uparrow}{2}, 1, 2, 1\}$$

- Its range is from – 2 to +2. Now decomposition means obtaining a particular sample from given sequence x(n).
- Suppose we want to obtain sample at n = – 2 that means x (– 2). Then take the multiplication of x (n) and δ (n + 2).
- We know that the value of δ (n + 2) is one only at n = – 2, while for all other values of n, it is zero.
- Thus if we take multiplication of x (n) and δ (n + 2) then we will get the sample x (– 2). Because all other terms become zero, after multiplication. This is shown in Fig. 3.4.1(e).



(H-304)Fig. 3.4.1(e) : Decomposed input signal sample at n = – 2

Similarly other decomposed input signal samples are obtained.



(H-305)**Fig. 3.4.1(f) : Decomposed sample at n = – 1**

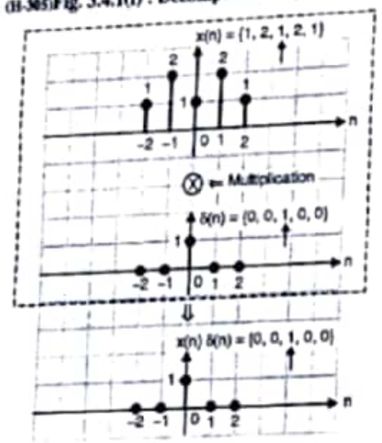


Fig. 3.4.1(g) : Decomposed sample at n = 0 (H-306)

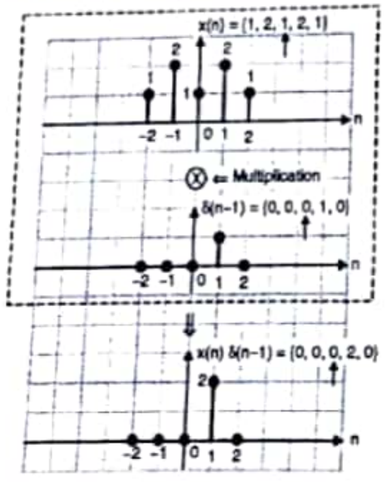
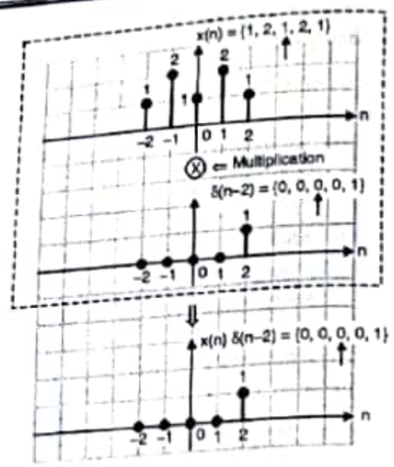


Fig. 3.4.1(h) : Decomposed sample at n = + 1



(H-307)**Fig. 3.4.1(i) : Decomposed sample at n = + 2**

- Now the original sequence x (n) can be obtained simply by adding all decomposed samples as shown in Fig. 3.4.2.

Mathematically we can write,

$$x(n) = x(n) \cdot \delta(n+2) + x(n) \cdot \delta(n+1) + x(n) \cdot \delta(n) + x(n) \cdot \delta(n-1) + x(n) \cdot \delta(n-2) \quad ...(3.4.6)$$

- Observe every decomposed sample. We get a decomposed sample only at n = k.
- For example sample x (– 2) is obtained by multiplication of x (n) and δ (n + 2) as shown in Fig. 3.4.1(e).
- This sample is present at n = k = – 2. Thus for simplicity replacing x (n) by x (k) in R.H.S. of Equation (3.4.6) we get,

$$x(n) = x(k)\,\delta(n+2) + x(k)\,\delta(n+1) + x(k)\,\delta(n) + x(k)\,\delta(n-1) + x(k)\,\delta(n-2) \quad ...(3.4.7)$$

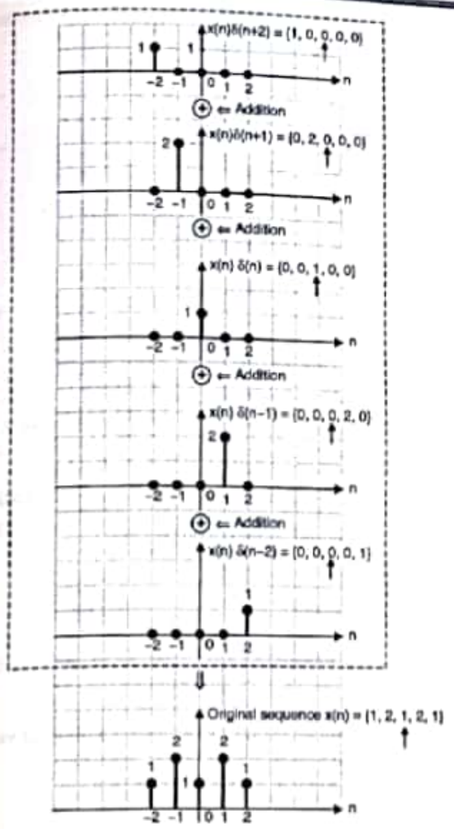
In terms of summation we can write,

$$x(n) = \sum_{k=-2}^{2} x(k)\,\delta(n-k) \quad ...(3.4.8)$$

In Equation (3.4.7), the amplitude of every 'δ' term is 1. We are multiplying every unit impulse (advanced or delayed) by corresponding amplitude of input sample. This is called as scaling of unit samples. Thus Equations (3.4.7) and (3.4.8) indicates that **input sequence x (n) is represented as sum of weighted (or scaled) unit impulses.**

In Equation (3.4.8) the limits of summation are from – 2 to + 2 ; this is because input sequence x (n) is present from n = – 2 to n = + 2. In general we can write Equation (3.4.8) as,

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \quad ...(3.4.9)$$



(H-308)**Fig. 3.4.2 : x (n) = Addition of all decomposed samples**

3.5 Linear Convolution (Convolution Sum) :

- Proof of linear convolution Or Proof of "LTI system is completely characterized by unit impulse response h (n)":
- Consider relaxed LSI (LTI) system. A relaxed system means if input x (n) is zero then output y (n) is zero.
- Let us say unit impulse δ (n) is applied to this system then its output is denoted by h (n). "h (n)" is called as impulse response of system.

Step 1 : $\delta(n) \xrightarrow{T} y(n) = h(n)$

Since the system is time invariant, if we delay input by 'k' samples then output should be delayed by same amount.

Step 2 : $\delta(n-k) \xrightarrow{T} y(n) = h(n-k)$

Now multiplying both sides by x (k) we get,

Step 3 : $x(k)\,\delta(n-k) \xrightarrow{T} y(n) = x(k)\,h(n-k)$

Since the system is linear; we can apply superposition theorem. So taking summation at both sides we get,

Step 4 : $\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \xrightarrow{T} y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$

Thus we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \quad ...(3.5.1)$$

This summation is called as **convolution sum**. In this case output y (n) is obtained by taking convolution of x (n) and h (n). This is called as linear convolution and it is denoted by symbol "*".

$$\therefore\ x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

In this case we can say that the behaviour of LTI system is completely characterized by its impulse response.

3.6 Methods of Convolution :

The different methods used for the computation of linear convolution are as follows :

1. Graphical method
2. Using mathematical equation of convolution
3. Tabulation method
4. Multiplication method

3.6.1 Graphical Method :

The linear convolution of two sequence is given by,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \quad ...(3.6.1)$$

We will have to calculate output for different values of 'n'

For n = 0 we get,

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) \quad ...(3.6.2)$$

Note that h (– k) indicates folding of h (k).

For n = 1 we get,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(1-k) \quad ...(3.6.3)$$

Here the term h (1 – k) can be written as h (– k + 1). Thus Equation (3.6.3) becomes,

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k+1) \quad ...(3.6.4)$$

Here h (– k + 1) indicates shifting of folded signal h (– k). It indicates that h (– k) is delayed by '1' sample. Similarly for other values of 'n' output y (n) is calculated.

Thus different operations involved in the calculation of linear convolution are as follows :

1. **Folding operation :** It indicates folding of sequence h (k).
2. **Shifting operation :** It indicates time shifting of h (− k) e.g. h (− k + 1).
3. **Multiplication :** It indicates multiplication of x (k) and h (n − k).
4. **Summation :** It indicates addition of all product terms obtained because of multiplication of x (k) and h (n − k).

Solved Examples :

Ex. 3.6.1 : For the following signals, determine and sketch convolution y (n) graphically :

$$x(n) = \frac{1}{3}n \quad 0 \le n \le 6$$
$$= 0 \quad \text{otherwise}$$
$$\text{and } h(n) = 1 \quad -2 \le n \le 2$$
$$= 0 \quad \text{otherwise}$$

Soln. : First we will write sequences x (n) and h (n) by putting different values of n.

Sequence x (n) :

Here range of x (n) is from n = 0 to n = 6 and $x(n) = \frac{1}{3}n$

For $n = 0 \Rightarrow x(0) = \frac{1}{3} \times 0 = 0$

For $n = 1 \Rightarrow x(1) = \frac{1}{3} \times 1 = \frac{1}{3}$

For $n = 2 \Rightarrow x(2) = \frac{1}{3} \times 2 = \frac{2}{3}$

For $n = 3 \Rightarrow x(3) = \frac{1}{3} \times 3 = \frac{3}{3}$

For $n = 4 \Rightarrow x(4) = \frac{1}{3} \times 4 = \frac{4}{3}$

For $n = 5 \Rightarrow x(5) = \frac{1}{3} \times 5 = \frac{5}{3}$

For $n = 6 \Rightarrow x(6) = \frac{1}{3} \times 6 = \frac{6}{3}$

Thus x (n) can be written as,

$$x(n) = \{x(0), x(1), x(2), x(3), x(4), x(5), x(6)\}$$

$$\therefore x(n) = \left\{\underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}\right\} \quad ...(1)$$

Now h (n) is given as,

$$h(n) = 1 \quad \text{for } -2 \le n \le 2$$
$$= 0 \quad \text{otherwise}$$

Thus range of h (n) is from − 2 to +2

$$\therefore h(n) = \{h(-2), h(-1), h(0), h(1), h(2)\}$$
$$\therefore h(n) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\} \quad ...(2)$$

Now the sequences x (k) and h (k) are obtained by replacing 'n' by 'k'.

$$\therefore x(k) = \left\{\underset{\uparrow}{0}, \frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}, \frac{5}{3}, \frac{6}{3}\right\} \quad ...(3)$$

$$\text{and } h(k) = \{\underset{\uparrow}{1}, 1, 1, 1, 1\} \quad ...(4)$$

The linear convolution of x (k) and h (k) is given by,

$$y(n) = x(k) * h(k)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \quad ...(5)$$

Range of 'n' :

Lower limit of y (n) i.e. $y_l = x_l + h_l = 0 + (-2) = -2$ (Addition of lower range of x (n) and h (n))

Higher limit of y (n) i.e. $y_h = x_h + h_h = 6 + 2 = 8$ (Addition of upper range of x (n) and h (n))

Thus range of n is from − 2 to + 8.

Range of 'k' :

Range of 'k' is similar to x (k). Thus range of 'k' is from 0 to 6.

Now using Equation (5) we will obtain n for different values of n from − 2 to 8.

$$y(n) = \sum_{k=0}^{6} x(k)\, h(n-k) \quad ...(6)$$

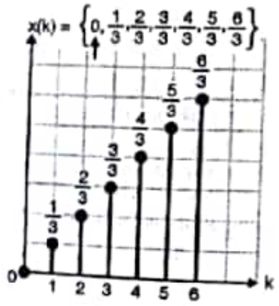
Calculation of y (− 2) :

Putting n = − 2 in Equation (6) we get,

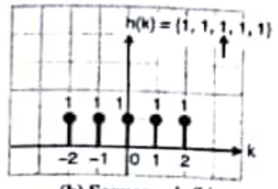
$$y(-2) = \sum_{k=0}^{6} x(k)\, h(-2-k) \quad ...(7)$$

Step 1 : First plot x (k) as shown in Fig. P. 3.6.1(a)

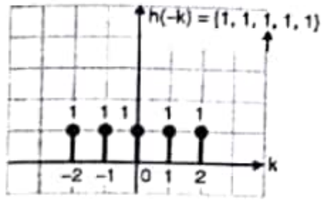
Plot of h (k) is shown in Fig. P. 3.6.1(b). From this obtain h (− k) which is folded version of as h (k) as shown in Fig. P. 3.6.1(c). Observe that h (k) and h (− k) are same in this case.



(H-309)Fig. P. 3.6.1(a) : Sequence x (k)



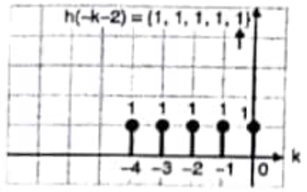
(b) Sequence h (k)



(c) Sequence h (− k)

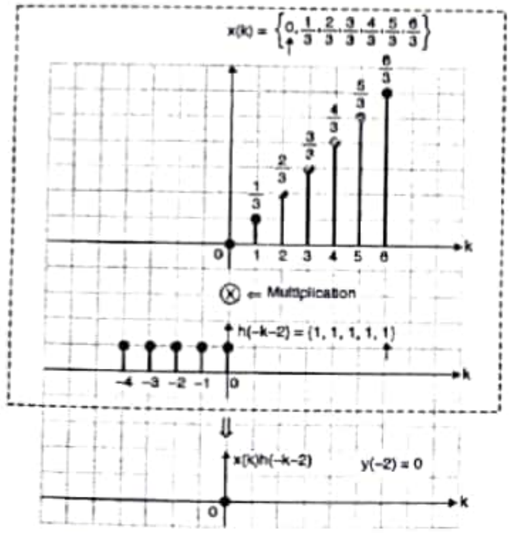
(H-309)Fig. P. 3.6.1

Step 2 : h (− 2 − k) is same as h (− k − 2). It indicates advance version of h (− k). It is obtained by shifting h (− k) towards left by '2' samples. This sequence is shown in Fig. P. 3.6.1(d).



(H-311)Fig. P. 3.6.1(d) : Sequence h (− k − 2)

Step 3 : The product of x (k) and h (− k − 2) is shown in Fig. P. 3.6.1(e)



(H-312)Fig. P. 3.6.1(e) : Product of x (k) and h (− k − 2)

Step 4 : y (− 2) is obtained by adding all product terms

$$\therefore y(-2) = 0$$

Similarly we can obtain the remaining values of y (n) as follows :

$$y(-1) = \sum_{k=0}^{6} x(k)\, h(-1-k) = 1/3 \quad ...(8)$$

$$y(0) = \sum_{k=0}^{6} x(k)\, h(-k) = 1 \quad ...(9)$$

$$y(1) = \sum_{k=0}^{6} x(k)\, h(1-k) = 2 \quad ...(10)$$

$$y(2) = \sum_{k=0}^{6} x(k)\, h(2-k) = 10/3 \quad ...(11)$$

$$y(3) = \sum_{k=0}^{6} x(k)\, h(3-k) = 5 \quad ...(12)$$

$$y(4) = \sum_{k=0}^{6} x(k)\, h(4-k) = 20/3 \quad ...(13)$$

$$y(5) = \sum_{k=0}^{6} x(k)\, h(5-k) = 6 \quad ...(14)$$

$$y(6) = \sum_{k=0}^{6} x(k)\, h(6-k) = 5 \quad ...(15)$$

$$y(7) = \sum_{k=0}^{6} x(k)\, h(7-k) = \frac{11}{3} \quad ...(16)$$

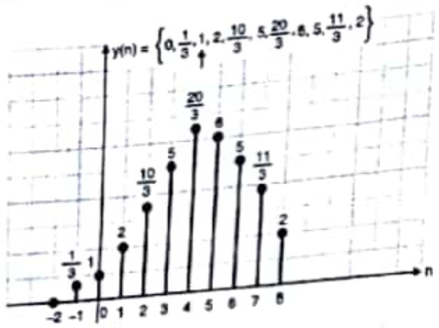
$$y(8) = \sum_{k=0}^{6} x(k)\, h(8-k) = 2 \quad ...(17)$$

$$\therefore y(n) = \{y(-2), y(-1), y(0), y(1), y(2), y(3), y(4), y(5), y(6), y(7), y(8)\}$$

$$\therefore y(n) = \left\{\underset{\uparrow}{0}, \frac{1}{3}, 1, 2, \frac{10}{3}, 5, \frac{20}{3}, 6, 5, \frac{11}{3}, 2\right\}$$

Note : Arrow always indicates the sample at n = 0. Thus arrow should be marked at sample y (0).

The sequence y (n) is graphically represented as shown in Fig. P. 3.6.1(f).



(H-332)Fig. P. 3.6.1(f) : Result of linear convolution

Ex. 3.6.2 : Prove and explain graphically the difference between relations :

1. $x(n)\,\delta(n-n_0) = x(n_0)$
2. $x(n) * \delta(n-n_0) = x(n-n_0)$

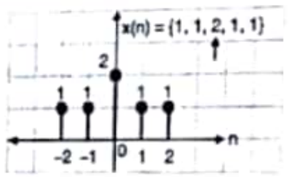
Soln. :

1. **$x(n)\,\delta(n-n_0) = x(n_0)$:**

Here x (n) is input sequence. Let the input sequence be given by,

$$x(n) = \{1, 1, \underset{\uparrow}{2}, 1, 1\}$$

This sequence is plotted as shown in Fig. P. 3.6.2(a).

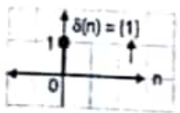


(H-333)Fig. P. 3.6.2(a) : Sequence x (n)

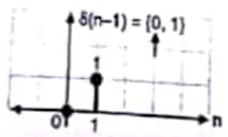
We know that δ (n) represent unit impulse. It is defined as,

$$\delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{elsewhere}$$

Sequence δ (n) is shown in Fig. P. 3.6.2(b).



(b) Unit impulse, δ (n)



(c) Sequence δ (n − 1)
(H-334)Fig. P. 3.6.2

Now δ (n − n_0) indicates delay of δ (n) by no units. So sequence δ (n − n_0) can be represented as,

$$\delta(n-n_0) = 1 \quad \text{for } n = n_0$$
$$= 0 \quad \text{for } n \neq n_0$$

For example, say $n_0 = 1$. Thus we get,

$$\delta(n-1) = 1 \quad \text{for } n = 1$$
$$= 0 \quad \text{for } n \neq n_0$$

The sequence δ (n − 1) is plotted as shown in Fig. P. 3.6.2(c).

Now multiplication of x (n) and δ (n − n_0) i.e. δ (n − 1) is shown in Fig. P. 3.6.2(d). Here we are getting the resultant sample only at n = n_0. We have assumed n_0 = 1. Thus result of multiplication is the sample at n = 1. This value is 1; which is the value of x (1).

$$\therefore\ x(n)\,\delta(n-1) = x(1)$$

Thus in general we can say,

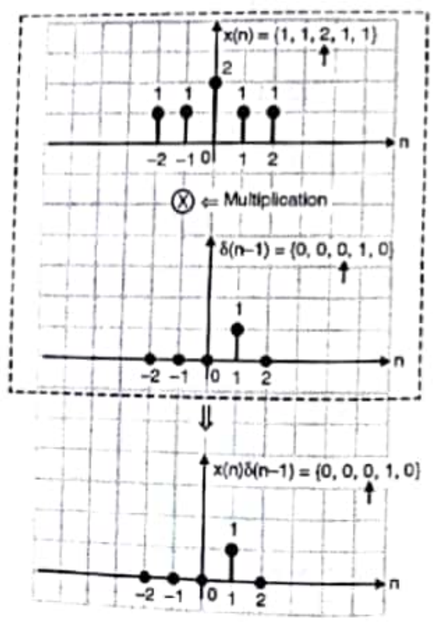
$$x(n)\,\delta(n-n_0) = x(n_0)$$

Hence proved.

2. **$x(n) * \delta(n-n_0) = x(n-n_0)$:**

Here '*' indicates convolution. So we have to calculate convolution of x (n) and δ (n − n_0). According to the definition of convolution,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ...(1)$$



(H-336)Fig. P. 3.6.2(d) : Product of x (n) δ (n − n_0)

Let $h(n) = \delta(n-n_0)$ ...(2)

Now assume two sequence as follows,

$$x(n) = \{1, 1, \underset{\uparrow}{1}\} \text{ and } h(n) = \delta(n-n_0) = \{0, \underset{\uparrow}{1}\}$$

also assume that value of $n_0 = 1$

Note : We cannot assume any arbitrary sequence for h (n). Since h (n) = δ (n − n_0); it should be in terms of delayed unit impulse.

Step 1 : Sequences x (k) and h (k) can be written by directly replacing 'n' by 'k' :

$$\therefore\ x(k) = \{1, 1, \underset{\uparrow}{1}\} \text{ and } h(k) = \delta(k-n_0) = \{0, \underset{\uparrow}{1}\}$$

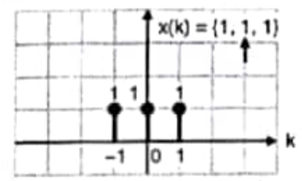
Thus different samples of x (k) and h (k) can be written as,

x (− 1) = 1 similarly, h (0) = 0

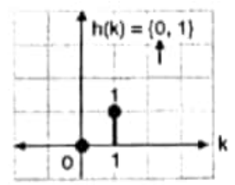
x (0) = 1 and h (1) = 1

and x (1) = 1

Plot of x (k) and h (k) is shown in Fig. P. 3.6.2(e).



(e) Sequence x (k)



(f) Sequence h (k)
(H-337)Fig. P. 3.6.2

Step 2 : Now range of 'n' is calculated as follows :

lower value of $n \Rightarrow y_l = x_l + h_l = -1 + 0 = -1$ and

higher value of $n \Rightarrow y_h = x_h + h_h = 1 + 1 = 2$

Thus range of 'n' is from − 1 to 2. That means we have to calculate sequence y (n) from y (− 1) to y (2).

Range of 'k' is same as x (k). Thus 'k' varies from − 1 to + 1.

Step 3 : Putting the value of 'k' Equation (1) can be written as,

$$y(n) = \sum_{k=-1}^{1} x(k)\,h(n-k) \qquad ...(3)$$

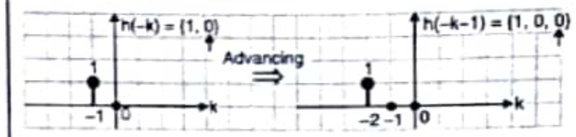
To calculate y (n); put values of 'n' from − 1 to + 2.

Calculation of y (− 1) :

Putting n = − 1 in Equation (3) we get,

$$y(-1) = \sum_{k=-1}^{1} x(k)\,h(-1-k) \qquad ...(4)$$

Here h (− 1 − k) is same as h (− k − 1) · h (− k) is folded version of h (k) and h (− k − 1) indicates advance of h (− k) by 1 sample. This sequence is shown in Fig. P. 3.6.2(g).



(H-339)Fig. P. 3.6.2(g) : Sequence h (− k − 1)

The multiplication of x (k) and h (− k − 1) is shown in Fig. P. 3.6.2(h).

y (− 1) is obtained by adding all product terms.

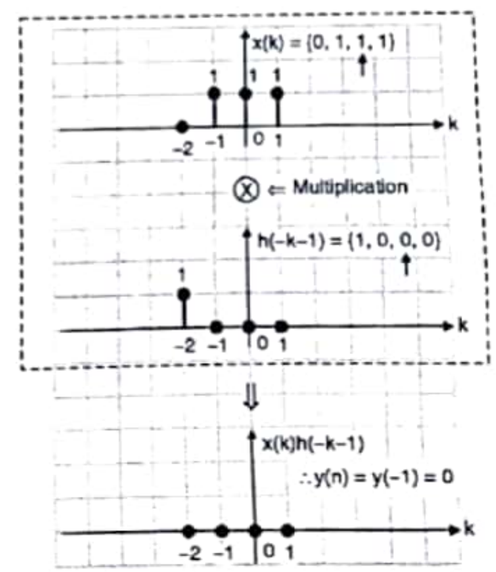
$$\therefore\ y(-1) = 0$$

Similarly we can calculate the remaining values of y (n).

$$y(0) = \sum_{k=-1}^{1} x(k)\,h(-k) = 1 \qquad ...(5)$$

$$y(1) = \sum_{k=-1}^{1} x(k)\,h(1-k) = 1 \qquad ...(6)$$

$$y(2) = \sum_{k=-1}^{1} x(k)\,h(2-k) = 1 \qquad ...(7)$$



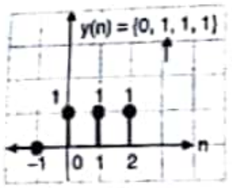
(H-340)Fig. P. 3.6.2(h) : Product of x (k) and h (− k − 1)

The output sequence y (n) is.

$$y(n) = \{y(-1), y(0), y(1), y(2)\}$$

$$\therefore \quad y(n) = \{\underset{\uparrow}{0}, 1, 1, 1\}$$

This sequence is plotted in Fig. P. 3.6.2(i).



(H-346)**Fig. P. 3.6.2(i) : Output y (n)**

Now the output sequence y (n) can also be written as,

$$\therefore \quad y(n) = \{\underset{\uparrow}{1}, 1, 1\}$$

Observe graph of x (k) shown in Fig. P. 3.6.2(e) and graph of y (i) shown in Fig. P. 3.6.2(i). It shows that output y (n) is obtained by delaying x (k) by '1' sample.

$$\therefore \quad y(n) = x(k-1)$$

But x (k) is same as x (n)

$$\therefore \quad y(n) = x(n-1)$$

Initially we have assumed $n_0 = 1$

$$\therefore \quad y(n) = x(n-n_0)$$

$$\therefore \quad y(n) = x(n) * \delta(n-n_0) = x(n-n_0)$$

Hence proved.

Note : This indicates that the convolution of input sequence x (n) with delayed unit impulse is equivalent to delay of input sequence by corresponding samples (n_0).

Ex. 3.6.3 : Use discrete convolution to find the response to the input $x(n) = a^n u(n)$ of the LTI system with impulse response $h(n) = b^n u(n)$.

Soln. :

Given : $x(n) = a^n u(n)$ and $h(n) = b^n u(n)$.

Sequences x (k) and h (k) can be directly written as,

$x(k) = a^k u(k)$ and $h(k) = b^k u(k)$

According to definition of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(1)$$

We have $h(k) = b^k u(k)$ ...(2)

Sequence h (n – k) is obtained by replacing k by n – k in Equation (2)

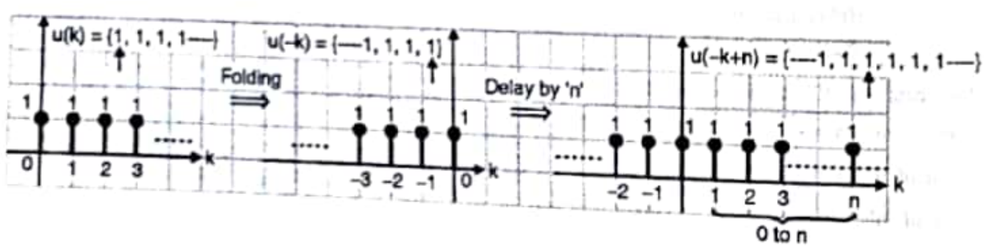
$$\therefore \quad h(n-k) = b^{n-k} u(n-k) \qquad ...(3)$$

Thus Equation (1) becomes,

$$y(n) = \sum_{k=-\infty}^{\infty} a^k u(k) \cdot b^{n-k} u(n-k)$$

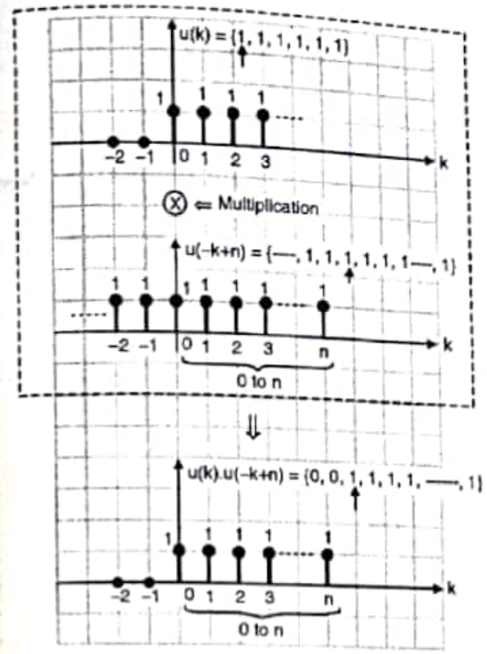
$$\therefore \quad y(n) = \sum_{k=-\infty}^{\infty} a^k \cdot b^{n-k} u(k)\, u(n-k) \qquad ...(4)$$

Sequence u (k) is unit step. While u (n – k) = u (– k + n) indicates delay of u (– k) by 'n' samples. This sequence is obtained as shown in Fig. P. 3.6.3.



(H-359)**Fig. P. 3.6.3 : Sequence u ((n – k)**

Now the product of u (k) and u (n – k) is shown in Fig. P. 3.6.3(a).



(H-360)**Fig. P. 3.6.3(a) : Product of u (k) and u (n – k)**

This multiplication i.e. u (k) · u ((n – k) indicates that the sequence is present only for the range 0 to n. So we will change the limits of summation in Equation (4). New limits of k will be from 0 to n.

$$\therefore \quad y(n) = \sum_{k=0}^{n} a^k \cdot b^{n-k} u(k) \cdot u(n-k) \qquad ...(5)$$

But the term u (k) · u (n – k) = 1.

$$\therefore \quad y(n) = \sum_{k=0}^{n} a^k \cdot b^{n-k} \qquad ...(6)$$

$$\text{Now } b^{n-k} = b^n \cdot b^{-k} = \frac{b^n}{b^k}$$

$$\therefore \quad y(n) = \sum_{k=0}^{n} a^k \frac{b^n}{b^k} \qquad ...(7)$$

Since the summation is in terms of 'k' we can take the term b^n outside the summation.

$$\therefore \quad y(n) = b^n \sum_{k=0}^{n} \frac{a^k}{b^k}$$

$$\therefore \quad y(n) = b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k \qquad ...(8)$$

Use standard equation of summation,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{For } a \neq 1 \\ N_2 - N_1 + 1 & \text{For } a = 1 \end{cases} \qquad ...(9)$$

Compare Equations (8) and (9).

Here $N_1 = 0$, $N_2 = n$ and $a = \left(\frac{a}{b}\right)$

$$\therefore \quad y(n) = b^n \cdot \left[\frac{\left(\frac{a}{b}\right)^0 - \left(\frac{a}{b}\right)^{n+1}}{1 + \frac{a}{b}}\right] \qquad ...\text{ for } \frac{a}{b} \neq 1$$

$$\therefore \quad y(n) = b^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}}\right] \qquad ...(10)$$

Equation (10) gives convolution of $a^n u(n)$ and $b^n u(n)$.

Ex. 3.6.4 : Obtain the expression for convolution of unit step sequence with finite duration sequence.

Soln. : We will assume some sequence x (n) as,

$$x(n) = \{\underset{\uparrow}{2}, 2, 1\} \qquad ...(1)$$

Note that this is finite duration sequence.

We have to obtain convolution of x (n) with unit step u (n).

So let, $h(n) = u(n)$ ...(2)

The equations of h (k) and x (k) can be directly written as,

$$\therefore \quad x(k) = \{\underset{\uparrow}{2}, 2, 1\}$$

$$\text{and } h(k) = u(k) = \{\underset{\uparrow}{1}, 1, 1, ...\infty\}$$

According to the definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(3)$$

To obtain range of y (n) :

Lowest index of $y(n) \Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest index of $y(n) \Rightarrow y_h = x_h + h_h = 2 + \infty = \infty$

Thus range of y (n) is from '0' to '∞'. That means we have to obtain y (n) for different values of n from 0 to ∞.

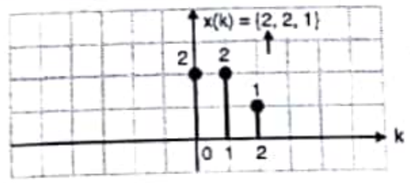
Now the range of k is same as x (k). So range of k is from 0 to 2.

Calculation of y (0) :

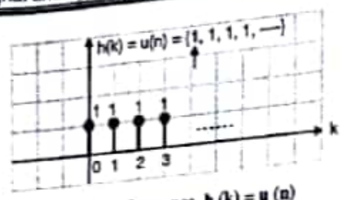
y (0) is calculated by putting n = 0 in Equation (3).

$$\therefore \quad y(0) = \sum_{k=0}^{2} x(k)\, h(-k) \qquad ...(4)$$

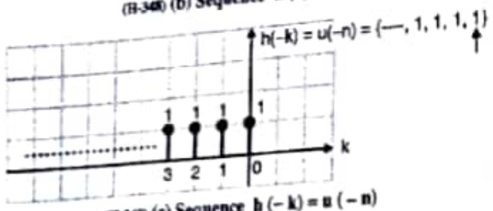
The sequence x (k) is shown in Fig. P. 3.6.4(a). Sequence h (k) = u (n) is shown in Fig. P. 3.6.4(b). Now h (– k) is folded version of h (k). It is obtained by folding unit step u (n). Sequence h (– k) is shown in Fig. P. 3.6.4(c).



(H-347)**Fig. P. 3.6.4(a) : Sequence x (k) = { 2, 2, 1}**



(H-348) (b) Sequence h (k) = u (n)



(H-349) (c) Sequence h (− k) = u (− n)

Fig. P. 3.6.4

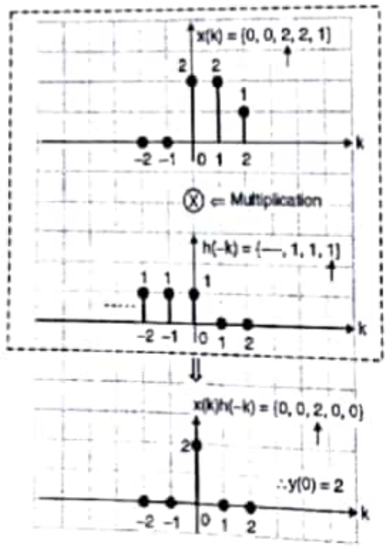
According to Equation (4) y (0) is obtained by multiplying x (k) by h (− k) and then adding all product terms. This is shown in Fig. P. 3.6.4(d).

$\therefore$ y (0) = 2

Calculation of y (1) :

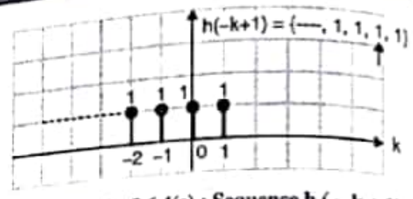
Putting n = 1 in Equation (3) we get,

$$y(n) = \sum_{k=0}^{2} x(k)\, h(1-k) \qquad ...(5)$$



(H-350) Fig. P. 3.6.4(d) : Product of x (k) and h (− k)

Here h (1 − k) is same as h (− k + 1). It indicates delay of h (− k) by '1' sample. This sequence is shown in Fig. P. 3.6.4(e).



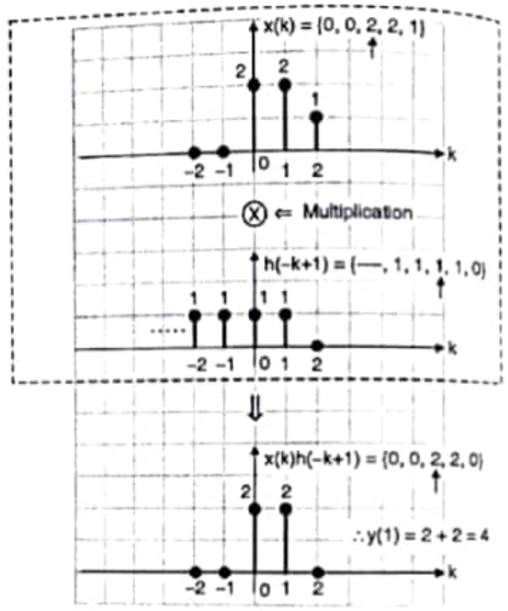
(H-351) Fig. P. 3.6.4(e) : Sequence h (− k + 1)

The multiplication of x (k) and h (− k + 1) is shown in Fig. P. 3.6.3(f).

Now y (1) is obtained by adding all product terms.

$\therefore$ y (1) = (2 × 1) + (2 × 1) = 2 + 2 = 4

$\therefore$ y (1) = 4



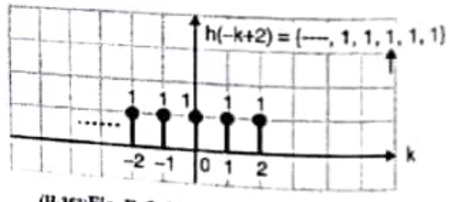
(H-352) Fig. P. 3.6.4(f) : Product of x (k) and h (− k + 1)

Calculation of y (2) :

Putting n = 2 in Equation (3) we get,

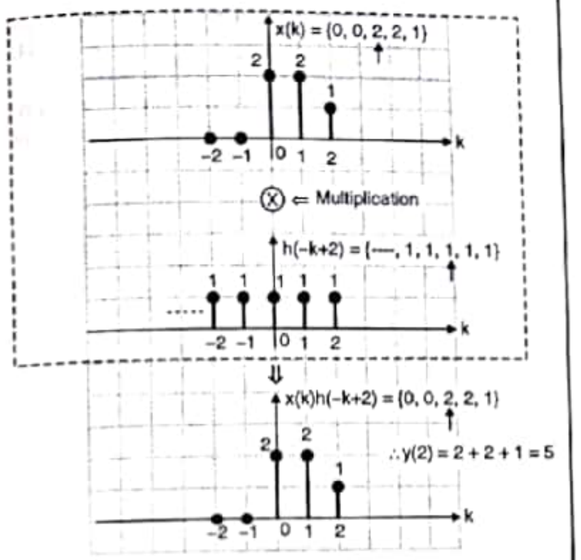
$$y(2) = \sum_{k=0}^{2} x(k)\, h(2-k) \qquad ...(6)$$

h (2 − k) is same as h (− k + 2). It is obtained by shifting h (− k) towards right by '2' samples as shown in Fig. P. 3.6.4(g).



(H-353) Fig. P. 3.6.4(g) : Sequence h (− k + 2)

The product of x (k) and h (− k + 2) is shown in Fig. P. 3.6.4(h).



(H-354) Fig. P. 3.6.4(h) : Product of x (k) and h (− k + 2)

$\therefore$ y (2) = (2 × 1) + (2 × 1) + (1 × 1)

= 2 + 2 + 1 = 5

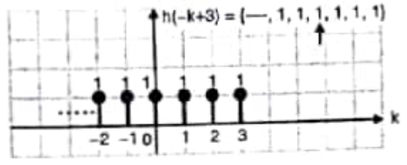
$\therefore$ y (2) = 5

Calculation of y (3) :

Putting n = 3 in Equation (3) we get,

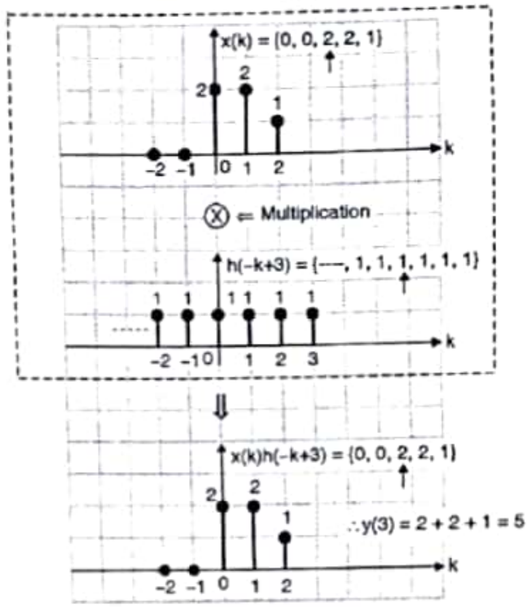
$$y(3) = \sum_{k=0}^{2} x(k)\, h(3-k) \qquad ...(7)$$

Sequence h (3 − k) is same as h (− k + 3). It is obtained by shifting h (− k) towards right by 3 samples. This sequence is shown in Fig. P. 3.6.4(i).



(H-355) Fig. P. 3.6.4(i) : Sequence h (− k + 3)

The product of x (k) and h (− k + 3) is shown in Fig. P. 3.6.4(j).



(H-356) Fig. P. 3.6.4(j) : Product of x (k) and h (− k + 3)

$\therefore$ y (3) = (2 × 1) + (2 × 1) + (1 × 1) = 2 + 2 + 1 = 5

$\therefore$ y (3) = 5

Since unit step is infinite sequence; it is clear that all the next samples of y (n) will have magnitude equal to 5. This is because every time we will shift h (− k) towards right by 1 sample. So for every value of n the product of x (k) and delayed h (− k) will remain same.

$\therefore$ y (4) = 5, y (5) = 5, y (6) = 5 ....

Now the resultant output y (n) can be written as,

y (n) = { y (0), y (1), y (2), y (3), ...∞ }

$\therefore$ y (n) = { 2, 4, 5, 5, 5, ...} ....(8)

↑

General expression :

The given sequence x (n) is ,

$\therefore$ x (n) = { 2, 2, 1} and h (n) = u (n) = { 1, 1, 1, 1, ...}

↑ ↑

$\therefore$ x (0) = 2 $\qquad\therefore$ h (0) = u (0) = 1

x (1) = 2 $\qquad$ h (1) = u (1) = 1

and x (2) = 1 $\qquad$ h (2) = u (2) = 1

h (3) = u (3) = 1 ....

Now observe Equations (6) and (7) carefully. We can write the generalized expression to obtain Equation (8) as follows.

$$y(n) = x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots \qquad ...(9)$$

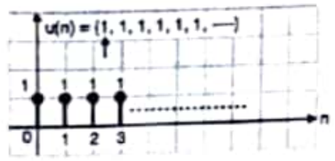
Let us verify this equation.

To calculate y (0) put n = 0 in Equation (9).

$\therefore$ y (0) = x (0) u (0) + x (1) u (− 1) + x (2) u (− 2)

$\therefore$ y (0) = (2 × 1) + (2 × 0) + (1 × 0) = 2

This is because u (0) = 1, u (− 1) = 0 and u (− 2) = 0. This can be observed from unit step shown in Fig. P. 3.6.4(k).



(II-357)Fig. P. 3.6.4(k) : Unit step u (n)

Similarly putting n = 1 in Equation (9) we get,

$$\therefore \; y(1) = x(0)\,u(1) + x(1)\,u(1) + x(2)\,u(-1)$$

$$\therefore \; y(1) = (2\times 1) + (2\times 1) + (1\times 0) = 2 + 2 + 0$$

$$\therefore \; y(1) = 4$$

Similarly other values can be verified.

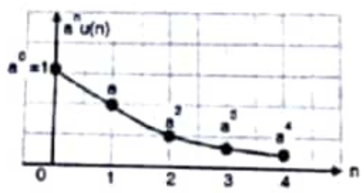
Ex. 3.6.5 : Obtain linear convolution of two discrete time signals given as :

$x(n) = u(n)$

$h(n) = a^n u(n),\; a < 1$

Soln. :

Given x (n) = u (n) i.e. unit step which is shown in Fig. P. 3.6.4(k). $h(n) = a^n u(n)$ is an exponential signal. Here a^n is exponential signal which is multiplied by u (n). We know that u (n) is present only for positive values of 'n'. That means from n = 0 to n = ∞. So this multiplication indicates that a^n is present for positive values of 'n'. It will not affect the magnitude of a^n, because magnitude of u (n) is 1. This sequence a^n is shown in Fig. P. 3.6.5(a).



(II-358)Fig. P. 3.6.5(a) : Exponential sequence a^n

Recall the equation of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ...(1)$$

In this equation h (n − k) is delayed version of h (k). Instead of that we can delay x (k) and keep h (k) as it is.

$$\therefore \; y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad ...(2)$$

This is commutative property of convolution which will be discussed later in this chapter. We will use Equation (2) in this example because it is easy to delay u (n) than a^n u (n).

Now we have derived generalized expression of convolution for finite sequence with an infinite sequence. This equation is,

$$y(n) = x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots \qquad ...(3)$$

When both sequences are infinite then Equation (3) can be easily modified as,

$$y(n) = \ldots\ldots x(-2)\,u(n+2) + x(-1)\,u(n+1) + x(0)\,u(n) + x(1)\,u(n-1) + x(2)\,u(n-2) + \ldots \qquad ...(4)$$

Here x (n) is some infinite sequence and h (n) = u (n).

Equation (4) is based on Equation (1). In this case input x (k) is kept as it is and h (k) is delayed. But if we want to use Equation (2) to obtain linear convolution then,

$$y(n) = \ldots\ldots h(-2)\,u(n+2) + h(-1)\,u(n+1) + h(0)\,u(n) + h(1)\,u(n-1) + h(2)\,u(n-2) + \ldots. \qquad ...(5)$$

In Equation (5) we are taking x (n) = u (n).

The given equation of h (n) is,

$$h(n) = a^n u(n) \qquad ...(6)$$

$$\therefore h(n) = \begin{cases} a^n & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases} \qquad ...(7)$$

Sequence h (n) can be written by putting values of n in Equation (7).

That means,

For $n = 0 \Rightarrow h(n) = h(0) = a^0 = 1$

For $n = 1 \Rightarrow h(n) = h(1) = a^1 = a$

For $n = 2 \Rightarrow h(n) = h(2) = a^2$

For $n = 3 \Rightarrow h(n) = h(3) = a^3$

$$\therefore h(n) = \{\underset{\uparrow}{1}, a, a^2, a^3 \ldots\} \qquad ...(8)$$

Putting values of h (n) in Equation (5) we get,

$$\therefore \; y(n) = 1\,u(n) + a\,u(n-1) + a^2 u(n-2) + a^3 u(n-3) + \ldots \qquad ...(9)$$

Equation (9) gives the convolution of given sequences. To calculate sequence y(n), put different values of n in Equation (9).

$$\therefore \; y(0) = u(0) + a\,u(-1) + a^2 u(-2) + a^3 u(-3)$$

Here $u(0) = 1,\; u(-1) = u(-2) = u(-3) = 0$

$$\therefore \; y(0) = 1$$

Putting n = 1 in Equation (9) we get,

$$y(1) = 1\,u(1) + a\,u(0) + a^2 u(-1) + a^3 u(-2)$$

Here $u(1) = u(0) = 1$ and $u(-1) = u(-2) = 0$

$$\therefore \; y(1) = 1 + a$$

Now putting n = 2 in Equation (9) we get,

$$y(2) = u(2) + a\,u(1) + a^2 u(0) + a^3 u(-1)$$

Here $u(2) = u(1) = u(0) = 1$ and $u(-1) = 0$

$$\therefore \; y(2) = 1 + a + a^2$$

Similarly for the nth sample, y (n) can be written as,

$$\therefore \; y(n) = 1 + a + a^2 + a^3 + \ldots\ldots + a^n \qquad ...(10)$$

In terms of summation Equation (10) can be written as,

$$y(n) = \sum_{k=0}^{n} a^k \qquad ...(11)$$

Now use standard equation of summation,

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{For } a \neq 1 \\ N_2 - N_1 + 1 & \text{For } a = 1 \end{cases} \qquad ...(12)$$

In the present example a < 1 thus a ≠ 1.

Now from Equation (11),

N_1 = lower limit of summation = 0

N_2 = upper limit of summation = n

Putting these values in the standard equation of summation (Equation (12)) we get,

$$y(n) = \frac{a^0 - a^{n+1}}{1-a}$$

$$\therefore \; y(n) = \frac{1 - a^{n+1}}{1-a}$$

Ex. 3.6.6 : Convolve $x[n] = \left(\frac{1}{3}\right)^n u[n]$ with $h[n] = \left(\frac{1}{2}\right)^n u[n]$ using convolution integral.

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Soln. :

Given : $x(n) = \left(\frac{1}{3}\right)^n u(n)$ and $h(n) = \left(\frac{1}{2}\right)^n u(n)$

According to definition of convolution

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\,h(n-K)$$

Here $x(K) = \left(\frac{1}{3}\right)^K u(K)$ and $h(n-K) = \left(\frac{1}{2}\right)^{n-K} u(n-K)$

$$\therefore h(n-K) = \left(\frac{1}{2}\right)^n \cdot 2^K u(-K+n)$$

$$\therefore \; y(n) = \sum_{K=-\infty}^{\infty} \left(\frac{1}{3}\right)^K u(K) \cdot \left(\frac{1}{2}\right)^n \cdot 2^K u(-K+n)$$

$$= \left(\frac{1}{2}\right)^n \sum_{K=-\infty}^{\infty} \left(\frac{2}{3}\right)^K u(K) \cdot u(-K+n)$$

Due to the term u (K) · u (− K + n) ; the limits of summation will be K = 0 to n.

$$\therefore \; y(n) = \left(\frac{1}{2}\right)^n \sum_{K=0}^{n} \left(\frac{2}{3}\right)^K$$

We have standard summation formula,

$$\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a}$$

Here $a = \frac{2}{3},\; N_1 = 0,\; N_2 = n$

$$\therefore \; y(n) = \left(\frac{1}{2}\right)^n \cdot \left[\frac{\left(\frac{2}{3}\right)^0 - \left(\frac{2}{3}\right)^{n+1}}{1 - \frac{2}{3}}\right]$$

$$\therefore \; y(n) = \left(\frac{1}{2}\right)^n \cdot 3\left[1 - \left(\frac{2}{3}\right)^{n+1}\right]$$

3.6.2 Calculation using Basic Equation of Convolution :

In the graphical method, we studied how to obtain convolution of two signals, using graphs. In that method, every time it is required to draw the plot of signal. So that is tedious method. One more simple method is to calculate the convolution of two signals using basic equation of convolution.

How to solve ?

According to the definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ...(3.6.5)$$

Step 1 : **Write down the equations of x (k) and h (k)**

Step 2 : **Decide range of 'n'**

We have studied, how to decide range of 'n'. Let us recall those equations.

Lowest index of 'n' for y (n) ⇒ Lowest index of x (n) + Lowest index of h (n)

$$\therefore \; y_l = x_l + h_l \qquad ...(3.6.6)$$

and Highest index of 'n' for y (n) ⇒ Highest index of x (n) + Highest index of h (n)

$$\therefore \; y_h = x_h + h_h \qquad ...(3.6.7)$$

Decide range of 'k' :

Range of k is same as range of x (k).

Step 3 : **Once the range of 'n' and 'k' is decided; then put different values of 'n' in the equation of y (n).**

Ex. 3.6.7 : Compute linear convolution of the following :

$x(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$ and $h(n) = \{\underset{\uparrow}{1}, 1, 1, 1\}$

Soln. : We have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ...(1)$$

Step 1 : **The sequences x (k) and h (k) can be written as,**

$x(k) = \{\underset{\uparrow}{1}, 1, 1, 1\}$ and $h(k) = \{\underset{\uparrow}{1}, 1, 1, 1\}$

∴ x (0) = 1	and	h (0) = 1
x (1) = 1		h (1) = 1
x (2) = 1		h (2) = 1
x (3) = 1		h (3) = 1

Step 2 : **Range of 'n' for y (n) is,**

Lowest index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 3 + 3 = 6$

Thus range of y (n) is from y (0) to y (6).

Now range of 'k' is same as x (k).

Thus range of 'k' is from k = 0 to k = 3.

Step 3 : **Putting the value of 'k' in Equation (1) we get,**

$$y(n) = \sum_{k=0}^{3} x(k)\,h(n-k) \qquad ...(2)$$

Calculation of y (0) :

Putting n = 0 in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} x(k)\,h(0-k) = \sum_{k=0}^{3} x(k)\,h(-k)$$

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Expanding the summation by putting values of 'k' we get,

$$y(0) = x(0)\,h(-0) + x(1)\,h(-1) + x(2)\,h(-2) + x(3)\,h(-3)$$

$$\therefore\ y(0) = (1\times 1) + (1\times 0) + (1\times 0) + (1\times 0) \ldots \text{ as } h(-1) = h(-2) = h(-3) = 0$$

$$\therefore\ y(0) = 1$$

Calculation of y (1) :

Putting n = 1 in Equation (2) we get,

$$y(1) = \sum_{k=0}^{3} x(k)\,h(1-k)$$

Expanding the summation we get,

$$y(1) = x(0)\,h(1) + x(1)\,h(0) + x(2)\,h(-1) + x(3)\,h(-2)$$

$$\therefore\ y(1) = (1\times 1) + (1\times 1) + (1\times 0) + (1\times 0).$$

$$\therefore\ y(1) = 1 + 1 \quad \ldots \text{ as } h(-1) = h(-2) = 0$$

$$\therefore\ y(1) = 2$$

Similarly we can calculate the remaining values as follows :

$$y(2) = \sum_{k=0}^{3} x(k)\,h(2-k) = 3$$

$$y(3) = \sum_{k=0}^{3} x(k)\,h(3-k) = 4$$

$$y(4) = \sum_{k=0}^{3} x(k)\,h(4-k) = 3$$

$$y(5) = \sum_{k=0}^{3} x(k)\,h(5-k) = 2$$

$$y(6) = \sum_{k=0}^{3} x(k)\,h(6-k) = 1$$

Step 4 : The result of convolution y (n) is given by,

$$y(n) = \{y(0), y(1), y(2), y(3), y(4), y(5), y(6)\}$$

$$\therefore\ y(n) = \{\underset{\uparrow}{1}, 2, 3, 4, 3, 2, 1\}$$

Note that arrow is marked at the sample y (0).

Ex. 3.6.8 : Compute the linear convolution of following :
x (n) = 1 and h (n) = {2, 1, 2, 1}

Soln. :

Step 1 : The sequences x (k) and h (k) can be written as,

$$x(k) = \{\underset{\uparrow}{1}\} \text{ and } h(k) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

$$\therefore\ x(0) = 1 \qquad \therefore\ h(0) = 2,\ h(1) = 1,\ h(2) = 2,\ h(3) = 1$$

Note that arrow is not present in the given sequences of x (n) and h (n). So by default it is at first (starting) position.

Step 2 : Range of 'n' for y (n) is,

Lowest index : $y_l = x_l + h_l = 0 + 0 = 0$

Highest index : $y_h = x_h + h_h = 0 + 3 = 3$

Thus range of y (n) is from y (0) to y (3).

Step 3 : According to definition of convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad \ldots(1)$$

In this case h (n – k) indicates folding and delaying h (– k). Since x (k) is smaller sequence than h (k); it will be convenient to use following equation,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad \ldots(2)$$

Here range of 'k' will be as per the sequence h(k). Thus range of 'k' is from 0 to 3.

Calculation of y (0) :

Putting n = 0 in Equation (2) we get,

$$y(0) = \sum_{k=0}^{3} h(k)\,x(-k)$$

Expanding the summation we get,

$$y(0) = h(0)\,x(0) + h(1)\,x(-1) + h(2)\,x(-2) + h(3)\,x(-3)$$

$$\therefore\ y(0) = (2\times 1) + (1\times 0) + (2\times 0) + (1\times 0)$$

$$\therefore\ y(0) = 2 + 0 + 0 + 0 \qquad \ldots \text{ as } x(-1) = x(-2) = x(-3) = 0$$

$$\therefore\ y(0) = 2$$

Similarly we can calculate the remaining values of y (n)

$$\therefore\ y(1) = 1$$

$$\therefore\ y(2) = 2$$

$$\therefore\ y(3) = 1$$

Step 4 : Now the result of convolution can be written as,

$$y(n) = \{y(0), y(1), y(2), y(3)\}$$

$$\therefore\ y(n) = \{\underset{\uparrow}{2}, 1, 2, 1\}$$

Note : Here given sequence x(n) = {1}. We know that δ (n) is the unit impulse and is expressed as, δ(n) = {1}. That means we have calculated convolution of {2, 1, 2, 1} with unit impulse.

Observe the result of convolution. It is the same as {2, 1, 2, 1}. Thus we can conclude that the convolution of any sequence with unit impulse produces the same sequence.

3.6.3 Tabulation Method of Linear Convolution :

This is the simplest method of performing linear convolution.

Let $x(n) = \{\underset{\uparrow}{x(0)}, x(1), x(2)\}$ and $h(n) = \{\underset{\uparrow}{h(0)}, h(1), h(2)\}$

Step 1 : Form the matrix as shown in Fig. 3.6.1 :

		x(n)		
		x(0)	x(1)	x(2)
h(n)	h(0)			
	h(1)			
	h(2)			

(II-365)**Fig. 3.6.1 : Matrix of x (n) and h (n)**

Note that we can interchange the positions of x (n) and h (n).

Step 2 : Multiply the corresponding elements of x (n) and h (n) as shown in Fig. 3.6.2 :

	x(0)	x(1)	x(2)
h(0)	h(0)x(0)	h(0)x(1)	h(0)x(2)
h(1)	h(1)x(0)	h(1)x(1)	h(1)x(2)
h(2)	h(2)x(0)	h(2)x(1)	h(2)x(2)

(II-366)**Fig. 3.6.2 : Multiplication of elements x (n) and h (n)**

Step 3 : Separate out the elements diagonally as shown in Fig. 3.6.3 :

	x(0)	x(1)	x(2)
h(0)	h(0)x(0)	h(0)x(1)	h(0)x(2)
h(1)	h(1)x(0)	h(1)x(1)	h(1)x(2)
h(2)	h(2)x(0)	h(2)x(1)	h(2)x(2)

(II-367)**Fig. 3.6.3 : Diagonally separation of elements**

Step 4 : Simply add the elements in that particular block. This addition gives corresponding values of y (n).

Here range of 'n' is : $y_l = x_l + h_l = 0 + 0 = 0$

and $y_h = x_h + h_h = 2 + 2 = 4$

Thus we get,

$$y(0) = h(0)\,x(0)$$

$$y(1) = h(1)\,x(1) + h(0)\,x(1)$$

$$y(2) = h(2)\,x(0) + h(1)\,x(1) + h(0)\,x(2)$$

$$y(3) = h(2)\,x(1) + h(1)\,x(2)$$

and $y(4) = h(2)\,x(2)$

After calculating all these values; write down the result of convolution as,

$$y(n) = \{\underset{\uparrow}{y(0)}, y(1), y(2), y(3), y(4)\}$$

Ex. 3.6.9 : Compute the convolution y (n) = x (n) * h (n) where $x(n) = \{\underset{\uparrow}{1}, 1, 0, 1, 1\}$ and $h(n) = \{\underset{\uparrow}{1}, -2, -3, 4\}$

Soln. :

Here x (n) contains '5' samples and h (n) has '4' samples. To make the length of x (n) and h (n) same; rewrite the sequence h (n) as follows :

$$h(n) = \{\underset{\uparrow}{1}, -2, -3, 4, 0\}$$

We can add zeros at the end or at the beginning of sequence. This will not affect the given sequence. This method of adding zeros, to adjust the length of sequence is called as **zero-padding.**

Range of 'n' :

The range of 'n' for y (n) is calculated as follows :

Lowest index of $y(n) \Rightarrow y_l = x_l + h_l = -2 + (-3) = -5$

and Highest index of $y(n) \Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 5) to y (3). Now using tabulation method the convolution is obtained as shown in Fig. P. 3.6.9.

		x(n)				
		1	1	0	1	1
h(n)	1					
	–2					
	–3					
	4					
	0					

Fig. P. 3.6.9(a) : Matrix of x (n) and h (n)

	1	1	0	1	1
1	1 × 1	1 × 1	1 × 0	1 × 1	1 × 1
–2	–2 × 1	–2 × 1	–2 × 0	–2 × 1	–2 × 1
–3	–3 × 1	–3 × 1	–3 × 0	–3 × 1	–3 × 1
4	4 × 1	4 × 1	4 × 0	4 × 1	4 × 1
0	0 × 1	0 × 1	0 × 0	0 × 1	0 × 1

(II-368)**Fig. P. 3.6.9(b) : Convolution of x (n) and h (n)**

	y(–5)	y(–4)	y(–3)	y(–2)		
	1	1	0	1	1	y(–1)
1	1	1	0	1	1	y(0)
–2	–2	–2	0	–2	–2	y(1)
–3	–3	–3	0	–3	–3	y(2)
4	4	4	0	4	4	y(3)
0	0	0	0	0	0	

(II-369)**Fig. P. 3.6.9(c) : Diagonally separation of elements**

Different values of y (n) are calculated by adding corresponding elements as follows :

$$y(-5) = 1$$
$$y(-4) = -2+1 = -1$$
$$y(-3) = -3-2+0 = -5$$
$$y(-2) = 4-3+0+1 = 2$$
$$y(-1) = 0+4+0-2+1 = 3$$
$$y(0) = 0+0-3-2 = -5$$
$$y(1) = 0+4-3 = 1$$
$$y(2) = 0+4 = 4$$
$$y(3) = 0$$

Thus result of convolution y (n) is ,

$$y(n) = \{y(-5), y(-4), y(-3), y(-2), y(-1), \underset{\uparrow}{y(0)}, y(1), y(2), y(3)\}$$

$$\therefore\ y(n) = \{1, -1, -5, 2, 3, -5, \underset{\uparrow}{1}, 4, 0\}$$

Neglecting the last '0' term we have,

$$y(n) = \{1, -1, -5, 2, 3, -5, \underset{\uparrow}{1}, 4\}$$

Ex. 3.6.10 : Compute $y(n) = x(n) * h(n)$ If $x(n) = h(n) = \{\underset{\uparrow}{1}, 2, -1, 3\}$

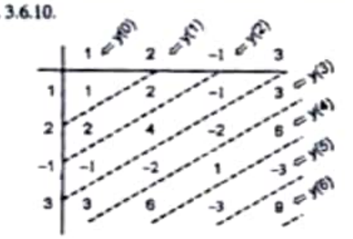
Soln. :

Range of n :

Lowest range of $y(n) \Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest range of $y(n) \Rightarrow y_h = x_h + h_h = 3 + 3 = 6$

Using tabular method; the convolution is obtained as shown in Fig. P. 3.6.10.



(H-370)Fig. P. 3.6.10 : Linear convolution of x (n) and h (n)

$$y(n) = \{\underset{\uparrow}{1}, 4, 2, 2, 13, -6, 9\}$$

Ex. 3.6.11 : The impulse response of linear time invarient is :

$$h(n) = \{\underset{\uparrow}{1}, 2, 1, -1\}$$

Determine the response of system to the input

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 1\}$$

Soln. : The response of the system is,

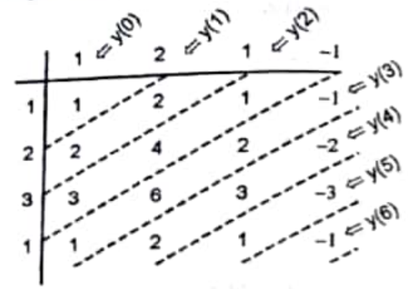
$$y(n) = x(n) * h(n)$$

Range of n :

Lowest range of $y(n) \Rightarrow y_l = x_l + h_l = 0 + 0 = 0$

Highest range of $y(n) \Rightarrow y_h = x_h + h_h = 3 + 3 = 6$

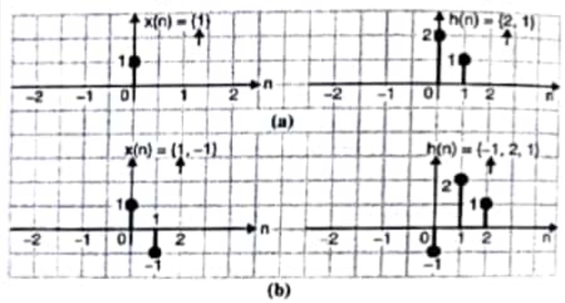
Using tabulation method; the convolution is obtained as shown in Fig. P. 3.6.11.



(H-371)Fig. P. 3.6.11 : Linear convolution of x (n) and h (n)

$$\therefore\ y(n) = \{\underset{\uparrow}{1}, 4, 8, 8, 3, -2, -1\}$$

Ex. 3.6.12 : Use discrete convolution to find the response to the input x (n) of the LTI system with impulse response h (n) for the corresponding x (n) shown in Fig. P. 3.6.12(a) and (b).



(H-373)Fig. P. 3.6.12 : Given sequence x (n) and h (n)

Soln. :

$$\therefore\ y(n) = \{-1, 3, -1, -1\}$$

Ex. 3.6.13 : Find out output of system if input x (n) and impulse response h (n) are given by, $x(n) = n, 0 \le n \le 3$, $h(n) = \{2, 3, 1\}$

Soln. :

Given $x(n) = n, 0 \le n \le 3$.

We will put values of n to obtain sequence x (n).

For $n = 0 \Rightarrow x(0) = 0$

For $n = 1 \Rightarrow x(1) = 1$

For $n = 2 \Rightarrow x(2) = 2$

For $n = 3 \Rightarrow x(3) = 3$

$$\therefore\ x(n) = \{\underset{\uparrow}{0}, 1, 2, 3\}$$

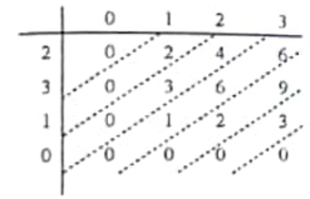
and $h(n) = \{\underset{\uparrow}{2}, 3, 1\} = \{\underset{\uparrow}{2}, 3, 1, 0\}$

Range of 'n' in y (n) is n = 0 to 6.

We will use tabular method to obtain output

$$y(n) = x(n) * h(n)$$

$$\therefore\ y(n) = \{\underset{\uparrow}{0}, 2, 7, 13, 11, 3, 0\}$$



(H-377) Fig. P. 3.6.13

This is the output of system.

Ex. 3.6.14 : Find the following convolution :

If $x(n) = \{\underset{\uparrow}{2}, 3, 1, 4\}$

$h(n) = \{\underset{\uparrow}{-1}, 2, 3\}$

and $y(n) = x(n) * h(n)$.

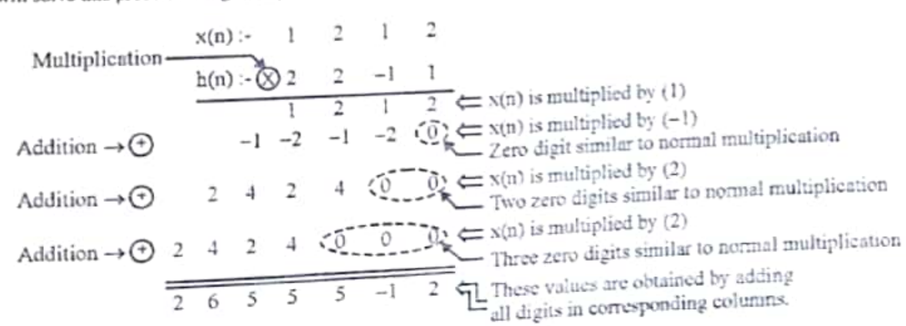
Soln. :

Given $x(n) = \{\underset{\uparrow}{2}, 3, 1, 4\}$

and $h(n) = \{\underset{\uparrow}{-1}, 2, 3\} = \{\underset{\uparrow}{-1}, 2, 3, 0\}$

Here range of 'n' in y (n) is n = – 3 to 2.

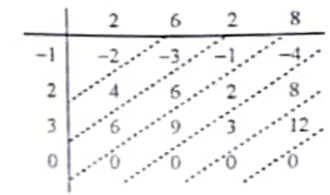
We will solve this problem using multiplication method.



(H-379)Fig. P. 3.6.15

We will use tabular method to obtain

$$y(n) = x(n) * h(n)$$



(H-378)Fig. P. 3.6.14

$$\therefore\ y(n) = \{-2, 1, 11, 7, 11, 12, \underset{\uparrow}{0}\}$$

3.6.4 Multiplication Method of Linear Convolution :

- This is another easy method to obtain convolution of two sequences.
- This method is similar to multiplication of multidigit numbers. Write down the two sequences, x (n) and h (n) and obtain the multiplication by usual method.
- The result of multiplication will be equal to the convolution of two sequences. We will solve one example by this method.

Ex. 3.6.15 : Obtain linear convolution of following sequences :

$x(n) = \{\underset{\uparrow}{1}, 2, 1, 2\}$ and $h(n) = \{2, 2, -1, \underset{\uparrow}{1}\}$

Soln. :

First we will decide the range of 'n' for y (n).

Lowest index $\Rightarrow y_l = x_l + h_l = -1 + (-2) = -3$

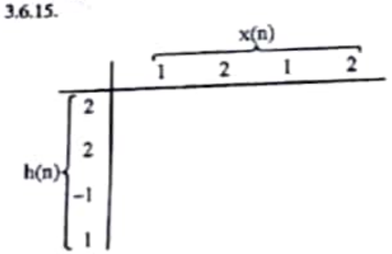
Highest index $\Rightarrow y_h = x_h + h_h = 2 + 1 = 3$

Thus y (n) varies from y (– 3) to y (3)

This result of multiplication is same as linear convolution. We know that range of y (n) is from y (– 3) to y (3). So the first term from L.H.S. represents the value of y (– 3). Now mark the arrow at the position of y (0). Thus output y (n) can be written as,

$$y(n) = \{2, 6, 5, 5, 5, -1, 2\}$$
$$\uparrow$$

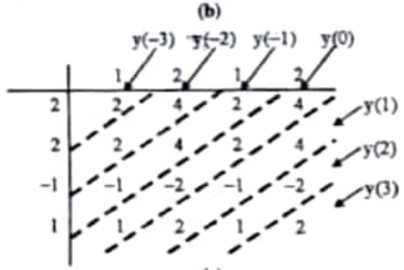
Let us verify this result using tabulation method shown in Fig. P. 3.6.15.



(a)

	1	2	1	2
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 × 2)
2	(2 × 1)	(2 × 2)	(2 × 1)	(2 × 2)
– 1	(– 1 × 1)	(– 1 × 2)	(– 1 × 1)	(– 1 × 2)
1	(1 × 1)	(1 × 2)	(1 × 1)	(1 × 2)

(b)



(c)

(H-381)**Fig. P. 3.6.15 : Convolution of x (n) = {1, 2, 1, 2} and**

$$h(n) = \{2, 2, -1, 1\}$$
$$\uparrow$$

Thus using tabulation method we get,

$y(-3) = 2$

$y(-2) = 2 + 4 = 6$

$y(-1) = -1 + 4 + 2 = 5$

$y(0) = 1 - 2 + 2 + 4 = 5$

$y(3) = 2$

$y(2) = 1 - 2 = -1$

$\therefore\ y(n) = \{2, 6, 5, 5, 5, -1, 2\}$
$\uparrow$

3.7 Properties of Linear Convolution

We have discussed how to obtain the convolution of two sequences. In this sub-section we will study some important properties of linear convolution.

These properties are :

1. Commutative property
2. Associative property
3. Distributive property.

1. Commutative property :

Statement : It states that linear convolution is commutative operation. That means

$$x(n) * h(n) = h(n) * x(n)$$

Proof : The linear convolution of x (n) and h (n) is given by,

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \qquad ...(3.7.1)$$

In this equation h (n – k) is folded delayed sequence; while x (k) is unchanged.

Put m = n – k as new index of summation in Equation (3.7.1).

$$\therefore\ k = n - m$$

Limits will change as follows :

When $k = -\infty \Rightarrow n - m = -\infty$

$\therefore\ -m = -\infty - n \qquad \therefore\ m = \infty$

When $k = +\infty \Rightarrow n - m = \infty$

$\therefore\ -m = \infty - n \qquad \therefore\ m = -\infty$

Putting these values in Equation (3.7.1) we get,

$$y(n) = \sum_{m=+\infty}^{-\infty} x(n-m)\, h(m) \qquad ...(3.7.2)$$

In case of summation operation limits from + ∞ to – ∞ are same as – ∞ to + ∞. Thus changing limits of summation we get,

$$y(n) = \sum_{m=-\infty}^{\infty} x(n-m)\, h(m) \qquad ...(3.7.3)$$

In Equation (3.7.3), m is dummy variable. It can be replaced by any other variable. So replacing 'm' by 'k' in Equation (3.7.3) we get,

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)\, h(k) \qquad ...(3.7.4)$$

In this equation x (n – k) is folded and delayed sequence; while sequence h (k) is unchanged. As per the definition of convolution Equation (3.7.4) can be written as,

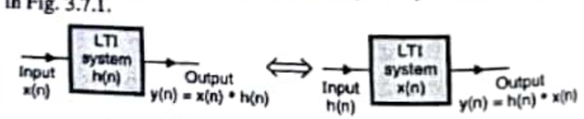
$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k$$
$$= h(n) * x(n) \qquad ...(3.7.5)$$

Thus from Equations (3.7.1) and (3.7.5) we get,

$$y(n) = x(n) * h(n) = h(n) * x(n) \qquad ...(3.7.6)$$

Hence proved that linear convolution is commutative.

Graphical presentation : Graphically it is presented as shown in Fig. 3.7.1.



(H-387)**Fig. 3.7.1 : x (n) * h (n) = h (n) * x (n)**

2. Associative property :

Statement :

It states that

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

Explanation :

We know that '*' indicates convolution operation. Every convolution operation indicates that certain input is applied to the LTI system and output is the convolution of input signal and the impulse response of system.

Now consider L.H.S. term.

That is the term, $[x(n) * h_1(n)] * h_2(n)$

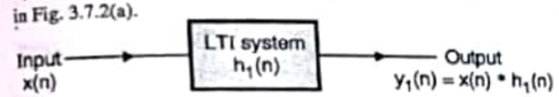
Let this output is y (n)

$$\therefore\ y(n) = [x(n) * h_1(n)] * h_2(n) \qquad ...(3.7.7)$$

There are two convolution operations. The bracket term indicates one convolution of x (n) and h_1 (n) Let this output be denoted by y_1 (n). Thus Equation (3.7.7) becomes,

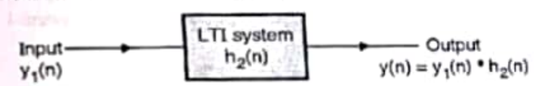
$$y(n) = y_1(n) * h_2(n) \qquad ...(3.7.8)$$

The convolution $y_1(n) = x(n) * h_1(n)$ indicates that, we are applying input to LTI system whose impulse response is h_1 (n). The output of system is y_1 (n). Such system is represented in Fig. 3.7.2(a).



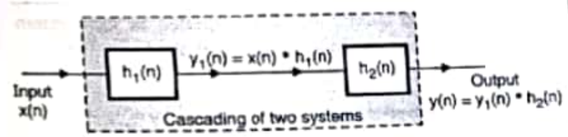
(H-388)**Fig. 3.7.2(a) : $y_1(n) = x(n) * h_1(n)$**

Now according to Equation (3.7.8); y (n) is obtained by computing convolution of y_1 (n) and h_2 (n). Here y_1 (n) acts as input signal to the LTI system whose impulse response is h_2 (n). The output of this system is y (n). Diagrammatically it is represented as shown in Fig. 3.7.2(b).



(H-389)**Fig. 3.7.2(b) : $y(n) = y_1(n) * h_2(n)$**

Thus to implement Equation (3.7.1) we have to connect (cascade) two systems shown in Figs. 3.7.1 and 3.7.2(a). This cascade connection is shown in Fig. 3.7.2(c).

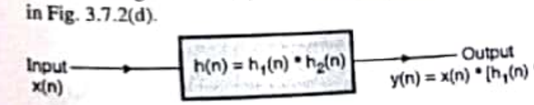


(H-390)**Fig. 3.7.2(c) : $y(n) = [x(n) * h_1(n)] * h_2(n)$**

Now consider the R.H.S. term x (n) * [h_1 (n) * h_2 (n)]. We can design one LTI system whose impulse response h (n) is the convolution of h_1 (n) and h_2 (n). That means convolution of two impulse responses. Thus $h(n) = h_1(n) * h_2(n)$.

$$\therefore\ x(n) * [h_1(n) * h_2(n)] = x(n) * h(n) \qquad ...(3.7.9)$$

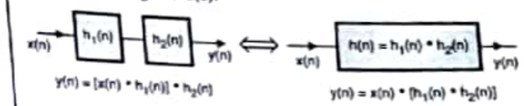
The diagrammatic representation of Equation (3.7.9) is shown in Fig. 3.7.2(d).



(H-391)**Fig. 3.7.2(d)**

Meaning of associative property :

This property indicates that we can replace cascade combination of LTI systems by a single system whose impulse response is convolution of individual impulse responses. It is represented in Fig. 3.7.2(e).



(H-392)**Fig. 3.7.2(e) : Meaning of associative property**

3. Distributive property :

Statement :

It states that linear convolution is distributive. That means

$$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)] \qquad ...(3.7.10)$$

Proof :

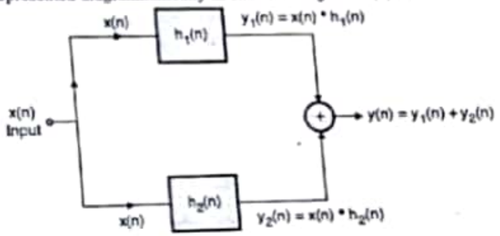
Consider the R.H.S. term of Equation (3.7.10). We will denote it by y (n).

$$\therefore\ y(n) = [x(n) * h_1(n)] + [x(n) * h_2(n)] \qquad ...(3.7.11)$$

Every bracket term in Equation (3.7.11) indicates the convolution operation. Thus we will denote output of each convolution as follows :

$$\therefore\ y_1(n) = x(n) * h_1(n) \qquad ...(3.7.12)$$
$$\text{and}\ \ y_2(n) = x(n) * h_2(n) \qquad ...(3.7.13)$$
$$\therefore\ y(n) = y_1(n) + y_2(n) \qquad ...(3.7.14)$$

Observe Equations (3.7.12) and (3.7.13). It indicates that there are two LTI systems whose impulse responses are h_1 (n) and h_2 (n). Important thing is that, the same input signal x (n) is applied to both systems. While Equation (3.7.14) shows that we have to add the outputs of these two LTI system. This process is represented diagrammatically as shown in Fig. 3.7.3(a).

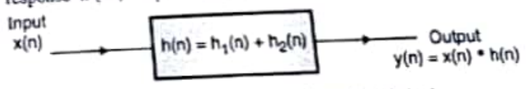


(H-393)**Fig. 3.7.3(a) : $y(n) = y_1(n) + y_2(n)$**
$= [x(n) * h_1(n)] + [x(n) * h_2(n)]$

As shown in Fig. 3.7.3(a), the same input is applied to both systems. So this is a parallel connection of two LTI systems. Now as shown in Fig. 3.7.3(a); two impulse responses are added.

$$\therefore\ h(n) = h_1(n) + h_2(n) \qquad ...(3.7.15)$$

Thus we can design a single LTI system having impulse response $h(n) = h_1(n) + h_2(n)$.

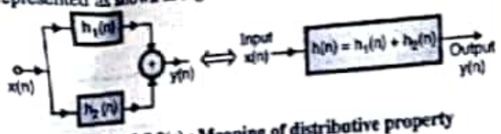


(H-394)**Fig. 3.7.3(b) : $y(n) = x(n) * h(n)$**
$= x(n) * [h_1(n) + h_2(n)]$

Meaning of distributive property :

This property indicates that, we can replace a parallel

combination of LTI systems by a single LTI system whose impulse response is the sum of individual impulse responses. It is represented as shown in Fig. 3.7.3(c).



(H-395)Fig. 3.7.3(c) : Meaning of distributive property

Summary of properties :

Table 3.7.1 shows the summary of properties of linear convolution.

Table 3.7.1

Sr. No.	Name of property	Mathematical equation
1.	Commutative	$x(n) * h(n) = h(n) * x(n)$
2.	Associative	$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3.	Distributive	$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)]$

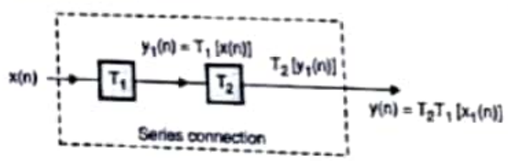
3.8 Impulse Response of Interconnected Systems :

In case of complicated operations; it is necessary to interconnect different subsystems. There are three types of interconnections as follows :

1. Cascade or series interconnection
2. Parallel interconnection
3. Feedback interconnection

1. Cascade or series interconnection :

In this case two systems are connected in series. That means output of one system is connected to the input of other system. This is shown in Fig. 3.8.1.



(H-396)Fig. 3.8.1 : Cascade or series interconnection

Here T_1 and T_2 represents impulse responses of systems 1 and 2 respectively. Thus output of system-1 is,

$$y_1(n) = T_1[x(n)]$$

This is applied as input to system 2. Thus total output is,

$$y(n) = T_2[y_1(n)]$$

$$\therefore \quad y(n) = T_2T_1[x(n)]$$

2. Parallel connection :

Two systems can be connected in parallel as shown in Fig. 3.8.2. Here same input, $x(n)$ is applied to two systems.

Thus output of system-1 is,

$$y_1(n) = T_1[x(n)]$$

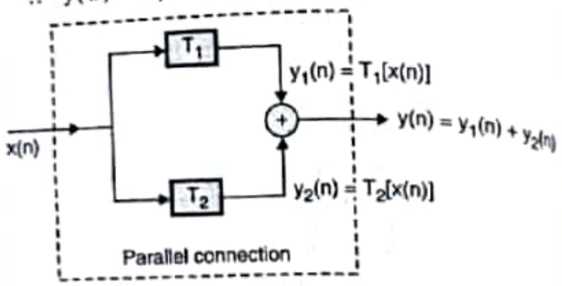
and output of system-2 is,

$$y_2(n) = T_2[x(n)]$$

Finally these two outputs are added by using adder block. Thus the total output is,

$$y(n) = y_1(n) + y_2(n)$$

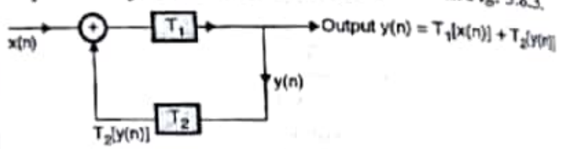
$$\therefore \quad y(n) = T_1[x(n)] + T_2[x(n)]$$



(H-397)Fig. 3.8.2 : Parallel interconnection

3. Feedback interconnection :

As the name indicates, a feedback connection is done from output to the input. This interconnection is as shown in Fig. 3.8.3.



(H-398)Fig. 3.8.3 : Feedback interconnection

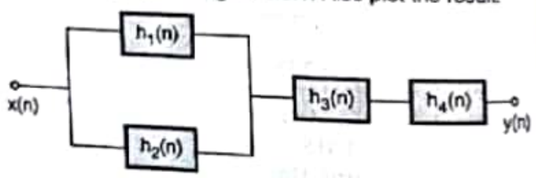
Here output $y(n)$ is applied to system-2 (T_2). Its output is $T_2[y(n)]$. This signal is added with input $x(n)$. So the output of adder is $x(n) + T_2[y(n)]$. This signal is applied to system-1 (T_1). Thus the total output is,

$$y(n) = T_1[x(n)] + T_2[y(n)]$$

Note : Continuous time systems are also interconnected using the same methods.

Solved Examples :

Ex. 3.8.1 : Find overall impulse response of the system shown in Fig. P. 3.8.1. Also plot the result.



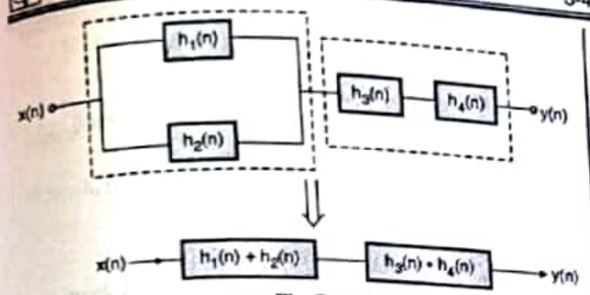
(H-399)Fig. P. 3.8.1

$$h_1(n) = \{\underset{\uparrow}{3}, 4, 2, 1\} \quad h_3(n) = \{\underset{\uparrow}{-2}, 4, 1\}$$

$$h_2(n) = \{\underset{\uparrow}{2}, 1, 3, 1\} \quad h_4(n) = \delta(n)$$

Soln. :

Here $h_1(n)$ and $h_2(n)$ are in parallel and $h_3(n)$ and $h_4(n)$ are in series; so applying the reduction rules the system can be drawn as shown in Fig. P. 3.8.1(a).



(H-400)Fig. P. 3.8.1(a)

But $h_4(n) = \delta(n)$ which is unit impulse. The convolution of any function with respect to unit impulse results the same function.

$$\therefore \quad h_3(n) * h_4(n) = h_3(n)$$

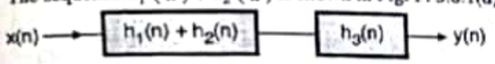
Thus the block schematic can be drawn as shown in Fig. P. 3.8.1(b).

Now two blocks are in series so we can draw the block diagram as shown in Fig. P. 3.8.1(c).

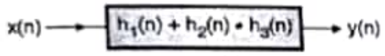
Thus $h(n) = [h_1(n) + h_2(n)] * h_3(n)$

Now we will calculate $h(n)$.

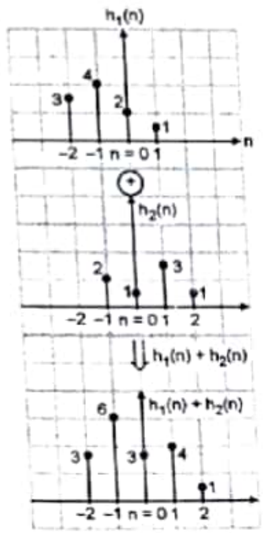
The sequence $h_1(n) + h_2(n)$ is shown in Fig. P. 3.8.1(d).



(b)



(H-401) (c)



(H-402) (d)

Fig. P. 3.8.1

$$\therefore \quad h_1(n) + h_2(n) = \{\underset{\uparrow}{3}, 6, 3, 4, 1\}$$

and we have $h_3(n) = \{\underset{\uparrow}{-2}, 4, 1\} = \{\underset{\uparrow}{-2}, 4, 1, 0, 0\}$

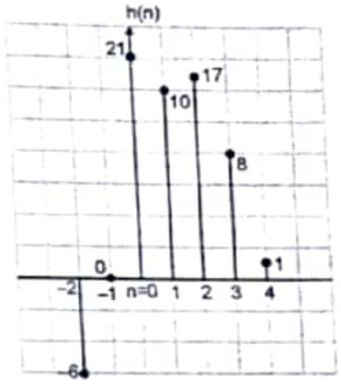
Using tabular method convolution of $[h_1(n) + h_2(n)]$ and $h_3(n)$ is obtained as follows.

Here the range of 'n' is h (n) is $n = -2$ to $n = 6$.

The plot of h (n) is shown in Fig. P. 3.8.1(f).

	3	6	3	4	1
−2	−6	−12	−6	−8	−2
4	12	24	12	16	4
1	3	6	3	4	1
0	0	0	0	0	0
0	0	0	0	0	0

(H-403) (e)



(H-404) (f)

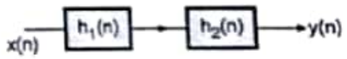
Fig. P. 3.8.1

$$\therefore \quad h(n) = \{-6, 0, \underset{\uparrow}{21}, 10, 17, 8, 1, 0, 0\}$$

$$\therefore \quad h(n) = \{-6, 0, \underset{\uparrow}{21}, 10, 17, 8, 1\}$$

This is the overall impulse response of the given system.

Ex. 3.8.2 : Two systems are cascaded as shown in Fig. P. 3.8.2.



(H-405) Fig. P. 3.8.2

If $h_1(n) = a^n u(n)$, $\quad a < 1$

and $h_2(n) = b^n u(n)$, $\quad b < 1$

Find overall impulse response of the system.

Soln. :

This is a cascade connection. Thus overall impulse response is,

$$h(n) = h_1(n) * h_2(n)$$

Here $h_1(n) = a^n u(n)$ and $h_2(n) = b^n u(n)$

Thus $h(n)$ is obtained by performing convolution of $a^n u(n)$ and $b^n u(n)$. We have already obtained this convolution in Ex. 3.6.3.

$$\therefore \quad h(n) = b^n \left[\frac{1 - \left(\frac{a}{b}\right)^{n+1}}{1 - \frac{a}{b}} \right]$$

Ex. 3.8.3 : Consider two LTI systems connected in series. Their impulse responses are $h_1[n]$ and $h_2[n]$ respectively. Find the output of the systems if x [n] is the input being applied to one of the systems.

$x[n] = \{1, 2\}$ $h_1[n] = \{1, 0, -1\}$

$h_2[n] = \{2, 1, -1\}$

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Soln. :

Given : $x(n) = \{1, 2\}$, $h_1(n) = \{1, 0, -1\}$

and $h_2(n) = \{2, 1, -1\}$

Since $h_1(n)$ and $h_2(n)$ are connected in series. Thus equivalent impulse response is,

$$h(n) = h_1(n) * h_2(n)$$

It is performed as follows :

	1	0	-1
2	2	0	-2
1	1	0	-1
-1	-1	0	1

(H-1646)

$\therefore h(n) = \{2, 1, -3, -1, 1\}$

Now final output is $y(n) = x(n) * h(n)$.

	2	1	-3	-1	1
1	2	1	-3	-1	1
2	4	2	-6	-2	2

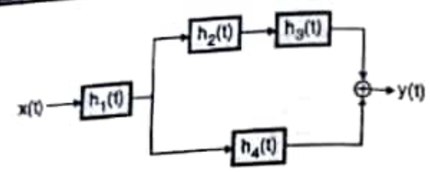
(H-1647)

$\therefore y(n) = \{2, 5, -1, -7, -1, 2\}$

Ex. 3.8.4 : For the LTI system in the Fig. P. 3.8.4.

(a) Express the system impulse response as a function of impulse responses of the subsystems.

(b) $h_1(t) = h_2(t) = 5\delta(t)$; $h_3(t) = h_4(t) = u(t)$. Find the impulse response of the system.

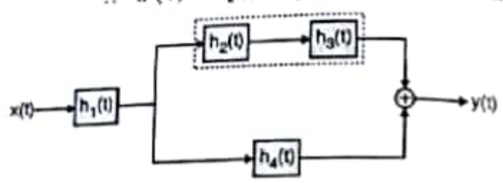


(H-406) Fig. P. 3.8.4

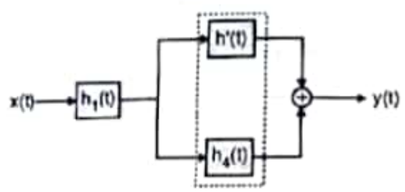
Soln. :

(a) Here, $h_2(t)$ and $h_3(t)$ are in series. Let the combination of $h_2(t)$ and $h_3(t)$ is $h'(t)$.

$$\therefore \quad h'(t) = h_2(t) * h_3(t) \qquad \text{...(1)}$$



(a)



(b)

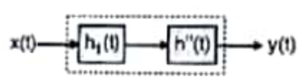
(H-407) Fig. P. 3.8.4

As shown in above Fig. P. 3.8.4(a), $h'(t)$ and $h_4(t)$ are in parallel. Let, combination is $h''(t)$

$$\therefore \quad h''(t) = h_4(t) + h'(t) \qquad \text{...(2)}$$

From Equation (1),

$$\therefore \quad h''(t) = h_4(t) + [h_2(t) * h_3(t)] \qquad \text{...(3)}$$



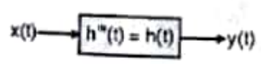
(H-408) Fig. P. 3.8.4(c)

As shown in above Fig. P. 3.8.4(c), $h_1(t)$ and $h''(t)$ is in series. Let, combination is $h'''(t)$.

$$\therefore \quad h'''(t) = h_1(t) * h''(t)$$

From Equation (3),

$$h'''(t) = h_1(t) * \{h_4(t) + [h_2(t) * h_3(t)]\} \qquad \text{...(4)}$$



(H-409) Fig. P. 3.8.4(d)

Thus, expression for system impulse response as a function of impulse responses of the system is,

$$h(t) = h_1(t) * \{h_4(t) + [h_2(t) * h_3(t)]\}$$

(b) Given :

$$h_1(t) = h_2(t) = 5\delta(t)$$
$$h_3(t) = h_4(t) = u(t)$$

First we will calculate $h_2(t) * h_3(t)$

$$\therefore \quad h_2(t) * h_3(t) = 5\delta(t) * u(t)$$

But convolution of any sequence with $\delta(t)$ produces the same sequence.

$$\therefore \quad h_2(t) * h_3(t) = 5[\delta(t) * u(t)] = 5u(t)$$

Now $h_4(t) + [h_2(t) * h_3(t)] = u(t) + 5u(t)$

$$= 6u(t)$$

$$h(t) = h_1(t) * \{h_4(t) + [h_2(t) * h_3(t)]\}$$
$$= 5\delta(t) * 6u(t)$$

But $\delta(t) * u(t) = u(t)$

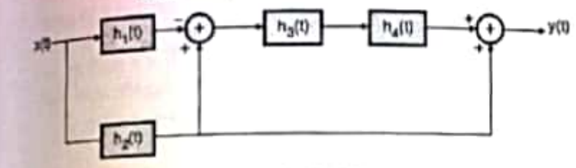
$$\therefore \quad h(t) = 30u(t)$$

Ex. 3.8.5 : For the inter connection of LTI systems depicted in Fig. P. 3.8.5, the impulse responses are :

$h_1(t) = \delta(t-1)$; $h_2(t) = e^{-2t}u(t)$

$h_3(t) = \delta(t-1)$; $h_4(t) = e^{-3(t+2)}u(t+2)$.

Obtain equation of overall impulse response of the system.



(H-753) Fig. P. 3.8.5

Soln. :

The path in which block is absent ; is considered as $\delta(t)$. Now if the systems are in parallel, they are added and if the systems are in cascade then convolution is performed. The total impulse response can be expressed as,

$$h_T(t) = \{[h_2(t) - h_1(t)] * [(h_3(t) * h_4(t)] + \delta(t)\} \qquad \text{...(1)}$$

Consider the term $h_3(t) * h_4(t)$

$$\therefore \quad h_3(t) * h_4(t) = \delta(t-1) * e^{-3(t+2)}u(t+2) \qquad \text{...(2)}$$

We have the standard identity

$$x(t) * \delta(t - t_0) = x(t - t_0)$$
$$\therefore \quad h_3(t) * h_4(t) = e^{-3(t+2)} \cdot u(t+2) * \delta(t-1)$$
$$= e^{-3(t+1)} \cdot u(t+1)$$

Putting this value in Equation (1) we get,

$$h_T(t) = \{[e^{-2t}u(t) - \delta(t-1)] * [e^{-3(t+1)} \cdot u(t+1) + \delta(t)]\}$$

Using distributive property we get,

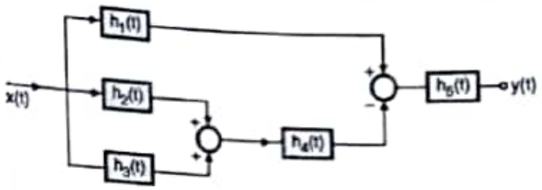
$$h_T(t) = e^{-2t}u(t) * e^{-3(t+1)} \cdot u(t+1) + e^{-2t}u(t) * \delta(t)$$
$$- e^{-3(t+1)} \cdot u(t+1) * \delta(t-1) - \delta(t) * \delta(t-1)$$

Now we have, $x(t) * \delta(t) = x(t)$.

$$\therefore \quad h_T(t) = [e^{-2t}u(t) * e^{-3(t+1)} \cdot u(t+1)]$$
$$+ e^{-2t}u(t) - e^{-3t}u(t) - \delta(t)$$

This is overall impulse response of system.

Ex. 3.8.6 : Find the expression for impulse response relating the input x(t) to the output y (t) in terms of impulse for each subsystem for the LTI system shown in Fig. P. 3.8.6.



(H-1451) Fig. P. 3.8.6

Soln. : Here $h_2(t)$ and $h_3(t)$ are in parallel. So these blocks can be combined by performing addition, that means $h_2(t) + h_3(t)$.

This block is in series with $h_4(t)$. So it can be combined by performing convolution. That means, $[h_2(t) + h_3(t)] * h_4(t)$.

This block is in parallel with $h_1(t)$. Thus system response is,

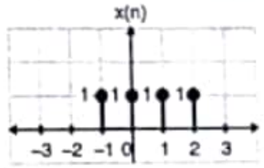
$h_1(t) - \{[h_2(t) + h_3(t)] * h_4(t)]$

Now this block is in series with $h_5(t)$. Thus overall impulse response is,

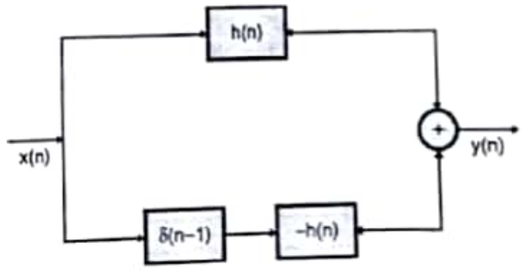
$h_T(t) = [h_1(t) - \{[h_2(t) + h_3(t)] * h_4(t)\}] * h_5(t)$

Ex. 3.8.7 : Consider an LTI system with input and output related by $y(n) = 0.8[x(n+1) + x(n)]$

1. Find impulse response h (n)
2. Is this system causal ? Why ?
3. Determine the system response y (n) for input shown in Fig. P. 3.8.7(a).
4. Consider interconnections of LTI system shown in Fig. P. 3.8.7(b). Find overall impulse response of the total system.
5. Solve for overall response of the total system for the same input x (n).



(a)



(b)

(H-417) Fig. P. 3.8.7

Soln. :

1. Given $y(n) = 0.8x(n+1) + 0.8x(n)$

Impulse response of a system means the response (output) of a system when unit impulse, $\delta(n)$ is applied at input.

$\therefore\ h(n) = 0.8\,\delta(n+1) + 0.8\,\delta(n)$

$\therefore\ h(n) = \{0.8, 0.8\ 0.8\}$
$\uparrow$

2. Impulse response h (n) contains advanced term so it is noncausal system. Because a system is causal if its impulse response contains present and past (delayed) terms.

3. Given $x(n) = \{1, 1, 1, 1\}$
$\uparrow$

We have $h(n) = \{0.8, 0.8\}$
$\uparrow$

Now response of system is.

$$y(n) = x(n) * h(n)$$

Here lower limit for $y(n)$ is $-1 + (-1) = -2$

	1	1	1	1
0.8	0.8	0.8	0.8	0.8
0.8	0.8	0.8	0.8	0.8
0	0	0	0	0
0	0	0	0	0

(II-418)

$\therefore\ y(n) = \{0.8, 1.6, 1.6, 1.6, 0.8\}$
$\uparrow$

4. As shown in Fig. P. 3.8.7(b), $\delta(n-1)$ and $-h(n)$ are in series and this series combination is in parallel with h (n). We will denote the total impulse response by $h_T(n)$.

$$\therefore\ h_T(n) = \{\delta(n-1) * (-h(n))\} + h(n)$$

Let $h_1(n) = \delta(n-1) * [-h(n)]$

$\therefore\ h_1(n) = \{0, 1\} * \{-0.8, -0.8\}$
$\uparrow \qquad \uparrow$

The lower limit of $h_1(n)$ is $0 + (-1) = -1$

	-0.8	0.8
0	0	0
1	-0.8	-0.8

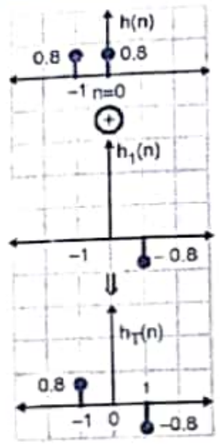
(II-419)

$\therefore\ h_1(n) = \{0, -0.8, -0.8\}$
$\uparrow$

Now $h_T(n) = h_1(n) + h(n)$

This addition is shown in Fig. P. 3.8.7(c).

$\therefore\ h_T(n) = \{0.8, 0, -0.8\}$
$\uparrow$



(II-420) Fig. P. 3.8.7(c)

5. We have $x(n) = \{1, 1, 1, 1\}$
$\uparrow$

Now $y(n) = x(n) * h_T(n)$

The lower limit of $y(n)$ is $-1 + (-1) = -2$

	1	1	1	1
0.8	0.8	0.8	0.8	0.8
0	0	0	0	0
-0.8	-0.8	-0.8	-0.8	-0.8
0	0	0	0	0

(II-421)

$\therefore\ y(n) = \{0.8, 0.8, 0, 0, -0.8, -0.8, 0\}$
$\uparrow$

3.9 System Properties in Terms of Impulse Response :

In this section we will study impulse response and different properties of LTI systems.

3.9.1 Causal or Non-causal :

We have studied that a discrete time system is causal if output of system depends only on present and past inputs and not on the present input. In this section we will prove that LTI system is causal if its impulse response $h(n) = 0$ for $n < 0$.

Statement :

A LTI system is causal if and only if $h(n) = 0$ for $n < 0$.

This is the necessary and sufficient condition for causality of LTI system.

In simple words we can write this statement as "A LTI system is causal if its impulse response, $h(n) = 0$ for negative values of n".

Proof : According to the equation of linear convolution we have,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k) \qquad ...(3.9.1)$$

This equation can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \qquad ...(3.9.2)$$

Here limits of summation are $k = -\infty$ to $+\infty$. This summation can be expressed into two parts as follows :

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k) + \sum_{k=-\infty}^{-1} h(k)\,x(n-k) \qquad ...(3.9.3)$$

Note that total range of k is from $-\infty$ to $+\infty$. In the first term we have taken the range of k from 0 to ∞. Thus for second term the range of k is from -1 to $-\infty$. Because we have covered $k = 0$ in the first term. So for the second summation the range of k should not be from $k = 0$ to $-\infty$ but it should be from $k = -1$ to $-\infty$.

Now expanding the terms of summation we get,

$$\begin{aligned} y(n) = {} & [h(0)x(0) + h(1)x(n-1) \\ & + h(2)x(n-2) + \ldots] \\ & + [h(-1)x(n+1) + h(-2)x(n+2) \\ & + h(-3)x(n+3) + \ldots] \end{aligned} \qquad ...(3.9.4)$$

Let us consider the 'present' condition is at $n = 0$.

Thus,

$x(0) \Rightarrow$ Present input

$x(n-1), x(n-2), \ldots \Rightarrow$ Delayed inputs that means past inputs.

and $x(n+1), x(n+2), \ldots \Rightarrow$ Advanced inputs that means future inputs.

We discussed that for the system to be causal we want only present and past inputs. We do not want future inputs. Thus second bracket in Equation (3.9.4) should be zero to make the system causal.

Here $x(n+1), x(n+2) \ldots$ are future inputs. We do not want to make input to the system zero. So the second bracket is zero if we adjust $h(-1), h(-2), h(-3) \ldots$ to be equal to zero. Then in this case we will have only first bracket terms. That means we will have only present and past input terms. And then the system will be causal.

Thus for causality,

$$h(-1) = h(-2) = h(-3) = \ldots = 0 \qquad ...(3.9.5)$$

That means,

$$h(n) = 0 \quad \text{for } n < 0 \qquad ...(3.9.6)$$

Similarly for C.T. system, the condition for causality is $h(t) = 0$ for $t < 0$.

Modification of convolution equation :

We have the equation of linear convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$\text{OR}\quad y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

We have derived that, the LTI system is causal then $h(n) = h(k) = 0$ for n (or k) < 0. Thus if LTI system is causal then the limits of summation will be from $k = 0$ to $+\infty$, because for negative values of 'k', $h(k)$ is zero. Thus for causal LTI system the equation of linear convolution is,

$$y(n) = \sum_{k=0}^{\infty} x(k)\,h(n-k)$$

$$\text{OR}\quad y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

And for C.T. system we get,

$$y(t) = \int_0^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$\text{or}\quad y(t) = \int_0^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$

3.9.2 Memory Less (Static) and With Memory (Dynamic) :

A system is memoryless if its outputs at any time depends only on the value of the input at that same time.

Discrete-time LTI system is memory less or static if

$$h(n) = 0 \quad \text{For } n \neq 0.$$

In this case the impulse response has the form,

$$h(n) = k\,\delta(n)$$

where $k = h(0)$ is a constants, and convolution sum reduces to,

$$y(n) = k\,x(n)$$

Dynamic or with memory :

A system is with memory or dynamic if systems present outputs depends on present input as well as input at other time.

Discrete-time LTI system is with memory or dynamic if $h(n) \neq 0$ For $n \neq 0$.

If a discrete time LTI system has an impulse response $h(n)$ that is not zero for $n \neq 0$ then the system has memory.

For C.T. system to be static its impulse response must be in the form.

$$h(t) = c\,\delta(t)$$

3.9.3 Stable Systems :

Earlier we have discussed the stability condition for discrete time system. It states that a discrete time system is stable if bounded input produces bounded output. Here the meaning of word "bounded" is; input and output signals should have some finite magnitude. The magnitude of these signals should not be infinity. Now in this section we will derive the stability condition applicable to LTI systems; in terms of unit sample response.

3.9.3.1 Proof of Stability Criteria For LTI (or LSI) Systems in Terms of Unit Impulse Response :

Statement : LTI system is stable if its impulse response is absolutely summable.

Proof : Consider a linear time invarient system having impulse response h (n). Let x (n) be the input applied to this system. Now according to the definition of convolution; the output of such system is expressed as,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \quad ...(3.9.7)$$

Equation (3.9.7) can also be written as,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \quad ...(3.9.8)$$

Now first we will define bounded sequence. A discrete time sequence is said to be bounded if the absolute value of every element is less than some finite number. Thus input x (n) will be bounded if | x (n) | is less than some finite number. Let us denote this finite number by M_x. Thus for input signal to be bounded.

$$|x(n)| \le M_x \quad ...(3.9.9)$$

Here M_x is a finite number; so definitely its value should be less than infinity. Thus Equation (3.9.9) can be written as,

$$|x(n)| \le M_x < \infty \quad ...(3.9.10)$$

Now taking absolute value of both sides of Equation (3.9.8) we get,

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \quad ...(3.9.11)$$

We will read R.H.S. of Equation (3.9.11) as "absolute value of summation of terms". If we take 'Σ' sign outside then the term becomes $\sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$. This is summation of absolute values of terms. Always absolute value of sum of terms is less than or equal to the sum of absolute values of terms.

Thus $$\left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right| \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \quad ...(3.9.12)$$

But from Equation (3.9.11); the L.H.S. term is | y (n) |.

$$\therefore |y(n) \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \quad ...(3.9.13)$$

Here x (n − k) is delayed input signal. If input is bounded then its delayed version is also bounded. This is because delay or folding is related to time shifting operations. By performing these operations; the magnitude is not changed. Now for bounded input we have,

$$|x(n)| \le M_x$$
$$\therefore |x(n-k)| \le M_x \quad ...(3.9.14)$$

Putting this value in Equation (3.9.13) we get,

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)| \cdot M_x$$

$$\therefore |y(n)| \le M_x \sum_{k=-\infty}^{\infty} |h(k)| \quad ...(3.9.15)$$

We know that M_x is a finite number. We want output y (n) to be bounded. That means | y (n) | should be finite. So from Equation (3.9.15), to obtain finite output we have,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad ...(3.9.16)$$

Here h (k) = h (n) is the impulse response of LTI system. Thus Equation (3.9.16) gives the condition of stability in terms of impulse response of the system.

Now the stability factor is denoted by 's'.

$$\therefore s = \sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad ...(3.9.17)$$

Thus LTI system is stable if its impulse response is absolutely summable.

3.9.3.2 Stability for CT Systems :

LTI system is BIBO stable if bounded input produces bounded output. We have equation of convolution integral,

$$y(t) = x(t) * h(t)$$

$$\therefore y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \quad ...(3.9.18)$$

Taking magnitude of both sides we get,

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \quad ...(3.9.19)$$

If we take magnitude sign inside integration then equality sign becomes ≤ .

$$\therefore |y(t)| \le \int_{-\infty}^{\infty} |h(\tau)x(t-\tau)d\tau|$$

$$\therefore |y(t)| \le \int_{-\infty}^{\infty} |h(\tau)||x(t-\tau)d\tau| \quad ...(3.9.20)$$

To check the stability we have to apply bounded (finite) input. We will assume that the value of input x (t) is less than or equal to some finite number M_x. In this case $M_x < \infty$. Thus we can write,

$$|y(t)| \le \int_{-\infty}^{\infty} |h(\tau)|M_x d\tau$$

$$\therefore |y(t)| \le M_x \int_{-\infty}^{\infty} |h(\tau)|d\tau \quad ...(3.9.21)$$

We know that M_x is finite number. Now to obtain bounded (finite) output; integration term in Equation (3.9.20) should be finite.

$$\therefore \int_{-\infty}^{\infty} |h(\tau)|d\tau < \infty \quad ...(3.9.22)$$

This is the condition for stability of LTI system. So LTI system is BIBO stable if its impulse response is absolutely summable.

3.9.3.3 Solved Examples on Stability :

Ex. 3.9.1 : Determine range of values of parameter 'a' for which the linear time invariant system with impulse response.

$$h(n) = a^n \quad n \ge 0 \text{ and } n \text{ even}$$
$$= 0 \quad \text{otherwise}$$

is stable

Soln. : We know that LTI system is BIBO stable when,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad ...(1)$$

Given equation is h (n) = a^n for n ≥ 0 and n even and h (n) = 0 otherwise. That means sequence h (n) is present from n = 0 to n = ∞ and for even values of n only. So limits of summation will change from n = 0 to n = ∞. We will also replace h (k) by h (n). Thus Equation (1) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} |a^n| \quad ...(2)$$

Since 'n' is even, to satisfy this condition we will put n = 2 p. Here value of p is 0, 1, 2 .... ∞. So 'n' is always even for any value of 'p'. Thus Equation (2) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{p=0}^{\infty} |a^p|$$

$$= \sum_{p=0}^{\infty} |a|^p$$

Expanding the summation we get,

$$\sum_{k=-\infty}^{\infty} |h(k)| = |a|^0 + |a|^1 + |a|^2 + |a|^3 + ...$$

$$= 1 + |a| + |a|^2 + |a|^3 + ... \quad ...(3)$$

This is geometric series. For geometric series we have,

$$\sum_{n=0}^{\infty} a^n = 1 + a + a^2 + a^3 + ...$$

$$= \frac{1}{1-a} \quad \text{if } |a| < 1$$

Thus Equation (3) becomes,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \frac{1}{1-|a|} \quad \text{if } |a| < 1$$

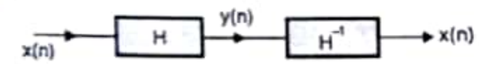
This condition (if | a | < 1) indicates that the given series converges to $\frac{1}{1-|a|}$ if and only if | a | < 1. Thus for the system to be stable | a | < 1 is the condition. Otherwise the system will be unstable.

3.9.4 Invertible Systems and Deconvolution :

A linear time invariant system generates output y (n) by taking convolution of input x(n) and its impulse response h(n). That means y(n) = x (n) * h (n). But in many applications it is required to generate the input signal from its output y (n). This system is called as inverse system.

Invertibility of LTI system :

A system is said to be invertible if input of a system can be recovered from the output of system. To achieve the same input consider two system connected in series; where second system is used to recover the input. The block schematic is shown in Fig. 3.9.1.



(H-413) **Fig. 3.9.1**

Here H represents discrete time system and produces y (n) when x (n) is the input. Then y (n) is applied to the second system. This is also discrete time system denoted by H^{-1}. Here H^{-1} denotes inverse operation.

$$\therefore y(n) = H[x(n)]$$

and $$H^{-1}[y(n)] = H^{-1}(H[x(n)])$$
$$= HH^{-1}[x(n)]$$

But $HH^{-1} = 1$

$$\therefore H^{-1}y(n) = x(n)$$

Here operator H^{-1} is the inverse operator and the associated system is called as inverse system.

3.9.4.1 Solved Examples on Invertible Systems :

Ex. 3.9.2 : Which of the following systems are invertible ? If invertible find the inverse.

1. y (t) = x (t − n)
2. y (t) = x (2t)
3. y (n) = n x (n)

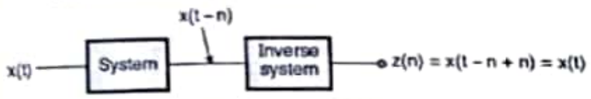
Soln. :

1. Given equation is, y (t) = x (t − n)

It indicates that signal x (t) is delayed by 'n' samples. So **it is invertible system.** We can design another system in which x (t) can be made advanced by 'n' samples. Thus inverse system is,

$$Z(t) = y(t+n)$$

This is shown in Fig. P. 3.9.2(a).



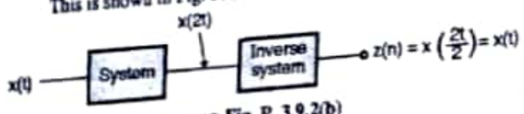
(H-414) **Fig. P. 3.9.2(a)**

2. Given, y (t) = x (2t)

Input is x (t); so this system indicates that output is obtained by compressing input by the factor 2. There can be another system which will expand input by 2. Thus **it is invertible system** and inverse system is,

$$Z(t) = y\left(\frac{t}{2}\right)$$

This is shown in Fig. P. 3.9.2(b).



(H-415) Fig. P. 3.9.2(b)

3. Given, $y(n) = n\,x(n)$

It indicates that input x (n) is multiplied by 'n'. We can design a system which will divide input by 'n'. Thus it is **invertible system** and inverse system is,

$$Z(n) = \frac{1}{n} y(n)$$

This is shown in Fig. P. 3.9.2(c).

(H-416) Fig. P. 3.9.2(c)

Ex. 3.9.3 : Consider an accumulator which computes the running sum of all inputs upto present time and expressed as :

$$y(n) = \sum_{m=0}^{n} x(m)$$

Is it an invertible system ? If yes, find inverse of this system. If no give reasons.

Soln. : The given equation is,

$$y(n) = \sum_{m=0}^{n} x(m) \quad ...(1)$$

Expanding the summation we get,

$$y(n) = x(0) + x(1) + x(2) + ... + x(n) \quad ...(2)$$

Replacing 'n' by n – 1 in Equation (1) we get,

$$y(n-1) = \sum_{m=0}^{n-1} x(m) \quad ...(3)$$

Expanding the summation we get,

$$y(n-1) = x(0) + x(1) + x(2) + ... + x(n-1) \quad ...(4)$$

Invertible system is designed to generate the original input.

If we subtract Equation (4) from Equation (3) then we can generate original input.

$$\therefore \; x(n) = y(n) - y(n-1) \quad ...(5)$$

Thus **it is invertible system.** Equation (5) indicates equation of inverse system.

3.9.5 Step Response in terms of Impulse Response :

It is the response of system when applied input to the system is unit step.

For Discrete Time System :

We have the equation of convolution.

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-K)$$

$$= \sum_{k=-\infty}^{\infty} h(K)\,x(n-K) \quad ...(3.9.23)$$

Here Input = $x(n) = u(n)$ = unit step = 1 for $n \geq 0$.

We know that $x(n-K)$ denotes delayed unit step.

$\therefore \; x(n-K) = u(n-K) = 1$ for $n \geq K$ that means for $K \leq n$.

Putting this value in Equation (3.9.23) and modifying the limits.

$$y(n) = \sum_{k=-\infty}^{n} h(K) \cdot 1$$

$$= \sum_{k=-\infty}^{n} h(K) \quad ...(3.9.24)$$

This is the step response of discrete time system.

For Continuous Time System :

We have the equation of convolution,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau \quad ...(3.9.25)$$

Again input applied is unit step.

Thus $x(t) = u(t) = 1$ for $t \geq 0$.

$\therefore \; x(t-\tau) = u(t-\tau)$

$= 1$ for $t \geq \tau$ that means $\tau \leq t$.

Putting this value and modifying the limits of Equation (3.9.24).

$$y(t) = \int_{-\infty}^{t} h(\tau) \cdot 1\,d\tau$$

$$\therefore \; y(t) = \int_{-\infty}^{t} h(\tau)\,d\tau \quad ...(3.9.26)$$

This is the step response for continuous time system.

3.9.5.1 Solved Examples on System Response :

Ex. 3.9.4 : Impulse response of DT-LTI system is given by $h(n) = n\left(\frac{1}{2}\right)^n u(n)$.

1. Determine whether the system is stable or not.
2. Justify whether the system is causal or anticipatory.

Soln. :

1. We have the condition for stability,

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty \quad ...(1)$$

Given equation is,

$$h(n) = n\left(\frac{1}{2}\right)^n u(n)$$

$$\therefore \; h(k) = k\left(\frac{1}{2}\right)^k u(k)$$

Thus condition for stability can be written as,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=-\infty}^{\infty} k\left(\frac{1}{2}\right)^k u(k) \quad ...(2)$$

Here $u(n) = u(k)$ = Unit step = 1 for k = 0 to ∞. Thus changing the limits of summation we get,

$$\sum_{k=-\infty}^{\infty} |h(k)| = \sum_{k=0}^{\infty} k\left(\frac{1}{2}\right)^k \quad ...(3)$$

We have the standard summation formula.

$$\therefore \; \sum_{k=0}^{\infty} k\,a^k = \frac{a}{(1-a)^2} \text{ if } |a| < 1.$$

$$\therefore \; \sum_{k=-\infty}^{\infty} |h(k)| = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2 < \infty.$$

So the given system is stable.

2. The given equation is,

$$h(n) = n\left(\frac{1}{2}\right)^n u(n).$$

Multiplication by u (n) indicates that h (n) is valid only for positive values of n. That means for the range n = 0 to ∞.

So this system is causal.

Ex. 3.9.5 : Impulse response of the RC circuit is given as $h(t) = \frac{1}{RC} e^{-t/RC} u(t)$

Find the unit step response of the circuit.

Soln. : Impulse response is the response of circuit when applied input is unit impulse δ (t). And unit step response means output of system when applied input is u (t).

Now step response is related to impulse response as,

$$s(t) = \int_{0}^{t} h(\tau)\,d\tau$$

$$\text{Give, } h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$\therefore \; h(\tau) = \frac{1}{RC} e^{-\tau/RC} u(\tau)$$

Due to multiplication of u(τ) the limits of integration will be from 0 to t

$$= \int_{0}^{t} \frac{1}{RC} e^{-\tau/RC}\,d\tau$$

$$= \frac{1}{RC}\left[\frac{e^{-\tau/RC}}{-\frac{1}{RC}}\right]_0^t = -\left[e^{-t/RC} - e^0\right]$$

$$\therefore \; s(t) = 1 - e^{-t/RC}$$

Ex. 3.9.6 : Find the impulse response of the following systems :

1. $y[n] = \frac{1}{4}\sum_{K=0}^{3} x[n-K]$

Represent your answer in terms of u[n].

2. $y(t) = \int_{-\infty}^{t} x(\lambda)\,d\lambda.$

Soln. :

1. Given $y(n) = \frac{1}{4}\sum_{K=0}^{3} x[n-K]$

Impulse response of system, h (n) means output of system when applied input is x (n) = δ (n).

$$\therefore \; h(n) = \frac{1}{4}\sum_{K=0}^{3} \delta[n-K]$$

$$\therefore \; h(n) = \frac{1}{4}[\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)]$$

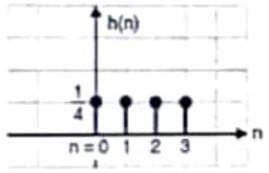
This signal is shown in Fig. P. 3.9.6.

$$\therefore \; h(n) = \left\{\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right\}$$

↑

In terms of unit step it can be expressed as,

$$h(n) = \frac{1}{4}[u(n) - u(n-3)]$$



(H-745) Fig. P. 3.9.6

2. Given $y(t) = \int_{-\infty}^{\infty} x(\lambda)\,d\lambda$

Consider convolution of x (t) and u (t). Using definition of convolution,

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\lambda)\,u(t-\lambda)\,d\lambda.$$

$$\therefore \; x(t) * u(t) = \int_{-\infty}^{\infty} x(\lambda)\,u(-\lambda + t)\,d\lambda$$

But u (– λ + t) is folded and delayed unit step which exists from – ∞ to t.

$$\therefore \; x(t) * u(t) = \int_{-\infty}^{t} x(\lambda)\,d\lambda$$

Now convolution is denoted by x (t) * h (t)

∴ Impulse response = h (t) = u (t)

Ex. 3.9.7 : Find the step response of the system whose impulse response is :

1. δ [n + 2] + δ [n – 3]
2. u [n]

Soln. :

1. **Given :** h (n) = δ [n + 2] + δ [n – 3]

Now step response means output of system when applied input is unit step that means x (n) = u (n).

Now $y(n) = x(n) * h(n)$

$\therefore\ y(n) = u(n) * [\delta(n+2) + \delta(n-3)]$

$\therefore\ y(n) = u(n-2) + u(n-3)$

2. Given : $h(n) = u(n)$

$\therefore\ y(n) = u(n) * u(n)$

Here $h(k) = u(k)$ and $x(k) = u(k)$

According to definition of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Here $h(n-k) = u(n-k) = u(-k+n)$

$$\therefore\ y(n) = \sum_{k=-\infty}^{\infty} u(k) \cdot u(-k+n)$$

The range of summation will be 0 to n

$$\therefore\ y(n) = \sum_{k=0}^{n} 1 = \sum_{k=0}^{n} (1)^k = (n+1)$$

Ex. 3.9.8 : Find the step response of the given LTI system :

$h(t) = t\,x(t+1)\,u(t)$

Soln. :

Given : $h(t) = t\,x(t+1)\,u(t)$

Now step response is output of system when applied input is unit step that means $x(t) = u(t)$.

$\therefore\ x(t+1) = u(t+1)$

$\therefore\ h(t) = t\,u(t+1)\,u(t)$

Now step response is given by,

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$\therefore\ s(t) = \int_{-\infty}^{t} \tau\, u(\tau+1) \cdot u(\tau)\, d\tau$$

The multiplication of $u(\tau+1) \cdot u(\tau)$ gives the range, $\tau = -1$ to ∞.

$$\therefore\ s(t) = \int_{-1}^{t} \tau\, d\tau = \left[\frac{\tau^2}{2}\right]_{-1}^{t}$$

$$\therefore\ s(t) = \frac{t^2}{2} - \frac{(-1)^2}{2}$$

$$\therefore\ s(t) = \frac{t^2-1}{2}$$

Ex. 3.9.9 : Test the stability of the LTI systems, whose impulse responses are given below :

1. $h(t) = e^{-5|t|}$ 2. $h(t) = e^{4t}\,u(t)$.

Soln. :

1. According to the condition of stability,

$$\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$$

Given, $h(t) = e^{-5|t|}$

$$\therefore \int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} e^{-5|t|}\, dt$$

$$= \int_{-\infty}^{0} e^{5t}\, dt + \int_{0}^{\infty} e^{-5t}\, dt$$

$$= \left[\frac{1}{5} \cdot e^{5t}\right]_{-\infty}^{0} + \left[\frac{-1}{5} \cdot e^{-5t}\right]_{0}^{\infty}$$

$$= \left[\frac{1}{5} - 0\right] + \left[0 + \frac{1}{5}\right]$$

$$= \frac{2}{5} < \infty$$

Output is bounded; so system is stable.

2. **Given :** $h(t) = e^{4t}\,u(t)$

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{-\infty}^{\infty} |e^{4t}\,u(t)|\, dt$$

$$= \int_{0}^{\infty} e^{4t}\, dt = \left[\frac{e^{4t}}{4}\right]_{0}^{\infty} = \infty$$

Since output is not bounded; system is unstable.

Ex. 3.9.10 : Find the step response of the first order recursive system with impulse response :

$$h[n] = \left(\frac{1}{3}\right)^n \cdot u[n].$$

Soln. : The step response of D.T. system is given by,

$$y(n) = \sum_{k=-\infty}^{n} h(k) \qquad ...(1)$$

$$\text{Given } h(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$\therefore\ h(k) = \left(\frac{1}{3}\right)^k u(k) \qquad ...(2)$$

Due to multiplication by $u(k)$; limits in equation becomes, $k = 0$ to n.

$$\therefore\ y(n) = \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k$$

We have standard summation formula.

$$\sum_{n=N_1}^{N_2} a^n = \begin{cases} \dfrac{a^{N_1} - a^{N_2+1}}{1-a} & \text{for } a \neq 1 \\ N_2 - N_1 + 1 & \text{for } a = 1 \end{cases}$$

Here $a = \frac{1}{3}, N_1 = 0, N_2 = n$

$$\therefore\ y(n) = \frac{\left(\frac{1}{3}\right)^0 - \left(\frac{1}{3}\right)^{n+1}}{1 - \frac{1}{3}} = \frac{1 - \left(\frac{1}{3}\right)^{n+1}}{\frac{2}{3}}$$

$$\therefore\ y(n) = \frac{3}{2}\left[1 - \left(\frac{1}{3}\right)^n \cdot \left(\frac{1}{3}\right)\right]$$

$$\therefore\ y(n) = \frac{3}{2} - \frac{1}{2} \cdot \left(\frac{1}{3}\right)^n$$

This is the step response of system.

Ex. 3.9.11 : Determine unit step response of the system whose impulse response is given as $h(t) = 3\,tu(t)$ **Dec. 12, 5 Marks**

Soln. : The step response is related to impulse response as,

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

Here $h(t) = 3\,t\,u(t)$

$\therefore\ h(\tau) = 3\,\tau\,u(\tau)$

Due to unit step, $u(\tau)$; the limits of integration will be from 0 to t.

$$\therefore\ s(t) = \int_{-\infty}^{t} 3\,\tau\, d\tau = 3\left[\frac{\tau^2}{2}\right]_{0}^{t}$$

$$\therefore\ s(t) = \frac{3}{2}t^2$$

This is the step response of system

3.10 Exponential Response :

- Consider an LTI system having impulse response h (t) driven by an input x (t) such that,

$$x(t) = e^{j2\pi ft} \qquad ...(3.10.1)$$

- Using Equation (3.10.2), the system response is given by,

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{j2\pi f(t-\tau)}\, d\tau \qquad ...(3.10.2)$$

$$\therefore\ y(t) = e^{j2\pi ft} \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau \qquad ...(3.10.3)$$

1. Let us define $H(f) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f\tau}\, d\tau$

$$\therefore\ y(t) = H(f)\, e^{j2\pi ft} \qquad ...(3.10.4)$$

Thus the response of an LTI system to a complex exponential function of frequency f is therefore the same complex exponential function multiplied by a constant coefficient H (f), which is called as the transfer function of the system.

The transfer function H (f) and the impulse function h (t) form a Fourier transform pair. This is indicated as follows :

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi ft}\, dt \qquad ...(3.10.5)$$

$$\text{and} \quad h(t) = \int_{-\infty}^{\infty} H(f) \cdot e^{j2\pi ft}\, df \qquad ...(3.10.6)$$

Alternative definition of transfer function

- The transfer function is defined alternatively as follows :

$$H(f) = \left.\frac{y(t)}{x(t)}\right|_{x(t) = e^{j2\pi ft}} \qquad ...(3.10.7)$$

- We know that in time domain,

$$y(t) = h(t) * x(t)$$

- Taking the Fourier transform of both sides we get,

$$Y(f) = H(f) \cdot X(f) \qquad ...(3.10.8)$$

Thus it is possible to describe an LTI system simply in the frequency domain.

Review Questions

Q. 1 Derive the equation of linear convolution.

Q. 2 Explain the properties of convolution sum.

Q. 3 How any arbitrary signal can be expressed in terms of summation of weighted unit impulses ?

Q. 4 Explain how we test BIBO stability of DTLTI system using impulse response ?

Q. 5 Define the term 'step response'. How it can be expressed in terms of impulse response for CT system ?

Q. 6 State and prove the condition of stability.

Q. 7 Prove that DTLTI system is causal if h (n) = 0 for n < 0.

□□□

CHAPTER 4

Module 2

Correlation and Spectral Density

Syllabus :

Correlation and spectral density : Auto correlation, Cross correlation, Analogy between correlation and convolution, Energy spectral density, Power spectral density, Relation between ESD, PSD with auto correlation.

4.1 Introduction :

- In chapter 1 we have discussed about the types of signals and their mathematical expressions.
- In this chapter we are going to discuss some important characteristics of signals such as correlation, energy and power spectral density, the relation between correlation and spectral density functions.

4.1.1 Correlation :

Correlation is a mathematical tool that can be used to compare two signals. From the value of correlation we can conclude upon the degree of similarity between the two waveforms being compared.

4.2 Spectral Density Functions :

- In this section, we will categorize signals and systems by concentrating on the "energy" or "power" of a signal. In order to do so let us get introduced to a new term called "spectral density". **The spectral density of a signal, is used for defining the distribution of energy or power per unit bandwidth as a function of frequency.**
- The spectral density of energy signals is called as "Energy Spectral Density (ESD)" while that of the power signals is called as "Power Spectral Density (PSD)". Let us discuss about them now.

4.2.1 Energy Spectral Density (ESD) :

According to Rayleigh's energy theorem, the total energy of a signal is given by,

$$E = \int_{-\infty}^{\infty} |X(f)|^2 df$$

This means that the energy of a signal x (t) is equal to the sum of energies contributed by all the spectral components of the signal x (t). Thus it is possible to obtain the energy of the signal x (t) by calculating the area under the square of its amplitude spectrum $|X(f)|$.

The Rayleigh energy theorem provides a useful method of evaluating the total energy and it also tells us that $|X(f)|^2$ can provide us the "distribution of energy" of signal x (t) in the frequency domain.

Definition of ESD :

The squared amplitude spectrum $|X(f)|^2$ is defined as the "Energy Spectral Density (ESD)" or "Energy Density Spectrum", of the signal x (t) and it is denoted by ψ (f).

$$\therefore \quad ESD = \psi(f) = |X(f)|^2 \qquad ...(4.2.1)$$

4.3 Power Spectral Density (PSD) :

- Let x (t) be a real valued power signal. Then the average normalized power of this signal is defined as,

$$P = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt \qquad ...(4.3.1)$$

- If the signal x (t) is periodic with a period T_0 then according to the Parseval's power theorem, the average signal power is given by,

$$P = \sum_{n=-\infty}^{\infty} |C_n|^2 \qquad ...(4.3.2)$$

- For a periodic signal, C_n is expressed as follows :

$$C_n = \frac{1}{T_0} X(n f_0) \qquad ...(4.3.3)$$

- Substituting this expression of C_n into Equation (4.3.2) we get,

$$P = \sum_{n=-\infty}^{\infty} \frac{1}{T_0^2} |X(n f_0)|^2$$

$$\therefore \quad P = \frac{1}{T_0^2} \sum_{n=-\infty}^{\infty} |X(n f_0)|^2 \qquad ...(4.3.4)$$

- But the Fourier transform of a periodic signal is not continuous, it is discrete and present only at frequencies $\pm f_0, \pm 2 f_0, \pm 3 f_0 \ldots$

- We now define the power spectral density of a periodic signal x (t) as follows :

Power spectral density :

$$S(f) = \frac{1}{T_0^2} \sum_{n=-\infty}^{\infty} |X(n f_0)|^2 \delta(f - n f_0) \qquad ...(4.3.5)$$

- Equation (4.3.5) shows that power spectral density is a discrete function of frequency. This is indicated by the term δ $(f - n f_0)$ in Equation (4.3.5). The power spectral density function is present only at the harmonic frequencies of f_0.
- The most important property of PSD is that the area under PSD is equal to the average power of the signal x (t).

$$\therefore \quad P = \int_{-\infty}^{\infty} S(f)\, df \qquad ...(4.3.6)$$

Let us prove this,

$$\int_{-\infty}^{\infty} S(f)\, df = \int_{-\infty}^{\infty} \frac{1}{T_0^2} \sum_{n=-\infty}^{\infty} |X(n f_0)|^2 \delta(f - n f_0)\, df \qquad ...(4.3.7)$$

- Using the sifting property of delta function we get,

$$\int_{-\infty}^{\infty} S(f)\, df = \frac{1}{T_0^2} \sum_{n=-\infty}^{\infty} |X(n f_0)|^2$$

$$\therefore \quad \int_{-\infty}^{\infty} S(f)\, df = P \quad \text{... from Equation (4.3.4)}$$

> Thus the average power of a periodic signal x (t) is given by the area under the power spectral density function.

4.4 Correlation of CT Energy Signals :

The correlation function is used to indicate the degree of similarity between two signals. Higher the value of correlation function is, more is the similarity between the two signals. The correlation function is of two types :

1. Auto-correlation function.
2. Cross-correlation function.

4.4.1 Auto-Correlation Function for the CT Energy Signals : MU : May 14

University Questions

Q. 1 Define and explain : Auto correlation **(May 14, 2 Marks)**

- The auto-correlation function for an energy signal x (t) provides the measure of similarity between the signal x (t) and its time delayed version x (t – τ).
- The auto-correlation function is denoted by R (τ) and it is defined as follows :

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x(t-\tau)\, dt \qquad \text{If } x(t) \text{ is real valued} \qquad ...(4.4.1)$$

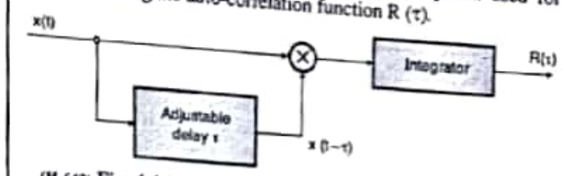
- The auto-correlation function can also be defined as,

$$R(\tau) = \int_{-\infty}^{\infty} x(t+\tau) \cdot x(t)\, dt \qquad \text{If } x(t) \text{ is real valued} \qquad ...(4.4.2)$$

- If the signal x (t) is complex valued then the auto-correlation function for it is defined as follows :

$$R(\tau) = \int_{-\infty}^{\infty} x(t) \cdot x^*(t-\tau)\, dt \qquad \text{If } x(t) \text{ is complex} \qquad ...(4.4.3)$$

- Fig. 4.4.1 shows the block diagram of a system used for measuring the auto-correlation function R (τ).



(H-642) **Fig. 4.4.1 : Arrangement to measure auto-correlation**

4.4.2 Analogy between Autocorrelation and Convolution :

- The convolution of two C. T. signals x (t) and y (t) is defined as follows :

$$x(t) * y(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \qquad ...(4.4.3(a))$$

- Now compare this equation with the definition of autocorrelation stated in Equation (4.4.1). You will find that the process of autocorrelation is very similar to that of convolution.

4.4.3 Properties of Auto-Correlation Function of CT Energy Signals :

The important properties of auto-correlation function of the energy signals are as follows :

Property 1 :

The auto-correlation function of a real valued energy signal x (t) is a real valued even (symmetric) function. That means,

$$R(\tau) = R(-\tau) \qquad ...(4.4.4)$$

Property 2 :

The value of the auto-correlation function of an energy signal x (t) at the origin is equal to the energy of the signal. That is,

$$R(0) = E \qquad ...(4.4.5)$$

Property 3 :

The maximum value of auto-correlation function of an energy signal x (t) occurs at the origin. That means,

$$|R(\tau)| \le R(0) \quad \text{... for all } \tau \qquad ...(4.4.6)$$

Interrelation between autocorrelation and ESD :

Property 4 :

For an energy signal x (t), the auto-correlation function and energy spectral density form a fourier transform pair. That is,

$$R(\tau) \xleftrightarrow{F} \psi(f) \qquad ...(4.4.7)$$

4.4.4 Auto-Correlation Function of CT Power Signals :

- We can develop a formula for the auto-correlation function of power signals by following a procedure similar to that described for the energy signals.

- The auto-correlation function of power signals is a measure of similarity between a periodic signal $x(t)$ and its delayed version $x(t-\tau)$. If the signal $x(t)$ is periodic with a period T_0, then the auto-correlation function over one complete period T_0 is defined as follows :

Auto-correlation function of periodic signals

$$R(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T/2} x(t)\cdot x(t-\tau)\,dt$$

For $x(t)$ to be real valued. ...(4.4.8)

- Note that this expression is same as that for the auto-correlation function of energy signals, except for the inclusion of $1/T_0$ and change in the limits of integration.
- For any period T it is defined as,

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)\cdot x(t-\tau)\,dt$$

For $x(t)$ to be real valued.

- For any period T with $x(t)$ complex,

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t)\cdot x^*(t-\tau)\,dt$$

- For any period T with $x(t)$ complex and τ in negative direction,

$$R(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} x(t+\tau)\cdot x^*(t)\,dt \quad ...(4.4.9)$$

- Both $x(t)$ and $x^*(t-\tau)$ are advanced by τ to get $x(t+\tau)$ and $x^*(t)$ respectively.

4.4.5 Properties of Auto-Correlation Function of a CT Periodic Signal :

The auto-correlation function of a power signal has properties that are similar to those of auto-correlation function of the energy signals.

Property 1 :

- The auto-correlation function of a real valued power signal $x(t)$ is a real valued even function. That means,

$$R(\tau) = R(-\tau) \quad ...(4.4.10)$$

- The auto-correlation function of complex valued signal $x(t)$ is given by,

$$R(\tau) = R^*(-\tau) \quad ...(4.4.11)$$

This is also called as the conjugate property.

Property 2 :

The value of auto-correlation function of a power signal $x(t)$ at the origin is equal to the average power P. That means,

$$R(0) = P \quad ...(4.4.12)$$

Property 3 :

The maximum value of the auto-correlation function of a power signal $x(t)$ occurs at the origin. That means,

$$|R(\tau)| \le R(0) \quad ...(4.4.13)$$

Interrelation between autocorrelation and PSD :

Property 4 :

For a power signal $x(t)$, the auto-correlation function and the power spectral density form a fourier transform pair. That means,

$$R(\tau) \overset{F}{\longleftrightarrow} S(f) \quad ...(4.4.14)$$

Ex. 4.4.1 : Find the autocorrelation of the continuous time signal given by, $x(t) = A\,\text{rect}(t/2)$.

May 11, 12 Marks, May 16, 5 Marks

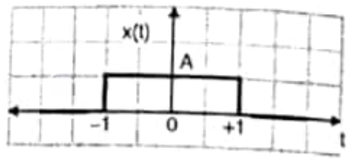
Soln. :

Step 1 : Represent the given signal graphically :

The given signal is represented graphically as shown in Fig. P. 4.4.1.

It is expressed mathematically as :

$$x(t) = A \quad \text{for } -1 \le t \le +1$$
$$= 0 \quad \text{elsewhere}$$



(H-643) **Fig. P. 4.4.1**

Step 2 : Obtain the autocorrelation function :

- Autocorrelation function of continuous time energy signals is given by,

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)\,dt$$

- Now refer Fig. P. 4.4.1(b), (c) and (d) which shows the shifted signal $x(t-\tau)$ and hence the possibility of overlap of $x(t)$ and $x(t-\tau)$.
- This shows that if τ is less than -2 then there will not be any overlap and if $\tau > 2$ then also there will not be any overlap of $x(t)$ and $x(t-\tau)$. And if there is no overlap, then the product $x(t)\cdot x(t-\tau) = 0$ and the autocorrelation also is zero.
- The actual overlapping as shown in Fig. P. 4.4.1(c) takes place for $t = -1$ to $t = (\tau + 1)$. Hence the autocorrelation function is given by,

$$R(\tau) = \underbrace{\int_{-\infty}^{-1} x(t)\,x(t-\tau)\,dt}_{\text{No overlap}} + \int_{-1}^{(\tau+1)} x(t)\cdot x(t-\tau)\,dt$$

$$+ \underbrace{\int_{\tau+1}^{\infty} x(t)\cdot x(t-\tau)\,dt}_{\text{No overlap}}$$

$$= \int_{-\infty}^{-1} 0\,dt + \int_{-1}^{\tau+1} A\cdot A\cdot dt + \int_{\tau+1}^{\infty} 0\,dt$$

$$= 0 + A^2[t]_{-1}^{\tau+1} + 0 = A^2[(\tau+1)-(-1)]$$

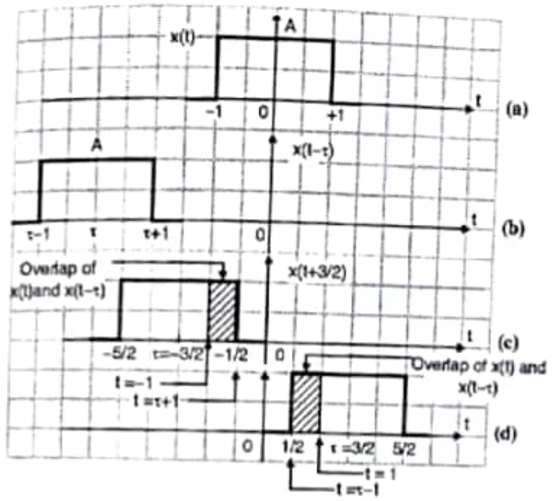
$$= A^2[\tau+2]$$

$$\therefore\ R(\tau) = A^2[\tau+2] \quad \text{... for } -2 \le \tau \le 0 \quad ...(1)$$

- Now consider Fig. P. 4.4.1(d) which shows that the overlapping will take place if $0 \le \tau \le 2$.

- The actual overlapping as shown in Fig. P. 4.4.1(d) takes place for $t = 1$ to $t = (\tau - 1)$. Hence the autocorrelation function is given by,

$$R(\tau) = \underbrace{\int_{-\infty}^{\tau-1} x(t)\,x(t-\tau)\,dt}_{\text{No overlap}} + \underbrace{\int_{\tau-1}^{1} x(t)\cdot x(t-\tau)\,dt}_{\text{Overlap}} + \underbrace{\int_{1}^{\infty} x(t)\,x(t-\tau)\,dt}_{\text{No overlap}}$$



(H-644) **Fig. P. 4.4.1**

$$\therefore\ R(\tau) = \int_{-\infty}^{(\tau-1)} 0\,dt + \int_{(\tau-1)}^{1} A\cdot A\cdot dt + \int_{1}^{\infty} 0\,dt$$

$$\therefore\ R(\tau) = 0 + A^2[t]_{\tau-1}^{1} + 0 = A^2[1-\tau+1]$$

$$= A^2[2-\tau] \quad \text{... for } 0 \le \tau \le +2 \quad ...(2)$$

- Hence the autocorrelation function $R(\tau)$ is mathematically expressed as :

$$R(\tau) = 0 \quad ... \tau < -2$$
$$= A^2(2+\tau) \quad ... -2 \le \tau \le 0$$
$$= A^2(2-\tau) \quad ... 0 \le \tau \le 2$$
$$= 0 \quad ... \tau > 2$$

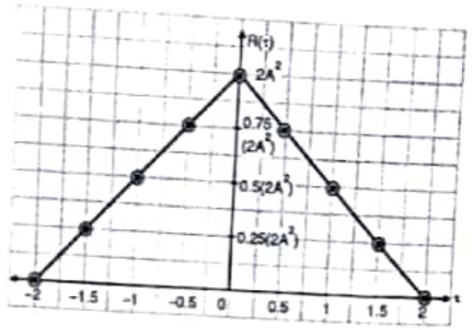
Step 3 : Graphical representation of R (τ) :

Table P. 4.4.1 shows the various values of $R(\tau)$ according to various ranges of τ and Fig. P. 4.4.1(e) shows the graphical representation of $R(\tau)$.

Table P. 4.4.1

τ	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$R(\tau)$	0	0.25 $(2A^2)$	0.5 $(2A^2)$	0.75 $(2A^2)$	$2A^2$	0.75 $(2A^2)$	0.75 $(2A^2)$	0.25 $(2A^2)$	0

$$R(\tau) = A^2(2+\tau) = 2A^2\left(1+\frac{\tau}{2}\right) \qquad R(\tau) = A^2(2-\tau) = 2A^2\left(1-\frac{\tau}{2}\right)$$



(H-645) **Fig. P. 4.4.1(e) : Graphical representation of R (τ)**

Ex. 4.4.2 : Find the autocorrelation of the following power signal :

$$x(t) = \sin\omega_0 t$$

Soln. :

- We know that the autocorrelation function of periodic signals with a period T_0 is given by,

$$R(\tau) = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,x(t-\tau)\,dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \sin(\omega_0 t)\sin\omega_0(t-\tau)\,dt$$

- But $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

Here let $A = \omega_0(t-\tau)$ and $B = \omega_0 t$

$$\therefore R(\tau) = \frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} [\cos(\omega_0 t - \omega_0\tau - \omega_0 t) - \cos(\omega_0 t - \omega_0\tau + \omega_0 t)]\,dt$$

$$= \underbrace{\frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} \cos(-\omega_0\tau)\,dt}_{(I)} - \underbrace{\frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} \cos(2\omega_0 t - \omega_0\tau)\cdot dt}_{(II)}$$

- Note that term II is a cosine wave of frequency $2f_0$ and phase shift of $\omega_0\tau$. If we integrate a sine or cosine wave over its one cycle period ($-T_0/2$ to $T_0/2$) then the result of integration is zero. Hence second term in the expression for $R(\tau)$ is zero.

$$\therefore\ R(\tau) = \frac{1}{2T_0}\int_{-T_0/2}^{T_0/2} \cos(-\omega_0\tau)\,dt$$

- Note that $\cos(-\omega_0\tau)$ is a constant and $\cos(-\omega_0\tau) = \cos(+\omega_0\tau)$

$$\therefore\ R(\tau) = \frac{\cos(\omega_0\tau)}{2T_0}[t]_{-T_0/2}^{T_0/2}$$

$$= \frac{\cos(\omega_0\tau)}{2T_0}\left[\frac{T_0}{2}+\frac{T_0}{2}\right] = \frac{1}{2}\cos(\omega_0\tau) \quad \textbf{...Ans.}$$

Ex. 4.4.3 : Obtain the power of signal $x(t) = \sin(\omega_0 t)$.

Soln. :

- We know that the power of a power signal is equal to its autocorrelation obtained at $\tau = 0$.

  i.e. $P = R(0)$

- Refer to Ex. 4.4.2 in which we have calculated the autocorrelation function, $R(\tau)$ of the given signal at

$$R(\tau) = \frac{1}{2}\cos(\omega_0 \tau)$$

- So substitute $\tau = 0$ to get,

$$P = R(0) = \frac{1}{2}\cos(0) = 1/2 \quad \text{...Ans.}$$

Ex. 4.4.4 : Find autocorrelation function and power spectral density of signal $x(t)$.

$x(t) = A\sin(\omega_1 t + \tau)$ where $\omega_1 = 2\pi / T$.

Soln. :

Step 1 : Find the autocorrelation, $R(\tau)$:

- The given signal is a phase shifted sinewave. But because we integrate over one complete cycle while obtaining the autocorrelation $R(\tau)$, this phase shift does not matter.
- So $R(\tau)$ or $A\sin\omega t$ and that of $A\sin(\omega t + \phi)$ will be the same.
- In Ex. 4.4.2 we have obtained the expression for $R(\tau)$ of $\sin\omega_0 t$ as,

$$R(\tau) = \frac{1}{2}\cos\omega_0 t$$

- Hence the autocorrelation function $R(\tau)$ of $x(t) = A\sin(\omega_1 t + \tau)$ is given by,

$$R(\tau) = \frac{A^2}{2}\cos\omega_1 \tau \quad \text{...Ans.}$$

Step 2 : Power spectral density $S(f)$:

$$S(f) = F[R(\tau)] = F\left[\frac{A^2}{2}\cos\omega_1\tau\right]$$

$$= \frac{A^2}{2}F[\cos\omega_1\tau] = \frac{A^2}{2}\times\frac{1}{2}[\delta(f-f_1)+\delta(f+f_1)]$$

$$= \frac{A^2}{4}[\delta(f-f_1)+\delta(f+f_1)] \quad \text{...Ans.}$$

Ex. 4.4.5 : Find the autocorrelation, power spectral density and power of the following signal :

$x(t) = 5 + 4\sin(10\pi t + 30^\circ)$

Soln. :

Step 1 : Find the FT of $x(t)$:

$$F[x(t)] = F[5 + 4\sin(10\pi t + 30^\circ)] \quad ...(1)$$

$$F[5] = 5\delta(f) \quad ...(2)$$

$$4\sin(10\pi t + \pi/6) = 4\left[\frac{e^{j(10\pi t+\pi/6)} - e^{-j(10\pi t+\pi/6)}}{2j}\right]$$

$$= \frac{2}{j}\left[e^{j\pi/6}e^{j10\pi t} - e^{-j\pi/6}e^{-j10\pi t}\right]$$

$$\therefore F[4\sin(10\pi t + \pi/6)] = F\left[\frac{2}{j}e^{j\pi/6}e^{j10\pi t} - \frac{2}{j}e^{-j\pi/6}e^{-j10\pi t}\right]$$

$$= \frac{2}{j}\left[e^{j\pi/6}F(e^{j10\pi t}) - e^{-j\pi/6}F(e^{-j10\pi t})\right]$$

But $e^{j10\pi t} \xleftrightarrow{F} \delta(f - f_0) = \delta(f-5)$

and $e^{-j10\pi t} \xleftrightarrow{F} \delta(f+5)$

$$\therefore F[4\sin(10\pi t + \pi/6)] = \frac{2}{j}e^{j\pi/6}\delta(f-5) - e^{-j\pi/6}\delta(f+5) \quad ...(3)$$

$\therefore$ F.T. of $x(t)$ is

$$X(f) = 5\delta(f) + \frac{2}{j}e^{j\pi/6}\delta(f-5) - e^{-j\pi/6}\delta(f+5) \quad ...(4)$$

Step 2 : Find $|X(f)|^2$:

$$X(f) = \underbrace{5\delta(f)}_{A} + \underbrace{\frac{2}{j}e^{j\pi/6}\delta(f-5)}_{B} + \underbrace{-\frac{2}{j}e^{-j\pi/6}\delta(f+5)}_{C}$$

$$\therefore |X(f)|^2 = (A+B+C)^2$$

$$= A^2 + B^2 + C^2 + 2AB + 2BC + 2AC$$

But $AB = BC = AC = 0$

Because, $\delta(f)\delta(f-5) = \delta(f-5)\delta(f+5) = \delta(f)\delta(f+5) = 0$

$$|e^{j\pi/6}| = |e^{-j\pi/6}| = 1$$

$$\therefore |X(f)|^2 = 25\delta(f) - \frac{4}{j^2}[\delta(f-5)] + \frac{4}{j^2}[\delta(f+5)]$$

But $PSD = |X(f)|^2$

$$= 25\delta(f) + 4[\delta(f-5) + \delta(f+5)] \quad \text{...Ans.}$$

Step 3 : Autocorrelation :

$$R(\tau) = F^{-1}[s(f)]$$

$$= F^{-1}[5\delta(f) + 4\delta(f-5) + \delta(f+5)]$$

$$= 5 + 4e^{j\pi\times 5\tau} + 4e^{-j2\pi 5\tau}$$

$$= 5 + 4\left[e^{j10\pi\tau} + e^{-j10\pi\tau}\right] = 5 + 8\,\frac{\left[e^{j10\pi\tau} + e^{-j10\pi\tau}\right]}{2}$$

$$= 5 + 8\cos(10\pi\tau) \quad \text{...Ans.}$$

Step 4 : Power :

Power $P = R(0) = 5 + 8\cos 0 = 13$ ...Ans.

Ex. 4.4.6 : Show that autocorrelation and ESD form Fourier pair of each other, for the following function : $x(t) = e^{-5t}\cdot u(t)$.

Soln. :

Given : $x(t) = e^{-5t}u(t)$

Step 1 : Calculation of autocorrelation :

We can write, $x(t-\tau) = e^{-5(t-\tau)}u(t-\tau)$

$$= e^{-5t}\cdot e^{5\tau}\cdot u(-\tau + t)$$

Now according to the definition of autocorrelation

$$R(\tau) = \int_{-\infty}^{\infty} x(t)\cdot x(t-\tau)\,dt$$

$$= \int_{-\infty}^{\infty} e^{-5t}\cdot e^{-5t}\cdot e^{5\tau}\cdot u(t)\cdot u(-\tau+t)\,dt$$

Due to the term $u(t)\cdot u(-\tau+t)$; the limits of integration will be τ to ∞.

$$\therefore R(\tau) = \int_{\tau}^{\infty} e^{-10t}\cdot e^{5\tau}\,dt = e^{5\tau}\int_{\tau}^{\infty} e^{-10t}\,dt$$

$$\therefore R(\tau) = e^{5\tau}\left[\frac{e^{-10t}}{-10}\right]_{\tau}^{\infty} = \frac{-e^{5\tau}}{10}[e^{-\infty} - e^{-10\tau}]$$

$$\therefore R(\tau) = \frac{e^{-5\tau}}{10} \quad ...(1)$$

Step 2 : Calculation of Fourier transform of $R(\tau)$:

$$\text{F.T.}\{R(\tau)\} = \text{F.T.}\left\{\frac{e^{-5\tau}}{10}\right\}$$

$$\therefore \text{F.T.}\{R(\tau)\} = \text{F.T.}\left\{\frac{e^{-5|\tau|}}{10}\right\}$$

This is because Equation (1) is valid only for positive values of τ.

$$\therefore \text{F.T.}\{R(\tau)\} = \int_{-\infty}^{\infty}\frac{e^{-5|t|}}{10}\cdot e^{-j2\pi ft}\,dt$$

$$= \frac{10}{10[25 + (2\pi f)^2]} = \frac{1}{25 + (2\pi f)^2} \quad ...(2)$$

Step 3 : Calculation of ESD :

ESD is given by,

$$\psi(f) = |X(f)|^2$$

Now $x(t) = e^{-5t}u(t)$

$$\therefore X(f) = \frac{1}{5 + j2\pi f}$$

$$\therefore |X(f)| = \frac{1}{\sqrt{25 + (2\pi f)^2}}$$

$$\therefore \psi(f) = |X(f)|^2 = \frac{1}{25 + (2\pi f)^2} \quad ...(3)$$

From Equations (1) and (3) ; we can say that autocorrelation and ESD forms a Fourier transform pair.

4.5 Cross Correlation of CT Signals :

- In the previous section we have discussed about the auto-correlation which is a measure of similarity between a signal and its shifted version.
- Now let us discuss the other type of correlation which is "cross correlation".
- Cross correlation measures the similarity between two completely different signals say $x(t)$ and $y(t)$.
- Cross correlation can be defined for the energy as well as power signals.

4.5.1 Cross-Correlation of CT Energy Signals :

MU : May 14

University Questions

Q. 1 Define and explain : Cross correlation **(May 14, 2 Marks)**

- Let $x_1(t)$ and $x_2(t)$ denote a pair of real valued energy signals. Then the cross-correlation function of this pair of signals is defined as follows :

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\cdot x_2(t-\tau)\,dt \quad ...(4.5.1)$$

- If there is some similarity between the two signals $x_1(t)$ and $x_2(t)$ then the value of the cross-correlation function $R_{12}(\tau)$ will be finite over some range of τ. Thus cross-correlation gives the measure of similarity or "coherence", between them.
- Equation (4.5.1) defines one possible value of the cross-correlation function for a specified value of the delay variable τ. We can define the second cross-correlation function of the energy signals $x_1(t)$ and $x_2(t)$ as follows :

$$R_{21}(\tau) = \int_{-\infty}^{\infty} x_2(t)\cdot x_1(t-\tau)\,dt \quad ...(4.5.2)$$

- If the signals $x_1(t)$ and $x_2(t)$ are complex valued signals of finite energy then the cross-correlation is defined as :

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\cdot x_2^*(t-\tau)\,dt \quad ...(4.5.3)$$

- It is important to note that the value of $R_{12}(\tau)$ will be finite if the two signals $x_1(t)$ and $x_2(t)$ have some similarity between them. If $R_{12}(\tau) = 0$ then the two signals will not have any similarity between them.

4.5.2 Analogy between Cross-Correlation and Convolution :

- Convolution between two C.T. signals $x_1(t)$ and $x_2(t)$ is defined as follows :

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\,x_2(t-\tau)\,d\tau$$

- Compare this equation with the definition of cross correlation stated in Equation (4.5.1). You will realize the similarity between cross correlation and convolution.

4.5.3 Properties of Cross-Correlation of the CT Energy Signals :

Property 1 :

- The energy signals $x_1(t)$ and $x_2(t)$ are said to be "orthogonal" over the entire time interval if $R_{12}(0)$ is zero. That means,

If $R_{12}(0) = 0$ ...$x_1(t)$ and $x_2(t)$ are orthogonal.

$$\text{But } R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\,x_2(t-\tau)\,dt$$

$$\therefore R_{12}(0) = \int_{-\infty}^{\infty} x_1(t)\,x_2(t-0)\,dt = \int_{-\infty}^{\infty} x_1(t)\,x_2(t)\,dt$$

$\therefore$ Condition for orthogonality is,

$$\int_{-\infty}^{\infty} x_1(t)\,x_2(t)\,dt = 0 \quad ...(4.5.4)$$

As $R_{12}(\tau) = 0$ for the orthogonal signals, there is no similarity between them.

Property 2 : The cross-correlation is generally not commutative. That means $R_{12}(\tau) \neq R_{21}(\tau)$ in fact.

$$R_{12}(\tau) = R_{21}(-\tau) \quad ...(4.5.5)$$

Property 3 : Correlation theorem :

- The cross-correlation of two energy signals corresponds to the multiplication of the fourier transform of one signal by the complex conjugate of the fourier transform of the other.
- That means :

$$R_{12}(\tau) \xleftrightarrow{F} X_1(f)\cdot X_2^*(f) \quad ...(4.5.6)$$

4.5.4 Cross-Correlation of CT Power Signals :

- Cross-correlation is defined for two different power signals $x_1(t)$ and $x_2(t)$. If $x_1(t)$ and $x_2(t)$ represents two different power signals then the cross-correlation between them is defined as follows :

$$R_{12}(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x_1(t) \cdot x_2^*(t-\tau)\, dt \qquad ...(4.5.7)$$

- Similarly we can define a second cross-correlation function as,

$$R_{21}(\tau) = \lim_{T \to \infty} \int_{-T/2}^{T/2} x_2(t) \cdot x_1^*(t-\tau)\, dt \qquad ...(4.5.8)$$

- If the two periodic signals $x_1(t)$ and $x_2(t)$ have the same time period T_0, then the cross-correlation is defined as,

$$R_{12}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_1(t) \cdot x_2^*(t-\tau)\, dt \qquad ...(4.5.9)$$

and the second cross-correlation for the two periodi signals is defined as,

$$R_{21}(\tau) = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x_2(t)\, x_1^*(t-\tau)\, dt \qquad ...(4.5.10)$$

- Note that the equations for the auto-correlation function and cross-correlation function are identical except for the fact that it has been defined for two different power signals $x_1(t)$ and $x_2(t)$ whereas the auto-correlation is defined for the same signal $x(t)$.

4.5.5 Properties of Cross-Correlation of CT Periodic Signals :

Property 1 :

- The pair of power signals $x_1(t)$ and $x_2(t)$ is said to be orthogonal over the entire interval if

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t)\, x_2^*(t)\, dt = 0 \qquad ...(4.5.11)$$

- That means if $R_{12}(0) = 0$. Thus if the cross-correlation $R_{12}(\tau)$ evaluated at $\tau = 0$ is zero, then the two signals are said to be orthogonal over the entire period.

Property 2 :

- The cross-correlation exhibits a conjugate symmetry. i.e.

$$\text{i.e.} \quad R_{12}(\tau) = R_{21}^*(\tau) \qquad ...(4.5.12)$$

- This property is identical to the corresponding property of cross-correlation of energy signals.
- The cross-correlation function is not cumulative. That means,

$$R_{12}(\tau) \neq R_{21}(\tau) \qquad ...(4.5.13)$$

Ex. 4.5.1 : Show that for an energy signal $x(t)$, the total area under its auto-correlation function $R(\tau)$ is equal to the value of its energy spectral density evaluated at $f = 0$.

Soln. : In this example we have to show that :

$$\int_{-\infty}^{\infty} R(\tau)\, d\tau = \psi(0)$$

1. According to the property of auto-correlation,

$$R(\tau) \overset{F}{\longleftrightarrow} \psi(f)$$

That means $\psi(f)$ is the fourier transform of $R(\tau)$ and $R(\tau)$ is the inverse fourier transform of $\psi(f)$.

$$\therefore \quad \psi(f) = \int_{-\infty}^{\infty} R(\tau) \cdot e^{-j2\pi f\tau}\, d\tau \qquad ...(1)$$

2. Now substitute $f = 0$ in Equation (1) to get,

$$\psi(0) = \int_{-\infty}^{\infty} R(\tau) \cdot e^{0}\, d\tau \qquad ...(2)$$

$$\therefore \quad \psi(f) = \int_{-\infty}^{\infty} R(\tau)\, d\tau \qquad ...(3)$$

The RHS of Equation (3) is area under the auto-correlation function.

Ex. 4.5.2 : Obtain the auto correlation function of sinc pulse A sinc (2Wt).

Soln. :

Step 1 : To obtain the fourier transform of A sinc (2Wt) :

- We know that,

$$A\,\text{rect}(t/T) \overset{F}{\longleftrightarrow} At\,\text{sinc}(Ft)$$

$$\therefore \quad \frac{A}{T}\,\text{rect}(t/T) \overset{F}{\longleftrightarrow} A\,\text{sinc}(fT)$$

- Now use the duality property to write,

$$A\,\text{sinc}(tT) \overset{F}{\longleftrightarrow} \frac{A}{T}\,\text{rect}(f/T)$$

Let $T = 2W$

$$\therefore \quad A\,\text{sinc}(2Wt) \overset{F}{\longleftrightarrow} \frac{A}{2W}\,\text{rect}(f/2W)$$

Thus $X(f) = \frac{A}{2W}\,\text{rect}[f/2W]$

Step 2 : To obtain the energy spectral density [$\psi(f)$] :

$$\psi(f) = |X(f)|^2 = \frac{A^2}{(2W)^2}\,\text{rect}^2[f/2W]$$

- But as the value of rect $[f/2W] = 1$, its square will also have the same value.

$$\therefore \quad \frac{A^2}{(2W)^2}\,\text{rect}^2[f/2W] = \frac{A^2}{(2W)^2}\,\text{rect}[f/2W]$$

$$\therefore \psi(f) = [A/2W]^2\,\text{rect}[f/2W]$$

Step 3 : To obtain the auto-correlation function R(τ) :

- As $R(\tau) \overset{F}{\longleftrightarrow} \psi(f)$, the auto-correlation function $R(\tau)$ can be obtained by taking the inverse fourier transform of $\psi(f)$.

$$\therefore \quad R(\tau) = \int_{-\infty}^{\infty} [A/2W]^2\,\text{rect}[f/2W] \cdot e^{j2\pi f\tau}\, df$$

But the $[A/2W]^2$ rect $(f/2W)$ is present only over the frequency range of – W to W and has an amplitude of $[A/2W]^2$.

$$\therefore R(\tau) = \int_{-W}^{W} [A/2W]^2 \cdot e^{+j2\pi f\tau}\, df = \left[\frac{A}{2W}\right]^2 \times \left[\frac{e^{j2\pi f\tau}}{j2\pi\tau}\right]_{-W}^{W}$$

$$= \left[\frac{A}{2W}\right]^2 \times \left[\frac{e^{j2\pi\tau W} - e^{-j2\pi\tau W}}{j2\pi\tau}\right]$$

$$= \frac{A^2}{(2W)^2 \times \pi\tau}\left[\frac{e^{j2\pi\tau W} - e^{-j2\pi\tau W}}{2j}\right] = \frac{A^2}{2W}\,\frac{\sin(2\pi W\tau)}{2\pi W\tau}$$

$$\therefore R(\tau) = \frac{A^2}{2W}\,\text{sinc}(2W\tau) \qquad \text{...Ans.}$$

Ex. 4.5.3 : Find and sketch the auto-correlation function of the signal $g(t) = e^{-at}$ and from it determine the energy spectral density function of $g(t)$.

Soln. :

In this case $g(t) = e^{-at}u(t)$ and $g(t-\tau) = e^{-a(t-\tau)}u(t-\tau)$.

The auto-correlation function is defined as the area under the product $g(t) \cdot g(t-\tau)$.

∴ Auto-correlation function

$$R(\tau) = \int_{-\infty}^{\infty} g(t) \cdot g(t-\tau)\, dt \qquad ...(1)$$

$$\therefore \quad R(\tau) = \int_{-\infty}^{\infty} e^{-at}u(t) \cdot e^{-a(t-\tau)} \cdot u(t-\tau)\, dt \qquad ...(2)$$

Due to the presence of the shifted unit step function $u(t-\tau)$, the integrations limits in Equation (2) will change to (τ, ∞).

$$\therefore \quad R(\tau) = \int_{\tau}^{\infty} e^{-at} e^{-a(t-\tau)}\, dt = \int_{\tau}^{\infty} e^{-at} \cdot e^{-at} \cdot e^{a\tau}\, dt$$

$$= e^{a\tau} \int_{\tau}^{\infty} e^{-2at}\, dt = \frac{e^{a\tau}}{-2a}\left[e^{-2at}\right]_{\tau}^{\infty}$$

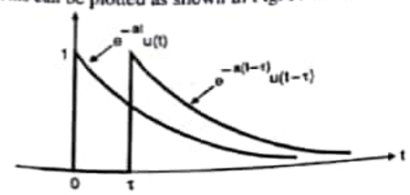
$$= \frac{e^{a\tau}}{-2a}\left[e^{-\infty} - e^{-2a\tau}\right] = \frac{e^{a\tau} \times -e^{-2a\tau}}{-2a}$$

$$\therefore \quad R(\tau) = \frac{e^{-a\tau}}{+2a} \qquad \text{...Ans.}$$

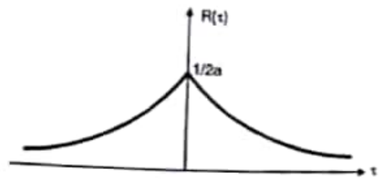
This is the desired expression for the auto-correlation function. But this expression is true for only positive values of τ. We can follow a similar procedure for negative values of τ. We know that if $g(t)$ is a real function then $R(\tau)$ is an even function of τ.

$$\therefore \quad R(\tau) = \frac{1}{2a}e^{-a|\tau|} \qquad ...(3)$$

This can be plotted as shown in Fig. P. 4.5.3(b).



(a)



(b)

(H-650) Fig. P. 4.5.3

Energy spectral density :

The energy spectral density $\psi(f)$ is the fourier transform of auto-correlation function i.e.

$$\psi(f) = F.T.[R(\tau)]$$

$$= \int_{-\infty}^{\infty} R(\tau) \cdot e^{-j2\pi f\tau}\, d\tau = \int_{-\infty}^{\infty} \frac{e^{-a|\tau|}}{2a} \cdot e^{-j2\pi f\tau}\, d\tau$$

The expression for $\psi(f)$ can be written as,

$$\psi(f) = \frac{2a}{2a\,[a^2 + (2\pi f)^2]} = \frac{1}{a^2 + (2\pi f)^2} \qquad \text{...Ans.}$$

Ex. 4.5.4 : Find the cross-correlation between the signals :

$x(t) = 4e^{-5t}u(t) \qquad y(t) = 10e^{-10t}u(t)$

Soln. :

The cross correlation between x(t) and y(t) is given by,

$$R_{12}(\tau) = \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, dt$$

Here $y(t) = 10e^{-10t}u(t)$

$$\therefore \quad y(t-\tau) = 10e^{-10(t-\tau)}u(t-\tau) = 10e^{-10t} \cdot e^{10\tau}u(t-\tau)$$

$$\therefore \quad R_{12}(\tau) = \int_{-\infty}^{\infty} 4e^{-5t}u(t) \cdot 10e^{-10t} \cdot e^{10\tau} \cdot u(t-\tau)\, dt$$

$$= 40e^{10\tau} \int_{-\infty}^{\infty} e^{-15t}u(t) \cdot u(t-\tau)\, dt$$

Due to the term $u(t) \cdot u(t-\tau)$; the range of integration becomes τ to ∞.

$$\therefore \quad R_{12}(\tau) = 40e^{10\tau} \int_{\tau}^{\infty} e^{-15t}\, dt = 40e^{10\tau}\left[\frac{e^{-15t}}{-15}\right]_{\tau}^{\infty}$$

$$= -2.67e^{10\tau}\left[e^{-15t}\right]_{\tau}^{\infty} = -2.67 \cdot e^{10\tau}\left[e^{-\infty} - e^{-15\tau}\right]$$

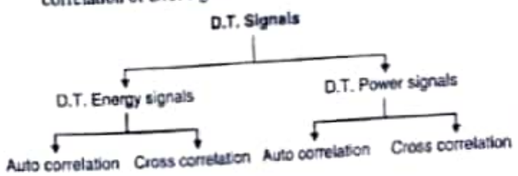
$$= -2.67e^{10\tau}\left[0 - e^{-15\tau}\right] = 2.67e^{-5\tau}$$

4.6 Correlation of Discrete Time Signals :

- The concept of correlation between the discrete time signals is the logical extension of the correlation of continuous time signals, discussed earlier.
- For discrete time (DT) signals, there are two types of correlations namely :
  1. Autocorrelation 2. Cross correlation
- Each of the two types of correlations are applicable to the two types of D.T. signals namely :

1. D.T. energy signals 2. D.T. power signals.

- Fig. 4.6.1 gives you the complete idea about the scope of correlation of D.T. signals.



(H-652) Fig. 4.6.1

4.6.1 Autocorrelation of D.T. Signals :

The autocorrelation is the measure of similarity between a sequence x (n) and its shifted version say x (n – k).

4.6.2 Autocorrelation of D.T. Energy Signals :

- The autocorrelation function between the real values D.T. energy signal x (n) and its time shifted version is given by,

$$R(k) = \sum_{n=-\infty}^{\infty} x(n) \cdot x(n-k) \quad ...(4.6.1)$$

or

$$R(k) = \sum_{n=-\infty}^{\infty} x(n) \cdot x(n+k) \quad ...(4.6.2)$$

Where n is the variable and k is the constant representing the shift.

- Note that the definition of autocorrelation of D.T. energy signals is same as that of the continuous time energy signal with "t" replaced by "n", "τ" replaced by "k" and the "integration sign" is replaced by the summation sign.
- Making the same changes we can state the properties of autocorrelation function of D.T. energy signals as follows.

4.6.3 Analogy between Correlation and Convolution :

- The convolution of two D. T. signals $x_1(n)$ and $x_2(n)$ is defined as follows :

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k).$$

- Now compare this expression with the definition of auto correlation, stated in Equation (4.6.1). You will notice the similarity between the two processes.

4.6.4 Properties of Autocorrelation Function of D.T. Energy Signals :

Property 1 : The autocorrelation function of a real valued D.T. energy signal x (n) is a real valued even function. That means

$$R(k) = R(-k) \quad ...(4.6.3)$$

Property 2 : The value of autocorrelation function of a D.T. energy signal x (n) at the origin is equal to the energy of the signal.

i.e. $R(0) = E$

Property 3 : The maximum value of autocorrelation function of a D.T. energy signal x (n) occurs at the origin i.e. at k = 0. That means,

$$|R(k)| \le R(0) \quad \text{... for all } k \quad ...(4.6.4)$$

Property 4 :

For a D.T. energy signal x (n), the autocorrelation function and energy spectral density form a Fourier transform pair.

$$\text{i.e. } R(k) \overset{F}{\leftrightarrow} \psi(f) \quad ...(4.6.5)$$

4.6.5 Autocorrelation of D.T. Power Signals :

Definition :

- The autocorrelation function of a real valued discrete time power signal x (n) is defined as follows :

$$R(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, x(n-k) \quad ...(4.6.6)$$

or

$$R(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)\, x(n+k) \quad ...(4.6.7)$$

4.6.6 Properties of Autocorrelation Function of a D.T. Power Signal :

Property 1 : The autocorrelation function of a real valued D.T. power signal x (n) is an even function. That means,

$$R(k) = R(-k) \quad ...(4.6.8)$$

Property 2 :

The value of autocorrelation function of a D.T. power signal at the origin i.e. at k = 0 is equal to the average power of the signal.

i.e. $R(0) = P$

Property 3 :

The maximum value of autocorrelation function of a D.T. power signal x (n) occurs at the origin.

That means, $|R(k)| \le R(0)$ ... for all k.

Property 4 :

For a D.T. power signal x (n), the autocorrelation function and power spectral density form a Fourier transform pair. That means

$$R(k) \overset{F}{\leftrightarrow} S(f)$$

4.6.7 Cross Correlation of D.T. Signals :

The cross correlation of two D.T. signals is a measure of similarity between them. We will define the cross correlation for energy and power D.T. signals.

4.6.8 Cross Correlation of D.T. Energy Signals :

Definition :

- Let $x_1(n)$ and $x_2(n)$ denote a pair of real valued D.T. energy signals. Then the cross correlation function of such a pair is defined as follows :

$$R_{12}(k) = \sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2(n-k) \quad ...(4.6.9)$$

- The second cross correlation function of $x_1(n)$ and $x_2(n)$ is defined as,

$$R_{21}(k) = \sum_{n=-\infty}^{\infty} x_1(n-k)\, x_2(n) \quad ...(4.6.10)$$

4.6.9 Properties of Cross Correlation of D.T. Energy Signals :

Property 1 :

The cross correlation of D.T. energy signals is generally not commutative. That means $R_{12}(k) \ne R_{21}(k)$. In fact,

$$R_{12}(k) = R_{21}(-k)$$

4.6.10 Cross Correlation of D.T. Power Signals :

- If $x_1(n)$ and $x_2(n)$ represent two different D.T. power signals, then the cross correlation between them is defined as,

$$R_{12}(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_1(n) \cdot x_2(n-k) \quad ...(4.6.11)$$

- The second cross correlation is defined as,

$$R_{21}(k) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x_1(n+k)\, x_2(n) \quad ...(4.6.12)$$

- The cross correlation function is generally not cumulative.
- The signals $x_1(n)$ and $x_2(n)$ are said to be orthogonal if the cross correlation between them is zero. i.e. for orthogonal signals.

$R_{12}(k) = 0$ ..... Condition of orthogonality

Positively correlated, negatively correlated and uncorrelated signals :

- If the correlation function has a nonzero value only for negative value of (for continuous time signals) or k (for discrete time signals), then the signals are said to be negatively correlated.
- If the correlation function has a nonzero value only for the positive values of τ (for C.T. signals) or k (for D.T. signals), then the signals are said to be positively correlated.
- If the correlation function is zero for all the values of τ or k, then the signals are said to be uncorrelated signals.

Ex. 4.6.1 : Determine the autocorrelation for the sequence $x(n) = [0, \underset{\uparrow}{1}, 2, 3]$.

Soln. :

- The sequence x (n) is a D.T. energy signal. The autocorrelation is defined as

$$R(k) = \sum_{n=-\infty}^{\infty} x(n) \cdot x(n-k)$$

For k = –4

$$R(-4) = \sum_{n=-\infty}^{\infty} x(n) \cdot x(n+4)$$

x (n + 4) is the advanced version of x (n) and the advance is by 4 places. The sequences x (n) and x (n + 4) are shown in Fig. P. 4.6.1(a).

- Since x (n) and x (n + 4) do not overlap as shown in Fig. P. 4.6.1(a).
- Therefore the product of x (n) and x (n + 4) is zero.

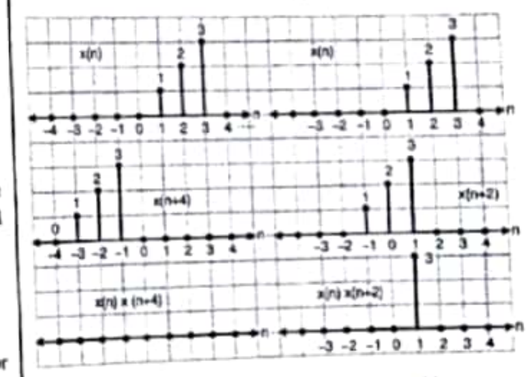
$$\therefore R(-4) = \sum_{n=-\infty}^{\infty} x(n)\, x(n+4) = \sum_{n=-\infty}^{\infty} 0 = 0$$

For k = – 3

$$R(-3) = \sum_{n=-\infty}^{\infty} x(n)\, x(n+3)$$

But the product sequence x (n) x (n + 3) = 0 for all n

$$\therefore R(-3) = \sum_{n=-\infty}^{\infty} 0 = 0$$



(H-653) Fig. P. 4.6.1

For k = – 2, Refer Fig. P. 4.6.1(b).

$$R(-2) = \sum_{n=-\infty}^{\infty} x(n)\, x(n+2)$$

$$= 0 + 0 + 0 + 3 + 0 + 0 + \ldots$$

$$\therefore R(-2) = 3$$

For k = – 1

$$R(-1) = \sum_{n=-\infty}^{\infty} x(n) \cdot x(n+1)$$

$$= x(0)\,x(1) + x(1)\,x(2) + x(2)\,x(3) + x(3)\,x(4) + \ldots$$

$$= (0 \times 1) + (1 \times 2) + (2 \times 3) + (3 \times 0) + \ldots$$

$$= 0 + 2 + 6 = 8$$

For k = 0

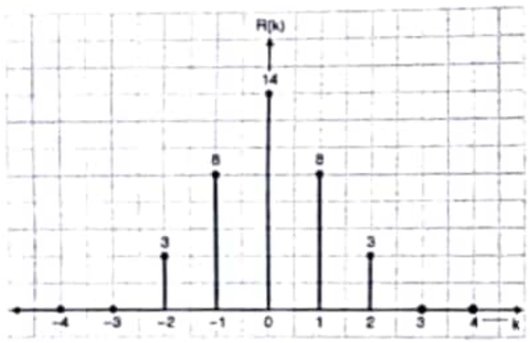
$$R(0) = \sum_{n=-\infty}^{\infty} x(n) \cdot x(n)$$
$$= x(0)\,x(0) + x(1)\,x(1) + x(2)\,x(2) + x(3)\,x(3) + \ldots$$
$$= (0 \times 0) + (1 \times 1) + (2 \times 2) + (3 \times 3)$$
$$= 0 + 1 + 4 + 9 = 14$$

For k = 1

$$R(1) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-1)$$
$$= x(0)\,x(-1) + x(1)\,x(0) + x(2)\,x(1) + x(3)\,x(2) + x(4)\,x(3) + \ldots$$
$$= 0 + (1 \times 0) + (2 \times 1) + (3 \times 2) + (0 \times 3) = 2 + 6 = 8$$

For k = 2

$$R(2) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-2)$$
$$= x(0)\,x(-2) + x(1)\,x(-1) + x(2)\,x(0) + x(3)\,x(1) + \ldots$$
$$R(2) = (0 \times 0) + (1 \times 0) + (2 \times 0) + (3 \times 1) = 3$$



(H-654) **Fig. P. 4.6.1(c) : Autocorrelation function**

For k = 3 $R(3) = \sum_{n=-\infty}^{\infty} x(n)\,x(n-3)$

$$= x(0)\,x(-3) + x(1)\,x(-2) + x(2)\,x(-1) + x(3)\,x(0) + \ldots$$
$$= (0 \times 0) + (1 \times 0) + (2 \times 0) + (3 \times 0) = 0$$

For all values of $k \geq 4$ the value of R (k) will be zero. The autocorrelation function is plotted as shown in Fig. P. 4.6.1(c).

Note : Note that the autocorrelation function R (k) is a symmetrical function. i.e. R (k) = R (– k).

Ex. 4.6.2 : Find the autocorrelation of the following signals :

1. $x_1[n] = [4\ 3, 2, 1]$ 2. $x_2[n] = u[n]$

Dec. 06. 8 Marks

Soln. :

1. **$x_1[n] = [4, 3, 2, 1]$:**

$$\text{Auto-correlation } R(k) = \sum_{n=-\infty}^{\infty} x_1(n)\,x_1(n-k)$$

Note that for $k = \pm 4$ there won't be any overlap between $x_1(n)$ and $x_1(n-4)$. So the product will be zero.

Hence we will find R (k) in the range $-3 \leq k \leq +3$

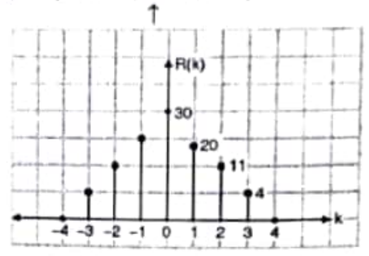
$$R(-3) = \sum_{n=-\infty}^{\infty} x_1(n)\,x_1(n+3)$$
$$= x_1(-1)\,x_1(2) + x_1(0)\,x_1(3) + x_1(1)\,x_1(4) + x_1(2)\,x_1(5)$$
$$\therefore\ R(-3) = (4 \times 1) + (3 \times 0) + (2 \times 0) + (1 \times 0) = 4$$

Similarly,

$$R(-2) = \sum x_1(n)\,x_1(n+2)$$
$$= x_1(-1)\,x_1(1) + x_1(0)\,x_1(2) + x_1(1)\,x_2(3) + x_1(2)\,x_1(4)$$
$$= (4 \times 2) + (3 \times 1) + (2 \times 0) + (1 \times 0) = 11$$
$$R(-1) = \sum x_1(n)\,x_1(n+1)$$
$$= x_1(-1)\,x_1(0) + x_1(0)\,x_1(1) + x_1(1)\,x_1(2) + x_1(2)\,x_1(3)$$
$$= (4 \times 3) + (3 \times 2) + (2 \times 1) + (1 \times 0) = 20$$
$$R(0) = \sum x_1(n)\,x_1(n)$$
$$= (4 \times 4) + (3 \times 3) + (2 \times 2) + (1 \times 1) = 30$$
$$R(1) = \sum x_1(n)\,x_1(n-1)$$
$$= x_1(-1)\,x_1(-2) + x_1(0)\,x_1(-1) + x_1(1)\,x_1(0) + x_1(2)\,x_1(1)$$
$$= (4 \times 0) + (3 \times 4) + (2 \times 3) + (1 \times 2) = 20$$
$$R(2) = (4 \times 0) + (3 \times 0) + (2 \times 4) + (1 \times 3) = 11$$
$$R(3) = (4 \times 0) + (3 \times 0) + (2 \times 0) + (1 \times 4) = 4$$

Fig. P. 4.6.2 shows R (k)

R (k) = {4, 11, 20, 30, 20, 11,4}



(H-655) **Fig. P. 4.6.2**

2. **$x_2[n] = u[n]$:**

$$\therefore\ \text{Auto-correlation} = R(k) = \sum_{n=-\infty}^{\infty} x_2(n)\,x_2(n-k)$$
$$\therefore\ R(k) = \sum_{n=-\infty}^{\infty} u(n)\,u(n-k)$$

But $u(n) \neq 0$ for $n \geq 0$. so limits of summation will extend from 0 to ∞.

For $k = -\infty$ to ∞, even though the overlapping changes, there won't be any change in R (k) because u (n) and u (n – k) both are

infinite sequences. so R (k) will be constant but will not have a finite value.

Ex. 4.6.3 : Obtain the cross correlation of the following sequences.

$x_1(n) = [\underset{\uparrow}{2}, 3, 4]$ and $x_2(n) = [\underset{\uparrow}{1}, 2, 3]$

using definition of cross correlation.

Soln. :

There are two methods of solving this example

Method I : Direct computation method :

- The cross correlation is defined as :

$$R_{12}(k) = \sum_{n=-\infty}^{\infty} x_1(n)\,x_2(n-k)$$

- Now let us obtain the values of $R_{12}(k)$ for different values of k. Since $x_1(n)$ exists from n = 0 to n = 2, the product term $x_1(n)\,x_2(n-k)$ will be non-zero only for these values of n.

$$\therefore\ R_{12}(k) = \sum_{n=0}^{2} x_1(n)\,x_2(n-k)$$

1. **$R_{12}(k)$ for k = – 3 :**

Let us arbitrarily start at k = – 3

$$\therefore\ R_{12}(-3) = \sum_{n=0}^{2} x_1(n)\,x_2(n+3)$$
$$= x_1(0)\,x_2(3) + x_1(1)\,x_2(4) + x_1(2)\,x_2(5)$$
$$= (2 \times 0) + (3 \times 0) + (4 \times 0) = 0$$

This shows that for $k \leq -3$ $R_{12}(k) = 0$

2. **$R_{12}(k)$ for k = – 2 :**

$$R_{12}(-2) = \sum_{n=0}^{2} x_1(n)\,x_2(n+2)$$
$$= x_1(0)\,x_2(2) + x_1(1)\,x_2(3) + x_1(2)\,x_2(4)$$
$$= (2 \times 3) + (3 \times 0) + (4 \times 0) = 6$$

3. **$R_{12}(k)$ for k = – 1 :**

$$R_{12}(-1) = \sum_{n=0}^{2} x_1(n)\,x_2(n+1)$$
$$= x_1(0)\,x_2(1) + x_1(1)\,x_2(2) + x_1(2)\,x_2(3)$$
$$= (2 \times 2) + (3 \times 3) + (4 \times 0) = 13$$

4. **$R_{12}(k)$ for k = 0 :**

$$R_{12}(0) = \sum_{n=0}^{2} x_1(n)\,x_2(n)$$
$$= x_1(0)\,x_2(0) + x_1(1)\,x_2(1) + x_1(2)\,x_2(2)$$
$$= (2 \times 1) + (3 \times 2) + (4 \times 3) = 20$$

5. **$R_{12}(k)$ for k = 1 :**

$$R_{12}(1) = \sum_{n=0}^{2} x_1(n)\,x_2(n-1)$$
$$= x_1(0)\,x_2(-1) + x_1(1)\,x_2(0) + x_1(2)\,x_2(1)$$
$$= (2 \times 0) + (3 \times 1) + (4 \times 2) = 11$$

6. **$R_{12}(k)$ for k = 2 :**

$$R_{12}(2) = \sum_{n=0}^{2} x_1(n)\,x_2(n-2)$$
$$= x_1(0)\,x_2(-2) + x_1(1)\,x_2(-1) + x_1(2)\,x_2(0)$$
$$= (2 \times 0) + (3 \times 0) + (4 \times 1) = 4$$

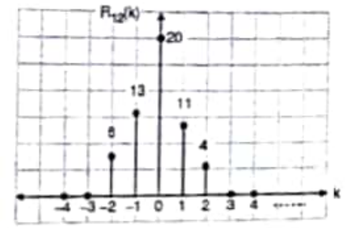
7. **$R_{12}(k)$ for k = 3 :**

$$R_{12}(3) = \sum_{n=0}^{2} x_1(n)\,x_2(n-3)$$
$$= x_1(0)\,x_2(-3) + x_1(1)\,x_2(-2) + x_1(2)\,x_2(-1)$$
$$= (2 \times 0) + (3 \times 0) + (4 \times 0) = 0$$

The value of $R_{12}(k) = 0$ for $k \geq 3$

Plot $R_{12}(k)$:

Fig. P. 4.6.3 shows the plot of $R_{12}(k)$ with respect to k.



(H-456) **Fig. P. 4.6.3 : Plot of $R_{12}(k)$**

$R_{12}(k)$ in the sequence form :

$R_{12}(k)$ in the sequence form is as follows :

$$R_{12}(k) = [6, 13, \underset{\uparrow}{20}, 11, 4]$$

Ex. 4.6.4 : Determine the cross-correlation between two sequences :

$x_1(n) = [\underset{\uparrow}{1}, 2, 3, 4]$ and $x_2(n) = \{\underset{\uparrow}{3}, 2, 1, 0\}$

Dec. 12. 8 Marks

Soln. :

We can obtain the cross correlation using the definition of cross correlation. We can also solve it using properties. One of the properties of cross correlation is; cross correlation is equivalent to convolution of $x_1(n)$ and folded version of $x_2(n)$.

That means, $R_{12}(K) = x_1(n) * x_2(-n)$

Here $x_2(-n) = \{0, 1, 2, \underset{\uparrow}{3}\}$

It is performed using tabular method as follows :

$$\therefore\ R_{12}(K) = \{0, 1, 4, \underset{\uparrow}{10}, 16, 17, 12\}$$

	1	2	3	4
0	0	0	0	0
1	1	2	3	4
2	2	4	6	8
3	3	6	9	12

(H-1235)

Ex. 4.6.5 : Find the auto-correlation of the DT signal :

$X[n] = [1, 2, 3, 0, -1]$.

Soln. :

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 0, -1\right\}$$

$$\therefore \quad x(-n) = \left\{-1, 0, 3, 2, \underset{\uparrow}{1}\right\}$$

Now $\quad R(\tau) = x(n) * x(-n)$.

This convolution is done using tabular method as follows :

$\therefore \quad R(\tau) = [1, 4, 10, 12, \underset{\uparrow}{7}, -4, -6, 0, 1]$.

	1	2	3	0	-1
1	1	2	3	0	-1
2	2	4	6	0	-2
3	3	6	9	0	-3
0	0	0	0	0	0
-1	-1	-2	-3	0	1

(H-1234)

4.7 Autocorrelation of Sum of Signals :

- The autocorrelation of sum of signals is equal to the sum of autocorrelation of the individual signal. That means

$$\text{If } x_3(t) = x_1(t) + x_2(t)$$
$$\text{Then } R_3(\tau) = R_1(\tau) + R_2(\tau)$$

- This concept will be clear after solving the following example.

Ex. 4.7.1 : Obtain the autocorrelation, power and power spectral density of the following signal.
$x(t) = 3\cos t + 4\cos 3t$. **May 14, 10 Marks**

Soln. :

Let $x_1(t) = 3\cos t$ and $x_2(t) = 4\cos 3t$. So their individual autocorrelation are denoted by $R_1(\tau)$ and $R_2(\tau)$.

Step 1 : Calculate $R_1(\tau)$:

$$R_1(\tau) = \text{Autocorrelation of } x_1(t) = \frac{1}{T_1}\int_{-T_1/2}^{T_1/2} 9\cos t \cos(t-\tau)\,dt$$

- But $f_1 = \frac{1}{2\pi}$ $\quad \therefore T_1 = 2\pi$

$$\therefore \quad R_1(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi} 9\cos t\cos(t-\tau)\,dt$$

- We know that $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$\therefore \quad 9\cos(t-\tau)\cos t = \frac{9}{2}[\cos(2t-\tau) + \cos(-\tau)]$$

$$\therefore \quad R_1(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{9}{2}[\cos(2t-\tau) + \cos(-z)]\,dt$$

$$\therefore \quad R_1(\tau) = \frac{9}{4\pi}\int_{-\pi}^{\pi}\cos(2t-\tau)\,dt + \frac{9}{4\pi}\int_{-\pi}^{\pi}\cos(-\tau)\,dt$$

$$= \frac{9}{4\pi}\left[\frac{\sin(2t-\tau)}{2}\right]_{-\pi}^{\pi} + \frac{9}{4\pi}\cos(-\tau)\,[t]_{-\pi}^{\pi}$$

$$= \frac{9}{8\pi}[\sin(2\pi-\tau) - \sin(-2\pi-\tau)] + \frac{9}{2}\cos(\tau)$$

$$= \frac{9}{8\pi}[\sin(2\pi-\tau) + \sin(+2\pi+\tau)] + \frac{9}{2}\cos(\tau)$$

$$\therefore R_1(\tau) = \frac{9}{8\pi}$$

$$[\sin 2\pi\cos\tau - \cos 2\pi\sin\tau + \sin 2\pi\cos\tau + \cos 2\pi\sin\tau] + \frac{9}{2}\cos\tau$$

$$\therefore \quad R_1(\tau) = \frac{9}{8\pi}\times 0 + \frac{9}{2}\cos\tau = \frac{9}{2}\cos\tau \qquad \text{...(1)}$$

Step 2 : Calculate $R_2(\tau)$:

$$R_2(\tau) = \frac{1}{12}\int_{-T_2/2}^{T_2/2} 16\cos 3t\cos 3(t-\tau)\,dt$$

But $\quad x_2(t) = 4\cos 3t \quad$ so $\quad f_2 = 3/2\pi$

$$\therefore \quad T_2 = 1/f_2 = 2\pi/3$$

$$\therefore \quad R_2(\tau) = \frac{3}{2\pi}\int_{-\frac{2\pi}{2\times 3}}^{2\pi/2\times 3} 16\cos 3t\cos 3(t-\tau)\,dt$$

$$\therefore \quad R_2(\tau) = \frac{24}{\pi}\int_{-\pi/3}^{\pi/3}\cos 3t\cos 3(t-\tau)\,dt$$

But $\quad 2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$$\therefore \quad 2\cos 3t\cos 3(t-\tau) = \cos(6t-3\tau) + \cos(3\tau)$$

$$\therefore \quad R_2(\tau) = \frac{12}{\pi}\int_{-\pi/3}^{\pi/3}[\cos(6t-3\tau) + \cos(3\tau)]\,dt$$

$$= \frac{12}{\pi}\int_{-\pi/3}^{\pi/3}\cos(6t-3\tau)\,dt + \frac{12}{\pi}\int_{-\pi/3}^{\pi/3}\cos(3\tau)\,dt$$

$$= \frac{12}{\pi}\left[\frac{\sin(6t-3\tau)}{6}\right]_{-\pi/3}^{\pi/3} + \frac{12}{\pi}\cos(3\tau)\,[t]_{-\pi/3}^{\pi/3}$$

$$= \frac{2}{\pi}[\sin(2\pi-3\tau) - \sin(-2\pi-3\tau)]$$
$$+ \frac{12}{\pi}\cos(3\tau)\left[\frac{\pi}{3} + \frac{\pi}{3}\right]$$

$$\therefore \quad R_2(\tau) = \frac{2}{\pi}[0] + \frac{12}{\pi}\times\frac{2\pi}{3}\cos(3\tau) = 8\cos(3\tau) \qquad \text{...(2)}$$

Step 3 : Calculate $R(\tau)$:

$$R(\tau) = R_1(\tau) + R_2(\tau) = \frac{9}{2}\cos(\tau) + 8\cos(3\tau) \quad \text{...Ans.}$$

Step 4 : Calculate power :

$$\text{Power } P = R(\tau)\big|_{\tau=0} = \frac{9}{2}\cos 0 + 8\cos 0 = 12.5 \quad \text{...Ans.}$$

Step 5 : Calculate PSD :

$$\text{PSD } S(f) = F[R(\tau)] = F\left[\frac{9}{2}\cos(\tau) + 8\cos(3\tau)\right]$$

But $\quad F[\cos\omega_0 t] = \frac{1}{2}[\delta(f+f_0) + \delta(f-f_0)]$

For $\cos\tau$, frequency $= 1/2\pi$ and for $\cos(3\tau)$, frequency $= 3/2\pi$

$$\therefore S(f) = \frac{9}{2}\left[\delta\left(f+\frac{1}{2\pi}\right) + \delta\left(f-\frac{1}{2\pi}\right)\right] + 8\left[\delta\left(f+\frac{3}{2\pi}\right) + \delta\left(f-\frac{3}{2\pi}\right)\right]$$

Ex. 4.7.2 : Obtain the cross correlation of $\delta(t)$ and $x(t)$.

Soln. :

- Let $x_1(t) = x(t)$ and $x_2(t) = \delta(t)$

$$\therefore \quad \text{Cross correlation } R_{12}(\tau) = \int_{-\infty}^{\infty} x_1(t)\cdot x_2(t-\tau)\,dt$$

$$\therefore \quad R_{12}(\tau) = \int_{-\infty}^{\infty} x(t)\cdot\delta(t-\tau)\,dt$$

By using sifting property we get,

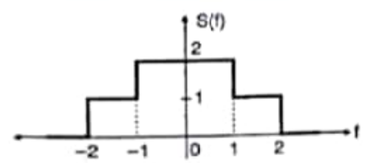
$$R_{12}(\tau) = x(\tau)$$

Cross correlation of sequences :

- For the cross correlation of sequences, we have to use the principle of cross correlation of discrete time signals.

4.8 Additional Solved Examples :

Ex. 4.8.1 : Fig. P. 4.8.1 shows the power spectral density of a power signal $x(t)$. Calculate the average power of the signal. **Dec. 07, 6 Marks**



(H-641) **Fig. P. 4.8.1**

Soln. :

Total average power of a signal is the area of curve under $S(f)$.

$$\therefore \quad P = \int_{-\infty}^{\infty} S(f)\,df = \int_{-2}^{-1} 1\,df + \int_{-1}^{1} 2\,df + \int_{1}^{2} 1\,df$$

$$= [1(-1+2) + 2(1+1) + 1(2-1)]$$

$$\therefore \quad P = 6 \quad \text{...Ans.}$$

Ex. 4.8.2 : Fig. P. 4.8.2 shows power spectral density of power signal $x(t)$. Calculate the average power of the signal.

(H-657) **Fig. P. 4.8.2**

Soln. :

Total average power is the area of the curve under $S(f)$.

$$\therefore \quad P = \int_{-\infty}^{\infty} S(f)\,df = \int_{-1}^{+1} 2\,d(f)$$

$$\therefore \quad P = 2[f]_{-1}^{+1} = 2[1-(-1)] = 4 \text{ Watts} \quad \text{...Ans.}$$

Review Questions

Q. 1 What is spectral density ?

Q. 2 Define energy spectral density (ESD).

Q. 3 State true or false and justify : The ESD can be a positive or negative valued function.

Q. 4 The total area under the curve of ESD of an energy signal represents : (a) Power (b) Signal energy (c) Bandwidth

Q. 5 State the expression for ESD at the output of a system in terms of system response and ESD at the system input.

Q. 6 What is the relation between ESD and auto correlation ?

Q. 7 Define PSD.

Q. 8 What is the significance of PSD ?

Q. 9 Area under the curve of PSD of a signal represents :
(a) Total signal energy
(b) Total signal power
(c) Power/unit BW.

Q. 10 PSD is a ..... function of frequency. (Continuous/Discrete)

Q. 11 State relation between auto correlation and PSD.

Q. 12 What is correlation ?

Q. 13 Define autocorrelation of energy signals.

Q. 14 Define cross correlation of energy signals.

Q. 15 Define auto and cross correlation of power signals.

□□□

CHAPTER 5

Module 3

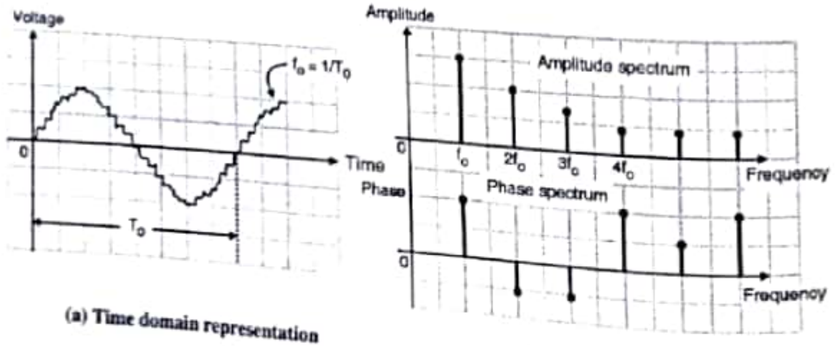
Review of Fourier Series

Syllabus :

Trignometric and exponential Fourier series representation of signals, Gibbs phenomenon, Discrete time Fourier series, Properties, Analogy between Continuous Time Fourier Series (CTFS) and Discrete Time Fourier Series (DTFS).

5.1 Frequency Domain Representation :

- All the signals till now were drawn with respect to time. That means time "t" was considered as a variable. The representation of signal with respect to time is called as its **time domain representation.**
- The time domain representation of the signal is not sufficient for its analysis. Hence we have to use the frequency domain representation of the signal for the sake of analysis.
- In the frequency domain representation, the variable plotted on the X-axis is frequency "f" rather than "t".
- The signal represented in the frequency domain is called as the **line spectrum.** The line spectrum consists of two graphs namely :
  1. **Amplitude spectrum :** A graph of amplitude on Y-axis versus frequency on X-axis.
  2. **Phase spectrum :** A graph of phase on Y-axis versus frequency on X-axis.
- The signal x(t) and its line spectrum are shown in Fig. 5.1.1.
- The graph of instantaneous signal voltage plotted on Y-axis versus time plotted on X-axis is called as the time domain representation.
- It is as shown in Fig. 5.1.1(a). The time domain representation gives us the following information :
  1. Shape of the signal
  2. Its frequency
  3. Type of the signal (periodic or nonperiodic).
  4. One cycle period.
- But we do not know anything about what frequency components are present and in what proportion they have been mixed in order to obtain the particular shape of the signal.
- All this information can be obtained from the line spectrum of a signal.
- Line spectrum [Fig. 5.1.1(b)] is the representation of the same signal x(t), now in the frequency domain.
- It can be obtained by using either Fourier series or Fourier transform. It consists of the amplitude and phase spectrums of the signal.
- The line spectrum indicates the amplitude and phase of various frequency components present in the given signal. The line spectrum enables us to analyze and synthesize a signal.



(a) Time domain representation (b) Line spectrum

(II-450) Fig. 5.1.1

5.1.1 How to Plot Line Spectrum ?

- The line spectrum is useful in knowing about the existence and amplitudes/phases of various frequency components present in a given signal x(t).
- The important conventions about the line spectra are as follows :
  1. In all the spectral drawings, the independent variable plotted on the x-axis is frequency "f" in Hz and not ω.
  2. Phase angle is always measured with respect to the cosine waves. That means it is measured with respect to the positive real axis of the phasor diagram. Hence it is necessary to convert sinewaves to cosines using the following standard identity :

$$\sin \omega t = \cos(\omega t - 90°) \quad ...(5.1.1)$$

  3. The amplitude is always considered as a "positive" quantity. So if negative signs appear for a phase angle, they should be absorbed in the phase change to keep amplitude positive. This is explained in the following expression,

$$-A \cos \omega t = A \cos(\omega t \pm 180°) \quad ...(5.1.2)$$

- The additional phase change of ± 180° converts the negative amplitude "– A" to positive amplitude "+ A". We can choose a phase shift of either + 180° or – 180°, as the effect is going to be the same.

Ex. 5.1.1 : Sketch the line spectrum of the following signal :

$$m(t) = 3 - 5\cos(40\pi t - 30°) + 4\sin 120\pi t$$

Soln. :

In the given signal, the first term represents a dc term which has a zero frequency. The other two terms can be written as follows :

1. First term : $3 = 3\cos 2\pi 0 t$ as $f = 0$
2. Second term : $-5\cos(40\pi t - 30°) = 5\cos(2\pi 20 t - 30° + 180°) = 5\cos(2\pi 20 t + 150°)$

   Thus the negative amplitude has been made positive by adding a phase angle of 180°.
3. Third term : $4\sin 120\pi t = 4\sin 2\pi 60 t = 4\cos(2\pi 60 t - 90°)$.

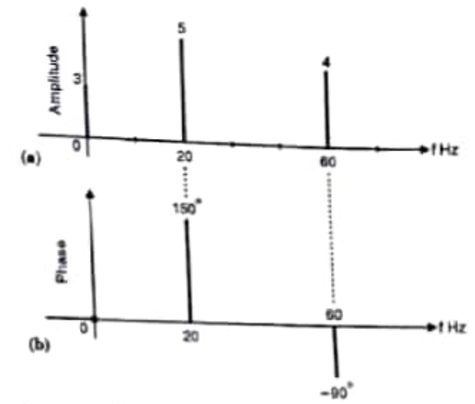
   Thus the sine term has been converted to the cosine term by adding a phase shift of – 90°.

The amplitudes, frequencies and phase angles of the three terms are listed in the Table P. 5.1.1.

Table P. 5.1.1

Sr. No.	Term	Amplitude	Frequency	Phase
1.	$3\cos 2\pi 0 t$	3V	0 Hz	0°
2.	$5\cos(2\pi 20 t + 150°)$	5V	20 Hz	150°
3.	$4\cos(2\pi 60 t - 90°)$	4V	60 Hz	– 90°

With the help of Table P. 5.1.1 we can plot the line spectra as shown in Fig. P. 5.1.1.



(a) Amplitude spectrum (b) Phase spectrum

(II-451) Fig. P. 5.1.1

5.1.2 Double Sided Line Spectrum :

- The line spectra drawn in section 5.1 are called as "one sided" or "positive frequency" line spectra.
- But the other spectral representation which involves negative frequencies as well proves to be more valuable in may communication applications.
- It is called as the double sided line spectrum. The double sided line spectra can be obtained by modifying the single sided spectrum as shown in Fig. 5.1.2. Both the line spectra shown in this figure correspond to a signal $x(t) = A\cos(2\pi f_0 t)$.

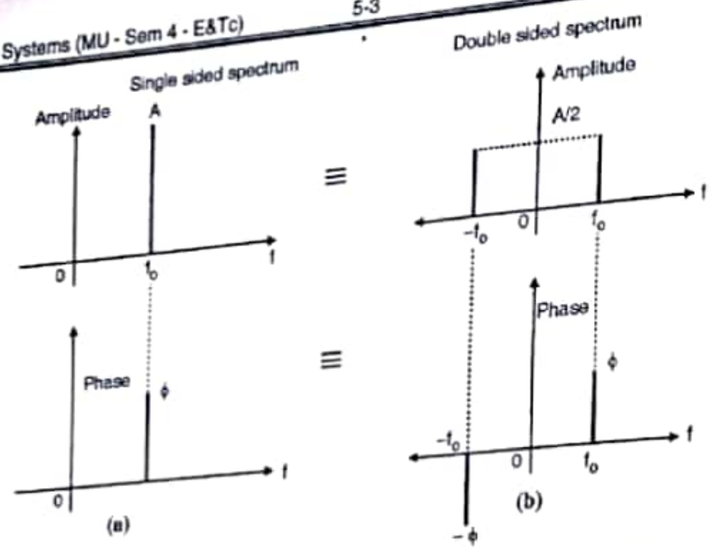
Conclusions :

The conclusions drawn from Fig. 5.1.2 are as follows :

1. Looking at Fig. 5.1.2(a) i.e. the amplitude spectrum, we conclude that in the single sided spectrum there is only one frequency component present at $f = f_0$ with an amplitude A.

   Whereas in double sided line spectra, two frequency components f_0 and $-f_0$ are present with amplitude (A/2) but no change in polarity.
2. The single sided phase spectrum contains only one component at f_0 with phase ϕ. But double sided spectrum contains two components at f_0 and $-f_0$ with phases equal to ϕ and $-\phi$ respectively. Thus phase shift remains unchanged but they have opposite signs.
3. From Fig. 5.1.2(a) it is clear that the double sided amplitude spectrum has an even symmetry and Fig. 5.1.2(b) shows that the double sided phase spectrum has an odd symmetry.

- The double sided line spectrum representation is very useful in mathematical analysis of a signal. The negative frequency components shown in the double sided spectrums are not practically present. They are fictitious frequency components.



(H-452) **Fig. 5.1.2 : Equivalence between single and double sided spectrums**

5.2 CT Fourier Series :

Need of CT Fourier series :

- Sinewaves and cosinewaves are the basic building functions for any periodic signal.
- That means any periodic signal basically consists of sinewaves having different amplitudes of different frequencies and having different relative phase shifts.
- Fourier series represents a periodic waveform in the form of sum of infinite number of sine and cosine terms. It is a representation of the signal in a time domain series form.
- Fourier series is a "tool" used to analyze any periodic signal. After the "analysis" we obtain the following information about the signal :
  1. What all frequency components are present in the signal ?
  2. Their amplitudes and
  3. The relative phase difference between these frequency components.

All the "frequency components" are nothing else but sinewaves at those frequencies.

Types of Fourier series :

- There are three types of Fourier series used for the analysis of periodic signals. They are :
  1. Trigonometric or quadrature Fourier series
  2. Polar Fourier series
  3. Exponential Fourier series.

Let us understand them one-by-one.

5.2.1 Trigonometric or Quadrature Fourier Series :

- A periodic signal x(t) with a period of "T_0" is represented by the trigonometric Fourier series as,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \quad ...(5.2.1)$$

where a_0, a_n and b_n are known as the Fourier coefficients and $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$

The expression for the Fourier coefficients are as follows :

1. The value of $$a_0 = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\,dt \quad ...(5.2.2)$$

This shows that a_0 is the average value of x (t). It is also called as the dc component of x(t).

2. The value of $$a_n = \frac{2}{T_0}\int_{t}^{t+T_0} x(t)\cdot\cos(n\omega_0 t)\,dt \quad ...(5.2.3)$$

3. The value of $$b_n = \frac{2}{T_0}\int_{t}^{t+T_0} x(t)\cdot\sin(n\omega_0 t)\,dt \quad ...(5.2.4)$$

Equation (5.2.1) can be expanded as follows :

$$x(t) = \underbrace{a_0}_{\text{DC value of } x(t)} + \underbrace{a_1\cos\omega_0 t}_{\text{Fundamental}} + \underbrace{a_2\cos 2\omega_0 t}_{\text{Second harmonic}} + \cdots + \underbrace{b_1\sin\omega_0 t}_{\text{Fundamental component}} + \underbrace{b_2\sin 2\omega_0 t}_{\text{Second harmonic}} + \cdots$$

(H-453)

Conclusions :

1. The above expression is suitable to plot the **line spectrum**.
2. The first term "a_0" has a zero frequency. Hence it is called as the **dc component of x(t)**. It is also called as the **average value** or dc value of x (t).
3. The sine and cosine terms at frequency f_0 are called fundamental components. The terms at frequency 2 f_0 are called as the second harmonic components and so on.

5.2.2 Polar Fourier Series :

The polar Fourier series is derived from the trigonometric Fourier series by combining the sine and cosine terms of same frequency. The polar Fourier series representation of x(t) is as follows :

$$x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos(n\omega_0 t + \phi_n) \quad ...(5.2.5)$$

where, $$C_n = \left[a_n^2 + b_n^2\right]^{1/2} \quad ...(5.2.6)$$

and $$\phi_n = -\tan^{-1}\left[\frac{b_n}{a_n}\right] \quad ...(5.2.7)$$

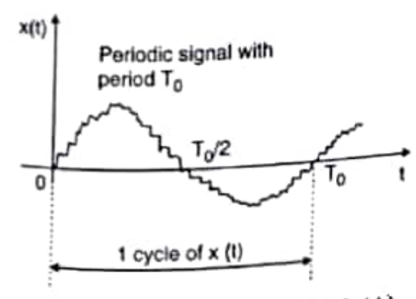
and C_0 = Average value of x (t) = a_0

Equation (5.2.5) can be expanded by opening the summation sign as follows :

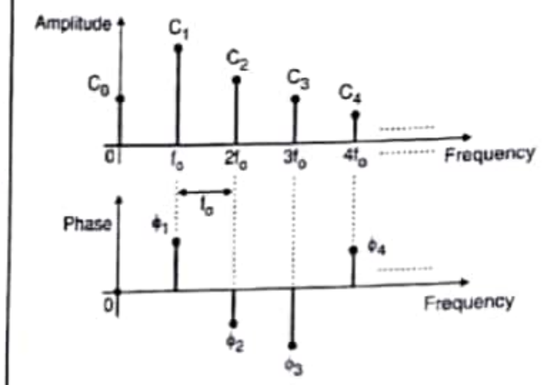
$$x(t) = \underbrace{C_0}_{\text{Average}} + \underbrace{C_1\cos(\omega_0 t + \phi_1)}_{\text{Fundamental}} + \underbrace{C_2\cos(2\omega_0 t + \phi_2)}_{\text{Second harmonic}} + \underbrace{C_3\cos(3\omega_0 t + \phi_3)}_{\text{Third harmonic}} + \cdots \quad ...(5.2.8)$$

Line spectrum :

- The line spectrum of x(t) can be plotted using Equation (5.2.8). A line spectrum of x(t) with arbitrary values of amplitudes and phases is shown in Fig. 5.2.1.



(a) Time domain representation of x(t)



(b) Line spectrum of x(t)

(H-454) Fig. 5.2.1 : Line spectrum using polar Fourier series

- As seen from Fig. 5.2.1(b) the frequency spectrum of a continuous signal is discrete in nature. The frequency components f_0, 2 f_0, 3 f_0— etc. are called as the "spectral components".
- The adjacent spectral components are spaced by "f_0" from each other. As the spectrum is consisting of vertical lines, (C_1, C_2, ...) this spectrum is called as the line spectrum.

5.2.3 Exponential Fourier Series [or Complex Exponential Fourier Series] :

- The sine and cosine terms can be expressed in terms of the exponential terms using Euler's equations as follows :

$$\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad ...(5.2.9)$$

and $$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2} \quad ...(5.2.10)$$

- Substituting the sine and cosine functions in terms of exponential function in the expression for the quadrature Fourier series i.e. in Equation (5.2.1), we can obtain another type of Fourier series called the exponential Fourier series.
- A periodic signal x (t) is expressed in the exponential Fourier series form as follows :

Exponential Fourier series :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n t/T_0} \quad ...(5.2.11)$$

$$\text{where, } C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\cdot e^{-j2\pi n t/T_0}\,dt \qquad ...(5.2.12)$$

As seen from the Equation (5.2.11), this Fourier series consists of exponential terms. Therefore it is called as the exponential Fourier series.

Concept of negative frequency :

- In Equation (5.2.11), observe that "n" is extending from $-\infty$ to $+\infty$ instead of from 0 to ∞. Due to this the frequencies in the frequency spectrum will extend from $-\infty$ to $+\infty$.
- Thus if we express the signal x (t) using exponential Fourier series then we obtain a double sided frequency spectrum.
- However the negative frequency signals do not exist physically. They are used as an important mathematical concept and for mathematical convenience.

Amplitude and phase spectrums :

- The amplitude spectrum of the signal x(t) is denoted by,

$$|C_n| = [(\text{Real part of } C_n)^2 + (\text{Imaginary part of } C_n)^2]^{1/2} \qquad ...(5.2.13)$$

- The amplitude spectrum is a graph of $|C_n|$ on the y-axis versus frequency $f = n f_0$ on the x-axis.

The phase spectrum of x (t) is denoted by,

$$\phi_n = \arg(C_n) = \tan^{-1}\left[\frac{\text{Imaginary part of } C_n}{\text{Real part of } C_n}\right] \qquad ...(5.2.14)$$

- The phase spectrum is a graph of ϕ_n on the y-axis versus frequency $f = n f_0$ on the x-axis.
- The amplitude spectrum is a symmetric or even function. That means $|C_n| = |C_{-n}|$. But the phase spectrum is an asymmetric or odd function. That means $\arg(C_n) = -\arg(C_{-n})$.

Advantages of exponential Fourier series :

1. Instead of finding a_0, a_n and b_n, we have to find only the value of one Fourier coefficient i.e. C_n.
2. It is easier to find and plot the amplitude and phase responses.
3. The exponential F.S. allows us to plot the double sided spectrum, because it is possible to plot the negative frequencies.

5.2.4 Dirichlet Conditions for the Existence of Fourier Series : MU : May 05, May 06, Dec. 09

University Questions

Q. 1 Give the Dirichlet's conditions for existence of a Fourier series. **(May 05, May 06, 4 Marks)**

Q. 2 Show that $\int_{-D}^{D} |x(t)|\,dt < \alpha$ is a sufficient condition for the existence of the Fourier transform of x(t). **(Dec. 09, 5 Marks)**

The Fourier series stated in the Equations (5.2.1) and (5.2.5) will exist if and only if the periodic signal x(t) satisfies the following conditions. These are known as the "Dirichlet" conditions. They are as follows :

1. The periodic signal x(t) and its integrals are finite and single valued in the interval (t to t + T_0), i.e. over a period of one cycle T_0.
2. x(t) must have only finite number of discontinuities in the given interval of time.
3. x (t) should have only finite number of maxima and minima in the given interval of time.
4. The function x(t) is absolutely integrable, that is

$$\int_{-T_0/2}^{T_0/2} |x(t)|\,dt < \infty$$

5.2.4.1 Knowledge of Waveform Symmetry used to Calculate Fourier Coefficient :

If the given waveform has some kind of a symmetry, then it becomes easy to calculate the Fourier coefficients. It is as explained below :

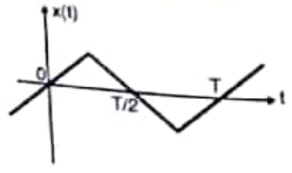
1. Signal with zero mean :

If the mean value of the given signal is zero.

$$\because \text{ i.e. if } \int_0^T x(t)\,dt = 0$$

$$\text{then } a_0 = 0$$

But there is no change in the expressions of a_n and b_n. A signal having zero mean value is shown in Fig. 5.2.2.



(H-543) Fig. 5.2.2

2. An even symmetry :

If the given signal x(t) is an even signal,

i.e. $x(t) = x(-t)$

Then $b_n = 0$ ... for all n

3. An odd symmetry :

If the given signal x (t) is an odd signal.

i.e. $x(t) = -x(-t)$

Then $a_0 = 0, a_n = 0$ ... for all n

4. Half wave symmetry :

If the given signal, x (t) has half wave symmetry

i.e. if $x(t) = -x\left(t + \frac{T_0}{2}\right)$

Then $a_0 = 0, a_n = 0$ ... for n even

$b_n = 0$ ... for n even

Thus if we know the symmetry of the given waveform, then the work involved in finding out the Fourier coefficients is reduced considerably.

Orthogonality :

- The orthogonality of complex sinusoids is very important in Fourier representation.
- Any two signals are said to be **orthogonal** if their inner product is equal to zero.
- For the DT periodic signals, the inner product is equal to the sum of the values in their products. If $\phi_k[n]$ and $\phi_m[n]$ are two N periodic signals then their inner product is given by,

$$I_{k,m} = \sum_{n=(N)} \phi_k[n]\,\phi_m^*[n] \qquad ...(5.2.15)$$

- If $I_{k,m} = 0$

...for $k \neq m$ then $\phi_k[n]$ and $\phi_m[n]$ are orthogonal signals.

- Similarly for CT signals, with period T, the inner product is defined as follows :

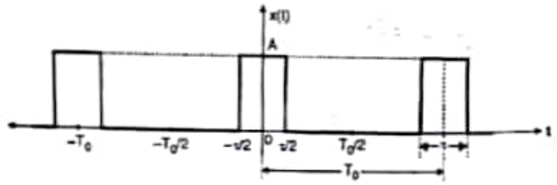
$$I_{k,m} = \int_{(T)} \phi_k(t)\,\phi_m^*(t)\,dt \qquad ...(5.2.16)$$

- If $I_{k,m} = 0$

... for $k \neq m$ then $\phi_k(t)$ and $\phi_m(t)$ are orthogonal signals.

5.2.5 Examples on Fourier Series :

Ex. 5.2.1 : Obtain the quadrature Fourier series for the rectangular pulse train shown in Fig. P. 5.2.1. **May 11, 5 Marks**



(H-455) Fig. P. 5.2.1

Soln. :

Let us obtain the quadrature Fourier series for the given rectangular pulse. The quadrature Fourier series is given by the following expression :

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \qquad ...(1)$$

Substituting $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$ into Equation (1) we get,

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left[\frac{2\pi n t}{T_0}\right] + \sum_{n=1}^{\infty} b_n \sin\left[\frac{2\pi n t}{T_0}\right] \qquad ...(2)$$

To find the Fourier coefficients, we must consider one complete cycle of x (t) for integration.

Here we will consider one cycle from $t = -\frac{T_0}{2}$ to $t = \frac{T_0}{2}$.

Let us obtain the Fourier coefficients now.

1. To obtain the value of a_0 :

$$a_0 = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,dt$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} x(t)\,dt \quad \text{...As x (t) exists from } -\tau/2 \text{ to } \tau/2 \text{ only}$$

$$= \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} A\,dt \quad \text{...As x (t) = A from } -\tau/2 \text{ to } \tau/2$$

$$\therefore a_0 = \frac{A\tau}{T_0} \qquad ...(3)$$

2. To obtain the value of a_n : From the Equation (5.2.3), a_n is given as,

$$a_n = \frac{2}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cdot\cos\left[\frac{2\pi n t}{T_0}\right]dt = \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} A\cos\left[\frac{2\pi n t}{T_0}\right]dt$$

$$= \frac{2A}{T_0} \cdot \frac{1}{\frac{2\pi n}{T_0}} \left[\sin\left(\frac{2\pi n t}{T_0}\right) \right]_{-\tau/2}^{\tau/2} = \frac{A}{\pi n} \left[\sin\left(\frac{2\pi n \tau}{2T_0}\right) - \sin\left(\frac{-2\pi n \tau}{2T_0}\right) \right]$$

$$= \frac{A}{\pi n} \left[\sin\left(\frac{\pi n \tau}{T_0}\right) + \sin\left(\frac{\pi n \tau}{T_0}\right) \right]$$

$$\therefore \quad a_n = \frac{2A}{\pi n} \sin\left[\frac{\pi n \tau}{T_0}\right] \qquad ...(4)$$

3. **To obtain the value of b_n :** From the Equation (5.2.4), b_n is given as,

$$b_n = \frac{2}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot \sin\left[\frac{2\pi n t}{T_0}\right] dt = \frac{2}{T_0} \int_{-\tau/2}^{\tau/2} A \sin\left[\frac{2\pi n t}{T_0}\right] dt$$

$$= \frac{-2A}{T_0} \cdot \frac{1}{\frac{2\pi n}{T_0}} \left[\cos\left(\frac{2\pi n t}{T_0}\right) \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-A}{\pi n} \left[\cos\left(\frac{2\pi n \tau}{2T_0}\right) - \cos\left(\frac{-2\pi n \tau}{2T_0}\right) \right]$$

$$\therefore \quad b_n = \frac{-A}{\pi n} \left[\cos\left(\frac{\pi n \tau}{T_0}\right) - \cos\left(\frac{\pi n \tau}{T_0}\right) \right]$$

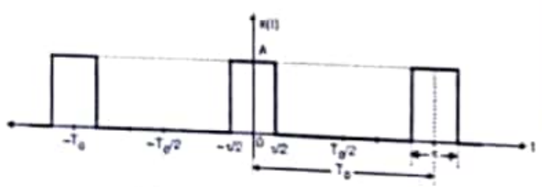
$$\therefore \quad b_n = 0 \qquad ...(5)$$

4. Substitute the values of a_0, a_n and b_n in the Equation (2) we get the quadrature Fourier series for $x(t)$ as,

$$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty} \frac{2A}{n\pi} \sin\left[\frac{n\pi\tau}{T_0}\right] \cos\left[\frac{2\pi n t}{T_0}\right] \qquad \textbf{...Ans.}$$

Ex. 5.2.2 : Obtain the exponential Fourier series for the rectangular pulse train shown in Fig. P. 5.2.2(a) and sketch the spectrum.

Dec. 04, 10 Marks. Dec. 06, 12 Marks. May 11, 5 Marks. May 13, 10 Marks. May 2015, 10 Marks



(H-456) Fig. P. 5.2.2(a) : A rectangular pulse train

Soln. : From the Equation (5.2.11), the exponential Fourier series is given as,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n t/T_0} \qquad ...(1)$$

$$\text{where, } C_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t) \cdot e^{-j2\pi n t/T_0} dt \qquad ...(2)$$

Let us consider one cycle of $x(t)$ extending from $t = \frac{-T_0}{2}$ to $t = \frac{T_0}{2}$.

1. **To obtain the value of C_n :**

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot e^{-j2\pi n t/T_0} dt$$

But $x(t) = A$, from $t = -\tau/2$ to $t = \tau/2$

$$\therefore C_n = \frac{1}{T_0} \int_{-\tau/2}^{\tau/2} A e^{-j2\pi n t/T_0} dt = \frac{-A}{j\frac{2\pi n}{T_0} \times T_0} \left[e^{-j2\pi n t/T_0} \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{-A}{j2\pi n} \left[e^{-j2\pi n \tau/2T_0} - e^{j2\pi n \tau/2T_0} \right] = \frac{A}{\pi n} \left[\frac{e^{jn\pi\tau/T_0} - e^{-jn\pi\tau/T_0}}{2j} \right]$$

$$\therefore \quad C_n = \frac{A}{\pi n} \sin\left(\frac{\pi n \tau}{T_0}\right) \qquad ...(3)$$

Multiply and divide the RHS of Equation (3) by (τ/T_0) we get,

$$C_n = \frac{A(\tau/T_0)}{(\pi n \tau/T_0)} \sin(\pi n \tau/T_0)$$

$$\therefore \quad C_n = A\left(\frac{\tau}{T_0}\right) \text{sinc}\left(\frac{n\tau}{T_0}\right) \qquad ...(4)$$

2. **To obtain the Fourier series :**

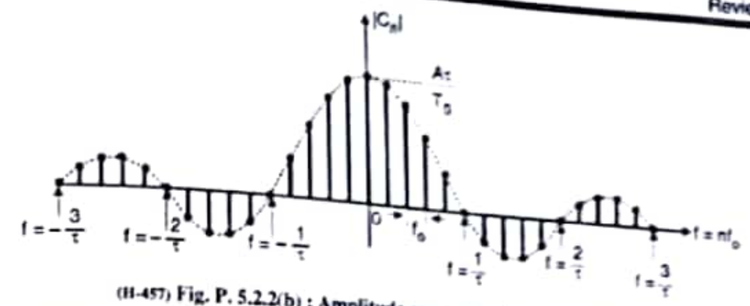
Substitute the value of C_n into Equation (1) to obtain the exponential Fourier series as,

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{\tau A}{T_0} \text{sinc}\left[\frac{n\tau}{T_0}\right] e^{+j2\pi n t/T_0} \qquad ...(5)$$

3. **Spectrum of the signal $x(t)$:**

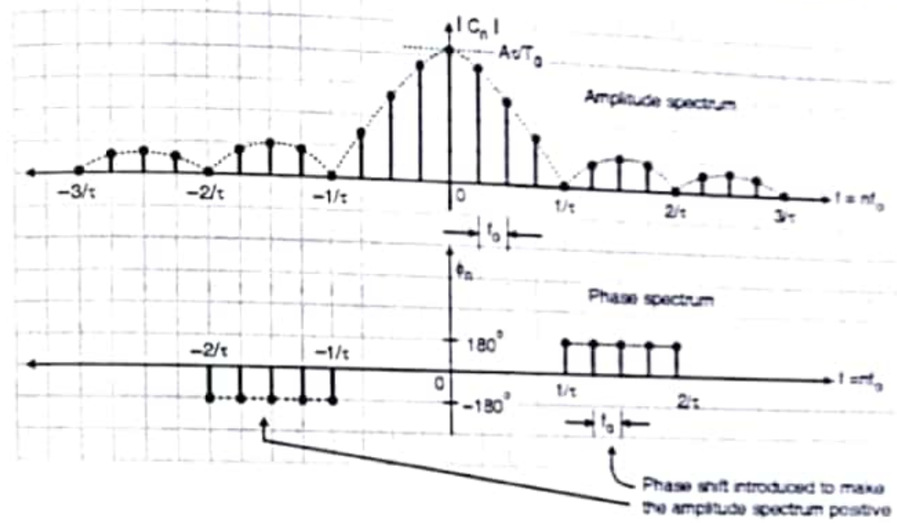
From the value of C_n in Equation (4), it is clear that C_n does not have any imaginary part. Therefore the amplitude spectrum of $x(t)$ is given as,

$$|C_n| = C_n = \frac{A\tau}{T_0} \text{sinc}\left[\frac{n\tau}{T_0}\right] \qquad ...(6)$$



(H-457) Fig. P. 5.2.2(b) : Amplitude spectrum of a rectangular pulse

Thus the amplitude spectrum of a rectangular pulse of duration τ is a sinc function. The spectrum is as shown in Fig. P. 5.2.2(b).



(H-458) Fig. P. 5.2.2(c) : Amplitude and phase spectrum of a periodic pulse train

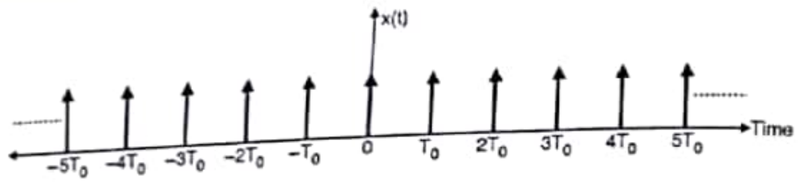
The imaginary part of C_n is zero therefore the phase spectrum is zero for all the values of f. The phase spectrum is shown in Fig. P. 5.2.2(c).

Conclusions :

Important points related to the amplitude and phase spectrums of Fig. P. 5.2.2(c) are :

1. The spacing between the adjacent spectral components is f_0.
2. The shape of the envelope of amplitude spectrum $|C_n|$ is determined by A, τ and f_0. The envelope is "sinc" shaped.
3. Zero crossings occur in the envelope of the amplitude spectrum at frequencies of $f = \pm 1/\tau, \pm 2/\tau, \pm 3/\tau$ ...etc.
4. The phase spectrum takes on values of 0° corresponding to the positive values of $|C_n|$. However it takes on values of ± 180° corresponding to the negative values of $|C_n|$ e.g. between $f = \pm 1/\tau, \pm 2/\tau$. The negative values in the amplitude spectrum are made positive by assigning phase shifts of ± 180°.
5. Note that the choice of phase shift + 180° or − 180° is arbitrary. However we have used both of them to preserve the "antisymmetry" of the phase spectrum.

Ex. 5.2.3 : Obtain the Fourier series of the unit impulse train shown in Fig. P. 5.2.3(a). Also plot the amplitude and phase spectrums for the same. **Dec. 10, 10 Marks**



(H-461) **Fig. P. 5.2.3(a) : Unit impulse train**

Soln. :

1. **Value of C_n for a train of rectangular pulses :**

In the Ex. 5.2.2, we have already obtained the exponential Fourier series for a train of rectangular pulses of duration τ and period T_0. Its value is,

$$C_n = \frac{A\tau}{T_0}\text{sinc}\left[\frac{n\tau}{T_0}\right] \quad ...(1)$$

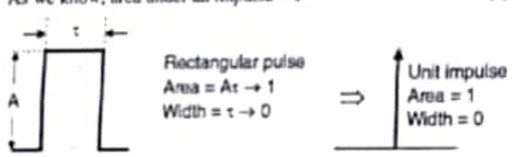
We will use it to find the value of C_n for an impulse train.

2. **Equivalence between a rectangular pulse and an impulse :**

The rectangular pulse of amplitude A and width τ is as shown in Fig. P. 5.2.3(b).

$\therefore$ Area under a rectangular pulse = $A\tau$ ...(2)

As we know, area under an impulse = 1 ...(3)



(H-462) **Fig. P. 5.2.3(b) : Equivalence between rectangular pulse and unit impulse**

Hence a rectangular pulse will be equivalent to a unit impulse as its area i.e. "$A\tau$" approaches unity and as its width τ approaches zero.

$$\therefore \quad A\tau \to 1 \text{ and } \tau \to 0 \quad ...(4)$$

3. **To obtain C_n for the unit impulse train :**

We can obtain the value of "C_n" for the unit impulse train by applying Equation (4) to Equation (1) as follows :

$$\therefore \quad C_n = \lim_{\substack{A\tau \to 1 \\ \tau \to 0}} \frac{A\tau}{T_0}\text{sinc}\left[\frac{n\tau}{T_0}\right]$$

$$= \lim_{\tau \to 0} \frac{1}{T_0}\text{sinc}\left[\frac{n\tau}{T_0}\right] \quad ...\text{as } A\tau = 1$$

$$\text{But } \lim_{\tau \to 0} \text{sinc}\left[\frac{n\tau}{T_0}\right] = \text{sinc}(0) = 1.$$

$$\therefore \quad C_n = \frac{1}{T_0} \quad ...(5)$$

This is the value of C_n for the unit impulse train.

4. **To obtain the exponential Fourier series :**

Substitute the value of "C_n" from Equation (5) into the standard expression of exponential Fourier series.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$$

$$\therefore \quad x(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{j2\pi nt/T_0} \quad ...(6)$$

This is the required Fourier series for the unit impulse train.

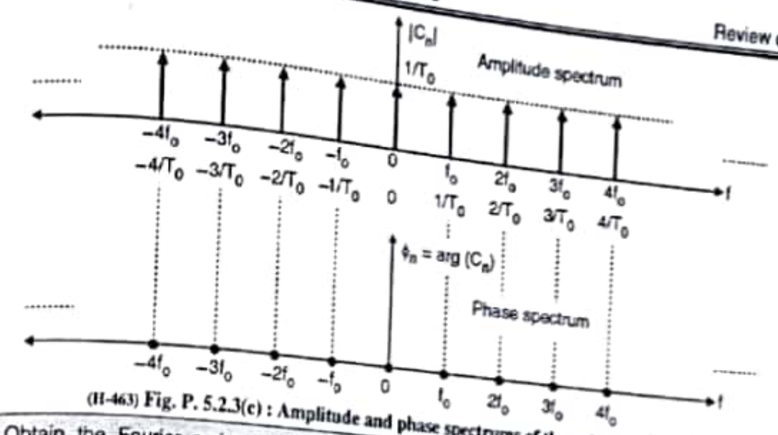
Amplitude spectrum :

Amplitude spectrum : $|C_n| = C_n = \frac{1}{T_0}$ ...(7)

This means that for every value of "n" the value of C_n is going to be the same, equal to ($1/T_0$). The amplitude spectrum also is train of impulses each having amplitude of ($1/T_0$), as shown in Fig. P. 5.2.3(c).

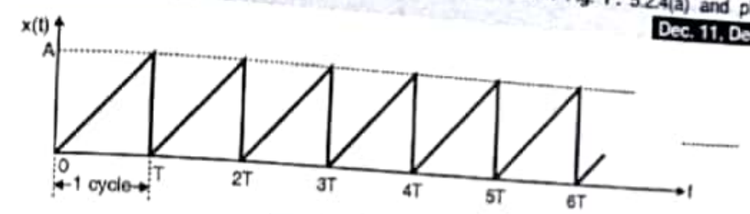
Phase spectrum :

The phase spectrum $\phi_n = \arg(C_n) = 0$ as C_n is constant. The phase spectrum is as shown in Fig. P. 5.2.3(c).



(H-463) **Fig. P. 5.2.3(c) : Amplitude and phase spectrums of the unit impulse train**

Ex. 5.2.4 : Obtain the Fourier series of the sawtooth waveform shown in Fig. P. 5.2.4(a) and plot its spectrum. **Dec. 11, Dec. 12, 10 Marks**



(H-464) **Fig. P. 5.2.4(a) : A sawtooth waveform**

Soln. :

1. **Representation of the signal :**

The sawtooth signal x (t) shown in Fig. P. 5.2.4(a) can be represented over one cycle as,

$$x(t) = \frac{A}{T}t \qquad \text{for } 0 < t < T \quad ...(1)$$

and one cycle period $T_0 = T$.

2. **To find C_n :**

In order to represent the signal in the form of an exponential Fourier series, let us find "C_n" using the following equation :

$$C_n = \frac{1}{T_0}\int_{T}^{t+T_0} x(t)\cdot e^{-j2\pi nt/T_0}dt \quad ...(2)$$

Substitute $T_0 = T$ and $x(t) = \frac{A}{T}t$ and integrate from t = 0 to t = T.

$$\therefore \quad C_n = \frac{1}{T}\int_0^T \frac{At}{T}e^{-j2\pi nt/T}dt \quad ...(3)$$

At this point, let us use a standard identity which states that,

$$\int x e^{ax}dx = e^{ax}\left[\frac{x}{a} - \frac{1}{a^2}\right] \quad ...(4)$$

Let x = t and $a = -j2\pi n/T$. Then "C_n" can be expressed as,

$$C_n = \frac{A}{T^2}\int_0^T t e^{-j2\pi nt/T}dt$$

Using Equation (4) we get,

$$= \frac{A}{T^2}\cdot\left[e^{at}\left(\frac{t}{a} - \frac{1}{a^2}\right)\right]_0^T$$

$$\therefore \quad C_n = \frac{A}{T^2}\left[e^{aT}\left(\frac{T}{a} - \frac{1}{a^2}\right) - e^0\left(0 - \frac{1}{a^2}\right)\right]$$

$$= \frac{A}{T^2}\left[e^{aT}\left(\frac{T}{a} - \frac{1}{a^2}\right) + \frac{1}{a^2}\right]$$

$$\therefore \quad C_n = \frac{A}{T^2}\left[\frac{e^{aT}(aT-1)+1}{a^2}\right] \quad ...(5)$$

$$= \frac{A}{a^2T^2}[e^{aT}(aT-1)+1]$$

Now substitute : $a = -j2\pi n/T$

$$\therefore \quad a^2 = j^2 4\pi^2 n^2/T^2 = -4\pi^2 n^2/T^2$$

and $aT = -j2\pi n$ in Equation (5) to get,

$$C_n = \frac{A}{-4\pi^2 n^2}[e^{-j2\pi n}(-j2\pi n - 1) + 1]$$

$$\text{But } e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1$$

Substituting we get, $C_n = \frac{A}{-4\pi^2 n^2}[-j2\pi n - 1 + 1] = \frac{j2\pi nA}{4\pi^2 n^2}$

$$\therefore \quad C_n = \frac{jA}{2\pi n} \quad [\text{for } n \neq 0] \quad ...(6)$$

At n = 0, let us find the value of C_0.

$$C_0 = \frac{1}{T}\int_0^T x(t)e^0 dt = \frac{1}{T}\int_0^T \frac{A}{T}t\,dt = \frac{A}{T^2}\times\left[\frac{t^2}{2}\right]_0^T$$

$$= \frac{A}{2T} \times T^2 \quad \therefore \; C_0 = \frac{A}{2} \qquad ...(6(a))$$

3. To obtain the exponential Fourier series :

The exponential Fourier series is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n t/T}$$

$$\therefore \; x(t) = \sum_{n=-\infty}^{\infty} \frac{-jA}{2\pi n} \cdot e^{j2\pi n t/T} \qquad ...(7)$$

4. Amplitude spectrum :

Amplitude spectrum : $|C_n| = \left[0 + \frac{A^2}{4\pi n^2}\right]^{1/2} = \frac{A}{2\pi n}$ ...(8)

The amplitude spectrum is as shown in Fig. P. 5.2.4(b).

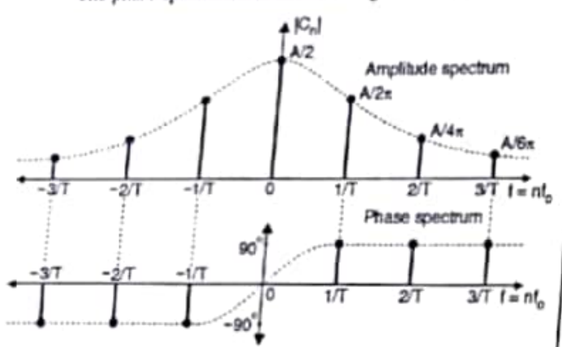
5. Phase spectrum :

Phase spectrum : $\phi_n = \arg(C_n) = \tan^{-1}\left[\frac{A/2\pi n}{0}\right] = \pm 90°$

$$\therefore \quad \phi_n = +90° \text{ for } n > 0$$

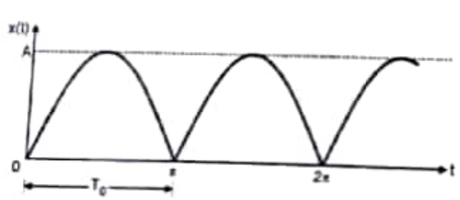
$$\text{and} \quad \phi_n = -90° \text{ for } n < 0$$

The phase spectrum is as shown in Fig. P. 5.2.4(b).



(H-465) Fig. P. 5.2.4(b) : Amplitude and phase spectrums

Ex. 5.2.5 : Find the quadrature Fourier series for the full wave rectified sine wave shown in Fig. P. 5.2.5.



(H-468) Fig. P. 5.2.5 : Full wave rectified sinewave

Soln. :

The quadrature Fourier series for a periodic signal x (t) is given by :

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \qquad ...(1)$$

So, we will have to find the values of the Fourier coefficients a_0, a_n and b_n.

1. To obtain the average value or a_0 :

$$a_0 = \frac{1}{T_0} \int_t^{t+T_0} x(t)\, dt \qquad ...(2)$$

In this equation substitutex (t)=A sin ($\pi f_0 t$) and $T_0 = \pi$

Let us integrate from t = 0 to t = π

$$\therefore \; a_0 = \frac{1}{\pi}\int_0^{\pi} A \sin(\pi f_0 t)\, dt = -\frac{A}{\pi}\left[\frac{\cos(\pi f_0 t)}{\pi f_0}\right]_0^{\pi} \qquad ...(3)$$

$$\text{Now,} \quad f_0 = \frac{1}{T_0} = \frac{1}{\pi}.$$

$$\therefore \quad \pi f_0 = 1.$$

$$\therefore \; a_0 = -\frac{A}{\pi}[\cos t]_0^{\pi} = -\frac{A}{\pi}[\cos\pi - \cos 0] = -\frac{A}{\pi}[-1-1]$$

$$\therefore \quad a_0 = \frac{2A}{\pi} \qquad ...(4)$$

2. To obtain the value of a_n :

$$a_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\cos(n\omega_0 t)\, dt \qquad ...(5)$$

$$= \frac{2}{\pi}\int_0^{\pi} A \sin(\pi f_0 t)\cos(2\pi n f_0 t)\, dt$$

Substituting $\pi f_0 = 1$ we get,

$$a_n = \frac{2}{\pi}\int_0^{\pi} A \sin(t)\cos(2nt)\, dt = \frac{A}{\pi}\int_0^{\pi} 2\cos(2nt)\sin(t)\, dt \qquad ...(6)$$

Use the identity,

$$2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

Applying this to Equation (6) we get,

$$a_n = \frac{A}{\pi}\int_0^{\pi}[\sin(2nt+t) - \sin(2nt-t)]\, dt$$

$$= \frac{A}{\pi}\int_0^{\pi}\sin(2n+1)t\, dt - \frac{A}{\pi}\int_0^{\pi}\sin(2n-1)t\, dt$$

$$= -\frac{A}{\pi}\left[\frac{\cos(2n+1)t}{(2n+1)}\right]_0^{\pi} + \frac{A}{\pi}\left[\frac{\cos(2n-1)t}{(2n-1)}\right]_0^{\pi}$$

$$= -\frac{A}{(2n+1)\pi}[\cos(2n+1)\pi - \cos 0]$$

$$+ \frac{A}{(2n-1)\pi}[\cos(2n-1)\pi - \cos 0]$$

$$= -\frac{A}{(2n+1)\pi}[\cos(2n\pi+\pi) - 1]$$

$$+ \frac{A}{(2n-1)\pi}[\cos(2n\pi-\pi) - 1]$$

$$= -\frac{A}{(2n+1)\pi}[\cos 2\pi n \cos\pi - 1]$$

$$+ \frac{A}{(2n-1)\pi}[\cos 2\pi n \cos\pi - 1]$$

But $\cos 2\pi n = 1$ and $\cos\pi = -1$

$$\therefore \quad a_n = \frac{-A}{(2n+1)\pi}(-2) + \frac{A}{(2n-1)\pi}(-2)$$

$$= \frac{2A}{\pi}\left[\frac{1}{(2n+1)} - \frac{1}{(2n-1)}\right]$$

$$\therefore \quad a_n = \frac{2A}{\pi}\left[\frac{2n-1-2n-1}{4n^2-1}\right] = \frac{-4A}{\pi(4n^2-1)} \qquad ...(7)$$

3. To obtain the value of b_n :

$$b_n = \frac{2}{T_0}\int_t^{t+T_0} x(t)\sin(n\omega_0 t)\, dt$$

$$= \frac{2}{\pi}\int_0^{\pi} A\sin(\pi f_0 t)\sin(2\pi n f_0 t)\, dt \qquad ...(8)$$

Substituting $\pi f_0 = 1$ we get,

$$b_n = \frac{2}{\pi}\int_0^{\pi} A\sin(t)\sin(2nt)\, dt$$

$$= \frac{A}{\pi}\int_0^{\pi} 2\sin(2nt)\sin(t)\, dt \qquad ...(9)$$

Use the standard identity,

$$2\sin A \sin B = \cos(A-B) - \cos(A+B)$$

$$\therefore \quad b_n = \frac{A}{\pi}\int_0^{\pi}[\cos(2nt-t) - \cos(2nt+t)]\, dt$$

$$= \frac{A}{\pi}\int_0^{\pi}\cos(2n-1)t\, dt - \frac{A}{\pi}\int_0^{\pi}\cos(2n+1)t\, dt$$

$$\therefore \; b_n = \frac{A}{\pi}\left[\frac{\sin(2n-1)t}{(2n-1)}\right]_0^{\pi} - \frac{A}{\pi}\left[\frac{\sin(2n+1)t}{(2n+1)}\right]_0^{\pi} = 0 \qquad ...(10)$$

4. To obtain the quadrature Fourier series :

Substitute the values of a_0, a_n and b_n into the Equation (1) to write the quadrature Fourier series as :

$$x(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} \frac{-4A}{\pi(4n^2-1)}\cos(n\omega_0 t) \qquad ...(11)$$

This is the desired result.

Ex. 5.2.6 : For the same full wave rectified sinewave of Example 5.2.5 find the exponential Fourier series and the spectrum.

Soln. : The signal x (t) is mathematically represented as :

$$x(t) = A\sin(\pi f_0 t) \qquad ...(1)$$

where, $T_0 = \frac{1}{f_0} = \pi$, as shown in Fig. P. 5.2.5.

$$= [\cos(2\pi n) - j\sin(2\pi n) + 1]$$

$$C_n = \frac{-2A}{\pi(4n^2-1)} \qquad ...(2)$$

$$\therefore \quad C_0 = \frac{2A}{\pi} \qquad ...(3)$$

Amplitude spectrum : The amplitude spectrum,

$$|C_n| = [C_n^2]^{1/2} = \frac{2A}{\pi(4n^2-1)} \qquad ...(4)$$

Phase spectrum :

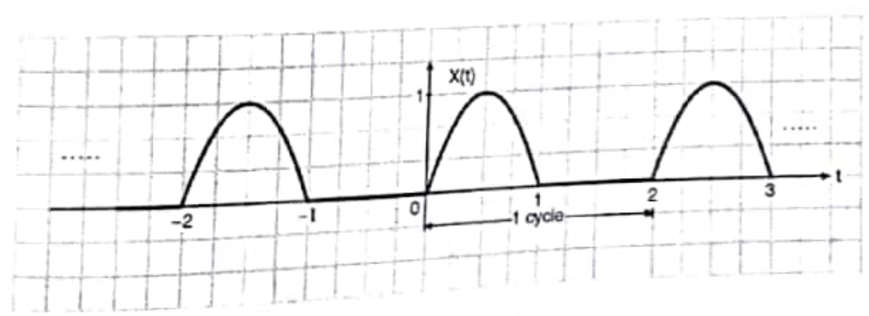
Phase spectrum $\phi_n = \arg(C_n) = 0$

Exponential Fourier series :

Substitute "C_n" in Equation (1) to get,

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{-2A}{\pi(4n^2-1)} \cdot e^{j2nt} \qquad ...(5)$$

Ex. 5.2.7 : Find the continuous time Fourier series in the exponential form for the signal shown in Fig. P.5.2.7.



(H-470) Fig. P. 5.2.7

Soln. : The exponential Fourier series is given by.

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$$

where $C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\cdot e^{-j2\pi nt/T_0}\,dt$

– The given signal can be mathematically expressed as

$$x(t) = \sin(2\pi f_0 t) \quad \ldots 0 \le t \le 1$$
$$= 0 \quad \ldots 1 < t \le 2$$

– But $f_0 = \frac{1}{T_0}$ and $T_0 = 2$ from Fig. P. 5.2.7

$$\therefore f_0 = \frac{1}{2}$$

$$\therefore x(t) = \sin(\pi t) \quad \ldots 0 \le t \le 1$$
$$= 0 \quad \ldots 1 < t \le 2$$

Step 1 : Calculate C_n :

$$C_n = \frac{1}{2}\int_0^1 \sin(\pi t)\, e^{-j2\pi nt/2}\,dt$$

$$= \frac{1}{2}\int_0^1 \sin(\pi t)\, e^{-j\pi nt}\,dt$$

Substitute $\sin(\pi t) = \frac{e^{j\pi t} - e^{-j\pi t}}{2j}$ to get

$$C_n = \frac{1}{4j}\int_0^1 \left[e^{j\pi t} - e^{-j\pi t}\right] e^{-j\pi nt}\,dt$$

$$= \frac{1}{4j}\int_0^1 \left[e^{j\pi t(1-n)} - e^{-j\pi t(1+n)}\right] dt$$

$$\therefore C_n = \frac{1}{4j\times j\pi(1-n)}\left[e^{j\pi(1-n)t}\right]_0^1 + \frac{1}{4j\times j\pi(1+n)}\left[e^{-j\pi(1-n)t}\right]_0^1$$

$$= \frac{1}{-4\pi(1-n)}\left[e^{j\pi(1-n)} - e^{0}\right] - \frac{1}{4\pi(1+n)}\left[e^{j\pi(1+n)} - e^{0}\right]$$

$$\therefore C_n = \frac{1}{4\pi(1-n)}\left[1 - e^{j\pi(1-n)}\right] + \frac{1}{4\pi(1+n)}\left[1 - e^{-j\pi(1+n)}\right]$$

$$= \frac{1}{4\pi(1-n)(1+n)}\left\{(1+n)(1-e^{j\pi(1-n)}) + (1-n)(1-e^{-j\pi(1+n)})\right\}$$

$$= \frac{1}{4\pi(1-n^2)}\left\{(1+n) - (1+n)e^{j\pi(1-n)} + (1-n) - (1-n)e^{-j\pi(1+n)}\right\}$$

$$= \frac{1}{4\pi(1-n^2)}\left\{2 - (1+n)e^{j\pi}e^{-j\pi n} - (1-n)e^{-j\pi}e^{-j\pi n}\right\}$$

$$= \frac{1}{4\pi(1-n^2)}\left\{2 - e^{-j\pi n}\left[(1+n)e^{j\pi} + (1-n)e^{-j\pi}\right]\right\}$$

$$= \frac{1}{4\pi(1-n^2)}\left\{2 - e^{-j\pi n}\left[e^{j\pi} + ne^{j\pi} + e^{-j\pi} - ne^{-j\pi}\right]\right\}$$

$$= \frac{1}{4\pi(1-n^2)}\left\{2 - e^{-j\pi n}\left[e^{j\pi} + e^{-j\pi} + n(e^{j\pi} - e^{-j\pi})\right]\right\}$$

$$= \frac{1}{4\pi(1-n^2)}\left\{2 - e^{-j\pi n}\left[2\cos\pi + 2jn\sin\pi\right]\right\}$$

But $\sin\pi = 0$ and $\cos\pi = -1$

$$\therefore C_n = \frac{1}{4\pi(1-n^2)}\left\{2 - e^{-j\pi n}\times -2\right\}$$

$$= \frac{2}{4\pi(1-n^2)}\left(1 + e^{-j\pi n}\right) = \frac{1 + e^{-j\pi n}}{2\pi(1-n^2)}$$

But $e^{-j\pi n} = \cos n\pi - j\sin n\pi$

$$\therefore C_n = \frac{1 + \cos(n\pi) - j\sin(n\pi)}{2\pi(1-n^2)}$$

$$= \frac{1}{2\pi(1-n^2)}\left[1 + \cos n\pi\right]$$

But $\sin(n\pi) = 0$ for all n

and $\cos(n\pi) = -1$ for odd n $= +1$ for even n

$\therefore C_n = 0$ for odd n

and $C_n = \frac{1}{\pi(1-n^2)}$ ... for n even

Step 2 : Obtain the Fourier series :

Substituting the value of C_n we get the exponential Fourier series as follows.

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\pi(1-n^2)} e^{+j\pi nt}$$

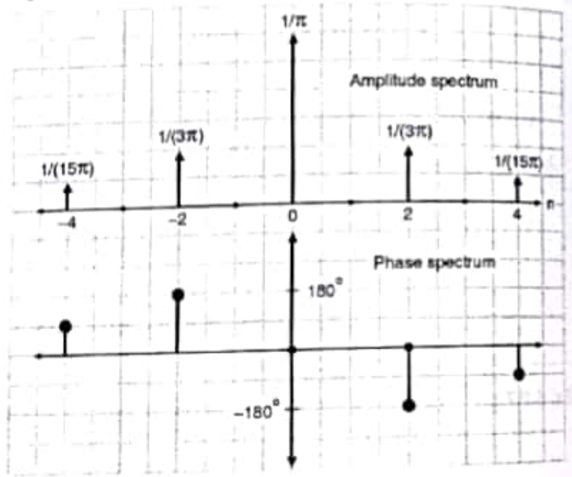
n even

$$= \ldots \frac{1}{\pi(1-4)}e^{-j2\pi t} + \frac{1}{\pi(1-0)}e^{0} + \frac{1}{\pi(1-4)}e^{j2\pi t} + \frac{1}{\pi(1-16)}e^{j4\pi t} \ldots$$

$$\therefore x(t) = \ldots -\frac{1}{3\pi}e^{-j2\pi t} + \frac{1}{\pi} - \frac{1}{3\pi}e^{j2\pi t} - \frac{1}{15\pi}e^{j4\pi t} + \ldots$$

Step 3 : Plot the spectrum :

The amplitude and phase spectrums are shown in Fig. P. 5.2.7(a).



(H-471) **Fig. P. 5.2.7(a) : Amplitude and phase spectrums**

Ex. 5.2.8 : Determine the Fourier series coefficients of :

1. $x(t) = \sin\omega_0 t$ 2. $x(t) = \cos\omega_0 t$.

Soln. :

1. $x(t) = \sin(\omega_0 t) = \sin(2\pi f_0 t)$:

Find a_0 :

$$\therefore a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}\int_0^{T_0} \sin(2\pi f_0 t)\,dt$$

$$= \frac{1}{T_0}\times\frac{-\left[\cos 2\pi f_0 t\right]_0^{T_0}}{2\pi f_0}$$

But $f_0 T_0 = 1$

$$\therefore a_0 = \frac{-1}{2\pi}\left[\cos 2\pi f_0 T_0 - \cos 0\right]$$

$$= \frac{-1}{2\pi}\left[\cos 2\pi - \cos 0\right] = 0$$

Find a_n :

$$a_n = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(2\pi n f_0 t)\,dt$$

$$= \frac{2}{T_0}\int_0^{T_0} \sin(2\pi f_0 t)\cos(2\pi n f_0 t)\,dt$$

But $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

$$\therefore a_n = \frac{2}{T_0}\int_0^{T_0}\left\{\sin\left[2\pi n f_0 t + 2\pi f_0 t\right] - \sin\left[2\pi n f_0 t - 2\pi f_0 t\right]\right\}dt$$

$$\therefore a_n = \frac{2}{10}\int_0^{T_0}\left\{\sin\left[2\pi f_0(n+1)t\right] - \sin\left[2\pi f_0(n-1)t\right]\right\}dt$$

$$\therefore a_n = \frac{-1}{T_0}\left[\frac{\cos 2\pi f_0(n+1)t}{(n+1)2\pi f_0}\right]_0^{T_0} + \frac{1}{T_0}\left[\frac{\cos 2\pi f_0(n-1)t}{(n-1)2\pi f_0}\right]_0^{T_0}$$

$$\therefore a_n = \frac{-\left[\cos 2\pi f_0(n+1)T_0 - \cos 0\right]}{2\pi(n+1)} + \frac{-\left[\cos 2\pi f_0(n-1)T_0 - \cos 0\right]}{2\pi(n-1)}$$

$$\therefore a_n = \frac{-\left[\cos 2\pi(n+1) - 1\right]}{2\pi(n+1)} + \frac{\left[\cos 2\pi(n-1) - 1\right]}{2\pi(n-1)}$$

$$\therefore a_n = \frac{(n+1)\left[\cos 2\pi(n-1) - 1\right] - (n-1)\left[\cos 2\pi(n+1) - 1\right]}{2\pi(n+1)(n-1)} \quad \ldots(1)$$

$a_n = 0$ for all n. Since $\cos 2\pi(n-1) = \cos 2\pi(n+1) = 1$ for all n. But a_1 can not be obtained from Equation (1) due to the presence of $(n-1)$ term in the denominator. So let us calculate a_1 separately.

$$a_1 = \frac{2}{T_0}\int_0^{T_0} \sin(2\pi f_0 t)\cos(2\pi n f_0 t)\,dt$$

$$\therefore a_1 = \frac{1}{T_0}\int_0^{T_0} \sin(4\pi f_0 t)\,dt = \frac{-\left[\cos(4\pi f_0 t)\right]_0^{T_0}}{T_0\times 4\pi f_0}$$

$$= \frac{-\left[\cos(4\pi f_0 T_0) - \cos 0\right]}{4\pi} = 0$$

Find b_n :

$$b_n = \frac{2}{T_0}\int_0^{T_0} \sin(2\pi f_0 t)\cos(2\pi n f_0 t)\,dt$$

But $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

$$\therefore b_n = \frac{2}{T_0}\int_0^{T_0} \cos\left[2\pi f_0(n-1)t\right] - \cos\left[2\pi f_0(n+1)t\right]dt$$

$$= \frac{2}{T_0}\left\{\frac{\sin\left[2\pi f_0(n-1)t\right]}{2\pi f_0(n-1)}\right\}_0^{T_0} - \frac{2}{T_0}\left\{\frac{\sin\left[2\pi f_0(n+1)t\right]}{2\pi f_0(n+1)}\right\}_0^{T_0}$$

$$= \frac{2}{2\pi(n-1)}\left\{\sin\left[2\pi(n-1)\right] - \sin 0\right\} - \frac{2}{2\pi(n+1)}\left\{\sin\left[2\pi(n+1)\right] - \sin 0\right\}$$

$$= \frac{\sin\left[2\pi(n-1)\right]}{\pi(n-1)} - \frac{\sin\left[2\pi(n+1)\right]}{\pi(n+1)}$$

$$= \frac{(n+1)\sin\left[2\pi(n-1)\right] - (n-1)\sin\left[2\pi(n-1)\right]}{\pi(n-1)(n+1)} \quad \ldots(2)$$

For $n \ge 2$ the value of $b_n = 0$

But we have to obtain b_1 separately.

$$b_1 = \frac{2}{T_0}\int_0^{T_0} \sin(2\pi f_0 t)\cos(2\pi f_0 t)\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0} 2\sin^2(2\pi f_0 t)\,dt$$

But $\cos 2\theta = 1 - 2\sin^2\theta \therefore 2\sin^2\theta = 1 - \cos 2\theta$

$$\therefore b_1 = \frac{1}{T_0}\int_0^{T_0} (1 - \cos 4\pi f_0 t)\,dt$$

$$\therefore b_1 = \frac{1}{T_0}\int_0^{T_0} 1\,dt - \frac{1}{T_0}\int_0^{T_0} \cos(4\pi f_0 t)\,dt$$

$$\therefore b_1 = \frac{1}{T_0}\left[t\right]_0^{T_0} - \frac{1}{T_0}\frac{\left[\sin(4\pi f_0 t)\right]_0^{T_0}}{4\pi f_0}$$

$$= \frac{1}{T_0}\left[T_0 - 0\right] - \frac{1}{4\pi}\left[\sin 4\pi - \sin 0\right] = 1 - 0 = 1$$

We can express $x(t) = \sin\omega_0 t$ as follows :

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n\sin(n\omega_0 t)$$

$$x(t) = 0 + 0 + \sin\omega_0 t$$

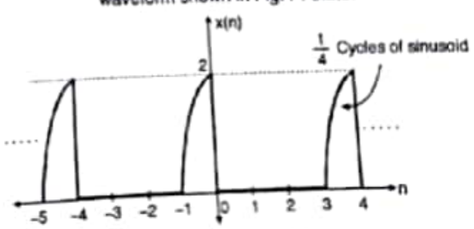
2. $x(t) = \cos\omega_0 t$:

Similar to the previous example, we can prove that,

$a_0 = 0, a_1 = 1, a_n = 0$ for $n \ge 2$

$b_n = 0$ for all n.

Ex. 5.2.9 : Find the trigonometric Fourier series for the waveform shown in Fig. P. 5.2.9.



(H-540) **Fig. P. 5.2.9**

Soln. :

Step 1 : Expression for x (t) :

From the given waveform, one cycle period, $T_0 = 4$ sec and Peak value = 2

Let us consider the cycle extending from $t = -1$ to $t = +3$

$\therefore\ x(t) = 2\sin(2\pi ft + 90^\circ)$

but $f = 1/T_0 = \frac{1}{4} = 0.25$ Hz.

$$\therefore\ x(t) = 2\sin(2\pi \times 0.25t + 90^\circ) = 2\cos\left(\frac{\pi}{2}t\right) \quad \ldots(1)$$

Step 2 : Find a_0 :

$$a_0 = \frac{1}{4}\int_{-1}^{0} 2\cos\left(\frac{\pi}{2}t\right)dt = \frac{2}{4}\times\frac{2}{\pi}\left[\sin\frac{\pi}{2}t\right]_{-1}^{0}$$

$$= \frac{1}{\pi}\left[\sin 0 - \sin\frac{\pi}{2}\times -1\right] = \frac{1}{\pi}\sin\frac{\pi}{2}$$

$$\therefore\ a_0 = \frac{1}{\pi}$$

Step 3 : Find a_n :

$$a_n = \frac{2}{4}\int_{-1}^{0} 2\cos\left(\frac{\pi t}{2}\right)\cos(n\,\omega_o t)\,dt$$

But $\omega_o = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \frac{\pi}{2}$

$$\therefore\ a_n = \frac{2\times 2}{4}\int_{-1}^{0}\cos\frac{\pi t}{2}\cos\frac{n\pi t}{2}\,dt$$

$$= \int_{-1}^{0}\cos\frac{n\pi t}{2}\cos\frac{\pi t}{2}\,dt$$

But $2\cos A\cos B = \cos(A-B) + \cos(A+B)$

$$\therefore\ a_n = \frac{1}{2}\int_{-1}^{0}\left\{\cos\left[\frac{\pi}{2}(n+1)t\right] + \cos\left[\frac{\pi}{2}(n-1)t\right]\right\}dt$$

$$a_n = \frac{1}{2}\int_{-1}^{0}\cos\left[\frac{\pi}{2}(n+1)t\right]dt + \frac{1}{2}\int_{-1}^{0}\cos\left[\frac{\pi}{2}(n-1)t\right]dt$$

$$= \frac{1}{2}\times\frac{2}{\pi(n+1)}\left[\sin\left[\frac{\pi}{2}(n+1)t\right]\right]_{-1}^{0}$$

$$+\frac{1}{2}\times\frac{2}{\pi(n-1)}\left[\sin\frac{\pi}{2}(n-1)t\right]_{-1}^{0}$$

$$= \frac{1}{\pi(n+1)}\left[\sin 0 - \sin\left(-\frac{\pi}{2}(n+1)\right)\right]$$

$$+\frac{1}{\pi(n-1)}\left[\sin 0 - \sin\left(-\frac{\pi}{2}(n-1)\right)\right]$$

$$\therefore\ a_n = \frac{1}{\pi(n+1)}\sin\frac{\pi}{2}(n+1) + \frac{1}{\pi(n-1)}\sin\frac{\pi}{2}(n-1) \quad \ldots(2)$$

For n even :

$\sin\frac{\pi}{2}(n+1) = +1$ and $\sin\frac{\pi}{2}(n-1) = +1$

$$\therefore\ a_n = \frac{1}{\pi(n+1)} + \frac{1}{\pi(n-1)} = \frac{n-1+n+1}{\pi(n^2-1)} = \frac{2n}{\pi(n^2-1)} \quad \ldots(3)$$

For n odd :

$\sin\frac{\pi}{2}(n+1) = 0$ and $\sin\frac{\pi}{2}(n-1) = 0$

$\therefore\ a_n = 0$ ... for n odd ...(4)

Step 4 : Find b_n :

$$b_n = \frac{2}{4}\int_{-1}^{0} 2\cos\left(\frac{\pi t}{2}\right)\sin(n\,\omega_o t)\,dt$$

But $\omega_o = \pi/2$

$$\therefore\ b_n = \frac{2\times 2}{4}\int_{-1}^{0}\cos\left(\frac{\pi t}{2}\right)\sin\left(\frac{n\pi t}{2}\right)dt$$

But $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$$\therefore\ b_n = \frac{1}{2}\int_{-1}^{0}\left\{\sin\left[(n+1)\frac{\pi}{2}t\right] + \sin\left((n+1)\frac{\pi}{2}t\right)\right\}dt$$

$$= \frac{1}{2}\int_{-1}^{0}\sin(n+1)\frac{\pi}{2}t\,dt + \frac{1}{2}\int_{-1}^{0}\sin(n-1)\frac{\pi}{2}t\,dt$$

$$\therefore\ b_n = \frac{1}{2}\times\frac{-2}{\pi(n+1)}\cos\left[\frac{\pi}{2}(n+1)t\right]_{-1}^{0}$$

$$-\frac{1}{2}\times\frac{2}{\pi(n-1)}\cos\left[\frac{\pi}{2}(n-1)\right]_{-1}^{0}$$

$$= \frac{-1}{\pi(n+1)}\left[\cos 0 - \cos\frac{-\pi}{2}(n+1)\right]$$

$$-\frac{1}{\pi(n-1)}\left[\cos 0 - \cos\frac{-\pi}{2}(n-1)\right]$$

$$= \frac{-1}{\pi(n+1)} - \frac{1}{\pi(n-1)} = \frac{-(n-1+n+1)}{\pi(n^2-1)}$$

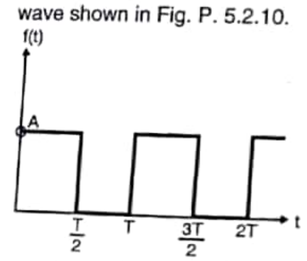
$$\therefore\ b_n = \frac{-2n}{\pi(n^2-1)} \quad \ldots(5)$$

Step 5 : Fourier series :

$$X(t) = \frac{1}{\pi} + \sum_{n=\text{even}}\frac{2n}{\pi(n^2-1)}\cos(n\,\omega_o t)$$

$$-\sum_{n=1}^{\infty}\frac{2n}{\pi(n^2-1)}\sin(n\,\omega_o t)$$

Ex. 5.2.10 : Find the trigonometric Fourier series for the wave shown in Fig. P. 5.2.10.



(H-544) **Fig. P. 5.2.10**

Soln. :

Step 1 : Find a_0 :

$$a_0 = \frac{1}{T}\int_{0}^{T/2} A\,dt = \frac{A}{T}[t]_0^{T/2} = \frac{A}{T}\times\frac{T}{2} = \frac{A}{2} \quad \ldots(1)$$

Step 2 : Find a_n and b_n :

$$a_n = \frac{2}{T}\int_{0}^{T/2} A\cos\left[\frac{2\pi nt}{T}\right]dt$$

$$= \frac{2}{T}A\times\frac{T}{2\pi n}\left[\sin\frac{2\pi nt}{T}\right]_0^{T/2}$$

$$\therefore\ a_n = \frac{A}{n\pi}\left[\sin\frac{2\pi nt}{2T} - \sin 0\right] = \frac{A}{n\pi}[\sin\pi n]$$

$\therefore\ a_n = 0$ ... [$\because \sin n\pi = 0$ for all n] ...(2)

$$\therefore\ b_n = \frac{2}{T}\int_{0}^{T/2} A\sin\left[\frac{2\pi nt}{T}\right]dt$$

$$= \frac{-2A}{T}\times\frac{T}{2\pi n}\left[\cos\frac{2\pi nt}{T}\right]_0^{T/2}$$

$$= \frac{-A}{n\pi}\left[\cos\frac{2\pi nT}{2T} - \cos 0\right] = \frac{-A}{n\pi}[\cos n\pi - 1]$$

For n even $\cos n\pi = 1$ $\quad\therefore\ b_n = 0$

For n odd $\cos n\pi = -1$ $\quad\therefore\ b_n = \frac{2A}{n\pi}$ ...(3)

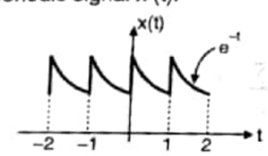
Step 3 : Write the Fourier series :

The trigonometric Fourier series for x (t) is

$$x(t) = a_0 + \sum_{n=\text{odd}} b_n\sin(n\,\omega_o t)$$

$$\therefore\ x(t) = \frac{A}{2} + \sum_{n\ \text{odd}}\frac{2A}{n\pi}(n\,\omega_o t) \quad \ldots \textbf{Ans.}$$

Ex. 5.2.11 : Find the exponential Fourier series and plot the magnitude and phase spectrum for the periodic signal x (t).



(H-1222) **Fig. P. 5.2.11**

Soln. : Exponential Fourier series is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$$

Where $C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\,e^{-j2\pi nt/T_0}\,dt$

The given signal is expressed as, $x(t) = e^{-t}, 0 < t < 1$

To obtain value of C_n :

$$C_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\,e^{-j2\pi nt/T_0}\,dt$$

Here $x(t) = e^{-t}$ for $0 < t < 1$ and $T_0 = 1$

$$\therefore\ C_n = \frac{1}{1}\int_0^1 e^{-t}\cdot e^{-j2\pi nt}\,dt = \int_0^1 e^{-(1+j2\pi n)t}\,dt$$

$$= \left[\frac{e^{-(1+j2\pi n)t}}{-(1+j2\pi n)}\right]_0^1 = \frac{1}{-(1+j2\pi n)}[e^{-(1+j2\pi n)} - e^0]$$

$$\therefore C_n = \frac{1}{(1+j2\pi n)}[1 - e^{-1}\cdot e^{j2\pi n}]$$

Here $e^{-1} = 0.632$ and

$e^{j2\pi n} = \cos 2\pi n + j\sin 2\pi n = 1$

$$\therefore\ C_n = \frac{0.632}{1+j2\pi n}$$

To obtain the Fourier series :

Putting the value of C_n in the equation of x (t) we get,

$$x(t) = \sum_{n=-\infty}^{\infty}\frac{0.632}{1+j2\pi n}e^{j2\pi nt/T_0}$$

Spectrum of signal :

Here $|C_n| = \frac{0.632}{\sqrt{1+(2\pi n)^2}}$

The phase spectrum is given by,

$$\phi_n = \tan^{-1}\left\{\frac{\text{Im}}{\text{Re}}\right\}$$

$$= \tan^{-1}\left\{\frac{\text{Im}}{\text{Re}}\right\}\text{for numerator} - \tan^{-1}\left\{\frac{\text{Im}}{\text{Re}}\right\}\text{for denominator}$$

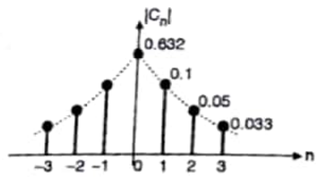
$$\therefore\ \phi_n = \tan^{-1}\{0\} - \tan^{-1}\{2\pi n\}$$

$$\therefore\ \phi_n = -\tan^{-1}\{2\pi n\}$$

The following table shows calculation of magnitude and phase

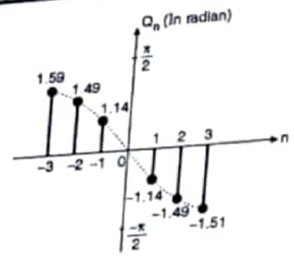
Value of n	$\lvert C_n \rvert$	ϕ_n (Radian)
– 3	0.033	1.51
– 2	0.05	1.49
– 1	0.1	1.14
0	0.632	0
1	0.1	– 1.14
2	0.05	– 1.49
3	0.033	– 1.51

The magnitude and phase spectrum are shown in Figs. P. 5.2.11(a) and P. 5.2.11(b) respectively.



(a)

Fig. 5.2.11(Contd...)



(b)

(H-1223) **Fig. 5.2.11**

Ex. 5.2.12 : Find trignometric fourier series coefficient of the following function. Also find its exponential series coefficient.

$x(t) = 4 + 2\cos(20\pi t) + \cos(40\pi) + 3\sin(20\pi t)$ **May 03, 5 Marks**

Soln. :

Given :

$$x(t) = 4 + 2\cos(20\pi t) + \cos(40\pi) + 3\sin(20\pi t)$$

We have,

$$\therefore X(\omega) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Here $a_0 = 2$

$$a_n = \frac{2}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)\cos n\omega_0 t\, dt$$

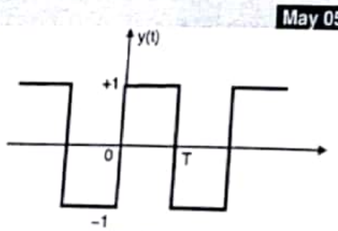
$$\text{and } b_n = \frac{2}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} x(t)\sin n\omega_0 t$$

Now exponential fourier series is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0}$$

$$\text{Here } C_n = \frac{1}{T_0}\int_{t}^{t+\tau_0} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

Ex. 5.2.13 : Find exponential Fourier series co-efficient of the signal shown Fig. P. 5.2.13. **May 05, 5 Marks**



(H-846) **Fig. P. 5.2.13**

Soln. : The signal y (t) can be represented as

$$y(t) = 1$$
$$\text{Period } T_0 = T.$$

The co-efficient of exponential series is given by,

$$C_n = \frac{1}{T}\int_{t}^{t+T_0} x(t)\cdot e^{-j2\pi nt/T_0} dt$$

Here $T_0 = T$

$$\therefore C_n = \frac{1}{T}\int_0^T 1\cdot e^{-j2\pi nt/T}\, dt$$

$$\therefore C_n = \frac{1}{T}\left[\frac{e^{-j2\pi nt/T}}{\frac{-j2\pi n}{T}}\right]_0^T$$

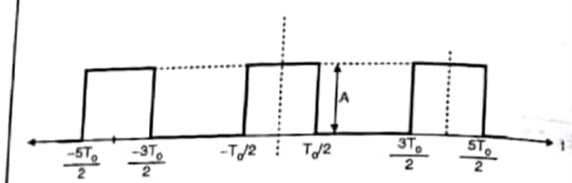
$$C_n = \frac{1}{T}\times\frac{T}{-j2\pi n}\left[e^{-j2\pi nt/T}\right]_0^T$$

$$= \frac{1}{-j2\pi n}\left[e^{-j2\pi nT/T} - e^{\frac{-j2\pi n0}{T}}\right]$$

$$\therefore C_n = \frac{1}{-j2\pi n}\left[e^{-j2\pi n} - 1\right]$$

Ex. 5.2.14 : Derive expression to find exponential Fourier series coefficients of a period square wave. **Dec. 03, 8 Marks**

Soln. : The exponential Fourier series is given as :



(H-848) **Fig. P. 5.2.14**

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi nt/T_0} \quad ...(1)$$

$$\text{Here } C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\, e^{-j2\pi nt/T_0}\, dt$$

$$\therefore C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} A\cdot e^{-j2\pi nt/T_0}\, dt$$

$$\therefore C_n = \frac{A}{T_0}\cdot\frac{1}{\frac{-j2\pi n}{T_0}}\left[e^{-j2\pi nt/T_0}\right]_{-T_0/2}^{T_0/2}$$

$$= \frac{A}{n\pi}\cdot\frac{1}{-2j}\left[e^{\frac{-j2\pi nt/T_0}{2T_0}} - e^{+j2\pi nT_0/2T_0}\right]$$

$$= \frac{A}{n\pi}\left[\frac{e^{+j\pi n} - e^{-j\pi n}}{2}\right]$$

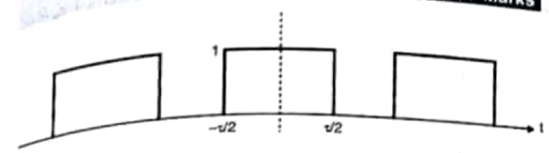
$$= \frac{A}{n\pi}\sin \pi n \quad \text{since } \sin x = \frac{e^{jx} - e^{-jx}}{2}$$

$$= A\frac{\sin \pi n}{n\pi} = A\cdot \text{sin c}(n\pi) \quad \text{since sin c}x = \frac{\sin(\pi\lambda)}{\pi\lambda}$$

Putting this value of C_n in Equation (1),

$$\therefore x(t) = \sum_{n=-\infty}^{\infty} A\,\text{sin c}(n\pi)\, e^{j2\pi nt/T_0}$$

Ex. 5.2.15 : Find the exponential Fourier series expansion for periodic gate function shown in Fig. P. 5.2.15. **May 04, 7 Marks**



(H-849) **Fig. P. 5.2.15**

Soln. :

The rectangular pulse shown in Fig. P. 5.2.15 can be expressed mathematically as,

$$\text{rect}(t/T) = 1 \quad \text{for } -T/2 \le t \le T/2$$
$$= 0 \quad \text{elsewhere}$$

This is also known as the gate function. Therefore the fourier transform will be,

$$F[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi ft}\, dt \text{ ...by definition of F.T.}$$

$$= \int_{-T/2}^{T/2} 1\, e^{-j2\pi ft}\, dt = \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right] = \frac{1}{j2\pi f}\left[e^{j\pi fT} - e^{-j\pi fT}\right]$$

$$= \frac{1}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right] \quad ...(1)$$

As per the Euler's theorem,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Applying this to Equation (1)

$$\text{we get, } F[x(t)] = \frac{1}{\pi f}[\sin(\pi fT)] \quad ...(2)$$

Multiply and divide the RHS of Equation (2) by T to get,

$$F[x(t)] = T\frac{\sin(\pi fT)}{\pi fT} \quad ...(3)$$

In the above equation,

$$T\frac{\sin(\pi fT)}{\pi fT} = \text{sinc}(fT)$$

$$\therefore F[x(t)] = T\,\text{sinc}(fT)$$

$$\therefore \text{rect}(t/T) \xleftrightarrow{F} T\,\text{sinc}(fT)$$

Thus the rectangular pulse transforms into a sinc function.

Amplitude spectrum : The amplitude spectrum of the rectangular function is as shown in Fig. P. 5.2.15(a).

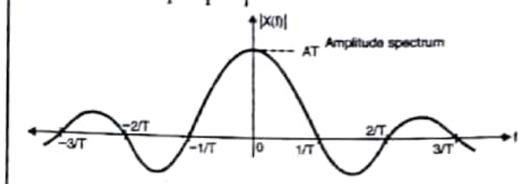
As we already know, sinc (0) = 1

$$\therefore T\,\text{sinc}(0) = T$$

The sinc function will have zero value for the following values of "fT":

$$\text{sinc}(fT) = 0 \quad \text{for } fT = \pm 1, \pm 2, \pm 3, ...$$

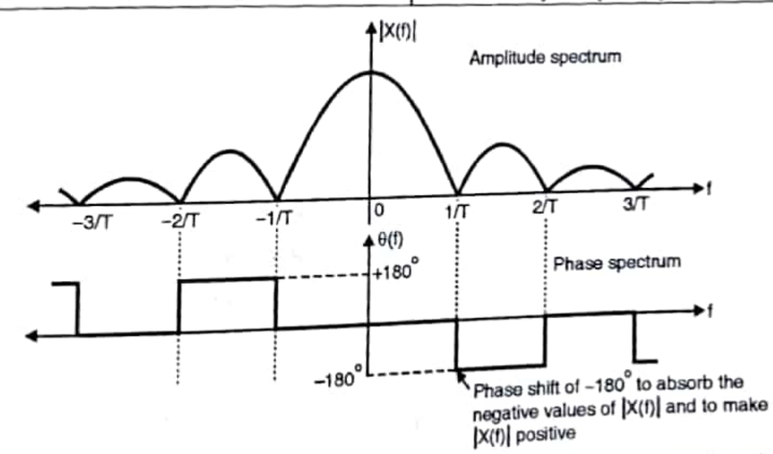
$$\text{i.e. for } f = \pm\frac{1}{T}, \pm\frac{2}{T}, \pm\frac{3}{T} .....$$



(H-850) **Fig. P. 5.2.15(a) : Amplitude spectrum of a rectangular pulse**

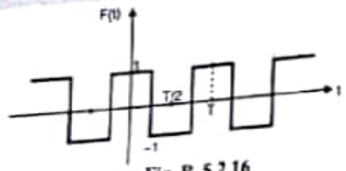
The phase spectrum has not been shown as it has zero value for all the values of f.

To absorb negative values of | X (f) | in the phase shift : The negative amplitude of the amplitude spectrum | X (f) | can be made positive by introducing a phase shift of ± 180° in the phase spectrum. This is as shown in Fig. P. 5.2.15(b). A negative phase shift for positive frequency and positive phase shift for the negative frequency is introduced in order to maintain symmetry of the phase spectrum.



(H-851) **Fig. P. 5.2.15(b) : Amplitude and phase spectrums for a rectangular pulse. Negative value of | X (f) | have been absorbed in the additional phase shift of ± 180° in the phase spectrum**

Ex. 5.2.16 : Find exponential Fourier series coefficient of the signal shown in Fig. P. 5.2.16. **May 05, 8 Marks**



(H-855) **Fig. P. 5.2.16**

Soln. :

The coefficient of exponential series is given by,

$$C_n = \frac{1}{T}\int_{t}^{t+T} x(t)\, e^{-j2\pi n t/T}\, dt$$

$$\therefore\ C_n = \frac{1}{T}\int_{-T/2}^{T/2} 1 \cdot e^{-j2\pi n t/T}\, dt$$

$$= \frac{1}{T}\cdot\frac{1}{\frac{-j2\pi n}{T}}\left[e^{-j2\pi n t/T}\right]_{-T/2}^{T/2}$$

$$= \frac{1}{n\pi - j2}\left[e^{-\frac{j2\pi n T}{2T}} - e^{j2\pi n T/2T}\right] = \frac{1}{n\pi}\left[\frac{e^{+j\pi n} - e^{-j\pi n}}{2j}\right]$$

$$= \frac{1}{n\pi}\sin \pi n \quad \text{...since } \sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$= \frac{\sin \pi n}{\pi n} = \text{sin c } n\pi \quad \text{...since sin c } x = \frac{\sin \pi \lambda}{\pi \lambda}$$

The exponential Fourier series is given by,

$$x(t) = \sum_{n=-\infty}^{\infty} \text{sin c}(n\pi)\, e^{j2\pi n t/T}$$

5.3 Properties of C.T. Fourier Series :

In this section we are going to discuss the following important properties of C.T. Fourier series :

1. Linearity
2. Time shifting
3. Time reversal
4. Time scaling
5. Multiplication
6. Conjugation and conjugate symmetry.
7. Parseval's relation.

Assumptions :

While discussing all the properties, we will have the following common assumptions.

1. x (t) is a periodic signal.
2. Let T_0 be the time period of x (t) and let the fundamental frequency be given by,

$$\omega_0 = 2\pi / T_0$$

3. Let the Fourier series coefficient of x(t) be denoted by a_n. Then the relation between x (t) and a_n is given by,

$$x(t) \xleftrightarrow{FS} a_n$$

5.3.1 Linearity :

- Let x (t) and y (t) denote two periodic signals both having a period T_0. Let their Fourier coefficients be denoted by a_n and b_n respectively.

That means,

$$x(t) \xleftrightarrow{FS} a_n$$

$$y(t) \xleftrightarrow{FS} b_n$$

- Then the linear combination z (t) or x(t) and y(t) will have Fourier coefficients C_n such that

$$z(t) \xleftrightarrow{FS} C_n$$

where $z(t) = Ax(t) + By(t)$

and $C_n = Aa_n + Bb_n$

That means,

$$z(t) = [Ax(t) + By(t)] \xleftrightarrow{FS} C_n = [Aa_n + Bb_n] \quad \text{...(5.3.1)}$$

Where A and B are arbitrary contents.

Proof :

Let x(t) and y(t) be expressed in terms of exponential Fourier series as

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t/T_0}$$

$$y(t) = \sum_{n=-\infty}^{\infty} b_n e^{j2\pi n t/T_0}$$

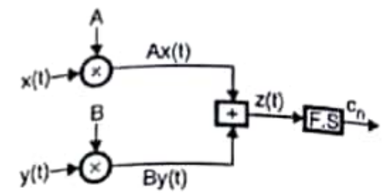
$$\therefore\ Z(t) = A\,x(t) + B\,y(t)$$

$$= A\sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t/T_0} + B\sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t/T_0}$$

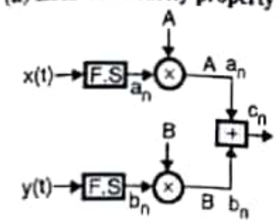
$$\therefore\ z(t) \xleftrightarrow{FS} C_n = Aa_n + B\,b_n \quad \text{...Proved.}$$

Graphical representation :

- The linearity property can be illustrated graphically as shown in Fig. 5.3.1.
- Fig. 5.3.1(a) shows the LHS. Here z(t) is first obtained as A x(t) + B y(t) and a F.S. of z(t) is obtained at the end.
- Fig. 5.3.1(b) shows the RHS of linearity property in which Fourier series of x(t) and y(t) is obtained first then the multiplication and addition takes place to obtain C_n.



(a) LHS of linearity property



(b) RHS of linearity property

(H-476) **Fig. 5.3.1 : Graphical illustration of linearity property**

Significance :

- This shows that the sequence in which operations carried out on a linear combination is not important.
- The linearity property can be extended easily to a combination of an arbitrary number of signals of period T_0.

5.3.2 Time Shifting :

- Let x(t) be a periodic signal with a period T_0 and Fourier coefficients a_n.

i.e. $x(t) \xleftrightarrow{F.S.} a_n$

- Then the time shifted signal $x(t - t_0)$ will have the Fourier coefficients of $e^{-jk\omega_0 t_0} a_n$. That means,

$$x(t - t_0) \xleftrightarrow{FS} .\ e^{-jk\omega_0 t_0} a_n \quad \text{...(5.3.2)}$$

- Note that when a time shift is applied to a periodic signal, the period T_0 of the signal should remain unchanged.

Proof :

- As per the definition of exponential Fourier series,

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t/T_0},$$

$$\text{Where } a_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\, e^{-j2\pi n t/T_0}\, dt \quad \text{...(5.3.3)}$$

- Similarly the Fourier series coefficients b_n of the time shifted signal $x(t - t_0)$ is given by,

$$b_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t - t_0)\, e^{-j2\pi n t/T_0}\, dt$$

- Let $(t - t_0) = \tau$

$$\therefore\ t = \tau + t_0$$

$$\therefore\ dt = d\tau$$

For $t = t,\ \tau = t - t_0$

For $t = t + T_0,\ \tau = t + T_0 - t_0$

$$\therefore\ b_n = \frac{1}{10}\int_{t-t_0}^{t-t_0+T_0} x(\tau)\cdot e^{-j2\pi n(\tau+t_0)/T_0}\, d\tau$$

$$= \frac{e^{-j2\pi n t_0/T_0}}{T_0}\int_{T_0} x(\tau)\cdot e^{-j2\pi n t}\, d\tau$$

$$b_n = e^{-j2\pi n t_0/T_0} a_n \quad \text{...Proved.}$$

Significance :

When a periodic signal is time shifted, then the magnitude of its Fourier series coefficients remain unchanged

i.e. $|a_n| = |b_n|$

5.3.3 Time Reversal :

This property states that if

$$x(t) \xleftrightarrow{FS} a_n$$

Then the time reversed version of x(t) i.e. x (– t) forms a Fourier series pair with a_{-n}

$$x(-t) \longleftrightarrow a_{-n}$$

Significance :

- The meaning of time reversal is that, the time reversal applied to C.T. signal results in the time reversal of the corresponding sequence of Fourier series coefficients.
- If x(t) is even, then the Fourier series coefficients will also be even. That means, if $x(t) = x(-t)$, then $a_n = a_{-n}$.
- Similarly if x(t) is odd, then a_n will also be odd. That means if $x(t) = -x(-t)$ then $a_n = -a_{-n}$.

Proof :

$$x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t/T_0} \quad \text{...Definition of Fourier series}$$

Replace t by – t to get

$$x(-t) = \sum_{n=-\infty}^{\infty} a_n e^{-j2\pi n t/T_0} \quad \text{Substitute } n = -m \text{ to get}$$

$$x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{j2\pi m t/T_0} \quad \text{...(5.3.4)}$$

This expression shows that the Fourier coefficients of a time reversed signal x (– t) are a_{-m}.

5.3.4 Time Scaling :

- x(t) is the original signal having a time period of T_0 and a fundamental frequency of $\omega_0 = 2\pi/T_0$.
- Then x(αt) represents its time scaled version. α is a positive real number. The time scaled signal x(αt) is periodic with a time period T_0/α and with fundamental frequency of $\alpha\,\omega_0$.

- Then the Fourier series of x (α t) is given by,

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n \alpha t/T_0}$$

Proof :

$$\text{Let } x(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t/T_0} \quad \text{...Definition}$$

Substitute t = α t to get,

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n \alpha t/T_0} \quad \text{...Proved.}$$

Conclusion (Significance) :

- Expand the expression for x (α t) as follows :

$$x(\alpha t) = \ldots a_{-1} e^{-j2\pi\alpha t/T_0} + \underbrace{a_0}_{\text{DC}} + \underbrace{a_1 e^{j2\pi\alpha t/T_0}}_{\text{Fundamental}} + \underbrace{a_2 e^{j4\pi\alpha t/T_0}}_{\text{Second harmonic}} + \ldots \quad \text{...(5.3.5)}$$

(H-1293)

- This shows that the Fourier coefficients remain unchanged, but time scaling of each harmonic component has taken place.

5.3.5 Frequency Shifting :

Statement :

$$\text{If } x(t) \overset{Fs}{\leftrightarrow} C_{xn}$$

$$\text{Then } x(t) \cdot e^{j2\pi n_0 t/T_0} = y(t) \overset{Fs}{\leftrightarrow} C_{yn} = C_{x(n-n_0)}$$

Meaning : Shifting of frequency component by 'n_0' positions is equivalent to multiplying signal x(t) by $e^{j2\pi n_0 t/T_0}$

Proof : According to the definition of C_n,

$$C_{yn} = \frac{1}{T_0} \int_{t}^{t+T_0} y(t)\, e^{-j2\pi n t/T_0}\, dt$$

$$\text{But } y(t) = x(t) \cdot e^{j2\pi n_0 t/T_0}$$

$$\therefore C_{yn} = \frac{1}{T_0} \int_{t}^{t+T_0} x(t)\, e^{j2\pi n_0 t/T_0} e^{-j2\pi n t/T_0}\, dt$$

$$= \frac{1}{T_0} \int_{t}^{t+T_0} x(t)\, e^{-j2\pi (n-n_0) t/T_0}\, dt$$

$$\therefore C_{yn} = C_{x(n-n_0)}$$

5.3.6 Differentiation :

Statement : If $x(t) \overset{Fs}{\leftrightarrow} C_n$

$$\text{then } \frac{d}{dt} x(t) \overset{Fs}{\leftrightarrow} j\frac{2\pi n}{T_0} \cdot C_n$$

Significance : Differentiation in time domain is equivalent to multiplying its Fourier coefficient by $\frac{j2\pi n}{T_0}$

Proof : According to the definition of Fourier series.

$$x(t) = \sum_{n=1}^{\infty} C_n \cdot e^{j2\pi n t/T_0}$$

Taking derivative with respect to time,

$$\frac{d}{dt} x(t) = \sum_{n=1}^{\infty} C_n \cdot \frac{d}{dt} [e^{j2\pi n t/T_0}]$$

$$= \sum_{n=1}^{\infty} C_n \cdot e^{j2\pi n t/T_0} \frac{j2\pi n}{T_0}$$

$$\frac{d}{dt} x(t) = \sum_{n=1}^{\infty} \left[\frac{j2\pi n}{T_0} C_n\right] \cdot e^{j2\pi n t/T_0}$$

Comparing it with the definition of Fourier series,

$$\frac{d}{dt} x(t) \overset{Fs}{\leftrightarrow} \frac{j2\pi n}{T_0} \cdot C_n$$

5.4 Applications of Fourier Series :

Some of the important applications of Fourier series are as follows :

1. We can approximate almost any kind of wave using the Fourier series.
2. It can be used in the compression technique such as MP3.
3. It is used in signal processing.
4. It is used in the approximation theory.
5. In the control theory.
6. In partial differential equation.
7. For comparative analysis of DNA sequences.
8. In the communication system.
9. In order to decompose any periodic waveform into weighted sum of simple sine waves.
10. Musical synthesizers.

5.5 Gibb's Phenomenon :

MU : May 04, Dec. 04, Dec. 05, Dec. 06, Dec. 14

University Questions

Q. 1 Explain Gibb's phenomenon. **(May 04, 4 Marks, Dec. 04, 5 Marks, Dec. 05, Dec. 06, 6 Marks)**

Q. 2 Explain Gibb's phenomenon. Also explain conditions necessary for the convergence of Fourier series. **(Dec. 14, 5 Marks)**

- For the periodic signal x(t), we may write the Fourier series as :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \quad \text{...exponential Fourier series.} \quad \text{...(5.5.1)}$$

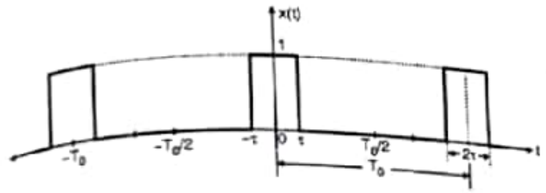
$$\text{where, } C_n = \frac{1}{T_0} \int x(t)\, e^{-jn\omega_0 t}\, dt \quad \text{...(5.5.2)}$$

- It is said that x (t) and C_n form a Fourier series pair. Their relationship is denoted as follows :

$$x(t) \overset{F.S;\ \omega_0}{\longleftrightarrow} C_n$$

- From the Fourier coefficients C_n we can determine x (t) and from x (t) we can find the value of C_n.
- Now consider the following example.

Ex 5.5.1 : Obtain the exponential Fourier series for the rectangular pulse train shown in Fig. P. 5.5.1 and sketch the spectrum.



(H-548) **Fig. P. 5.5.1 : A rectangular pulse train**

Soln. :

From the Equation (5.5.1), the exponential Fourier series is given as,

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n t/T_0} \quad \text{...(1)}$$

$$\text{where, } C_n = \frac{1}{T_0} \int_{t}^{t+T_0} x(t) \cdot e^{-j2\pi n t/T_0}\, dt \quad \text{...(2)}$$

Let us consider one cycle of x (t) extending from $t = \frac{-T_0}{2}$ to $t = \frac{T_0}{2}$.

1. Value of C_n :

$$C_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) \cdot e^{-j2\pi n t/T_0}\, dt$$

But x (t) = 1, from t = – τ/2 to t = τ/2

$$\therefore C_n = \frac{1}{T_0} \int_{-\tau}^{\tau} 1 \cdot e^{-j2\pi n t/T_0}\, dt$$

$$= \frac{-1}{j\frac{2\pi n}{T_0} \times T_0} \left[e^{-j2\pi n t/T_0}\right]_{-\tau}^{\tau}$$

$$= \frac{-1}{j2\pi n} \left[e^{-j2\pi n \tau/T_0} - e^{j2\pi n \tau/T_0}\right]$$

$$= \frac{1}{\pi n}\left[\frac{e^{j2\pi n \tau/T_0} - e^{-j2\pi n \tau/T_0}}{2j}\right]$$

$$\therefore C_n = \frac{1}{\pi n} \sin\left(\frac{2\pi n \tau}{T_0}\right) \quad \text{...(3)}$$

Multiply and divide the RHS of Equation (3) by (2τ/T_0) we get,

$$C_n = \frac{2(\tau/T_0)}{(2\pi n\tau/T_0)} \sin(2\pi n\tau/T_0)$$

Substitute $1/T_0 = f_0$, to get,

$$C_n = \frac{2\tau f_0}{2\pi n \tau f_0} \sin(2\pi n \tau f_0)$$

Substitute $2\pi f_0 = \omega_0$, to get,

$$C_n = \frac{2\tau f_0}{n\tau\omega_0} \sin(n\tau\omega_0)\ n \neq 0.$$

$$\therefore C_n = \frac{2 f_0}{n\omega_0} \sin(n\tau\omega_0)$$

But, $f_0 = 1/T_0$

$$\therefore C_n = \frac{2}{n T_0 \omega_0} \sin(n\tau\omega_0) n \neq 0 \quad \text{...(4)}$$

For n = 0, we get,

$$C_n = \frac{1}{T_0} \int_{-\tau}^{\tau} dt = \frac{1}{T_0} [t]_{-\tau}^{\tau} = \frac{2\tau}{T_0} \quad \text{...(5)}$$

- Using L'Hospital's rule, it can be shown that,

$$\lim_{n \to 0} \frac{2\sin(n\tau\omega_0)}{n T_0 \omega_0} = \frac{2\tau}{T_0}$$

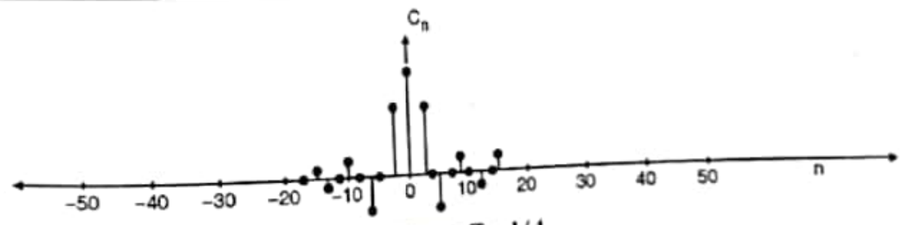
- Thus Equation (5) becomes valid for all the values of n.

$$\therefore C_n = \frac{2\sin(n\tau\omega_0)}{n T_0 \omega_0}$$

- Now substitute $\omega_0 = 2\pi/T_0$ to get C_n as a function of the ratio τ/T_0.

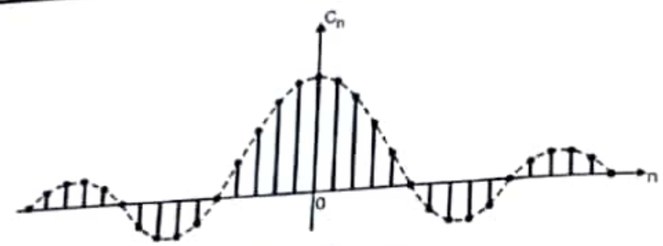
$$\therefore C_n = \frac{2\sin(2\pi n\tau/T_0)}{n T_0 \times 2\pi/T_0} = \frac{2\sin\left(n\frac{2\pi\tau}{T_0}\right)}{n 2\pi} \quad \text{...(6)}$$

- Fig. P. 5.5.1(a) shows the plot of C_n for – 50 ≤ n ≤ + 50 for τ/T_0 = 1/4 and τ/T_0 = 1/16. From Fig. P. 5.5.1(b), it is evident that as τ/T_0 decreases, C_n becomes more concentrated.



(a) For τ/T_0 = 1 / 4

Fig. P. 5.5.1(Contd...)



(b) For $\tau/T_0 = 1/16$

(H-1273) **Fig. P. 5.5.1 : The FS coefficient C_n for $-50 \le n \le +50$**

- We know that $\text{sinc}(u) = \dfrac{\sin(\pi u)}{\pi u}$
- So multiply and divide Equation (6) by τ/T_0 to get,

$$C_n = \frac{2\tau/T_0 \sin[\pi\, 2n\, \tau/T_0]}{(\pi\, 2n\tau/T_0)}$$

$$= \frac{2\tau}{T_0}\,\text{sinc}\left[\frac{2n\tau}{T_0}\right] \qquad \ldots(7)$$

- We can represent x (t) in the Fourier series form as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= C_0 + \sum_{m=1}^{\infty} C_m e^{jm\omega_0 t} + C_{-m} e^{-jm\omega_0 t}$$

$$= C_0 + \sum_{m=1}^{\infty} 2C_m \frac{e^{jm\omega_0 t} + e^{-jm\omega_0 t}}{2}$$

$$= C_0 + \sum_{m=1}^{\infty} 2C_m \cos(m\omega_0 t) \qquad \ldots(8)$$

- If we define $B_0 = C_0$ and $B_n = 2C_m$, $n \ne 0$ then

$$x(t) = \sum_{n=0}^{\infty} B_n \cos(n\omega_0 t)$$

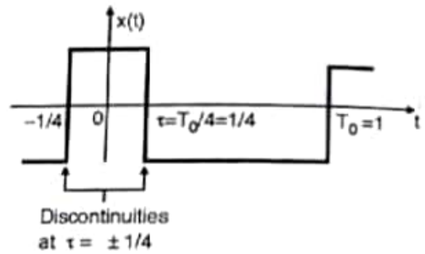
- Now we will define the partial sum approximation to the Fourier series representation for the square wave as follows :

$$\hat{x}_J(t) = \sum_{n=0}^{J} B_n \cos(n\omega_0 t) \qquad \ldots(9)$$

- Now assume that $T_0 = 1$ and $\tau/T_0 = 1/4$. Then,

$$B_n = \begin{cases} 1/2 & \text{for } n = 0 \\ \dfrac{2(-1)^{(n-1)/2}}{n\pi}, & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases}$$

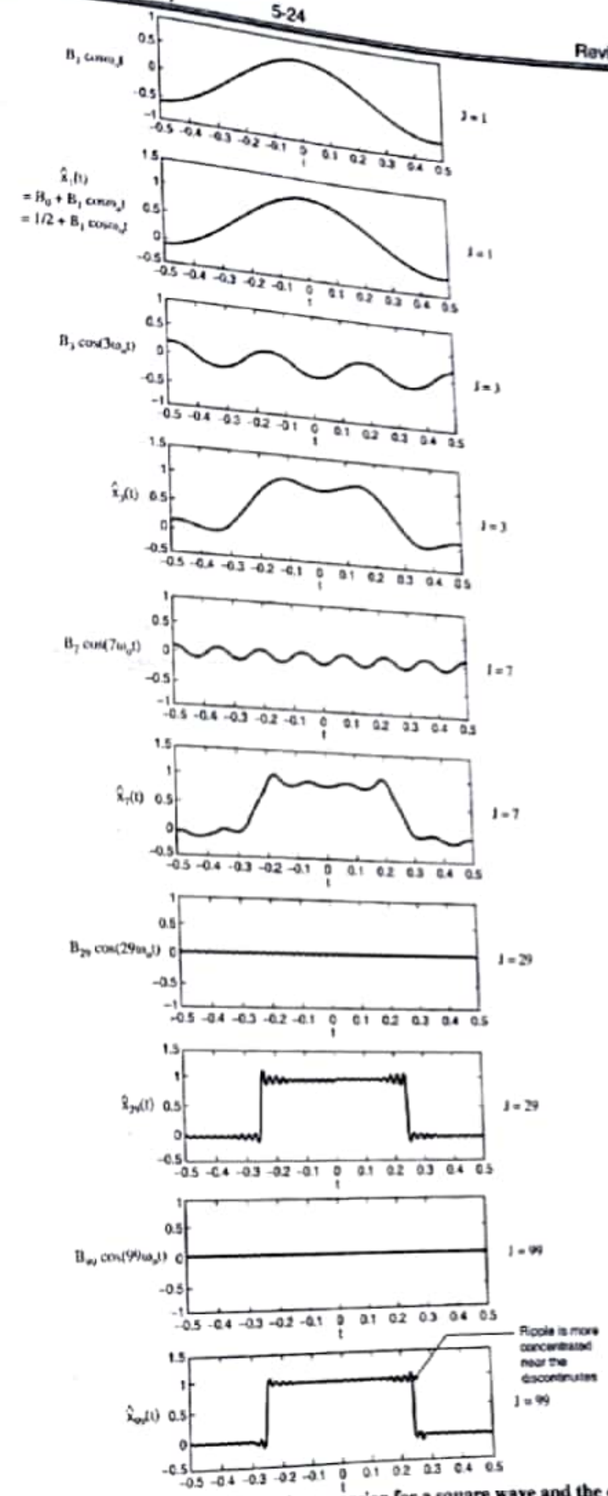
- Let us now find the waveforms for $\hat{x}_J(t)$ for J = 1, 3, 7, 29 and 99. Fig. P. 5.5.1(c) shows the partial sums and individual terms.
- Note that the behaviour of the partial sum approximation near the square wave discontinuities at $t = \pm 1/4$ (see Fig. P. 5.5.1(c)) is of particular interest.
- Note that each partial sum approximation passes through the average value (i.e. 1 / 2) of the discontinuity.
- On each side of discontinuity, the partial sum approximation exhibits ripple.
- As J increases, there is no change in the maximum value of the ripple. In fact, it can be proved that for any value of J the value of maximum ripple is only 9% of the discontinuity.



(H-1274) **Fig. P. 5.5.1(c)**

Gibbs phenomenon :

- The ripple near discontinuities in the partial sum Fourier series approximations is called as the Gibbs phenomenon, in the name of the physicist J. Willard Gibbs because he explained this phenomenon.
- With increase in the value of J, the ripple in the partial sum approximations becomes more and concentrated near the discontinuities.
- Therefore for any value of J, the accuracy of the partial sum approximation becomes better as we move away from the instant of discontinuities.



(H-1275) **Fig. P. 5.5.1(d) : Individual terms in Fourier series expansion for a square wave and the corresponding partial sum approximations. The J = 0 term is $\hat{x}_0(t) = 1/2$ and it is not shown.**

5.6 Fourier Series for Discrete Time Periodic Signals (DTFS) :

Consider x (n) is a discrete time periodic signal i.e.

$$x(n) = x(n+N) \quad \text{... for all n.}$$

The FS representation for x (n) consists of N harmonically related exponential function.

$$e^{j2\pi kn/N} \quad ...k = 0, 1, ...N-1$$

$$\therefore \quad x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N}$$

where C_k = Coefficients in series expansion.

The equation for fourier coefficient is,

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...k = 0, ... N-1$$

This equation is called analysis equation

∴ Synthesis equation,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} \quad ...(5.6.1)$$

Analysis equation

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad ...k = 0, 1... N-1 \quad ...(5.6.2)$$

Equation (5.6.1) provide discrete time fourier series. C_k provides amplitude and phase associated with frequency component of x (n) in the frequency domain.

5.6.1 Periodicity Condition :

The FS coefficients, C_k also satisfy periodicity condition,

$$\therefore \quad C_{k+N} = C_k$$

Proof :

$$C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi(k+N)n}{N}}$$

$$\therefore \quad C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \cdot e^{j2\pi kn/N} \cdot e^{-j2\pi n}$$

$$e^{-j2\pi k} = \cos 2\pi K - j \sin 2\pi K = 1 \text{ always}$$

$$\therefore \quad C_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{j2\pi kn/N}$$

$$\therefore \quad C_{k+N} = C_k$$

Note that fourier series is possible if f = k/N is rational number.

e.g. If $x(n) = \cos\sqrt{2}\pi n$

then $\omega = \sqrt{2}\pi$

$\therefore \quad 2\pi f = \sqrt{2}\pi$

$$\therefore \quad f = \frac{1}{\sqrt{2}}$$

∴ f is not rational number, the signal is not periodic.

∴ signal cannot be expanded in fourier series.

5.6.2 Properties of DTFS :

The properties of DTFS are summerized in Table 5.6.1.

Table 5.6.1

Properties of Discrete time fourier series		
Property	**Periodic signal**	**Fourier series coefficients**
	x (n) periodic with period N and y (n) fundamental frequency $\omega_0 = 2\pi/N$	a_k b_k periodic with period N
Linearity	A x (n) + B y (n)	$A a_k + B b_k$
Time shifting	$x(n - n_0)$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency shifting	$e^{jm(2\pi/N)n} x(n)$	$X(k - m)$
Conjugation	$x^*(n)$	a^*_{-k}
Time reversal	$x(-n)$	a_{-k}

5.7 Properties of DTFS :

5.7.1 Linearity :

Statement :

If $x_1(n) \xleftrightarrow{DTFS} X_1(k)$

and $x_2(n) \xleftrightarrow{DTFS} X_2(k)$ then,

$$A x_1(n) + B x_2(n) \xleftrightarrow{DTFS} AX_1(k) + B X_2(k)$$

Proof

According to the definition of DTFS,

$$X(k) \text{ or } C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}} \quad ...k = 0 \text{ to } N-1$$

$$\therefore \text{ DTFS } [Ax_1(n) + B x_2(n)]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} [Ax_1(n) + B x_2(n)] e^{-\frac{j2\pi kn}{N}}$$

$$= A \cdot \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) e^{-\frac{j2\pi kn}{N}} + B \cdot \frac{1}{N}\sum_{n=0}^{N-1} x_2(n) e^{-\frac{j2\pi kn}{N}}$$

$$= AX_1(k) + BX_2(k)$$

Meaning :

DFS of linear combination of two or more sequence is equal to sum of linear combination of DFS of individual signal.

5.7.2 Time Shifting :

Statement :

If $x(n) \xleftrightarrow{DTFS} X(k)$ then,

$$x(n-m) \xleftrightarrow{DTFS} e^{-\frac{j2\pi km}{N}} X(k)$$

Proof :

According to the definition of inverse DTFS,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{\frac{j2\pi kn}{N}} \text{ or } \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi kn}{N}} \quad ...(5.7.1)$$

$$\therefore \text{ IDTFS}\left\{e^{-\frac{j2\pi km}{N}} \cdot X(k)\right\} = \sum_{k=0}^{N-1} X(k) e^{-\frac{j2\pi km}{N}} \cdot e^{\frac{j2\pi kn}{N}}$$

$$= \sum_{k=0}^{N-1} X(k) e^{\frac{j2\pi k(n-m)}{N}} \quad ...(5.7.2)$$

Comparing Equations (5.7.1) and (5.7.2) we can write,

$$\text{IDTFS } \{x(n-m)\} = e^{-\frac{j2\pi km}{N}} \cdot Y(k)$$

Meaning :

Shifting of sequence in time domain is equivalent to multiplying its Fourier series by $e^{-\frac{j2\pi km}{N}}$

5.7.3 Frequency Shift :

Statement :

If $x(n) \xleftrightarrow{DTFS} X(k)$ then,

$$e^{jk_0 \cdot \frac{2\pi kn}{N}} x(k) \longleftrightarrow X(k - k_0)$$

Proof :

According to definition of DTFS,

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}}$$

$$\therefore \text{ DTFS}\left\{e^{jk_0 \cdot \frac{2\pi kn}{N}} \cdot x(n)\right\}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} e^{jk_0 \cdot \frac{2\pi kn}{N}} \cdot x(n) e^{-\frac{j2\pi kn}{N}}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-j(k-k_0) \cdot \frac{2\pi n}{N}}$$

$$= X(k - k_0)$$

Meaning :

Multiplying the sequence in time domain by $e^{jk_0 \cdot \frac{2\pi kn}{N}}$ is equivalent is shifting in frequency domain by k_0.

5.7.4 Convolution :

Statement :

If $x(n) \xleftrightarrow{DTFS} X(k)$ and $h(n) \xleftrightarrow{DTFS} H(k)$ then,

$$x(n) * h(n) \xleftrightarrow{DTFS} N \cdot X(k) \cdot H(k)$$

Here '*' denotes convolution.

Proof :

According to the definition of DTFS,

$$X(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) e^{-\frac{j2\pi kn}{N}}$$

$$\therefore \text{ DTFS } [x(n) * h(n)] = \frac{1}{N}\sum_{n=0}^{N-1} [x(n) * h(n)] e^{-\frac{j2\pi kn}{N}}$$

Using the formula of convolution we get,

$$\text{DTFS } [x(n) * h(n)] = \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(l) h(n-l)\right] \cdot e^{-\frac{j2\pi kn}{N}}$$

Put $n - l = m \quad \therefore n = m + l$

$$\therefore \text{DTFS } [x(n) * h(n)] = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=-\infty}^{\infty} x(l) h(m) e^{-\frac{j2\pi k}{N}(m+l)}$$

$$= \frac{1}{N}\left[\sum_{n=0}^{N-1} x(l) \sum_{m=-\infty}^{\infty} h(m) e^{-\frac{j2\pi km}{N}} \cdot e^{-\frac{j2\pi kl}{N}}\right]$$

$$= \frac{1}{N}\left[\sum_{n=0}^{N-1} x(l) \cdot e^{-\frac{j2\pi kl}{N}} \cdot \sum_{m=-\infty}^{\infty} h(m) e^{-\frac{j2\pi km}{N}}\right]$$

Using definition of DTFS, for first summation we can write N · X (k) and if we perform the summation for n = 0 to N − 1, in case of second summation then it is N · H (k).

$$\therefore \text{ DTFS } [x(n) * h(n)] = \frac{1}{N}[N \cdot X(k) \cdot N \cdot H(k)]$$

$$\therefore \text{ DTFS } [x(n) * h(n)] = N X(k) \cdot H(k)$$

Meaning :

Convolution in time domain is equivalent to multiplication in Fourier domain.

5.7.5 Modulation or Multiplication :

Statement :

If $x(n) \xleftrightarrow{DTFS} X(k)$, $h(n) \xleftrightarrow{DTFS} H(k)$ then,

$$x(n) \cdot h(n) \xleftrightarrow{DTFS} X(k) * H(k).$$

Proof :

Using definition of DTFS,

$$\text{DTFS } [x(n) \cdot h(n)] = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \cdot h(n) \cdot e^{-\frac{j2\pi kn}{N}} \quad ...(5.7.3)$$

Using the definition of inverse DTFS,

$$x(n) = \sum_{k=0}^{N-1} x(l) e^{\frac{j2\pi kl}{N}} \quad ...(5.7.4)$$

Putting Equation (5.7.4) in Equation (5.7.3) we get,

$$\text{DTFS}\{x(n)\cdot h(n)\} = \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1} x(l)\, e^{-\frac{j2\pi kl}{N}}\right]\cdot e^{-\frac{j2\pi kn}{N}}\, h(n)$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x(l)\cdot \sum_{k=0}^{N-1} h(n)\, e^{-j(n-l)\cdot\frac{2\pi k}{N}}$$

The second summation and $\frac{1}{N}$ indicates DTFS of $h(n-l)$.

$$\therefore \text{DTFS}\{x(n)\cdot h(n)\} = \sum_{n=0}^{N-1} x(l)\, H(n-l)$$

The RHS term is equivalent to convolution.

$$\therefore \text{DTFS}\{x(n)\cdot h(n)\} = X(k) * H(k)$$

Meaning : The multiplication of two signals in time domain is equivalent to convolution of DTFS of two signals.

5.7.6 Parseval's Theorem :

Statement :

If $x(n) \xleftrightarrow{\text{DTFS}} X(k)$ then the average power of periodic signal $x(n)$ is given by,

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |x(k)|^2$$

Proof : The average power of signal is given by,

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

Now $|x(n)|^2$ can be expressed as, $|X(n)|^2 = x(n)\cdot x^*(n)$

$$\therefore P = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cdot x^*(n) \qquad ...(5.7.5)$$

According to the definition of IDTFS,

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$$

$$x^*(n) = \sum_{k=0}^{N-1} X^*(k)\, e^{-\frac{j2\pi kn}{N}} \qquad ...(5.7.6)$$

Putting Equation (5.7.6) in Equation (5.7.5) we get,

$$P = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\cdot \sum_{k=0}^{N-1} X^*(k)\, e^{-\frac{j2\pi kn}{N}}$$

$$= \sum_{k=0}^{N-1} X^*(k)\cdot \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}} \qquad ...(5.7.7)$$

Now the term $\frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-\frac{j2\pi kn}{N}}$ indicates DTFS of $x(n)$. It is denoted by $X(k)$.

$$\therefore P = \sum_{k=0}^{N-1} X^*(k)\cdot X(k)$$

$$\therefore P = \sum_{k=0}^{N-1} |X(k)|^2$$

Meaning :

The term $|X(k)|^2$ indicates the distribution of power as a function of frequency.

5.7.7 Duality :

Statement :

If $x(n) \xleftrightarrow{\text{DTFS}} X(k)$ then,

$X(n) \xleftrightarrow{\text{DTFS}} \frac{1}{N}x(-k)$

Proof :

Using definition of IDTFS,

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$$

Replace 'n' by '– n'

$$\therefore x(-n) = \sum_{k=0}^{N-1} X(k)\, e^{-\frac{j2\pi kn}{N}}$$

Multiplying both sides by $\frac{1}{N}$.

$$\therefore \frac{1}{N}x(-n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{-\frac{j2\pi kn}{N}}$$

Interchange variables 'n' and 'k' we get,

$$\frac{1}{N}x(-k) = \frac{1}{N}\sum_{n=0}^{N-1} X(n)\, e^{-\frac{j2\pi kn}{N}}$$

$$\therefore \text{DTFS}\{X(n)\} = \frac{1}{N}x(-k).$$

Meaning :

The shape of signals in time and frequency domain can be interchanged.

5.7.8 Symmetry :

If $x(n)$ is real then $X(-k) = X^*(k)$

Consider that even part of $x(n)$ is $x_e(n)$ and odd part of $x(n)$ is $x_o(n)$. That means,

$x(n) = x_e(n) + x_o(n)$ then,

$x_e(n) \xleftrightarrow{\text{DTFS}} \text{Re}\{X(k)\}$

and $x_o(n) \xleftrightarrow{\text{DTFS}} \text{Im}\{X(k)\}$

Thus if $x(n)$ is real and even then its Fourier is real and if $x(n)$ is real and odd then its Fourier is imaginary.

5.7.9 Summary of Properties of DTFS :

Sr. No.	Name of property	Time domain equation	Fourier domain equation
1.	Linearity	$A\,x_1(n) + B\,x_2(n)$	$A\,X_1(k) + B\,X_2(k)$
2.	Time shifting	$x(n-m)$	$e^{-\frac{j2\pi km}{N}} X(k)$
3.	Frequency shift	$e^{jk_0\cdot\frac{2\pi kn}{N}} x(n)$	$X(k-k_0)$
4.	Convolution	$x(n) * h(n)$	$N\,X(k)\cdot H(k)$
5.	Modulation	$x(n)\cdot h(n)$	$X(k) * H(k)$
6.	Parseval's theorem	$\frac{1}{N}\sum_{n=0}^{N-1} \lvert x(n)\rvert^2$	$\sum_{k=0}^{N-1} \lvert X(k)\rvert^2$
7.	Duality	$X(n)$	$\frac{1}{N}x(-k)$
8.	Symmetry	$x_e(n)$ and $x_o(n)$	$\text{Re}\{X(k)\}$ $\text{Im}\{X(k)\}$

5.8 Solved Examples on DTFS :

Ex. 5.8.1 : Determine the spectra of periodic signal $x(n) = \{1, 1, 1, 0\}$ with period $N = 4$.

Soln. :

We have to determine the amplitude and phase spectra of given periodic sequence. This information is provided by coefficient C_k. We have,

$$C_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N} \;...\; k = 0, 1, ...N-1 \qquad ...(1)$$

Here period of sequence = N = 4

$$\therefore C_k = \frac{1}{4}\sum_{n=0}^{3} x(n)\, e^{-j2\pi kn/N} \;....\; k = 0, 1, 2, 3 \qquad ...(2)$$

Expanding the summation we get,

$$C_k = \frac{1}{4}\left[x(0)\,e^0 + x(1)\,e^{-j2\pi k/4} + x(2)\,e^{-j4\pi k/4} + x(3)\,e^{-j6\pi k/4}\right]$$

Now given sequence is,

$x(n) = \{1, 1, 1, 0\}$

$\therefore x(0) = 1, x(1) = 1, x(2) = 1$ and $x(3) = 0$

$$\therefore C_k = \frac{1}{4}\left[1 + 1\,e^{-j\pi k/2} + e^{-j\pi k} + e^{-j3\pi k/2}\right] \qquad ...(3)$$

We know that range of k is k = 0 to 3.

For $k = 0 : C_0 = \frac{1}{4}[1 + e^0 + e^0 + 0]$

$$\therefore C_0 = \frac{1}{4}[1 + 1 + 1]$$

$\therefore C_0 = 3/4$ That means magnitude $= \frac{3}{4}$ and phase = 0

For $k = 1 : C_1 = \frac{1}{4}[1 + e^{-j\pi/2} + e^{-j\pi}]$

Using identity, $e^{-j\theta} = \cos\theta - j\sin\theta$ we get,

$$\therefore C_1 = \frac{1}{4}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi\right]$$

$$\therefore C_1 = \frac{1}{4}[1 + 0 - 1j + (-1) - 0]$$

Note : The calculations are done in radian mode.

$$\therefore C_1 = \frac{1}{4}[-1j] = 0 - \frac{1}{4}j$$

Converting into polar form we get,

$C_1 = 0.25 \angle -1.57$

That means magnitude = 0.25 and angle = – 1.57 radians $= \frac{-\pi}{2}$.

For $k = 2 : C_2 = \frac{1}{4}[1 + 1\cdot e^{-j\pi} + 1\cdot e^{-j2\pi}]$

$$\therefore C_2 = \frac{1}{4}[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi]$$

$$\therefore C_2 = \frac{1}{4}[1 - 1 - 0 + 1 + 0] = \frac{1}{4}$$

That means magnitude is $\frac{1}{4}$ and angle = 0

For $k = 3 : \therefore C_3 = \frac{1}{4}[1 + e^{-j3\pi/2} + 1\cdot e^{-j3\pi}]$

$$C_3 = \frac{1}{4}\left[1 + \cos\left(\frac{3\pi}{2}\right) - j\sin\left(\frac{3\pi}{2}\right) + \cos(3\pi) - j\sin(3\pi)\right]$$

$$\therefore C_3 = \frac{1}{4}[1 + 0 + j - 1 - 0]$$

$$\therefore C_3 = \frac{j}{4} = 0 + j\,0.25 = 0.25 \angle +1.57$$

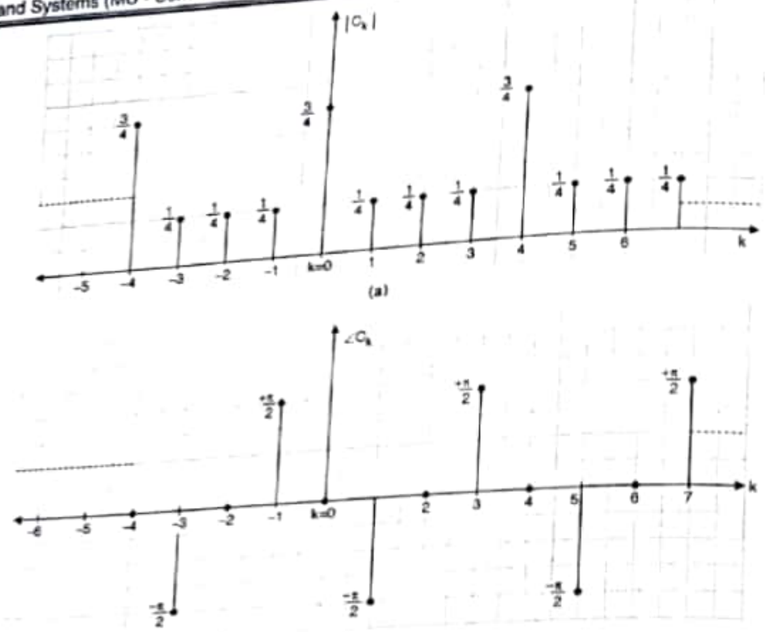
That means magnitude = 0.25 and angle = 1.57, radians $= \frac{\pi}{2}$

Table P. 5.8.1 shows the summary of coefficients.

Table P. 5.8.1

Sr. No.	Coefficient	Magnitude	Phase
1.	$C_0 = \frac{3}{4}$	$\frac{3}{4}$	0
2.	$C_1 = -\frac{1}{4}j$	$\frac{1}{4}$	$\frac{-\pi}{2}$
3.	$C_2 = \frac{1}{4}$	$\frac{1}{4}$	0
4.	$C_3 = \frac{j}{4}$	$\frac{1}{4}$	$\frac{\pi}{2}$

The magnitude response is shown in Fig. P. 5.8.1(a) and the phase response is shown in Fig. P. 5.8.1(b).



(H-1462) **Fig. P. 5.8.1**

Since the sequence is periodic; we have calculated $|C_k|$ and $\angle C_k$ only for the range k = 0 to k = 3. Actual response is periodic in nature.

Ex. 5.8.2 : Sketch the magnitude and phase spectra of,

$$x(n) = \cos\left(\frac{\pi}{8}n + \phi\right)$$

Soln. :

Step I : First we will check that the given sequence is periodic or not. Given,

$$x(n) = \cos\left(\frac{\pi}{8}n + \phi\right) \quad ...(1)$$

We have standard equation,

$$x(n) = \cos(2\pi fn + \phi) \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$2\pi f = \frac{\pi}{8}$$

$$\therefore \quad f = \frac{1}{16} \quad ...(3)$$

This is the ratio of two integer. If the frequency (f) is $\frac{k}{N}$, where k and N are integers then the sequence is periodic. Thus given sequence is periodic in nature with period N. That means from Equation (3), $F = \frac{k}{N} = \frac{1}{16}$; the period is N = 16.

Step II : We have the equation of as,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{+j2\pi kn/N} \quad ...(4)$$

Now we can expressing given equation x(n) using Euler's identity as,

$$\therefore \quad x(n) = e^{-j\left(\frac{\pi}{8}n + \phi\right)} + e^{-j\left(\frac{\pi}{8}n + \phi\right)}$$

$$\therefore \quad x(n) = \frac{1}{2}\left[e^{\frac{-j\pi n}{8}} \cdot e^{-j\phi}\right] + \frac{1}{2}\left[e^{\frac{j\pi n}{8}} \cdot e^{j\phi}\right] \quad ...(5)$$

We know that ϕ denotes the phase angle. Now comparing first term of Equation (5) with Equation (4) we get,

$$e^{\frac{j\pi n}{8}} = e^{\frac{j2\pi kn}{N}}$$

$$\therefore \quad -\frac{j\pi}{8}n = \frac{j2\pi kn}{16}$$

$$\therefore \quad k = -1 \quad ...(6)$$

That means we will get term (1) when k = –1. From Equations (4) and (5) we can write,

$$\therefore \quad C_k = C_{-1} = \frac{1}{2}e^{-j\phi} \quad ...(7)$$

Similarly comparing term (2) of Equation (5) with Equation (4) we get,

$$e^{\frac{j\pi n}{8}} = e^{\frac{j2\pi kn}{N}}$$

$$\therefore \quad e^{\frac{j\pi n}{8}} = e^{\frac{j2\pi kn}{16}}$$

$$\therefore \quad k = 1 \quad ...(8)$$

That means we will get term (2) when k = + 1

From Equations (4) and (5) we can write,

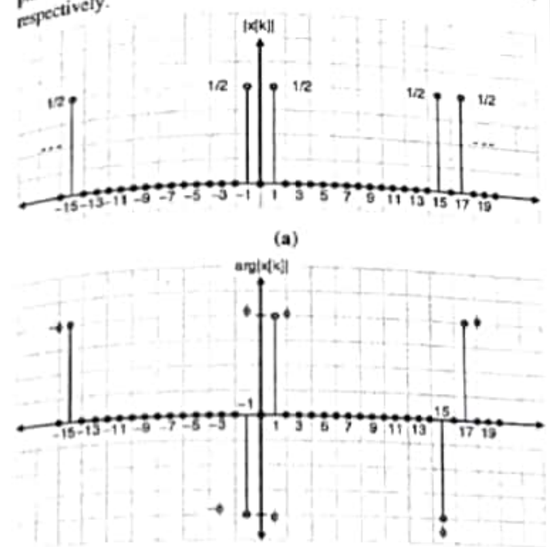
$$\therefore \quad C_k = C_1 = \frac{1}{2}e^{j\phi} \quad ...(9)$$

Step III : Since C_k exists for negative value of k (k = – 1) and total period is N = 16 ; we can conclude that the range of k is k = –7 to 8. But from Equation (5) we can say that x (n) has only two terms.

That means C_k is present at k = – 1 and k = + 1; for other values of k; C_k is zero.

$$C_k = \begin{cases} \frac{1}{2}e^{j\phi}, k = -1 \\ \frac{1}{2}e^{j\phi}, k = +1 \\ 0 \quad , k = -7, -6, -5, -4, -3, -2, 0, 2, 3, 4, 5, 6, 7, 8 \end{cases}$$

That means magnitude is $\frac{1}{2}$ at k = – 1 and phase is – ϕ. While magnitude is $\frac{1}{2}$ at k = + 1 and phase is + ϕ. The magnitude and phase response are shown in Fig. P. 5.8.2(a) and P. 5.8.2(b) respectively.



(H-1463) **Fig. P. 5.8.2**

Ex. 5.8.3 : Consider the signal

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi n}{N} + \frac{\pi}{2}\right)$$

Determine Fourier series coefficient this signal. **Dec. 15, 10 Marks**

Soln. :

Given equation is,

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}\right)n + 3\cos\left(\frac{2\pi}{N}\right)n + \cos\left(\frac{4\pi n}{N} + \frac{\pi}{2}\right)$$

Here $\omega_0 = \frac{2\pi}{N}$ thus $\frac{4\pi}{N} = 2\omega_0$ ...(1)

We have Euler's identity,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \text{ and } \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Thus Equation (1) can be written as,

$$x(n) = 1 + \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] + \frac{3}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right] + \frac{1}{2}\left[e^{j\left(2\omega_0 n + \frac{\pi}{2}\right)} + e^{-j\left(2\omega_0 n + \frac{\pi}{2}\right)}\right]$$

$$\therefore \quad x(n) = 1 + \frac{1}{2j}e^{j\omega_0 n} - \frac{1}{2j}e^{-j\omega_0 n} + \frac{3}{2}e^{j\omega_0 n} + \frac{3}{2}e^{-j\omega_0 n} + \frac{1}{2}\left[e^{j2\omega_0 n}\, e^{j\frac{\pi}{2}} + e^{-j2\omega_0 n}\, e^{-j\frac{\pi}{2}}\right]$$

Rearranging the terms we get,

$$x(n) = 1 + \frac{1}{2}\left(3 + \frac{1}{j}\right)e^{j\omega_0 n} + \frac{1}{2}\left(3 - \frac{1}{j}\right)e^{-j\omega_0 n} + \frac{1}{2}\left(e^{j2\omega_0 n}\, e^{j\frac{\pi}{2}} + e^{-j2\omega_0 n}\, e^{-j\frac{\pi}{2}}\right) \quad ...(2)$$

Now $\frac{1}{j} = -1$ and $e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = 1j = 1$

Thus Equation (2) becomes,

$$x(n) = 1 + \frac{1}{2}(3 - j)e^{j\omega_0 n} + \frac{1}{2}(3 + j)e^{-j\omega_0 n} + \frac{j}{2}e^{j2\omega_0 n} - \frac{j}{2}e^{-j2\omega_0 n} \quad ...(3)$$

Now we have the equation,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} = \sum_{k=0}^{N-1} C_k e^{j\omega_0 kn}$$

Here range of k can be from $-\frac{N}{2}$ to $+\frac{N}{2}$. From equation we can say the range is from – 2 to + 2

$$x(n) = \sum_{k=-2}^{2} C_k e^{j\omega_0 kn} \quad ...(4)$$

Expanding Equation (4) we get,

$$x(n) = C_{-2}e^{-j2\omega_0 n} + C_{-1}e^{-j\omega_0 n} + C_0 e^{j0} + C_1 e^{j\omega_0 n} + C_2 e^{j2\omega_0 n} \quad ...(5)$$

Comparing like coefficients of Equations (3) and (5) we get,

$$C_2 = \frac{-j}{2}, C_{-1} = \frac{3+j}{2}, C_0 = 1, C_1 = \frac{3-j}{2}, C_2 = \frac{j}{2}$$

5.9 Convergence of DTFS :

MU : Dec. 14

University Questions

Q. 1 Explain Gibb's phenomenon. Also explain conditions necessary for the convergence of Fourier series. **(Dec. 14, 5 Marks)**

- The convergence of DTFS is similar to the convergence and Gibb's phenomenon of CTFS.
- In case of DTFS, always a discrete summation is preformed and then Fourier coefficients are calculated.
- There is one to one correspondence between discrete signal and its corresponding discrete Fourier coefficients.
- In case of DTFS, while calculating the partial sum; the mean square error reduces as the value of N increases.
- For larger values of 'N', there is no Gibb's phenomenon in DTFS.

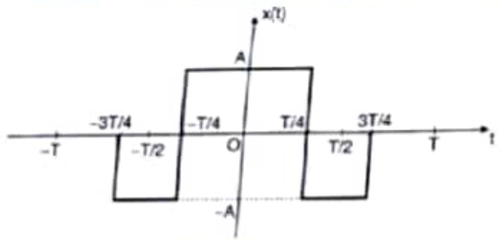
- Basically the discrete time function has no continuity; so there is no concept of convergence at the discontinuity for DT signals.

5.9.1 Analogy between CTFS and DTFS :

- Continues time fourier series (CTFS) consists of infinite number of frequency components.
- In CTFS, the spacing between two successive frequency components is F_s ; where $F_s = \frac{1}{T_s}$ and T_s is the fundamental period.
- In case of CTFS, the frequency range is from $-\infty$ to $+\infty$.
- While the frequency range of discrete time fourier series (DTFS) is from $-\pi$ to $+\pi$ or 0 to 2π.
- If the fundamental period of DTFS is N then it contains frequency components separated by $\frac{2\pi}{N}$ radians.
- DTFS can contain, maximum N frequency components.

5.10 Additional Solved Examples :

Ex. 5.10.1 : Determine Fourier series representation of the following signal : **May 14, 10 Marks**



(H-1526) Fig. P. 5.10.1

Soln. :

This signal is symmetric. So, its average value is zero.

$\therefore \quad a_0 = 0$

and $x(t) = x(-t)$, that means only cosine terms will be present.

$\therefore \quad b_n = 0$

The given signal can be represented as,

$$x(t) = \begin{cases} -A, -\frac{T}{2} < t < -\frac{T}{4} \\ A, -\frac{T}{4} < t < \frac{T}{4} \\ -A, \frac{T}{4} < t < \frac{T}{2} \end{cases}$$

Now the coefficient a_n is given by,

$$a_n = \frac{2}{T_0} \int_{\frac{-T_0}{2}}^{\frac{T_0}{2}} x(t) \cos n\omega_0 t \, dt \qquad ...(1)$$

Here, T_0 = Fundamental period $= -\frac{T}{2} t_0 + \frac{T}{2} = T$

and $\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} = \frac{2\pi}{T}$

Thus Equation (1) becomes,

$$a_n = \frac{2}{T} \int_{\frac{-T}{2}}^{\frac{-T}{4}} (-A) \cos\left(\frac{n2\pi t}{T}\right) dt + \frac{2}{T} \int_{\frac{-T}{4}}^{\frac{T}{4}} A \cos\left(\frac{n2\pi t}{T}\right) dt$$

$$+ \frac{2}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} -A \cos\left(\frac{n2\pi t}{T}\right) dt$$

$$= \frac{2A}{T} \left\{ \left[\frac{-\sin\left(\frac{n2\pi t}{T}\right)}{\frac{n2\pi}{T}} \right]_{-T/2}^{-T/4} + \left[\frac{\sin\left(\frac{n2\pi t}{T}\right)}{\frac{n2\pi}{T}} \right]_{-T/4}^{T/4} + \left[\frac{-\sin\left(\frac{n2\pi t}{T}\right)}{\frac{n2\pi}{T}} \right]_{T/4}^{T/2} \right\}$$

$$= \frac{2A}{T} \cdot \frac{T}{n2\pi} \left\{ \left[-\sin\left(\frac{n2\pi}{T} \cdot \left(-\frac{T}{4}\right)\right) + \sin\left(\frac{n2\pi}{T} \cdot \left(-\frac{T}{2}\right)\right) \right] \right.$$

$$+ \left[\sin\left(\frac{n2\pi}{T}\left(\frac{T}{4}\right)\right) - \sin\left(\frac{n2\pi}{T}\left(-\frac{T}{4}\right)\right) \right]$$

$$\left. + \left[-\sin\left(\frac{n2\pi}{T}\left(\frac{T}{2}\right)\right) + \sin\left(\frac{n2\pi}{T}\left(\frac{T}{4}\right)\right) \right] \right\}$$

$$= \frac{A}{n\pi} \left\{ \left[\sin\left(\frac{n\pi}{2}\right) - \sin(n\pi) \right] + \left[\sin\left(\frac{n\pi}{2}\right) + \sin\left(\frac{n\pi}{2}\right) \right] \right.$$

$$\left. + \left[-\sin(n\pi) + \sin\left(\frac{n\pi}{2}\right) \right] \right\}$$

$$\therefore \quad a_n = \frac{4A}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

Ex. 5.10.2 : For the periodic signal $x(t) = e^{-t}$ with a fundamental period $T_0 = 1$ second, find the exponential form of Fourier series. Also plot the Fourier spectrum (Magnitude and phase spectrum). **Dec. 14, 10 Marks**

Soln. : Refer Ex. 5.2.11.

Ex. 5.10.3 : For the periodic signal x[n] given below, find out Fourier series coefficient :

$$x[n] = 1 + \sin\left(\frac{2\pi}{n}\right) n + 3 \cos\left[\frac{2\pi}{N}\right] n + \cos\left(\frac{4\pi}{N} n + \frac{\pi}{2}\right)$$

Dec. 14, 10 Marks

Soln. :

Given equation is,

$$x(n) = 1 + \sin\left(\frac{2\pi}{N}\right) n + 3 \cos\left(\frac{2\pi}{N}\right) n + \cos\left(\frac{4\pi n}{N} + \frac{\pi}{2}\right)$$

Here $\omega_0 = \frac{2\pi}{N}$ thus $\frac{4\pi}{N} = 2\omega_0$ ...(1)

We have Euler's identity,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \text{ and } \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Thus Equation (1) can be written as :

$$x(n) = 1 + \frac{1}{2j}\left[e^{j\omega_0 n} - e^{-j\omega_0 n}\right] + \frac{3}{2}\left[e^{j\omega_0 n} + e^{-j\omega_0 n}\right]$$

$$+ \frac{1}{2}\left[e^{j\left(2\omega_0 n + \frac{\pi}{2}\right)} + e^{-j\left(2\omega_0 n + \frac{\pi}{2}\right)}\right]$$

$$\therefore \quad x(n) = 1 + \frac{1}{2j} e^{j\omega_0 n} - \frac{1}{2j} e^{-j\omega_0 n} + \frac{3}{2} e^{j\omega_0 n} + \frac{3}{2} e^{-j\omega_0 n}$$

$$+ \frac{1}{2}\left[e^{j2\omega_0 n} e^{j\frac{\pi}{2}} + e^{-j2\omega_0 n} e^{-j\frac{\pi}{2}}\right]$$

Rearranging the terms we get,

$$x(n) = 1 + \frac{1}{2}\left(3 + \frac{1}{j}\right) e^{j\omega_0 n} + \frac{1}{2}\left(3 - \frac{1}{j}\right) e^{-j\omega_0 n}$$

$$+ \frac{1}{2}\left(e^{j2\omega_0 n} e^{j\frac{\pi}{2}} + e^{-j2\omega_0 n} e^{-j\frac{\pi}{2}}\right) \qquad ...(2)$$

Now $\frac{1}{j} = -1$ and $e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = 1j = 1$

Thus Equation (2) becomes,

$$x(n) = 1 + \frac{1}{2}(3 - j) e^{j\omega_0 n} + \frac{1}{2}(3 + j) e^{-j\omega_0 n}$$

$$+ \frac{j}{2} e^{j2\omega_0 n} - \frac{j}{2} e^{-j2\omega_0 n} \qquad ...(3)$$

Now we have the equation,

$$x(n) = \sum_{k=0}^{N-1} C_k e^{j2\pi kn/N} = \sum_{k=0}^{N-1} C_k e^{j\omega_0 kn}$$

Here range of k can be from $-\frac{N}{2}$ to $+\frac{N}{2}$. From equation we can say the range is from -2 to $+2$

$$x(n) = \sum_{k=-2}^{2} C_k e^{j\omega_0 kn} \qquad ...(4)$$

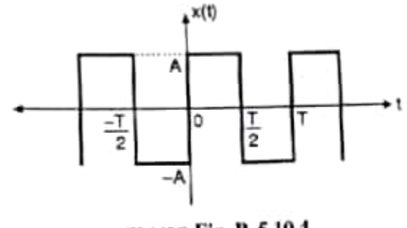
Expanding Equation (4) we get,

$$x(n) = C_{-2} e^{-2j\omega_0 n} + C_{-1} e^{-j\omega_0 n} + C_0 e^{j0} + C_1 e^{j\omega_0 n} + C_2 e^{2j\omega_0} \qquad ...(5)$$

Comparing like coefficients of Equations (3) and (5) we get,

$$C_2 = \frac{-j}{2}, C_{-1} = \frac{3+j}{2}, C_0 = 1, C_1 \frac{3-j}{2}, C_2 = \frac{j}{2}$$

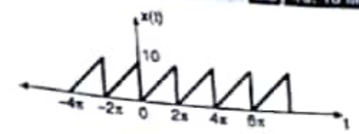
Ex. 5.10.4 : Determine the trigonometric form of Fourier series of the waveform shown in Fig. P. 5.10.4. **Dec. 15, 10 Marks**



(H-1656) Fig. P. 5.10.4

Soln. : Similar to Ex. 5.2.1

Ex. 5.10.5 : Find the trigonometric Fourier series for the waveform shown in Fig. P. 5.10.5. **May 16, 10 Marks**



(H-1660) Fig. P. 5.10.5

Soln. :

We have, $a_0 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\, dt$

Here $T_0 = 2\pi$ and $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2\pi} = 1$

$$\therefore \quad a_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} dt = \frac{1}{2\pi}\left[\frac{t^2}{4\pi}\right]_0^{2\pi} = \frac{1}{8\pi^2}\{4\pi^2 - 0\} = \frac{1}{2}$$

$$a_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \cos\left(\frac{2\pi nt}{T_0}\right) dt$$

$$= \frac{2}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \cos\left(\frac{2\pi nt}{2\pi}\right) = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2\pi} \cos nt\, dt$$

$$= \frac{1}{2\pi^2}\left[1 \frac{\sin nt}{n} + \frac{\cos nt}{n^2}\right]_0^{2\pi}$$

$$\therefore \quad a_n = 0$$

$$b_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) \sin\left(\frac{2\pi nt}{T_0}\right) dt$$

$$= \frac{2}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} \sin\left(\frac{2\pi nt}{2\pi}\right) dt$$

$$\therefore \quad b_n = \frac{1}{\pi} \int_0^{2\pi} \frac{1}{2\pi} \sin(nt)\, dt$$

$$= \frac{1}{2\pi^2}\left[-1 \frac{\cos nt}{n} + \frac{\sin nt}{n^2}\right]_0^{2\pi}$$

$$= -\frac{1}{\pi n}$$

Ex. 5.10.6 : For the following discrete time signals with a fundamental period of N = 6, determine the Fourier series coefficients :

(a) $x[n] = 1 + \cos\left(\frac{2\pi n}{6}\right)$

(b) $y[n] = \sin\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right)$

May 16, 5 Marks

Soln. :

Given :

(a) $$x(n) = 1 + \cos\left(\frac{2\pi n}{6}\right)$$

$$\therefore x(n) = 1 + \frac{e^{j\frac{2\pi n}{6}} + e^{-j\frac{2\pi n}{6}}}{2}$$

$$= 1 + \frac{1}{2}e^{j\frac{2\pi n}{6}} + \frac{1}{2}e^{-j\frac{2\pi n}{6}}$$

Now $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\therefore x(n) = 1 + \frac{1}{2}e^{jk\omega_0 n} + \frac{1}{2}e^{-jk\omega_0 n} \quad ...(1)$$

Using definition of DTFS

$$x(n) = \sum_{k=-2}^{3} C_k e^{jk\omega_0 kn}$$

$$\therefore x(n) = C_{-2}e^{-j2\omega_0 n} + C_{-1}e^{-j\omega_0 n} + C_0 + C_1 e^{j\omega_0 n} + C_2 e^{j2\omega_0 n} + C_3 e^{j3\omega_0 n} \quad ...(2)$$

Comparing Equations (1) and (2),

$$C_0 = 1, C_1 = \frac{1}{2}, C_{-1} = \frac{1}{2}$$

(b) $$x(n) = \sin\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right)$$

$$= \frac{1}{2j}\left[e^{j\left(\frac{2\pi n}{6}+\frac{\pi}{4}\right)} - e^{-j\left(\frac{2\pi n}{6}+\frac{\pi}{4}\right)}\right]$$

$$x(n) = \frac{1}{2j}e^{j\frac{\pi n}{3}} \cdot e^{j\frac{\pi}{4}} - \frac{1}{2j}e^{-j\frac{\pi n}{3}} \cdot e^{j\frac{\pi}{4}} \quad ...(3)$$

Here $\omega_0 = \frac{2\pi}{6} = \frac{\pi}{3}$

$$\therefore x(n) = \frac{1}{2j}e^{jk\omega_0 n} \cdot e^{j\frac{\pi}{4}} - \frac{1}{2j}e^{-jk\omega_0 n} \cdot e^{j\frac{\pi}{4}} \quad ...(4)$$

Using definition of DTFS,

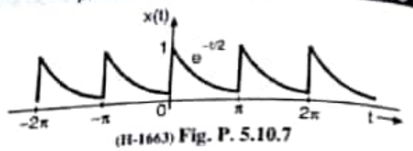
$$x(n) = \sum_{k=-2}^{3} C_k e^{jk\omega_0 kn}$$

$$\therefore x(n) = C_{-2}e^{-j2\omega_0 n} + C_{-1}e^{-j\omega_0 n} + C_0 + C_1 e^{j\omega_0 n} + C_2 e^{j2\omega_0 n} + C_3 e^{j3\omega_0 n} \quad ...(5)$$

Comparing Equations (4) and (5),

$$C_{-1} = -\frac{1}{2j}e^{j\frac{\pi}{4}} \text{ and } C_1 = \frac{1}{2j}e^{j\frac{\pi}{4}}$$

Ex. 5.10.7 : Find the compact trigonometric Fourier Series for the signal x(t) shown in Fig. P. 5.10.7. Sketch the amplitude and phase spectra for x(t). **Dec. 06, 10 Marks**



(H-1663) Fig. P. 5.10.7

Soln. :

We have,

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt$$

Here $T_0 = \pi$

$$\therefore a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\,dt = \frac{1}{\pi}\left[\frac{e^{-t/2}}{-\frac{1}{2}}\right]_0^{\pi}$$

$$\therefore a_0 = -\frac{2}{\pi}\left[e^{-\frac{\pi}{2}} - e^0\right] = -\frac{2}{\pi}[0.2078 - 1]$$

$$\therefore a_0 = 0.5 = \frac{2}{T_0}\int_0^{T_0} x(t)\cos(2\pi n f_0 t)\,dt$$

Here $T_0 = \pi$ and $f_0 = \frac{1}{T_0} = \frac{1}{\pi}$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos\left(2\pi n \cdot \frac{1}{\pi}t\right)dt$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\cos(2nt)\,dt$$

We have, $\int e^{-ax}\cos bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\cos bx + b\sin bx)$

$$\therefore a_n = \frac{2}{\pi}\left\{\frac{e^{0.5t}}{0.25+4n^2}[-0.5\cos(2nt) + 2n\sin(2nt)]\right\}_0^{\pi}$$

$$\therefore a_n = \frac{2}{\pi}\left\{\frac{e^{0.5\pi}}{0.25+4n^2}[-0.5\cos(2\pi n) + 2n\sin(2\pi n)] - [-0.5\cos(0) + 2n\sin(0)]\right\}$$

Now $\cos(2\pi n) = 1$, $\sin(2\pi n) = 0$, $\cos(0) = 1$, $\sin(0) = 0$

$$\therefore a_n = \frac{2}{\pi}\left\{\frac{e^{0.5\pi}}{0.25+4n^2}[-0.5 + 0.5]\right\}$$

$$\therefore a_n = 0$$

$$\text{Now } b_n = \frac{2}{T_0}\int_0^{T_0} x(t)\sin(2\pi n F_0 t)\,dt$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin\left(2\pi n \cdot \frac{1}{\pi}t\right)dt$$

$$= \frac{2}{\pi}\int_0^{\pi} e^{-t/2}\sin(2nt)\,dt$$

We have, $\int e^{-ax}\sin bx\,dx = \frac{e^{-ax}}{a^2+b^2}(-a\sin bx - b\cos bx)$

Here $a = 0.5$, $b = 2n$, $x = t$

$$\therefore b_n = \frac{2}{\pi}\left\{\frac{e^{-0.5t}}{0.25+4n^2}(-0.5\sin 2nt - 2n\cos 2nt)\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left\{\frac{e^{-0.5\pi}}{0.25+4n^2}[[-0.5\sin(2\pi n) - 2n\cos(2\pi n)] - [-0.5\sin(0) - 2n\cos(0)]]\right\}$$

$$= \frac{2}{\pi}\left\{\frac{e^{-0.5\pi}}{0.25+4n^2}[[0 - 2n] - [0 - 2n]]\right\}$$

$$\therefore b_n = 0$$

$$\therefore x(t) = a_0 = 0.5$$

Thus magnitude spectrum remains constant.

Review Questions

Q. 1 What do you mean by frequency analysis of a signal ?

Q. 2 What is the use of analysis of a signal ?

Q. 3 What is a phase spectrum ?

Q. 4 What is the line spectrum ?

Q. 5 State and prove following properties of CTFS :

1. Time shifting 2. Scaling
3. Differentiation 4. Parseval's theorem

Q. 6 Explain analogy between CTFS and DTFS.

□□□

CHAPTER 6

Module 3

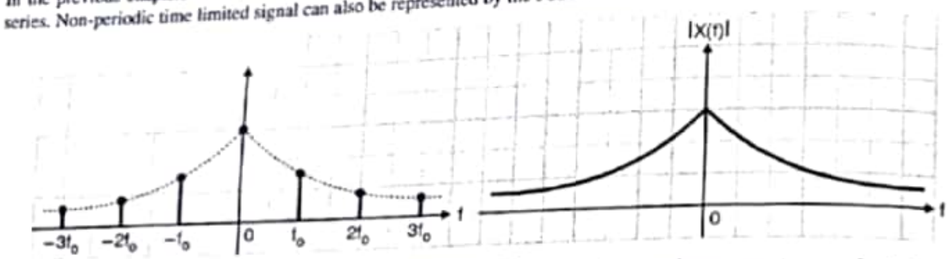
Fourier Transform

Syllabus :

Fourier transform and inverse Fourier transform on periodic and non periodic signals, Limitations of CT/DT Fourier transform and need for Laplace/Z-transform.

6.1 C.T. Fourier Transform :

- In the previous chapter, we have seen how to represent the periodic signals extended over the interval $(-\infty, \infty)$, using the Fourier series. Non-periodic time limited signal can also be represented by the Fourier series.



(a) Line spectrum showing vertical spectral lines at f_0 $2f_0$, ... **(b) Continuous spectrum as $f_0 \to 0$**

(H-477) **Fig. 6.1.1**

- However the non-periodic signals which extend from $-\infty$ to ∞ can be represented more conveniently using the "Fourier Transform" in the frequency domain.
- It is possible to find the Fourier transform of periodic signal as well. For the non-periodic signals $T_0 \to \infty$. Hence the frequency $f_0 = \frac{1}{T_0} \to 0$. Therefore the difference between the spectral components which is f_0 (as seen in the line spectrum) becomes extremely small and they come very close to each other. Due to this the frequency spectrum of non-periodic signals appears to be continuous as shown in Figs. 6.1.1(a) and (b).

6.1.1 Necessity of Fourier Transform :

- Any signal is made up by addition of elementary signals (sine and cosine signals) which are at different frequencies, have different amplitudes and relative phases.
- Using the Fourier transform we can plot the amplitude and phase spectrums of the given signal which provide us all the information about amplitudes and relative phases of such elementary signals.
- Thus Fourier transform can be used for the "analysis" of a signal. It is used for transformation from the time domain to frequency domain.
- The F.T. can also be used for analysis of LTI systems as discussed later on.

6.1.2 Definition of Fourier Transform :

The Fourier transform of a signal x (t) is defined as follows :

Fourier transform : $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

OR $$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt \quad ...(6.1.1)$$

These equations are known as "analysis" equations.

6.1.3 Definition of Inverse Fourier Transform :

The signal x (t) can be obtained back from its Fourier transform X (f) by using the inverse Fourier transform. The inverse Fourier transform (IFT) is defined as follows :

Inverse Fourier transform :

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Or $$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi ft}\, df \quad ...(6.1.2)$$

Representation :

The signal x(t) and its Fourier transform X(f) form a Fourier transform pair which can be represented as,

$$x(t) \xleftrightarrow{F} X(f) \quad ...(6.1.3)$$

The other way to represent it is as follows :

$$X(t) = F[x(t)] \quad ...(6.1.4)$$

or it can be represented as,

$$X(t) = F^{-1}|X(f)| \quad ...(6.1.5)$$

The Fourier transform is a complex function of frequency f. Therefore it is possible to express it in the complex exponential form as follows :

$$X(f) = |X(f)| \cdot e^{j\theta(f)} \quad ...(6.1.6)$$

In this expression :

$|X(f)|$ = The amplitude spectrum of x (t) and

$\theta(f)$ = The phase spectrum.

The amplitude spectrum is a graph of amplitude versus frequency, whereas the phase spectrum is a graph of phase angle versus frequency.

6.1.4 Conditions for the Existence of Fourier Transform :

- We have studied the **Dirichlet conditions** for Fourier series. Similarly these conditions should be satisfied by a signal x (t), then only it is possible to obtain the Fourier transform of x (t).
- For the periodic signals the integration is obtained over one period however for the periodic signals, it will be obtained over a range $-\infty$ to ∞.
- The signal x(t) will have to satisfy the following conditions so that it's Fourier transform can be obtained :

1. The function x (t) should be single valued in any finite time interval T.
2. It should have a finite number of discontinuities in any finite interval T.
3. The function x (t) should have a finite number of maxima and minima in any finite interval of time T.
4. The function x (t) should be an absolutely integrable function.

That means $$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \quad ...(6.1.7)$$

The conditions stated above are sufficient conditions, but they are not the necessary conditions.

6.1.5 Amplitude and Phase Spectrums :

- The amplitude and phase spectrums are continuous and not discrete in nature. Out of them, the amplitude spectrum of a real valued function x (t) exhibits an even symmetry.

$$\therefore \quad X(f) = X(-f) \quad ...(6.1.8)$$

- And the phase spectrum has an odd symmetry. That means,

$$\theta(f) = -\theta(-f) \quad ...(6.1.9)$$

6.2 Properties of Fourier Transform :

University Questions **Dec. 04**

Q. 1 State properties of Fourier transform. Give proof of convolution property. **(Dec. 04, 10 Marks)**

Some of the important properties of the Fourier transform are listed as follows :

- Linearity or superposition
- Time scaling
- Duality or symmetry
- Time shifting
- Frequency shifting
- Area under x (t)
- Area under X (f)
- Differentiation in time domain
- Integration in time domain
- Conjugate function
- Multiplication in time domain (Multiplication theorem)
- Convolution theorem.

Let us understand these properties one-by-one.

6.2.1 Property 1 : Linearity or Superposition :

- If $x_1(t) \xleftrightarrow{F} X_1(f)$ and $x_2(t) \xleftrightarrow{F} X_2(f)$ represent the Fourier transform pairs and if a_1 and a_2 are constants then we can write,

$$[a_1 x_1(t) + a_2 x_2(t)] \xleftrightarrow{F} [a_1 X_1(f) + a_2 X_2(f)] \quad ...(6.2.1)$$

- That means the linear combination of inputs gets transformed into linear combination of their Fourier transforms.
- This property can be used to obtain the Fourier transform of a complicated function say x(t) by decomposing it in the form of sum of simpler functions, say $x_1(t)$ and $x_2(t)$.

Proof of the property :

$$F[a_1 x_1(t) + a_2 x_2(t)] = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)]\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} a_1 x_1(t)\, e^{-j2\pi ft}\, dt + \int_{-\infty}^{\infty} a_2 x_2(t) \cdot e^{-j2\pi ft}\, dt$$

$$= a_1 X_1(f) + a_2 X_2(f) \quad \text{...Proved.}$$

6.2.2 Property 2 : Time Scaling :

MU : Dec. 14, Dec. 15

University Questions

Q. 1 State and prove time shifting and time scaling property of continuous time Fourier transform. **(Dec. 14, 4 Marks)**

Q. 2 Prove and explain time scaling and amplitude scaling property of continuous time Fourier transform. (Dec. 15, 5 Marks)

Let x (t) and X (f) form a Fourier transform pair and let "α" be a constant. Then the time scaling property states that,

$$x(\alpha t) \xleftrightarrow{F} \frac{1}{|\alpha|} X(f/\alpha) \qquad ...(6.2.2)$$

Meaning :

1. x (αt) represents a time scaled signal and X (f / α) represents the frequency scaled signal or scaled frequency spectrum.
2. For α < 1, x (αt) represents a compressed signal but X (f / α) represents an expanded version of X (f).
3. And for α > 1, x (α t) will be an expanded signal in the time domain. But its Fourier transform X (f / α) represents a compressed version of X (f).

Thus compression in the time domain results in expansion of frequency spectrum whereas expansion in the time domain results in compression of the frequency spectrum.

Proof of scaling property :

"α" being a constant, can be positive or negative. i.e. α > 0 or α < 0. Let us find the F.T. considering both the possibilities.

1. "α" positive (α > 0) :

$$F[x(\alpha t)] = \int_{-\infty}^{\infty} x(dt)\, e^{-j2\pi ft}\, dt$$

Substitute $\tau = \alpha t$, then $d\tau = \alpha\, dt$

$$\therefore \quad dt = \frac{d\tau}{\alpha}$$

$$\therefore \quad F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \frac{\tau}{\alpha}} \frac{d\tau}{\alpha}$$

$$= \frac{1}{\alpha}\int_{-\infty}^{\infty} x(\tau) e^{-j2\pi (f/\alpha)\tau}\, d\tau$$

$$= \frac{1}{\alpha} X(f/\alpha) \qquad ...(6.2.3)$$

2. α < 0 i.e. "α" negative :

$$F[x(\alpha t)] = \int_{-\infty}^{\infty} x(\alpha t)\, e^{-j2\pi ft}\, d\tau$$

Substitute $\tau = \alpha t$ $\quad \therefore \; d\tau = \alpha\, dt$

$$\therefore \quad dt = d\tau/\alpha$$

$$\therefore \quad F[x(\alpha t)] = \frac{1}{\alpha}\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi (f/\alpha)\tau}\, d\tau$$

But as α < 0 i.e. negative,

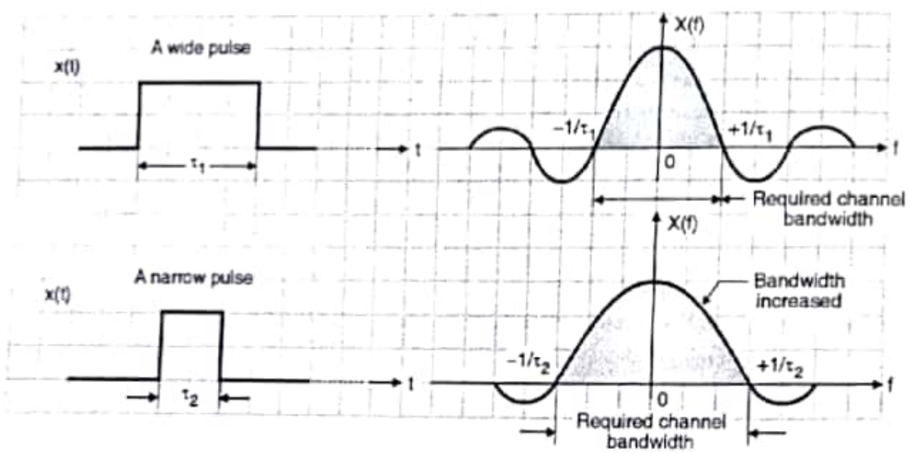
$$F[x(\alpha t)] = -\frac{1}{\alpha} X(f/\alpha) \qquad ...(6.2.4)$$

Combining the results obtained in Equations (6.2.3) and (6.2.4) we can write,

$$F[x(\alpha t)] = \frac{1}{|\alpha|} X(f/\alpha) \qquad ...(6.2.5)$$

Significance of scaling property in communication system :

- In digital communication systems a train of digital pulses is transmitted. The number of such pulses transmitted per second is called as the signalling rate. The signalling rate should be as high as possible in order to transmit as much information as possible.
- But with increase in signalling rate, the width of each pulse will reduce i.e. compression in time domain will take place.
- To transmit narrow pulses without introducing signal distortion, the channel bandwidth needs to be increased. i.e. expansion in the frequency domain will take place.



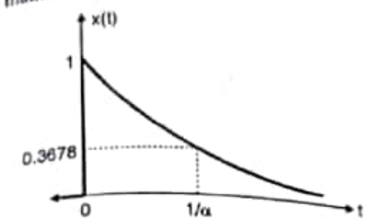
Compression in time domain ≡ Expansion in frequency domain

(H-478) **Fig. 6.2.1 : Significance of time scaling property**

Ex. 6.2.1 : Find the fourier transform of the decaying exponential pulse shown in Fig. P. 6.2.1.

Soln. :

The exponential pulse shown in the Fig. P. 6.2.1 can be represented mathematically as follows :



(H-479) **Fig. P. 6.2.1 : Decaying exponential pulse**

$$x(t) = e^{-\alpha t} \quad \text{for } t \geq 0$$
$$= 0 \quad \text{for } t < 0 \qquad ...(1)$$

It can be represented in an alternate way as,

$$x(t) = e^{-\alpha t} u(t) \qquad ...(2)$$

The meaning of both the Equations (1) and (2) is the same. This is because u (t) = 1 for t ≥ 0. So multiplying by u (t) does not affect the original function.

To find the fourier transform :

$$F[x(t)] = \int_{-\infty}^{\infty} e^{-\alpha t} u(t)\, e^{-j2\pi ft}\, dt \quad \text{...as per definition of FT.}$$

$$= \int_{0}^{\infty} e^{-\alpha t} e^{-j2\pi ft}\, dt = \int_{0}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt$$

$$= \frac{-1}{(\alpha + j2\pi f)}\left[e^{-(\alpha + j2\pi f)t}\right]_0^{\infty} = \frac{-1}{(\alpha + j2\pi f)}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{(\alpha + j2\pi f)}[0 - 1] = \frac{1}{(\alpha + j2\pi f)}$$

This is the required result.

$$\therefore \quad e^{-\alpha t} u(t) \xleftrightarrow{F} \frac{1}{(\alpha + j2\pi f)} \qquad ...(3)$$

Ex. 6.2.2 : Find the fourier transform of the exponential pulse shown in Fig. P. 6.2.2(a). Also find the amplitude and phase spectrums for the same.

Soln. :

The pulse shown in Fig. P. 6.2.2(a) can be represented as,

$$x(t) = e^{\alpha t} \quad \text{for } t \leq 0$$
$$= 0 \quad \text{for } t > 0 \qquad ...(1)$$

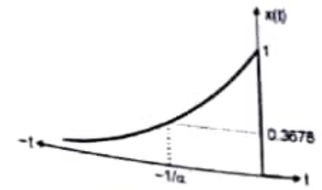
It can also be represented as,

$$x(t) = e^{\alpha t} u(-t) \qquad ...(2)$$

It's fourier transform is given as,

$$X(f) = \int_{-\infty}^{\infty} e^{\alpha t} u(-t) \cdot e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{0} e^{\alpha t} e^{-j2\pi ft}\, dt = \int_{-\infty}^{0} e^{(\alpha - j2\pi f)t}\, dt$$



(H-480) **Fig. P. 6.2.2(a)**

$$= \frac{1}{(\alpha - j2\pi f)}\left[e^{(\alpha - j2\pi f)t}\right]_{-\infty}^{0}$$

$$\therefore \quad X(f) = \frac{1}{(\alpha - j2\pi f)}\left[e^{0} - e^{-\infty}\right]$$

$$= \frac{1}{(\alpha - j2\pi f)}[1 - 0] = \frac{1}{(\alpha - j2\pi f)} \qquad ...(3)$$

This is the desired result. Thus,

$$e^{\alpha t} u(-t) \xleftrightarrow{F} \frac{1}{(\alpha - j2\pi f)} \qquad ...(4)$$

To find the amplitude spectrum :

The amplitude spectrum is denoted by | X (f) | and is given by,

$$|X(f)| = [(\text{Real part of } X(f))^2 + (\text{Imaginary part of } X(f))]^{1/2}$$

We have found out that $X(f) = \dfrac{1}{(\alpha - j2\pi f)}$

$$= \frac{1}{(\alpha - j2\pi f)} \times \frac{(\alpha + j2\pi f)}{(\alpha + j2\pi f)}$$

$$= \frac{\alpha + j2\pi f}{\alpha^2 + (2\pi f)^2}$$

$$\therefore \quad \text{Real part of } x(f) = \frac{\alpha}{\alpha^2 + 4\pi^2 f^2} \qquad ...(5)$$

$$\text{and imaginary part of } X(f) = \frac{2\pi f}{\alpha^2 + 4\pi^2 f^2} \qquad ...(6)$$

Therefore the amplitude spectrum is given as,

$$|X(f)| = \left[\frac{\alpha^2}{(\alpha^2 + 4\pi^2 f^2)^2} + \frac{4\pi^2 f^2}{(\alpha^2 + 4\pi^2 f^2)^2}\right]^{1/2}$$

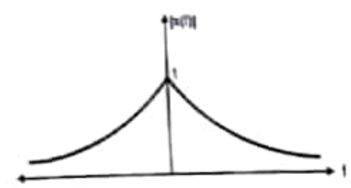
$$\therefore \quad |X(f)| = \frac{1}{(\alpha^2 + 4\pi^2 f^2)^{1/2}} \qquad ...(7)$$

Substitute different values of "f" in Equation (6.2.5) and plot the amplitude spectrum. It looks as shown in Fig. P. 6.2.2(b).

To obtain the phase spectrum :

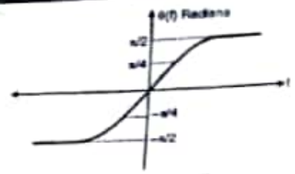
The phase spectrum is denoted by θ (f) and is expressed as,

$$\theta(f) = \tan^{-1}\left[\frac{\text{Imaginary part of } X(f)}{\text{Real part of } X(f)}\right] \qquad ...(8)$$



(b) Amplitude spectrum of $e^{\alpha t} \cdot u(-t)$

(H-481) **Fig. P. 6.2.2 Contd...**



(c) Phase spectrum of $e^{\alpha t} \cdot u(-t)$

(H-481) Fig. P. 6.2.2

Substituting the values of real and imaginary values of X(f) we get,

The phase spectrum : $\theta(f) = \tan^{-1}\left[\frac{2\pi f}{\alpha}\right]$ ...(9)

Substitute different values of "f" in the Equation (7) and plot the phase spectrum. It looks as shown in Fig. P. 6.2.2.

Ex. 6.2.3 : Obtain the fourier transform of the double exponential pulse shown in the Fig. P. 6.2.3(a). **Dec. 11, 4 Marks**

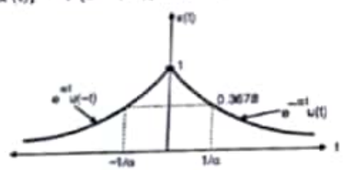
Soln. : The signal x (t) is the sum of two exponential pulses

i.e. $x(t) = x_1(t) + x_2(t)$

$= e^{\alpha t}(-t) + e^{-\alpha t} u(t)$

Taking the fourier transform of both sides,

$F[x(t)] = F[e^{\alpha t}(-t)] + F[e^{-\alpha t} u(t)]$



(H-482) Fig. P. 6.2.3(a) : Double exponential pulse

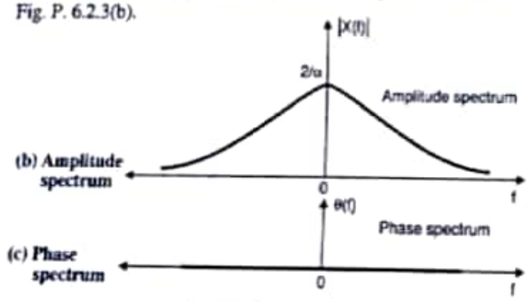
The fourier transforms of the two terms on the RHS have already been found in the Ex. 6.2.1 and Ex. 6.2.2. Substituting them we get,

$$X(f) = \frac{1}{\alpha - j2\pi f} + \frac{1}{\alpha + j2\pi f} = \frac{\alpha + j2\pi f + \alpha - j2\pi f}{\alpha^2 + (2\pi f)^2}$$

$$\therefore \quad X(f) = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$$

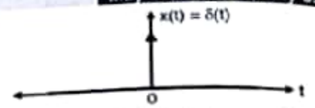
1. The amplitude spectrum $|X(f)| = \frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
2. The phase spectrum $\theta(f) = 0$.

The amplitude and phase spectrums are plotted as shown in Fig. P. 6.2.3(b).



(H-483) Fig. P. 6.2.3(b)

Ex. 6.2.4 : Obtain the fourier transform of the delta function shown in Fig. P. 6.2.4(a). **May 03, 2 Marks, May 05, 4 Marks**



(H-484) Fig. P. 6.2.4(a) : Delta function

Soln. :

1. By the definition of fourier transform,

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi ft}\, dt \quad ...(1)$$

We cannot substitute the value of δ (t) directly in the Equation (1) because it is infinitely large at t = 0. Therefore let us use the shifting property of the delta function.

2. **Shifting property of delta function :**

The shifting property states that

$$\int_{-\infty}^{\infty} f(t)\, \delta(t - t_d)\, dt = f(t_d) \quad ...(2)$$

Let us use this property in Equation (1) as follows :

3. In Equation (2) assume that $t_d = 0$ and $f(t) = e^{-j2\pi ft}$

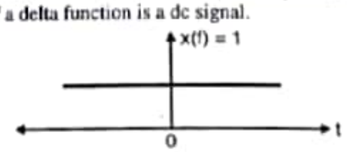
$$\therefore \quad X(f) = \int_{-\infty}^{\infty} e^{-j2\pi ft} \cdot \delta(t - 0) \quad \therefore \text{ by using Equation (2),}$$

$$X(f) = e^{-j2\pi ft_d}, \quad \text{but } t_d = 0$$

$$\therefore \quad X(f) = e^{-j2\pi \cdot 0} = 1 \quad ...(3)$$

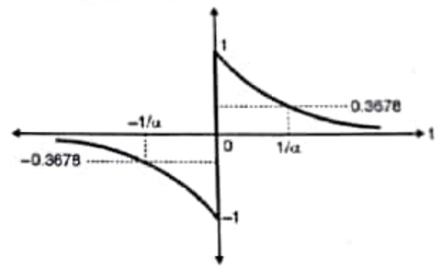
$$\text{Thus } \delta(t) \xleftrightarrow{F} 1$$

The amplitude spectrum of the delta function is as shown in the Fig. P. 6.2.4(b). This shows that the delta function contains all the frequencies from − ∞ to ∞ with equal amplitudes. The fourier transform of a delta function is a dc signal.



(H-485) Fig. P. 6.2.4(b) : Amplitude spectrum of delta function

Ex. 6.2.5 : Obtain the F.T. of the antisymmetrical pulse shown in the Fig. P. 6.2.5.



(H-486) Fig. P. 6.2.5

Soln. :

The antisymmetric pulse can be represented as,

$$x(t) = e^{-\alpha t} \qquad t > 0$$
$$= |1| \qquad t = 0$$
$$= -e^{\alpha t} \qquad t < 0$$

Therefore the fourier transform is given by,

$$F[x(t)] = \int_{-\infty}^{0^-} -e^{\alpha t} e^{-j2\pi ft}\, dt + \int_{0^-}^{0^+} |1|\, e^{-j2\pi ft}\, dt + \int_{0^+}^{\infty} e^{-\alpha t} e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{0^-} -e^{(\alpha - j2\pi f)t}\, dt + \int_{0^+}^{\infty} e^{-(\alpha + j2\pi f)t}\, dt$$

$$= \frac{-1}{\alpha - j2\pi f}\left[e^{(\alpha - j2\pi f)t}\right]_{-\infty}^{0^-} + \frac{1}{-(\alpha + j2\pi f)}\left[e^{-(\alpha + j2\pi f)t}\right]_{0^+}^{\infty}$$

$$= \frac{-1}{(\alpha - j2\pi f)}\left[e^0 - e^{-\infty}\right] - \frac{1}{(\alpha + j2\pi f)}\left[e^{-\infty} - e^0\right]$$

$$= \frac{-1}{(\alpha - j2\pi f)} + \frac{1}{(\alpha + j2\pi f)} = \frac{-\alpha - j2\pi f + \alpha - j2\pi f}{\alpha^2 + (2\pi f)^2}$$

$$\therefore \quad X(f) = \frac{-j4\pi f}{\alpha^2 + (2\pi f)^2} \quad ...(1)$$

Ex. 6.2.6 : Find the fourier transform of the signum function shown in Fig. P. 6.2.6. **Dec. 05, 3 Marks, May 06, 8 Marks**

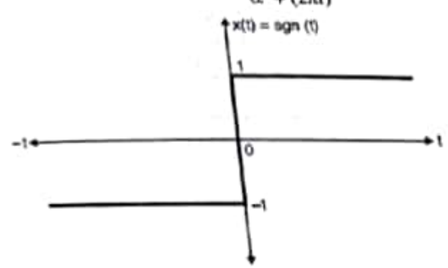
Soln. :

In the Ex. 6.2.5, we have found FT of an antisymmetrical exponential pulse. The antisymmetrical pulse can be represented as,

$$x(t) = e^{-|\alpha t|}\, \text{sgn}(t) \quad ...(1)$$

Therefore as proved in Ex. 6.2.5, the FT of the antisymmetric exponential pulse is given by,

$$F[e^{-|\alpha t|}\, \text{sgn}(t)] = \frac{-j4\pi f}{\alpha^2 + (2\pi f)^2} \quad ...(2)$$



(H-487) Fig. P. 6.2.6 : Signum function

In the Equation (2), substitute α = 0 to get,

$$F[e^0\, \text{sgn}(t)] = \frac{-j4\pi f}{4\pi^2 f^2}$$

$$\therefore \quad F[\text{sgn}(t)] = \frac{-j}{\pi f}$$

$$\therefore \text{ Signum function sgn (t)} \xleftrightarrow{F} \frac{-j}{\pi f} \quad \text{...Ans.}$$

Ex. 6.2.7 : Obtain the fourier transform of a unit step function.

Soln. :

A unit step function is mathematically defined as,

$$u(t) = 1 \qquad \text{for } t \geq 0$$
$$= 0 \qquad \text{elsewhere}$$

Using the definition of the fourier transform we get,

$$F[u(t)] = \int_{-\infty}^{\infty} u(t) \cdot e^{-j2\pi ft}\, dt$$

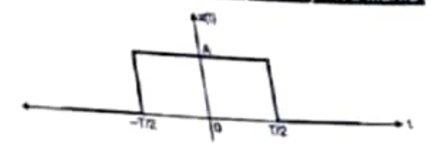
But unit step function is present only for $t \geq 0$

$$\therefore \quad F[u(t)] = \int_{0}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt = \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_0^{\infty}$$

$$= \frac{1}{-j2\pi f}\left[e^{-\infty} - e^0\right] = \frac{1}{-j2\pi f}[0 - 1] = \frac{1}{j2\pi f}$$

$$\therefore \quad u(t) \xrightarrow{F} \frac{1}{j2\pi f} \quad \text{...Ans.}$$

Ex. 6.2.8 : Obtain the fourier transform of a rectangular pulse of duration T and amplitude A as shown in Fig. P. 6.2.8(a). **May 03, May 11, 5 Marks**



(H-488) Fig. P. 6.2.8(a) : Rectangular pulse

Soln. :

The rectangular pulse shown in Fig. P. 6.2.8(a) can be expressed mathematically as,

$$\text{rect}(t/T) = A \qquad \text{for } -T/2 \leq +\leq T/2$$
$$= 0 \qquad \text{elsewhere}$$

This is also known as the gate function.

Therefore the fourier transform will be,

$$F[x(t)] = X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft}\, dt \quad \text{...by definition of F.T.}$$

$$= \int_{-T/2}^{T/2} A e^{-j2\pi ft}\, dt = \frac{A}{-j2\pi f}\left[e^{-j2\pi ft}\right]_{-T/2}^{T/2}$$

$$= \frac{A}{-j2\pi f}\left[e^{-j\pi fT} - e^{j\pi fT}\right] = \frac{A}{j2\pi f}\left[e^{j\pi fT} - e^{-j\pi fT}\right]$$

$$= \frac{A}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right] \quad ...(1)$$

As per the Euler's theorem,

$$\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

Applying this to Equation (1),

$$\text{we get, } F[x(t)] = \frac{A}{\pi f}[\sin(\pi fT)] \quad ...(2)$$

Multiply and divide the RHS of Equation (2) by T to get,

$$F[x(t)] = AT\frac{\sin(\pi fT)}{\pi fT} \quad ...(3)$$

In the above equation,

$$\frac{\sin(\pi f T)}{\pi f T} = \text{sinc}(fT) \qquad ...(4)$$

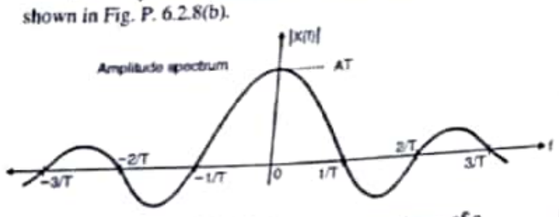
$$\therefore \quad F[x(t)] = AT\,\text{sinc}(fT)$$

$$\therefore \quad A\,\text{rect}(t/T) \xleftrightarrow{F} AT\,\text{sinc}(fT)$$

Thus the rectangular pulse transforms into a sinc function.

Amplitude spectrum :

The amplitude spectrum of the rectangular function is as shown in Fig. P. 6.2.8(b).



(H-489) **Fig. P. 6.2.8(b) : Amplitude spectrum of a rectangular pulse**

As we already know,

$$\text{sinc}(0) = 1 \qquad \therefore \quad AT\,\text{sinc}(0) = AT$$

The sinc function will have zero value for the following values of "fT" :

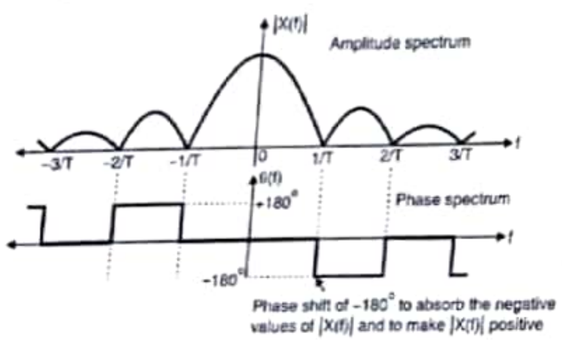
$$\text{sinc}(fT) = 0 \qquad \text{for} \qquad fT = \pm 1, \pm 2, \pm 3, ...$$

$$\text{i.e. for } f = \pm\frac{1}{T}, \pm\frac{2}{T}, \pm\frac{3}{T} .....$$

The phase spectrum has not been shown as it has zero value for all the values of f.

To absorb negative values of | X(f) | in the phase shift :

The negative amplitude of the amplitude spectrum | X (f) | can be made positive by introducing a phase shift of ± 180° in the phase spectrum. This is as shown in Fig. P. 6.2.8(c). A negative phase shift for positive frequency and positive phase shift for the negative frequency is introduced in order to maintain symmetry of the phase spectrum.



(H-490) **Fig. P. 6.2.8(c) : Amplitude and phase spectrums for a rectangular pulse. Negative values of | X(f) | have been absorbed in the additional phase shift of ± 180° in the phase spectrum**

Note : The choice of the phase shift of ± 180° is completely arbitrary.

6.2.3 Property 3 : Duality or Symmetry Property :

This property states that, if $x(t) \xleftrightarrow{F} X(-f)$

$$\text{Then} \quad X(t) \xleftrightarrow{F} x(-f) \qquad ...(6.2.6)$$

i.e. t and f can be interchanged.

Meaning :

1. In the term x(t), "x" represents the shape of the signal and "t" shows that the variable is time.
2. And in the term X (– f), "X" represents the shape of the spectrum and "f" shows that the variable is frequency.
3. The Duality theorem tell us that if $x(t) \xleftrightarrow{F} X(-f)$ then the shape of the signal in the time domain and the shape of the spectrum can be interchanged.

Proof :

– By definition of the inverse Fourier transform,

$$x(t) = \int_{-\infty}^{\infty} x(t)\, e^{j2\pi ft}\, df$$

– Substitute t = – t in the above equation to get,

$$x(-t) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft}\, df$$

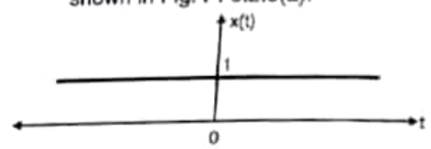
– Interchange t and f to get,

$$x(-f) = \int_{-\infty}^{\infty} X(t)\, e^{-j2\pi ft}\, df = F[X(t)]$$

$$\therefore \quad x(-f) \xleftrightarrow{F} X(t) \qquad \text{...Proved.}$$

The interchanging of variables is possible because the Fourier transform is a definite integral where the variables of integration are dummy variables.

Ex. 6.2.9 : Obtain the fourier transform of a dc signal shown in Fig. P. 6.2.9(a).



(H-491) **Fig. P. 6.2.9(a) : A dc signal**

Soln. :

We have already obtained the fourier transform of the unit impulse or delta function δ (t).

$$\delta(t) \xleftrightarrow{F} 1 \qquad ...(1)$$

To solve this example we are going to use the duality property.

Duality property : It states that

$$\text{If} \quad x(t) \xleftrightarrow{F} X(-f) \text{ then } X(t) \xleftrightarrow{F} x(-f).$$

Step 1 : To verify the condition $x(t) \xleftrightarrow{F} X(-f)$:

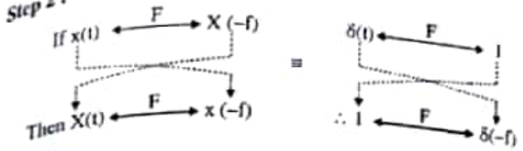
$$\delta(t) \xleftrightarrow{F} 1$$

Hence X (f) = 1. As 1 does not depend on f, X (f) = X (– f) = 1

$$\therefore \quad \delta(t) \xleftrightarrow{F} X(-f) \qquad \text{...Proved.}$$

As this condition is verified, we can apply duality theorem.

Step 2 : Apply duality theorem :



(H-492)

Consider the statement $1 \xleftrightarrow{F} \delta(-f)$. As the "δ" function is a symmetrical function,

$$\delta(-f) = \delta(f)$$

$$\therefore \quad 1 \xleftrightarrow{F} \delta(f) \qquad \text{...Ans.}$$

Thus the dc signal of unity amplitude is transformed into a delta function in the frequency domain as shown in Fig. P. 6.2.9(b). Hence we can use the duality property as,

$$X(t) \xleftrightarrow{F} x(-f) \qquad ...(2)$$

In Equation (2), X (t) = 1 and x (– f) = δ (– f)

$$\therefore \quad 1 \xleftrightarrow{F} \delta(-f) \qquad ...(3)$$

But as the delta function is a symmetrical (even) function

$$\delta(-f) = \delta(f)$$

$$\text{Therefore } 1 \xleftrightarrow{F} \delta(f) \qquad \text{...Ans.}$$

Meaning :

The above equation states that a dc signal of unity amplitude is transformed into a delta function in the frequency domain.

Important deduction :

The dc signal shown in Fig. P. 6.2.9(a) is represented mathematically as,

$$x(t) = 1 \text{ for } -\infty < t < \infty \qquad ...(4)$$

Therefore its fourier transform is given as,

$$X(f) = \int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt \qquad ...(5)$$

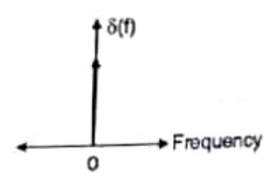
But we have already obtained the fourier transform of a dc signal

$$\therefore \quad X(f) = \delta(f) \qquad ...(6)$$

Substituting this in Equation (5) we get,

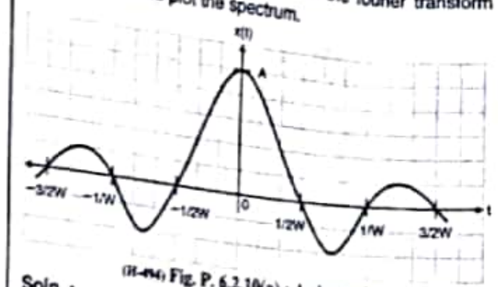
$$\int_{-\infty}^{\infty} 1 \cdot e^{-j2\pi ft}\, dt = \delta(f) \qquad ...(7)$$

The spectrum of the dc signal is as shown in Fig. P. 6.2.9(b).



(H-493) **Fig. P. 6.2.9(b) : Spectrum of a dc signal**

Ex. 6.2.10 : For the sinc function shown in Fig. P. 6.2.10(a), obtain the fourier transform and plot the spectrum.



(H-494) **Fig. P. 6.2.10(a) : A sinc pulse**

Soln. :

– The sinc signal shown in Fig. P. 6.2.10(a) can be expressed mathematically as :

$$x(t) = A\,\text{sinc}(2Wt) \qquad ...(1)$$

– To evaluate the fourier transform of this function, we are going to apply the duality and time scaling properties of the fourier transform. We have obtained the fourier transform of a rectangular pulse of amplitude A and duration T, as :

$$A\,\text{rect}[t/T] \xleftrightarrow{F} AT\,\text{sinc}(fT) \qquad ...(2)$$

– Using the duality property we can write that,

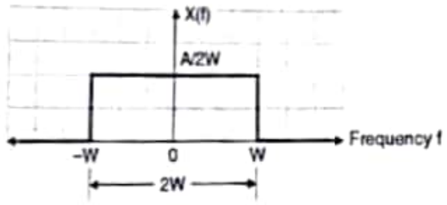
$$AT\,\text{sinc}(tT) \xleftrightarrow{F} A\,\text{rect}[f/T] \qquad ...(3)$$

Compare the LHS of Equation (3) with the RHS of Equation (1) which states the expression for x (t), it is observed that T = 2W. Substituting this into Equation (3) we get,

$$2AW\,\text{sinc}(2Wt) \xleftrightarrow{F} A\,\text{rect}[f/2W]$$

$$\therefore \quad A\,\text{sinc}(2Wt) \xleftrightarrow{F} \frac{A}{2W}\,\text{rect}[f/2W] \qquad \text{...Ans.}$$

Thus a sinc pulse in the time domain is transformed into a rectangular pulse in the frequency domain. The spectrum of sinc pulse is shown in Fig. P. 6.2.10(b).



(H-495) **Fig. P. 6.2.10(b) : Spectrum of a sinc pulse**

6.2.4 Property 4 : Time Shifting :

MU : May 11, May 14, Dec. 14, Dec. 15

University Questions

Q. 1 Explain and prove time shifting and frequency shifting property of Fourier transform. **(May 11, 6 Marks)**

Q. 2 State and prove the following properties of Fourier transform : Time shifting property **(May 14, 2 Marks)**

Q. 3 State and prove time shifting and time scaling property of continuous time Fourier transform. **(Dec. 14, 4 Marks)**

Q. 4 Prove and explain time scaling and amplitude scaling property of continuous time Fourier transform. **(Dec. 15, 5 Marks)**

The time shifting property states that if x (t) and X (f) form a Fourier transform pair then,

$$x(t - t_d) \xrightarrow{F} e^{-j2\pi f t_d} X(f) \qquad ...(6.2.7)$$

Here the signal $x(t - t_d)$ is a time shifted signal. It is the same signal x (t) only shifted in time.

Proof :

$$F[x(t - t_d)] = \int_{-\infty}^{\infty} x(t - t_d) \cdot e^{-j2\pi f t} dt \qquad ...(6.2.8)$$

Let $(t - t_d) = \tau$,

$\therefore \quad t = t_d + \tau$

$\therefore \quad dt = d\tau$.

Substituting these values in Equation (6.2.8) we get,

$$F[x(t - t_d)] = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j2\pi f (t_d + \tau)} d\tau$$

$$= e^{-j2\pi f t_d} \int_{-\infty}^{\infty} x(\tau) e^{-j2\pi f \tau} d\tau$$

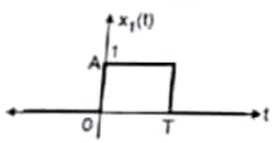
$$\therefore F[x(t - t_d)] = e^{-j2\pi f t_d} X(f) \qquad \textbf{...Proved.}$$

This shows that the time shifting does not have any effect on the amplitude spectrum, but there is an additional phase shift of $-2\pi f t_d$, which is denoted by the term $e^{-j2\pi f t_d}$.

Significance of time shifting in communication systems :

- If signal x(t) is transmitted by a transmitter, then due to the distance travelled, this signal becomes a time delayed signal $x(t - t_d)$ when it reaches the receiver.
- The time delay "t_d" is dependent on the distance between the transmitter and the receiver.
- The time shifting property explains the effect of such time shifting on the spectrum of the signal. It tells us that there is no effect of time shifting on the amplitude spectrum but there is an additional phase shift of $-2\pi f t_d$.

Ex. 6.2.11 : Obtain the fourier transform of the rectangular pulse as shown in Fig. P. 6.2.11.



(H-496) **Fig. P. 6.2.11 : A shifted rectangular pulse**

Soln. :

1. A pulse which has an amplitude A and which extends from $t = -T/2$ to $t = +T/2$ is expressed as follows :

$$x(t) = A \text{ rect } (t / T) \qquad ...(1)$$

2. The rectangular pulse of Fig. P. 6.2.11 is the delayed version of the standard rectangular pulse expressed in Equation (2), with a time delay of T/2. Hence it can be mathematically represented as,

$$x_1(t) = A \text{ rect}\left[\frac{t - T/2}{T}\right] \qquad ...(2)$$

3. Using the property of "time shifting" we get its fourier transform as,

$$F[x_1(t)] = X_1(f) = e^{-j2\pi f T/2} X(f)$$

$$= e^{-j\pi f T} \cdot AT \text{ sinc } (fT)$$

$$\therefore \quad X_1(f) = AT e^{-j\pi f T} \text{ sinc } (fT) \qquad \textbf{...Ans.}$$

Ex. 6.2.12 : Obtain the fourier transform of the rectangular pulse as shown in Fig. P. 6.2.12.

(H-497) **Fig. P. 6.2.12 : A shifted rectangular pulse**

Soln. :

1. The rectangular pulse in Fig. P. 6.2.12 is the advanced version of the standard rectangular pulse, with an advance of T/2. Hence the signal is represented as,

$$x_2(t) = A \text{ rect}\left[\frac{t + T/2}{T}\right] \qquad ...(1)$$

2. Using the property of time shifting, the fourier transform of this pulse is given as,

$$X_2(f) = e^{-j2\pi f(-T/2)} \cdot AT \text{ sinc } (fT)$$

$$\therefore \quad X_2(f) = e^{+j\pi f T} AT \text{ sinc } (fT) \qquad \textbf{...Ans.}$$

6.2.5 Property 5 : Area under x(t) :

This property states that the area under the curve x(t) equals the value of its Fourier transform at f = 0.

i.e. if $x(t) \xleftrightarrow{F} X(f)$ then,

$$\text{Area under } x(t) = \int_{-\infty}^{\infty} x(t)\, dt = X(0) \qquad ...(6.2.9)$$

$$\text{Area under } x(t) = \int_{-\infty}^{\infty} x(t)\, dt = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \text{ at } f = 0.$$

$$= X(f) \text{ at } f = 0$$

$$\therefore \quad \text{Area under } x(t) = X(0). \qquad \textbf{...Proved.}$$

6.2.6 Property 6 : Area under X (f) :

If $x(t) \xleftrightarrow{F} X(f)$ then the area under X (f) is equal to the value of signal x(t) at t = 0.

That means if $x(t) \xleftrightarrow{F} X(f)$ then,

$$\text{Area under } X(f) = \int_{-\infty}^{\infty} X(f) \cdot df$$

$$= x(0) \qquad ...(6.2.10)$$

Proof :

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{j2\pi f t} df$$

Substitute t = 0 in this equation to get,

$$x(0) = \int_{-\infty}^{\infty} X(f) \cdot e^{0} df = \int_{-\infty}^{\infty} X(f)\, dt$$

The RHS of this equation is the area under X (f). Hence the property is proved.

Ex. 6.2.13 : Show that the total area under the curve of sinc function is equal to 1. That means

$$\int_{-\infty}^{\infty} \text{sinc } (t)\, dt = 1 \qquad ...(1)$$

Soln. :

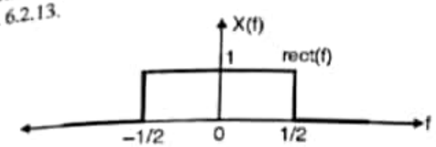
1. The fourier transform of a sinc pulse is given by,

$$F[\text{sinc } (t)] = \int_{-\infty}^{\infty} \text{sinc}(t) \cdot e^{-j2\pi f t} dt \qquad ...(2)$$

2. If we substitute f = 0 in Equation (2) then we will get the area under the sinc function. Therefore let us write the expression for the fourier transform of sinc (t) and then substitute f = 0 into it to get the area.

$$\therefore \quad F[\text{sinc } (t)] = \text{rect } (f) \qquad ...(3)$$

Thus fourier transform is a rectangular pulse as shown in Fig. P. 6.2.13.



(H-498) **Fig. P. 6.2.13**

3. $\therefore$ Area under sinc (t) = rect (0) ...(4)

From Fig. P. 6.2.13, rect (0) = 1

$$\therefore \quad \text{Area under sinc } (t) = \int_{-\infty}^{\infty} \text{sinc } (t)\, dt = 1 \textbf{...Proved.}$$

6.2.7 Property 7 : Frequency Shifting :

MU : May 11

University Questions

Q. 1 Explain and prove time shifting and frequency shifting property of Fourier transform. **(May 11, 6 Marks)**

The frequency shifting characteristics states that if x(t) and X(f) form a Fourier transform pair then,

$$e^{j2\pi f_c t} x(t) \xleftrightarrow{F} X(f - f_c) \qquad ...(6.2.11)$$

Here f_c is a real constant.

Proof :

$$F\left[e^{j2\pi f_c t} x(t)\right] = \int_{-\infty}^{\infty} e^{j2\pi f_c t} x(t) e^{-j2\pi f t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f - f_c) t} dt$$

$$= X(f - f_c) \qquad \textbf{...Proved.}$$

The term $X(f - f_c)$ represents a shifted frequency spectrum. The whole spectrum is thus shifted right by "f_c" in the frequency domain, when the signal x (t) is multiplied by $e^{j2\pi f_c t}$ in the time domain.

6.2.8 Property 8 : Differentiation in Time Domain :

MU : May 06, Dec. 06, Dec. 11

University Questions

Q. 1 Explain and prove time shifting and frequency shifting property of Fourier transform. **(May 06, 8 Marks, Dec. 06, 4 Marks)**

Q. 2 State and prove following properties of Fourier transform :

1. Convolution in time domain.
2. Differentiation in time domain. **(Dec. 11, 6 Marks)**

Some processing techniques involve differentiation and integration of the signal x(t). **This property is applicable if and only if the derivative of x (t) is Fourier transformable.**

Statement : Let $x(t) \xleftrightarrow{F} X(f)$ and let the derivative of x(t) be Fourier transformable. Then,

$$\frac{d}{dt} x(t) \xleftrightarrow{F} j2\pi f X(f) \qquad ...(6.2.12)$$

Proof : By the definition of inverse Fourier transform,

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{+j2\pi f t} df$$

$$\text{Therefore} \quad \frac{d}{dt} x(t) = \frac{d}{dt}\left[\int_{-\infty}^{\infty} X(f) e^{+j2\pi f t} df\right]$$

$$= \int_{-\infty}^{\infty} X(f) \left(\frac{d}{dt} e^{+j2\pi f t}\right) df$$

$$\frac{d}{dt} x(t) = \int_{-\infty}^{\infty} [X(f) \cdot j2\pi f] e^{+j2\pi f t} df$$

As per the definition of the inverse Fourier transform the term inside the square bracket must be the Fourier transform of $\frac{d}{dt} x(t)$.

$$\therefore \quad F\left[\frac{d}{dt} x(t)\right] = j2\pi f X(f)$$

$$\text{OR} \quad \frac{d}{dt} x(t) \xleftrightarrow{F} j2\pi f X(f). \qquad \textbf{...Proved.}$$

Meaning :

Differentiating the signal in time domain is equivalent to multiplying its Fourier transform by (j2πf) in the frequency domain. Thus differentiation will enhance the high frequency components since $|j2\pi f X(f)| > |X(f)|$.

6.2.9 Property 9 : Integration in Time Domain :

Integration in time domain is equivalent to dividing the Fourier transform by (j2πf).

i.e. if $x(t) \overset{F}{\longleftrightarrow} X(f)$ and provided that $X(0) = 0$ then,

$$\int_{-\infty}^{t} x(\lambda)\, d\lambda \overset{F}{\longleftrightarrow} \frac{1}{j2\pi f} X(f) \qquad ...(6.2.13)$$

Proof :

In order to avoid confusion let us use a different variable than "t". Therefore let us express x(t) as,

$$x(t) = \frac{d}{dt}\left[\int_{-\infty}^{t} x(\lambda)\, d\lambda\right] \qquad ...(6.2.14)$$

Taking the Fourier transform of both the sides,

$$X(f) = F\left[\frac{d}{dt}\left\{\int_{-\infty}^{t} x(\lambda)\, d\lambda\right\}\right]. \qquad ...(6.2.15)$$

Using the property of differentiation we get,

$$X(f) = j\,2\pi f\, F\left[\int_{-\infty}^{t} x(\lambda)\, d\lambda\right]$$

$$\therefore \quad F\left[\int_{-\infty}^{t} x(\lambda)\, d\lambda\right] = \frac{1}{j\,2\pi f} X(f) \qquad ...(6.2.16)$$

This is the desired result. Equation (6.2.16) shows that the integration suppresses the high frequency components because X (f) is being divided by 2πf.

6.2.10 Property 10 : Multiplication in Time Domain (Multiplication Theorem) :

The multiplication theorem states that : If $x_1(t) \overset{F}{\longleftrightarrow} X_1(f)$ and $x_2(t) \overset{F}{\longleftrightarrow} X_2(f)$ are the two Fourier transform pairs then,

$$x_1(t) \cdot x_2(t) \overset{F}{\longleftrightarrow} \int_{-\infty}^{\infty} X_1(\tau) \cdot X_2(f-\tau)\, d\tau \qquad ...(6.2.17)$$

This means that the multiplication of two signals $x_1(t)$ and $x_2(t)$ in the time domain gets transformed into convolution of their Fourier transforms in the frequency domain.

$$\therefore \quad x_1(t) \cdot x_2(t) \overset{F}{\longleftrightarrow} X_1(f) * X_2(f) \qquad ...(6.2.18)$$

Proof : The Fourier transform of the product signal is given by,

$$F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) \cdot x_2(t)]\, e^{-j2\pi f t}\, dt \qquad ...(6.2.19)$$

Let us express $x_2(t)$ as follows :

By definition of IFT, $x_2(t) = \int_{-\infty}^{\infty} X_2(f') \cdot e^{+j2\pi f' t}\, df'$

Here f and f′ are not same. This is just to avoid confusion.

Now substitute this value of $x_2(t)$ in the Equation (6.2.19) to get,

$$F[x_1(t) \cdot x_2(t)] = \int_{-\infty}^{\infty} x_1(t) \cdot \int_{-\infty}^{\infty} X_2(f')\, e^{+j2\pi f' t}\, df' \cdot e^{-j2\pi f t}\, dt \qquad ...(6.2.20)$$

$$= \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(f')\, e^{-j2\pi(f-f')t}\, df'\, dt \qquad ...(6.2.21)$$

Now substitute $\tau = f - f'$

$$= \int_{-\infty}^{\infty} x_1(t) \int_{-\infty}^{\infty} X_2(f-\tau)\, e^{-j2\pi\tau t}\, d\tau\, dt \qquad ...(6.2.22)$$

$$= \int_{-\infty}^{\infty} X_2(f-\tau)\, d\tau \cdot \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi f t}\, dt \qquad ...(6.2.23)$$

The second term in the above equation represents the Fourier transform of $x_1(t)$. Therefore Equation (6.2.23) can be written as,

$$F[x_1(t)\, x_2(t)] = \int_{-\infty}^{\infty} X_2(f-\tau)\, d\tau\, X_1(\tau)$$

$$\text{OR} = \int_{-\infty}^{\infty} X_1(\tau) \cdot X_2(f-\tau) \cdot d\tau$$

Thus $[x_1(t) \cdot x_2(t)] \overset{F}{\longleftrightarrow} X_1(f) * X_2(f)$ **...Proved.**

This is the desired result.

6.2.11 Property 11 : Convolution in the Time Domain (Convolution Theorem) :

MU : Dec. 04, Dec. 11, May 14

University Questions

Q. 1 State properties of Fourier transform. Give proof of convolution property. **(Dec. 04, 10 Marks)**

Q. 2 State and prove following properties of Fourier transform :
1. Convolution in time domain.
2. Differentiation in time domain.

(Dec. 11, 10 Marks)

Q. 3 State and prove the following properties of Fourier transform : Convolution property **(May 14, 2 Marks)**

This property states that the convolution of signals in the time domain will be transformed into the multiplication of their Fourier transforms in the frequency domain.

$$\text{i.e. } [x_1(t) * x_2(t)] \overset{F}{\longleftrightarrow} X_1(f)\, X_2(f) \qquad ...(6.2.24)$$

Proof :

The convolution of the two signals in the time domain is defined as,

$$X_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\lambda) \cdot x_2(t-\lambda)\, d\lambda \qquad ...(6.2.25)$$

Taking the Fourier transform of the convolution.

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\lambda) \cdot x_2(t-\lambda)\, d\lambda\right] e^{-j2\pi f t}\, dt \qquad ...(6.2.26)$$

Multiply and divide the RHS of the Equation (6.2.26) by $e^{-j2\pi f\lambda}$ to get,

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\lambda) \cdot e^{-j2\pi f\lambda}\, d\lambda \cdot$$

$$\int_{-\infty}^{\infty} x_2(t-\lambda) \cdot e^{-j2\pi f t} \cdot e^{j2\pi f\lambda}\, dt$$

$$= \int_{-\infty}^{\infty} x_1(\lambda)\, e^{-j2\pi f\lambda}\, d\lambda \int_{-\infty}^{\infty} x_2(t-\lambda)\, e^{-j2\pi f(t-\lambda)}\, dt \qquad ...(6.2.27)$$

Let $(t - \lambda) = m$ in Equation (6.2.26)

$$\therefore \quad F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} x_1(\lambda) \cdot e^{-j2\pi f\lambda}\, d\lambda \int_{-\infty}^{\infty} x_2(m) \cdot e^{-j2\pi f m}\, dm$$

Using the definition of the Fourier transform to the RHS we get,

$$F[x_1(t) * x_2(t)] = X_1(f)\, X_2(f) \qquad ...(6.2.28)$$

This is the required result.

Significance of convolution in a communication system :

Consider a communication system with input x(t), output y(t) and impulse response h(t), as shown in Fig. 6.2.2.

(H-499) **Fig. 6.2.2**

The relation between x(t), h(t) and y(t) is as follows,

$$y(t) = x(t) * h(t) \qquad ...(6.2.29)$$

That means y(t) is obtained by taking convolution of x(t) and h (t).

Method I :

Output of a system y(t) can be obtained by taking the convolution of input x(t) and impulse response of the system h (t).

$$y(t) = x(t) * h(t)$$

Method II :

The output y(t) can be obtained using the Fourier transform.

Let $y(t) \overset{F}{\longleftrightarrow} Y(f)$, $x(t) \overset{F}{\longleftrightarrow} X(f)$ and $h(t) \overset{F}{\longleftrightarrow} H(f)$. Then taking the F.T. of Equation (6.2.29) we get

$$Y(f) = X(f) \cdot H(f)$$

Now take IFT to get $y(t) = \text{IFT}[Y(f)]$

Multiplication and taking IFT is simpler than obtaining the convolution. Hence in practice we can use this method to obtain y (t) i.e. output of a system.

A signal cannot be band limited and time limited simultaneously :

- To justify this statement let us first consider a time limited signal and its Fourier transform. A rectangular pulse of duration T and amplitude A is an excellent example of a time limited signal.
- We know that its Fourier transform is a sinc pulse which extends from $f = -\infty$ to $f = +\infty$ eventhough the amplitude of the sinc function goes on decreasing with increase in frequency.
- Thus a time limited signal results into an unlimited spectrum in frequency domain.
- Now consider a rectangular pulse of width "f" in the frequency domain. This is a bandlimited signal. Due to the principle of duality, the inverse Fourier transform of this rectangular pulse is a sinc function in time domain which extends from $t = -\infty$ to $t = +\infty$.
- Thus band limiting in the frequency domain results in a "time unlimited" signal in the time domain. Thus a signal cannot be band limited or time limited simultaneously.

Ex. 6.2.14 : State and prove the convolution property of Fourier transform and find the Fourier transform of the following signal using convolution property :

x (t) = rect (t) * rect (t)

where

$$\text{rect}(t) = \begin{cases} 1 & -\frac{1}{2} \le t \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Soln. :

We know that $\text{rect}(t) \overset{F}{\longleftrightarrow} \text{sinc}(f)$.

$$\therefore [\text{rect}(t) * \text{rect}(t)] \overset{F}{\longleftrightarrow} \text{sinc}(f)\,\text{sinc}(f) = \text{sinc}^2(f)$$ **...Ans.**

6.2.12 Conjugate Functions :

This property states that, if $x(t) \overset{F}{\longleftrightarrow} X(f)$ and if x (t) is a complex valued function then,

$$x^*(t) \overset{F}{\longleftrightarrow} X^*(f)$$

Proof :

As per the definition of IFT, we have

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df$$

Now take the complex conjugate of both the sides to get,

$$x^*(t) = \left[\int_{-\infty}^{\infty} X(f) \cdot e^{+j2\pi f t}\, df\right]^*$$

$$\therefore \quad x^*(t) = \int_{-\infty}^{\infty} X^*(f)\, e^{-j2\pi f t}\, df \qquad ...(6.2.30)$$

Now replace f by – f to get,

$$x^*(t) = \int_{-\infty}^{\infty} X^*(-f)\, e^{+j2\pi f t}\, df \qquad ...(6.2.31)$$

The RHS of above equation is nothing but the inverse Fourier

transform of $X^*(f)$ ...(6.2.32)

$$\therefore \quad x^*(t) = F^{-1}[X^*(-f)]$$

$$\therefore \quad x^*(t) \xleftrightarrow{F} X^*(-f)$$ ...Proved.

Real and imaginary parts of a time function :

Let x(t) be a complex-valued function which can be represented in terms of its real and imaginary parts as follows :

$$x(t) = Re[x(t)] + j\,Im[x(t)]$$ ...(6.2.33)

where Re = Real part of x(t) and

Im = Imaginary part of x(t).

The complex conjugate of x (t) is given by,

$$x^*(t) = Re[x(t)] - j\,Im[x(t)]$$ ...(6.2.34)

Add Equations (6.2.33) and (6.2.34) to get,

$$2\,Re[x(t)] = x(t) + x^*(t)$$

$$\therefore \quad Re[x(t)] = \frac{1}{2}\left[x(t) + X^*(t)\right]$$ ...(6.2.35)

Subtract Equations (6.2.34) from (6.2.33) to get,

$$2j\,Im[x(t)] = x(t) - x^*(t)$$

$$\therefore \quad Im[x(t)] = \frac{1}{2j}\left[x(t) - X^*(t)\right]$$ ...(6.2.36)

Therefore applying the conjugate property we can write that,

$$F\{Re[x(t)]\} = \frac{1}{2}\left\{F[x(t)] + F[x^*(t)]\right\}$$

$$\therefore \quad F\{Re[x(t)]\} = \frac{1}{2}\left[X(f) + X^*(-f)\right]$$ ...(6.2.37)

Similarly,

$$F\{Im[x(t)]\} = \frac{1}{2j}\left\{F[x(t)] - F[x^*(t)]\right\}$$

$$\therefore \quad F\{Im[x(t)]\} = \frac{1}{2j}\left[X(f) - X^*(-f)\right]$$ ...(6.2.38)

Conclusion : From Equation (6.2.38), it is clear that imaginary part of x(t) for a real valued signal x(t) will be 0. Therefore $X(f) = X^*(-f)$, that means X(f) exhibits a "conjugate symmetry".

6.2.13 Parseval's Power Theorem :

This theorem relates the average power P of a periodic signal to its "fourier series" coefficients. The Parseval's power theorem states that the total average power of a periodic signal x(t) is equal to the sum of the average powers of the individual fourier coefficients i.e. C_n.

$\therefore$ Average power of x (t) = (Power of C_1) + (Power of C_2) + ...

...(6.2.39)

$$\text{or total average power : } P = \sum_{n=-\infty}^{\infty} |C_n|^2 \quad ...(6.2.40)$$

Proof of the theorem :

The total average normalized power of signal x (t) is given by,

$$P = \frac{1}{T_0}\int_{t}^{t+T_0} |x(t)|^2\,dt \quad ...(6.2.41)$$

Substitute $|x(t)|^2 = x(t)\,x^*(t)$ in the Equation (6.2.41) where $x^*(t)$ is the complex conjugate of x (t).

$$\therefore \quad P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\,x^*(t)\cdot dt \quad ...(6.2.42)$$

We have defined x(t) using the exponential fourier series as follows :

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}$$

Therefore the complex conjugate of x(t) is given as,

$$x^*(t) = \left[\sum_{n=-\infty}^{\infty} C_n e^{j2\pi n f_0 t}\right]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} C_n^* e^{-j2\pi n f_0 t}\right] \quad ...(6.2.43)$$

Substitute the value of $x^*(t)$ into the Equation (6.2.42) to get,

$$P = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cdot\left[\sum_{n=-\infty}^{\infty} C_n^* e^{-j2\pi n f_0 t}\right]dt$$

Rearranging the equation we get,

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} C_n^*\left[\int_{-T_0/2}^{T_0/2} x(t)\cdot e^{-j2\pi n f_0 t}\,dt\right]$$

$$P = \sum_{n=-\infty}^{\infty}\left[\frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cdot e^{-j2\pi n f_0 t}\,dt\right]C_n^* \quad ...(6.2.44)$$

The term inside the square bracket in Equation (6.2.44) is nothing but "C_n".

$$\therefore \quad P = \sum_{n=-\infty}^{\infty} C_n\cdot C_n^*$$

But $C_n\cdot C_n^* = |C_n|^2$

$$\therefore \quad P = \sum_{n=-\infty}^{\infty} |C_n|^2$$ ...Proved. ...(6.2.45)

The interpretation of the Parseval's theorem is that the total average power of the signal x (t) can be found by squaring and adding the heights $|C_n|$ of the amplitude lines in the spectrum of the periodic signal x(t). Thus the Parseval's theorem implies "superposition" of the average powers.

6.2.14 Solved University Examples on Properties of F.T. :

Ex. 6.2.15 : Determine Fourier transform of an impulse function. What is the effect on its Fourier transform it impulse is delayed in time by 3 sec ? **May 03, 4 Marks**

Soln. : For the first part of Example refer Ex. 6.2.4.

If the impulse is delayed in time by 3 sec.

Shifting property of delta function :

Assume $t_d = 3$ and $f(t) = e^{-j2\pi t}$

$$x(t - t_d) \xleftrightarrow{F} e^{-j2\pi t_d} X(f)$$

$$\delta(t)1 \xleftrightarrow{F} 1$$

$$\therefore \quad X(f) = 1$$

$$\therefore \quad \delta(t-3) \xleftrightarrow{F} e^{-j2\pi f(3)} X(f)$$

$$\therefore \quad \delta(t-3) \xleftrightarrow{F} e^{-j6\pi f} X(f)$$

$$\therefore \quad \delta(t-3) \xleftrightarrow{F} e^{-j6\pi f}$$ ...Ans.

Ex. 6.2.16 : Prove that if Fourier transform of a function f (t) is F (ω) then Fourier transform of $-jt\,f(t)$ is $\frac{d}{d\omega}F(\omega)$ **May 03, 8 Marks**

Soln. :

Given :

$$f(t) \longleftrightarrow F(\omega)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt$$

Differentiating with respect to ω

$$\frac{d}{d\omega}F(\omega) = \int_{-\infty}^{\infty} f(t)\cdot(-jt)\,e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} [-jt\,f(t)]\,e^{-j\omega t}\,dt$$

The right hand side equation is basically Fourier transform of $-jt\,f(t)$. Hence Fourier transform

$$(F\cdot T)[-jt\,f(t)] = \frac{d}{d\omega}F(\omega)$$

Ex. 6.2.17 : If Fourier transform of a function x (t) is given by X (f) prove that : **Dec. 03, 8 Marks**

1. $\frac{d}{dt}x(t) \xleftrightarrow{F} j\,2\pi f\,X(f)$
2. $x(t)\cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2}[X(f-f_0) + X(f+f_0)]$

Soln. :

1. $\frac{d}{dt}x(t) \xleftrightarrow{F} j\,2\pi f\,X(f)$:

1. By the definition of inverse Fourier transform,

$$x(t) = \int_{-\infty}^{\infty} X(f)\,e^{j2\pi f t}\,df$$

$$\text{Therefore } \frac{d}{dt}x(t) = \frac{d}{dt}\left[\int_{-\infty}^{\infty} X(f)\,e^{j2\pi f t}\,df\right]$$

$$= \int_{-\infty}^{\infty} X(f)\left(\frac{d}{dt}e^{j2\pi f t}\right)df$$

$$\frac{d}{dt}x(t) = \int_{-\infty}^{\infty}[X(f)\cdot j2\pi f]\,e^{j2\pi f t}\,df$$

As per the definition of the inverse Fourier transform the term inside the square bracket must be the Fourier transform of $\frac{d}{dt}x(t)$.

$$\therefore \quad F\left[\frac{d}{dt}x(t)\right] = j\,2\pi f\,X(f)$$

$$\text{OR} \quad \frac{d}{dt}x(t) \xleftrightarrow{F} j\,2\pi f\,X(f)$$ ...Proved.

2. $x(t)\cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2}[X(f-f_0) + X(f+f_0)]$:

By Eulers identity we can write,

$$\cos(2\pi f_0 t) = e^{j2\pi f_0 t} + e^{-j2\pi f_0 t} \quad ...(1)$$

Fourier transform of $x(t)\cdot\cos(2\pi f_0 t)$ is given as

$$X(f) = \int_{-\infty}^{\infty} x(t)\cdot\cos(2\pi f_0 t)\cdot e^{-j2\pi f t}\,dt \quad ...(2)$$

Substituting value of $\cos(2\pi f_0 t)$ in Equation (2),

$$X(f) = \int_{-\infty}^{\infty} x(t)\cdot\left[\frac{e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}}{2}\right]e^{-j2\pi f t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty}\left(e^{-j2\pi(f-f_0)t} + e^{-j2\pi(f+f_0)t}\right)\cdot x(t)\,dt$$

$$\therefore \quad X(f) = \frac{1}{2}\int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi(f-f_0)t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f+f_0)t}\cdot x(t)\,dt$$

$$\int_{-\infty}^{\infty} x(t)\cdot e^{-j2\pi f t}\,dt = x(f)$$

and using frequency shifting property we get,

$$\therefore \quad X(f) \longleftrightarrow \frac{1}{2}X(f-f_0) + \frac{1}{2}X(f+f_0)$$

$$X(f) = \frac{1}{2}[X(f-f_0) + X(f+f_0)]$$

Ex. 6.2.18 : Find Fourier transform of X (t) where the function X (t) is given by x (t) = U (t) and using properties of Fourier transform find Fourier transform of y (t) where

y (t) = U (2t) + U (t – 1) **Dec. 03, 7 Marks**

Soln. :

1. A unit step function is mathematically defined as

$$U(t) = 1 \text{ for } t > 0$$
$$= 0 \text{ elsewhere.}$$

Using the definition of Fourier transform,

$$F[U(t)] = \int_{-\infty}^{\infty} U(t)\cdot e^{-j2\pi f t}\,dt$$

But unit step function is present only for $t \geq 0$,

$$\therefore F[U(t)] = \int_0^{\infty} 1 \cdot e^{-j2\pi ft}\,dt = \frac{1}{-j2\pi f}\left[e^{-j2\pi ft}\right]_0^{\infty}$$

$$= \frac{1}{-j2\pi f}\left[e^{-\infty} - e^{0}\right] = \frac{1}{-j2\pi f}[0 - 1] = \frac{1}{-j2\pi f}$$

$$\therefore U(t) \xleftrightarrow{F} \frac{1}{j2\pi f}$$

2. $y(t) = U(2t) + U(t-1)$

Use the scaling property and time shifting properties of Fourier transform which are given as follows.

Time scaling $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$

$$\therefore \text{for } U(at) \longleftrightarrow \frac{1}{a} \cdot \frac{1}{j2\pi}\left(\frac{f}{a}\right)$$

$$\therefore U(2t) \xleftrightarrow{F} \frac{1}{2} \cdot \frac{1}{j2\pi}\left(\frac{f}{2}\right)$$

$$= \frac{1}{j2\pi f}$$

Time shifting property

$$x(t - t_0) \longleftrightarrow X(f) \cdot e^{-j2\pi ft_0}$$

$$\therefore U(t - t_0) \longleftrightarrow \frac{1}{j2\pi f} \cdot e^{-j2\pi ft_0}$$

$$\therefore U(t-1) \longleftrightarrow \frac{1}{j2\pi f} \cdot e^{-j2\pi f}$$

Using linearity property

$$y(t) = U(2t) + U(t-1)$$

$$\therefore F[y(t)] = F[U(2t)] + F[U(t-1)]$$

$$y(t) \xleftrightarrow{F} \frac{1}{j2\pi f} + \frac{e^{-j2\pi f}}{j2\pi f}$$

Ex. 6.2.19 : Using the time shifting property, show that if $f(t) \leftrightarrow F(\omega)$ then $F(t+T) + F(t-T) \leftrightarrow 2F(\omega)\cos T\omega_0$

May 04, 4 Marks

Soln. :

According to time shifting property we know that,

$$F[x(t-T)] = e^{-j2\pi fT} X(F) \quad ...(1)$$

$$= e^{-j\omega_0 T} F(\omega)$$

$$\therefore F[x(t+T)] = e^{+j2\pi fT} X(F) \quad ...(2)$$

$$= e^{+j\omega_0 T} F(\omega)$$

Using linearity property.

$$F[x(t+T)] + F[x(t-T)] = (e^{+j2\pi fT} - e^{-j2\pi fT}) X(F)$$

But by relation, $\cos\omega_0 T = \dfrac{e^{j\omega_0 T} - e^{-j\omega_0 T}}{2}$

where $\omega_0 = 2\pi f$

Substituting, $2\cos\omega_0 T$ for $(e^{+j2\pi fT} - e^{-j2\pi fT})$

$$F[(t+T)] + F(t-T) = 2\cos\omega_0 T\, F(\omega)$$

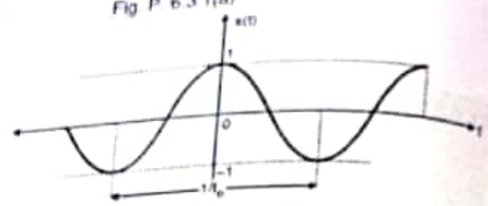
Ex. 6.2.20 : If $f(t)$ has transform $F(\omega)$, then prove that $f(at)$ has fourier transfrom $= \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$

Dec. 04, 10 Marks

Ans. : This is same as scaling property. So refer Section 6.2.2.

6.3 Solved Examples on C.T. Fourier Transform :

Ex. 6.3.1 : Obtain the fourier transform of a cosine wave having a frequency f_o and peak amplitude of unity and plot its spectrum. Refer Fig. P. 6.3.1(a)



(H-502) **Fig. P. 6.3.1(a) : Given wave**

Soln. :

A cosine wave can be mathematically represented as,

$$x(t) = A\cos(2\pi f_o t)$$

But $A = 1$

$$\therefore x(t) = \cos(2\pi f_o t) \quad ...(1)$$

By Euler's identity we can write,

$$\therefore x(t) = \cos(2\pi f_o t) = \frac{e^{+j2\pi f_o t} + e^{-j2\pi f_o t}}{2} \quad ...(2)$$

The fourier transform of $x(t)$ is given by,

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2\pi ft}\,dt$$

Substituting the value of $x(t)$ we get,

$$X(f) = \int_{-\infty}^{\infty} \left[\frac{e^{+j2\pi f_o t} + e^{-j2\pi f_o t}}{2}\right] e^{-j2\pi ft}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \left[e^{-j2\pi(f-f_o)t} + e^{-j2\pi(f+f_o)t}\right] dt$$

$$\therefore X(f) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_o)t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} e^{-j2\pi(f+f_o)t}\,dt \quad ...(3)$$

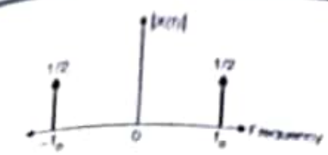
Refer to Fig. 6.3.1(a) in which we have found the fourier transform of a dc signal. In that example we have proved that,

$$\int_{-\infty}^{\infty} e^{-j2\pi ft} = \delta(f) \quad ...(4)$$

Using this property for the RHS of Equation (3) we get,

$$X(f) = \frac{1}{2}\delta(f - f_o) + \frac{1}{2}\delta(f + f_o) \quad \text{...Ans.}$$

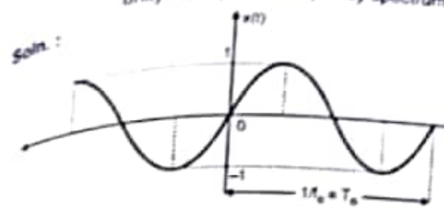
The frequency spectrum is as shown in the Fig. P. 6.3.1(b) which shows that two impulses are present one at f_o and the other at $-f_o$.



(H-503) **Fig. P. 6.3.1(b) : Spectrum of a cosine wave**

Ex. 6.3.2 : Find the Fourier transform of a sinewave having frequency of f_o and peak amplitude of unity. Also plot its frequency spectrum.

Soln. :



(H-504) **Fig. P. 6.3.2(a) : Given waveform**

This example is identical to the previous example. Express the signal $x(t)$ as,

$$x(t) = \sin(2\pi f_o t) \quad ...(1)$$

Using Euler's identity express $x(t)$ as follows :

$$x(t) = \sin(2\pi f_o t) = \frac{e^{+j2\pi f_o t} - e^{-j2\pi f_o t}}{2j} \quad ...(2)$$

The Fourier transform of $x(t)$ is given by,

$$X(f) = \int_{-\infty}^{\infty} \sin(2\pi f_o t)\, e^{-j2\pi ft}\,dt$$

Substituting the value of $\sin(2\pi f_o t)$ we get,

$$X(f) = \int_{-\infty}^{\infty} \frac{1}{2j}\left[e^{j2\pi f_o t} - e^{-j2\pi f_o t}\right] e^{-j2\pi ft}\,dt$$

$$= \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j2\pi(f-f_o)t}\,dt - \frac{1}{2j}\int_{-\infty}^{\infty} e^{-j2\pi(f+f_o)t}\,dt \quad ...(3)$$

We know that, $\displaystyle\int_{-\infty}^{\infty} e^{-j2\pi ft}\,dt = \delta(f)$

Hence Equation (3) can be written as follows :

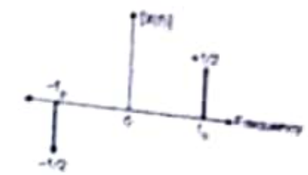
$$X(f) = \frac{1}{2j}[\delta(f - f_o) - \delta(f + f_o)] \quad ...(4)$$

The amplitude spectrum of $\sin(2\pi f_o t)$ is represented as,

$$|X(f)| = \frac{1}{2}[\delta(f - f_o) - \delta(f + f_o)] \quad ...(5)$$

The term $\delta(f - f_o)$ is a delta function in the frequency domain which is shifted right by frequency "f_o". Its amplitude is + 1/2 and the term $\delta(f + f_o)$ represents a delta function which is shifted left by frequency "f_o". Its amplitude is – 1/2.

Therefore the spectrum is as shown in Fig. P. 6.3.2(b).



(H-505) **Fig. P. 6.3.2(b) : Spectrum of a sine wave**

Ex. 6.3.3 : If $x(t) \xleftrightarrow{F} X(f)$ then show that

$$x(t)\cos\omega_0 t \xleftrightarrow{F} \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0)$$

Soln. : The above expression is called as modulation theorem.

$$\therefore F[x(t)\cos\omega_0 t] = \int_{-\infty}^{\infty} [x(t)\cos\omega_0 t]\, e^{-j\omega t}\,dt$$

Using Euler's identity, $\cos\omega_0 t = \dfrac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$$\therefore X(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} x(t)\, e^{+j\omega_0 t}\, e^{-j\omega t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} x(t)\, e^{-j\omega_0 t}\, e^{-j\omega t}\,dt$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\,dt + \frac{1}{2}\int_{-\infty}^{\infty} x(t)\, e^{-j(\omega+\omega_0)t}\,dt$$

$$\therefore X(\omega) = \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0)$$

Thus $F[x(t)\cos\omega_0 t] \xrightarrow{F} \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega + \omega_0)$

...Proved.

Note : The terms $X(\omega - \omega_0)$ and $X(\omega + \omega_0)$ represents the shifted spectrums centered about ω_0 and $-\omega_0$ respectively.

Ex. 6.3.4 : Verify the following expression :

$$u(t) \xrightarrow{F} \frac{1}{2}\delta(f) + \frac{1}{2j\pi f}$$

Soln. : We can represent $u(t)$ as follows :

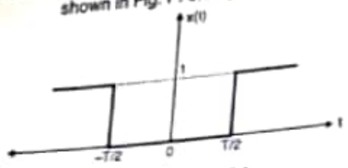
$$u(t) = \frac{1}{2} + \frac{1}{2}\,\text{sgn}(t) \quad ...(1)$$

Taking FT of both the sides we get,

$$F[u(t)] = F[1/2] + F[(1/2)\,\text{sgn}(t)] = \frac{1}{2}F[1] + \frac{1}{2}F[\text{sgn}(t)]$$

$$\therefore F[u(t)] = \frac{1}{2}\delta(f) + \frac{1}{2j\pi f} \quad \text{...Proved.}$$

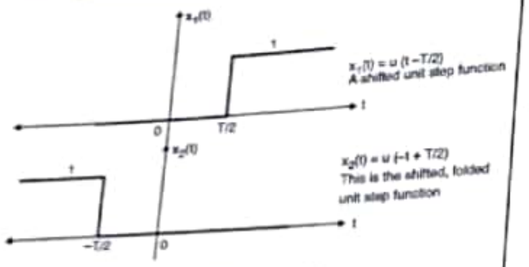
Ex. 6.3.5 : Find the Fourier transform of the waveform shown in Fig. P. 6.3.5(a).



(H-506) **Fig. P. 6.3.5(a)**

Soln. :

The waveform given in Fig. P. 6.3.5(a) can be expressed as addition of two waveforms as shown in the Fig. P. 6.3.5(b).



(H-507) **Fig. P. 6.3.5(b)**

$$x(t) = x_1(t) + x_2(t) \quad ...(1)$$

$$\therefore \quad x(t) = u(t - T/2) + u(-t + T/2) \quad ...(2)$$

Taking the Fourier transform of both the sides.

$$X(f) = F[u(t - T/2)] + F[u(-t + T/2)]$$

Using the property of time shifting we get,

$$X(f) = e^{-j2\pi fT/2} \cdot F[u(t)] + e^{j2\pi fT/2} \cdot F[u(-t)]$$

$$= e^{-j2\pi fT/2} \cdot \frac{1}{j2\pi f} + e^{+j2\pi fT/2} \cdot \frac{1}{-j2\pi f}$$

$$= -\frac{1}{j2\pi f}\left[e^{j2\pi fT/2} - e^{-j2\pi fT/2}\right]$$

$$= -\frac{1}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right] \quad ...(3)$$

The term inside the square bracket is sin (πfT)

$$\therefore \quad X(f) = -\frac{1}{\pi f}\sin(\pi fT) \quad ...(4)$$

Multiply and divide by T to get,

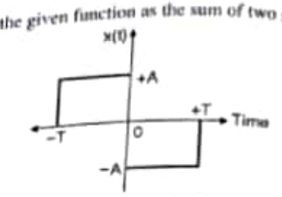
$$X(f) = \frac{-T\sin(\pi fT)}{(\pi fT)}$$

$$\therefore \quad X(f) = -T\,\text{sinc}(fT) \quad \textbf{...Ans.}$$

Ex. 6.3.6 : Obtain the fourier transform of the signal shown in Fig. P. 6.3.6 and obtain its amplitude spectrum.

Soln. :

Let us express the given function as the sum of two signals.



(H-508) **Fig. P. 6.3.6**

$$\therefore \quad x(t) = x_1(t) + x_2(t)$$

where, $x_1(t) = +A$ for $-T < t < 0$ ...(1)

and $x_2(t) = -A$ for $0 < t < T$ ...(2)

Using the property of superposition we can write that,

$$F[x(t)] = F[x_1(t)] + F[x_2(t)]$$

i.e. $X(f) = X_1(f) + X_2(f)$ ...(3)

Let us obtain $X_1(f)$ and $X_2(f)$ separately.

1. To obtain $X_1(f)$:

$$X_1(f) = \int_{-T}^{0} A e^{-j2\pi ft}\,dt = \frac{-A}{j2\pi f}\left[e^{-j2\pi ft}\right]_{-T}^{0}$$

$$= \frac{-A}{j2\pi f}\left[e^{0} - e^{+j2\pi fT}\right] = \frac{-A}{j2\pi f}\left[1 - e^{j2\pi fT}\right]$$

$$\therefore \quad X_1(f) = \frac{-A}{j2\pi f}\left[e^{j2\pi fT} - 1\right] \quad ...(4)$$

In Equation (4) make two substitutions as,

$$e^{j2\pi fT} = e^{j2\pi fT/2} \cdot e^{j2\pi fT/2} = e^{+j\pi fT} \cdot e^{+j\pi fT}$$

and $1 = e^{-j\pi fT} \cdot e^{j\pi fT}$

Now substitute these in Equation (4) to get,

$$X_1(f) = \frac{A}{j2\pi f}\left[e^{+j\pi fT} e^{+j\pi fT} - e^{-j\pi fT} \cdot e^{j\pi fT}\right]$$

$$\therefore \quad X_1(f) = \frac{A e^{+j\pi fT}}{\pi f}\left[\frac{e^{+j\pi fT} - e^{-j\pi fT}}{2j}\right] = \frac{A e^{+j\pi fT}}{\pi f}\sin(\pi fT)$$

Multiply and divide by T to get,

$$X_1(f) = AT e^{+j\pi fT}\frac{\sin(\pi fT)}{\pi fT}$$

$$\therefore \quad X_1(f) = AT e^{+j\pi fT}\,\text{sinc}(fT) \quad ...(5)$$

2. To obtain $X_2(f)$:

$$X_2(f) = \int_{0}^{T} -A e^{-j2\pi ft}\,dt = \frac{A}{j2\pi f}\left[e^{-j2\pi ft}\right]_{0}^{T}$$

$$\therefore \quad X_2(f) = \frac{A}{j2\pi f}\left[e^{-j2\pi fT} - 1\right] \quad ...(6)$$

Substitute $e^{-j2\pi fT} = e^{-j\pi fT} \cdot e^{-j\pi fT}$

and $1 = e^{+j\pi fT} \cdot e^{-j\pi fT}$

$$\therefore \quad X_2(f) = \frac{A}{j2\pi f}\left[e^{-j\pi fT} e^{-j\pi fT} - e^{j\pi fT} \cdot e^{-j\pi fT}\right]$$

$$= \frac{-A e^{-j\pi fT}}{\pi f}\left[\frac{e^{j\pi fT} - e^{-j\pi fT}}{2j}\right]$$

$$\therefore \quad X_2(f) = \frac{-A e^{-j\pi fT}}{\pi f}\sin(\pi fT)$$

Multiply and divide by T to get,

$$X_2(f) = -AT e^{-j\pi fT}\frac{\sin(\pi fT)}{\pi fT}$$

$$\therefore \quad X_2(f) = -AT e^{-j\pi fT}\,\text{sinc}(fT) \quad ...(7)$$

3. To obtain X (f) :

$$X(f) = X_1(f) + X_2(f)$$

Substitute values from the Equations (5) and (7) to get,

$$X(f) = AT e^{j\pi fT}\,\text{sinc}(fT) - AT e^{-j\pi fT}\,\text{sinc}(fT)$$

$$= AT\,\text{sinc}(fT)\left[e^{j\pi fT} - e^{-j\pi fT}\right]$$

Multiply and divide by 2j to get,

$$\therefore \quad X(f) = 2j\,AT\,\text{sinc}(fT)\frac{\left[e^{j\pi fT} - e^{-j\pi fT}\right]}{2j}$$

$$\therefore \quad X(f) = 2j\,AT\,\text{sinc}(fT)\sin(\pi fT) \quad \textbf{...Ans.}$$

4. Amplitude spectrum :

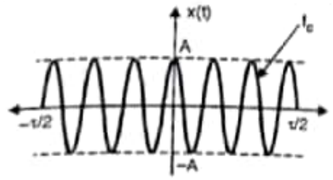
The amplitude spectrum is given by,

$$|X(f)| = 2\,AT\,\text{sinc}(fT)\sin(\pi fT)$$

Ex. 6.3.7 : Obtain the Fourier transform and amplitude spectrum of the RF pulse shown in Fig. P. 6.3.7(a).

Soln. :

The pulse shown in Fig. P. 6.3.7(a) is termed as RF pulse when f_c falls in the radio frequency band. From the waveform of Fig. P. 6.3.7(a) we can write the expression for x (t) as,



(H-511) **Fig. P. 6.3.7(a) : RF pulse**

$$x(t) = A\,\text{rect}(t/\tau) \cdot \cos\omega_c t \quad ...(1)$$

Substitute $\cos\omega_c t = \dfrac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$ to get,

$$x(t) = \frac{A}{2}\,\text{rect}(t/\tau) \cdot e^{j2\pi f_c t} + \frac{A}{2}\,\text{rect}(t/\tau)\,e^{-j2\pi f_c t}$$

$$\therefore \quad x(t) = x_1(t) + x_2(t)$$

Let us obtain the Fourier transforms of $x_1(t)$ and $x_2(t)$ separately.

$$\therefore \quad F[x_1(t)] = X_1(f) = \int_{-\infty}^{\infty} \frac{A}{2}\,\text{rect}(t/\tau) \cdot e^{j2\pi f_c t} \cdot e^{-j2\pi ft}\,dt$$

$$= \int_{-\infty}^{\infty} \frac{A}{2}\,\text{rect}(t/\tau)\,e^{-j2\pi(f - f_c)t}\,dt$$

We know the Fourier transform of a rectangular pulse. In this case it will be shifted by f_c on the right hand side.

$$\therefore \quad X_1(f) = \frac{A\tau}{2}\,\text{sinc}[(f - f_c)\tau] \quad ...(2)$$

Similarly we will get $X_2(f)$ as,

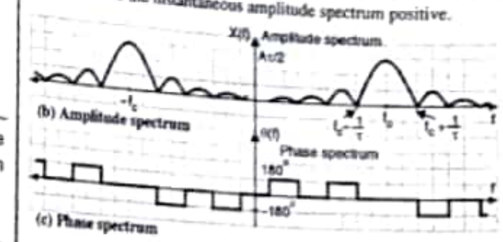
$$X_2(f) = \frac{A\tau}{2}\,\text{sinc}[(f + f_c)\tau] \quad ...(3)$$

Adding Equations (2) and (3) we get,

$$X(f) = X_1(f) + X_2(f)$$

$$\therefore \quad X(f) = \frac{A\tau}{2}\,\text{sinc}[(f - f_c)\tau] + \frac{A\tau}{2}\,\text{sinc}[(f + f_c)\tau]$$

The amplitude spectrum is as shown in Fig. P. 6.3.7(b) which shows sinc pulses shifted by + f_c and – f_c. Fig. P. 6.3.7(c) shows the phase spectrum where the ± 180° phase shifts are introduced in order to make the instantaneous amplitude spectrum positive.



(H-512) **Fig. P. 6.3.7**

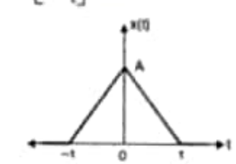
Ex. 6.3.8 : Obtain the Fourier transform of the triangular pulse shown in Fig. P. 6.3.8(a).

Soln. :

The triangular waveform of Fig. P. 6.3.8(a) can be represented as follows :

$$x(t) = A\left[\frac{t}{\tau} + 1\right] \quad \text{...for } -\tau \le t \le 0$$

and $x(t) = A\left[1 - \frac{t}{\tau}\right]$ ...for $0 \le t \le \tau$



(H-513) **Fig. P. 6.3.8(a) : Triangular pulse**

Therefore, using the definition of Fourier transform we get,

$$X(f) = \int_{-\tau}^{0} A\left[1 + \frac{t}{\tau}\right] \cdot e^{-j2\pi ft}\,dt + \int_{0}^{\tau} A\left[1 - \frac{t}{\tau}\right] \cdot e^{-j2\pi ft}\,dt$$

Solving it, we get,

$$X(f) = \frac{A}{2\pi^2 f^2 \tau}\{1 - \cos(2\pi f\tau)\}$$

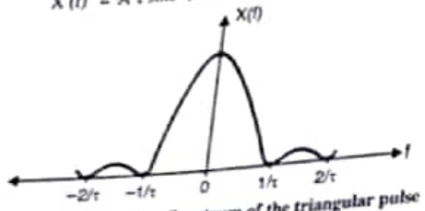
$$= \frac{2A}{2\pi^2 f^2 \tau} \sin^2(\pi f \tau)$$

Multiply and divide by τ to get,

$$X(f) = \frac{A\tau}{\pi^2 f^2 \tau^2} \sin^2(\pi f \tau)$$

So by definition of sinc function, ...Ans.

$$X(f) = A\tau \, \text{sinc}^2(f\tau)$$



(H-514) Fig. P. 6.3.8(b) : Spectrum of the triangular pulse

The spectrum is as shown in Fig. P. 6.3.8(b).

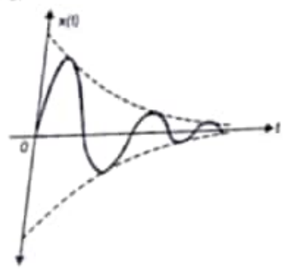
Ex. 6.3.9 : Consider an exponentially damped sinusoidal signal defined as :

$$x(t) = \begin{cases} e^{-t} \cdot \sin(2\pi f_c t), & t > 0 \\ 0, & t \le 0 \end{cases}$$

Obtain its Fourier transform using the frequency shifting property.

Soln. :

The exponentially damped sinusoidal signal is as shown in Fig. P. 6.3.9. Note that the sinewave has a frequency f_c and its amplitude is progressively decreasing due to the multiplying factor of e^{-t} for $t > 0$.



(H-515) Fig. P. 6.3.9 : Exponentially damped sinusoidal signal

1. Let us express the given sinusoidal signal as,

$$\sin(2\pi f_c t) = \frac{1}{2j}\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right] \quad ...(1)$$

2. The Fourier transform of the signal x (t) is given by,

$$X(f) = \int_{-\infty}^{\infty} \frac{e^{-t}}{2j}\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right] \cdot e^{-j2\pi f t} \, dt$$

But the given function is non-zero only for $t > 0$

$$\therefore \quad X(f) = \frac{1}{2j}\int_0^{\infty} \left[e^{j2\pi f_c t} \cdot e^{-t}\right] \cdot e^{-j2\pi f t} \, dt$$

$$- \frac{1}{2j}\int_0^{\infty} \left[e^{-j2\pi f_c t} \cdot e^{-t}\right] \cdot e^{-j2\pi f t} \, dt \quad ...(2)$$

3. Now let us use the frequency shifting property which states that,

$$F\left[e^{j2\pi f_c t} \cdot x(t)\right] = X(f - f_c) \quad ...(3)$$

Applying the property to both the terms of Equation (2) we get,

$$F\left[e^{-t}\sin(2\pi f_c t)\right] = \frac{1}{2j}\left[\frac{1}{1 + 2\pi(f - f_c)} - \frac{1}{1 + 2\pi(f + f_c)}\right] \quad ...\text{Ans.}$$

This result has been obtained because we have assumed $x(t) = e^{-t}$ and

$$F[e^{-t}] = X(f) = \frac{1}{(1 + 2\pi f)}$$

Because of the multiplying terms of $e^{j2\pi f_c t}$ and $e^{-j2\pi f_c t}$ in Equation (2) we have obtained the shifted frequency spectrum terms of $\frac{1}{1 + 2\pi(f - f_c)}$ and $\frac{1}{1 + 2\pi(f + f_c)}$ respectively in the answer.

Ex. 6.3.10 : Find and sketch the Fourier transform i.e. amplitude spectrum and phase spectrum of the following signals :

1. 6 sin c (0.5t)
2. 6 rect [(t − 4)/3]
3. 10 sin (100πt).

Soln. :

1. $x(t) = 6 \text{ sinc}(0.5\,t) = A \text{ sinc}(2\,Wt)$

$\therefore \; A = 6$ and $2W = 0.5 \quad \therefore \; W = 0.25$

$$A \text{ sinc } 2Wt \xleftrightarrow{F} \frac{A}{2W} \text{rect}[f/2W]$$

$$\therefore \; 6 \text{ sinc}(0.5\,t) \xleftrightarrow{F} \frac{6}{0.5} \text{rect}[f/0.5]$$

$$\therefore \; X(f) = 12 \text{ rect}[2f]$$

Refer Ex. 6.2.10 and solve.

2. $x(t) = 6 \text{ rect}[(t - 4)/3] = A \text{ rect}\left[\frac{(t - t_d)}{T}\right]$

$\therefore \; A = 6$, delay $t_d = 4$ and $T = 3$

Use the time shifting property and solve.

3. $x(t) = 10 \sin(100\pi t) = 10 \sin(2 \times \pi \times 50\,t)$

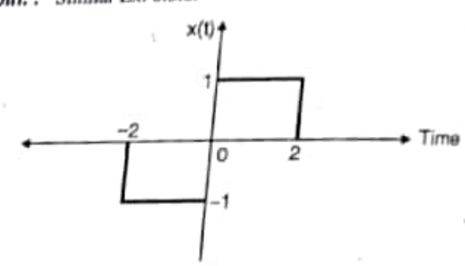
$= A \sin(2\pi f t)$

$\therefore \; A = 10, \quad f = 50.$

Refer Ex. 6.3.2 and solve.

Ex. 6.3.11 : Obtain the Fourier transform of the signal shown in Fig. P. 6.3.11, using linearity property (Refer Fig. P. 6.3.11).

Soln. : Similar Ex. 6.3.6.



(H-526) Fig. P. 6.3.11

Ex. 6.3.12 : Prove the time scaling property of Fourier transform and hence find the Fourier transform of $f(t) = e^{-0.5t}u(t)$

Soln. : For time scaling property of Fourier transform refer section 6.2.2.

$$\text{F.T. of } f(t) = e^{-0.5t}u(t)$$

We know that

$$e^{-\alpha t}u(t) \xleftrightarrow{F} \frac{1}{(\alpha + j2\pi f)}$$

$$\therefore \quad e^{-0.5t}u(t) \xleftrightarrow{F} \frac{1}{(0.5 + j2\pi f)} \quad ...\text{Ans.}$$

Ex. 6.3.13 : State and prove the convolution property of the Fourier transform and find the Fourier transform of the following signal :

$$y(t) = e^{-at}u(t) * u(t)$$

Soln. :

For the convolution property of F.T. Refer section 6.2.11.

$$y(t) = e^{-at} \cdot u(t) * u(t)$$

Step 1 : Fourier transform of $e^{-at}u(t)$:

Let $x_1(t) = e^{-at}u(t)$

$$\therefore \quad X_1(f) = \int_0^{\infty} e^{-at} e^{-j2\pi f t} \, dt = \int_0^{\infty} e^{-(a + j2\pi f)t} \, dt$$

$$= \frac{-1}{(a + j2\pi f)}\left[e^{-(a + j2\pi f)t}\right]_0^{\infty}$$

$$\therefore \quad X_1(f) = \frac{-1}{(a + j2\pi f)}\left[e^{-\infty} - e^{0}\right] = \frac{1}{(a + j2\pi f)} \quad ...(1)$$

Step 2 : F.T. of u (t) :

Let $x_2(t) = u(t)$

$$\therefore \quad X_2(f) = \int_0^{\infty} 1 \, e^{-j2\pi f t} \, dt = \frac{-1}{j2\pi f}\left[e^{-j2\pi f t}\right]_0^{\infty}$$

$$\therefore \quad X_2(f) = \frac{-1}{j2\pi f}\left[e^{-\infty} - e^{0}\right] = \frac{1}{(j2\pi f)} \quad ...(2)$$

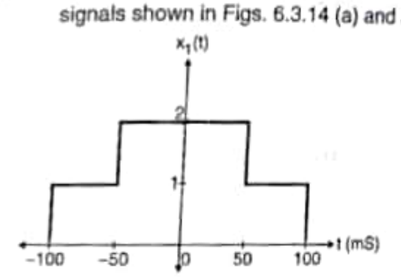
Step 3 : F.T. of y (t) :

$$Y(f) = F[e^{-at}u(t) * u(t)]$$
$$= X_1(f)\,X_2(f)$$

$$\therefore \quad Y(f) = \frac{1}{(a + j2\pi f)} \times \frac{1}{(j2\pi f)} \quad ...\text{Ans.}$$

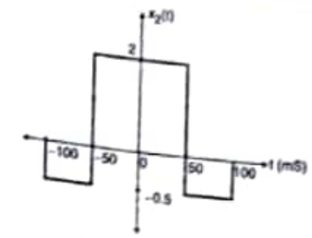
Ex. 6.3.14 : Draw the amplitude spectrum of the following signals shown in Figs. 6.3.14 (a) and (b)

1



(H-541) (a)

2.



(H-542) (b) Fig. P. 6.3.14

Soln. :

1. **Given signal**

$$X_1(t) = 1 \quad ... \; -100 \le t \le -50$$
$$= 2 \quad ... \; -50 \le t \le +50$$
$$= 1 \quad ... \; +50 \le t \le 100$$

$$\therefore \quad X_1(f) = \int_{-\infty}^{\infty} x_1(t)\, e^{-j2\pi f t} \, dt$$

$$= \int_{-100}^{-50} 1\, e^{-j2\pi f t} \, dt + \int_{-50}^{+50} 2\, e^{-j2\pi f t} \, dt + \int_{50}^{100} 1\, e^{-j2\pi f t} \, dt$$

$$\therefore \quad X_1(f) = \frac{-1}{j2\pi f}\left\{e^{j100\pi f} - e^{200\pi f} + 2(e^{-j100\pi f} - e^{+j100}) + (e^{-j200\pi f} - e^{-j200\pi f})\right\}$$

$$= \frac{-1}{j2\pi f}\left\{e^{j100\pi f} - e^{j100\pi f} - e^{j200\pi f} + e^{-j200\pi f} + 2(e^{j100\pi f} - e^{j100})\right\}$$

$$= \frac{-(e^{j100\pi f} - e^{-j100\pi f})}{(\pi f)\, j^2} + \frac{1}{\pi f}\frac{(e^{j200\pi f} - e^{-j200\pi f})}{2j} + \frac{2}{\pi f}\frac{(e^{j100\pi f} - e^{-j100\pi f})}{2j}$$

$$\therefore \quad X_1(f) = \frac{\sin(100\pi f)}{\pi f} + \frac{\sin(200\pi f)}{\pi f} + \frac{2\sin(100\pi f)}{\pi f}$$

$$= \frac{\sin 200\pi f}{\pi f} + \frac{\sin 100\pi f}{\pi f}$$

$$= \frac{200 \sin(200\pi f)}{200\pi f} + \frac{100 \sin(100\pi f)}{100\pi f}$$

$$\therefore \quad X_1(f) = 200 \text{ sinc}(200 f) + 100 \text{ sinc}(100 f) \quad ...\text{Ans.}$$

The amplitude spectrum is shown in Fig. P. 6.3.14(b).

2.

$$x_2(t) = -0.5 \quad ... \; -100 \le t \le -50$$
$$= 2 \quad ... \; -50 \le t \le 50$$
$$= -0.5 \quad ... \; 50 \le t \le 100$$

$$\therefore \quad x_2(f) = \int_{-\infty}^{\infty} x_2(t)\, e^{-j2\pi f t} \, dt$$

$$= \int_{-100}^{-50} -0.5\, e^{-j2\pi f t} \, dt + \int_{-50}^{50} 2\, e^{-j2\pi f t} \, dt - \int_{50}^{100} 0.5\, e^{-j2\pi f t} \, dt$$

$$= \frac{+1}{j2\pi f}\left[\frac{(e^{-j100\pi t} - e^{j200\pi t})}{2} - 2(e^{-j100\pi t} - e^{j100\pi t}) + \frac{(e^{-j200\pi t} - e^{j100\pi t})}{2}\right]$$

$$= \frac{(e^{j100\pi f} - e^{-j100\pi f})}{(2\pi f)(2j)} + \frac{2(e^{+j100\pi f} - e^{-j100\pi f})}{(\pi f)(2j)} - \frac{(e^{j200\pi f} - e^{-j200\pi f})}{(2\pi f)(2j)}$$

$$= \frac{\sin(100\pi f)}{2\pi f} + \frac{2\sin(100\pi f)}{\pi f} - \frac{\sin(200\pi f)}{2\pi f}$$

$$= \frac{2.5\sin(100\pi f)}{\pi f} - \frac{\sin(200\pi f)}{2\pi f}$$

$$= \frac{250\sin(100\pi f)}{100\pi f} - \frac{100\sin(200\pi f)}{200\pi f}$$

$$\therefore X_2(f) = 250\,\text{sinc}(100f) - 100\,\text{sinc}(200f) \quad \text{...Ans.}$$

6.4 Fourier Transform of a Periodic Signal :

- The Fourier transform of the periodic signals extending from $-\infty$ to ∞ on the time scale cannot be obtained using the equations defined for the non-periodic signals.
- The reason for this is that such functions are not absolutely integrable. Function x(t) should be absolutely integrable is one of the conditions which must be satisfied.

i.e. $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$

But for a periodic signal :

$$\int_{-\infty}^{\infty} |x(t)| = \infty$$

- Therefore x(t) is not absolutely integrable and hence the Fourier transform for the periodic signal does not exist.

How to get it from an exponential Fourier series ?

However the Fourier transform for a periodic function can be obtained over one cycle period i.e. $-T_0/2$ to $T_0/2$. The procedure to obtain the Fourier transform for periodic signals is as follows :

1. Express the periodic signal $x_p(t)$ in the form of an exponential Fourier series.
2. Take the Fourier transform of both the sides.
3. The Fourier transform of the periodic signal $x_p(t)$ will then be the summation of the Fourier transforms of the individual terms in the Fourier series expansion.

After following this procedure we get the Fourier transform of a periodic signal as follows :

Step 1 : Let $x_p(t)$ be a periodic signal with period of T_0 seconds. It is expressed in the exponential Fourier series form as follows :

$$x_p(t) = \sum_{-\infty}^{\infty} C_n e^{j2\pi n f_0 t} \text{ where } C_n$$

$$= \frac{1}{T_0}\int_{t}^{t+T_0} x_p(t)\, e^{-j2\pi n f_0 T} \quad \text{...(6.4.1)}$$

Step 2 : Take the Fourier transform of both the sides of Equation (6.4.1) to get,

$$F[x_p(t)] = F\left[\sum_{-\infty}^{\infty} C_n e^{j2\pi n f_0 t}\right]$$

$$= \int_{-\infty}^{\infty}\left[\sum_{-\infty}^{\infty} C_n e^{j2\pi n f_0 t}\right]\cdot e^{-j2\pi ft}\,dt$$

$$= \sum_{-\infty}^{\infty} C_n \int_{-\infty}^{\infty} e^{j2\pi n f_0 t}\cdot e^{-2\pi ft}\,dt$$

$$= \sum_{-\infty}^{\infty} C_n \int_{-\infty}^{\infty} e^{-j2\pi(f-nf_0)t}\,dt \quad \text{...(6.4.2)}$$

But $\int e^{-j2\pi ft}\,dt = \delta(f)$

$$\therefore F[x_p(t)] = \sum_{-\infty}^{\infty} C_n \cdot \delta(f - nf_0) \quad \text{...(6.4.3)}$$

where $C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x_p(t)\cdot e^{-j2\pi n f_0 t}\,dt$ ...(6.4.4)

Conclusions from Equation (6.4.3) :

1. Equation (6.4.3) tells us that the Fourier transform of a periodic signal $x_p(t)$ is in the form of impulses weighted by the factor C_n.

   i.e. $X(f) = \ldots\ldots C_{-1}\,\delta(f+f_0) + C_0\,\delta(f) + C_1\,\delta(f-f_0) + C_2\,\delta(f-2f_0) + \ldots\ldots$

2. Thus Fourier transform of periodic signal is discrete and we get a discrete frequency spectrum.

Definition of inverse Fourier transform for a periodic signal :

The original periodic signal $x_p(t)$ can be recovered back from the Fourier transform, by using the inverse Fourier transform. It is defined as,

$$x_p(t) = F^{-1}[X(f)]$$

$$= \int_{-\infty}^{\infty}\left[\sum_{n=-\infty}^{\infty} C_n\,\delta(f-nf_0)\right]\cdot e^{j2\pi ft}\cdot df$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_{-\infty}^{\infty} e^{j2\pi ft}\cdot\delta(f-nf_0)\,df \quad \text{...(6.4.5)}$$

According to the shifting property of delta function which states that,

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_m)\,dt = x(t_m)$$

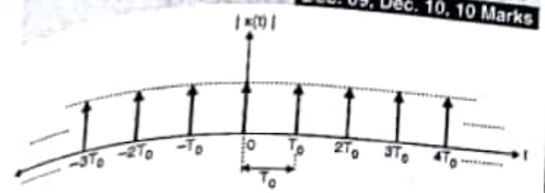
Applying this property to Equation (6.4.5) we get,

$$x_p(t) = \sum_{n=-\infty}^{\infty} C_n\cdot e^{j2\pi nf_0 t} \quad \text{...(6.4.6)}$$

This is the expression for the inverse Fourier transform of a periodic signal.

Ex. 6.4.1 : Find the FT of an impulse train as shown in Fig. P.6.4.1(a). The impulse train extends from $-\infty$ to ∞ and has a period T_0 as shown in Fig. P. 6.4.1(a).

Dec. 09, Dec. 10, 10 Marks



(H-516) **Fig. P. 6.4.1(a) : Train of impulses**

Soln. :

1. **To represent the signal x (t) mathematically :**

This train of impulses is also known as the sampling function or Dirac Delta Comb function. It is represented mathematically as follows :

$$x(t) = \sum_{a=-\infty}^{\infty} \delta(t - aT_0) \quad \text{...(1)}$$

2. **FT of the signal :**

This signal is a periodic signal. Therefore let us use the definition of FT for the periodic signal stated in Equation (6.4.1).

$$\therefore X(f) = \sum_{n=-\infty}^{\infty} C_n\,\delta(f - nf_0) \quad \text{...(2)}$$

We do not know the value of C_n in this equation. So let us obtain it.

3. **To obtain C_n :**

As defined in Equation (6.4.2), C_n is given as,

$$C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\cdot e^{-j2\pi nt/T_0}\,dt \quad \text{...(3)}$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)\cdot e^{-j2\pi nf_0 t}\,dt \quad \text{...(4)}$$

Note here that we have substituted $x(t) = \delta(t)$ and not $\delta(t - aT_0)$. This is because in the range $-T_0/2$ to $T_0/2$, $x(t) = \delta(t)$ as shown in the Fig. P. 6.4.1(a).

Since $\delta(t)$ exists only at $t = 0$, we can change the limits of integration in Equation (4) to $-\infty$ to $+\infty$ without changing the value of the equation.

$$\therefore C_n = \frac{1}{T_0}\int_{-\infty}^{\infty} \delta(t)\cdot e^{-j2\pi nf_0 t}\,dt \quad \text{...(5)}$$

The delta functions have a standard property which states that,

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_d)\,dt = x(t-t_d) \quad \text{...(6)}$$

In the Equation (5), let : $t_d = 0$ and $x(t) = e^{-j2\pi nf_0 t}$

$$\therefore C_n = \frac{1}{T_0}e^{-j2\pi n_d}$$

$$\therefore C_n = \frac{1}{T_0} \quad \text{...(7)}$$

Now substitute the value of "C_n" in Equation (2) to get the fourier transform of x (t).

The fourier transform : $X(f) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0}\delta(f - nf_0)$ ...(8)

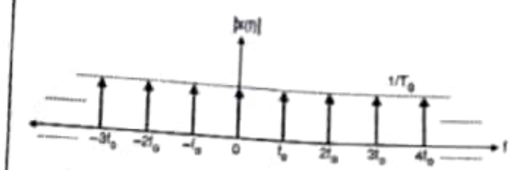
This is the desired result.

4. **To plot the amplitude spectrum :**

The amplitude spectrum is given as,

$$|X(f)| = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0) \quad \text{...(9)}$$

The amplitude spectrum is a train of impulses of amplitude $(1/T_0)$ as shown in Fig. P. 6.4.1(b).



(H-517) **Fig. P. 6.4.1(b) : Amplitude spectrum of dirac delta comb function**

Ex. 6.4.2 : Obtain the fourier transform of a periodic gate function of amplitude A, period T_0 and width τ assuming that the function is centred around the origin.

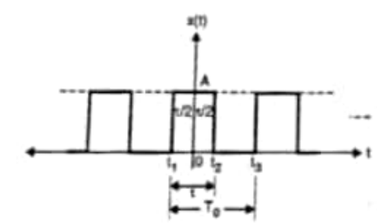
May 05, 8 Marks, May 08, 4 Marks, Dec. 11, 10 Marks

Soln. :

Step 1 : Express the given waveform mathematically :

One cycle of the given periodic signal can be expressed as :

$$x(t) = A \ldots \text{for } -\tau/2 \le t \le \tau/2$$
$$= 0 \ldots \text{for } \tau/2 < t \le t_3$$



(H-518) **Fig. P. 6.4.2(a)**

Step 2 : To obtain the expression for C_n :

$$C_n = \frac{1}{T_0}\int_{t}^{t+T_0} x(t)\cdot e^{-j2\pi nf_0 t}\,dt$$

$$= \frac{1}{T_0}\int_{t_1}^{t_2} x(t)\cdot e^{-j2\pi n f_o t}\,dt$$

$$= \frac{1}{T_0}\left[\int_{-\tau/2}^{\tau/2} A\,e^{-j2\pi n f_o t}\,dt + \int_{\tau/2}^{t_2} 0\,e^{-j2\pi n f_o t}\,dt\right]$$

$$= \frac{A}{T_0}\cdot\frac{1}{-j2\pi n f_o}\cdot\left[e^{-j2\pi n f_o t}\right]_{-\tau/2}^{\tau/2}$$

But $T_0 \times f_o = 1$

$$\therefore\ C_n = \frac{A}{-j2\pi n} = \left[e^{-j2\pi n f_o \tau/2} - e^{+j2\pi n f_o \tau/2}\right]$$

$$= \frac{A\left[e^{+j\pi n f_o \tau} - e^{-j\pi n f_o \tau}\right]}{j2\pi n}$$

$$\therefore\ C_n = \frac{A}{\pi n}\cdot\sin(\pi n f_o \tau)$$

Multiply and divide by "τf_o" to get,

$$C_n = \tau A f_o \cdot \frac{\sin(\pi n f_o \tau)}{(\pi n f_o \tau)}$$

$$\therefore\ C_n = \tau A f_o \cdot \text{sinc}(n f_o \tau)$$

$$\therefore\ C_n = \frac{A\tau}{T_0}\sin(n f_o \tau) \qquad ...(1)$$

Step 3 : To obtain the fourier transform :

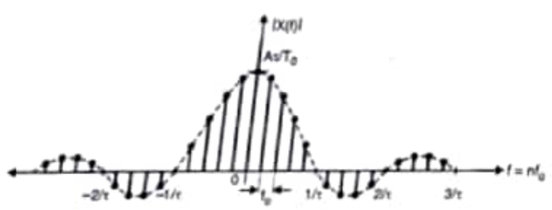
$$X(f) = \sum_{n=-\infty}^{\infty} C_n \cdot \delta(f - n f_o)$$

Substitute the value of C_n from Equation (1) to write,

$$X(f) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T_0}\text{sinc}(n f_o \tau)\cdot\delta(f - n f_o) \qquad ...(2)$$

Step 4 : To draw the spectrum :

The amplitude spectrum $|X(f)|$ of the gate function is shown in Fig. P. 6.4.2(b).



(H-519) **Fig. P. 6.4.2(b) : Spectrum of the gate function**

Ex. 6.4.3 : Obtain the fourier transform of a periodic rectangular pulse train with a period $T_0 = 1$ and width $\tau = 0.1$ symmetrically placed around the origin.

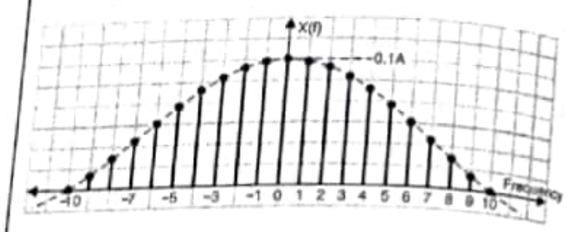
Soln. : A periodic rectangular pulse train is nothing but the gate function discussed in Ex. 6.4.2. The fourier transform of this function is given by,

$$X(f) = \sum_{n=-\infty}^{\infty} \frac{A\tau}{T_0}\cdot\text{sinc}(n\tau f_o)\cdot\delta(f - n f_o) \qquad ...(1)$$

Substitute the values of T_0 and τ to get,

$$X(f) = \sum_{n=-\infty}^{\infty} 0.1\tau\,\text{sinc}(0.1\,n)\,\delta(f - n) \qquad ...(2)$$

1. sinc $(0.1\,n) = 0$ at $0.1\,n = \pm 1, \pm 2, \pm 3$ .....
   i.e. at $n = \pm 10, \pm 20, \pm 30$.....
2. The first zero of the spectrum will occur at $n = \pm 10$ or when $f = n f_o = \pm 10 f_o$.
3. The separation between consecutive spectral components in the spectrum will be $f_o = 1/T_0 = 1$.
4. The maximum amplitude of the sinc function is $\frac{A\tau}{T_0} = 0.1\,A$ at $f = 0$ as sinc $(0) = 1$. The spectrum is as shown in Fig. P. 6.4.3.



(H-520) **Fig. P. 6.4.3**

Ex. 6.4.4 : Derive the expression for the inverse fourier transform of a periodic signal $x_p(t)$ having period T_0 and with its fourier transform given by the following expression :

$$X(f) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(n f_o)\cdot\delta(f - n f_o)$$

Soln. :

Step 1 : As per the definition of inverse fourier transform,

$$x_p(t) = F^{-1}[X(f)]$$

$$= \int_{-\infty}^{\infty}\left[\frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(n f_o)\,\delta(f - n f_o)\right]\cdot e^{j2\pi n}\,df$$

$$= \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(n f_o)\int_{-\infty}^{\infty} e^{j2\pi f t}\,\delta(f - n f_o)\,df \qquad ...(1)$$

Step 2 : By using the shifting property of a delta function we write that,

$$x_p(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(n f_o)\,e^{j2\pi n f_o t} \qquad ...(2)$$

This is the required expression for the inverse fourier transform.

We can conclude that the fourier transform pair of a periodic signal $x_p(t)$ with a period of T_0 seconds is given by,

1. Fourier transform of a periodic signal

$$X(f) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(n f_o)\cdot\delta(f - n f_o)$$

2. Inverse fourier transform of a periodic signal

$$x_p(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} X(n f_o)\cdot e^{j2\pi n f_o t} \qquad ...(3)$$

Ex. 6.4.5 : Prove the "modulation theorem" which is given by the following expression :

$$x(t)\cdot\cos(\omega_c t + \phi) \xleftrightarrow{F} \frac{e^{j\phi}}{2}X(f - f_c) + \frac{e^{-j\phi}}{2}X(f + f_c).$$

Soln. :

Modulation theorem :

The modulation theorem stated in the above equation tells us that if we multiply the carrier signal $\cos(\omega_c t + \phi)$ by the modulating signal $x(t)$ in the time domain then this process will translate the spectrum of $x(t)$, up and down in frequency by f_c.

Proof :

We want to obtain the fourier transform of the following signal :

$$x(t)\cdot\cos(\omega_c t + \phi)$$

Substitute $\cos(\omega_c t + \phi) = \dfrac{e^{j(\omega_c t + \phi)} + e^{-j(\omega_c t + \phi)}}{2}$

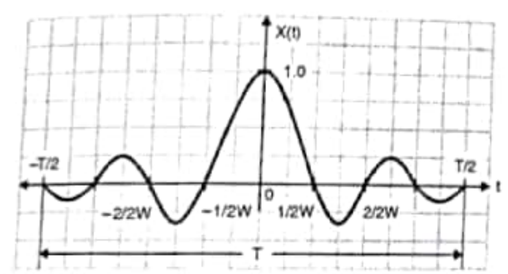
$$\therefore\ F[x(t)\cdot\cos(\omega_c t + \phi)] = \int_{-\infty}^{\infty}\frac{1}{2}x(t)\,e^{j(\omega_c t + \phi)}\cdot e^{-j2\pi f t}\,dt + \int_{-\infty}^{\infty}\frac{1}{2}x(t)\cdot e^{-j(\omega_c t + \phi)}\cdot e^{-j2\pi f t}\,dt$$

Applying the frequency shifting property we get,

$$F[x(t)\cdot\cos(\omega_c t + \phi)] = \frac{1}{2}e^{j\phi}\cdot X(f - f_c) + \frac{1}{2}e^{-j\phi}\cdot X(f + f_c)$$

This is the required result.

Ex. 6.4.6 : Obtain the fourier transform of a truncated sinc pulse shown in Fig. P. 6.4.6.



(H-521) **Fig. P. 6.4.6 : Truncated sinc pulse**

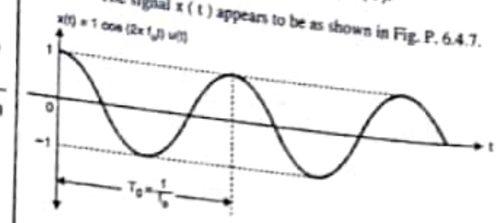
Soln. :

$$\therefore\ X(f) = \frac{1}{2\pi W}[\text{Si}(\pi f T + \pi T W) - \text{Si}(\pi f T - \pi T W)]$$

This is the desired result.

Ex. 6.4.7 : Find the Fourier transform of the periodic signal : $x(t) = \cos(2\pi f_o t)\,u(t)$.

Soln. : The signal $x(t)$ appears to be as shown in Fig. P. 6.4.7.



(H-523) **Fig. P. 6.4.7**

The FT of this signal is given by,

$$F[x(t)] = \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi n}\,dt$$

$$= \int_{-\infty}^{\infty}\cos(2\pi f_o t)\cdot u(t)\cdot e^{-j2\pi n}\,dt$$

Due to the presence of $u(t)$, the integration limits will be from 0 to ∞ instead of $-\infty$ to ∞.

$$\therefore\ X(f) = \int_{0}^{\infty}\cos(2\pi f_o t)\cdot e^{-j2\pi n}\,dt$$

Now substitute : $\cos(2\pi f_o t) = \left[\dfrac{e^{j2\pi f_o t} + e^{-j2\pi f_o t}}{2}\right]$

$$\therefore\ X(f) = \frac{1}{2}\int_{0}^{\infty}\left[e^{j2\pi f_o t} + e^{-j2\pi f_o t}\right]\cdot e^{-j2\pi n}\,dt$$

$$= \frac{1}{2}\int_{0}^{\infty}\left[e^{-j2\pi(f - f_o)t} + e^{-j2\pi(f + f_o)t}\right]dt$$

$$= \frac{1}{2}\left\{\frac{1}{-j2\pi(f - f_o)}\left[e^{-j2\pi(f - f_o)t}\right]_0^{\infty} - \frac{1}{j2\pi(f + f_o)}\left[e^{-j2\pi(f + f_o)t}\right]_0^{\infty}\right\}$$

$$= \frac{1}{2}\left[\frac{1}{j2\pi(f - f_o)} + \frac{1}{j2\pi(f + f_o)}\right]$$

$$= \frac{A}{j4\pi}\left[\frac{f + f_o + f - f_o}{(f - f_o)(f + f_o)}\right] = \frac{2f}{j4\pi(f^2 - f_o^2)}$$

$$\therefore\ X(f) = \frac{1}{j2\pi}\,\frac{f}{(f^2 - f_o^2)} \qquad \text{...Ans.}$$

Ex. 6.4.8 : Find spectrum of $y(t) = \sin 2\pi f_1 t \cos 2\pi f_2 t$

Soln. :

Given : $y(t) = \sin(2\pi f_1 t)\cos(2\pi f_2 t)$

$$= \frac{1}{2}[\sin 2\pi(f_2 + f_1)t + \sin 2\pi(f_2 - f_1)t]$$

$$\therefore \quad y(t) = \frac{1}{2}\sin 2\pi f_H t + \frac{1}{2}\sin 2\pi f_L t \qquad ...(1)$$

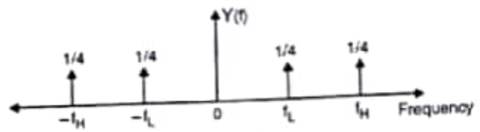
We know that

$$\sin \omega_0 t \overset{F}{\leftrightarrow} \frac{1}{2}[\delta(f - f_0) - \delta(f + f_0)]$$

$$\therefore Y(f) = F\left[\frac{1}{2}\sin 2\pi f_H t\right] + F\left[\frac{1}{2}\sin 2\pi f_L t\right]$$

$$= \frac{1}{4}[\delta(f - f_H) - \delta(f + f_H)] + \frac{1}{4}[\delta(f - f_L) - \delta(f + f_L)] \qquad ...\text{Ans.}$$

The spectrum is shown in Fig. P. 6.4.8.



(H-524) **Fig. P. 6.4.8 : Spectrum of y(t)**

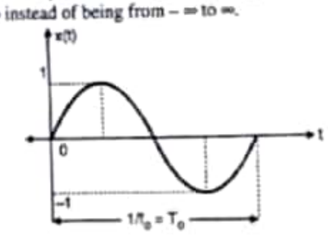
Ex. 6.4.9 : Find Fourier transform of x(t) : $x(t) = \sin \omega_0 t \cdot u(t)$. **Dec. 09, 10 Marks**

Soln. : The signal x(t) is as shown in the Fig. P. 6.4.9.

The FT of the signal x(t) is given as,

$$X(f) = \int_{-\infty}^{\infty} u(t) \cdot \sin(2\pi f_0 t) \cdot e^{-j2\pi f t}\, dt \qquad ...(1)$$

Due to the presence of u(t), the integration limits will be from 0 to ∞ instead of being from −∞ to ∞.



(H-525) **Fig. P. 6.4.9**

$$\therefore X(f) = \int_{0}^{\infty} \sin(2\pi f_0 t) \cdot e^{-j2\pi f t}\, dt \qquad ...(2)$$

Substitute : $\sin(2\pi f_0 t) = \dfrac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}$ in Equation (2)

$$\therefore X(f) = \int_{0}^{\infty}\left[\frac{\left(e^{-j2\pi(f - f_0)t} - e^{-j2\pi(f + f_0)t}\right)}{2j}\right] dt$$

$$= \frac{1}{2j}\left\{\frac{1}{-j2\pi(f - f_0)}\left[e^{-j2\pi(f - f_0)t}\right]_0^{\infty} + \frac{1}{j2\pi(f + f_0)}\left[e^{-j2\pi(f - f_0)t}\right]_0^{\infty}\right\}$$

$$= \frac{1}{2j}\left\{\frac{1}{-j2\pi(f - f_0)}(-1) + \frac{1}{j2\pi(f - f_0)}(-1)\right\}$$

$$= \frac{1}{4j^2\pi}\left[\frac{1}{(f - f_0)} - \frac{1}{(f + f_0)}\right]$$

But $j^2 = -1$, therefore changing the signs inside the bracket

$$\therefore X(f) = \frac{1}{4\pi}\left[\frac{1}{(f + f_0)} - \frac{1}{(f - f_0)}\right]$$

$$\therefore X(f) = \frac{1}{4\pi}\left[\frac{f - f_0 - f - f_0}{(f^2 - f_0^2)}\right] = \frac{-1}{2\pi}\left[\frac{f_0}{f^2 - f_0^2}\right]$$

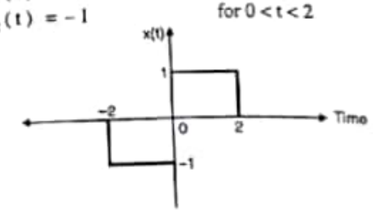
Ex. 6.4.10 : Obtain the Fourier transform of the signal shown in Fig. P. 6.4.10 and obtain its amplitude spectrum.

Soln. : Let us express the given function as the sum of two signals :

$$\therefore x(t) = x_1(t) + x_2(t) \qquad ...(1)$$

where, $x_1(t) = +1$ for $-2 < t < 0$

and $x_2(t) = -1$ for $0 < t < 2$ ...(2)



(H-526) **Fig. P. 6.4.10**

Using the property of superposition we can write that,

$$F[x(t)] = F[x_1(t)] + F[x_2(t)]$$

$$\text{i.e.} \quad X(f) = X_1(f) + X_2(f) \qquad ...(3)$$

Let us obtain $X_1(f)$ and $X_2(f)$ separately.

1. **To obtain $X_1(f)$:**

$$X_1(f) = \int_{-2}^{0} -1\, e^{-j2\pi f t}\, dt = \frac{+1}{j2\pi f}\left[e^{-j2\pi f t}\right]_{-2}^{0}$$

$$= \frac{+1}{j2\pi f}\left[e^{0} - e^{j4\pi f}\right] = \frac{+1}{j2\pi f}$$

$$\therefore X_1(f) = \frac{+1}{j2\pi f}\left[1 - e^{j4\pi f}\right] \qquad ...(4)$$

In Equation (4) make two substitutions as,

$$e^{j4\pi f} = e^{j2\pi f} \cdot e^{j2\pi f} = e^{+j2\pi f} \cdot e^{+j2\pi f}$$

and $1 = e^{-j2\pi f} \cdot e^{j2\pi f}$

Now substitute these in Equation (4) to get,

$$X_1(f) = \frac{1}{j2\pi f}\left[e^{-j2\pi f} \cdot e^{j2\pi f} - e^{+j2\pi f} \cdot e^{j2\pi f}\right]$$

$$= \frac{e^{j2\pi f}}{\pi f} = \left[\frac{e^{-j2\pi f} - e^{j2\pi f}}{2j}\right]$$

$$= \frac{-e^{j2\pi f}}{\pi f}\left[\frac{e^{j2\pi f} - e^{-j2\pi f}}{2j}\right] = \frac{-e^{j2\pi f}}{\pi f}\sin(2\pi f)$$

Multiply and divide by 2 to get,

$$X_1(f) = -2e^{j2\pi f}\frac{\sin(2\pi f)}{2\pi f} = -2e^{j2\pi f}\,\text{sinc}(2f) \qquad ...(5)$$

2. **To obtain $X_2(f)$:**

$$X_2(f) = \int_{0}^{2} +e^{-j2\pi f t}\, dt = \frac{1}{j2\pi f}\left[e^{-j2\pi f t}\right]_0^2$$

$$\therefore X_2(f) = \frac{-1}{j2\pi f}\left[e^{-j4\pi f} - 1\right] \qquad ...(6)$$

Substitute $e^{-j4\pi f} = e^{-j2\pi f} \cdot e^{-j2\pi f}$

and $1 = e^{j2\pi f} \cdot e^{-j2\pi f}$

$$\therefore X_2(f) = \left[e^{j2\pi f} \cdot e^{-j2\pi f} - e^{-j2\pi f} \cdot e^{-j2\pi f}\right]$$

$$= \frac{+e^{-j2\pi f}}{\pi f}\left[\frac{e^{j2\pi f} - e^{-j2\pi f}}{2j}\right]$$

$$\therefore X_2(f) = \frac{+e^{-j2\pi f}}{\pi f}\sin(2\pi f)$$

Multiply and divide by 2 to get,

$$X_2(f) = +2e^{-j2\pi f}\frac{\sin(2\pi f)}{2\pi f}$$

$$\therefore X_2(f) = +2e^{-j2\pi f}\,\text{sinc}(2f) \qquad ...(7)$$

3. **To obtain X (f) :**

$$X(f) = X_1(f) + X_2(f)$$

Substitute values from the Equations (5) and (7) to get,

$$X(f) = -2e^{j2\pi f}\,\text{sinc}(2f) + 2e^{-j2\pi f}\,\text{sinc}(2f)$$

$$= AT\,\text{sinc}(fT)\left[e^{j2\pi f} - e^{-j2\pi f}\right]$$

$$= -2\,\text{sinc}(2f)\left[e^{j2\pi f} - e^{-j2\pi f}\right]$$

Multiply and divide by 2j to get,

$$\therefore X(f) = -2j \times 2\,\text{sinc}(2f)\frac{\left[e^{j2\pi f} - e^{-j2\pi f}\right]}{2j}$$

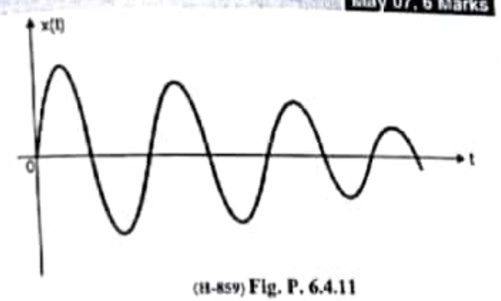
$$\therefore X(f) = -4j\,\text{sinc}(2f)\sin(2\pi f) \qquad ...\text{Ans.}$$

4. **Amplitude spectrum :**

The amplitude spectrum is given by,

$$|X(f)| = -4\,\text{sinc}(2f)\sin(2\pi f)$$

Ex. 6.4.11 : Find out the Fourier transform of damped sinusoid shown in Fig. P. 6.4.11. **May 07, 6 Marks**



(H-859) **Fig. P. 6.4.11**

Soln. : The damped sinusoid is given as,

$$X(t) = e^{-at} \cdot \sin \omega t \cdot u(t)$$

Fourier transform is,

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

Putting value of X (t) in above equation,

$$X(\omega) = \int_{-\infty}^{\infty} e^{-at} \cdot \sin \omega t \cdot u(t)\, e^{-j\omega t} \cdot dt$$

As, $u(t) = 1$ for $t \geq 0$

$= 0$ for $t < 0$

$$X(\omega) = \int_{0}^{\infty} e^{-at} \cdot \sin \omega t \cdot e^{-j\omega t} \cdot dt$$

$$\therefore X(\omega) = \int_{0}^{\infty} e^{-(a + j\omega)t} \cdot \sin \omega t \cdot dt$$

6.5 Application of Fourier Series and Fourier Transform to the System Analysis :

In the following sections we will study the frequency response and differential equations used for analysis of CT-LTI systems.

6.5.1 Frequency Response of an LTI System :

- Consider an LTI system having impulse response h (t) with an input x (t) such that,

$$x(t) = e^{j2\pi f t} \qquad ...(6.5.1)$$

- The system response is given by,

$$y(t) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{j2\pi f(t - \tau)}\, d\tau \qquad ...(6.5.2)$$

$$\therefore y(t) = e^{j2\pi f t}\int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f \tau}\, d\tau \qquad ...(6.5.3)$$

- Let us define

$$H(f) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j2\pi f \tau}\, d\tau$$

$$\therefore y(t) = H(f)\, e^{j2\pi f t} \qquad ...(6.5.4)$$

> Thus the response of an LTI system to a complex exponential function of frequency f is the same complex exponential function multiplied by a constant coefficient H (f), which is called as the **transfer function** of the system.

- The transfer function H (f) and the impulse function h (t) form a fourier transform pair. This is indicated as follows :

$$H(f) = \int_{-\infty}^{\infty} h(t)\, e^{-j2\pi f t}\, dt \qquad ...(6.5.5)$$

$$\text{and} \quad h(t) = \int_{-\infty}^{\infty} H(f) \cdot e^{j2\pi f t}\, df \qquad ...(6.5.6)$$

Alternative definition of transfer function :

- The transfer function is defined alternatively as follows :

$$H(f) = \left.\frac{y(t)}{x(t)}\right|_{x(t) = e^{j2\pi f t}} \qquad ...(6.5.7)$$

- We know that in time domain,

$$y(t) = h(t) * x(t)$$

- Taking the fourier transform of both sides we get,

$$Y(f) = H(f) \cdot X(f) \qquad ...(6.5.8)$$

Thus it is possible to describe an LTI system simply in the frequency domain.

6.5.2 Amplitude and Phase Response of a System :

1. The transfer function H (f) is a complex quantity expressed as,

$$H(f) = |H(f)|\, e^{j\theta(f)} \qquad ...(6.5.9)$$

where $|H(f)|$ = Amplitude response

and $\theta(f)$ = Phase response.

The phase response is given by,

$$\theta(f) = \arg |H(f)| \quad ...(6.5.10)$$

- For a linear system with real valued impulse response h (t) the transfer function H (f) exhibits a conjugate symmetry. That means,

$$|H(f)| = |H(-f)| \quad ...(6.5.11)$$

and the phase spectrum

$$\theta(f) = -\theta(-f) \quad ...(6.5.12)$$

- Thus the amplitude spectrum is an even function of frequency, whereas phase spectrum is an odd function of frequency.
- The graphs of amplitude response | H (f) | and phase response θ (f) versus frequency f represent the frequency domain description of the system. Therefore H (f) is also referred to as "Frequency Response" of the system.

6.5.3 Differential Equations :

Many physical processes can be easily modelled by n^{th} order constant coefficient differential equations. Consider such a differential equation,

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t) \quad ...(6.5.13)$$

According to differentiation property of fourier transform,

$$\frac{d}{dt} x(t) \xleftrightarrow{F.T.} j\omega X(\omega)$$

Applying differentiation property to Equation (6.5.13) we get,

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(\omega) \quad ...(6.5.14)$$

We know that $\frac{Y(\omega)}{X(\omega)}$ gives system transfer function, H (ω).

Thus from Equation (6.5.14) we can write,

$$\frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k} \quad ...(6.5.15)$$

This equation is used to calculate the frequency response and impulse response of system.

Solved Examples :

Ex. 6.5.1 : LTI system , which is initially at rest is described by the differential equation.

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$

Calculate system transfer function and impulse response of the system.

Soln. : The given difference equation is,

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t) \quad ...(1)$$

Taking fourier transform of both sides (applying differential property)

$$(j\omega)^2 Y(\omega) + 3(j\omega) Y(\omega) + 2Y(\omega) = j\omega X(\omega) + 2X(\omega)$$

$$Y(\omega)[(j\omega)^2 + 3(j\omega) + 2] = X(\omega)[3 + j\omega]$$

$$\therefore \frac{Y(\omega)}{X(\omega)} = H(\omega) = \frac{3 + j\omega}{(j\omega)^2 + 3(j\omega) + 2} \quad ...(2)$$

First we will obtain roots of denominator as follows :

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2}$$

$$\therefore \text{roots} = \frac{-3+1}{2}, \frac{-3-1}{2} = -1, -2$$

$$\therefore H(\omega) = \frac{3 + j\omega}{(j\omega + 1)(j\omega + 2)} \quad ...(3)$$

In the partial fraction expansion we can write,

$$H(\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 2} \quad ...(4)$$

The values of coefficients A and B are calculated as follows :

$$A = (j\omega + 1) \cdot H(\omega)|_{j\omega = -1}$$

$$\therefore A = \left.\frac{(j\omega + 1) \cdot (3 + j\omega)}{(j\omega + 1)(j\omega + 2)}\right|_{j\omega = -1} = \frac{3-1}{-1+2} = 2$$

and $B = (j\omega + 2) H(\omega)|_{j\omega = -2}$

$$\therefore B = \left.\frac{(j\omega + 2) \cdot (3 + j\omega)}{(j\omega + 1)(j\omega + 2)}\right|_{j\omega = -2} = \frac{3-2}{-2+1} = \frac{1}{-1} = -1$$

Putting values of A and B in Equation (4) we get,

$$H(\omega) = \frac{2}{j\omega + 1} - \frac{1}{j\omega + 2} \quad ...(5)$$

This is the system transfer function. To calculate we will take inverse fourier transform of H(ω).

$$\therefore h(t) = IFT\left\{\frac{2}{j\omega + 1}\right\} - IFT\left\{\frac{1}{j\omega + 2}\right\}$$

We have standard fourier transform pair,

$$e^{-at} u(t) \xleftrightarrow{F.T.} \frac{1}{a + j\omega}$$

that means $IFT\left\{\frac{1}{a + j\omega}\right\} = e^{-at} u(t)$

$$\therefore h(t) = 2e^{-t} u(t) - e^{-2t} u(t)$$

This is impulse response of system.

Ex. 6.5.2 : LTI system is described by the differential equation,

$$\frac{d^2 y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = x(t)$$

Calculate output y (t) if input $x(t) = e^{-3t} u(t)$ is applied to the system.

Soln. : First we will calculate transfer function H (ω). Given difference equation is,

$$\frac{d^2 y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = x(t)$$

Taking fourier transform of both sides we get,

$$(j\omega)^2 Y(\omega) + 3j\omega Y(\omega) + 2Y(\omega) = X(\omega)$$

$$\therefore Y(\omega)[(j\omega)^2 + 3j\omega + 2] = X(\omega)$$

$$\therefore H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$= \frac{1}{(j\omega)^2 + 3j\omega + 2} \quad ...(1)$$

Now given input signal is,

$$x(t) = e^{-3t} u(t) \quad ...(2)$$

We have standard fourier transform pair,

$$e^{-at} u(t) \xleftrightarrow{F.T.} \frac{1}{a + j\omega}$$

Thus taking fourier transform of Equation (2) we get,

$$X(\omega) = F.T.\{e^{-3t} u(t)\} = \frac{1}{3 + j\omega} \quad ...(3)$$

But from Equation (1),

$$Y(\omega) = \frac{1}{(j\omega)^2 + 3j\omega + 2} \cdot X(\omega)$$

$$\therefore Y(\omega) = \frac{1}{(j\omega + 1)(j\omega + 2)} \cdot \frac{1}{(3 + j\omega)}$$

$$\therefore Y(\omega) = \frac{1}{(j\omega + 1)(j\omega + 2)(j\omega + 3)} \quad ...(4)$$

Output y (t) is obtained by taking inverse fourier transform of Y (ω). In partial fraction expansion from Equation (4) can be written as,

$$Y(\omega) = \frac{A}{j\omega + 1} + \frac{B}{j\omega + 2} + \frac{C}{j\omega + 3} \quad ...(5)$$

The coefficients A,B and C are calculated as follows :

$$A = Y(\omega) \cdot (j\omega + 1)|_{j\omega = -1}$$

$$\therefore A = (j\omega + 1) \cdot \left.\frac{1}{(j\omega + 1)(j\omega + 2)(j\omega + 3)}\right|_{j\omega = -1}$$

$$\therefore A = \frac{1}{(-1+2)(-1+3)} = \frac{1}{1(2)} = \frac{1}{2}$$

$$B = (j\omega + 2) Y(\omega)|_{j\omega = -2}$$

$$\therefore B = (j\omega + 2) \cdot \left.\frac{1}{(j\omega + 1)(j\omega + 2)(j\omega + 3)}\right|_{j\omega = -2}$$

$$\therefore B = \frac{1}{(-2+1)(-2+3)} = \frac{1}{(-1)(1)} = -1$$

$$\text{and } C = (j\omega + 3) \cdot \left.\frac{1}{(j\omega + 1)(j\omega + 2)(j\omega + 3)}\right|_{j\omega = -3}$$

$$\therefore C = \frac{1}{(-3+1)(-3+2)} = \frac{1}{(-2)(-1)} = \frac{1}{2}$$

Putting these values in Equation (5) we get,

$$Y(\omega) = \frac{1}{2}\frac{1}{(j\omega + 1)} - \frac{1}{(j\omega + 2)} + \frac{1}{2} \cdot \frac{1}{(j\omega + 3)}$$

Taking inverse fourier transform of both sides,

$$y(t) = \frac{1}{2} e^{-t} u(t) - e^{-2t} u(t) + \frac{1}{2} e^{-3t} u(t)$$

This is the output of system.

Ex. 6.5.3 : Consider a linear time invariant system with impulse response : $h(t) = \frac{0.5 \sin(2t)}{t}$.

Find the system response y (t) if the input is x(t) = cos t + sin 3t using Fourier transforms.

Soln. :

Given :

$$h(t) = \frac{0.5 \sin(2t)}{t}$$

We have standard F.T. pair, $\sin\frac{Wt}{\pi t} \longleftrightarrow 1$ for $|\omega| < W$.

h (t) can be expressed as,

$$h(t) = \frac{0.5\pi \sin(2t)}{\pi t}$$

Thus its fourier transform is,

$$H(F) = 0.5\pi \quad \text{For } |\omega| < 2$$

$$= 0.5\pi \quad \text{For } |2\pi f| < 2$$

$$\therefore H(F) = 0.5\pi \quad \text{For } |f| < \frac{1}{\pi} \quad ...(1)$$

Given input is,

$$x(t) = \cos t + \sin 3t$$

We have standard F.T. pairs

$$\cos f_0 t \longleftrightarrow \pi[\delta(f - f_0) + \delta(f + f_0)]$$

$$\text{and } \sin f_0 t \longleftrightarrow \pi[\delta(f + f_0) - \delta(f - f_0)]$$

$$\therefore X(F) = \pi[\delta(F-1) + \delta(f+1)] + \pi[\delta(f+3) - \delta[f-3]]$$

Now $H(F) = \frac{Y(F)}{X(F)}$

$$\therefore Y(F) = H(F) \cdot X(F)$$

$$\therefore Y(F) = 0.5\pi^2[\delta(f-1) + \delta(f+1)] + 0.5\pi^2[\delta(f+3) - \delta(f-3)]$$

Taking IFT we get,

$$y(t) = \frac{1}{0.5\pi}\cos t + \frac{1}{0.5\pi}\sin 3t$$

Ex. 6.5.4 : Find Fourier transform of the following function using properties :

$$y(t) = \frac{d}{dt}\{te^{-3t} \cdot u(t) \times e^{-2t} \cdot u(t)\}.$$

Soln. :

$$\text{Given, } y(t) = \frac{d}{dt}\{te^{-3t} u(t) \times e^{-2t} u(t)\}$$

$$\text{Let } x(t) = te^{-3t} u(t) \quad ...(1)$$

$$\text{and } h(t) = e^{-2t} u(t) \quad ...(2)$$

$$\therefore y(t) = \frac{d}{dt}\{x(t) \times h(t)\}$$

By using differentiation and convolution property we get,

$$y(j\omega) = j\omega \times (j\omega) \cdot H(j\omega) \quad ...(3)$$

Here $x(j\omega) = x(j2\pi f)$, is fourier transformation.

Now we have,

$$F.T.\{e^{-3t} u(t)\} = \frac{1}{3 + j\omega}$$

According to differentiation in frequency domain property.

$$F.T.[tx(t)] = j\frac{d}{d\omega}[X(j\omega)]$$

$$\therefore F.T.\{te^{-3t} u(t)\} = j\frac{d}{d\omega}\left[\frac{1}{3 + j\omega}\right]$$

$$= j\frac{d}{d\omega}[(3 + j\omega)^{-1}]$$

$$= j[(-1) \cdot j \cdot (3 + j\omega)^{-2}]$$

$$\therefore F.T.\{te^{-3t} u(t)\} = \frac{1}{(3 + j\omega)^2} \quad ...(4)$$

Now H(jω) is fourier transform of Equation (2).

$$\therefore H(j\omega) = \frac{1}{2 + j\omega} \quad ...(5)$$

Putting Equations (4) and (5) in Equation (3) we get,

$$Y(j\omega) = \frac{1}{(3+j\omega)^2} \cdot \frac{1}{(2+j\omega)}$$

Ex. 6.5.5 : The impulse response of an LTI system is given by $h(t) = e^{-2t} u(t)$ and the input applied is $x(t) = 2 \cos 2t$ for all time t. Then find : Step response of the system.

Soln. :

Step response of the system :

Step response means applied input is unit step.

$$\therefore \quad x(t) = u(t)$$

$$\therefore \quad X(f) = \frac{1}{j2\pi f}$$

$$\text{Now} \quad h(t) = e^{-2t} u(t)$$

$$\therefore \quad H(f) = \frac{1}{2 + j2\pi f}$$

$$\text{We have} \quad H(f) = \frac{Y(f)}{X(f)}$$

$$\therefore \quad Y(f) = H(f) \cdot X(f)$$

$$\therefore \quad Y(f) = \frac{1}{j2\pi f} \cdot \frac{1}{j2\pi f + 2}$$

$$\therefore \quad Y(f) = \frac{1}{(j2\pi f - 0)(j2\pi f + 2)}$$

In partial fraction expansion form we can write

$$Y(f) = \frac{A_1}{j2\pi f - 0} + \frac{A_2}{j2\pi f + 2}$$

$$\text{Here} \quad A_1 = (j2\pi f - 0) \cdot \frac{1}{(j2\pi f - 0)(j2\pi f + 2)} \Bigg|_{j2\pi f = 0}$$

$$\therefore \quad A_1 = \frac{1}{2}$$

$$\text{and} \quad A_2 = (j2\pi f + 2) \cdot \frac{1}{(j2\pi f - 0)(j2\pi f + 2)} \Bigg|_{j2\pi f = -2}$$

$$\therefore \quad A_2 = -\frac{1}{2}$$

$$\therefore \quad Y(f) = \frac{1}{2} \cdot \frac{1}{j2\pi f} - \frac{1}{2} \cdot \frac{1}{j2\pi f + 2}$$

Taking inverse fourier we get,

$$y(t) = \frac{1}{2} \cdot u(t) - \frac{1}{2} e^{-2t} u(t)$$

This is the step response of system.

6.6 Merits of Fourier Transform :

1. It is possible to uniquely recover the original time function x (t).
2. We can evaluate the convolutional integrals using F.T.
3. F.T. is very useful in communication systems.

6.7 Limitations of Fourier Transform and Need of LT and ZT :

- The most important limitation of F.T. is that there are many time functions for which the F.T. does not exist, because such functions are not absolutely integrable.
- For the LTI system analysis, if we use the Fourier transform then it becomes a little difficult to obtain the convolution of x (t) and h (t) to obtain the system response.
- Another problem with the Fourier transform is that there are some functions for which the Fourier Transform fails to exist.
- These problems can be overcome by using the Laplace transform. Laplace transform is a general form of representation of signals in the exponential form.
- System analysis becomes simpler if we use Laplace transform, because then convolution of x (t) and h (t) becomes multiplication of their Laplace transforms.
- The solution of differential equations also, becomes simpler using Laplace transform because they become simple algebraic equations.
- Z-transform is another method for representation of signals and systems. Z-transform representation is more general and most important is that it is applicable to stable, unstable, causal, non-causal systems as well.
- It is easier to obtain the system response of an LTI system using the Z-transform because we have to compute the multiplication X (z) and H (z) of the Z-transforms of x (n) and h (n) instead of computing the convolution sum of x (n) and h (n) in the time domain.

6.8 Introduction to Discrete Time Fourier Transform :

The DSP processors are used to perform frequency analysis of signals. To do the frequency analysis, we should convert the time domain signal into the frequency domain. For this, the different frequency transformation techniques are used.

What does a Fourier transform do ?

Any signal is build up by addition of elementary signals which are at different frequencies, have different amplitudes and relative phases. Using the fourier transform we can plot the amplitude and phase spectrums which provide us all the information about amplitudes and relative phases of such elementary signals. This fourier transform can be used for the "analysis" of a signal. It is used for transformation from the time domain to frequency domain.

6.9 Analogy between CTFT and DTFT :

- Similar to Continuous Time Fourier Transform (CTFT) ; a Discrete Time Fourier Transform (DTFT) involves a Fourier transform of time domain signal.
- For CTFT, the spectrum is in the range $-\infty$ to $+\infty$ while DTFT has a spectrum in the range 0 to 2π or $-\pi$ to $+\pi$.
- DTFT is periodic with a period of 2π.
- In case of DTFT ; since the signal is discrete in nature a summation term is used instead of integration term.

6.10 Fourier Transform of Discrete Time Aperiodic Signals (DTFT) :

Basically the fourier transform of a periodic, finite energy discrete time signal is called as DTFT.

Mathematical equation : Mathematically it is defined as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \text{...(6.10.1)}$$

Here $X(\omega)$ = Fourier transform of x (n)

and x (n) = Discrete time aperiodic signal

We can obtain discrete time signal x (n) from X (ω) by taking inverse fourier transform. It is given by,

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \qquad \text{...(6.10.2)}$$

Notation : The fourier transform of x (n) is denoted by,

$$x(n) \overset{F}{\longleftrightarrow} X(\omega)$$

Equation (6.10.1) is called as Analysis Formula (FT) and Equation (6.10.2) is called as Synthesis Formula (IFT).

6.10.1 Periodic Nature of X (ω) MU : May 04

University Questions

Q. 1 Prove that for a DT system, the frequency range is 2π. **(May 04, 5 Marks)**

The frequency range of ω is from $-\pi$ to $+\pi$ or from 0 to 2π. So X (ω) becomes periodic with period 2π.

Proof : According to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \text{...(6.10.3)}$$

Replacing ω by $(\omega + 2\pi k)$ we get,

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega + 2\pi k) n}$$

$$\therefore \quad X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \times e^{-j2\pi k n} \qquad \text{...(6.10.4)}$$

According to Euler's identity we have,

$$e^{-jx} = \cos x - j \sin x$$

$$\therefore \quad e^{-j2\pi k n} = \cos(2\pi k n) - j \sin(2\pi k n)$$

Now 'k' and 'n' both are integers

$$\therefore \quad \cos(2\pi k n) = 1 \quad \text{and} \quad \sin(2\pi k n) = 0$$

$$\therefore \quad e^{-j2\pi k n} = 1$$

Putting this value in Equation (6.10.2) we get,

$$X(\omega + 2\pi k) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \text{...(6.10.5)}$$

Comparing Equations (6.10.1) and (6.10.3) we get,

$$X(\omega) = X(\omega + 2\pi k) \qquad \text{...(6.10.6)}$$

Thus X (ω) is periodic with period 2π. So we can obtain x (n) from X (ω) as,

$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$. Thus **x (n) and X (ω) are inverse of each other.**

Condition for existence of FT : The fourier transform is convergent if,

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

That means fourier transform exists only if discrete time signal is absolutely summable. This is necessary and sufficient condition for the existence of fourier transform.

6.11 Discrete Fourier Transform of Standard Signals :

We will obtain the discrete fourier transform of some standard signals with the help of examples.

Ex. 6.11.1 : Obtain DTFT of unit impulse δ (n).

Soln. : Here discrete time sequence is,

$$x(n) = \delta(n) \qquad \text{...(1)}$$

According to the definition of DTFT we have,

$$\text{DTFT}\{x(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\therefore \quad X(\omega) = \sum_{n=-\infty}^{\infty} \delta(n) \times e^{-j\omega n} \qquad \text{...(2)}$$

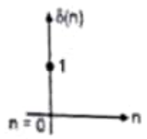
But we know that δ (n) exists only at n = 0. It is shown in Fig. P. 6.11.1

$$\therefore \quad \delta(n) = 1 \quad \text{for } n = 0$$
$$= 0 \quad \text{otherwise}$$

Thus Equation (2) becomes,

$$X(\omega) = \sum_{\text{at } n=0} 1 \times e^{-j\omega n} = 1 \times e^{0} = 1$$

$$\therefore \quad \delta(n) \overset{\text{DTFT}}{\longleftrightarrow} 1$$



(B-554) Fig. P. 6.11.1

This is the standard fourier transform pair.

Ex. 6.11.2 : Obtain DTFT of unit step u (n).

Soln. :

$$\text{Here } x(n) = u(n)$$

$$\text{According to definition of DTFT, } X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

$$\text{DTFT}\{u(n)\} = X(\omega) = \sum_{n=-\infty}^{\infty} u(n) e^{-j\omega n} \qquad \text{...(1)}$$

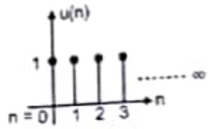
But we know that u (n) exists only for positive values of n. It is shown in Fig. P. 6.11.2.

$$\therefore \quad u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0$$

Thus we can write the limits of summation, from n = 0 to $n = \infty$.

$$\therefore\ X(\omega) = \sum_{n=0}^{\infty} 1 \times e^{-j\omega n}$$

$$\therefore\ X(\omega) = \sum_{n=0}^{\infty} \left[e^{-j\omega}\right]^n \qquad \ldots(2)$$



(H-555) **Fig. P. 6.11.2**

Now, we have the geometric series formula,

$$\sum_{n=0}^{\infty} A^n = A^0 + A + A^2 + A^3 + \ldots$$

$$= \frac{1}{1-A} \text{ if } |A| < 1$$

Let $A = e^{-j\omega}$. Thus Equation (2) can be written as,

$$X(\omega) = \frac{1}{1-e^{-j\omega}} \qquad \ldots(3)$$

Now, if we put $\omega = 0$ in Equation (3) then we get, $X(\omega) = \frac{1}{0} = \infty$. That means at $\omega = 0$; the DTFT obtained by Equation (3) is not valid. So we should obtain $X(\omega)$ for different values of 'ω' other than zero. Rearranging Equation (3) we get,

$$X(\omega) = \frac{1}{e^{-j\omega/2} \times e^{-j\omega/2} - e^{-j\omega/2} \times e^{-j\omega/2}} \qquad \ldots(4)$$

Note : $e^{-j\omega/2} \cdot e^{j\omega/2} = e^0 = 1$
and $e^{-j\omega/2} \cdot e^{-j\omega/2} = e^{-j\omega}$

Taking $e^{-j\omega/2}$ common from the denominator we get,

$$X(\omega) = \frac{1}{e^{-j\omega/2}\left[e^{j\omega/2} - e^{-j\omega/2}\right]} \qquad \ldots(5)$$

According to Euler's identity we have,

$$\frac{e^{j\omega/2} - e^{-j\omega/2}}{2} = j \sin\frac{\omega}{2}$$

Thus Equation (5) becomes,

$$X(\omega) = \frac{1}{e^{-j\omega/2} \times 2j \sin\frac{\omega}{2}}$$

$$\therefore\ X(\omega) = \frac{e^{j\omega/2}}{2j \sin \omega/2}$$

Thus $\quad u(n) \xleftrightarrow{\text{DTFT}} \dfrac{e^{j\omega/2}}{2j \sin \omega/2}$

Ex. 6.11.3 : Obtain DTFT of rectangular pulse,

$x(n) = A \quad$ for $0 \le n \le L-1$
$\quad\ = 0 \quad$ otherwise

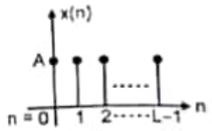
Soln. :

This pulse is shown in Fig. P. 6.11.3.

Here 'A' indicates the area of rectangular pulse and 'L' is the length of rectangular pulse.

According to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \ldots(1)$$



(H-557) **Fig. P. 6.11.3**

But $x(n)$ is present only in the range $n = 0$ to $n = L-1$

$$\therefore\ X(\omega) = \sum_{n=0}^{L-1} A\, e^{-j\omega n} \qquad \ldots(2)$$

Here area 'A' is constant and independent of limits of summation.

$$\therefore\ X(\omega) = A \sum_{n=0}^{L-1} \left(e^{-j\omega}\right)^n \qquad \ldots(3)$$

Now use the standard summation formula,

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here $A = e^{-j\omega}$, $N_1 = 0$ and $N_2 = L-1$

Thus Equation (3) becomes,

$$X(\omega) = A\left[\frac{(e^{-j\omega})^0 - (e^{-j\omega})^{L-1+1}}{1-e^{-j\omega}}\right]$$

$$\therefore\ X(\omega) = A\left[\frac{1-e^{-j\omega L}}{1-e^{-j\omega}}\right] \qquad \ldots(4)$$

Let us rearrange the terms to use Euler's identity.

For the numerator put $1 = e^{j\omega L/2} \times e^{-j\omega L/2}$ and $e^{-j\omega L} = e^{-j\omega L/2} \times e^{-j\omega L/2}$ and for the denominator put $1 = e^{j\omega/2} \times e^{-j\omega/2}$ and $e^{-j\omega} = e^{-j\omega/2} \times e^{-j\omega/2}$.

$$\therefore\ X(\omega) = A\left[\frac{e^{j\omega L/2} \times e^{-j\omega L/2} - e^{-j\omega L/2} \times e^{-j\omega L/2}}{e^{j\omega/2} \times e^{-j\omega/2} - e^{-j\omega/2} \times e^{-j\omega/2}}\right]$$

$$\therefore\ X(\omega) = A\,\frac{e^{-j\omega L/2}}{e^{-j\omega/2}}\left[\frac{e^{j\omega L/2} - e^{-j\omega L/2}}{e^{j\omega/2} - e^{-j\omega/2}}\right] \qquad \ldots(5)$$

According to Euler's identity we have,

$$\sin x = \frac{e^{jx} - e^{-jx}}{2} \quad \therefore\ e^{jx} - e^{-jx} = 2 \sin x$$

$$\therefore\ X(\omega) = A \cdot e^{-j\omega L/2} \cdot e^{j\omega/2}\left[\frac{2 \sin \omega L/2}{2 \sin \omega/2}\right]$$

$$\therefore\ X(\omega) = A\, e^{-j\omega(L-1)/2}\left[\frac{\sin \omega L/2}{\sin \omega/2}\right]$$

$$\therefore\ \text{'A'} \xleftrightarrow{\text{DTFT}} A\, e^{-j\omega(L-1)/2}\left[\frac{\sin \omega L/2}{\sin \omega/2}\right]$$

Ex. 6.11.4 : Determine the discrete time signal $x(n)$ whose fourier transform is,

$$X(e^{j\omega}) = 1 \quad -\omega_c \le \omega \le \omega_c$$
$$\quad\ = 0 \quad -\omega_c \le |\omega| \le \pi$$

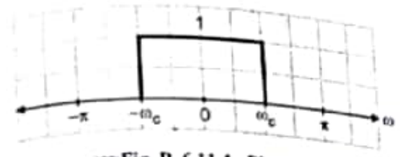
Soln. : The given signal is plotted as shown in Fig. P. 6.11.4.

We have to obtain $x(n)$ which means we have to take Inverse Fourier Transform (IFT) of given signal.

$$\therefore\ x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} cX(e^{j\omega}) \times e^{j\omega n}\, d\omega \qquad \ldots(1)$$

But $X(e^{j\omega}) = 1$ for the range $-\omega_c$ to ω_c

$$\therefore\ x(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega n}\, d\omega$$



(H-558) **Fig. P. 6.11.4 : Plot of $X(e^{j\omega})$**

$$\therefore\ x(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

According to Euler's identity,

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\therefore\ x(n) = \frac{1}{2\pi n} \times 2 \sin \omega_c n$$

$$\therefore\ x(n) = \frac{\sin \omega_c n}{\pi n}$$

Summary of standard DTFT pairs :

Table P. 6.11.4 shows DTFT of standard signals.

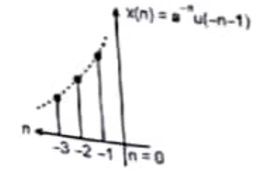
Table P. 6.11.4

Sr. No	Signal	Discrete time	DTFT
1.	Unit impulse	$\delta(n)$	1
2.	Unit step	$u(n)$	$\dfrac{e^{j\frac{\omega}{2}}}{2j \sin\frac{\omega}{2}}$
3.	Right handed exponential	$a^n u(n)$	$\dfrac{1}{1-a e^{-j\omega}}$
4.	Left handed exponential	$a^{-n} u(-n-1)$	$\dfrac{a e^{j\omega}}{1-a e^{j\omega}}$
5.	Double sided exponential	$a^n u(n) + a^{-n} u(-n-1)$	$\dfrac{1}{1-a e^{-j\omega}} + \dfrac{a e^{j\omega}}{1-a e^{j\omega}}$
6.	Rectangular pulse	$x(n) = A$	$A e^{\frac{-j\omega(L-1)}{2}}\left[\dfrac{\sin\frac{\omega L}{2}}{\sin\frac{\omega}{2}}\right]$

Ex. 6.11.5 : Obtain DTFT of left handed exponential signal, $x(n) = a^{-n} u(-n-1)$.

Soln. :

We know that $u(-n)$ is folded unit step and $u(-n-1)$ is folded unit step advanced by 1 sample. Similar to $u(n)$, the multiplication of signal by $u(-n-1)$ will not change the amplitude. But the range of signal becomes, $n = -1$ to $n = -\infty$; because $x(-n-1)$ is present from $n = 1$ to $n = -\infty$. It is shown in Fig. P. 6.11.5.



(H-556) **Fig. P. 6.11.5**

According to the definition of DTFT we have,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \qquad \ldots(1)$$

New limits of summation will be from $n = -1$ to $n = -\infty$.

$$\therefore\ X(\omega) = \sum_{n=-1}^{-\infty} a^{-n} e^{-j\omega n} \qquad \ldots(2)$$

The limits are negative; to make it positive put $n = -l$. Then the limits will change as follows :

when $\quad n = -1 \quad \Rightarrow -1 = -l \quad \therefore l = 1$
and when $\quad n = -\infty \quad \Rightarrow -\infty = -l \quad \therefore l = \infty$

$$\therefore\ X(\omega) = \sum_{l=1}^{\infty} a^l \cdot e^{j\omega l}$$

$$\therefore\ X(\omega) = \sum_{l=1}^{\infty} \left(a e^{j\omega}\right)^l \qquad \ldots(3)$$

We have the standard geometric series for the summation from '0' to '∞'. So we will adjust the summation as follows :

$$X(\omega) = \sum_{l=0}^{\infty} \left(a e^{j\omega}\right)^l - 1$$

Using the formula of geometric series,

$$X(\omega) = \frac{1}{1-a e^{j\omega}} - 1 = \frac{1-1+a e^{j\omega}}{1-a e^{j\omega}}$$

$$\therefore\ X(\omega) = \frac{a e^{j\omega}}{1-a e^{j\omega}}$$

$$\therefore\ a^{-n} u(-n-1) \xleftrightarrow{\text{DTFT}} \frac{a e^{j\omega}}{1-a e^{j\omega}}$$

Ex. 6.11.6 : Obtain the Fourier transform of single sided exponential pulse.

$x(n) = a^n u(n)$

Soln. : Here 'a' is some constant and $a^n u(n)$ is an exponential signal which exists from 0 to ∞. That means only for the positive values of 'n'. This is also called as right handed exponential sequence.

Note : When any signal is multiplied by unit step $u(n)$ then the result of multiplication is positive signal. This is because $u(n)$ exists from $n = 0$ to $n = \infty$, that means for positive values of 'n'. But since the amplitude of $u(n)$ is 1; it will not change the amplitude of signal after multiplication. So while

doing the calculations we will not consider u (n). It is just used to indicate the range of signal.

According to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad ...(1)$$

Here $x(n) = a^n u(n)$. The multiplication of a^n by $u(n)$ indicates that the signal is present from $n = 0$ to $n = \infty$. So we will change the limits of summation as :

$$\therefore \quad X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} \quad ...(2)$$

Note that in Equation (2) we have not put $x(n) = a^n u(n)$. Because we have adjusted the limits accordingly.

$$\text{Now} \quad X(\omega) = \sum_{n=0}^{\infty} (a e^{-j\omega})^n \quad ...(3)$$

Recall the geometric series formula,

$$\sum_{n=0}^{\infty} A^n = A^0 + A^1 + A^2 + A^3 + \ldots$$

$$= \frac{1}{1-A} \quad \text{if } |A| < 1$$

Let $A = a e^{-j\omega}$, thus Equation (3) can be written as,

$$X(\omega) = \frac{1}{1 - a e^{-j\omega}}$$

$$\therefore a^n u(n) \xleftrightarrow{FT} \frac{1}{1 - a e^{-j\omega}}$$

Ex. 6.11.7 : Obtain DTFT of double sided exponential signal.

$$x(n) = a^n u(n) + a^{-n} u(-n-1)$$

Soln. :

We have already obtained the DTFT of single sided exponential sequence separately. So by adding two DTFT's we can directly obtain the DTFT of double sided exponential signal.

$$\therefore a^n u(n) + a^{-n} u(-n-1) \xleftrightarrow{FT} \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}}$$

Ex. 6.11.8 : Obtain DTFT of a signal

$$x(n) = r^n \cos(\omega_0 n) u(n), \; r < 1$$

Soln. : According to the definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad ...(1)$$

The given sequence is multiplied by unit step, u (n). So we can write the limits of summation from $n = 0$ to $n = \infty$.

$$\therefore X(\omega) = \sum_{n=0}^{\infty} r^n \cos \omega_0 n \, e^{-j\omega n} \quad ...(2)$$

According to Euler's identity we have,

$$\cos \omega_0 n = \frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2}$$

$$\therefore X(\omega) = \sum_{n=0}^{\infty} r^n \left(\frac{e^{j\omega_0 n} + e^{-j\omega_0 n}}{2} \right) \times e^{-j\omega n}$$

$$\therefore X(\omega) = \frac{1}{2} \left[\sum_{n=0}^{\infty} r^n e^{j\omega_0 n} \times e^{-j\omega n} + \sum_{n=0}^{\infty} r^n e^{-j\omega_0 n} \times e^{-j\omega n} \right]$$

$$\therefore X(\omega) = \frac{1}{2} \left[\sum_{n=0}^{\infty} r^n e^{-j(\omega - \omega_0) n} + \sum_{n=0}^{\infty} r^n e^{-j(\omega + \omega_0) n} \right]$$

$$\therefore X(\omega) = \frac{1}{2} \left\{ \sum_{n=0}^{\infty} \left[r e^{-j(\omega - \omega_0)} \right]^n + \sum_{n=0}^{\infty} \left[r e^{-j(\omega + \omega_0)} \right]^n \right\} \quad ...(3)$$

Use the standard geometric series formula,

$$\sum_{n=0}^{\infty} A^n = \frac{1}{1-A}$$

$$\therefore X(\omega) = \frac{1}{2} \left[\frac{1}{1 - r e^{-j(\omega - \omega_0)}} + \frac{1}{1 - r e^{-j(\omega + \omega_0)}} \right] \quad ...(4)$$

Now we will simplify Equation (4) as follows,

$$X(\omega) = \frac{1}{2} \left\{ \frac{1 - r e^{-j(\omega + \omega_0)} + 1 - r e^{-j(\omega - \omega_0)}}{\left[1 - r e^{-j(\omega - \omega_0)} \right] \left[1 - r e^{-j(\omega + \omega_0)} \right]} \right\} \quad ...(5)$$

Consider numerator term only. It can be simplified as,

$$\text{Numerator} = 2 - r e^{-j\omega} \times e^{-j\omega_0} - r e^{-j\omega} \times e^{j\omega_0}$$

$$= 2 - r e^{-j\omega} \left(e^{-j\omega_0} + e^{j\omega_0} \right)$$

$$\text{But we have, } \cos x = \frac{e^{jx} + e^{-jx}}{2}$$

$$\therefore \text{Numerator} = 2 - r e^{-j\omega} (2 \cos \omega_0) = 2 - 2 r \cos \omega_0 e^{-j\omega} \quad ...(6)$$

Now consider the denominator term,

$$\text{Denominator} = \left(1 - r e^{-j\omega} \times e^{j\omega_0} \right) \left(1 - r e^{-j\omega} \times e^{-j\omega_0} \right)$$

$$= 1 - r e^{-j\omega} \times e^{-j\omega_0} - r e^{-j\omega} \times e^{j\omega_0} + r^2 e^{-j\omega} \times e^{j\omega_0} \times e^{-j\omega} \times e^{-j\omega_0}$$

$$= 1 - r e^{-j\omega} \left(e^{-j\omega_0} + e^{j\omega_0} \right) + r^2 \times e^{-j2\omega}$$

$$= 1 - r e^{-j\omega} \cdot 2 \cos \omega_0 + r^2 e^{-j2\omega}$$

$$= 1 - 2r \cos \omega_0 e^{-j\omega} + r^2 e^{-j2\omega} \quad ...(7)$$

Putting Equations (6) and (7) in Equation (5) we get,

$$X(\omega) = \frac{1}{2} \left[\frac{2 - 2r \cos \omega_0 e^{-j\omega}}{1 - 2r \cos \omega_0 e^{-j\omega} + r^2 e^{-j2\omega}} \right]$$

$$\therefore X(\omega) = \frac{1 - r \cos \omega_0 e^{-j\omega}}{1 - 2r \cos \omega_0 e^{-j\omega} + r^2 e^{-j2\omega}}$$

This is the required DTFT. From this we can write the standard DTFT pair as,

$$a^n \cos \omega_0 n \, u(n) \xleftrightarrow{FT} \frac{1 - r \cos \omega_0 e^{-j\omega}}{1 - 2r \cos \omega_0 e^{-j\omega} + r^2 e^{-j2\omega}}$$

Ex. 6.11.9 : Obtain DTFT and plot the magnitude and phase response of h(n) = {0, 1, 1, 1}

May 13, 5 Marks

Soln. : According to the definition of DTFT,

$$X(\omega) = \sum_{n=0}^{3} x(n) e^{-j\omega n}$$

$$= x(0) + x(1) e^{-j\omega} + x(2) e^{-2j\omega} + x(3) e^{-j3\omega}$$

$$\therefore X(\omega) = e^{-j\omega} + e^{-j2\omega} + e^{-j3\omega}$$

$$X(\omega) = e^{-j2\omega} \left[e^{j\omega} + 1 + e^{-j\omega} \right]$$

$$\text{Now } e^{j\omega} + e^{-j\omega} = 2 \cos \theta$$

$$\therefore X(\omega) = e^{-j2\omega} [1 + 2 \cos \omega]$$

Thus magnitude is,

$$|X(\omega)| = 1 + 2 \cos \omega$$

And phase is,

$$\angle X(\omega) = -2\omega \text{ for } |X(\omega)| > 0$$

$$= -2\omega + \pi \text{ for } |X(\omega)| < 0$$

By putting different values of 'ω' the values of magnitude and phase response are calculated.

Value of ω	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$-\frac{\pi}{3}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
$\lvert X(\omega)\rvert$	-1	-0.41	1	2	2.41	3	2.41	2	1	-0.41	-1
$\angle X(\omega)$	3π	$\frac{5\pi}{2}$	π	$\frac{2\pi}{3}$	$\frac{\pi}{2}$	0	$\frac{+\pi}{2}$	$\frac{+2\pi}{3}$	$+\pi$	$\frac{+5\pi}{2}$	3π

The magnitude and phase response are shown in Fig. P. 6.11.9 and Fig. P. 6.11.9(a).

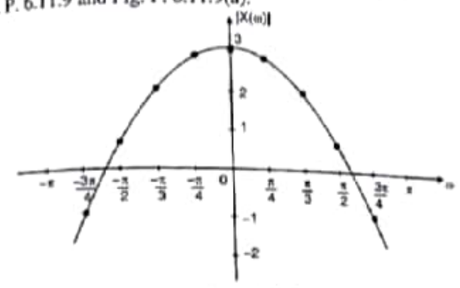
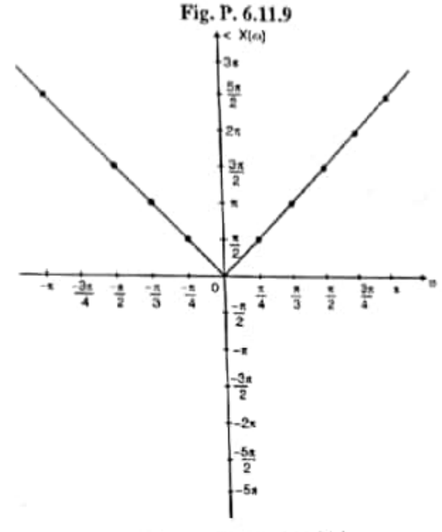


Fig. P. 6.11.9



(B-1525) **Fig. P. 6.11.9(a)**

Ex. 6.11.10 : Find the DTFT of x(n) = {2, 1, 2} and compute its magnitude at ω = 0 and $\omega = \frac{\pi}{2}$

Dec. 12, 5 Marks

Soln. :

Given, $x(n) = \{2, \underset{\uparrow}{1}, 2\}$

Using definition of DTFT,

$$X(\omega) = \sum_{n=0}^{3} x(n) e^{-j\omega n} = x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega}$$

$$= 2 + e^{-j\omega} + 2 e^{-j2\omega} = 2 [1 + e^{-j2\omega}] + e^{-j\omega}$$

$$= 2 [e^{j\omega} \cdot e^{-j\omega} + e^{-j\omega} \cdot e^{-j\omega}] + e^{-j\omega}$$

$$= 2 e^{-j\omega} [e^{j\omega} + e^{-j\omega}] + e^{-j\omega}$$

$$\therefore X(\omega) = 4 \cos \theta \cdot e^{-j\omega} + e^{-j\omega}$$

$$\therefore X(\omega) = e^{-j\omega} [1 + 4 \cos \omega]$$

$$|X(\omega)| = 1 + 4 \cos \omega$$

$$\text{At } \omega = 0 \Rightarrow |X(0)| = 1 + 4 \cos(0) = 5$$

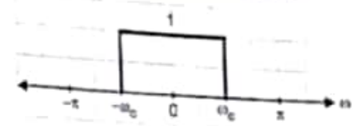
$$\text{At } \omega = \frac{\pi}{2} \Rightarrow X\left(\frac{\pi}{2}\right) = 1 + 4 \cos\left(\frac{\pi}{2}\right) = 1$$

Ex. 6.11.11 : Determine the discrete time signal x (n) whose Fourier transform is,

$$X(e^{j\omega}) = 1, \quad -\omega_c \le \omega \le \omega_c$$

$$= 0, \quad -\omega_c \le |\omega| \le \pi$$

Soln. : The given signal is plotted as shown in Fig. P. 6.11.11.



(B-1666) **Fig. P. 6.11.11 : Plot of X ($e^{j\omega}$)**

We have to obtain x (n) which means we have to take inverse Fourier transform (IFT) of given signal.

$$\therefore x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} cX(e^{j\omega}) \times e^{j\omega n} d\omega \quad ...(1)$$

But $X(e^{j\omega}) = 1$ for the range $-\omega_c$ to ω_c

$$\therefore x(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1 \times e^{j\omega n} d\omega$$

$$\therefore x(n) = \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn} \left[e^{j\omega_c n} - e^{-j\omega_c n} \right]$$

According to Euler's identity,

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j} \quad \therefore x(n) = \frac{1}{2\pi n} \times 2 \sin \omega_c n$$

$$\therefore x(n) = \frac{\sin \omega_c n}{\pi n}$$

6.11.1 Summary of Properties of DTFT :

Table 6.11.1 shows summary of properties of DTFT.

Table 6.11.1

Sr. No.	Name of property	Time domain representation	Frequency domain representation
1.	Linearity	$a_1 x_1(n) + a_2 x_2(n)$	$a_1 X_1(\omega) + a_2 X_2(\omega)$
2.	Time shifting	$x(n-k)$	$e^{-j\omega k} X(\omega)$
3.	Time reversal	$x(-n)$	$X(-\omega)$
4.	Convolution theorem	$x_1(n) * x_2(n)$	$X_1(\omega) \cdot X_2(\omega)$
5.	Correlation theorem	$r_{x_1 x_2}(n)$	$X_1(\omega) \cdot X_2(-\omega)$
6.	Frequency shifting	$e^{j\omega_0 n} \cdot x(n)$	$X(\omega - \omega_0)$
7.	Multiplication of two sequences	$x_1(n) \cdot x_2(n)$	$X_3(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda)\, d\lambda$
8.	Parseval's theorem	$x_1(n) \cdot x_2^*(n)$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\omega)\, x_2^*(\omega)\, d\omega$
9.	Differentiation in Frequency domain	$nx(n)$	$j \frac{d}{d\omega} X(\omega)$

6.12 Additional Solved Examples :

Ex. 6.12.1 : The impulse response of an LTI system $h[n] = \{1, 2, 1, -1\}$. Find the response y [n] of the system for the input x [n] = {1, 2, 3, 1} using discrete time Fourier transform.

May 14, 10 Marks

Soln. :

Using definition of DTFT,

$$H(\omega) = \sum_{n=0}^{3} h(n) e^{-j\omega n}$$

$$= h(0) e^0 + h(1) e^{-j\omega} + h(2) e^{-j2\omega} + h(3) e^{-j3\omega}$$

$$\therefore H(\omega) = 1 + 2e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega}$$

Similarly, $X(\omega) = \sum_{n=0}^{3} x(n) e^{-j\omega n}$

$$\therefore X(\omega) = x(0) e^0 + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + x(3) e^{-j3\omega}$$

$$\therefore X(\omega) = 1 + 2e^{-j\omega} + 3e^{-j2\omega} + e^{-j3\omega}$$

Now output is given by,

$$y(n) = x(n) * h(n)$$

In DTFT, convolution becomes multiplication

$$\therefore Y(\omega) = X(\omega) \cdot H(\omega)$$

$$\therefore Y(\omega) = \left[1 + 2e^{-j\omega} + e^{-j2\omega} - e^{-j3\omega}\right] \cdot \left[1 + 2e^{-j\omega} + 3e^{-j2\omega} + e^{-j3\omega}\right]$$

$$\therefore Y(\omega) = 1 + 2e^{-j\omega} + 3e^{-j2\omega} + e^{-j3\omega} + 2e^{-j\omega} + 4e^{-j2\omega} + 6e^{-j3\omega} + 2e^{-j4\omega}$$
$$+ e^{-j2\omega} + 2e^{-j3\omega} + 3e^{-j4\omega} + e^{-j3\omega} - e^{-j5\omega} - 2e^{-j4\omega} - 3e^{-j5\omega} - e^{-j6\omega}$$

$$\therefore Y(\omega) = 1 + 4e^{-j\omega} + 8e^{-j2\omega} + 8e^{-j3\omega} + 3e^{-j4\omega} - 2e^{-j5\omega} - e^{-j6\omega} \quad ...(1)$$

Now using definition of DTFT we can write,

$$Y(\omega) = \sum_{n=0}^{6} Y(n) e^{-j\omega n}$$

$$\therefore Y(\omega) = Y(0) + y(1) e^{-j\omega} + y(2) e^{-j2\omega} + y(3) e^{-j3\omega}$$
$$+ y(4) e^{-j4\omega} + y(5) e^{-j5\omega} + y(6) e^{-j6\omega} \quad ...(2)$$

Comparing like coefficients of Equations (1) and (2) we get,

$Y(0) = 1,\quad Y(1) = 4,\quad Y(2) = 8,$

$Y(3) = 8,\quad Y(4) = 3,\quad Y(5) = -2,$

$Y(6) = -1$

$$\therefore Y(n) = \{1, 4, 8, 8, 3, -2, -1\}$$

Ex. 6.12.2 : Consider a continuous time LTI system described by $\frac{dy(t)}{dt} + 2y(t) = x(t)$. Using the Fourier transform, find out output to each of the following input signals :

1. $x(t) = e^{-t} u(t)$ 2. $x(t) = u(t)$

Dec. 14, 10 Marks

Soln. :

Given : $\frac{d}{dt} y(t) + 2y(t) = x(t)$

Taking Fourier transform,

$$(j\omega) y(j\omega) + 2y(j\omega) = x(j\omega)$$

$$\therefore y(j\omega)[(j\omega) + 2] = x(j\omega)$$

$$\therefore y(j\omega) = \frac{x(j\omega)}{j\omega + 2} \quad ...(1)$$

1. $x(t) = e^{-t} u(t)$:

$$\therefore x(j\omega) = \frac{1}{j\omega + 1}$$

Thus from Equation (1),

$$y(j\omega) = \frac{1}{(j\omega + 1)(j\omega + 2)} = \frac{A_1}{j\omega + 1} + \frac{A_2}{j\omega + 2}$$

Here $A_1 = \frac{1}{-1+2} = 1$ and $A_2 = \frac{1}{-2+1} = -1$

$$\therefore y(j\omega) = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2}$$

Taking inverse Fourier,

$$y(t) = e^{-t} u(t) - e^{-2t} u(t)$$

2. $x(t) = u(t)$:

$$\therefore x(j\omega) = \frac{1}{j\omega}$$

From Equation (1),

$$y(j\omega) = \frac{1}{(j\omega)(j\omega + 2)} = \frac{A_1}{j\omega} + \frac{A_2}{j\omega + 2}$$

Here $A_1 = \frac{1}{0+2} = 0.5$

and $A_2 = \frac{1}{-2} = -0.5$

$$\therefore y(j\omega) = \frac{0.5}{j\omega} - \frac{0.5}{j\omega + 2}$$

Taking inverse Fourier,

$$y(t) = 0.5\, u(t) - 0.5\, e^{-2t} u(t)$$

Ex. 6.12.3 : Find out Fourier transform of f (t) = 10 δ (t – 2). Sketch its amplitude and phase spectrum.

May 15, 5 Marks

Soln. :

Given : $f(t) = 10\,\delta(t-2)$

We have, $\delta(t) \leftrightarrow 1$

Using time shifting property,

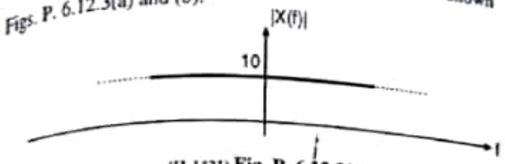
$$\delta(t-2) \leftrightarrow e^{-j2\pi f \times 2} \cdot 1$$

$$\therefore F(s) = F\{10\,\delta(t-2)\} = 10\, e^{-j4\pi f}$$

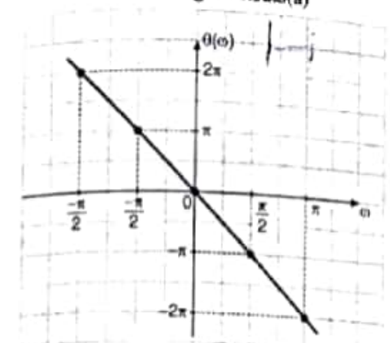
Here Amplitude $= |F(s)| = 10$

and Phase $= -4\pi f = -2\omega$

The amplitude and phase spectrum are shown in Figs. P. 6.12.3(a) and (b).



(H-1421) **Fig. P. 6.12.3(a)**



(H-1422) **Fig. P. 6.12.3(b)**

Ex. 6.12.4 : Convolve $x[n] = \left(\frac{1}{3}\right)^n u[n]$ with $h[n] = \left(\frac{1}{2}\right)^n u[n]$ using Fourier transform.

May 15, 10 Marks

Soln. :

Given : $x(n) = \left(\frac{1}{3}\right)^n u(n)$ and $h(n) = \left(\frac{1}{2}\right)^n u(n)$

Convolution becomes multiplication in Fourier domain. Now we have standard formula,

$$a^n u(n) \leftrightarrow \frac{1}{1 - ae^{-j\omega}}$$

$$\therefore X(\omega) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}} \quad \text{and} \quad H(\omega) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$\text{Let } Y(\omega) = X(\omega) \cdot H(\omega) = \frac{1}{1 - \frac{1}{3}e^{-j\omega}} \cdot \frac{1}{1 - \frac{1}{2}e^{-j\omega}}$$

$$= \frac{e^{j\omega}}{e^{j\omega} - \frac{1}{3}} \cdot \frac{e^{j\omega}}{e^{j\omega} - \frac{1}{2}} = \frac{e^{j2\omega}}{\left(e^{j\omega} - \frac{1}{3}\right)\left(e^{j\omega} - \frac{1}{2}\right)}$$

$$Y(\omega) = \frac{A}{e^{j\omega} - \frac{1}{3}} + \frac{B}{e^{j\omega} - \frac{1}{2}}$$

$$\text{Here } A = \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2} - \frac{1}{3}\right)} = \frac{\frac{1}{4}}{\frac{1}{3}} = \frac{3}{4}$$

$$\text{and } B = \frac{\left(\frac{1}{3}\right)^2}{\left(\frac{1}{3} - \frac{1}{2}\right)} = \frac{\frac{1}{9}}{-\frac{1}{6}} = -\frac{2}{3}$$

$$\therefore Y(\omega) = \frac{\frac{3}{4}}{e^{j\omega} - \frac{1}{3}} - \frac{\frac{2}{3}}{e^{j\omega} - \frac{1}{3}}$$

Taking inverse Fourier,

$$y(n) = \frac{3}{4}\left(\frac{1}{3}\right)^n u(n) - \frac{2}{3}\left(\frac{1}{3}\right)^n u(n)$$

Ex. 6.12.5 : State duality property of Fourier transform. If Fourier transform of $e^{-t} u(t)$ is $\frac{1}{1 + j\Omega}$, then find the Fourier transform of $\frac{1}{1+t}$ using duality property.

Dec. 15, 10 Marks

Soln. :

Refer section 6.2.3 for duality property of Fourier transform.

Given : $e^{-t} u(t) \xleftrightarrow{FT} \frac{1}{1 + j\Omega}$

Using duality t is replaced by jΩ and jΩ is replaced by 't'. Similarly L.H.S. and R.H.S. are interchanged.

$$\therefore \frac{1}{1+t} \xleftrightarrow{FT} e^{-j\Omega} u(j\Omega)$$

Ex. 6.12.6 : Consider a causal LTI system with $H(j\omega) = \frac{1}{j\omega + 2}$. For a particular input x (t), this system produces output $y(t) = e^{-2t} u(t) - e^{-3t} u(t)$. Find out x (t) using Fourier transform.

May 16, 10 Marks

Soln. :

Given : $H(j\omega) = \frac{1}{j\omega + 2}$

and $y(t) = e^{-2t} u(t) - e^{-3t} u(t)$

$$\therefore Y(j\omega) = \frac{1}{j\omega + 2} - \frac{1}{j\omega + 3}$$

We have, $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$

$$\therefore X(j\omega) = \frac{Y(j\omega)}{H(j\omega)}$$

$$\therefore X(j\omega) = \frac{\frac{1}{j\omega + 2} - \frac{1}{j\omega + 3}}{\frac{1}{j\omega + 2}} = 1 - \frac{j\omega + 2}{j\omega + 3}$$

$$\therefore X(j\omega) = 1 - \frac{j\omega}{j\omega + 3} - \frac{2}{j\omega + 3}$$

Taking inverse Fourier transform,

$$x(t) = \delta(t) - \frac{d}{dt}(e^{-3t}) - 2e^{-3t} u(t)$$

$$\therefore x(t) = \delta(t) - (-3)e^{-3t} u(t) - 2e^{-3t} u(t)$$

$$= \delta(t) + 3e^{-3t}u(t) - 2e^{-3t}u(t)$$

$$\therefore x(t) = \delta(t) + e^{-3t}u(t)$$

Ex. 6.12.7 : Find out DTFT of the following :

1. $x[n] = \{1, -1, 2, 2\}$
2. $x[n] = \sin\left[\frac{\pi n}{2}\right] u[n]$

May 16, 5 Marks

Soln. :

1. $x(n) = \{1, -1, 2, 2\}$

Using definition of DTFT,

$$X(\omega) = \sum_{n=0}^{3} x(n)e^{-j\omega n}$$

$$= x(0) + x(1)e^{-j\omega} + x(2)e^{-j2\omega} + x(3)e^{-j3\omega}$$

$$\therefore X(\omega) = 1 - 1e^{-j\omega} + 2e^{-j2\omega} + 2e^{-j3\omega}$$

2. $x(n) = \sin\left(\frac{\pi n}{2}\right) u(n)$

$$\therefore x(n) = \frac{e^{j\frac{\pi n}{2}} - e^{-j\frac{\pi n}{2}}}{2j} \cdot u(n)$$

$$= \frac{1}{2j}\left[\left(e^{j\frac{\pi}{2}}\right)^n u(n) - \left(e^{-j\frac{\pi}{2}}\right)^n u(n)\right]$$

We have, $a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1 - ae^{-j\omega}}$

$$\therefore X(\omega) = \frac{1}{2j}\left[\frac{1}{1 - e^{j\frac{\pi}{2}} \cdot e^{-j\omega}} - \frac{1}{1 - e^{-j\frac{\pi}{2}} \cdot e^{-j\omega}}\right]$$

Now $e^{j\frac{\pi}{2}} = \cos\frac{\pi}{2} + j\sin\frac{\pi}{2} = 1$ and $e^{-j\frac{\pi}{2}} = -1$

$$\therefore X(\omega) = \frac{1}{2j}\left[\frac{1}{1 - e^{-j\omega}} - \frac{1}{1 + e^{-j\omega}}\right]$$

$$\therefore X(\omega) = \frac{1}{2j}\left[\frac{1 + e^{-j\omega} - 1 + e^{-j\omega}}{1 - e^{-j2\omega}}\right]$$

$$\therefore X(\omega) = \frac{1}{2j}\left[\frac{2e^{-j\omega}}{1 - e^{-j2\omega}}\right]$$

$$\therefore X(\omega) = \frac{e^{-j\omega}}{j - je^{-j2\omega}}$$

Review Questions

Q. 1 What do you mean by frequency analysis of a signal ?

Q. 2 What is the use of analysis of a signal ?

Q. 3 What is a phase spectrum ?

Q. 4 What is the line spectrum ?

Q. 5 What is double sided line spectrum ?

Q. 6 What does a Fourier transform do ?

Q. 7 Define Fourier transform ?

Q. 8 Define inverse Fourier transform ?

Q. 9 What are Dirichlet conditions ?

Q. 10 What is the relationship between time domain and frequency domain ?

Q. 11 What are the different properties of Fourier transforms ?

Q. 12 What are the different types of Fourier series ?

Q. 13 What is the necessity of Fourier series ?

Q. 14 State and prove the following properties of Fourier transform :

1. Time shifting
2. Frequency shifting
3. Convolution in time domain

Q. 15 State and prove the following properties of Fourier Transform :

1. Time scaling 2. Duality
3. Differentiation in time domain

Q. 16 Define unit impulse function and find its Fourier transform.

Q. 17 List out various properties of the Fourier transform and prove modulation theorem stating its significance in analog communication system.

Q. 18 List the merits and limitations of Fourier transform.

Q. 19 Show that compression in the time domain is equivalent to expansion in the frequency domain and vice versa.

Q. 20 State and prove the Frequency translation theorem.

Q. 21 What is the difference between time domain representation and frequency domain representation of signals ?

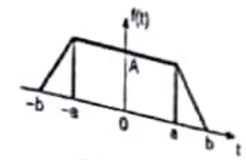
Q. 22 Discuss the effects of the pulse width and the time period of the gated pulse on the bandwidth and the shape and the density of the spectrum.

Q. 23 Fourier transform is a limiting case of Fourier series by letting the period of periodic function infinite. Justify the statement with an example of periodic gate function.

Q. 24 Explain "a signal cannot be band limited and time limited simultaneously".

6.13 University Questions and Answers :

Q. 1 Determine the Fourier transform of the trapezoidal function shown in Fig. Q. 1. **(Dec. 2016, 10 Marks)**



(H-1664) Fig. Q. 1

Ans. :

Mathematical given waveform can be expressed as,

$$x(t) = t + b \quad \text{for } -b < t < -a$$

$$= A \quad \text{for } -a < t < a$$

$$= -t + b \quad \text{for } a < t < b$$

Using definition of fourier transform,

$$X(F) = \int_{-b}^{-a} (t+b)e^{-2\pi ft}dt + \int_{-a}^{a} Ae^{-j2\pi ft}dt + \int_{a}^{b} (-t+b)e^{-j2\pi ft}dt$$

$$\therefore X(F) = \int_{-b}^{-a} te^{-2\pi ft}dt + \int_{-b}^{-a} be^{-j2\pi ft}dt + \int_{-a}^{a} Ae^{-j2\pi ft}dt - \int_{a}^{b} te^{-j2\pi ft}dt + \int_{a}^{b} be^{-j2\pi ft}dt$$

Using integration formula, $\int xe^{ax} = \frac{e^{ax}}{a^2}(ax - 1)$

$$\therefore X(F) = \left[\frac{e^{-j2\pi ft}}{(-j2\pi f)^2}(-j2\pi ft - 1)\right]_{-b}^{-a} + b\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-b}^{-a} + A\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{-a}^{a} - \left[\frac{e^{-j2\pi ft}}{(-j2\pi f)^2}(-j2\pi ft - 1)\right]_{a}^{b} + b\left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{a}^{b}$$

$$X(F) = \left[\frac{e^{j2\pi fa}}{-4\pi^2 f^2}(j2\pi fa - 1) + \frac{e^{j2\pi fb}}{+4\pi^2 f^2}(j2\pi fb - 1)\right] + b\left[\frac{e^{j2\pi fa} - e^{j2\pi fb}}{-j2\pi f}\right] + A\left[\frac{e^{j2\pi fa} - e^{-j2\pi fa}}{-j2\pi f}\right]$$

$$- \left[\frac{e^{-j2\pi fb}}{-4\pi^2 f^2}(-j2\pi fb - 1) - \frac{e^{-j2\pi fa}}{-4\pi^2 f^2}(-j2\pi fa - 1)\right] + b\left[\frac{e^{-j2\pi fb} - e^{-j2\pi fa}}{-j2\pi f}\right]$$

$$\therefore X(F) = \frac{1}{-4\pi^2 f^2}\left[j2\pi fa\, e^{j2\pi fa} - j2\pi fa\, e^{-j2\pi fa}\right] + \frac{1}{-4\pi^2 f^2}\left[e^{j2\pi fa} - e^{-j2\pi fa}\right] + \frac{1}{-4\pi^2 f^2}\left[e^{j2\pi fb} \cdot j2\pi fb - e^{-j2\pi fb} \cdot j2\pi fb\right]$$

$$+ \frac{1}{4\pi^2 f^2}\left[-e^{j2\pi fb} - e^{-j2\pi fb}\right] + \frac{b}{-j2\pi f}\left[e^{j2\pi fa} - e^{-j2\pi fa}\right] + \frac{b}{-j2\pi f}\left[e^{j2\pi fb} + e^{-j2\pi fb}\right] + \frac{A}{\pi f}\left[\frac{e^{j2\pi fa} - e^{-j2\pi fa}}{2s}\right]$$

Use identities, $e^{j\theta} - e^{-j\theta} = 2j\sin\theta$ and $e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$$\therefore X(F) = \frac{(j2\pi fa)(2j)}{-4\pi^2 f^2}[\sin(2\pi fa)] + \frac{1}{4\pi^2 f^2}[2\cos(2\pi fa)] + \frac{j2\pi fb}{4\pi^2 f^2}[2j\sin(2\pi fb)] - \frac{1}{4\pi^2 f^2}[2\cos(2\pi fb)]$$

$$+ \frac{b}{-\pi f}[\sin(2\pi fa)] + \frac{b}{\pi f}[\sin 2\pi fb] + \frac{A}{\pi f}\sin(2\pi fb)$$

$$\therefore X(F) = \frac{a}{\pi f}\sin(2\pi fa) + \frac{1}{2\pi^2 f^2}\cos(2\pi fa) - \frac{b}{2\pi}\sin(2\pi fb) - \frac{1}{2\pi^2 f^2}\cos(2\pi fb) - \frac{b}{\pi f}\sin(2\pi fa) + \frac{b}{\pi f}\sin(2\pi fb) + \frac{A}{\pi f}\sin 2\pi fa$$

$$\therefore X(F) = \sin(2\pi fa)\left[\frac{a}{\pi f} - \frac{b}{\pi f}\right] + \frac{1}{2\pi^2 f^2}[\cos(2\pi fa) - \cos(2\pi fb)] + \frac{A}{\pi f}\sin 2\pi fa$$

Q. 2 Find out DTFT of the following

1. $x(n) = \{1, -1, 2, 2\}$
2. $x(n) = -a^n u(-n-1)$, where $|a| < 1$

(Dec. 2016, 10 Marks)

Ans. :

Given, $x(n) = \{1, -1, 2, 2\}$

According to definition of DTFT,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{3} x(n) e^{-j\omega n}$$

$$\therefore X(\omega) = x(0) + x(1) e^{-j\omega} + x(2) e^{-j2\omega} + x(3) e^{-j3\omega}$$

$$\therefore X(\omega) = 1 - e^{-j\omega} + 2e^{-j2\omega} + 2e^{-j3\omega}$$

□□□

CHAPTER 7

Module 3

Laplace Transform

Syllabus :

Overview of Laplace transform : Need of Laplace Transform, review of unilateral and bilateral Laplace Transform, properties, inverse of Laplace Transform, concept of Region of Convergence (ROC) poles and zeros, relation between continuous time Fourier Transform and Laplace Transform.

7.1 Introduction :

- The laplace transform is named in honor of great French mathematician, **Pierre Simon De Laplace.**
- It is a mathematical technique used for solving differential equations.
- Laplace transform changes one signal into another by some fixed set of equations.
- Basically laplace transform changes a signal from time domain in 's' domain.

7.1.1 Need of Laplace Transform (Application of Laplace Transform) :

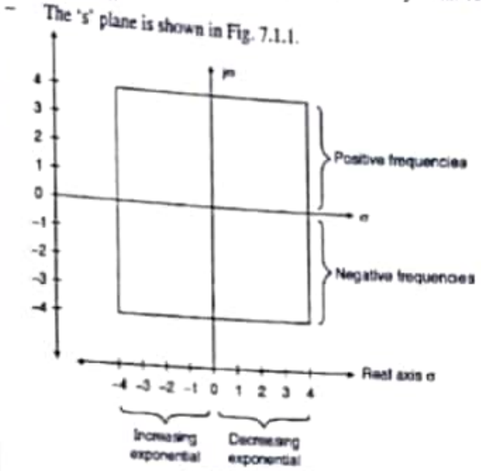
It is an obvious in question that, if we can solve all equations in time domain then why to select laplace domain ?

1. Laplace transform reduces the problem of solving differential equations to an algebraic problem. So solution of differential equations become simple.
2. Many mechanical and electrical forces are discontinuous in nature or they may be present only for short amount of time period. So these forces are not sine or cosine functions. That means solution of these functions become difficult. Laplace transform converts such functions into sine and cosine function; so solution becomes simple.
3. Both complementary and particular solution can be obtained in one operation using laplace transform. So it gives complete solution of differential equations.
4. The knowledge of both magnitude and frequency can be obtained using laplace transformation.

After reading these advantages; you may ask what is laplace plane ? and how to represent any equation in laplace.

7.1.2 What is the 's' Domain ?

- The 's' domain is a complex plane that means there are real numbers along horizontal axis and imaginary numbers along vertical axis.
- The distance along real axis is expressed by variable 'σ' and distance along imaginary axis is expressed by variable 'ω'. It is a natural frequency.
- In this co-ordinate system, the location of any point is specified by providing values of 'σ' and 'ω'.
- Thus using complex notation, each location is represented by complex variable 's' and $s = \sigma + j\omega$.
- The 's' plane is continuous and extends to infinity in all four directions.
- The 's' plane is shown in Fig. 7.1.1.



(II-1459) Fig. 7.1.1

After discussing some basics of laplace transform; now we will define the laplace transform.

7.1.3 Definition of Laplace Transform :

Consider $x(t)$ is a function of 't' for all $t > 0$. Then the laplace transform of $x(t)$ is denoted by $L\{x(t)\}$ and it is defined as,

$$L\{x(t)\} = \int_0^{\infty} x(t) e^{-st} dt \qquad ...(7.1.1)$$

The laplace transform of $x(t)$ is also denoted by $X(s)$. Here notation 'L' is called as laplace transform operator.

Important note :

If any equation is function of time like $x(t)$ then always small letters are used. Once the laplace transform is calculated; then capital letters are used. That means laplace transform of $x(t)$ is denoted by $X(s)$.

Notation :

$L[x(t)] = X(s)$. That it is denoted as,

$$x(t) \xleftrightarrow{L.T.} X(s) \quad ...(7.1.2)$$

$$\therefore X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt \quad ...(7.1.3)$$

- The laplace transform exists if integration in Equation (7.1.3) exists. That means laplace transform exists if integral in Equation (7.1.3) is **convergent**. The meaning of word **convergent** is "answer of integration must be finite".

After reading Equation (7.1.2), an obvious question is **"Why a bidirectional arrow is used in the notation?"** The answer of this question is simple. We can always calculate inverse laplace from X (s). That means we can obtain x (t) from X (s). Inverse laplace is denoted by L^{-1}.

$$\therefore x(t) = L^{-1} X(s) \quad ...(7.1.4)$$

Other notations : Some authors use other notations to represent laplace transform of x (t). The other notations are $\phi(s)$ or $\bar{X}(s)$.

We will use notation X (s) throughout our discussion.

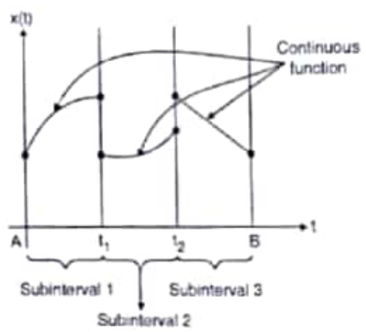
7.2 Sufficient Conditions for Existance of Laplace Transform :

The sufficient conditions for existance of laplace transform are as follows :

1. Function x (t) is piece-wise continuous.
2. Function x (t) is function of exponential order.

Meaning of condition 1 :

- Let us consider that function x (t) is present in the time internal A to B. Then this time interval is divided into subintervals say A to t_1, t_1 to t_2, t_2 to B.
- In each subinterval; the function must be continuous. That means at every value of 't' there must be some value of x (t).
- Every subinterval should have finite limits.



(H-1460) **Fig. 7.2.1**

Meaning of condition 2 :

Consider the constants α and m. Then the function x (t) must be of exponential order of m. That means,

$$|x(t)| \propto e^{\alpha t}$$

> **Note :** These two conditions are sufficient for existence of laplace transform but these are not necessary conditions. That means laplace transform exists even if these conditions are not satisfied.

7.2.1 Region of Convergence :

> **University Question**
>
> **Q. 1** Explain in brief ROC (Region of Convergence) conditions of Laplace transform. **(Dec. 16, 4 Marks)**

According to the definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \quad ...(7.2.1)$$

Thus laplace transform will exist only if,

$$\int_{-\infty}^{\infty} |x(t)\, e^{-st}|\, dt < \infty \quad ...(7.2.2)$$

Now we have,

$$s = \sigma + j\omega$$

$$\therefore e^{-st} = e^{-\sigma t} \cdot e^{-j\omega t} \quad ...(7.2.3)$$

We have $e^{-j\omega t} = \cos \omega t - j \sin \omega t$. Thus the value of $e^{-j\omega t}$ is always in the range + 1 to – 1. So we can modify the condition of existence of laplace transform as,

$$\int_{-\infty}^{\infty} |x(t)\, e^{-\sigma t}|\, dt < \infty \quad ...(7.2.4)$$

Equation (7.2.4) gives sufficient condition for the existence of laplace transform.

Definition :

The range of values of σ for which Equation (7.2.4) attains some finite value is called as region of convergence (ROC).

7.2.2 Pole - Zero Concept :

The laplace transform of any time domain signal can be expressed in terms of ratio of numerator polynomial and denominator polynomial.

$$\therefore X(s) = \frac{N(s)}{D(s)} \quad ...(7.2.5)$$

Here N (s) = Numerator polynomial

D (s) = Denominator polynomial

We can factorize the numerator and denominator. Thus after obtaining roots of numerator and denominator; Equation (7.2.5) can be written as,

$$X(s) = \frac{a_0 \cdot (s - z_1)(s - z_2) \ldots (s - z_n)}{b_0 (s - p_1)(s - p_2) \ldots (s - p_m)} \quad ...(7.2.6)$$

Here a_0 and b_0 are constants.

Zeros : If 's' takes values $z_1, z_2 \ldots z_n$ then according to Equation (7.2.6); laplace transform X (s) vanishes. Thus such complex frequencies are called as zeros of laplace transform. Zeros are denoted by the mark '0' in the pole zero diagram.

Poles : When 's' takes the values $p_1, p_2 \ldots p_m$ then according to Equation (7.2.6); laplace transform X (s) becomes infinity. Such complex frequencies are called as the poles of laplace transform. Poles are denoted by the mark 'X' in the pole-zero diagram.

Example :

Consider that the laplace transform of signal X (t) is given by,

$$X(s) = \frac{s^2 - s - 2}{s^2 - s - 6}$$

By obtaining the roots of numerator and denominator we can write,

$$X(s) = \frac{(s+1)(s-2)}{(s+2)(s-3)}$$

Zeros

Zeros are obtained by equating each bracket of numerator term to zero.

$\therefore$ $s + 1 = 0 \Rightarrow s = -1$ that means $z_1 = -1$

and $s - 2 = 0 \Rightarrow s = 2$ that means $z_2 = 2$

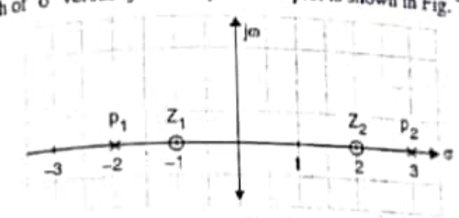
Poles

Poles are obtained by equating each bracket of denominator term to zero.

$\therefore$ $s + 2 = 0 \Rightarrow s = -2$, that means $p_1 = -2$

$s - 3 = 0 \Rightarrow s = 3$, that means $p_2 = 3$

The pole-zero plot is a graph drawn in s-plane. That means it is graph of real part of 's' versus imaginary part of 's'. So it is the graph of 'σ' versus 'jω'. The pole-zero plot is shown in Fig. 7.2.2.



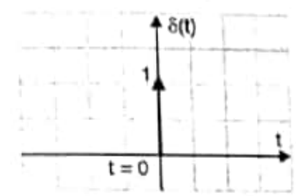
(H-559) **Fig. 7.2.2**

7.3 Laplace Transform of Elementary Signals :

In this section we will obtain the laplace transform of some basic signals. Such signals include delta function unit step, exponential signal etc.

7.3.1 Delta Function :

The delta function is shown in Fig. 7.3.1.



(H-560) **Fig. 7.3.1**

It is defined as,

$$\delta(t) \neq 0 \qquad \text{for } t = 0 \quad ...(7.3.1)$$

According to the definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt \quad ...(7.3.2)$$

Here x (t) = δ (t) and its value is 1 only at t = 0; it is not necessary to take integration. Thus Equation (7.3.2) becomes,

$$X(s) = 1 \cdot e^0 = 1 \quad ...(7.3.3)$$

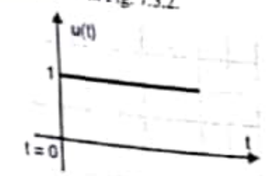
$$\therefore \delta(t) \xleftrightarrow{L.T.} 1$$

ROC :

Since 's' term is absent in Equation (7.3.3); ROC is entire s-plane.

7.3.2 Unit step :

Unit step is as shown in Fig. 7.3.2.



(H-561) **Fig. 7.3.2**

It is defined as,

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad ...(7.3.4)$$

Now we have,

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt \quad ...(7.3.5)$$

But here x (t) = u (t) = 1 in the range t = 0 to t = ∞

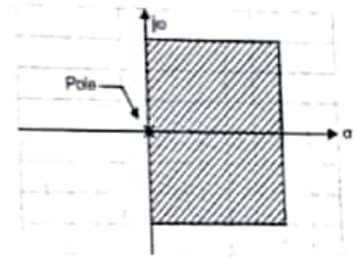
$$\therefore X(s) = L[u(t)] = \int_{-\infty}^{\infty} 1 \cdot e^{-st}\, dt$$

$$\therefore X(s) = \left[\frac{-e^{-st}}{s}\right]_0^{\infty} = -\frac{1}{s}[e^{-\infty} - e^0] = \frac{1}{s}$$

$$\therefore u(t) \xleftrightarrow{LT} \frac{1}{s} \quad ...(7.3.6)$$

ROC :

The laplace transform of unit step is $\frac{1}{s}$ and it is for the range $R_e\{s\} > 0$, that means for σ > 0. Thus ROC is σ > 0. It has a pole at s = 0 that means at origin. The sketch of ROC is shown in Fig. 7.3.3.



(H-562) **Fig. 7.3.3**

> **Note :** Simple rules to decide ROC.
>
> - If signal is multiplied with u (t); it is called as causal signal.
> - If signal is multiplied with u (– t); it is called as anticausal signal.
> - Once the laplace transform of any equation is calculated; then obtain roots of denominator (poles).
> - If signal is causal then ROC is greater than value of pole.
> - If signal is anticausal then ROC is less than value of pole.

- If signal is combination of many causal terms then ROC is greater than the largest pole.
- If signal is combination of many anticausal terms then ROC is less than the smallest pole.
- If signal is combination of causal and anticausal terms then ROC is the region between two poles.

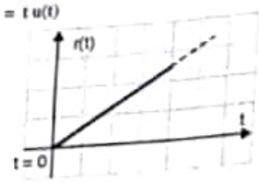
7.3.3 Unit Ramp :

Unit ramp signal is shown in Fig. 7.3.4.

It is defined as,

$$r(t) = t \quad \text{for } t \geq 0 \qquad \text{...(7.3.7)}$$
$$= 0 \quad \text{otherwise}$$

$$r(t) = t\,u(t)$$



(H-563) Fig. 7.3.4

We have, $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ ...(7.3.8)

Here $x(t) = r(t) = t$ for the range $t = 0$ to $t = \infty$. Thus Equation (7.3.8) becomes,

$$X(s) = \int_{0}^{\infty} t\, e^{-st}\, dt \qquad \text{...(7.3.9)}$$

Recall the integration formula, $\int t\, e^{at}\, dt = \frac{e^{at}}{a^2}(at - 1)$

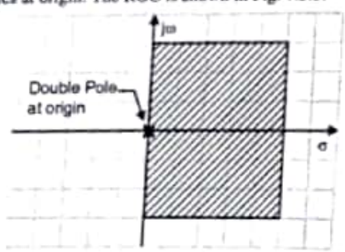
$$\therefore \quad X(s) = \left[\frac{e^{-st}}{s^2}(-st - 1)\right]_0^{\infty} = -\left[\frac{e^{-st}}{s^2}(st + 1)\right]_0^{\infty}$$

$$= \frac{-1}{s^2}\left[e^{-st} \cdot st + e^{-st}\right]_0^{\infty}$$

$$= -\frac{1}{s^2}\left\{\left[e^{-st} \cdot st\right]_0^{\infty} + \left[e^{-st}\right]_0^{\infty}\right\}$$

$$= -\frac{1}{s^2}\left\{\left[\frac{st}{e^{st}}\right]_0^{\infty} + \left[\frac{1}{e^{st}}\right]_0^{\infty}\right\} = -\frac{1}{s^2}[0 - 1] = \frac{1}{s^2}$$

ROC : Here laplace transform is $\frac{1}{s^2}$ and it is valid for the range $R_e\{s\} > 0$; that means $\sigma > 0$. Thus ROC is $\sigma > 0$. This function has two poles at origin. The ROC is shown in Fig. 7.3.5.

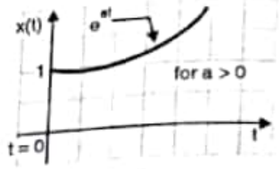


(H-564) Fig. 7.3.5

Note : This is causal signal and the poles are at origin. So, ROC is greater than the pole, that means $\sigma > 0$.

7.3.4 Positive Sided Growing Exponential Pulse :

This is also called as right handed growing exponential pulse. Such a pulse is represented as $F(t) = e^{at}\, u(t)$. Here a is arbitrary constant. For $a > 0$, the nature of pulse is as shown in Fig. 7.3.6.



(H-565) Fig. 7.3.6

We have, $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ ...(7.3.10)

Here $x(t) = e^{at} \cdot u(t)$ ...(7.3.11)

Multiplication by unit step u (t) indicates that the exponential pulse e^{at} is present in the range $t = 0$ to $t = \infty$.

Thus limits of integration in Equation (7.3.10) will be from $t = 0$ to $t = \infty$.

$$\therefore \quad X(s) = \int_{0}^{\infty} e^{at} \cdot e^{-st}\, dt \quad \therefore \quad X(s) = \int_{0}^{\infty} e^{-(s-a)t}\, dt$$

$$\therefore \quad X(s) = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$\therefore \quad X(s) = -\frac{1}{s-a}\left[\frac{1}{e^{(s-a)t}}\right]_0^{\infty} = \frac{-1}{s-a}[0 - 1]$$

$$\therefore \quad X(s) = \frac{1}{s-a} \qquad \text{...(7.3.12)}$$

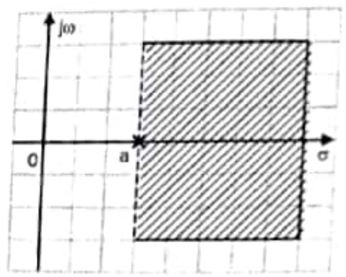
$$\therefore \quad e^{at}\, u(t) \xleftrightarrow{\text{L.T.}} \frac{1}{s-a}$$

ROC :

The laplace transform is $\frac{1}{s-a}$. It has pole as + 'a'. Thus ROC is $Re\{s - a\} > 0$, that means $Re\{s\} > a$. But $Re\{s\}$ means σ.

∴ ROC is $\sigma > a$

This ROC is shown in Fig. 7.3.7.

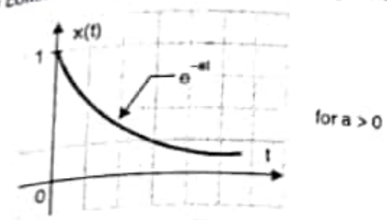


(H-566) Fig. 7.3.7

Note : This is causal signal and pole is at + a. So ROC is $\sigma > a$.

7.3.5 Positive Sided Decaying Exponential Signal :

This is also called as right handed decaying exponential signal. It is given by $x(t) = e^{-at}\, u(t)$. Here 'a' is some positive arbitrary constant. Such a function is shown in Fig. 7.3.8.



(H-567) Fig. 7.3.8

According to definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad \text{...(7.3.13)}$$

We have $x(t) = e^{-at}\, u(t)$. Multiplication by unit step u (t) indicates that the signal is only in the range $t = 0$ to $t = \infty$. Thus Equation (7.3.13) becomes,

$$X(s) = \int_{0}^{\infty} e^{-at} \cdot e^{-st}\, dt$$

$$\therefore \quad X(s) = \int_{0}^{\infty} e^{-(s+a)t} \cdot dt$$

$$\therefore \quad X(s) = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} = \frac{1}{-(s+a)}\left[\frac{1}{e^{+(s+a)t}}\right]_0^{\infty}$$

$$\therefore \quad X(s) = \frac{1}{-(s+a)}[0 - 1] = \frac{1}{s+a}$$

$$\therefore \quad e^{-at}\, u(t) \xleftrightarrow{\text{L.T.}} \frac{1}{s+a} \quad \text{The pole is at } s = -a$$

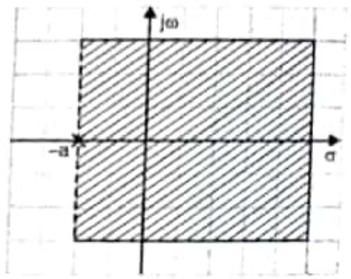
ROC :

The ROC is,

$Re\{s + a\} > 0$; that means $Re\{s\} > -a$

∴ ROC is $\sigma > -a$

This ROC is shown in Fig. 7.3.9.



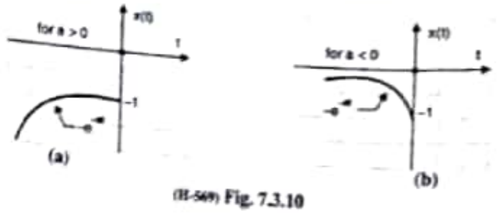
(H-568) Fig. 7.3.9

7.3.6 Negative Sided (Left Handed) Exponential Signal :

As the name indicates; such exponential signal is in the range $t = -\infty$ to $t = 0$. This signal is given by,

$$x(t) = -e^{-at}\, u(-t)$$

Here 'a' is some arbitrary constant. For $a > 0$ the signal is shown in Fig. 7.3.10(a) and for $a < 0$, the signal is shown in Fig. 7.3.10(b).



(H-569) Fig. 7.3.10

We have, $X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$ ...(7.3.14)

Since the given signal is in the range $t = -\infty$ to $t = 0$ we get,

$$X(s) = \int_{-\infty}^{0} -e^{-at} \cdot e^{-st} \cdot dt = -\int_{-\infty}^{0} e^{-(s+a)t}\, dt$$

$$\therefore \quad X(s) = -\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{-\infty}^{0} = \left[\frac{e^{-(s+a)t}}{s+a}\right]_{-\infty}^{0}$$

$$\therefore \quad X(s) = \frac{e^{0}}{s+a} - \frac{e^{(s+a)\infty}}{s+a}$$

$$\therefore \quad X(s) = \frac{1}{s+a} - \frac{e^{(s+a)\infty}}{s+a} \qquad \text{...(7.3.15)}$$

In Equation (7.3.15) if the power of exponent of second term is negative then we will get $\frac{1}{\infty}$ that means this term becomes zero. Thus we can write the laplace transform.

$$X(s) = \frac{1}{s+a}, \text{ if } s + a \text{ is negative} \qquad \text{...(7.3.16)}$$

$$\therefore \quad -e^{-at}\, u(-t) \xleftrightarrow{\text{L.T.}} \frac{1}{s+a}$$

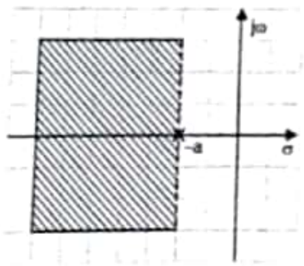
The pole is at $s = -a$

ROC :

To obtain this laplace transform, the condition is $R_e\{(s + a)\} < 0$. That means $R_e\{s\} < -a$.

∴ ROC is $\sigma < -a$

This ROC is shown in Fig. 7.3.11.



(H-570) Fig. 7.3.11

Note : The laplace transform of $e^{-at}u(t)$ and $-e^{-at}u(-t)$ is same; but the ROC is different.

7.3.7 Laplace Transform of Two Sided Exponential Signal :

Consider two sided exponential signal given by,

$$x(t) = e^{-at}u(t) - e^{+at}u(-t)$$

Here 'a' is some constant. If a > 1; then sketch of such signal is shown in Fig. 7.3.12.

We have, $$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \qquad ...(7.3.17)$$

$$\therefore X(s) = \int_{-\infty}^{\infty} [e^{-at}u(t) - e^{at}u(-t)] \cdot e^{-st}\, dt$$

$$\therefore X(s) = \int_{-\infty}^{\infty} e^{-at} \cdot e^{-st} \cdot u(t)\, dt - \int_{-\infty}^{\infty} e^{at} \cdot e^{-st}\, u(-t)\, dt \qquad ...(7.3.18)$$

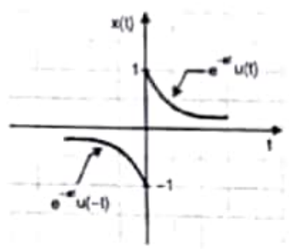
In the first integration, we have u (t). So limits of integration will be from 0 to ∞ and for second summation; because of u (– t) the limits of integration will be from – ∞ to 0.

$$\therefore X(s) = \int_{0}^{\infty} e^{-at} e^{-st}\, dt - \int_{-\infty}^{0} e^{at} \cdot e^{-st}\, dt$$

$$\therefore X(s) = \int_{0}^{\infty} e^{-(a+s)t}\, dt - \int_{-\infty}^{0} e^{(a-s)t}\, dt$$

$$\therefore X(s) = \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^{\infty} - \left[\frac{e^{(a-s)t}}{(a-s)}\right]_{-\infty}^{0}$$

$$\therefore X(s) = -\frac{1}{a+s}\left[\frac{1}{e^{(a+s)t}}\right]_0^{\infty} - \frac{1}{a-s} \cdot \left[e^{(a-s)t}\right]_{-\infty}^{0}$$



(H-571) **Fig. 7.3.12**

$$X(s) = -\frac{1}{a+s}[0-1] - \frac{1}{a-s}[1-0]$$

$$X(s) = \frac{1}{a+s} - \frac{1}{a-s} = \frac{a-s-a-s}{(a+s)(a-s)}$$

$$= \frac{-2s}{(a+s)(a-s)} = \frac{2s}{(-1)(a+s)(a-s)}$$

$$\therefore X(s) = \frac{2s}{(s+a)(s-a)} \qquad ...(7.3.19)$$

Here the poles are at – a and + a.

ROC : We have,

$$X(s) = \frac{1}{s+a} - \frac{1}{a-s} = \frac{1}{s+a} + \frac{1}{s-a} \qquad ...(7.3.20)$$

The first term is causal and it has a pole at – a. Thus ROC is

$$\therefore \sigma > -a$$

The second term is anticausal and it has pole at + a. Thus ROC is,

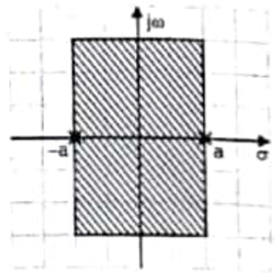
$$\therefore \sigma < a$$

Thus combined ROC is,

$$\sigma > -a \text{ and } \sigma < a$$

$$\therefore \text{ROC} : -a < \sigma < a$$

This ROC is as shown in Fig. 7.3.13.



(H-572) **Fig. 7.3.13**

7.3.8 Laplace Transform of Sine Wave :

Consider an input signal,

$$x(t) = A \sin \omega t\, u(t) \qquad ...(7.3.21)$$

From the definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$\therefore X(s) = \int_{-\infty}^{\infty} A \sin \omega t\, u(t)\, e^{-st}\, dt \qquad ...(7.3.22)$$

Because of the multiplication of u (t); the limits of integration will be from 0 to ∞.

$$\therefore X(s) = \int_{0}^{\infty} A \sin \omega t\, e^{-st}\, dt \qquad ...(7.3.23)$$

We have the trignometric identity,

$$\sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$\therefore X(s) = A \int_{0}^{\infty} \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right] e^{-st}\, dt$$

$$\therefore X(s) = \frac{A}{2j}\left[\int_{0}^{\infty} e^{j\omega t} \cdot e^{-st}\, dt - \int_{0}^{\infty} e^{-j\omega t} \cdot e^{-st}\, dt\right]$$

$$\therefore X(s) = \frac{A}{2j}\left\{L[e^{j\omega t}] - L[e^{-j\omega t}]\right\} \qquad ...(7.3.24)$$

We have the standard laplace transform pairs,

$$e^{at}u(t) \xleftrightarrow{L} \frac{1}{s-a}, \quad \text{ROC} : \sigma > a$$

$$\therefore L\{e^{j\omega t}\} \rightarrow \frac{1}{s-j\omega}, \quad \text{ROC} : \sigma > j\omega$$

Similarly we have,

$$e^{-at}u(t) \xleftrightarrow{L} \frac{1}{s+a}, \text{ROC} : \sigma > -a$$

$$\therefore L\{e^{-j\omega t}\} = \frac{1}{s+j\omega}, \text{ROC} : \sigma > -j\omega$$

Putting these values in Equation (7.3.24) we get,

$$X(s) = \frac{A}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right] = \frac{A}{2j} \cdot \frac{2j\omega}{s^2+\omega^2}$$

$$\therefore X(s) = \frac{A\omega}{s^2+\omega^2}$$

ROC :

For the first term ROC is, σ > jω that means σ > (0 + jω) for the ROC, only real part is considered. Thus ROC of first part is σ > 0.

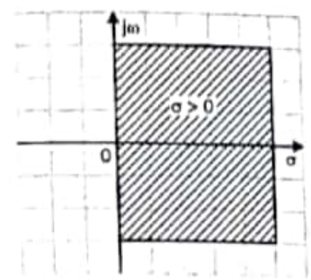
For the second term ROC is, σ > – jω that means σ > (0 – jω). In this case also, considering only real part, ROC is σ > 0.

Thus the combined ROC is σ > 0.

This is the standard laplace transform pair,

$$A \sin \omega t\, u(t) \xleftrightarrow{L.T.} \frac{A\omega}{s^2+\omega^2}, \text{ROC} : \sigma > 0$$

This ROC is shown in Fig. 7.3.14.



(H-573) **Fig. 7.3.14**

7.3.9 Laplace Transform of Cosine Wave :

Consider a signal,

$$x(t) = A \cos \omega t\, u(t) \qquad ...(7.3.25)$$

From the definition of laplace transform,

$$X(F) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(F) = \int_{-\infty}^{\infty} A \cos \omega t\, u(t)\, e^{-st}\, dt$$

$$X(F) = \int_{0}^{\infty} A \cos \omega t\, e^{-st}\, dt \qquad ...(7.3.26)$$

We have the trigonometric identity,

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$\therefore X(s) = \int_{0}^{\infty} A\left[\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right] e^{-st}\, dt$$

$$\therefore X(s) = \frac{A}{2}\left[\int_{0}^{\infty} e^{j\omega t} \cdot e^{-st}\, dt + \int_{0}^{\infty} e^{-j\omega t} \cdot e^{-st}\, dt\right] \qquad ...(7.3.27)$$

We have the standard Laplace transform pair

$$e^{at}u(t) \xleftrightarrow{L} \frac{1}{s-a}, \qquad \text{ROC} : \sigma > a$$

$$\therefore L\{e^{j\omega t}u(t)\} = \frac{1}{s-j\omega}, \qquad \text{ROC} : \sigma > j\omega$$

Similarly, $$e^{-at}u(t) \xleftrightarrow{L} \frac{1}{s+a} \qquad \text{ROC} : \sigma > -a$$

$$\therefore L\{e^{-j\omega t}u(t)\} = \frac{1}{s+j\omega} \qquad \text{ROC} : \sigma > -j\omega$$

Putting these values in Equation (7.3.27) we get,

$$X(s) = \frac{A}{2}\left[\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right]$$

$$\therefore X(s) = \frac{A}{2}\left[\frac{2s}{s^2+\omega^2}\right] = \frac{As}{s^2+\omega^2}$$

ROC :

For the first term ROC is σ > jω that means σ > 0 + jω. Considering only real part, ROC is σ > 0.

For the second term ROC is σ > – jω that means σ > 0 – jω. Considering only real part, ROC is σ > 0.

Thus the combined ROC is σ > 0. This ROC is same as shown in Fig. 7.3.14.

This is the standard laplace transform pair.

$$\therefore A \cos \omega t\, u(t) \xleftrightarrow{L.T.} \frac{As}{s^2+\omega^2}, \text{ROC} : \sigma > 0$$

7.3.10 Laplace Transform of Damped Sine Wave :

Consider a damped sine wave given by,

$$x(t) = e^{-at} \sin \omega t \cdot u(t) \qquad ...(7.3.28)$$

From the definition of laplace transform we have,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(s) = \int_{-\infty}^{\infty} [e^{-at} \sin \omega t \cdot u(t)]\, e^{-st}\, dt \qquad ...(7.3.29)$$

But we have,

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$X(s) = \int_{0}^{\infty}\left[e^{-at}\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)\right] e^{-st}\, dt$$

$$X(s) = \frac{1}{2j}\int_{0}^{\infty}[e^{-(a-j\omega)t} - e^{-(a+j\omega)t}]\, e^{-st}\, dt$$

$$X(s) = \frac{1}{2j}\left\{\int_{0}^{\infty} e^{-(a-j\omega)t} \cdot e^{-st}\, dt - \int_{0}^{\infty} e^{-(a+j\omega)t} \cdot e^{-st}\, dt\right\} \qquad ...(7.3.30)$$

We have the standard laplace transform pair.

$$e^{-at} \xleftrightarrow{L} \frac{1}{s+a}, \qquad \text{ROC} : \sigma > a$$

$$\therefore L\{e^{-(a-j\omega)t}\} = \frac{1}{s+(a-j\omega)}, \qquad \text{ROC} : \sigma > -(a-j\omega)$$

$$\text{and } L\{e^{-(a+j\omega)t}\} = \frac{1}{s+(a+j\omega)}, \qquad \text{ROC} : \sigma > -(a+j\omega)$$

Putting these values in Equation (7.3.29) we get,

$$X(s) = \frac{1}{2j}\left[\frac{1}{s+(a-j\omega)} - \frac{1}{s+(a+j\omega)}\right] = \frac{1}{2j}\cdot\frac{2j\omega}{(s+a)^2+\omega^2}$$

$$X(s) = \frac{\omega}{(s+a)^2+\omega^2}$$

ROC

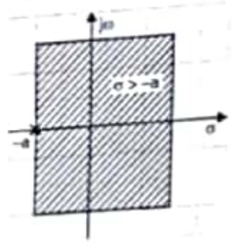
For the first term ROC is $\sigma > -(a - j\omega)$ that means $\sigma > -a + j\omega$. Thus it is $\sigma > -a$.

For the second term ROC is $\sigma > -(a + j\omega)$ that means $\sigma > -a - j\omega$. Thus it is $\sigma > -a$.

Thus the combined ROC is $\sigma > -a$

$$\therefore e^{-at}\sin\omega t \cdot u(t) \xleftrightarrow{L.T.} \frac{\omega}{(s+a)^2+\omega^2}, \quad ROC : \sigma > -a$$

This ROC is shown in Fig. 7.3.15.



(H-573) Fig. 7.3.15

7.3.11 Laplace Transform of Damped Cosine Wave :

Consider a damped cosine wave given by,

$$x(t) = e^{-at}\cos\omega t \cdot u(t) \quad ...(7.3.31)$$

From the definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$X(s) = \int_{-\infty}^{\infty} [e^{-at}\cos\omega t \cdot u(t)]\, e^{-st}\, dt \quad ...(7.3.32)$$

We have,

$$\cos\omega t = \frac{e^{j\omega t}+e^{-j\omega t}}{2}$$

$$X(s) = \int_0^{\infty} e^{-at}\left[\frac{e^{j\omega t}+e^{-j\omega t}}{2}\right] e^{-st}\, dt$$

$$X(s) = \frac{1}{2}\left\{\int_0^{\infty} e^{-(s-j\omega)t}\cdot e^{-st}\, dt + \int_0^{\infty} e^{-(s+j\omega)t}\, dt\right\} \quad ...(7.3.33)$$

We have the standard laplace transform pair,

$$e^{-at} \xleftrightarrow{L} \frac{1}{s+a} \qquad ROC : \sigma > -a$$

$$\therefore L\{e^{-(a-j\omega)t}\} = \frac{1}{s+(a-j\omega)} \qquad ROC : \sigma > -(a - j\omega)$$

$$\text{and } L\{e^{-(a+j\omega)t}\} = \frac{1}{s+(a+j\omega)}. \qquad ROC : \sigma > -(a + j\omega)$$

Putting these values in Equation (7.3.33) we get,

$$X(s) = \frac{1}{2}\cdot\left[\frac{1}{s+(a-j\omega)} + \frac{1}{s+(a+j\omega)}\right] = \frac{1}{2}\cdot\frac{2(s+a)}{(s+a)^2+\omega^2}$$

$$X(s) = \frac{(s+a)}{(s+a)^2+\omega^2}$$

ROC :

For the first term ROC is,

$\sigma > -(a - j\omega)$ that means $\sigma > -a + j\omega$

Thus ROC is $\sigma > -a$

For the second term ROC is,

$\sigma > -(a + j\omega)$ that means $\sigma > -a - j\omega$

Thus ROC is $\sigma > -a$

The combined ROC is $\sigma > -a$. This is same as shown in Fig. 7.3.15.

$$\therefore e^{-at}\cos\omega t \cdot u(t) \xleftrightarrow{L.T.} \frac{s+a}{(s+a)^2+\omega^2}, \quad ROC : \sigma > -a$$

7.3.12 x (t) = sinh at :

According to the definition of laplace transform,

$$L\{x(t)\} = X(s) \int_0^{\infty} x(t)\, e^{-st}\, dt$$

$$\therefore X(s) = \int_0^{\infty} \sinh at\, e^{-st}\, dt \quad ...(7.3.34)$$

Now we have standard identity,

$$\sinh at = \frac{e^{at}-e^{-at}}{2}$$

Putting this value in Equation (7.3.34) we get,

$$X(s) = \int_0^{\infty} \frac{e^{at}-e^{-at}}{2}\cdot e^{-st}\cdot dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{at}\cdot e^{-st}\, dt - \frac{1}{2}\int_0^{\infty} e^{-at}\cdot e^{-st}\, dt \quad ...(7.3.35)$$

Now we have standard laplace transformation formula,

$$L\{e^{at}\} = \int_0^{\infty} e^{at}\cdot e^{-st}\, dt = \frac{1}{s-a} \qquad \text{if } \sigma > a$$

$$\text{and } L\{e^{-at}\} = \int_0^{\infty} e^{-at}\cdot e^{-st}\, dt = \frac{1}{s+a} \qquad \text{if } \sigma > -a$$

Using these results, Equation (7.3.35) becomes,

$$F(s) = \frac{1}{2}\cdot\frac{1}{s-a} - \frac{1}{2}\cdot\frac{1}{s+a} \qquad \text{if } \sigma > a \text{ and } \sigma > -a$$

$$F(s) = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \qquad \text{if } \sigma > a \text{ and } \sigma > -a$$

Note : Here the conditions are $\sigma > a$ and $\sigma > -a$. Thus the combined condition can be written as $\sigma > |a|$.

$$\therefore F(s) = \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \qquad \text{if } \sigma > |a|$$

$$\therefore F(s) = \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right] \qquad \text{if } \sigma > |a|$$

$$\therefore F(s) = \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] \qquad \text{if } \sigma > |a|$$

$$\therefore F(s) = \frac{a}{s^2-a^2} \qquad \text{if } \sigma > |a|$$

$$\therefore L\{\sinh at\} = \frac{a}{s^2-a^2} \qquad \text{if } s > |a|$$

7.3.13 x (t) = cosh at :

According to the definition of laplace transform,

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} \cosh at \cdot e^{-st}\cdot dt \quad ...(7.3.36)$$

We have the standard identity,

$$\cosh at = \frac{e^{at}+e^{-at}}{2}$$

$$\therefore X(s) = \int_0^{\infty} \frac{e^{at}+e^{-at}}{2}\cdot e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{at}\cdot e^{-st}\, dt + \frac{1}{2}\int_0^{\infty} e^{-at}\cdot e^{-st}\, dt \quad ...(7.3.37)$$

Now we have standard laplace transformation formulae,

$$L\{e^{at}\} = \int_0^{\infty} e^{at}\cdot e^{-st}\, dt = \frac{1}{s-a} \qquad \text{if } \sigma > a$$

$$\text{and } L\{e^{-at}\} = \int_0^{\infty} e^{-at}\, dt = \frac{1}{s+a} \qquad \text{if } \sigma > -a$$

Using these results in Equation (7.3.37) we get,

$$X(s) = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\cdot\frac{1}{s+a} \qquad \text{if } \sigma > a \text{ and } \sigma > -a$$

The combined condition can be written as $s > |a|$

$$\therefore X(s) = \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] \qquad \text{if } \sigma > |a|$$

$$\therefore X(s) = \frac{1}{2}\left[\frac{s-a+s+a}{(s-a)(s+a)}\right] \qquad \text{if } \sigma > |a|$$

$$\therefore X(s) = \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right] \qquad \text{if } \sigma > |a|$$

$$\therefore X(s) = \frac{s}{s^2-a^2} \qquad \text{if } \sigma > |a|$$

$$\therefore L\{\cosh at\} = \frac{s}{s^2-a^2} \qquad \text{if } \sigma > |a|$$

7.3.14 x (t) = t^n :

According to the definition of laplace transform,

$$L\{x(t)\} = X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

$$\therefore X(s) = \int_0^{\infty} t^n e^{-st}\, dt \quad ...(7.3.38)$$

Put, $st = u$ that means $t = \frac{u}{s}$

Differentiating both sides we get,

$$s\, dt = du$$

$$\therefore dt = \frac{du}{s} \quad ...(7.3.39)$$

The limits will change as follows,

When $t = 0 \Rightarrow s \times 0 = u \quad \therefore u = 0$

When $t = \infty \Rightarrow s \times \infty = u \quad \therefore u = \infty$

Thus Equation (7.3.38) becomes,

$$X(s) = \int_0^{\infty} \left(\frac{u}{s}\right)^n \cdot e^{-u}\cdot\frac{du}{s} = \int_0^{\infty} \frac{(u)^n}{(s)^n}\cdot e^{-u}\cdot du\cdot\frac{1}{s}$$

$$= \int_0^{\infty} \frac{1}{(s)^{n+1}}\cdot u^n\cdot e^{-u}\, du$$

$$\therefore X(s) = \frac{1}{(s)^{n+1}}\int_0^{\infty} e^{-u}\cdot u^n\, du \quad ...(7.3.40)$$

We have the standard formula,

$$\Gamma n = \int_0^{\infty} e^{-u}\cdot u^{n-1}\, du \qquad \text{if } n > 0$$

$$\therefore \Gamma(n+1) = \int_0^{\infty} e^{-u}\cdot u^n\cdot du \qquad \text{if } n > 0$$

Using this result Equation (7.3.40) can be written as,

$$X(s) = \frac{1}{(s)^{n+1}}\cdot\Gamma(n+1) \qquad \text{if } \sigma > 0 \text{ and } n > 0$$

Now if n is positive integer then $\Gamma(n+1) = n!$

$$\therefore X(s) = \frac{1}{(s)^{n+1}}\cdot n! \text{ if } \sigma > 0 \text{ and } n > 0$$

$$\therefore L\{t^n\} = \frac{1}{(s)^{n+1}}\cdot n! \qquad \text{if } \sigma > 0 \quad ...(7.3.41)$$

Note : We will obtain the results for different values of n as follows :

1. If $n = 0 \Rightarrow L\{t^0\} = L\{1\} = \frac{1}{s}\cdot 0!$

   But $0! = 1$

   $\therefore L\{1\} = \frac{1}{s}$

2. If $n = 1 \Rightarrow L\{t\} = \frac{1}{(s)^{1+1}}\cdot 1! = \frac{1}{s^2}$

3. If $n = 2 \Rightarrow L\{t^2\} = \frac{1}{(s)^{2+1}}\cdot 2! = \frac{2}{s^3}$

7.4 Standard Laplace Transform Pairs :

Table 7.4.1 shows the standard laplace transform pairs.

Table 7.4.1

Sr. No.	Time domain signal F (t)	Laplace transform X (s)	ROC
1.	$\delta(t)$	1	Entire s-plane
2.	$u(t)$	$\frac{1}{s}$	$\sigma > 0$
3.	$r(t)$	$\frac{1}{s^2}$	$\sigma > 0$
4.	$e^{at} u(t)$	$\frac{1}{s-a}$	$\sigma > a$
5.	$e^{-at} u(t)$	$\frac{1}{s+a}$	$\sigma > -a$
6.	$-e^{-at} u(-t)$	$\frac{1}{s+a}$	$\sigma < -a$
7.	$e^{-at} u(t) - e^{at} u(-t)$	$\frac{2s}{(s+a)(s-a)}$	$-a < \sigma < +a$
8.	$A \sin \omega t \, u(t)$	$\frac{A\omega}{s^2+\omega^2}$	$\sigma > 0$
9.	$A \cos \omega t \, u(t)$	$\frac{As}{s^2+\omega^2}$	$\sigma > 0$
10.	$e^{-at} \sin \omega t \cdot u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\sigma > -a$
11.	$e^{-at} \cos \omega t \cdot u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\sigma > -a$
12.	t^n	$\frac{n!}{s^{n+1}}$	$\sigma > a$

7.5 Properties of Laplace Transform :

Properties of laplace transform are helpful to obtain the laplace transform of composite functions. In this section we will discuss some important properties of laplace transform.

7.5.1 Linearity :

Statement : Consider two time domain signals $x_1(t)$ and $x_2(t)$.

If $x_1(t) \xleftrightarrow{LT} X_1(s)$ and $x_2(t) \xleftrightarrow{LT} X_2(s)$

Then $[a_1 x_1(t) + a_2 x_2(t)] \xleftrightarrow{LT} a_1 X_1(s) + a_2 X_2(s)$

Where a_1 and a_2 are constants.

Proof : According to the definition of Laplace transform.

$$X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

Here $x(t) = a_1 x_1(t) + a_2 x_2(t)$

$$\therefore L\{a_1 x_1(t) + a_2 x_2(t)\} = \int_{-\infty}^{\infty} [a_1 x_1(t) + a_2 x_2(t)] e^{-st} dt$$

$$= a_1 \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + a_2 \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$$

Comparing each term of R.H.S. with the definition of laplace transform we get,

$$L\{a_1 x_1(t) + a_2 x_2(t)\} = a_1 X_1(s) + a_2 X_2(s)$$

ROC :

The resultant ROC is an intersection of ROC of $X_1(s)$ and $X_2(s)$

Use of linearity property :

We can express the given signal in the form of two or more elementary signals. Then by finding their individual laplace transforms and by adding them; we can simplify the solution.

> **Meaning of linearity property :** It states that laplace transform of linear combination of signals is same as linear combination of their individual laplace transforms. While the resultant ROC is intersection or overlap of individual ROC.

7.5.1.1 Solved Examples on Linearity Property :

Ex. 7.5.1 : Calculate laplace transform of :

$$x(t) = e^{-2t} u(t) - e^{2t} u(-t)$$

and plot the ROC.

Soln. :

Let $x(t) = x_1(t) + x_2(t)$ ...(1)

The given equation is,

$$x(t) = e^{-2t} u(t) - e^{2t} u(-t) \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$x_1(t) = e^{-2t} u(t) \text{ and } x_2(t) = -e^{2t} u(-t)$$

Then using linearity property we can write,

$$X(s) = X_1(s) + X_2(s) \quad ...(3)$$

Now recall standard laplace transform pair,

$$e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad \text{ROC } \sigma > -a$$

$$\therefore \quad e^{-2t} u(t) \xleftrightarrow{LT} \frac{1}{s+2} \qquad \text{ROC } \sigma > -2 \quad ...(4)$$

Similarly we have standard laplace transform pair,

$$-e^{-at} u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad \text{ROC } \sigma < -a$$

$$\therefore \quad -e^{-2t} u(-t) \xleftrightarrow{LT} \frac{1}{s-2} \qquad \text{ROC } \sigma < 2 \quad ...(5)$$

> **Note :** Here the function is $-e^{2t} u(-t)$ while the standard equation is for $-e^{-at} u(-t)$. Thus in this case $a = -2$. That means the given equation can be written as $-e^{-(-2)t} u(-t)$.

Putting Equations (4) and (5) in Equation (3) we get,

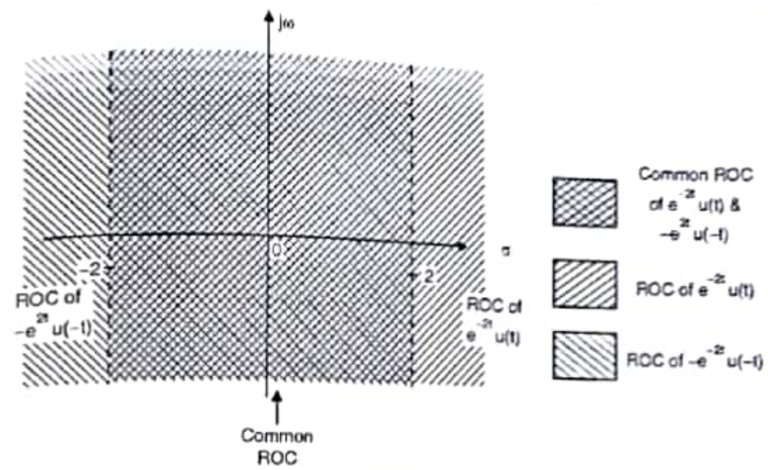
$$X(s) = \frac{1}{s+2} + \frac{1}{s-2}$$

ROC : ROC is intersection of two ROCs. That means.

ROC : $\sigma > -2$ and $\sigma < 2$ $\therefore -2 < \sigma < 2$

$$\therefore \quad X(s) = \frac{1}{s+2} + \frac{1}{s-2} \qquad \text{ROC} : -2 < \sigma < 2$$

This ROC is shown in Fig. P. 7.5.1.



(H-575)Fig. P. 7.5.1

Ex. 7.5.2 : Find the Laplace Transform of $f(t) = e^{-4t} u(-t) + e^{-6t} u(t)$. Does the Laplace Transform exist. Show the ROC.

May 07-8 Marks

Soln. :

$$f(t) = e^{-4t} u(-t) + e^{-6t} u(t)$$

Assuming, $x(t) = x_1(t) + x_2(t)$ ...(1)

The given equation is,

$$x(t) = e^{-4t} u(-t) + e^{-6t} u(t) \quad ...(2)$$

Comparing Equations (1) and (2),

$$x_1(t) = e^{-4t} u(-t)$$

$$x_2(t) = e^{-6t} u(t)$$

Then using linearity property,

$$X(s) = x_1(s) + x_2(s)$$

$$e^{-6t} u(t) \xleftrightarrow{LT} \frac{1}{s+6} \qquad \text{ROC } \sigma > -6$$

$$e^{-4t} u(-t) \xleftrightarrow{LT} \frac{1}{s-4} \qquad \text{ROC } \sigma < 4$$

$$\therefore X(s) = \frac{1}{s-4} + \frac{1}{s+6}$$

ROC : ROC is intersection of two ROCs that means

ROC : $\sigma > -6$ and $\sigma < 4$

$$-6 < \sigma < 4$$

$$\therefore X(s) = \frac{1}{s-4} + \frac{1}{s+6} \qquad \text{ROC} : -6 < \sigma < 4$$

Ex. 7.5.3 : Find the L.T of $x(t) = 5e^{4t} + 6t^3 - 3\sin 5t + 2\cos 2t$ using linearity property.

Soln. :

$$x(t) = 5e^{4t} + 6t^3 - 3\sin 5t + 2\cos 2t$$

Taking LT on both sides and using linearity property,

$$L\{x(t)\} = 5L\{e^{4t}\} + 6L\{t^3\} - 3L\{\sin 5t\} + 2L\{\cos 2t\}$$

We know, $e^{at} \xleftrightarrow{LT} \frac{1}{s-a}$

Remember one important laplace transform pair

$$t^n \xleftrightarrow{LT} \frac{n!}{s^{n+1}}$$

Similarly we know,

$$\sin at \xleftrightarrow{LT} \frac{a}{a^2+s^2}$$

$$\cos at \xleftrightarrow{LT} \frac{s}{a^2+s^2}$$

$$\therefore L\{x(t)\} = 5\left(\frac{1}{s-4}\right) + 6\left[\frac{3!}{s^{3+1}}\right] - 3\left[\frac{5!}{5^2+s^2}\right] + 2\left[\frac{s}{2^2+s^2}\right]$$

$$\therefore x(s) = \left(\frac{5}{s-4}\right) + \left(\frac{36}{s^4}\right) - \left(\frac{15}{s^2+25}\right) + \left(\frac{2s}{s^2+4}\right)$$

Ex. 7.5.4 : Find the laplace transform of :

$$x(t) = e^{-a|t|}$$

Soln. : Here 'a' is some constant. First we will decide the nature of x (t).

Let $a = \frac{1}{2}$. We will plot the given function x (t) for some values of t.

For, $t = 0 \Rightarrow x(0) = e^{-\frac{1}{2}|0|} = e^0 = 1$

For, $t = 1 \Rightarrow x(1) = e^{-\frac{1}{2}|1|} = e^{-1/2} = 0.606$

For, $t = 2 \Rightarrow x(2) = e^{-\frac{1}{2}|2|} = e^{-1} = 0.37$

For, $t = 3 \Rightarrow x(3) = e^{-\frac{1}{2}|3|} = 0.223$

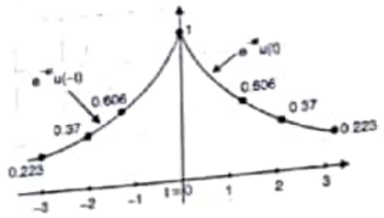
Now we will calculate x (t) for some negative values of 't'

For, $t = -1 \Rightarrow x(-1) = e^{-\frac{1}{2}|-1|} = e^{-1/2} = 0.606$

For $t = -2 \Rightarrow x(-2) = e^{-\frac{1}{2}|-2|} = e^{-1} = 0.37$

For $t = -3 \Rightarrow x(-3) = e^{-\frac{1}{2}|-3|} = e^{-3/2} = 0.223$

The plot of x (t) is shown in Fig. P. 7.5.4.



(H-576)Fig. P. 7.5.4

Thus the given function can be expressed as. ...(1)

$$x(t) = e^{-t}u(t) + e^{t}u(-t) \quad ...(2)$$

Let $x(t) = x_1(t) + x_2(t)$

Using linearity property we can write. ...(3)

$$X(s) = X_1(s) + X_2(s)$$

Comparing Equations (1) and (2) we get.

$$x_1(t) = e^{-t}u(t) \quad \text{and} \quad x_2(t) = e^{t}u(-t)$$

Consider $x_1(t) = e^{-t}u(t)$

We have standard laplace transform pair

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad ROC : \sigma > -a$$

$$\therefore X_1(s) = \frac{1}{s+a} \qquad ROC : \sigma > -a$$

Now consider $X_2(s) = e^{at}u(-t)$

We have standard laplace transform pair

$$-e^{-at}u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad ROC : \sigma < -a$$

$$\therefore e^{-at}u(-t) \xleftrightarrow{LT} -\frac{1}{s+a} \qquad ROC : \sigma < -a$$

$$\therefore e^{at}u(-t) \xleftrightarrow{LT} -\frac{1}{s-a} \qquad ROC : \sigma < +a$$

$$\therefore X_2(s) = -\frac{1}{s-a} \qquad ROC : \sigma < +a$$

Putting these values in Equation (3) we get,

$$X(s) = \frac{1}{s+a} - \frac{1}{s-a} = \frac{-2a}{s^2 - a^2}$$

ROC : For $X_1(s)$ ROC is $\sigma > -a$ and for $X_2(s)$ ROC is $\sigma < +a$.

Thus combined ROC is,

$$\sigma > -a \text{ and } \sigma < a$$

$\therefore$ ROC is $-a < \sigma < +a$

$$\therefore X(s) = \frac{-2a}{s^2 - a^2} \qquad ROC : -a < \sigma < +a$$

This ROC is same as shown in Fig. P. 7.5.1.

7.5.2 Time Shifting (Shifting in Time Domain) :

If x (t) is input signal whose ROC is 'R' then

$$L\{x(t - t_0)\} = e^{-st_0} \cdot X(s) \qquad ROC = R$$

Proof : From the definition of L.T. we have

$$L\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \cdot e^{-st} \cdot dt$$

$$\therefore L\{x(t - t_0)\} = \int_{-\infty}^{+\infty} x(t - t_0) \cdot e^{-st} \cdot dt$$

Let $k = t - t_0$, $\therefore dk = dt$ and $t = k + t_0$

$$\therefore L\{x(t - t_0)\} = \int_{-\infty}^{+\infty} x(k) \cdot e^{-s(k + t_0)} \cdot dk$$

$$= e^{-st_0} \int_{-\infty}^{+\infty} x(k) \cdot e^{-s(k)} \cdot dk$$

$$\therefore L\{x(t - t_0)\} = e^{-st_0} \cdot X(s) \qquad ROC = R \quad ...(7.5.1)$$

Meaning of time shifting property : Here $x(t - t_0)$ indicates that the sequence x (t) is shifted in time domain (delayed). Thus shifting the signal in time domain by 't_0' corresponds to multiplication by e^{-st_0} in the frequency domain. That means it introduces a phase shift in the frequency domain.

7.5.3 Shifting in s - Domain :

If $x(t) \xleftrightarrow{LT} X(s)$; $\quad ROC = R$

then, $e^{s_0 t}x(t) \xleftrightarrow{LT} X(s - s_0) \qquad ROC : R + R_e(s_0) \quad ...(7.5.2)$

Proof

From definition of laplace transform,

$$L[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\therefore L[e^{s_0 t}x(t)] = \int_{-\infty}^{\infty} x(t) e^{-st} \cdot e^{s_0 t} dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-(s - s_0)t} dt = X(s - s_0) \qquad ROC = R + R_e(s_0)$$

Meaning of shifting in 's' domain property : Here $X(s - s_0)$ indicates the frequency shift by s_0. Thus frequency shift by s_0 is equivalent to multiplying time domain signal, x (t) by $e^{s_0 t}$. The ROC is also shifted by $R_e(s_0)$.

7.5.3.1 Solved Examples on Shifting Property :

Ex. 7.5.5 : Using shifting property find laplace transform of $\sin \omega t \cdot u(t)$ and $\cos \omega t \cdot u(t)$.

Soln. : Using Euler's identity we have,

$$\sin \omega t = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} \quad ...(1)$$

$$\text{and} \quad \cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \quad ...(2)$$

Now consider unit step signal u (t). We have standard laplace transform pair,

$$u(t) \xleftrightarrow{LT} \frac{1}{s} \qquad ROC : \sigma > 0 \quad ...(3)$$

According to the time shifting property,

$$\therefore e^{s_0 t}x(t) \xleftrightarrow{LT} X(s - s_0)$$

$$\therefore e^{j\omega t}u(t) \xleftrightarrow{LT} \frac{1}{s - j\omega} \qquad ROC = \sigma > 0 \quad ...(4)$$

Note : Here $s_0 = j\omega$ and we know that using time shifting property, ROC is unaffected.

Similarly, $e^{-j\omega t} \cdot u(t) \xleftrightarrow{LT} \frac{1}{s + j\omega}$ ROC : $\sigma > 0$ ...(5)

Now from Equation (1) we can write,

$$L\{\sin \omega t \cdot u(t)\} = \frac{1}{2j}\left[L\{e^{j\omega t} \cdot u(t)\} - L\{e^{-j\omega t} \cdot u(t)\}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right] = \frac{1}{2j}\left[\frac{s + j\omega - s + j\omega}{s^2 - j^2\omega^2}\right]$$

$$\therefore L\{\sin \omega t \cdot u(t)\} = \frac{\omega}{s^2 + \omega^2} \text{ ROC} : \sigma > 0$$

This is the standard laplace transform pair. From Equation (2) we can write,

$$L\{\cos \omega t \cdot u(t)\} = \frac{1}{2}\left[L\{e^{j\omega t} \cdot u(t)\} + L\{e^{-j\omega t} \cdot u(t)\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s - j\omega} + \frac{1}{s + j\omega}\right] = \frac{1}{2}\left[\frac{s + j\omega + s - j\omega}{s^2 - j^2\omega^2}\right]$$

$$\therefore L\{\cos \omega t \cdot u(t)\} = \frac{s}{s^2 + \omega^2} \quad ROC : \sigma > 0$$

Ex. 7.5.6 : Find laplace transform of :

$$x(t) = \cos\left(t - \frac{2\pi}{7}\right)$$

Soln. :

$$x(t) = \cos\left(t - \frac{2\pi}{7}\right)$$

Taking L.T on both sides.

$$L\{x(t)\} = L\left\{\cos\left(t - \frac{2\pi}{7}\right)\right\}$$

We know that,

$$\cos at\, u(t) \xleftrightarrow{LT} \frac{s}{a^2 + s^2}$$

$$\cos t \xleftrightarrow{LT} \frac{s}{s^2 + 1}$$

Now use **time shifting property.**

We get,

$$L\{x(t - t_0)\} = e^{-st_0} \cdot X(s)$$

Here $\quad t_0 = \frac{2\pi}{7}$

$$\therefore X(s) = e^{-\frac{2\pi}{7}s} L\{\cos t\}$$

$$\therefore X(s) = e^{-\frac{2\pi}{7}s}\left[\frac{s}{s^2 + 1}\right] = \frac{s}{s^2 + 1}e^{-2\pi s/7}$$

Ex. 7.5.7 : Find Laplace transform of :

$$x(t) = (t - 3)^2$$

Soln. : Here $\quad x(t) = (t - 3)^2 \quad$ for $t > 3$

Taking L.T on both sides.

$$X(s) = L\{(t - 3)^2\}$$

We know $\quad t^n \xleftrightarrow{LT} \frac{n!}{s^{n+1}}$

$$\therefore t^2 \xleftrightarrow{LT} \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

Now using **time shifting property.**

$$L\{x(t - t_0)\} = e^{-st_0} \cdot X(s)$$

Here $\quad t_0 = 3$

$$\therefore X(s) = e^{-3s} \times L\{t^2\}$$

$$\therefore X(s) = e^{-3s} \times \frac{2}{s^3}$$

$$\therefore X(s) = \frac{2e^{-3s}}{s^3}$$

7.5.4 Time Scaling :

If $\quad x(t) \xleftrightarrow{L} X(s)$; ROC = R

Then,

$$L\{x(at)\} = \frac{1}{|a|}X\left(\frac{s}{a}\right) \quad \text{with ROC } R_1 = \frac{R}{a}$$

Proof : From the definition of L.T., we have

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t) e^{-st} \cdot dt$$

$$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} \cdot dt \quad ...(7.5.3)$$

Put at = P $\therefore t = \frac{P}{a}$ and $dt = \frac{1}{a}dP$

$$\therefore L\{x(at)\} = \int_{-\infty}^{\infty} X(P) e^{-\left(\frac{s}{a}\right)P} \cdot \frac{dP}{a}$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} X(P) e^{-\left(\frac{s}{a}\right)P} \cdot dP$$

Comparing R.H.S. with the definition of laplace transform,

$$L\{x(at)\} = \frac{1}{a}X\left(\frac{s}{a}\right)$$

Similarly for negative values of 'a' we get,

$$L\{x(-at)\} = \frac{1}{a}X\left(\frac{s}{-a}\right) \quad ROC : \frac{R}{-a}$$

In general we can write,

$$L\{x(at)\} = \frac{1}{|a|}X\left(\frac{s}{a}\right) \text{ with ROC } R_1 = \frac{R}{a}$$

Meaning of time scaling property : In time domain we have the term x (at). It indicates multiplication of time with some constant 'a'. Thus it is expansion of time axis. While in frequency domain we have the term $X\left(\frac{s}{a}\right)$. It indicates compression in frequency domain. Thus expanding time axis is equivalent to compression in frequency domain. The ROC is also expanded or compressed depending on value of 'a'.

7.5.5 Time Differentiation (Differentiation in Time Domain) :

Statement :

If $x(t) \xleftrightarrow{LT} X(s)$ ROC : R

then $\frac{d}{dt}x(t) \xleftrightarrow{LT} s\,X(s)$, ROC : R

Proof : For the proof of this statement, we will use the definition of inverse laplace. It is,

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\,e^{st}\,ds \qquad ...(7.5.4)$$

Differentiating both sides with respect to 't' we get.

$$\frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\cdot\frac{d}{dt}(e^{st})\,ds$$

$$= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)\cdot s\,e^{st}\,ds$$

$$\therefore \frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} [s\,X(s)]\,e^{st}\,ds \qquad ...(7.5.5)$$

Comparing Equations (7.5.4) and (7.5.5), we can conclude that L.H.S. of Equation (7.5.5) is laplace inverse of s X (s).

$$\therefore \frac{d}{dt}x(t) \xleftrightarrow{LT} s\,X(s)$$

ROC

ROC of $\frac{d}{dt}x(t)$ is contained in the ROC of x (t). In general we can write this result as.

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{LT} s^n X(s);\ \text{ROC : containing R}$$

> **Meaning of time differentiation property :** Differentiating the signal in time domain is equivalent to multiplying its Laplace transform by 's' in the frequency domain.

7.5.6 Differentiation in s-Domain :

Statement :

If $x(t) \xleftrightarrow{LT} X(s)$, ROC : R

then $-t\,x(t) \xleftrightarrow{LT} \frac{d}{ds}X(s)$, ROC : R

Proof : According to the definition of laplace transform,

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt \qquad ...(7.5.6)$$

Differentiating both sides with respect to 's' we get,

$$\frac{d}{ds}X(s) = \int_{-\infty}^{\infty} x(t)\cdot\frac{d}{ds}(e^{-st})\,dt$$

$$\therefore \frac{d}{ds}X(s) = \int_{-\infty}^{\infty} [-t\,x(t)]\,e^{-st}\,dt \qquad ...(7.5.7)$$

Comparing Equations (7.5.6) and (7.5.7), we can conclude that the laplace transform of $[-t\,x(t)]$ is equal to $\frac{d}{dt}X(s)$.

That means $-t\,x(t) \xleftrightarrow{LT} \frac{d}{ds}X(s)$

ROC : In this case, the ROC is unchanged. In general we can write this result as,

$$(-t)^n x(t) \xleftrightarrow{LT} \frac{d^n}{ds^n}X(s),\ \text{ROC : R}$$

> **Meaning of differentiation in s-domain property :** Differentiating the sequence in 's' domain is equivalent to multiplying the sequence in time domain by (– t).

7.5.6.1 Solved Examples on Scaling and Differentiation Property :

Ex. 7.5.8 : Given that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}(1/s)$

Find the L.T of function $\left[\frac{\sin at}{t}\right]$

Soln. : Given that,

$$L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\left(\frac{1}{s}\right)$$

Referring to time scaling property.

$$x(at) \xleftrightarrow{LT} \frac{1}{|a|}X\left(\frac{s}{a}\right)$$

$$\therefore L\left\{\frac{\sin at}{at}\right\} = \frac{1}{|a|}\tan^{-1}\left(\frac{1}{s/a}\right)$$

$$\therefore \frac{1}{a}\cdot L\left\{\frac{\sin at}{t}\right\} = \frac{1}{|a|}\tan^{-1}\left(\frac{a}{s}\right)$$

$$\therefore L\left\{\frac{\sin at}{at}\right\} = \tan^{-1}\left(\frac{a}{s}\right)$$

Ex. 7.5.9 : If $x(t) = \delta(t) - \delta(t-2)$ then find laplace transform of signal $\frac{d^2x(t)}{dt^2}$.

Soln. : Given equation is,

$$x(t) = \delta(t) - \delta(t-2) \qquad ...(1)$$

Taking laplace transform of both sides,

$$L\{x(t)\} = X(s) = L\{\delta(t)\} - L\{\delta(t-2)\} \qquad ...(2)$$

We have, $L\{\delta(t)\} = 1$

According to the time shifting property,

$$L\{x(t-t_0)\} = e^{-st_0}\cdot X(s)$$

$$\therefore L(\delta(t-2)) = e^{-2s}\cdot 1 = e^{-2s}$$

Putting these values in Equation (2) we get,

$$X(s) = 1 - e^{-2s} \qquad ...(3)$$

Now from the time differentiation property,

$$\frac{d^n}{dt^n}x(t) \xleftrightarrow{LT} s^n\cdot X(s)$$

$$\therefore \frac{d^2x(t)}{dt^2} \xleftrightarrow{LT} s^2\cdot[1 - e^{-2s}]$$

ROC : ROC is same as ROC of original signal δ (n). That means ROC is entire s-plane.

7.5.7 Convolution Property :

Statement :

If $x(t) \xleftrightarrow{LT} X(s)$, ROC : R_1

and $h(t) \xleftrightarrow{LT} H(s)$ ROC : R_2

then $x_1(t) * h(t) \xleftrightarrow{LT} X(s)\cdot H(s)$ ROC : Intersection of R_1 and R_2

Soln. :

Proof : From the definition of convolution,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

Taking Laplace transform on both sides.

$$Y(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau\right]\cdot e^{-st}\,dt$$

Changing the order of integration and noting the fact that x (τ) does not depend on 't'.

$$Y(s) = \int_{-\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty}[h(t-\tau)\,e^{-st}\,dt]\,d\tau$$

The second integration indicates laplace transform of h (t – τ)

$$\therefore Y(s) = \int_{-\infty}^{\infty} x(\tau)\,L\{h(t-\tau)\}\,d\tau$$

If $h(t) \xleftrightarrow{L} H(s)$

Then from shifting property

$$h(t-t_0) \xleftrightarrow{L} e^{-st_0}H(s) \quad \therefore\ h(t-\tau) \leftrightarrow e^{-s\tau}\cdot H(s)$$

$$\therefore Y(s) = \int_{-\infty}^{\infty} x(\tau)\,H(s)\,e^{-s\tau}\,d\tau = H(s)\int_{-\infty}^{\infty} x(\tau)\,e^{-s\tau}\,d\tau$$

$$Y(s) = H(s)\,X(s) \qquad \text{ROC atleast } R_1 \cap R_2$$

Hence proved.

> **Meaning of convolution property :** The convolution of two signals in time domain becomes a simple multiplication process in the transformed domain. We know that the time domain response of LTI system means the convolution of input signal and its impulse response. Thus to find the response of LTI system it is always better to go in laplace domain than the time domain.

7.5.8 Time Integration (Integration in Time Domain) :

Statement :

If $x(t) \xleftrightarrow{LT} X(s)$, ROC : R

then $\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{LT} \frac{X(s)}{s}$, ROC : R ∩ [σ > 0]

Here symbol '∩' indicates intersection.

Proof : Consider signal x (t) and unit step u (t). According to the definition of convolution.

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau \qquad ...(7.5.8)$$

The R.H.S. indicates multiplication of signal x (τ) by u (t – τ). Here u (t – τ) indicates delayed unit step. It is defined as.

$$u(t-\tau) = 1 \quad \text{for } t \ge \tau \text{ that means } \tau \le t$$
$$= 0 \quad \text{otherwise} \qquad ...(7.5.9)$$

The value of u (t – τ) is 1 for the range τ ≤ t: that means for the range – ∞ to t.

$$\therefore x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau \qquad ...(7.5.10)$$

Taking laplace transform of both sides.

$$L\{x(t) * u(t)\} = L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

Using convolution property we can write the L.H.S. term as.

$$L\{x(t)\}\cdot L\{u(t)\} = L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$\therefore X(s)\cdot\frac{1}{s} = L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right]$$

$$\therefore L\left[\int_{-\infty}^{t} x(\tau)\,d\tau\right] = \frac{X(s)}{s} \quad \text{ROC : R} \cap [\sigma > 0]$$

> **Meaning of time integration property :** Integral of time domain signal corresponds to division by 's' in the laplace domain.

7.5.9 Integration in s-domain (Division by t) :

Statement :

If $x(t) \xleftrightarrow{LT} X(s)$, ROC : R

then $\frac{x(t)}{t} \xleftrightarrow{LT} \int_{s}^{\infty} X(s)\,ds$, ROC : R

Proof : According to the definition of laplace transform

$$X(s) = \int_{-\infty}^{\infty} x(t)\,e^{-st}\,dt \qquad ...(7.5.11)$$

Integrating both sides from s to ∞ we get.

$$\int_s^{\infty} X(s)\cdot ds = \int_s^{\infty}\left[\int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt\right] ds$$

Changing the order of integration of RHS term we get.

$$\int_s^{\infty} X(s)\cdot ds = \int_{-\infty}^{\infty} x(t)\left[\int_s^{\infty} e^{-st}\, ds\right] dt$$

$$\therefore \int_s^{\infty} X(s)\, ds = \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt = \int_{-\infty}^{\infty} x(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right] dt$$

But $e^{-\infty} = \frac{1}{e^{\infty}} = 0$

$$\therefore \int_s^{\infty} X(s)\, ds = \int_{-\infty}^{\infty}\left[\frac{x(t)}{t}\right] e^{-st}\cdot dt \qquad ...(7.5.12)$$

Comparing R.H.S. with the definition of laplace transform.

$$\int_s^{\infty} X(s)\, ds = L\left\{\frac{x(t)}{t}\right\}$$

$$\therefore \frac{x(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} X(s)\, ds,\ \text{ROC : R}$$

Meaning of integration in s-domain property : If the laplace transform of the function x (t) exists then the integral of corresponding transform with respect to 's' in the complex frequency domain will correspond to division by 't' in the time domain.

7.5.10 Laplace Transform of Periodic Signal :

Statement :

If x (t) is the periodic function with fundamental period 'T_0' that means $x(t) = x(t + T_0)$ then

$$L\{x(t)\} = \frac{1}{1 - e^{-sT_0}} X(s)$$

Proof : From the definition for unilateral Laplace transform

We have.

$$L\{x(t)\} = \int_s^{\infty} x(t)\, e^{-st}\, dt$$

$$= \underbrace{\int_0^{T_0} x(t)\, e^{-st}\, dt}_{\text{Term 1}} + \underbrace{\int_{T_0}^{2T_0} x(t)\, e^{-st}\, dt}_{\text{Term 2}} + \underbrace{\int_{2T_0}^{3T_0} x(t)\, e^{-st}\, dt}_{\text{Term 3}} + \ldots \quad ...(7.5.13)$$

We know signal x (t) is periodic

$\therefore\ x(t) = x(t + T_0) = x(t + 2T_0) = ...$

Put $t = t + T_0$ in term 2 of Equation (7.5.13) and

$t = t + 2T_0$ in term 3 of Equation (7.5.13) .... etc.

$\therefore$ Equation (7.5.13) becomes,

$$L\{x(t)\} = \int_0^{T_0} x(t)\, e^{-st}\, dt + \int_0^{T_0} x(t + T_0)\, e^{-s(t+T_0)}\, dt + \int_0^{T_0} x(t + 2T_0)\, e^{-s(t+2T_0)}\, dt + \ldots$$

$$= \int_0^{T_0} x(t)\, e^{-st}\, dt + e^{-sT_0}\int_0^{T_0} x(t)\, e^{-st}\, dt + e^{-2sT_0}\int_0^{T_0} x(t)\, e^{-st}\, dt$$

$$= \int_0^{T_0} x(t)\, e^{-st}\, dt\,[1 + e^{-sT_0} + e^{-2sT_0} + \ldots]$$

$$\text{Put } [1 + e^{-sT_0} + e^{-2sT_0} + \ldots] = \frac{1}{1 - e^{sT_0}}$$

$$\therefore\ L\{x(t)\} = \frac{\int_0^{T_0} x(t)\, e^{-st}\, dt}{1 - e^{-sT_0}}$$

$$\therefore\ L\{x(t)\} = \frac{X(s)}{1 - e^{-sT_0}}$$

Meaning of this property : If the function is periodic with period T_0 then calculate laplace transform of first term only. The total laplace transform is obtained by multiplying the laplace transform of first term by the factor $\frac{1}{1 - e^{sT_0}}$.

7.5.11 Conjugation :

$$\text{If } x(t) \xleftrightarrow{LT} X(s)$$

$$\text{then } x^*(t) \xleftrightarrow{LT} X^*(s^*)$$

Here '*' indicates complex conjugate operation.

7.5.12 Initial Value Theorem :

$$\text{If } x(t) \xleftrightarrow{LT} X(s)$$

then initial value theorem is stated as.

$$L\left\{\lim_{t\to 0} x(t)\right\} = \lim_{s\to\infty} s\,X(s)$$

7.5.13 Final Value Theorem :

MU - Dec. 04, May 07, May 11

University Question

Q. 1 List the properties of Laplace Transform.
(Dec. 04, 5 Marks, May 07, May 11, 4 Marks)

$$\text{If } x(t) \xleftrightarrow{LT} X(s)$$

then final value theorem is stated as.

$$L\left\{\lim_{t\to\infty} x(t)\right\} = \lim_{s\to 0} s\,X(s)$$

7.5.14 Summary of Laplace Transform Properties :

Table 7.5.1 shows the summary of laplace transform properties.

Table 7.5.1

Sr. No.	Name of property	Mathematical equation	ROC
1.	Linearity	$a_1 x_1(t) + a_2 x_2(t) \xleftrightarrow{LT} a_1 X_1(s) + a_2 X_2(s)$	$R_1 \cap R_2$
2.	Time shifting	$x(t - t_0) \xleftrightarrow{LT} e^{-st_0} X(s)$	R
3.	Shifting in 's' domain	$e^{s_0 t} x(t) \xleftrightarrow{LT} X(s - s_0)$	$R + R_e(s_0)$
4.	Time scaling	$x(at) \xleftrightarrow{LT} \frac{1}{\lvert a\rvert} X\left(\frac{s}{a}\right)$	$\frac{R}{a}$
5.	Differentiation in time domain	$\frac{d}{dt} x(t) \xleftrightarrow{LT} s\,X(s)$	R
6.	Differentiation in s-domain	$-t\,x(t) \xleftrightarrow{LT} \frac{d}{ds} X(s)$	R
7.	Convolution	$x_1(t) * x_2(t) \xleftrightarrow{LT} X_1(s)\cdot X_2(s)$	$R_1 \cap R_2$
8.	Integration in time domain	$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{LT} \frac{X(s)}{s}$	$R \cap (\sigma > 0)$
9.	Integration in s-domain	$\frac{x(t)}{t} \xleftrightarrow{LT} \int_s^{\infty} X(s)\, ds$	R
10.	Periodic function x (t) with period T_0	$x(t) \xleftrightarrow{LT} \frac{1}{1 - e^{-sT_0}} X(s)$	—
11.	Conjugation	$f^*(t) \xleftrightarrow{LT} F^*(s^*)$	—
12.	Initial value theorem	$L\left\{\lim_{t\to 0} x(t)\right\} = \lim_{s\to\infty} s\,X(s)$	—
13.	Final value theorem	$L\left\{\lim_{t\to\infty} x(t)\right\} = \lim_{s\to\infty} s\,X(s)$	—

7.5.14.1 Solved Examples on Properties of Laplace Transform :

Ex. 7.5.10 : If $X(s) = \frac{s+2}{s^2 + 4s + 5}$

Find Laplace transform of following signals :

1. $y(t) = x(2t - 1)\, u(2t - 1)$
2. $y(t) = t\,x(t)$
3. $y(t) = x(t) \cos 7t$

Soln. :

1. We have

$$x(t) \xleftrightarrow{LT} X(s) = \frac{s+2}{s^2 + 4s + 5}$$

Given $y(t) = x(2t - 1)\, u(2t - 1)$

According to time scaling property,

$$x(at) \xleftrightarrow{LT} \frac{1}{\lvert a\rvert} X\left(\frac{s}{a}\right)$$

$$\therefore\ L\{x(2t)\} = \frac{1}{2} X\left(\frac{s}{2}\right)$$

$$\therefore\ L\{x(2t)\} = \frac{1}{2}\cdot\frac{\frac{s}{2} + 2}{\frac{s^2}{4} + \frac{4s}{2} + 5} = \frac{1}{2}\cdot\frac{\frac{s}{2} + 2}{\frac{s^2}{4} + 2s + 5}$$

$$\therefore\ L\{x(2t)\} = \frac{\frac{s}{4} + 1}{\frac{s^2 + 8s + 20}{4}} = \frac{s + 4}{s^2 + 8s + 20}$$

Now according to time shifting property,

$$x(t - t_0) \xleftrightarrow{LT} e^{-st_0}\cdot X(s)$$

$$\therefore\ L\{x(2t - 1)\} = e^{-s}\cdot\left[\frac{s + 4}{s^2 + 8s + 20}\right] \quad \text{--[Here } t_0 = 1]$$

2. According to differentiation in s domain property,

$$-t\,x(t) \xleftrightarrow{LT} \frac{d}{ds} X(s)$$

$$\therefore\ t\,x(t) \xleftrightarrow{LT} -\frac{d}{ds}\left[\frac{s+2}{s^2 + 4s + 5}\right]$$

$$\therefore\ L\{t\,x(t)\} = -\left[\frac{(s^2 + 4s + 5)\cdot\frac{d}{ds}(s + 2) - (s + 2)\frac{d}{ds}(s^2 + 4s + 5)}{(s^2 + 4s + 5)^2}\right]$$

$$= -\left[\frac{(s^2 + 4s + 5)\cdot 1 - (s + 2)(2s + 4)}{(s^2 + 4s + 5)^2}\right] = -\frac{(s^2 + 4s + 5) + (2s^2 + 8s + 8)}{(s^2 + 4s + 5)^2}$$

$$\therefore\ L\{t\,x(t)\} = \frac{s^2 + 4s + 3}{(s^2 + 4s + 5)^2}$$

3. Given $y(t) = x(t) \cos 7t$

One of the important properties of laplace transform is Time cos property. It states that,

$$\cos(\alpha t)\, x(t) \xleftrightarrow{LT} 0.5\,[X(s + j\alpha) - X(s - j\alpha)]$$

Here $\alpha = 7$ and $X(s) = \frac{s+2}{s^2 + 4s + 5}$

$$\therefore\ L\{x(t)\cos 7t\} = 0.5\left[\frac{s + j7 + 2}{(s + j7)^2 + 4(s + j7) + 5} - \frac{s - j7 + 2}{(s - j7)^2 + 4(s - j7) + 5}\right]$$

Ex. 7.5.11 : Find Laplace transform of following signals. Draw ROC in each case :

1. $x(t) = \cos\left(3t + \frac{\pi}{4}\right) u(t)$
2. $x(t) = t\,u(t) * e^{-2t}\, u(t)$
   where * is for convolution.

Soln. :

1. $x(t) = \cos\left(3t + \frac{\pi}{4}\right) u(t)$

We have standard Laplace transform pair.

$$\cos \omega t\, u(t) \xleftrightarrow{LT} \frac{s}{s^2 + \omega^2}, \quad \sigma > 0$$

$$\therefore \cos 3t\, u(t) \xleftrightarrow{LT} \frac{s}{s^2 + 9}$$

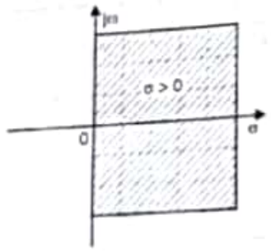
According to time shifting property.

$$x(t - t_0) \xleftrightarrow{LT} e^{-s t_0} X(s)$$

$$\therefore L\left\{\cos\left(3t + \frac{\pi}{4}\right)\right\} = e^{+s\frac{\pi}{4}} \cdot \frac{s}{s^2 + 9}$$

Here ROC remains unchanged.

Thus ROC is $\sigma > 0$. It is shown in Fig. P. 7.5.11.



(II-5766(a)) **Fig. P. 7.5.11**

2. $x(t) = t\, u(t) * e^{-3t} u(t)$

Let $x_1(t) = t\, u(t)$ and $x_2(t) = e^{-3t} u(t)$

$\therefore \quad x(t) = x_1(t) * x_2(t)$

According to convolution property.

$$X(s) = X_1(s) \cdot X_2(s)$$

Here $X_1(s) = L[t\, u(t)]$

We have standard Laplace transform pair.

$$r(t) = t\, u(t) \xleftrightarrow{LT} \frac{1}{s^2}, \quad \sigma > 0$$

Now consider, $x_2(t) = e^{-3t} u(t)$

We have standard Laplace transform pair.

$$e^{-at} u(t) \xleftrightarrow{LT} \frac{1}{s + a} \qquad \sigma > -a$$

$$\therefore \quad X_2(s) = L\{e^{-3t} u(t)\} = \frac{1}{s+3} \sigma > -3$$

$$\therefore \quad X(s) = \frac{1}{s^2} \cdot \frac{1}{s+3} = \frac{1}{s^3 + 3s^2}$$

ROC of X (s) is the combination of two ROCs. Thus combined ROC is $\sigma > 0$. It is same as ROC shown in Fig. P. 7.5.11.

Ex. 7.5.12 : Given the Laplace transform of :

$$X(t) \xleftrightarrow{LT} \frac{2s}{s^2 + 2}$$

where $x(t) = 0$ for $t < 0$.

Determine the Laplace transform of the following signals :

1. $x(3t)$
2. $x(t) * \frac{d\,x(t)}{dt}$
3. $x(t - 2)$
4. $e^{-t} x(t)$
5. $\int_0^t x(3\tau)\, d\tau$
6. $2t\, x(t)$.

Soln. :

Given $x(t) \xleftrightarrow{L.T.} \frac{2s}{s^2 + 2}$ that means $X(s) = \frac{2s}{s^2 + 2}$.

1. **x (3t)**

According to time scaling property.

$$x(at) \xleftrightarrow{} \frac{1}{|a|} \times \left(\frac{s}{a}\right)$$

$$\therefore \quad x(3t) \xleftrightarrow{L.T.} \frac{1}{3} \cdot \frac{2 \cdot s/3}{(s/3)^2 + 2} = \frac{2}{9} \cdot \frac{s}{\frac{s^2}{9} + 2}$$

2. $x(t) * \frac{dx(t)}{dt}$

According to convolution property.

$$x(t) * h(t) \xleftrightarrow{L.T.} X(s) \cdot H(s).$$

$$\therefore \quad x(t) * \frac{d}{dt} x(t) \xleftrightarrow{L.T.} X(s) \cdot L.T.\left\{\frac{d}{dt} x(t)\right\}$$

But according to differentiation property.

$$\frac{d}{dt} x(t) \xleftrightarrow{L.T.} s \cdot X(s)$$

$$\therefore \quad x(t) * \frac{d}{dt} x(t) \xleftrightarrow{L.T.} \frac{2s}{s^2 + 2} \cdot s \cdot \frac{2s}{s^2 + 2} = \frac{4s^3}{(s^2 + 2)^2}$$

3. **x (t – 2)**

According to time shifting property.

$$x(t - t_0) \xleftrightarrow{L.T.} e^{-s t_0} X(s)$$

$$\therefore \quad x(t - 2) \xleftrightarrow{L.T.} e^{-2s} \cdot \frac{2s}{s^2 + 2}$$

4. $e^{-t} x(t)$

According to frequency shifting property.

$$e^{s_0 t} x(t) \longleftrightarrow X(s - s_0)$$

Here $\quad s_0 = -1$

$$\therefore \quad e^{-t} x(t) \longleftrightarrow X(s + 1) = \frac{2(s+1)}{(s+1)^2 + 2}$$

5. $\int_0^t x(3\tau)\, d\tau$

According to integration property.

$$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{L.T.} \frac{X(s)}{s} = \frac{2(s/3)}{3s\left(\frac{s^2}{9} + 2\right)}$$

6. **2t x(t)**

According to frequency differentiation property.

$$-t\, x(t) \xleftrightarrow{L.T.} \frac{d}{ds} X(s)$$

Multiplying both sides by – 2

$$\therefore \quad 2t\, x(t) \xleftrightarrow{L.T.} -2 \cdot \frac{d}{ds} X(s)$$

$$= -2 \cdot \frac{d}{ds}\left[\frac{2s}{s^2 + 2}\right] = -2\left[\frac{(s^2 + 2) \cdot 2 - 2s\,(2s)}{(s^2 + 2)^2}\right]$$

$$= -2\left[\frac{2s^2 + 4 - 4s^2}{(s^2 + 2)^2}\right] = -\frac{-4s^2 - 8 + 8s^2}{(s + 2)^2} = \frac{4s^2 - 8}{(s + 2)^2}$$

Ex. 7.5.13 : Find the Laplace transform and associated ROC for each of the following signals :

1. $e^{-2t}[u(t) - u(t - 5)]$
2. $\sum_{K=0}^{\infty} \delta(t - KT)$
3. $\delta(at + b)$, a, b real constants.

Soln. :

1. $x(t) = e^{-2t}[u(t) - u(t - 5)]$

$\therefore \quad x(t) = e^{-2t} u(t) - e^{-2t} u(t - 5)$

Let $\quad x_1(t) = e^{-2t} u(t)$ and $x_2(t) = e^{-2t} u(t - 5)$

$\therefore \quad x(t) = x_1(t) - x_2(t)$

$\therefore \quad x_1(s) = \frac{1}{s + 2}$

Consider $x_2(t)$.

Using time shifting property

$$L.T.\{u(t - 5)\} = e^{-5s} \cdot \frac{1}{s}$$

Using shifting in 'S' domain property,

$$e^{s_0 t} x(t) \longleftrightarrow x(s - s_0)$$

$$\therefore \quad e^{-2t} u(t - 5) \longleftrightarrow e^{-5(s+2)} \cdot \frac{1}{s + 2}$$

$\therefore \quad x(s) = x_1(s) - x_2(s)$

$$\therefore \quad x(s) = \frac{1}{s + 2} - e^{-5(s+2)} \cdot \frac{1}{s + 2} = \frac{1 - e^{-5(s+2)}}{s + 2}$$

2. $x(t) = \sum_{K=0}^{\infty} \delta(t - KT)$

$\therefore \quad x(t) = \delta(t) + \delta(t - T) + \delta(t - 2T) + \ldots$

$$\therefore \quad x(s) = 1 + e^{-Ts} e^{-2Ts} + \ldots = \sum_{T=0}^{\infty} e^{-Ts}$$

3. $x(t) = \delta(at + b)$

$L.T.\{\delta(at)\} = \frac{1}{a}$

$$\therefore L.T.\{\delta(at + b)\} = e^{bs} \cdot \frac{1}{a}$$

Ex. 7.5.14 : Using properties of Laplace transform, find it :

$$X(s) = \frac{4s}{s^2 + 2s + 1}$$

1. $x(5t)$ 2. $x(t) * u(t)$

Soln. :

Given : $\quad X(S) = \frac{4s}{s^2 + 2s + 1}$

1. According to time scaling property.

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

$$\therefore \quad x(5t) \leftrightarrow \frac{1}{5} \cdot \frac{4\frac{s}{5}}{\left(\frac{s}{5}\right)^2 + 2\left(\frac{s}{5}\right) + 1}$$

$$\therefore \quad L\{x(5t)\} = \frac{0.16\,s}{\frac{s^2}{25} + \frac{2s}{5} + 1} = \frac{4s}{s^2 + 10s + 25}$$

2. According to convolution property.

$$x(t) * u(t) \leftrightarrow X(s) \cdot L\{u(t)\}$$

$$= \frac{4s}{s^2 + 2s + 1} \cdot \frac{1}{s} = \frac{4}{s^2 + 2s + 1}$$

Ex. 7.5.15 : Find Laplace transform of the following signals using properties of Laplace transform.

1. $x(t) = u(t - 3)$
2. $x(t) = t^2 e^{-3t} u(t)$
3. $x(t) = 3e^{-2t} \cos(4t) - 5e^{-3t} \sin 2t + 2$

Soln. :

1. Given : $x(t) = u(t - 3)$

We have, $\quad L\{u(t)\} = \frac{1}{s}$

According to time shifting property.

$$x(t - t_0) \xleftrightarrow{LT} e^{-s t_0} X(s)$$

$$\therefore \quad L\{u(t - 3)\} = e^{-3s} \cdot \frac{1}{s}$$

2. Given : $x(t) = t^2 e^{-3t} u(t)$

Consider the term $e^{-3t} u(t)$

$$\therefore \quad L\{e^{-3t} u(t)\} = \frac{1}{s + 3}$$

According to differentiation in 's' domain property.

$$(-t)^n x(t) \xleftrightarrow{LT} \frac{d^n}{ds^n} X(s)$$

For n = 2 we get.

$$\therefore \quad t^2 x(t) \longleftrightarrow \frac{d^2}{ds^2} X(s)$$

$$\therefore \quad t^2 e^{-3t} u(t) \longleftrightarrow \frac{d^2}{ds^2}\left[\frac{1}{s + 3}\right]$$

$$\therefore \quad L\{t^2 e^{-3t} u(t)\} = \frac{d}{ds} \cdot \left[\frac{d}{ds}\left(\frac{1}{s + 3}\right)\right]$$

$$= \frac{d}{ds}\left[\frac{(s + 3) \cdot 0 - 1 \cdot (1)}{(s + 3)^2}\right] = \frac{d}{ds}\left[\frac{-1}{(s + 3)^2}\right]$$

$$= \frac{(s + 3)^2 (0) - (-1) \cdot 2(s + 3)}{(s + 3)^4}$$

$\therefore$ L $[t^2 e^{-3t} u(t)] = \frac{2(s+3)}{(s+3)^4} = \frac{2}{(s+3)^3}$

3. Given : $x(t) = 3e^{-2t}\cos(4t) - 5e^{-3t}\sin(2t) + 2$

Let $x_1(t) = 3e^{-2t}\cos(4t)$, $x_2(t) = 5e^{-3t}\sin 2t$,

$x_3(t) = 2\,\delta(t)$

$\therefore\ x(t) = x_1(t) - x_2(t) + x_3(t)$

According to linearity property,

$X(s) = X_1(s) - X_2(s) + X_3(s)$

If we consider cos (4t) and sin (2t) then it is not Laplace transformable; because limits will be $-\infty$ to $+\infty$. So consider,

$x_1(t) = 3e^{-2t}\cos(4t)\,u(t)$ and $x_2(t) = 5e^{-3t}\sin 2t\,u(t)$

We have standard Laplace transform pairs

$e^{-at}\cos\omega t\,u(t) \longleftrightarrow \frac{s+a}{(s+a)^2+\omega^2}$

At $e^{-at}\sin\omega t\,u(t) \longleftrightarrow \frac{\omega}{(s+a)^2+\omega^2}$

$\therefore\ X_1(s) = 3\cdot\frac{s+2}{(s+2)^2+(4)^2}$, $X_2(s)$

$= \frac{5\times 2}{(s+3)^2+(2)^2}$ and $X_3(s) = 2$

$\therefore\ X_2(s) = 3\cdot\frac{s+2}{(s+2)^2+16} - \frac{10}{(s+3)^2+4} + 2$

Ex. 7.5.16 : Determine the laplace transform of following using properties of laplace transform.

$x(t) = (t^2 - 2t)\,u(t-1)$. **Dec. 11. 4 Marks**

Soln. :

Given : $x(t) = (t^2 - 2t)\,u(t-1)$

$\therefore\ x(t) = t^2 u(t-1) - 2t\,u(t-1)$

Let $x_1(t) = t^2 u(t-1)$ and $x_2(t) = 2t\,u(t-1)$ ...(1)

$\therefore\ x(t) = x_1(t) - x_2(t)$

Now we have, $u(t) \leftrightarrow \frac{1}{s}$

Using time shifting property

$u(t-1) \leftrightarrow e^{-s}\cdot\frac{1}{s}$

According to differentiation in s domain property,

$-t\,x(t) \leftrightarrow \frac{d}{ds}X(s)$

$\therefore\ t^2 x(t) \leftrightarrow -\frac{d^2}{ds^2}X(s)$

Here $X(s) = \frac{e^{-s}}{s}$

$\therefore\ \frac{d}{ds}X(s) = \frac{s\cdot(-1)e^{-s} - e^{-s}\cdot 1}{s^2}$

$\therefore\ \frac{d}{ds}X(s) = \frac{-s\cdot e^{-s} - e^{-s}}{s^2}$ ...(2)

$\therefore\ \frac{d^2}{ds^2}X(s) = \frac{s^2\frac{d}{ds}[-s\cdot e^{-s} - e^{-s}] - (-s\cdot e^{-s} - e^{-s})\frac{d}{ds}s^2}{s^4}$

$= \frac{s^2[[-s\cdot(-1)e^{-s} + e^{-s}\cdot(-1)] - [(-1)e^{-s}]] + (s e^{-s} + e^{-s})\cdot 2s}{s^4}$

$\therefore\ \frac{d^2}{ds^2}X(s) = \frac{s^2[-s e^{-s} - e^{-s} + e^{-s}] + 2s^2\cdot e^{-s} + 2 s e^{-s}}{s^4}$

$\therefore\ \frac{d^2}{ds^2}X(s) = \frac{-s^3 e^{-s} + 2s^2 e^{-s} + 2s e^{-s}}{s^4}$

$\therefore\ \frac{d^2}{ds^2}X(s) = \frac{e^{-s}[-s^2 + 2s + 2]}{s^3}$ ...(3)

$X_1(s) = \text{L.T.}\,[t^2 u(t-1)] = \frac{-e^{-s}[-s^2 + 2s + 2]}{s^3}$

and $X_2(s) = \text{L.T.}\,[2t\,u(t-1)] = \frac{2 s e^{-s} + e^{-s}}{s^2}$

$\therefore\ X(s) = X_1(s) - X_2(s)$

$\therefore\ X(s) = \frac{-e^{-s}[-s^2 + 2s + 2]}{s^3} + \frac{2 s e^{-s} + e^{-s}}{s^2}$ ...Ans.

7.6 Laplace Transform used for Waveform Synthesis :

Any time domain function can be expressed in terms of singular functions. There are additions or subtractions of a singular function to an existing function. Generally there are three possibilities.

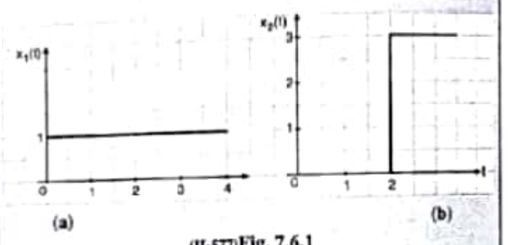
1. Step ± Step
2. Step ± Ramp
3. Ramp ± Ramp

Case 1 : Step ± Step :

The addition or subtraction of a step to a step results in a step function. The magnitude of the resultant is the algebraic addition or subtraction respectively of the two steps. The change in the magnitude occurs at the instant of addition.

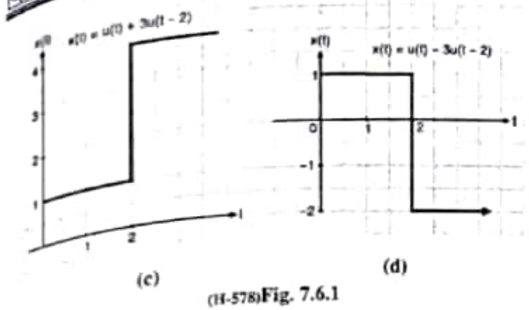
DEMO

Consider $x(t) = u(t) + 3u(t-2)$
$= x_1(t) + x_2(t)$ say



(H-577)Fig. 7.6.1

For $t < 0$, $x_1(t)$ and $x_2(t)$ have value zero. For $0 < t < 2$, $x_2(t)$ is zero. Therefore resultant will have only value of $x_1(t)$ i.e. 1. At $t = 2$, we have addition of case 1 - step plus step. The resultant is a step. The magnitude is the addition of two steps, i.e. $1 + 3 = 4$. The change in the resultant occurs at the instant of addition i.e. at $t = 2$ as shown in Fig. 7.6.1(c).



(H-578)Fig. 7.6.1

If instead we add a wave $-3u(t-2)$ the resultant would have magnitude $1 + (-3) = -2$, at $t = 2$ i.e. as shown in Fig. 7.6.1(d).

The student should note that in any term such as $M\,u(t-N)$, M refers to the magnitude (either positive or negative) and N to the instant in time.

Case 2 : Step ± Ramp :

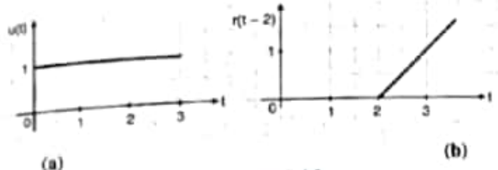
When the addition of a step is done to a ramp function, the result is a ramp function, shifted by an amount equal to the step.

DEMO

Consider $x(t) = u(t) + r(t-2)$

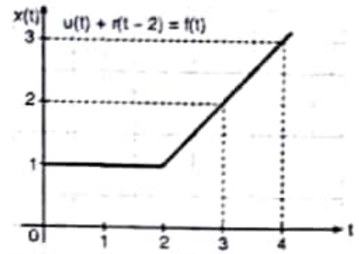
For $t < 0$ $x(t) = 0$

For $0 < t < 2$ $x(t) = 1$



(H-579)Fig. 7.6.2

At $t = 2$, case 2 addition occurs. The resultant is a ramp. It is shifted by the magnitude of step i.e. + 1. This is done at $t = 2$. This gives the wave as shown in Fig. 7.6.2(c).

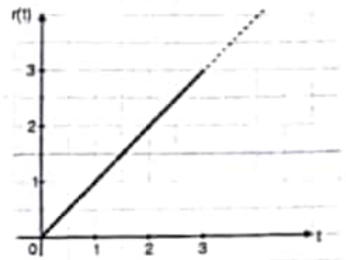


(H-580)Fig. 7.6.2(c)

Case 3 : Ramp ± Ramp :

The addition of a ramp to a ramp gives a ramp. The slope of this ramp is the algebraic addition of two slopes. The change in slope occurs at the instant of addition.

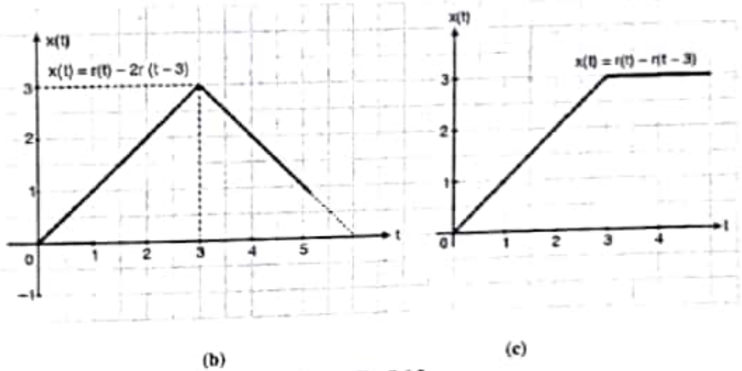
A special case is an addition of two equal and opposite slopes. **This results in a ramp of zero slope i.e. a horizontal line.** Consider, $x(t) = r(t) - 2r(t-3)$



(H-581)Fig. 7.6.3(a)

$r(t)$ is shown in Fig. 7.6.3(b). At $t = 3$, there is a case 3 addition. The new ramp has negative slope twice the magnitude as the first one. the algebraic addition gives a resultant of negative unit magnitude $[1 + (-2) = -1]$. So the resultant ramp has a slope of -1 units. This change in slope occurs at $t = 3$.

$x(t) = r(t) - 2r(t-3)$



(H-582)Fig. 7.6.3

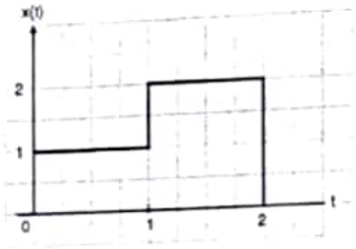
If $x(t) = r(t) - r(t-3)$, then the resultant would be a ramp of slope zero $[1 + (-1) = 0]$ and in this case the line would be horizontal as shown in Fig. 7.6.3(c).

In waveforms synthesis, we have to usually decode a waveform into component parts. The problems will illustrate this.

Note : To make a ramp as horizontal ; subtract delayed ramp from it.

Solved Examples :

Ex. 7.6.1 : Find the Laplace transform of Fig. P. 7.6.1.



(H-583)Fig. P. 7.6.1

Soln. :

Step 1 :

$$\therefore \quad x(t) = u(t) + u(t-1) - 2u(t-2)$$

Note : You can draw separate diagrams for each terms and then perform algebraic operations.

Step 2 : We have,

$$L\{x(t)\} = X(s) \text{ then}$$

$$\therefore \quad L\{x(t-a)\} = e^{-as} X(s)$$

$$\text{For } x(t) = u(t) + u(t-1) - 2u(t-2)$$

$$\therefore \quad L\{u(t-1)\} = e^{-s} \cdot \frac{1}{s} = \frac{e^{-s}}{s}$$

$$\text{and } L\{-2u(t-2)\} = -2 \cdot e^{-2s} \cdot \frac{1}{s} = -\frac{2e^{-2s}}{s}$$

$$\text{Hence } X(s) = \frac{1}{s} + \frac{e^{-s}}{s} + \frac{(-2)e^{-2s}}{s} = \frac{1}{s}\{1 + e^{-s} - 2e^{-2s}\} \quad \text{...Ans.}$$

Ex. 7.6.2 : Find the Laplace transform of Fig. P. 7.6.2.

(H-584)Fig. P. 7.6.2

Soln. :

$$\therefore \quad x(t) = u(t) + u(t-1) - u(t-2) - u(t-3)$$

$$\text{The Laplace of } u(t) = \frac{1}{s}$$

$$\text{Laplace of } u(t-1) = \frac{e^{-s}}{s}$$

$$\text{Laplace of } -u(t-2) = -\frac{e^{-2s}}{s}$$

$$\text{Laplace of } -u(t-3) = -\frac{e^{-3s}}{s}$$

$$\therefore \quad X(s) = \frac{1}{s} + \frac{1}{s}e^{-s} - \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s} \quad \text{...Ans.}$$

Ex. 7.6.3 : Find the Laplace transform x (t) of Fig. P. 7.6.3.

Soln. :

Step 1 : $x(t) = r(t) - r(t-1) + u(t-2) - 2u(t-3)$.

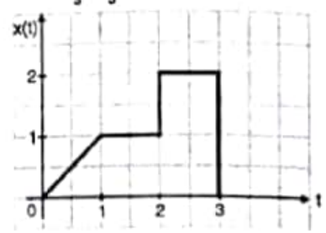
Step 2 : Laplace of $r(t) = \frac{1}{s^2}$

$$\text{Laplace of } r(t-1) = \frac{1}{s^2}e^{-s}$$

$$\text{Laplace of } (t-2) = \frac{1}{s}e^{-2s}$$

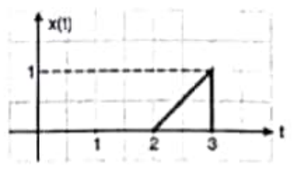
$$\text{Laplace of } -2u(t-3) = -\frac{2}{s}e^{-3s}$$

$$\therefore \quad X(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} + \frac{1}{s}e^{-2s} - \frac{2}{s}e^{-3s} \quad \text{...Ans.}$$



(H-585)Fig. P. 7.6.3

Ex. 7.6.4 : Find the Laplace transform of Fig. P. 7.6.4.



(H-594)Fig. P. 7.6.4

Soln. : This signal is generated as shown in Fig. P. 7.6.4(a).

$$\therefore \quad x(t) = r(t-2) - r(t-3)\,u(t-3)$$

We have,

$$L\{r(t)\} = \frac{1}{s^2}$$

$$\therefore L\{r(t-2)\} = e^{-2s} \cdot \frac{1}{s^2}$$

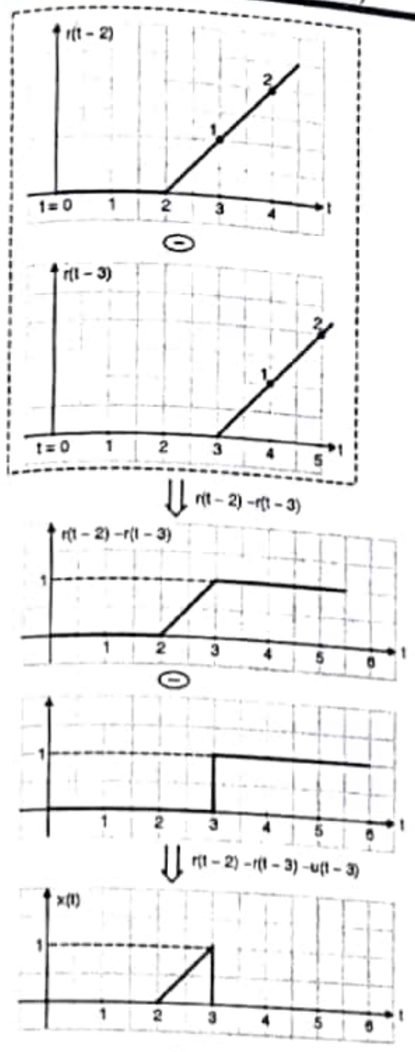
$$\text{and } L\{r(t-3)\} = e^{-3s} \cdot \frac{1}{s^2}$$

Similarly we have,

$$L\{u(t)\} = \frac{1}{s}$$

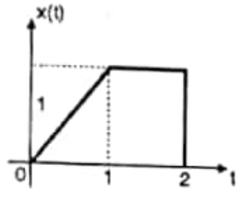
$$\therefore L\{u(t-3)\} = e^{-3s} \cdot \frac{1}{s}$$

$$\therefore \quad X(s) = \frac{e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}$$



(H-595)Fig. P. 7.6.4(a)

Ex. 7.6.5 : Obtain Laplace transform of the waveform shown in Fig. P. 7.6.5.



(H-783)Fig. P. 7.6.5

Soln. : For the given waveform we can write the mathematical equation as,

$$x(t) = r(t) - r(t-1) - u(t-2)$$

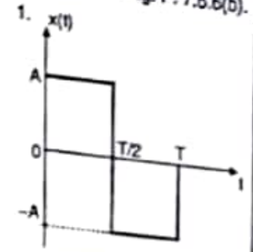
We have, $L\{r(t)\} = \frac{1}{s^2}$

$$L\{r(t-1)\} = \frac{1}{s^2} \cdot e^{-s}; \quad L\{u(t)\} = \frac{1}{s}$$

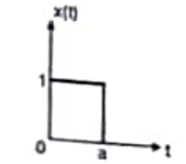
$$\text{and } L\{u(t-2)\} = \frac{1}{s}e^{-2s}$$

$$\therefore \quad X(S) = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-2s}$$

Ex. 7.6.6 : Determine the laplace transform of the signals shown in Fig. P. 7.6.6(a) and Fig. P. 7.6.6(b). **Dec. 11. 6 Marks**



(H-1217)Fig. P.7.6.6(a)



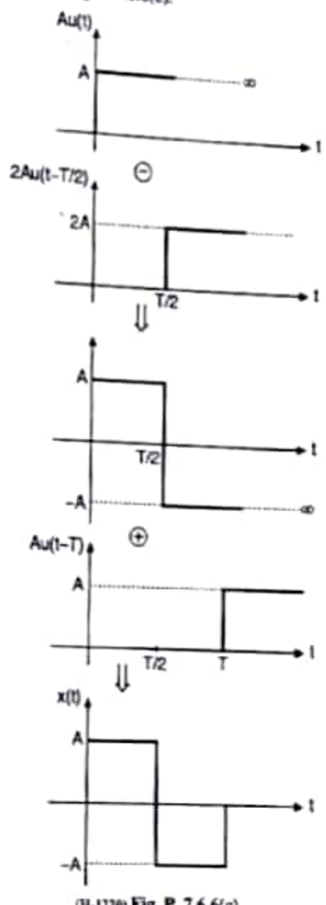
(H-1218)Fig. P.7.6.6(b)

Soln. :

1. The given signal can be expressed as

$$x(t) = Au(t) - A\,2u\left(t - \frac{T}{2}\right) + Au(t-T)$$

It is shown in Fig. P.7.6.6(c).



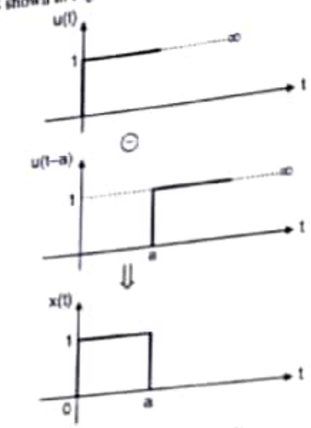
(H-1220) Fig. P. 7.6.6(c)

$$\therefore X(s) = \frac{A}{s} - \frac{2A}{s} e^{-s \cdot \frac{T}{2}} + \frac{A}{s} e^{-Ts}$$ ...Ans.

2. The given signal can be expressed as

$$x(t) = u(t) - u(t-a)$$

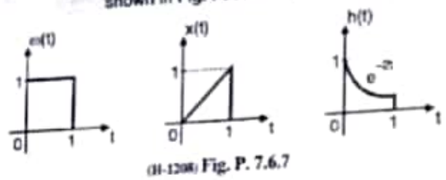
It is shown in Fig. P.7.6.6(d).



(H-1221) Fig. P. 7.6.6(d)

$$\therefore X(s) = \frac{1}{s} - e^{-as}\frac{1}{s}$$ ...Ans.

Ex. 7.6.7 : Find the Laplace transform of the signals shown in Fig. P. 7.6.7. **Dec. 10, 10 Marks**



(H-1208) Fig. P. 7.6.7

Soln. :

1. Here $x(t) = 1$ for $0 \le t \le 1$

We have $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$

$$\therefore X(s) = \int_0^1 1 e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^1$$

$$\therefore X(s) = -\frac{1}{s}[e^{-s} - e^{-0}] = -\frac{1}{s}[e^{-s} - 1]$$

$$\therefore X(s) = \frac{1}{s} - \frac{e^{-s}}{s}$$

2. Here $x(t) = t$ for $0 \le t \le 1$

$$\therefore X(s) = \int_0^1 t e^{-st} dt$$

We have $\int t e^{at} dt = \frac{e^{at}}{a^2}(at - 1)$

$$\therefore X(s) = \left[\frac{e^{-st}}{s^2}(-st - 1)\right]_0^1$$

$$\therefore X(s) = \left[\frac{e^{-s}}{s^2}(-s - 1) - \frac{e^{-0}}{s^2}(0 - 1)\right]$$

$$\therefore X(s) = \frac{-e^{-s}(s+1) + 1}{s^2}$$

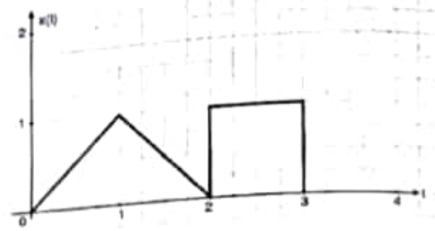
3. Here $h(t) = e^{-2t}$ for $0 \le t \le 1$.

$$\therefore H(s) = \int_0^1 e^{-2t} e^{-st} dt = \int_0^1 e^{-t(s+2)} dt$$

$$\therefore H(s) = \left[\frac{e^{-t(s+2)}}{-(s+2)}\right]_0^1 = \left[\frac{e^{-(s+2)}}{-(s+2)} + \frac{1}{(s+2)}\right]$$

$$\therefore H(s) = \frac{1 - e^{-(s+2)}}{s+2}$$

Ex. 7.6.8 : Find the Laplace transform of shown in Fig. P. 7.6.8.



(H-586) Fig. P. 7.6.8

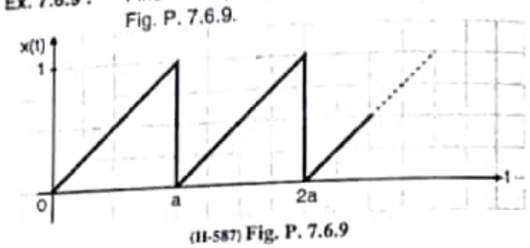
Soln. :

Step 1 : $x(t) = r(t) - 2r(t-1) + r(t-2) + u(t-2) - u(t-3)$

Step 2 : The Laplace transform is.

$$X(s) = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-3s}$$ ...Ans.

Ex. 7.6.9 : Find the laplace transform of wave shown in Fig. P. 7.6.9.



(H-587) Fig. P. 7.6.9

Soln. :

Step 1 : The given waveform is, periodic.

For a single saw tooth function.

$$x(t) = r(t) - r(t-a) - u(t-a)$$

Step 2 : Its laplace is,

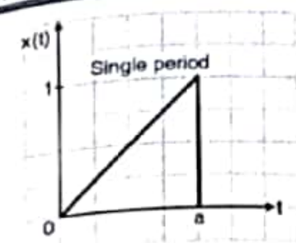
$$X(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \frac{1}{s}e^{-as}$$

Step 3 : Laplace of periodic waveform is,

$$X(s) = \frac{X_1(s)}{1 - e^{-T_0 s}}$$ where T_0 is the periodic

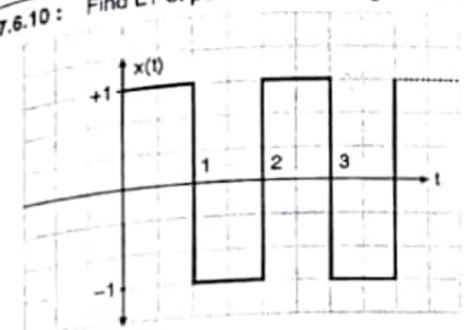
Here $T_0 = a$

$$\therefore X_1(s) = \frac{\left[\frac{1}{s^2} - \frac{1}{s^2}e^{-as} - \frac{1}{s}e^{-as}\right]}{1 - e^{-as}}$$ ...Ans.



(H-588) Fig. P. 7.6.9(a)

Ex. 7.6.10 : Find LT of periodic wave of Fig. P. 7.6.10.



(H-589) Fig. P. 7.6.10

Soln. :

Step 1 : $x_1(t)$ for one period of given waveform is as shown in Fig. P. 7.6.10(a).

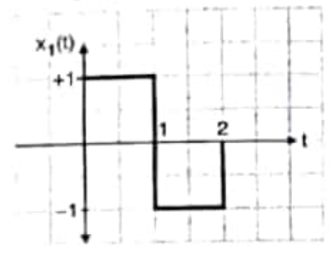
$$x_1(t) = u(t) - 2u(t-1) + u(t-2)$$

Step 2 : Its Laplace is,

$$X_1(s) = \frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}$$

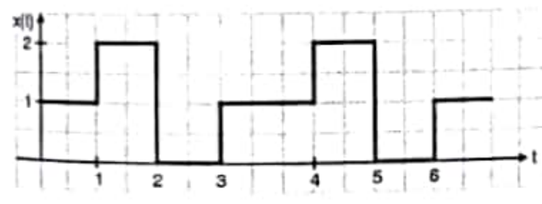
Step 3 : Laplace of periodic wave of period $T_0 = 2$ is.

$$\therefore X(s) = \frac{1}{1 - e^{-2s}}\left[\frac{1}{s} - \frac{2}{s}e^{-s} + \frac{1}{s}e^{-2s}\right]$$



(H-590) Fig. P. 7.6.10(a)

Ex. 7.6.11 : Find LT of periodic wave of Fig. P. 7.6.11.



(H-591) Fig. P. 7.6.11

Soln. :

Step 1 : $x(t)$ is given as,

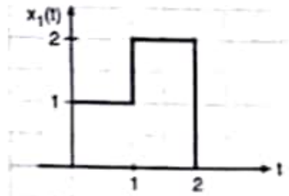
$$x(t) = u(t) + u(t-1) - 2u(t-2)$$

for one wave only. Call it $x_1(t)$

The period, $T_0 = 3$

Step 2 : Laplace of $x_1(t)$ is.

$$X_1(s) = \frac{1}{s} + \frac{1}{s}e^{-s} - \frac{2}{s}e^{-2s}$$



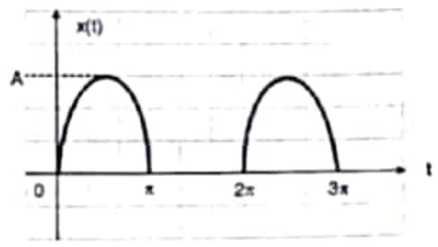
(H-592) Fig. P. 7.6.11(a)

Step 3 : Laplace of given periodic wave X (s) is.

$$X(s) = \frac{X_1(s)}{1 - e^{-Ts}}$$

$$\therefore X(s) = \frac{\left(\frac{1}{s} + \frac{1}{s}e^{-s} - \frac{2}{s}e^{-2s}\right)}{1 - e^{-3s}}$$ ...Ans.

Ex. 7.6.12 : Find the LT of function shown in Fig. P. 7.6.12.



(H-593) Fig. P. 7.6.12

Soln. : From Fig. P. 7.6.12.

$$x(t) = A \sin t \quad \text{for } 0 < t < \pi$$
$$= 0 \quad \text{for } \pi < t < 2\pi$$

$$\because L[x(t)] = \frac{\int_0^{T_0} x(t) e^{-st} dt}{1 - e^{-sT_0}}$$ ...periodicity property

$$= \frac{\int_0^{\pi} A \sin t \, e^{-st} dt}{1 - e^{-sT_0}} = \frac{A}{1 - e^{-sT_0}} \int_0^{\pi} \left(\frac{e^{jt} - e^{-jt}}{2j}\right) e^{-st} dt \quad \text{... } \sin\theta$$

$$= \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

$$= \frac{A}{2j(1 - e^{-sT_0})}\left[\int_0^{\pi} e^{(j-s)t} dt - \int_0^{\pi} e^{-(j+s)t} dt\right]$$

$$= \frac{A}{2j(1 - e^{-sT_0})}\left\{\left[\frac{e^{(j-s)t}}{j-s}\right]_0^{\pi} + \left[\frac{e^{-(j+s)t}}{j+s}\right]_0^{\pi}\right\}$$

$$= \frac{A}{2j(1-e^{-sT_0})}\left\{\frac{e^{(j-s)\pi}}{j-s} - \frac{e^{0}}{j-s} + \frac{e^{-(j+s)\pi}}{j+s} - \frac{e^{0}}{j+s}\right\}$$

$$= \frac{A}{2j(1-e^{-sT_0})}\left\{\frac{e^{j\pi}\cdot e^{-s\pi}}{j-s} - \frac{1}{j-s} + \frac{e^{-j\pi}\cdot e^{-s\pi}}{j+s} - \frac{1}{j+s}\right\}$$

But $e^{j\pi} = 1$ and $e^{-j\pi} = -1$

$$\therefore L[x(t)] = \frac{A}{2j(1-e^{-sT_0})}\left\{\frac{+e^{-s\pi}}{j-s} - \frac{e^{-s\pi}}{j+s} + \frac{2j}{1+s^2}\right\}$$

$$= \frac{A}{1-e^{-sT_0}}\left\{\frac{2j\,e^{-s\pi}+2j}{s^2+1}\right\}$$

$$\therefore X(s) = \frac{A}{s^2+1}\left[\frac{1+e^{-s\pi}}{1-e^{-sT_0}}\right]$$

This is nothing but L.T. of half wave rectified periodic signal. We can find for full wave rectified periodic signal also in same manner.

7.7 One Sided (Unilateral) Laplace Transform :

We have studied **Bilateral Laplace Transform** $(-\infty < t < +\infty)$. Most of practical systems are causal and in majority of cases $x(t) = 0, t < 0$.

The **one-sided (unilateral)** laplace transform of causal signals is defined as.

$$X(s) = L[x(t)] \quad \text{for } t = 0 \text{ to } \infty$$

$$\therefore X(s) = \int_{0^-}^{\infty} x(t)\,e^{-st}\,dt \qquad ...(7.7.1)$$

In unilateral Laplace transform the integration is carried from $t = 0^-$ to $t = \infty$

Equation (7.7.1) is referred as unilateral or one sided L.T. of the signal x (t).

Here 0^- is called as initial condition. It indicated value of x (t) be fore t = 0.

7.7.1 Properties of U.T. (Unilateral Laplace Transform) :

Similar property of BLT and ULT as follows

1. Linearity
2. S-domain shifting
3. Time sculing
4. Conjugation
5. Differentiation in s-domain
6. Initial and final value theorem
7. Convolution property

Difference in property of unilateral and bilateral Laplace transform

Differentiation property :

(a) Differentiation in time domain.

(b) Differentiation in s domain.

Thus

- ULT is interpreted as BLT of signal [Provided values prior to t = 0 must be zero].
- ULT is useful for analyzing linear constant coefficient differential equations with non-zero initial condition.

7.7.2 Initial Value Theorem :

Statement : If $x(t) \xleftrightarrow{L.T.} X(s)$ then initial value of x(t) is given as

$$x(0^+) = \lim_{t\to 0^+} x(t) = \lim_{s\to\infty}[s\,X(s)]$$

Proof : The differentiation property of unilateral laplace transform is given by.

$$\frac{dx(t)}{dt} \xleftrightarrow{L.T.} s\,X(s) - x(0^-) \qquad ...(7.7.2)$$

Here $x(0^-)$ is initial condition. that means initial value of x (t)

Taking limits as $s \to \infty$ both sides we get,

$$\lim_{s\to\infty} L\left\{\frac{d}{dt}x(t)\right\} = \lim_{s\to\infty}\{s\,X(s) - x(0^-)\} \qquad ...(7.7.3)$$

Using definition of unilateral laplace transform; L.H.S. of Equation (7.7.3) can be given as,

$$\lim_{s\to\infty} L\left\{\frac{d}{dt}x(t)\right\} = \lim_{s\to\infty}\int_{0^-}^{\infty}\frac{d}{dt}x(t)\,e^{-st}\,dt$$

$$\text{But } \lim_{s\to\infty} e^{-st} = 0$$

Thus L.H.S term of Equation (7.7.3) is zero.

$$\therefore 0 = \lim_{s\to\infty}\{s\,X(s) - x(0^-)\}$$

$$\therefore 0 = \lim_{s\to\infty} s\,X(s) - \lim_{s\to\infty} x(0^-) \qquad ...(7.7.4)$$

But in the second term of Equation (7.7.4), 's' term is not present. So we can neglect $\lim_{s\to\infty}$. Thus Equation (7.7.4) becomes,

$$\therefore 0 = \lim_{s\to\infty} s\,X(s) - x(0^-)$$

$$\therefore x(0^-) = \lim_{s\to\infty} s\,X(s)$$

Here $x(0^-)$ is the value of x(t) just before t = 0 and $x(0^+)$ is the value of x(t) just after t = 0. But if the signal is continuous then $x(0^-) = x(0^+)$.

$$\therefore x(0^+) = \lim_{s\to\infty} s\,X(s)$$

7.7.3 Final Value Theorem :

Statement : If $x(t) \xleftrightarrow{L.T.} X(s)$ then

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}[s\,X(s)]$$

Proof : Differentiation property of unilateral laplace transform gives,

$$L\left[\frac{d}{dt}x(t)\right] = s\,X(s) - x(0^-) \qquad ...(7.7.5)$$

Taking limits of both sides as $s \to 0$ we get

$$\lim_{s\to 0} L\left[\frac{d}{dt}x(t)\right] = \lim_{s\to 0}\{s\,X(s) - x(0^-)\}$$

$$= \lim_{s\to 0} s\,X(s) - \lim_{s\to 0} x(0^-) \qquad ...(7.7.6)$$

But the second term does not contain 's'. So we can neglect the limit sign for second term at the R.H.S.

$$\therefore \lim_{s\to 0} L\left[\frac{d}{dt}x(t)\right] = \lim_{s\to 0} s\,X(s) - x(0^-) \qquad ...(7.7.7)$$

Using the definition of unilateral laplace transform, L.H.S. term of Equation (7.7.7) can be expressed as,

$$\lim_{s\to 0} L\left[\frac{d}{dt}x(t)\right] = \lim_{s\to 0}\int_{0^-}^{\infty}\frac{d}{dt}x(t)\,e^{-st}\,dt$$

$$\text{But } \lim_{s\to 0} e^{-st} = e^0 = 1$$

$$\therefore \lim_{s\to 0} L\left[\frac{d}{dt}x(t)\right] = \int_{0^-}^{\infty}\frac{d}{dt}x(t)\,dt = [x(t)]_{0^-}^{\infty} = x(\infty) - x(0^-)$$

The term $x(\infty)$ can be written as $\lim_{t\to\infty} x(t)$.

$$\lim_{s\to 0} L\left[\frac{d}{dt}x(t)\right] = \lim_{t\to\infty} x(t) - x(0^-) \qquad ...(7.7.8)$$

Putting Equation (7.7.7) in Equation (7.7.8) we get,

$$\lim_{s\to 0} s\,X(s) - x(0^-) = \lim_{t\to\infty} x(t) - x(0^-)$$

$$\therefore \lim_{t\to\infty} x(t) = \lim_{s\to 0}[s\,X(s)]$$

$$\therefore x(\infty) = \lim_{s\to 0}[s\,X(s)]$$

Hence proved.

Solved Examples :

Ex. 7.7.1 : Find initial and final value of : $X(s) = \frac{1}{s+1}$

Soln. : Initial value is given by,

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} s\cdot\frac{1}{s+1}$$

$$= \lim_{s\to\infty}\frac{s}{s+1} = \lim_{s\to\infty}\frac{1}{\frac{1}{s}+1}$$

$$\therefore x(0^+) = 1$$

The final value is given by,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\,X(s)$$

$$= \lim_{s\to 0} s\cdot\frac{1}{s+1} = \lim_{s\to 0}\frac{s}{s+1}$$

$$\therefore \lim_{t\to\infty} x(t) = 1$$

Ex. 7.7.2 : Find initial and final value of :

$$X(s) = \frac{2(s^2+1)}{s(s+2)(s+5)}$$

Soln. : Initial value is given by,

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty}\frac{s\cdot 2(s^2+1)}{s(s+2)(s+5)}$$

$$= \lim_{s\to\infty}\frac{2(s^2+1)}{(s+2)(s+5)} = \lim_{s\to\infty}\frac{2s^2+2}{s^2+7s+10}$$

$$= \lim_{s\to\infty}\frac{2+2/s^2}{1+\frac{7}{s}+\frac{10}{s^2}}$$

$$\therefore x(0^+) = 2$$

The final value is given by,

$$\lim_{t\to\infty} x(t) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty}\frac{s\cdot 2(s^2+1)}{s(s+2)(s+5)}$$

$$= \lim_{s\to\infty}\frac{2(s^2+1)}{(s+2)(s+5)} = \frac{2}{(2)(5)}$$

$$\therefore \lim_{t\to\infty} x(t) = \frac{1}{5}$$

Ex. 7.7.3 : Find initial and final value of :

$$f(t) = \frac{0.8}{s(s^2+0.6s+0.2)}$$

Soln. :

$$f(t) = \frac{0.8}{s(s^2+0.6s+0.2)}$$

Initial value is,

$$x(0^+) = \lim_{s\to\infty} s\times(s) = \lim_{s\to\infty} s\cdot\frac{0.8}{s(s^2+0.6s+0.2)}$$

$$= \lim_{s\to\infty}\frac{0.8}{(s^2+0.6s+0.2)} = 0$$

Final value is given by,

$$x(\infty) = \lim_{s\to 0} s\times(s) = \lim_{s\to 0}\frac{s\cdot 0.8}{s(s^2+0.8s+0.2)}$$

$$= \lim_{s\to 0}\frac{0.8}{s^2+0.8s+0.2} = \frac{0.8}{0.2}$$

$$\therefore x(\infty) = 4$$

Ex. 7.7.4 : Determine initial value of the following Laplace transform : $X(s) = \frac{7s+10}{s(s+2)}$

Soln. : Initial value is given by,

$$x(0^+) = \lim_{s\to\infty} s\,X(s) = \lim_{s\to\infty} s\cdot\frac{7s+10}{s(s+2)}$$

$$= \lim_{s\to\infty}\frac{7s+10}{s+2} = \lim_{s\to\infty}\frac{7+10/s}{1+2/s}$$

$$\therefore x(0^+) = 7.$$

Ex. 7.7.5 : Find f(∞), final value of the function whose Laplace transform is given as :

$$f(s) = \frac{5}{s} - \frac{1}{s-4}.$$

Soln. :

Given, $f(s) = \frac{5}{s} - \frac{1}{s-4}$

$$\therefore f(s) = \frac{5(s-4)-s}{s(s-4)} = \frac{5s-20-s}{s(s-4)} = \frac{4s-20}{s(s-4)}$$

According to final value theorem,

$$f(\infty) = \lim_{s\to 0} s\cdot f(s) = \lim_{s\to 0} s\cdot\left[\frac{4s-20}{s(s-4)}\right]$$

$$= \lim_{s\to 0}\frac{4s-20}{s-4} = \frac{20}{4}$$

$$\therefore f(\infty) = 5$$

7.8 Inverse Laplace Transform :

- The laplace transform is used to convert function x (t) into X (s).
- That means using laplace transform; we can convert time domain function. x (t) into laplace domain function X (s).
- The basic advantage of converting any function into laplace; is the mathematical calculation are reduced.

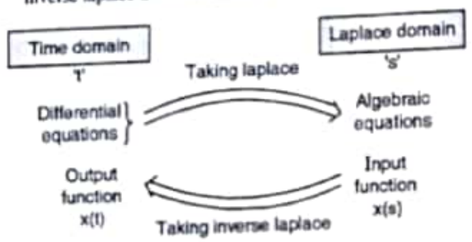
- So the solution of differential equations become simple. As well as many electrical networks can be reduced using laplace transform.

7.8.1 Necessity of Inverse Laplace Transform :

- The given function, x (t) is converted into laplace X (s). That means the operator is 's' and not 't'. Operator 't' indicates the time domain operation.

Note : Meaning of word "domain" is region where function is present. So, time domain means function is changing with respect to time and laplace domain (or 's' domain) means function is changing with respect to variable 's'.

- All operations, that we are doing in our daily life are with reference to time. That means it is time domain.
- In laplace transform, equations are with respect to variable 's'.
- The end user cannot understand the language of laplace; so for designing of any system output must be with respect to time. That means it must be in time domain.
- To convert laplace domain functions into time domain; inverse laplace transform is used.



(B-1461) Fig. 7.8.1

7.8.2 Definition of Inverse Laplace Transform :

The laplace transform of function x (t) is denoted by X (s). It is given by,

$$L\{x(t)\} = X(s) = \int_0^{\infty} x(t)\, e^{-st}\, dt \qquad ...(7.8.1)$$

Here the function x (t) is called as inverse laplace transform of X (s).

Notation :

- The laplace transform is denoted by notation 'L' and inverse laplace transform is denoted by L^{-1}.
- Inverse laplace of X (s) is x (t). It is denoted by,

$$L^{-1}\{X(s)\} = x(t) \qquad ...(7.8.2)$$

7.8.3 Calculation of Inverse Laplace using Duality :

- We have calculated the laplace transform of some standard functions. These results can be directly used to calculate inverse laplace transform.

The following table gives the values of inverse laplace of some standard functions.

Sr. No.	Function x (t)	Laplace transform X (s)	Inverse Laplace x (t)
1.	1	$\frac{1}{s}$	$\therefore L^{-1}\left\{\frac{1}{s}\right\} = 1$
2.	e^{at}	$\frac{1}{s-a}$	$\therefore L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
3.	e^{-at}	$\frac{1}{s+a}$	$\therefore L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}$
4.	a^t	$\frac{1}{s-\log a}$	$\therefore L^{-1}\left\{\frac{1}{s-\log a}\right\} = a^t$
5.	sin at	$\frac{a}{s^2+a^2}$	$\therefore L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$
6.	cos at	$\frac{a}{s^2+a^2}$	$\therefore L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$
7.	sin hat	$\frac{a}{s^2-a^2}$	$\therefore L^{-1}\left\{\frac{a}{s^2-a^2}\right\} = \sinh at$
8.	cosh at	$\frac{s}{s^2-a^2}$	$\therefore L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$
9.	t^n	$\frac{n!}{s^{n+1}}$ if n is positive integer	$\therefore L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$ if n is positive

Similarly we have obtained the laplace transform of some special functions like unit step and unit impulse. The following table gives values of inverse laplace of these functions.

Sr. No.	Function f (t)	Laplace transform F (s)	Inverse Laplace f (t)
1.	U (t)	$\frac{1}{s}$	$\therefore L^{-1}\left\{\frac{1}{s}\right\} = U(t)$
2.	U (t – a)	$\frac{e^{-as}}{s}$	$\therefore L^{-1}\left\{\frac{e^{-as}}{s}\right\} = U(t-a)$
3.	δ (t)	1	$\therefore L^{-1}\{1\} = \delta(t)$
4.	δ (t – a)	e^{-as}	$\therefore L^{-1}\{e^{-as}\} = \delta(t-a)$

7.8.3.1 Solved Examples on Inverse Laplace using Duality :

Ex. 7.8.1 : Find inverse laplace of following functions :

1. $\frac{5}{s^3}$ 2. $\frac{6s}{s^2+4}$

3. $\frac{1}{s^2+16}$ 4. $\frac{4s}{s^2-16}$

Soln. :

1. $F(s) = \frac{5}{s^3}$

We have

$$L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \quad \text{if } n \geq 0$$

The given equation can be rearranged as,

$$F(s) = \frac{5}{s^3} = \frac{5}{2!}\,\frac{2!}{s^{2+1}}$$

Note that in this case n = 2.

$$\therefore L^{-1}F(s) = L^{-1}\left\{\frac{5}{2!}\,\frac{2!}{s^{2+1}}\right\} = \frac{5}{2!}L^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = \frac{5}{2!}\cdot t^2$$

$$\therefore L^{-1}F(s) = f(t) = \frac{5}{2}t^2$$

2. Given $F(s) = \frac{6s}{s^2+4}$

We have,

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

The given equation can be rearranged as,

$$F(s) = 6\cdot\frac{s}{s^2+2^2}$$

Here a = 2

$$\therefore L^{-1}\{F(s)\} = L^{-1}\left\{6\cdot\frac{s}{s^2+2^2}\right\} = 6L^{-1}\left\{\frac{a}{s^2+2^2}\right\}$$

$$\therefore L^{-1}\{F(s)\} = f(t) = 6\cos 2t$$

3. Given, $F(s) = \frac{1}{s^2+16}$

We have, $L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

The given equation can be rearranged as,

$$F(s) = \frac{1}{s^2+4^2} = \frac{1}{4}\cdot\frac{1}{s^2+4^2}$$

Here a = 4

$$\therefore L^{-1}\{F(s)\} = \frac{1}{4}L^{-1}\left\{\frac{4}{s^2+4^2}\right\}$$

$$\therefore L^{-1}\{F(s)\} = f(t) = \frac{1}{4}\sin 4t$$

4. Given $F(s) = \frac{4s}{s^2-16}$

We have $L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cosh at$

The given equation can be rearranged as,

$$F(s) = 4\cdot\frac{s}{s^2-4^2}$$

Here a = 4

$$\therefore L^{-1}\{F(s)\} = 4L^{-1}\left\{\frac{s}{s^2-4^2}\right\}$$

$$\therefore L^{-1}\{F(s)\} = f(t) = 4\cdot\cosh 4t$$

7.8.4 Partial Fraction Expansion Method (P.F.E.) :

This method is suitable for the laplace transforms which are rational in nature. That means they are expressed as the ratio of two polynomials.

$$\therefore X(s) = \frac{N(s)}{D(s)}$$

Here N (s) = Numerator polynomial

D (s) = Denominator polynomial

Steps to follow in partial fraction expansion :

1. Factorize the denominator and obtain the roots. Then the denominator will be in the form.

$$(s-P_1)(s-P_2)\ldots.(s-P_N)$$

Here $P_1, P_2 \ldots P_N$ are called as poles.

2. Write down the equation in partial fraction expansion form as,

$$X(s) = \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2} + \ldots. + \frac{A_N}{s-P_N}$$

Here $A_1, A_2 \ldots. A_N$ are the coefficients of P.F.E.

The coefficient A_K is calculated as,

$$A_K = (s-p_K)\cdot X(s)\Big|_{s=p_K}$$

3. After calculating $A_1, A_2 \ldots. A_N$ use the standard laplace transform pairs.

1. $e^{at}u(t) \xleftrightarrow{LT} \frac{1}{s-a}$ ROC : σ > a
2. $-e^{at}u(-t) \xleftrightarrow{LT} \frac{1}{s-a}$ ROC : σ < a

4. The nature of time domain signal; that means whether it is right handed (causal) or left-handed (anticausal), is decided using following rules :
   1. When the poles lie to the L.H.S. of ROC then corresponding signal is right handed (causal). That means it is multiplied by u (t).
   2. When the poles lie to the R.H.S. of ROC then corresponding signal is left handed (anticausal). That means it is multiplied by u (– n).
   3. For bilateral ILT, the poles must lie on both sides of ROC

7.8.4.1 Solved Examples on P.F.E. :

While solving the numericals we will consider different cases depending on the nature of poles.

Case I : When there are simple poles :

Ex. 7.8.2 : Obtain inverse laplace transform of :

$$X(s) = \frac{s-\frac{1}{2}}{s^2+\frac{3}{4}s+\frac{1}{8}} \quad \text{ROC} : \sigma > -\frac{1}{4}$$

Soln. : First we will obtain the roots (poles) of denominator term

$$\therefore X(s) = \frac{s-\frac{1}{2}s}{(s+1/2)(s+1/4)}$$

Step 1 : In the partial fraction expansion form we can write the equation of X (s) as,

$$X(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2}$$

Here $P_1 = -\frac{1}{2}$, $P_2 = -\frac{1}{4}$

$$\therefore X(s) = \frac{A_1}{\left(s+\frac{1}{2}\right)} + \frac{A_2}{\left(s+\frac{1}{4}\right)}$$

Now we will calculate the coefficients A_1 and A_2.

We have, $A_K = (s-P_K)\cdot X(s)\Big|_{s=P_K}$

$$\therefore A_1 = (s-P_1)X(s)\Big|_{s=P_1}$$

$$\therefore A_1 = \left(s+\frac{1}{2}\right)\cdot\frac{s-\frac{1}{2}}{\left(s+\frac{1}{2}\right)\left(s+\frac{1}{4}\right)}\Bigg|_{s=-\frac{1}{2}}$$

$$\therefore A_1 = \frac{-1}{-\frac{1}{2}+\frac{1}{4}} = \frac{-1}{-1/4} = 4$$

and $A_2 = (s-P_2)\,X(s)\Big|_{s=P_2}$

$$\therefore A_2 = \left(s+\frac{1}{4}\right)\cdot\frac{s-\frac{1}{2}}{\left(s+\frac{1}{2}\right)\left(s+\frac{1}{4}\right)}\Bigg|_{s=-\frac{1}{4}}$$

$$\therefore A_2 = \frac{-\frac{1}{4}-\frac{1}{2}}{-\frac{1}{4}+\frac{1}{2}} = \frac{-3/4}{1/4} = -3$$

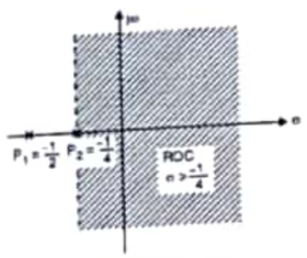
Putting these values in the equation of X (s) we get,

$$X(s) = \frac{4}{s+\frac{1}{2}} - \frac{3}{s+\frac{1}{4}}$$

Step 2 : The pole-zero plot and given ROC is shown in Fig. P. 7.8.2.

Here both the poles are to the left side of ROC. Hence both terms corresponds to right handed signal (multiplied by u (t)).

$$\therefore x(t) = L^{-1}[X(s)] = L^{-1}\left\{\frac{4}{s+\frac{1}{2}}\right\} - L^{-1}\left\{\frac{3}{s+\frac{1}{4}}\right\}$$



(H-596) **Fig. P. 7.8.2**

Now we have standard laplace transform pair,

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \qquad ROC : \sigma > -a$$

that means $L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t) \quad ROC : \sigma > -a$

$$\therefore L^{-1}\left\{\frac{4}{s+\frac{1}{2}}\right\} = 4e^{-\frac{1}{2}t}u(t)$$

and $L^{-1}\left\{\frac{3}{s+\frac{1}{4}}\right\} = 3e^{-\frac{1}{4}t}u(t)$

$$\therefore x(t) = 4e^{-\frac{1}{2}t}u(t) - 3e^{-\frac{1}{4}t}u(t)$$

Ex. 7.8.3 : If $X(s) = \frac{-2}{s+1} + \frac{4}{s+2}$ ROC : $-2 < s < -1$ then find x (t) which is inverse laplace of X (s).

Soln. : Given equation is,

$$X(s) = \frac{-2}{s+1} + \frac{4}{s+2} \quad ROC : -2 < s < -1$$

This equation is already in the form of P.F.E. Here the poles are at $P_1 = -1$ and $P_2 = -2$. The plot of poles and ROC is as shown in Fig. P. 7.8.3.

Now the inverse laplace of given expression is,

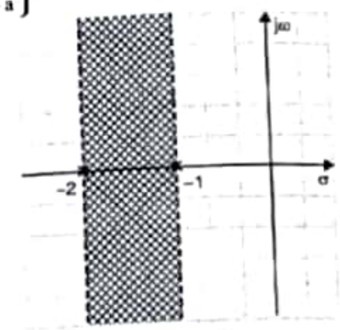
$$x(t) = L^{-1}[X(s)] = L^{-1}\left\{\frac{-2}{s+1}\right\} + L^{-1}\left\{\frac{4}{s+2}\right\}$$

Consider the first term only that is $L^{-1}\left\{\frac{-2}{s+1}\right\}$. Here the pole is at – 1. Observe Fig. P. 7.8.3. This pole is to the right of ROC. Hence corresponding term is left handed sequence. That means it is multiplied by u (– t).

We have standard laplace transform pair,

$$-e^{-at}u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \quad \text{for } \sigma < -a$$

$$\therefore L^{-1}\left\{\frac{1}{s+a}\right\} = -e^{-at}u(-t)$$



(H-597) **Fig. P. 7.8.3**

This term is $\frac{-2}{s+1}$ that means,

$$L^{-1}\left\{\frac{2}{s+1}\right\} = -(-2)e^{-t}u(-t) = 2e^{-t}u(-t)$$

Consider the second term that is $L^{-1}\left\{\frac{4}{s+2}\right\}$. Here the pole is at – 2. This pole is to the left of ROC. Hence corresponding time domain term is right handed sequence. That means it is multiplied by u (t).

We have standard laplace transform pair,

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \quad \text{for } \sigma > -a$$

$$\therefore L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t) \therefore L^{-1}\left\{\frac{1}{s+2}\right\} = 4e^{-2t}u(t)$$

Thus laplace inverse of given sequence is,

$$x(t) = 2e^{-t}u(-t) + 4e^{-2t}u(t)$$

Ex. 7.8.4 : Find the inverse Laplace Transform of the following.

(i) $X(S) = \frac{s-3}{s^2+4s+13}$

May 2015, 5 Marks

Soln. :

(i) $X(s) = \frac{s-3}{s^2+4s+13}$:

$$= \frac{(s+2)-5}{(s+2)^2+3^2}$$

$$\therefore x(t) = L^{-1}\left[\frac{s+2}{(s+2)^2+3^2}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right]$$

We have, $L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos ab$ and $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin ab$

and $L\{e^{-ab}f(b)\} = f(s+a)$

$$\therefore x(t) = e^{-2t}\cos 3t - 5e^{-2t}\cdot\frac{1}{3}\cdot\sin 3t$$

$$\therefore x(t) = e^{-2t}\left[\cos 3t - \frac{5}{3}\sin 3t\right]$$

Ex. 7.8.5 : Find inverse laplace transform of :

$$X(s) = \frac{-3}{(s+2)(s-1)}$$

if ROC is,

1. $-2 < Re\{s\} < 1$,
2. $Re\{s\} > 1$,
3. $Re\{s\} < -2$

Soln. :

Step 1 : The given equation is,

$$X(s) = -\frac{3}{(s+2)(s-1)}$$

The poles are at $P_1 = -2$ and $P_2 = 1$.

Step 2 : In PFE form we can write,

$$X(s) = \frac{A_1}{s+2} + \frac{A_2}{s-1}$$

We will calculate A_1 and A_2 as follows :

$$A_1 = (s+2)\cdot X(s)\Big|_{s=-2}$$

$$\therefore A_1 = (s+2)\cdot\frac{-3}{(s+2)(s-1)}\Bigg|_{s=-2}$$

$$\therefore A_1 = \frac{-3}{-2-1} = \frac{-3}{-3} = 1$$

and $A_2 = (s-1)\,X(s)\Big|_{s=1}$

$$\therefore A_2 = (s-1)\cdot\frac{-3}{(s+2)(s-1)}\Bigg|_{s=1}$$

$$\therefore A_2 = \frac{-3}{1+2} = \frac{-3}{3} = -1$$

Putting these values in the equation of X (s) we get,

$$X(s) = \frac{1}{s+2} - \frac{1}{s-1}$$

Now we will consider different cases depending on the given ROC.

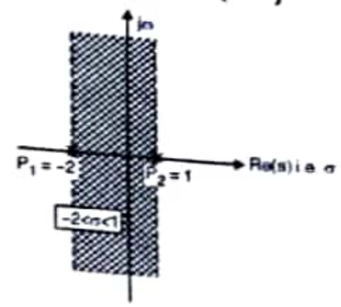
Case 1 : When ROC is $-2 < Re(s) < 1$

That means ROC is $-2 < \sigma < 1$

This ROC and the plot of poles is shown in Fig. P. 7.8.5.

Now inverse laplace of X (s) is,

$$x(t) = L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$



(H-598) **Fig. P. 7.8.5**

Consider first term only. That is $L^{-1}\left\{\frac{1}{s+2}\right\}$. Here the pole is at – 2. This pole is to the left of ROC. Hence corresponding time domain sequence is right handed. That means multiplied by u (t).

Now recall standard laplace transform pair,

$$e^{-at}u(t) \xleftrightarrow{LT} \frac{1}{s+a} \quad \text{for } \sigma > -a$$

$$\therefore L^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}u(t)$$

$$\therefore L^{-1}\left\{\frac{4}{s+2}\right\} = e^{-2t}u(t)$$

Now consider the second term. That is $L^{-1}\left\{\frac{1}{s-1}\right\}$. Here the pole is at + 1. This pole is to the right side of ROC. Hence corresponding time domain sequence is left handed. That means multiplied by u (– t). Recall standard laplace transform pair,

$$-e^{-at}u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \quad \text{for } \sigma < -a$$

$$\therefore L^{-1}\left\{\frac{1}{s-1}\right\} = L^{-1}\left\{\frac{1}{s+(-1)}\right\} = -e^{t}u(-t)$$

We have $x(t) = L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$

Thus for this case inverse laplace transform is,

$$x(t) = e^{-2t}u(t) + e^{t}u(-t)$$

Case 2 : When ROC is $Re(s) > 1$ i.e. $\sigma > 1$.

This ROC and the plot of poles is shown in Fig. P. 7.8.5(a).

Here both the poles lie at the L.H.S. of given ROC. Hence corresponding time domain terms are right sided. That means multiplied by u (t). Thus using standard laplace transform pairs we get,

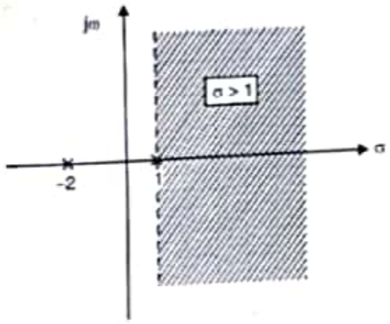
$$L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}u(t)$$

and $L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}u(t)$

$$\therefore x(t) = L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\therefore x(t) = e^{-2t}u(t) - e^{t}u(t)$$

$$\therefore x(t) = (e^{-2t} - e^{t})\,u(t)$$



(H-599) Fig. P. 7.8.5(a)

Case 3 :

When ROC is $R_c(s) < -2$ i.e. $\sigma < -2$. The ROC and position of poles are shown in Fig. P. 7.8.5(b).

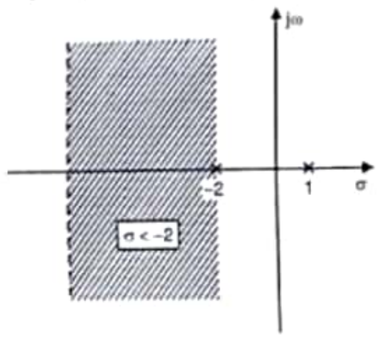
Both the poles lie at the R.H.S. of given ROC. Hence the corresponding time domain term are left handed. That means multiplied by $u(-t)$.

We have standard laplace transform pair.

$$-e^{-at}u(-t) \xleftrightarrow{LT} \frac{1}{s+a} \quad \text{for } \sigma < -a$$

$$\therefore L^{-1}\left\{\frac{1}{s+2}\right\} = -e^{-2t}u(-t)$$

$$\text{and} \quad L^{-1}\left\{\frac{1}{s-1}\right\} = L^{-1}\left\{\frac{1}{s+(-1)}\right\} = -e^{t}u(-t)$$



(H-600) Fig. P. 7.8.5(b)

$$\text{We have, } x(t) = L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\therefore x(t) = -e^{-2t}u(-t) + e^{t}u(-t)$$

$$\therefore x(t) = (-e^{-2t} + e^{t})u(-t)$$

Ex. 7.8.6 : Obtain inverse Laplace transform of the function :

$$X(s) = \frac{3s+7}{s^2-2s-3}$$

Write down and sketch possible ROCs. Find out inverse Laplace for all the possible ROCs. **MU - May 2015, 10 Marks**

Soln. :

$$X(s) = \frac{3s+7}{s^2-2s-3}$$

$$= \frac{3s+7}{(s-3)(s+1)}$$

$$X(s) = \frac{A_1}{s-3} + \frac{A_2}{s+1}$$

$$\text{Here } A_1 = \frac{9+7}{3+1} = 4$$

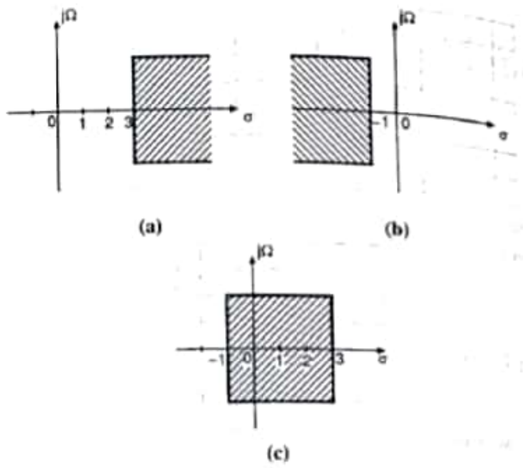
$$\text{and } A_2 = \frac{-3+7}{-1-3} = -1$$

$$\therefore X(s) = \frac{4}{s-3} - \frac{1}{s+1} \quad \text{...(1)}$$

The possible ROC are as follows :

1. $\sigma > 3$ 2. $\sigma < -1$ 3. $-1 < \sigma < 3$

The sketch of ROC is shown in Figs. P. 7.8.6(a), P. 7.8.6(b) and P. 7.8.6(c).



(H-1420) Fig. P. 7.8.6

Inverse Laplace for these possible ROC are as follows :

1. $x(t) = 4e^{3t} - e^{-t}$ 2. $x(t) = -4e^{3t} + e^{-t}$

3. $x(t) = -4e^{3t} - e^{-t}$

Ex. 7.8.7 : Obtain inverse Laplace transform of the function $X(s) = \frac{3s+7}{s^2-s-12}$ for following ROCs. Also comment on the stability and causality of the system for each of the ROC conditions. Support your answer with appropriate sketches of ROCs.

1. $Rs(s) > 4$
2. $Re(s) < -3$
3. $-3 < Re(s) < 4$

May 2016, 10 Marks

Soln. :

$$\text{Given : } X(s) = \frac{3s+7}{s^2-s-12}$$

$$\therefore X(s) = \frac{3s+7}{(s-4)(s+3)} = \frac{A_1}{s-4} + \frac{A_2}{s+3}$$

$$\text{Here } A_1 = \left.\frac{3s+7}{s+3}\right|_{s=4} = \frac{12+7}{4+3} = \frac{19}{7}$$

$$\text{and } A_2 = \left.\frac{3s+7}{s-4}\right|_{s=-3} = \frac{-9+7}{-3-4} = \frac{2}{7}$$

$$\therefore X(s) = \frac{19/7}{s-4} + \frac{2/7}{s+3}$$

1. $Re\{s\} > 4$

$$\therefore x(t) = \frac{19}{7} \cdot e^{4t}u(t) + \frac{2}{7}e^{-3t}u(t)$$

2. $Re\{s\} < -3$

$$\therefore x(t) = -\frac{19}{7}e^{4t}u(-t) - \frac{2}{7}e^{-3t}u(-t)$$

3. $-3 < Re\{s\} < 4$

$$\therefore x(t) = -\frac{19}{7}e^{4t}u(-t) + \frac{2}{7}e^{-3t}u(t)$$

Ex. 7.8.8 : Find the inverse Laplace transform of : $X(s) = \frac{s^3+2s^2+6}{s^2+3s}$

Soln. : $X(s) = \frac{s^3+2s^2+6}{s^2+3s}$

Here degree of numerator is greater than degree of denominator so we will perform the long division as follows.

$$\begin{array}{r|l} & s-1 \\ \hline s^2+3s & s^3+2s^2+6 \\ & s^3+3s^2 \\ & - \quad - \\ \hline & -s^2+6 \\ & -s^2-3s \\ & + \quad + \\ \hline & 3s+6 \end{array}$$

$$\therefore X(s) = (s-1) + \frac{3s+6}{s^2+3s}$$

$$\text{Let } X_1(s) = (s-1) \quad \text{and} \quad X_2(s) = \frac{3s+6}{s^2+3s}$$

First we will calculate inverse Laplace of $X_1(s)$

$$\text{We have, } \delta(t) \xleftrightarrow{L.T.} 1$$

According to differentiation property,

$$\frac{d}{dt}x(t) \xleftrightarrow{L.T.} s \cdot X(s)$$

$$\therefore \frac{d}{dt}\delta(t) \xleftrightarrow{L.T.} s \cdot 1$$

$$\therefore L^{-1}\{s\} = \frac{d}{dt}\delta(t) = \delta'(t)$$

$$\therefore x_1(t) = \delta'(t) - \delta(t)$$

$$\text{Now } X_2(s) = \frac{3s+6}{s^2+3s} = \frac{3s+6}{s(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+3}$$

$$\text{Here } A_1 = s \cdot \left.\frac{3s+6}{s(s+3)}\right|_{s=0} = \frac{6}{3} = 2$$

$$\text{and } A_2 = (s+3) \cdot \left.\frac{3s+6}{s(s+3)}\right|_{s=-3}$$

$$\therefore A_2 = \frac{-9+6}{-3} = 1$$

$$\therefore x_2(s) = \frac{2}{s} + \frac{1}{s+3}$$

$$\therefore x_2(t) = 2u(t) + e^{-3t}u(t)$$

$$\therefore x(t) = x_1(t) + x_2(t)$$

$$\therefore x(t) = \delta'(t) - \delta(t) + 2u(t) + e^{-3t}u(t)$$

Ex. 7.8.9 : Find inverse Laplace transform of following function : $X(s) = \frac{s^2+s-3}{s^2+3s+2}$

Soln. :

In this case the degree of numerator and denominator is same. So the function is not in the proper form. To obtain the function in proper form, we will perform the long division as follows :

$$\begin{array}{r|l} & 1 \\ \hline s^2+3s+2 & s^2+s-3 \\ & s^2+3s+2 \\ & - \quad - \quad - \\ \hline & -2s-5 \end{array}$$

$$\therefore X(s) = 1 - \frac{2s+5}{s^2+3s+2}$$

$$\text{Let } X_1(s) = \frac{2s+5}{s^2+3s+2} \quad \text{and} \quad X_2(s) = 1$$

$$\therefore X(s) = X_2(s) - X_1(s) \quad \text{...(1)}$$

Now we will obtain Laplace inverse of $X_1(s)$ using partial fraction expansion as follows :

$$X_1(s) = \frac{2s+5}{s^2+3s+2}$$

$$\therefore X_1(s) = \frac{2s+5}{(s+1)(s+2)}$$

$$\therefore X_1(s) = \frac{A_1}{s+1} + \frac{A_2}{s+2} \quad \text{...(2)}$$

$$\text{Here } A_1 = \left.\frac{2s+5}{(s+1)(s+2)} \cdot (s+1)\right|_{s=-1}$$

$$\therefore A_1 = \frac{-2+5}{1} = 3$$

$$\text{And } A_2 = \left.\frac{2s+5}{(s+1)(s+2)} \cdot (s+2)\right|_{s=-2}$$

$$\therefore A_2 = \frac{-4+5}{-2+1} = -1$$

$$\therefore X_1(s) = \frac{3}{s+1} - \frac{1}{s+2}$$

Now we have standard Laplace transform equation.

$$e^{-at}u(t) \xleftrightarrow{L.T.} \frac{1}{s+a}$$

$$\therefore x_1(t) = [3 \cdot e^{-t} - e^{-2t}]u(t)$$

Now consider $X_2(s)$

$$X_2(s) = 1$$

$$\text{We have } \delta(t) \xleftrightarrow{L.T.} 1$$

$$\therefore x_2(t) = \delta(t)$$

Putting these values in Equation (1) we get.

$$x(t) = \delta(t) - 3e^{-t}u(t) + 2e^{-2t}u(t)$$

Case 2 : When there are complex roots

If there are complex roots then the equation of X (s) can be expressed as,

$$X(s) = \frac{A_1}{s-\alpha-j\beta} + \frac{A_2}{s-\alpha+j\beta} + \ldots$$

Here the poles are at $P_1 = \alpha + j\beta$ and $P_2 = \alpha - j\beta$. That means the poles are complex conjugate of each other. But the procedure of obtaining the time domain signal is same as that of case-1.

Ex. 7.8.10 : Find inverse laplace transform of :

$$X(s) = \frac{2s+3}{s^2+s+1}; \text{ x (t) is right handed sequence.}$$

Soln. :

Step 1 : First we will obtain the roots of denominator.

The denominator term is $s^2 + s + 1$.

$$\therefore P_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \text{ and } P_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

Here the poles are complex conjugate of each other.

Step 2 : In the P.F.E. form the equation of X (s) can be written as,

$$X(s) = \frac{A_1}{s-P_1} + \frac{A_2}{s-P_2}$$

$$\therefore X(s) = \frac{A_1}{\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)} + \frac{A_2}{\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}$$

$$\text{Now } A_1 = (s-P_1)X(s)\Big|_{s=P_1}$$

$$\therefore A_1 = \frac{\left(s+\frac{1}{2}-\frac{j\sqrt{3}}{2}(2s+3)\right)}{\left(s+\frac{1}{2}-\frac{j\sqrt{3}}{2}\right)\left(s+\frac{1}{2}+\frac{j\sqrt{3}}{2}\right)}\Bigg|_{s=-\frac{1}{2}+\frac{j\sqrt{3}}{2}}$$

$$\therefore A_1 = \frac{-1+j\sqrt{3}+3}{-\frac{1}{2}+\frac{j\sqrt{3}}{2}+\frac{1}{2}+\frac{j\sqrt{3}}{2}} = \frac{2+j\sqrt{3}}{j\sqrt{3}}$$

$$\text{Similarly, } A_2 = (Z-P_2)X(s)\Big|_{s=P_2}$$

$$\therefore A_2 = \frac{\left(s+\frac{1}{2}+\frac{j\sqrt{3}}{2}(2s+3)\right)}{\left(s+\frac{1}{2}-\frac{j\sqrt{3}}{2}\right)\left(s+\frac{1}{2}+\frac{j\sqrt{3}}{2}\right)}\Bigg|_{s=-\frac{1}{2}-\frac{j\sqrt{3}}{2}}$$

$$\therefore A_2 = \frac{-1-j\sqrt{3}+3}{-\frac{1}{2}-\frac{j\sqrt{3}}{2}+\frac{1}{2}-\frac{j\sqrt{3}}{2}} = \frac{2-j\sqrt{3}}{-j\sqrt{3}}$$

Putting the values of A_1 and A_2 in the equation of X (s) we get,

$$X(s) = \frac{\left(\frac{2+j\sqrt{3}}{j\sqrt{3}}\right)}{s+\frac{1}{2}-\frac{j\sqrt{3}}{2}} + \frac{\left(\frac{2-j\sqrt{3}}{-j\sqrt{3}}\right)}{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}}$$

$$\therefore X(s) = \frac{1}{j\sqrt{3}}\left[\frac{2+j\sqrt{3}}{s+\frac{1}{2}-\frac{j\sqrt{3}}{2}} - \frac{2-j\sqrt{3}}{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}}\right]$$

Let us rearrange the equation of X (s) as follows

$$X(s) = \frac{1}{j\sqrt{3}}\left\{\frac{2}{s+\frac{1}{2}-\frac{j\sqrt{3}}{2}} \frac{j\sqrt{3}}{s+\frac{1}{2}-\frac{j\sqrt{3}}{2}} - \frac{2}{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}} + \frac{j\sqrt{3}}{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}}\right\}$$

$$\therefore X(s) = \frac{1}{j\sqrt{3}}\left\{2\left[\frac{1}{s+\frac{1}{2}-\frac{j\sqrt{3}}{2}} - \frac{1}{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}}\right] + j\sqrt{3}\left[\frac{1}{s+\frac{1}{2}-\frac{j\sqrt{3}}{2}} + \frac{1}{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}}\right]\right\}$$

$$\therefore X(s) = \frac{1}{j\sqrt{3}}\left\{2\left[\frac{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}-s-\frac{1}{2}+\frac{j\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right] + j\sqrt{3}\left[\frac{s+\frac{1}{2}+\frac{j\sqrt{3}}{2}+s+\frac{1}{2}-\frac{j\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]\right\}$$

$$\therefore X(s) = \frac{1}{j\sqrt{3}}\left\{2\left[\frac{j\sqrt{3}}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right] + j\sqrt{3}\left[\frac{2s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]\right\}$$

$$\therefore X(s) = \frac{2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}} + \frac{(2s+1)}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}$$

Step 3 : The laplace inverse of X (s) is,

$$x(t) = L^{-1}\left\{\frac{2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} + L^{-1}\left\{\frac{2s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

Consider the first term only. That is $L^{-1}\left\{\frac{2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$.

We have standard laplace transform pair,

$$e^{-at}\sin\omega t \xleftrightarrow{L.T.} \frac{\omega}{(s+a)^2+\omega^2} \quad \text{for } \sigma > -a$$

Let us rearrange this term as follows,

$$L^{-1}\left\{\frac{2}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} = \left\{\frac{2\cdot\frac{\sqrt{3}}{2}\cdot\frac{2}{\sqrt{3}}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{4}{\sqrt{3}}L^{-1}\left\{\frac{\frac{\sqrt{3}}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= \frac{4}{\sqrt{3}}\cdot e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}\cdot t$$

Now consider the second term. That is,

$$L^{-1}\left\{\frac{2s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\}$$

We have standard Laplace transform pair,

$$e^{-at}\cos\omega t \xleftrightarrow{L.T.} \frac{s+a}{(s+a)^2+\omega^2} \quad \text{for } \sigma > -a$$

Let us rearrange the second term as follows,

$$L^{-1}\left\{\frac{2s+1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right\} = 2L^{-1}\left\{\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right\}$$

$$= 2e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}\cdot t$$

Thus complete time domain expression is,

$$x(t) = \frac{4}{\sqrt{3}}\cdot e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}\cdot t + 2e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$$

Case 3 : When there are repeated poles

When there are repeated poles then we have to add some extra terms in the equation of X (s). For such extra terms the coefficients A_K are calculated by taking derivative of X (s). Let as solve some numericals related to repeated poles.

Ex. 7.8.11 : Find inverse laplace transform of :

$$X(s) = \frac{s^2}{(s-1)\left(s-\frac{1}{2}\right)^2}$$

for the causal time domain signal.

Soln. :

Step 1 : The given equation is,

$$X(s) = \frac{s^2}{(s-1)\left(s-\frac{1}{2}\right)^2}$$

There is a single pole at + 1; while there are repeated poles at $+\frac{1}{2}$.

Step 2 : For the repeated poles; in partial fraction expansion form equation of X (s) can be written as,

$$X(s) = \frac{A_1}{s-1} + \frac{A_2}{s-\frac{1}{2}} + \frac{A_3}{\left(s-\frac{1}{2}\right)^2}$$

Note that the second term in the equation of X (s) is an added term. The coefficients A_1 and A_3 are calculated using earlier method. But extra coefficient A_2 is calculated by taking the derivative as follows :

$$A_2 = \frac{d}{ds}\left[\left(s-\frac{1}{2}\right)^2 X(s)\right]\Bigg|_{s=\frac{1}{2}}$$

$$\therefore A_2 = \frac{d}{ds}\left[\left(s-\frac{1}{2}\right)^2\frac{s^2}{(s-1)\left(s-\frac{1}{2}\right)^2}\right]\Bigg|_{s=\frac{1}{2}}$$

$$\therefore A_2 = \frac{d}{ds}\left[\frac{s^2}{s-1}\right]\Bigg|_{s=\frac{1}{2}}$$

$$\therefore A_2 = \left[s^2\cdot\frac{d}{ds}(s-1)^{-1} + (s-1)^{-1}\cdot\frac{d}{ds}s^2\right]\Bigg|_{s=\frac{1}{2}}$$

$$\therefore A_2 = \left[s^2(-1)\times(s-1)^{-2} + (s-1)^{-1}\cdot 2s\right]\Bigg|_{s=\frac{1}{2}}$$

$$\therefore A_2 = \left[\frac{1}{4}(-1)\times\frac{1}{\left(\frac{1}{2}-1\right)^2} + \frac{1}{\left(\frac{1}{2}-1\right)}\cdot 2\cdot\frac{1}{2}\right]$$

$$\therefore A_2 = -3$$

$$\text{Now } A_1 = (s-1)\cdot X(s)\Big|_{s=1}$$

$$\therefore A_1 = (s-1)\cdot\frac{s^2}{(s-1)\left(s-\frac{1}{2}\right)^2}\Bigg|_{s=1}$$

$$\therefore A_1 = \frac{s^2}{\left(s-\frac{1}{2}\right)^2}\Bigg|_{s=1}$$

$$\therefore A_1 = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4$$

$$\text{And } A_3 = \left(s-\frac{1}{2}\right)^2\cdot X(s)\Big|_{s=\frac{1}{2}}$$

$$\therefore A_3 = \frac{\left(s-\frac{1}{2}\right)^2\cdot s^2}{(s-1)\left(s-\frac{1}{2}\right)^2}\Bigg|_{s=\frac{1}{2}}$$

$$\therefore A_3 = \frac{s^2}{s-1}\Bigg|_{s=\frac{1}{2}} = \frac{1/4}{\frac{1}{2}-1} = -\frac{1}{2}$$

Putting the values of A_1, A_2 and A_3 in the equation of X (s),

$$X(s) = \frac{4}{s-1} - \frac{3}{s-\frac{1}{2}} - \frac{1/2}{\left(s-\frac{1}{2}\right)^2}$$

Step 3 : The laplace inverse of X (s) can be written as,

$$x(t) = 4L^{-1}\left\{\frac{1}{s-1}\right\} - 3L^{-1}\left\{\frac{1}{s-\frac{1}{2}}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{(s-1/2)^2}\right\}$$

For first two terms we have standard laplace transform pair,

$$e^{-at}u(t) \xleftrightarrow{L.T.} \frac{1}{s-a} \quad \text{For } \sigma > a$$

$$\therefore L^{-1}\left\{\frac{1}{s-1}\right\} = e^{t}u(t)$$

$$\text{and } L^{-1}\left\{\frac{1}{s-\frac{1}{2}}\right\} = e^{\frac{1}{2}t}u(t)$$

For the third term use the standard laplace transform pair,

$$t^n\cdot e^{at} \xleftrightarrow{L.T.} \frac{n!}{(s-a)^{n+1}}$$

We have the term,

$$L^{-1}\left\{\frac{1}{\left(s-\frac{1}{2}\right)^2}\right\} = L^{-1}\left\{\frac{1!}{\left(s-\frac{1}{2}\right)^{1+1}}\right\}$$

$$= te^{\frac{1}{2}t}$$

Thus the time domain sequence can be written as.

$$x(t) = 4e^{t}u(t) - 3e^{\frac{1}{2}t}u(t) - \frac{1}{2}te^{\frac{1}{2}t}u(t)$$

Ex. 7.8.12 : Find the inverse Laplace transform of $\frac{s-2}{s(s+1)^3}$

Dec. 2014. 5 Marks

Soln. :

Given : $X(s) = \frac{s-2}{s(s+1)^3} = \frac{(s+1)-3}{s(s+1)^3}$

$$= \frac{1}{s(s+1)^2} - \frac{3}{s(s+1)^3} \quad ...(1)$$

Consider first term only.

$$L^{-1}\left[\frac{1}{s(s+1)^2}\right]$$

Now $L^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}\cdot t$

$$\therefore L^{-1}\left\{\frac{1}{s(s+1)^2}\right\} = \int_0^t e^{-t}\cdot t\,dt = \left[t(-e^{-t}) - \int -e^{-t}dt\right]_0^t$$

$$= \left[-te^{-t} - e^{-t}\right]_0^t$$

$$= -te^{-t} - e^{-t} + 1 \quad ...(2)$$

Consider second term.

$$-3L^{-1}\left[\frac{1}{(s+1)^3}\right] = -3e^{-t}\cdot L^{-1}\left\{\frac{1}{s^3}\right\} = -3/2\int_0^t e^{-t}\cdot t^2\,dt$$

$$= -3/2\left[t^2(-e^{-t}) - 2t(e^{-t}) + 2(-e^{-t})\right]_0^t$$

$$= -3\left[1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right)\right] \quad ...(3)$$

$$\therefore x(t) = -te^{-t} - e^{-t} + 1 + 3\left[1 - e^{-t}\left(\frac{t^2}{2} + t + 1\right)\right]$$

Ex. 7.8.13 : Find the inverse Laplace transform of the following :

$$X(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}$$

May 2015. 5 Marks

Soln. :

$$X(s) = \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3}:$$

$$\therefore \frac{5s^2 - 15s - 11}{(s+1)(s-2)^3} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{(s-2)^2} + \frac{D}{(s-2)^3}$$

$$\therefore 5s^2 - 15s - 11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1) \quad ...(1)$$

Put $s = -1$,

$$\therefore 5 + 15 - 11 = A(-3)^2 \qquad \therefore A = -\frac{1}{3}$$

Put $s = 2$

$$\therefore 20 - 30 - 11 = D(3) \qquad \therefore D = -7$$

Equating coefficients of s^3 on both sides of Equation (1).

$$0 = As^3 + Bs^3$$

$$\therefore A + B = 0$$

$$\therefore A = -B \qquad \therefore B = \frac{1}{3} \quad ...(2)$$

Now equating coefficients of s^2.

$$5s^2 = A(-6s^2) + B(-2s^2 + s^2) + C(s^2)$$

$$\therefore 5 = -6A - B + C$$

$$\therefore C = \frac{10}{3}$$

Thus, $X(s) = \frac{-\frac{1}{3}}{s+1} + \frac{\frac{1}{3}}{s-2} + \frac{\frac{10}{3}}{(s-2)^2} - \frac{7}{(s-2)^3}$

Taking inverse Laplace.

$$x(t) = -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + \frac{10}{3}e^{2t}\cdot t - 7e^{2t}\cdot\frac{t^2}{2}$$

Ex. 7.8.14 : Obtain inverse laplace transform of the function. Write down and sketch possible ROCs.

$$X(s) = \frac{8}{(s+2)^3(s+4)}$$

Dec. 2015. 10 Marks

Soln. :

Given : $F(s) = \frac{8}{(s+2)^3(s+4)}$

This is the case of repeated roots. So, in P.F.E it is expressed as,

$$F(s) = \frac{A_1}{s+2} + \frac{A_2}{(s+2)^2} + \frac{A_3}{(s+2)^3} + \frac{A_4}{(s+4)}$$

Here, $A_1 = \frac{d^2}{ds^2}\left[\frac{8}{s+4}\right]\Bigg|_{s=-2} = \frac{d}{ds}\left[-\frac{8}{(s+4)^2}\right]\Bigg|_{s=-2}$

$$\therefore A_1 = \frac{16}{(s+4)^3}\Bigg|_{s=-2} = \frac{16}{(-2+4)^3} = 2$$

$$A_2 = \frac{d}{ds}\left[\frac{8}{s+4}\right]\Bigg|_{s=-2}$$

$$= \frac{-8}{(s+4)^2}\Bigg|_{s=-2} = \frac{8}{(-2+4)^2} = -2$$

$$A_3 = \frac{8}{s+4}\Bigg|_{s=-2} = \frac{8}{-2+4} = 4$$

and $A_4 = \frac{8}{(s+2)^3}\Bigg|_{s=-4} = \frac{8}{(-4+2)^3} = -1$

$$\therefore F(s) = \frac{2}{s+2} - \frac{2}{(s+2)^2} + \frac{4}{(s+2)^3} - \frac{1}{s+4}$$

We have formula, $L^{-1}\left\{\frac{n!}{(s-a)^{n+1}}\right\} = t^n e^{at}$

Considering causal sequence inverse laplace is, $F(t) = 2e^{-2t}u(t) - 2te^{-2t}u(t) + 4t^2e^{-2t}u(t) - e^{-4t}u(t)$.

ROCs :

The possible ROCs are

1. $\sigma < -2$, 2. $\sigma > -4$,
3. $-2 < \sigma < -4$

These ROCs are shown in Fig. P. 7.8.14(a)(a), (b) and (c) respectively.

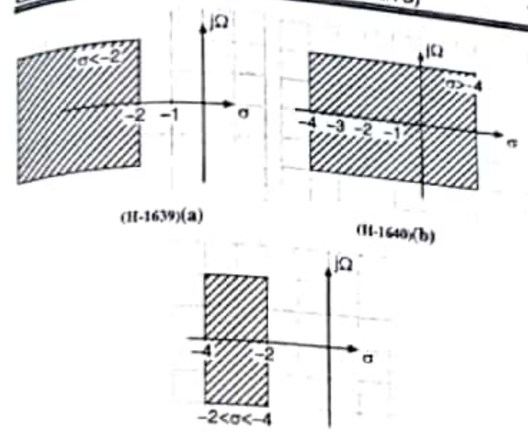


Fig. P. 7.8.14(a)

Ex. 7.8.15 : Determine inverse Laplace Transform of

$$X(s) = \frac{3s^2 + 8s + 23}{(s+3)(s^2 + 2s + 10)}$$

Dec. 2016. 10 Marks

Soln. :

Given. $X(s) = \frac{3s^2 + 8s + 23}{(s+3)(s^2 + 2s + 10)}$

$$= \frac{3s^2 + 8s + 23}{(s+3)(s+1-3j)(s+1+3j)}$$

$$\therefore X(s) = \frac{A_1}{s+3} + \frac{A_2}{(s+1-3j)} + \frac{A_3}{(s+1+3j)}$$

Here $A_1 = \frac{3(-3)^2 + 8(-3) + 23}{(-3+1-3i)(-3+1+3i)} = \frac{26}{13} = 2$

$$A_2 = \frac{3(-1+3j)^2 + 8(-1+3j) + 23}{(-1+3j+3)(-1+3j+1+3j)}$$

$$= \frac{3(-8-6j) - 8 + 24j + 23}{-18 + 12j} = 0.5$$

Similarly $A_3 = 0.5$

$$\therefore X(s) = \frac{2}{s+3} + \frac{0.5}{(s+1-3j)} + \frac{0.5}{(s+1+3j)}$$

Taking inverse laplace

$$x(t) = 2e^{-3t}u(t) + 0.5e^{-(1-3j)t}u(t) + 0.5e^{-(1+3j)t}u(t)$$

$$= 2e^{-3t}u(t) + 0.5e^{-t}\cdot e^{3jt}u(t) + 0.5e^{-t}\cdot e^{-3jt}u(t)$$

$$= 2e^{-3t}u(t) + 0.5e^{-t}\left[e^{3jt} + e^{-3jt}\right]u(t)$$

Use identity. $e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$$\therefore x(t) = 2e^{-3t}u(t) + e^{-t}\cdot\cos(3t)\,u(t)$$

Ex. 7.8.16 : The impulse response of a CT system is given below. Determine the unit step response of the system using convolution theorem of Laplace Transform.

$$h(t) = a(t+2) + u(t-2)$$

Dec. 2016. 10 Marks

Soln. :

Given : $h(t) = u(t+2) + u(t-2)$

$$\therefore H(s) = e^{2s}\cdot\frac{1}{s} + e^{-2s}\cdot\frac{1}{s}$$

Now unit step response means applied input is $x(t) = u(t)$

$$\therefore X(s) = \frac{1}{s}$$

We have, $H(s) = \frac{Y(s)}{X(s)}$

$$\therefore Y(s) = H(s)\cdot X(s) = \left(e^{2s}\cdot\frac{1}{s} + e^{-2s}\cdot\frac{1}{s}\right)\cdot\frac{1}{s}$$

$$Y(s) = e^{2s}\frac{1}{s^2} + e^{-2s}\cdot\frac{1}{s^2}$$

Laplace transform of ramp is.

$$L[t\,u(t)] = \frac{1}{s^2}$$

According to shifting property of Laplace,

If $x(t) \longrightarrow x(s)$ then

$$x(t - t_0) \longrightarrow e^{-st_0}x(s)$$

Thus inverse laplace of $Y(s)$ is.

$$y(t) = (t+2)u(t+2) + (t-2)u(t-2)$$

This is unit step response of system.

7.9 Relationship between Laplace Transform and C.T.F.T. :

MU - May 04. May 05. May 07. May 11

University Questions

Q. 1 Find the relationship between Laplace Transform and Fourier Transform.

(May 04, 5 Marks, May 11, 7 Marks)

Q. 2 Derive the relation between Laplace transform and Fourier transform. **(May 05, 4 Marks)**

Q. 3 Derive the relationship between Laplace Transform and Fourier Transform.

(May 07, 6 Marks, Dec. 16, 4 Marks)

F.T of CT signal is a special case of Laplace transform. The relation between L.T and F.T are explained below,

We have, $X(s) = L\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt$

where, $s = \sigma + j\omega$ is complex variable.

i.e. $X(\sigma + j\omega) = \int_{-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$

$$\therefore X(\sigma + j\omega) = \int_{-\infty}^{\infty}\left[x(t)e^{-\sigma t}\right]e^{-j\omega t}dt$$

But we know the definition of F.T.

$$F.T\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$\therefore X(\sigma + j\omega) = F.T\{x(t)e^{-\sigma t}\}$

i.e. L.T of x (t) is equal to F.T. of C.T. signal $x(t)e^{-\sigma t}$.

If s = 0, then L.T reduces to F.T.

Review Questions

Q. 1 Explain the necessity of laplace transform.

Q. 2 Obtain laplace transform of unit step and unit impulse.

Q. 3 State and prove time shifting property of laplace transform.

Q. 4 List out the properties of laplace transform.

Q. 5 State and prove convolution property of laplace transform.

Q. 6 State initial and final value theorem.

Q. 7 Define the transfer function. How to obtain impulse response from transfer function ?

Q. 8 Obtain the equation of system response to complex exponential input signal.

Q. 9 What do you mean by unilateral laplace transform ? How it is used to calculate initial and final values ?

Q. 10 Derive the relationship between laplace and fourier transform.

7.10 Examples for Practice :

Ex. 7.10.1 : Find Laplace transform of following signals. Draw ROC in each case.

$x(t) = t^2 u(t-1)$

where * is for convolution.

May 03. 8 Marks

Ans. :

$$\therefore X(s) = \frac{2}{s^3} \cdot \frac{e^{-s}}{s} = \frac{2e^{-s}}{s^3} \quad \text{...Ans.}$$

Ex. 7.10.2 : Using Laplace transform find output of the system of input is unit step function and impulse response is given by :

$h(t) = e^{-t} \sin(20\pi t)\, u(t)$ **Dec. 03. 8 Marks**

Ans. :

$$\therefore y(s) = \frac{1}{s} \cdot \frac{20\pi}{(s+1)^2 + (20\pi)^2} \quad \text{...Ans.}$$

Ex. 7.10.3 : Find the initial and final values, if they exist, of the signals with transforms as given :

$$X(s) = \frac{s}{(s+2)(s+3)^2}$$

May 05. 4 Marks

Ans. :

$X(0^+) = 0$ ...Ans.

$X(\infty) = 0$ ...Ans.

Ex. 7.10.4 : Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s+2)(s+3)^2}$$

1. $-3 < Re(s) < -2$
2. $Re(s) > -2$

May 05. 6 Marks

Ans. :

1. $-3 < Re(s) < -2$:

$X(t) = 2e^{-3t}u(t) - 2e^{-2t}u(t) + 3t \cdot e^{-3t}u(t)$ ...Ans.

2. $Re(s) > -2$:

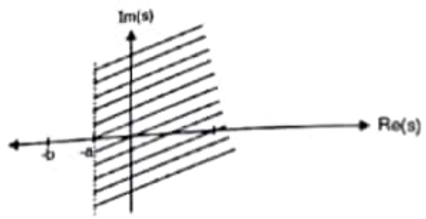
$X(t) = -2e^{-2t}u(t) + 2e^{-3t}u(t) + 3t \cdot e^{-3t}u(t)$ ...Ans.

Ex. 7.10.5 : If $f(t) = \begin{cases} e^{-at} & t > 0 \\ e^{-bt} & t < 0 \end{cases}$ show ROC of its Laplace transform.

May 05. 6 Marks

Ans. :

$$\therefore f(s) = -\frac{e^{+(s+b)}}{(s+b)} + \frac{1}{s+a}$$



(II-921) Fig. P. 7.10.5

Ex. 7.10.6 : Find the Laplace transform of $f(t) = e^{-4t}u(-t) + e^{-6t}u(t)$. Does the Laplace transform exist. Show the ROC.

May 07. 8 Marks

Ans. :

$$\therefore X(s) = \frac{1}{s-4} + \frac{1}{s+6} \quad ROC: -6 < \sigma < 4 \quad \text{...Ans.}$$

Ex. 7.10.7 : If $F(s) = \frac{1}{(s+2)(s+3)}$

1. Determine the final value by application of final value theorem.
2. Verify the result by finding f (t).

May 07. 8 Marks

Ans. :

$$\therefore \lim_{t \to \infty} f(t) = 0 \quad \text{...Ans.}$$

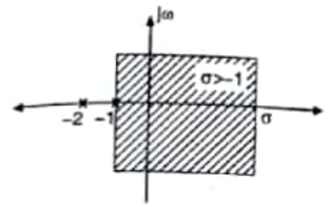
Ex. 7.10.8 : Find the ROC of the given signal

$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$.

May 10. 5 Marks

Ans. :

$$X(s) = 3 \cdot \frac{1}{s+2} - 2 \cdot \frac{1}{s+1}$$



(II-1198) Fig. P. 7.11.8

Ex. 7.10.9 : Find inverse Laplace transform of following function

$$X(s) = \frac{s^2 + s - 3}{s^2 + 3s + 2}$$

Ans. :

$x(t) = \delta(t) - 3e^{-t}u(t) + 2e^{-2t}u(t)$ ...Ans.

Ex. 7.10.10 : Find the final value of y (t) and y_1 (t) if its Laplace transform Y (s) is given by,

1. $Y(s) = \frac{s+3}{s^2 + 4s + 5}$ 2. $Y_1(s) = \frac{1}{s} Y(s)$

Ans. :

1. 0
2. $\frac{3}{5}$

...Ans.

Ex. 7.10.11 : Find initial and final value of signal

$$X(s) = \frac{s+10}{s^2 + 2s + 2}$$

Ans. :

$\therefore x(0^+) = 1$ and $\therefore \lim_{t \to \infty} x(t) = 0$ ...Ans.

Ex. 7.10.12 : Find the function x (t) if its Laplace transform is given by, $X(s) = \frac{10s}{(s+1)(s+3)} \cdot e^{-s}$

Ans. :

$$-5e^{-(t-1)}u(t) + 15e^{-3(t-1)}u(t)$$

□□□

CHAPTER 8

Module 4

Z-Transform

Syllabus :

Need of Z-Transform, definition of unilateral and bilateral Z-Transform. Z-Transform of finite and infinite duration sequences, properties, Inverse Z-Transform, relation between discrete time Fourier Transform and Z-Transform, Z-Transform of standard signals, ROC for ZT, plotting poles and zeros of transfer function. Analysis of discrete time LTI systems using Z-Transform : Transfer Function, causality and stability of systems, frequency response (impulse and step), relation between Laplace Transform and Z-Transform.

8.1 Need of Z-transform :

Fourier transform provides a valuable technique for frequency domain analysis and design of continuous time signals and LTI systems. While the Z-transform provides a valuable technique for analysis and design of discrete time signals and discrete time LTI systems.

The Z-transform has real and imaginary parts like fourier transform. A plot of imaginary part versus real part is called as Z-plane or complex Z-plane. The poles and zeros of discrete time system are plotted in the complex Z-plane. The pole-zero plot is main characteristic of discrete time LTI systems. We can also check the stability of system using pole-zero plot.

8.1.1 Advantages of Z-transform :

The importance of Z-transform is as follows :

1. Discrete time signals and LTI systems can be completely characterized using Z-transform.
2. The stability of LTI system can be determined using Z-transform.
3. Mathematical calculations are reduced using Z-transform. For example convolution operation is transformed into simple multiplication operation.
4. By calculating Z-transform of given signal, DFT and FT can be determined.
5. Entire family of digital filters can be obtained from one proto–type design using Z-transform.
6. The solution of differential equations can be simplified using Z-transform.

8.2 Definition of Uniliteral and Bilateral Z-transform :

There are two types of Z-transform :

1. Single sided Z-transform
2. Double sided Z-transform.

1. Single sided Z-transform :

Definition :

A single sided Z-transform of discrete time signal $x(n)$ is defined as.

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \text{...(8.2.1)}$$

Here "Z" is a complex variable. In Equation (8.2.1), limits of summation are from 0 to ∞. So while expanding the summation we will put only positive values of n (from n = 0 to n = ∞). So this is single sided or one-sided Z-transform.

2. Double sided Z-transform :

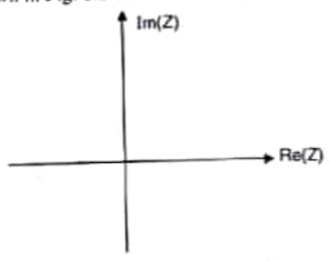
Definition : A double sided Z-transform of discrete time signal $x(n)$ is defined as,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{...(8.2.2)}$$

While expanding the summation, we will put both positive and negative values of 'n'. Thus this is double-sided Z- transform.

What is Z-domain ?

Observe Equations (8.2.1) and (8.2.2). Here we are transforming discrete time sequence $x(n)$ into $X(Z)$. We know that discrete time sequence $x(n)$ is drawn by plotting amplitude versus 'n'. This is called as discrete domain. In case of Z-transform we are plotting real part of Z on X-axis and imaginary part of Z on Y-axis as shown in Fig. 8.2.1. This is called as Z-domain.



(H-985) Fig. 8.2.1 : Z-domain

How to denote Z-transform ?

The relationship between $x(n)$ and $X(Z)$ is indicated as follows :

$$x(n) \xrightarrow{Z} X(Z) \quad \text{...(8.2.3)}$$

Note that $X(Z)$ is 'Z' transform of $x(n)$. Always, when Z-transform of sequence is obtained then it is denoted by capital letter, 'X'(Z). Here the arrow is bidirectional. This is because we can also obtain $x(n)$ from $X(Z)$ using inverse Z-transform.

The 'Z' transform of $x(n)$ is also denoted as,

$$X(Z) = Z\{x(n)\} \quad \text{...(8.2.4)}$$

Here $x(n)$ and $X(Z)$ are called as Z-transform pairs.

8.2.1 Region of Convergence (ROC) (Meaning and Significance) :

In case of Z-transform; the limits of summation are from $n = -\infty$ to $n = \infty$. So if we will expand this summation then we will get an infinite power series. This infinite power series (that means Z-transform) will exist only for those values of Z for which the series attains a finite value. (That means the series converges).

Definition of ROC :

The region of convergence (ROC) of $X(Z)$ is set for all the values of Z for which $X(Z)$ attains a finite value. Everytime when we find the Z-transform, we must indicate its ROC.

Significance of ROC :

1. ROC will decide whether a system (filter) is stable or unstable.
2. ROC also determines the type of sequence that means :
   1. Causal or non-causal
   2. Finite or infinite.

8.2.1.1 Solved Examples using Definition of Z :

Ex. 8.2.1 : Obtain the Z-transform of following finite duration sequences :

1. $x(n) = \{1, 2, 4, 5, 0, 7\}$
2. $x(n) = \{1, 2, 4, 5, 0, \underset{\uparrow}{7}\}$
3. $x(n) = \{1, 2, 4, \underset{\uparrow}{5}, 0, 7\}$

Soln. :

1. Here arrow is not mentioned. So by default it is at first position.

$$\therefore \; x(n) = \{\underset{\uparrow}{1}, 2, 4, 5, 0, 7\} \quad \text{...(1)}$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{...(2)}$$

But $x(n)$ is present from n = 0 to n = 5. The different values are as follows :

$$x(0) = 1$$
$$x(3) = 5$$
$$x(1) = 2$$
$$x(4) = 0$$
$$x(2) = 4$$
$$x(5) = 7$$

Thus we will change the limits of summation from n = 0 to n = 5.

$$\therefore \; X(Z) = \sum_{n=0}^{5} x(n) Z^{-n} \quad \text{...(3)}$$

Expanding the summation we get,

$$\therefore X(Z) = x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + x(4)Z^{-4} + x(5)Z^{-5}$$

$$\therefore X(Z) = 1Z^{0} + 2Z^{-1} + 4Z^{-2} + 5Z^{-3} + 0 \cdot Z^{-4} + 7Z^{-5}$$

But $Z^{0} = 1$

$$X(Z) = 1 + \frac{2}{Z} + \frac{4}{Z^2} + \frac{5}{Z^3} + \frac{7}{Z^5} \quad \text{...(4)}$$

ROC : This is finite duration sequence. For this sequence we will check for which values of 'Z': the value of $X(Z)$ becomes infinity. The simple method to decide the ROC is, put Z = 0 and Z = ∞ in the equation of $X(Z)$.

1. Putting Z = 0 in Equation (4) we get,

$$X(Z) = 1 + \frac{2}{0} + \frac{4}{0} + \frac{5}{0} + \frac{7}{0} = 1 + \infty = \infty$$

We know that $\frac{2}{0}, \frac{4}{0}$ etc. are equal to ∞. And 1 + ∞ is again ∞. Thus Z = 0 is not valid as it results $X(Z) = \infty$.

2. Putting Z = ∞ in Equation (4) we get,

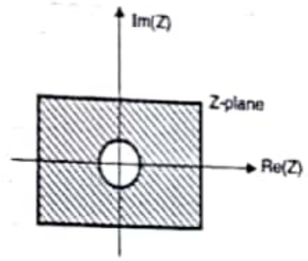
$$X(Z) = 1 + \frac{2}{\infty} + \frac{4}{\infty} + \frac{5}{\infty} + \frac{7}{\infty} = 1 + 0 = 1$$

Here $\frac{2}{\infty}, \frac{4}{\infty} = 0$. Thus putting Z = ∞ we get finite value. So Z = ∞ is allowed.

Now ROC of given sequence is written as follows :

ROC : Entire Z-plane except | Z | = 0

This ROC is shown in Fig. P. 8.2.1(a).



(H-986) Fig. P. 8.2.1(a) : ROC

Note : This sequence is causal, since $x(n)$ is present only for positive values of n. Thus for a causal finite duration sequence ROC is entire Z-plane except | Z | = 0.

2. Given sequence is,

$$x(n) = \{\underset{\uparrow}{1}, 2, 4, 5, 0, 7\} \quad \text{...(5)}$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(6)$$

Note that the range of given sequence is from n = – 5 to n = 0.

The different values of x (n) are :

$x(-5) = 1 \qquad x(-2) = 5$

$x(-4) = 2 \qquad x(-1) = 0$

$x(-3) = 4 \qquad x(0) = 7$

We will change the limits of summation from n = – 5 to n = 0.

$$\therefore \; X(Z) = \sum_{n=-5}^{0} x(n) Z^{-n} \quad ...(7)$$

Expanding the summation we get,

$$X(Z) = x(-5)Z^{+5} + x(-4)Z^{+4} + x(-3)Z^{+3} + x(-2)Z^{+2} + x(-1)Z^{+1} + x(0)Z^{0}$$

$$\therefore \; X(Z) = 1Z^{5} + 2Z^{4} + 4Z^{3} + 5Z^{2} + 0Z^{1} + 7Z^{0}$$

$$\therefore \; X(Z) = Z^{5} + 2Z^{4} + 4Z^{3} + 5Z^{2} + 7 \quad ...(8)$$

ROC : We will determine ROC by putting Z = 0 and Z = ∞.

1. Putting Z = 0 in Equation (8) we get.

   $X(Z) = 0 + 0 + 0 + 0 + 7 = 7$

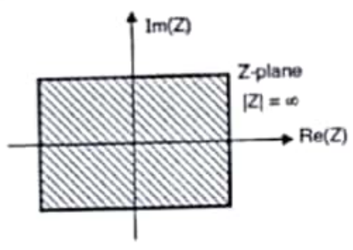
   This is a finite value. So Z = 0 is allowed.

2. Putting Z = ∞ in Equation (8) we get.

   $X(Z) = \infty + \infty + \infty + \infty + 7 = \infty + 7 = \infty$

   This is because ∞ + 7 etc. = ∞. So Z = ∞ is not allowed.

Thus ROC is **entire Z-plane except | Z | = ∞**. This ROC is shown in Fig. P. 8.2.1(b).



(II-987) **Fig. P. 8.2.1(b) : ROC**

> **Note :** The given sequence x (n) is anticausal. This is because x (n) is present only for negative values of 'n'. Thus for anticausal finite duration sequence, ROC is entire Z-plane except | Z | = ∞.

3. Given sequence is.

$$x(n) = \{1, 2, 4, 5, 0, 7\} \quad ...(9)$$
$$\uparrow$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(10)$$

But x (n) is present from n = – 2 to n = 3. The different values of x (n) are as follows :

$x(-2) = 1 \qquad x(1) = 5$

$x(-1) = 2 \qquad x(2) = 0$

$x(0) = 4 \qquad x(3) = 7$

We will change the limits of summation from n = – 2 to n = 3.

$$\therefore \; X(Z) = \sum_{n=-2}^{n=3} x(n) Z^{-n} \quad ...(11)$$

Expanding the summation we get,

$$X(Z) = x(-2)Z^{+2} + x(-1)Z^{+1} + x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$\therefore X(Z) = 1Z^{2} + 2Z^{1} + 4Z^{0} + 5Z^{-1} + 2Z^{-2} + 7Z^{-3}$$

$$\therefore \; X(Z) = Z^{2} + Z + 4 + \frac{5}{Z} + \frac{2}{Z^{2}} + \frac{7}{Z^{3}} \quad ...(12)$$

ROC :

We will determine ROC by putting Z = 0 and Z = ∞ in Equation (12).

1. Putting Z = 0 in Equation (12) we get,

   $$X(Z) = 0 + 0 + 4 + \frac{5}{0} + \frac{2}{0} + \frac{7}{0} = 4 + \infty = \infty$$

   This is because $\frac{5}{0}, \frac{2}{0}$ and $\frac{7}{0} = \infty$

   Thus Z = 0 is not allowed.

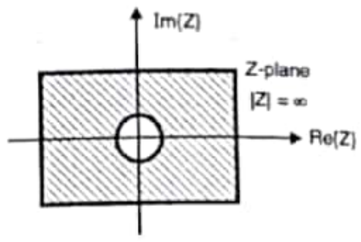
2. Putting Z = ∞ in Equation (12) we get,

   $$X(Z) = \infty + \infty + 4 + \frac{5}{\infty} + \frac{2}{\infty} + \frac{7}{\infty} = \infty$$

   Thus Z = ∞ is not allowed.

ROC : ROC is entire Z-plane except | Z | = 0 and | Z | = ∞.

This ROC is shown in Fig. P. 8.2.1(c).



(II-988) **Fig. P. 8.2.1(c) : ROC**

> **Note :** The given sequence is a bothsided sequence. This is because x (n) is present for both positive and negative values of 'n'. Thus for bothsided finite duration sequence the ROC is entire Z-plane except | Z | = 0 and | Z | = ∞.

Ex. 8.2.2 : Determine the Z transform and the ROC of the discrete time signal.

$$x[n] = \{2, 10, 1, \underset{\uparrow}{2}, 2, 5, 7, 2\}$$

MU - May 2014, 5 Marks

Soln. :

1. Given. $x[n] = \{2, 10, 1, \underset{\uparrow}{2}, 5, 7, 2\}$

According to definition of Z transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$\therefore \; X(Z) = \sum_{n=-3}^{3} x(n) Z^{-n}$$

$$\therefore X(Z) = x(-3)Z^{3} + x(-2)Z^{2} + x(-1)Z^{1} + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

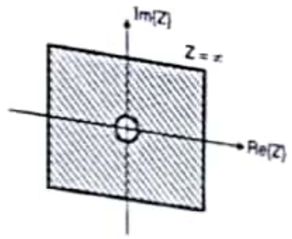
$$\therefore \; X(Z) = 2Z^{3} + 10Z^{2} + Z + 2 + 5Z^{-1} + 7Z^{-2} + 2Z^{-3}$$

$$= 2Z^{3} + 10Z^{2} + Z + 2 + \frac{5}{Z} + \frac{7}{Z^{2}} + \frac{2}{Z^{3}}$$

ROC :

1. Put Z = 0, ∴ X (Z) = ∞
2. Put Z = ∞, ∴ X (Z) = ∞

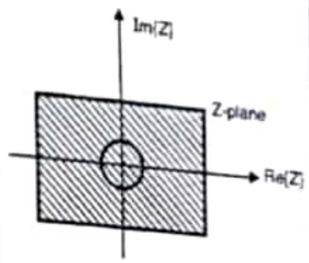
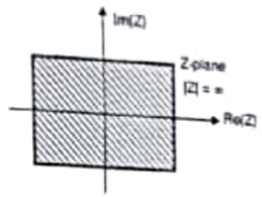
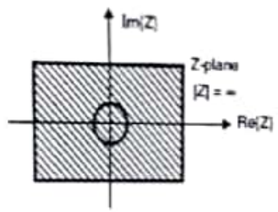
Thus ROC is entire Z plane except Z = 0 and Z = ∞. It is shown in Fig. P. 8.2.2.



(II-1520) **Fig. P. 8.2.2**

Summary of ROC : Table 8.2.1 shows the ROC for different finite duration sequences.

Table 8.2.1

Sr. No.	Finite duration sequence	ROC (Diagram)	ROC (Statement)
1.	Causal sequence	 (II-989)	Entire Z-plane except | Z | = 0
2.	Anticausal sequence	 (II-990)	Entire Z-plane except | Z | = ∞
3.	Two sided sequence	 (II-991)	Entire Z-plane except | Z | = 0 and | Z | = ∞

8.2.2 Z-transform of Standard Sequences :

In this section we will obtain Z-transform of some standard sequences. Most of the standard sequences are having infinite duration. But there are some exceptions like unit impulse, $\delta(n)$.

1. Z-transform of unit impulse $\delta(n)$:

Unit impulse $\delta(n)$ is shown in Fig. 8.2.2. It is given by,

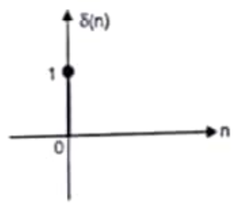
$$\delta(n) = 1 \quad \text{Only at } n = 0$$
$$= 0 \quad \text{Otherwise}$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.2.5)$$

Here $x(n) = \delta(n)$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} \delta(n) Z^{-n} \quad ...(8.2.6)$$



(II-992) **Fig. 8.2.2 : Unit impulse $\delta(n)$**

Since $\delta(n)$ is present only at $n = 0$ we can directly write,

$$X(Z) = \delta(0) Z^0$$

But $\delta(0) = 1$ and $Z^0 = 1$

$$\therefore X(Z) = 1 \quad ...(8.2.7)$$

ROC : In Equation (8.2.7), there is no 'Z' term. So ***ROC is entire Z-plane.*** That means Z can have any value.

Thus the Z-transform pair is

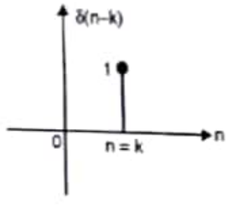
$$\delta(n) \xleftrightarrow{Z} 1$$

2. Z-transform of delayed unit impulse, $\delta(n-k)$:

Here $\delta(n-k)$ is a delayed unit impulse. It indicates that $\delta(n)$ is delayed by 'k' samples. It is shown in Fig. 8.2.3.

It is given by,

$$\delta(n-k) = 1 \quad \text{only at } n = k \text{ and } k > 0$$
$$= 0 \quad \text{otherwise}$$



(II-992) **Fig. 8.2.3 : Delayed unit impulse**

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.2.8)$$

Here $x(n) = \delta(n-k), k > 0$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} \delta(n-k) Z^{-n}$$

But $\delta(n-k) = 1$ only at $n = k$. Thus we get,

$$X(Z) = 1 \cdot Z^{-k}$$

$$\therefore X(Z) = Z^{-k}$$

ROC : Here $X(Z) = Z^{-k} = \frac{1}{Z^k}$. Since k is positive, $(k > 0)$ for any value of 'Z' (except $Z = 0$) we will get finite value of $X(Z)$. Thus ***ROC is entire Z-plane except Z = 0.*** This is because if we put $Z = 0$ then $X(Z) = \infty$. Thus the Z-transform pair is,

$$\delta(n-k) \xleftrightarrow{Z} Z^{-k}$$

3. Z-transform of advanced unit impulse, $\delta(n+k)$, $k > 0$:

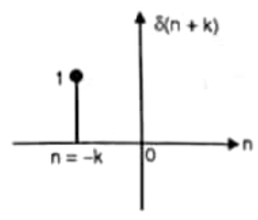
Here $\delta(n+k)$, $k > 0$ is an advanced unit impulse. It indicates that $\delta(n)$ is advanced by '+ k' samples. It is shown in Fig. 8.2.4.

It is given by,

$$\delta(n+k) = 1 \quad \text{only at } n = -k, \ k > 0$$
$$= 0 \quad \text{otherwise}$$

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$



(II-994) **Fig. 8.2.4 : Advanced unit impulse**

Here $x(n) = \delta(n+k)$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} \delta(n+k) Z^{-n}$$

But $\delta(n+k) = 1$ only at $n = k$. Thus we get,

$$X(Z) = 1 \cdot Z^{k}$$

$$\therefore X(Z) = Z^{k}$$

ROC : Here $k > 0$. So we will get finite values of $X(Z)$ for all 'Z' except $Z = \infty$. Because for $Z = \infty$ we get, $X(Z) = (\infty)^k = \infty$. Thus ***ROC is entire Z-plane except Z = ∞***

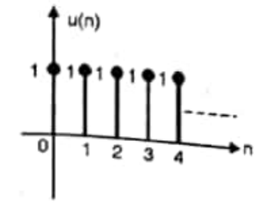
Thus Z-transform pair is,

$$\delta(n+k) \xleftrightarrow{Z} Z^{+k}$$

4. Z-transform of unit step, $u(n)$:

We know that $u(n)$ is unit step as shown in Fig. 8.2.5. It is given by,

$$u(n) = 1 \quad \text{for } n \geq 0$$
$$= 0 \quad \text{otherwise}$$



(II-995) **Fig. 8.2.5 : Unit step**

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.2.9)$$

Here $x(n) = u(n)$. Since $u(n)$ is present from $n = 0$ to $n = \infty$ we will change the limits of summation.

$$\therefore X(Z) = \sum_{n=0}^{\infty} u(n) Z^{-n} \quad ...(8.2.10)$$

But the magnitude of $u(n)$ is always 1.

$$\therefore X(Z) = \sum_{n=0}^{\infty} 1 \cdot Z^{-n}$$

$$\therefore X(Z) = \sum_{n=0}^{\infty} (Z^{-1})^n \quad ...(8.2.11)$$

The standard equation of geometric series is,

$$\sum_{n=0}^{\infty} A^n = A^0 + A + A^2 + A^3 + ... = 1 + A + A^2 + A^3 + ...$$

$$= \frac{1}{1-A} \text{ if } |A| < 1 \quad ...(8.2.12)$$

Let $A = Z^{-1}$. Thus Equation (8.2.11) becomes,

$$X(Z) = \frac{1}{1-A}$$

$$\text{if } |A| < 1 \quad = \frac{1}{1-Z^{-1}} \quad \text{if } |Z^{-1}| < 1$$

$\therefore$ Multiplying numerator and denominator by Z we get,

$$X(Z) = \frac{Z}{Z - Z^{-1} Z} \quad \text{if } |Z^{-1}| < 1$$

$$\therefore X(Z) = \frac{Z}{Z-1} \quad \text{if } |Z^{-1}| < 1$$

In Equation (8.2.12), the condition "if $|A| < 1$" indicates that if this condition is satisfied then only the geometric series is convergent otherwise not. In the given example we get this condition as, $|Z^{-1}| < 1$. That means this is the condition of ROC.

Thus ROC : $|Z^{-1}| < 1$

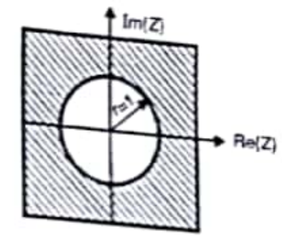
that means $\left|\frac{1}{Z}\right| < 1$

$$\therefore |1| < |Z| \cdot 1$$

$\therefore$ ROC is $|Z| > 1$

This ROC is plotted as shown in Fig. 8.2.6.

Thus ROC is exterior part of circle having radius 1.



(II-996) **Fig. 8.2.6 : ROC of unit step**

Thus the Z-transform pair is,

$$u(n) \xleftrightarrow{Z} \frac{Z}{Z-1} \text{ if } |Z| > 1$$

Why to draw circle only ?

The given ROC is $|Z| > 1$. In case of circle having radius 1, every point outside the periphery of circle satisfies this condition $(|Z| > 1)$. For other diagrams like square or triangle this condition cannot be satisfied.

> **Note :** The unit step is an infinite sequence and as shown in Fig. 8.2.6, the range of $u(n)$ is from $n = 0$ to $n = \infty$. That means this sequence is present only for positive values of n. So it is a causal sequence.

5. Z-transform of unit ramp :

We know that unit ramp sequence is denoted by $r(n)$. It is as shown in Fig. 8.2.7.

Its magnitude is as follows :

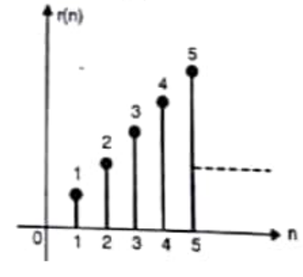
at $n = 0$, $r(n) = 0$
at $n = 1$, $r(n) = 1$
at $n = 2$, $r(n) = 2$
at $n = 3$, $r(n) = 3$ ...

Thus it is expressed as,

$$r(n) = n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{otherwise} \quad ...(8.2.13)$$

Since this sequence is again causal sequence we can write

$$r(n) = n\,u(n) \quad ...(8.2.14)$$



(II-997) **Fig. 8.2.7 : Unit ramp**

Note : Whenever any sequence is multiplied by u (n) then the magnitude of that sequence is not changed. But since u (n) is present, only for positive values of n (n ≥ 0); the result of multiplication is a causal sequence. Thus multiplication of any sequence with u (n) always results in causal sequence. According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.2.15)$$

But here $x(n) = r(n) = n\,u(n)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} n\,u(n) Z^{-n} \quad ...(8.2.16)$$

Just now we discussed that the multiplication by u (n) results in a causal sequence. So the limits of summation will be from n = 0 to n = ∞. Remember that since the magnitude of u (n) is 1; it will not affect the calculation. It is just used to change the limits. So now onwards we will not write u (n) in the calculations.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} n Z^{-n}$$

$$= \sum_{n=0}^{\infty} n (Z^{-1})^{n} \quad ...(8.2.17)$$

Now use standard summation formula.

$$\sum_{n=0}^{\infty} n A^{n} = \frac{A}{(1-A)^2} \quad \text{if } |A| < 1$$

Let $A = Z^{-1}$.

Thus Equation (8.2.17) becomes,

$$X(Z) = \frac{Z^{-1}}{(1-Z^{-1})^2} \quad \text{if } |Z^{-1}| < 1$$

To convert this equation into positive powers of 'Z' we will multiply numerator and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2 \cdot Z^{-1}}{Z^2 (1-Z^{-1})^2} \quad \text{if } |Z^{-1}| < 1$$

$$= \frac{Z}{Z^2 (1 - 2Z^{-1} + Z^{-2})} \quad \text{if } |Z^{-1}| < 1$$

$$= \frac{Z}{Z^2 - 2Z + 1} \quad \text{if } |Z^{-1}| < 1$$

$$\therefore X(Z) = \frac{Z}{(Z-1)^2} \quad \text{if } |Z^{-1}| < 1$$

ROC : Here the condition "$|Z^{-1}| < 1$" indicates the ROC.

$$\therefore \text{ ROC is } \left|\frac{1}{Z}\right| < 1 \Rightarrow 1 < |Z|$$

$$\therefore \text{ ROC is } |Z| > 1$$

Thus ROC is exterior part of circle having radius 1. This ROC is same as shown in Fig. 8.2.6.

Thus the Z-transform pair is

$$n u(n) \xleftrightarrow{Z} \frac{Z}{(Z-1)^2} \quad \text{if } |Z| > 1$$

6. Z-transform of right hand exponential sequence :

As the name indicates, it is an exponential sequence present at the right hand that means only for positive values of 'n'. So it is a causal exponential sequence. Thus it is expressed as,

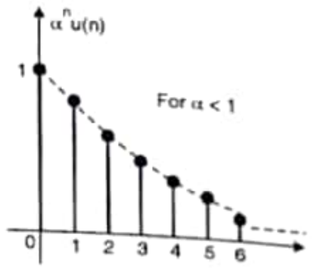
$$x(n) = \alpha^n u(n) = \alpha^n \quad \text{for } n \geq 0$$
$$= 0 \quad \text{for } n < 0 \quad ...(8.2.18)$$

This exponential sequence is shown in Fig. 8.2.8.

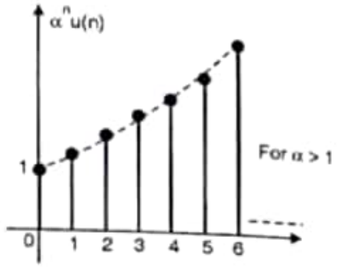
According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.2.19)$$

Here $x(n) = \alpha^n u(n)$. Since the sequence is multiplied by u (n), the limits of summation will change. It will be n = 0 to n = ∞.



(II-998) **Fig. 8.2.8(a)**



(b)

(II-998) **Fig. 8.2.8 : Exponential sequence $\alpha^n u(n)$**

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} \alpha^n Z^{-n} \quad ...(8.2.20)$$

Let us bring it in the form of A^n.

$$\therefore \quad X(Z) = \sum_{n=0}^{\infty} (\alpha Z^{-1})^n \quad ...(8.2.21)$$

Now we have the equation of geometric series,

$$\sum_{n=0}^{\infty} A^n = \frac{1}{1-A} \quad \text{if } |A| < 1$$

Let $A = \alpha Z^{-1}$.

Thus Equation (8.2.21) becomes,

$$X(Z) = \frac{1}{1-\alpha Z^{-1}} \quad \text{if } |\alpha Z^{-1}| < 1 \quad ...(8.2.22)$$

To convert it into positive powers of Z; multiply numerator and denominator by Z.

$$\therefore \quad X(Z) = \frac{Z}{Z(1-\alpha Z^{-1})} \quad \text{if } |\alpha Z^{-1}| < 1$$

$$\therefore X(Z) = \frac{Z}{Z-\alpha} \quad \text{if } |\alpha Z^{-1}| < 1$$

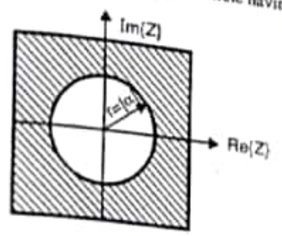
ROC :

This condition $|\alpha Z^{-1}| < 1$ indicates the ROC.

$$\therefore \text{ ROC is } |\alpha Z^{-1}| < 1 \Rightarrow \left|\frac{\alpha}{Z}\right| < 1 \Rightarrow |\alpha| < |Z|$$

Thus ROC is $|Z| > |\alpha|$.

It indicates that ROC is exterior part of circle having radius α.



(II-999) **Fig. 8.2.9 : ROC of $\alpha^n u(n)$**

This ROC is shown in Fig. 8.2.9.

Thus we can write Z-transform pair as

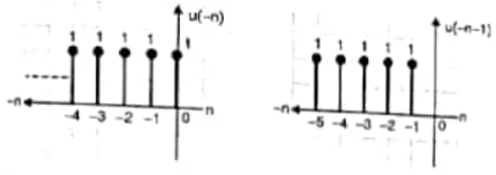
$$\therefore \quad \alpha^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-\alpha} \quad \text{if } |Z| > |\alpha|$$

7. Z-transform of left handed exponential sequence :

As the name indicates; it is an exponential sequence present on left hand side; that means only at negative values of 'n'. So it is anticausal or non-causal exponential sequence. It is expressed as,

$$x(n) = -\alpha^n u(-n-1) \quad ...(8.2.23)$$

Here u (− n) is folded unit step as shown in Fig. 8.2.10(a).



(a) Folded unit step, u (− n) **(b) Sequence u (− n − 1)**

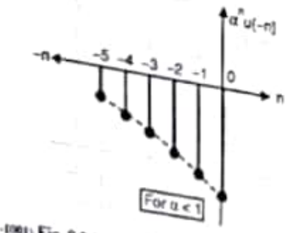
(II-1000) **Fig. 8.2.10**

While u (− n − 1) indicates the advancing of u (− n) by '1' sample. It is obtained by shifting Fig. 8.2.10(a) towards left by 1 sample. This sequence is shown in Fig. 8.2.10(b).

Note that this sequence is having magnitude equals to 1. From n = − 1 to n = − ∞. So we can express it as,

$$u(-n-1) = 1 \quad \text{for } n \leq -1$$
$$= 0 \quad \text{for } n \geq 0 \quad ...(8.2.24)$$

Clearly this is an anticausal sequence. So ***multiplication of any sequence with u (− n − 1) gives anticausal sequence.*** And since its magnitude is unity; it will not change the magnitude of sequence after multiplication. Now anticausal exponential sequence is shown in Fig. 8.2.11.



(II-1001) **Fig. 8.2.11 : Anticausal exponential sequence**

According to definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.2.25)$$

But $x(n) = -\alpha^n u(-n-1)$

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{\infty} -\alpha^n u(-n-1) Z^{-n} \quad ...(8.2.26)$$

Here multiplication by u (− n − 1) gives anticausal sequence present from n = − 1 to n = − ∞. So we will change the limits of summation. And since the magnitude of ($-\alpha^n$) is not changed with this multiplication; we will not consider the term u (− n − 1) in the calculations.

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{-1} (-\alpha^n) Z^{-n}$$

Here also we will have to bring it in the form A^n.

$$\therefore \quad X(Z) = \sum_{n=-\infty}^{-1} -(\alpha^{-1} Z^{-1})^n \quad ...(8.2.27)$$

The limits of summation are negative, so to make it positive we will substitute

$$l = -n \qquad \therefore \quad n = -l$$

∴ when n = − ∞, we get − ∞ = − l ; That means l = ∞

and when n = − 1, we get − 1 = − l; That means l = 1.

$$\therefore \quad X(Z) = -\sum_{l=1}^{\infty} -(\alpha^{1} Z^{-1})^{-l} \quad ...(8.2.28)$$

Take negative sign outside and change the term inside the bracket.

$$\therefore \quad X(Z) = \sum_{l=1}^{\infty} (\alpha^{-1} Z)^{l} \quad ...(8.2.29)$$

Now put $A = \alpha^{-1} Z$

Thus $\sum_{l=1}^{\infty} A^l = A + A^2 + A^3 + A^4 + \ldots$

$= A(1 + A + A^2 + A^3 + \ldots)$

The term inside the bracket is standard geometric series which converges to $\frac{1}{1-A}$ if $|A| < 1$.

Thus we get, $\sum_{l=1}^{\infty} A^l = A \times \frac{1}{1-A}$ if $|A| < 1$ ...(8.2.30)

Thus Equation (8.2.29) becomes,

$$X(Z) = -(\alpha^{-1}Z) \times \frac{1}{1-\alpha^{-1}Z} \quad \text{if } |\alpha^{-1}Z| < 1$$

$$= \frac{-(Z/\alpha)}{1-(Z/\alpha)} \quad \text{if } \left|\frac{Z}{\alpha}\right| < 1$$

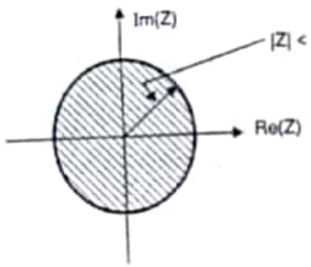
$$\therefore X(Z) = \frac{Z/\alpha}{Z/\alpha - 1} \quad \text{if } |Z| < \alpha$$

Multiplying numerator and denominator by α we get,

$$X(Z) = \frac{Z}{Z-\alpha} \quad \text{if } |Z| < |\alpha|$$

ROC :

Here ROC is $|Z| < |\alpha|$. That means ROC is interior part of circle having radius α. This ROC is shown in Fig. 8.2.12.



(II-1002) **Fig. 8.2.12 : ROC of $-\alpha^n u(-n-1)$**

Thus Z-transform pair is

$$-\alpha^n u(-n-1) \xleftrightarrow{Z} \frac{Z}{Z-\alpha} \quad \text{if } |Z| < |\alpha|$$

> **Note :** This sequence is an infinite exponential sequence and it is anticausal. Thus ROC of infinite anticausal sequence is interior part of circle having radius $|\alpha|$

Comment :

Z-transforms of $\alpha^n u(n)$ and $-\alpha^n u(-n-1)$ are same but their ROC's are different.

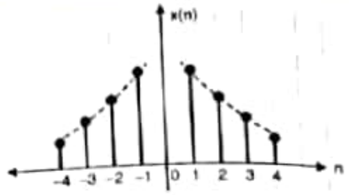
8. Z-transform of two sided exponential :

Consider two sided exponential sequence, which is addition of causal and non-causal exponential sequences. Thus,

$$x(n) = \alpha^n u(n) + \beta^n u(-n-1) \quad \ldots(8.2.31)$$

This two sided exponential sequence is shown in Fig. 8.2.13(a).

Let $x_1(n) = \alpha^n u(n)$ and $x_2(n) = \beta^n u(-n-1)$.



(II-1003) **Fig. 8.2.13(a) : Two sided exponential**

Now $X(Z) = X_1(Z) + X_2(Z)$ ...(8.2.32)

We have obtained the Z-transform for such sequences separately.

Z-transform of $\alpha^n u(n)$:

We have, $\alpha^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-\alpha}$ if $|Z| > |\alpha|$

$$\therefore X_1(Z) = \frac{Z}{Z-\alpha} \quad \text{if } |Z| > |\alpha| \quad \ldots(8.2.33)$$

Z-transform of $\beta^n u(-n-1)$:

We have, $-\alpha^n u(-n-1) \xleftrightarrow{Z} \frac{Z}{Z-\alpha}$ if $|Z| < |\alpha|$

From this we can write,

$$\beta^n u(-n-1) = -\frac{Z}{Z-\beta} \quad \text{if } |Z| < |\beta|$$

$$\therefore X_2(Z) = \frac{-Z}{Z-\beta} \quad \text{if } |Z| < |\beta| \quad \ldots(8.2.34)$$

Putting Equations (8.2.33) and (8.2.34) in Equation (8.2.32) we get,

$$X(Z) = \frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$$

ROC : $|Z| > |\alpha|$ and $|Z| < |\beta|$

Deciding combined ROC : We have obtained ROC separately. Now for the total sequence we will find ROC for which $X(Z) = X_1(Z) + X_2(Z)$ will convergent. In other words there should be some overlap of two ROC's. Now this will depend on the values of α and β.

We will consider two cases as follows :

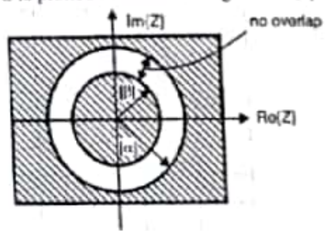
Case I : If $|\beta| < |\alpha|$

This ROC is plotted as shown in Fig. 8.2.13(b).

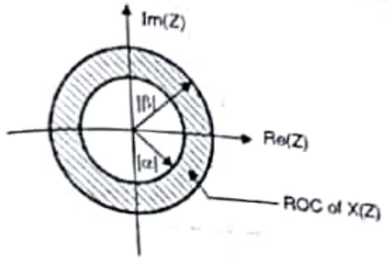
We know that for $X_1(Z)$, ROC is $|Z| > |\alpha|$; which is exterior part of circle having radius α, that means exterior part of outer circle. And for $X_2(Z)$, ROC is $|Z| < |\beta|$; which is interior part of circle having radius β. That means interior part of inner circle. Now as shown in Fig. 8.2.13(a), the two ROC's will not overlap. There is a gap between two ROC's. For this portion, X (Z) will not convergent. That X (Z) will not exist.

Case II : If $|\beta| > |\alpha|$

This ROC is plotted as shown in Fig. 8.2.13(c).



(II-1004) **Fig. 8.2.13(b) : ROC for $|\beta| < |\alpha|$**



(c) **ROC for $|\beta| > |\alpha|$**

(II-1004) **Fig. 8.2.13**

In this case there is a ring in the Z-plane. In this case both power series converge simultaneously. That means X (Z) exist.

$$\therefore X(Z) = X_1(Z) + X_2(Z) = \frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta}$$

$$\therefore X(Z) = \frac{Z}{Z-\alpha} + \frac{Z}{\beta - Z} \quad \text{for } |\alpha| < |Z| < |\beta|$$

ROC :

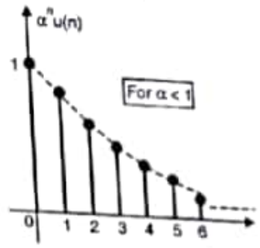
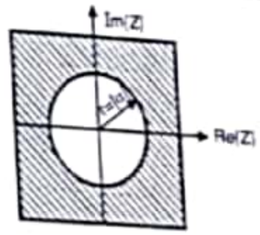
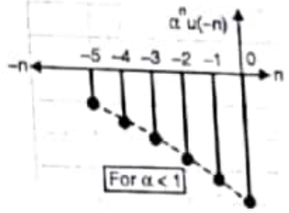
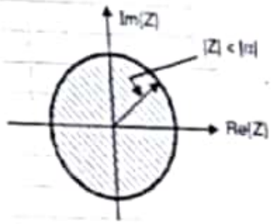
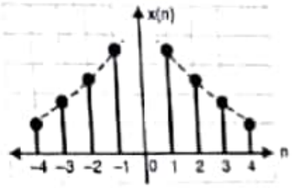
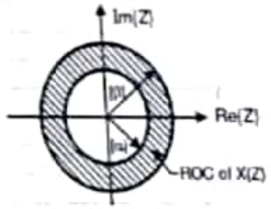
Thus combined ROC is $|\alpha| < |Z| < |\beta|$

> **Note :** When we combine many Z-transforms the ROC of the overall Z-transform is at least the intersection of the individual Z-transforms.

Summary of ROC for infinite duration sequences :

Table 8.2.2 shows the ROC for infinite duration sequence.

Table 8.2.2

Sr. No.	Infinite duration sequence	ROC
1.	Causal : e.g. $x(n) = \alpha^n u(n)$  (II-1005)	 (II-1006)
2.	Anticausal : e.g. $x(n) = \alpha^n u(-n-1)$  (II-1007)	 (II-1008)
3.	Two sided e.g. $x(n) = \alpha^n u(n) + \beta^n u(-n-1)$  (II-1009)	 (II-1010)

8.2.3 Properties of ROC : MU - Dec. 10, Dec. 11

University Questions

Q. 1 State and explain properties of region of convergence for the Z-transform **(Dec. 10, 10 Marks)**

Q. 2 Explain properties of region of convergence (ROC) of Z-transform. **(Dec. 11, 4 Marks)**

We have solved some examples using Z transform and we have discussed ROC in each case. Based on this: the properties of ROC are summarized as follows :

1. The ROC is a ring, whose center is at origin.
2. ROC cannot contain any pole.
3. If ROC of X(Z) includes unit circle then and then only the fourier transform of D.T. sequence x (n) converges.
4. The ROC must be a connected region.
5. For a finite duration sequence, x (n); the ROC is entire Z plane except Z = 0 and Z = ∞.
6. If x (n) is causal then ROC is exterior part of circle of radius say 'α'.
7. If x (n) is anticausal then ROC is interior part of circle of radius say 'α'.
8. If x (n) is two sided sequence then ROC is intersection of two circles of radii 'α' and 'β'.

8.3 Properties of Z-transform : MU - Dec. 10

University Question

Q. 1 State and explain properties of region of convergence for the Z-transform.**(Dec. 10, 10 Marks)**

In this section we will discuss different properties of Z-transform. These properties are applicable to both double sided as well as single sided Z-transform.

8.3.1 Linearity :

Statement : It states that if $x(n) = a_1 x_1(n) + a_2 x_2(n)$

and if $x_1(n) \xrightarrow{Z} X_1(Z)$ and $x_2(n) \xrightarrow{Z} X_2(Z)$ then,

$$x(n) \xrightarrow{Z} X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$$

Where a_1 and a_2 are constants.

Proof : According to definition of Z-transform.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.3.1)$$

Here $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] Z^{-n} \quad ...(8.3.2)$$

Writing two terms separately we get,

$$X(Z) = \sum_{n=-\infty}^{\infty} a_1 x_1(n) Z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) Z^{-n}$$

Here a_1 and a_2 are constants. So we can take it outside the summation sign.

$$\therefore X(Z) = a_1 \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n} \quad ...(8.3.3)$$

Comparing Equation (8.3.3) with Equation (8.3.1) we get,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$$

Comment on ROC :

The combined ROC is the overlap or intersection of the individual ROC's of $X_1(Z)$ and $X_2(Z)$.

Use of this property :

We can express the given signal in the form of sum of two or more elementary signals. Then find their individual Z-transforms and add them. This will simplify the solution.

> **Note :** Linearity property states that Z-transform of linear combination of signals is same as linear combination of Z-transform. While the ROC of X(Z) is intersection or overlap of individual region of convergence of $X_1(Z)$ and $X_2(Z)$.

8.3.1.1 Solved Examples on Linearity Property

Ex. 8.3.1 : Determine Z-transform of $x(n) = (n+1)u(n)$.

Soln. : The given function is,

$$x(n) = (n+1)u(n)$$

$$\therefore x(n) = n u(n) + u(n) \quad ...(1)$$

Let $x_1(n) = n u(n)$ and $x_2(n) = u(n)$

$$\therefore x(n) = x_1(n) + x_2(n) \quad ...(2)$$

Here $n\,u(n)$ is ramp sequence. We have standard Z transform pair,

$$n u(n) \xrightarrow{Z} \frac{Z}{(Z-1)^2} \quad ROC : |Z| > 1$$

$$\therefore X_1(Z) = Z\{n u(n)\} = \frac{Z}{(Z-1)^2}, \quad |Z| > 1 \quad ...(3)$$

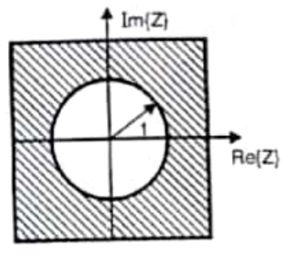
Now $u(n)$ is a unit step and we have,

$$u(n) \xrightarrow{Z} \frac{Z}{Z-1} \quad ROC : |Z| > 1$$

$$\therefore X_2(Z) = Z\{u(n)\} = \frac{Z}{Z-1} \; ROC : |Z| > 1 \quad ...(4)$$

From Equation (2) we can write,

$$X(Z) = X_1(Z) + X_2(Z) \quad ...(5)$$



(II-1011) **Fig. P. 8.3.1**

Putting Equations (3) and (4) in Equation (5) we get,

$$X(Z) = \frac{Z}{(Z-1)^2} + \frac{Z}{Z-1} \quad ROC : |Z| > 1$$

Thus ROC is exterior part of unit circle as shown in Fig. P. 8.3.1.

Ex. 8.3.2 : Determine Z-transform and ROC of signal : $x(n) = [3(4^n) - 5(3^n)]u(n)$.

Soln. :

Let us bring it into the known form that is,

$$x(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$\therefore x(n) = 3(4^n)u(n) - 5(3^n)u(n) \quad ...(1)$$

Comparing Equations (1) and (2) we can write, ...(2)

$$x_{1(n)} = 4^n u(n) \text{ and } x_{2(n)} = 3^n u(n)$$

Thus using linearity property,

$$X(Z) = 3X_1(Z) - 5X_2(Z) \quad ...(3)$$

Now recall the standard Z-transform pair,

$$\alpha^n u(n) \xrightarrow{Z} \frac{Z}{Z-\alpha} \quad \text{and} \quad ROC \text{ is } |Z| > |\alpha|$$

Here $x_{1(n)} = 4^n u(n) \quad \therefore \alpha = 4$

$$\therefore X_1(Z) = \frac{Z}{Z-4}, \quad ROC\,|Z| > 4 \quad ...(4)$$

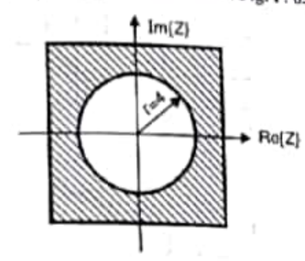
And $x_{2(n)} = 3^n u(n) \quad \therefore \alpha = 3$

$$\therefore X_2(Z) = \frac{Z}{Z-3}, \quad ROC\,|Z| > 3 \quad ...(5)$$

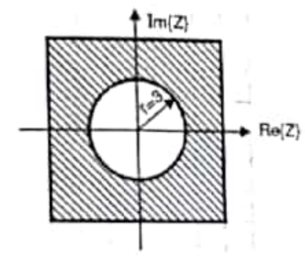
Putting Equations (4) and (5) in Equation (3) we get,

$$X(Z) = 3 \times \frac{Z}{Z-4} - 5 \times \frac{Z}{Z-3}$$

ROC : For $X_1(Z)$, ROC is $|Z| > 4$ and for $X_2(Z)$, ROC is $|Z| > 3$. Now we have to decide the common ROC such that both $X_1(Z)$ and $X_2(Z)$ will converge. We have discussed that the combined ROC is overlap or intersection of two ROC's. The common ROC is $|Z| > 4$. It is as shown in Fig. P. 8.3.2.

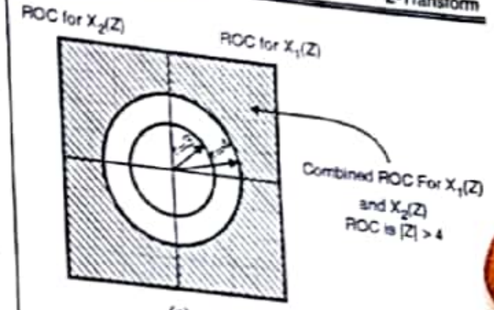


(a)



(b)

(II-1012) **Fig. P. 8.3.2**



(c)

(II-1012) **Fig. P. 8.3.2 : ROC of given signal**

> **Note :** We cannot take combined ROC as $|Z| > 3$. Because for $X_2(Z)$ ROC is $|Z| > 4$. So in this case $X_2(Z)$ will not converge. But if we take ROC $|Z| > 4$ then by default it is > 3; so both sequences will converge. That means in this case $|Z| > 4$ is the common ROC for both sequences.

Ex. 8.3.3 : Determine the Z-transform and sketch the ROC of :

$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^n, & n \geq 0 \\ \left(\frac{1}{2}\right)^{-n}, & n < 0 \end{cases}$$

Soln. : For the range $n > 0$, we will get a causal sequence and for the range $n < 0$), we will get anticausal sequence. Thus the given expression can be written as,

$$x(n) = \left(\frac{1}{3}\right)^n u(n) + \left(\frac{1}{2}\right)^{-n} u(-n-1) \quad ...(1)$$

The known form of sequence $x(n)$ is,

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \quad ...(2)$$

Comparing Equations (1) and (2) we get,

$$x_1(n) = \left(\frac{1}{3}\right)^n u(n)$$

$$\text{and } x_2(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$$

Using linearity property,

$$X(Z) = X_1(Z) + X_2(Z) \quad ...(3)$$

We have standard Z-transform pair for causal sequence.

$$\alpha^n u(n) \xrightarrow{Z} \frac{Z}{Z-\alpha} \quad ROC \text{ is } |Z| > |\alpha|$$

Here $x_1(n) = \left(\frac{1}{3}\right)^n u(n)$

$$\therefore \alpha = \frac{1}{3}$$

$$\therefore X_1(Z) = \frac{Z}{Z-\frac{1}{3}} \quad ROC \text{ is } |Z| > \left|\frac{1}{3}\right| \quad ...(4)$$

Now $x_2(n) = \left(\frac{1}{2}\right)^{-n} u(-n-1)$ ...(5)

Recall the standard Z-transform pair for anticausal sequence.

$$-\alpha^n u(-n-1) \xleftrightarrow{Z} \frac{Z}{Z-\alpha}$$

ROC is $|Z| < |\alpha|$

From this we can write,

$$+\alpha^n u(-n-1) \xleftrightarrow{Z} -\frac{Z}{Z-\alpha} \quad \text{ROC is } |Z| < |\alpha|$$

Now Equation (5) can be written as.

$$x_2(n) = \left[\left(\frac{1}{2}\right)^{-1}\right]^n u(-n-1)$$

$$= \left[\left(\frac{1}{1/2}\right)^n\right] u(-n-1)$$

$$\therefore x_2(n) = 2^n u(-n-1)$$

Thus, here $\alpha = 2$.

$$\therefore X_2(Z) = \frac{Z}{Z-2} \quad \text{ROC is } |Z| < |2| \qquad ...(6)$$

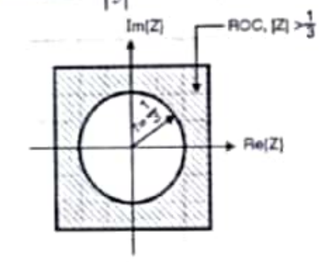
Putting Equations (4) and (5) in Equation (3) we get.

$$X(Z) = \frac{Z}{Z-\frac{1}{3}} + \frac{Z}{Z-2}$$

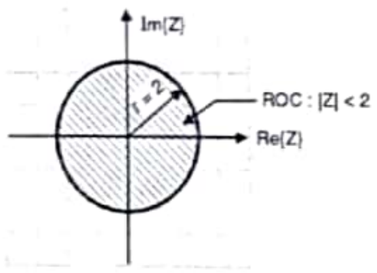
ROC :

For $X_1(Z)$, ROC is $|Z| > \frac{1}{3}$ and for $X_2(Z)$ ROC is $|Z| < 2$. We know that the combined ROC is overlap or intersection of two ROC's. This ROC is shown in Fig. P. 8.3.3.

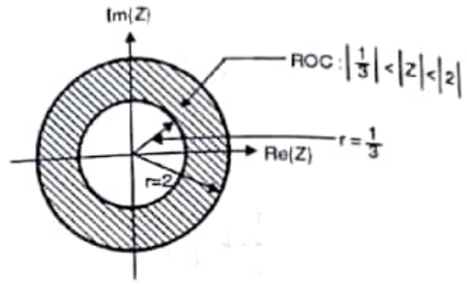
Thus combined ROC is $\left|\frac{1}{3}\right| < |Z| < |2|$.



(a)



(H-1013) Fig. P. 8.3.3(b)



(H-1014) Fig. P. 8.3.3(c) : ROC of given sequence

Ex. 8.3.4 : Find Z transform of following function along with ROC, $x(n) = a^n u(n) + \delta(n-5)$.

Soln. :

The given function is.

$$x(n) = a^n u(n) + \delta(n-5) \qquad ...(1)$$

$$\text{Let } x_1(n) = a^n u(n) \qquad ...(2)$$

$$\text{and } x_2(n) = \delta(n-5) \qquad ...(3)$$

$$\therefore x(n) = x_1(n) + x_2(n) \qquad ...(4)$$

First we will calculate Z transform of $x_1(n)$.

We have, $a^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-a}$; ROC : $|Z| > |a|$

$$\therefore Z\{a^n u(n)\} = X_1(Z) = \frac{Z}{Z-a};$$

ROC : $|Z| > |a|$

Now we will calculate Z transform of $x_2(n)$.

We have standard Z transform pair.

$$\delta(n-k) \xleftrightarrow{Z} Z^{-k};$$

ROC : Entire Z plane except Z = 0.

$$\therefore Z\{\delta(n-5)\} = X_2(Z) = Z^{-5};$$

From Equation (4) we can write,

$$X(Z) = X_1(Z) + X_2(Z)$$

$$X(Z) = \frac{Z}{Z-a} + Z^{-5}$$

The combined ROC is $|Z| > |a|$.

Ex. 8.3.5 : Determine the Z-transform of : $x(n) = (\cos \omega_0 n)\, u(n)$.

Soln. :

According to Euler's identity we have,

$$\cos\theta = \frac{1}{2} e^{j\theta} + \frac{1}{2} e^{-j\theta} \qquad \therefore \cos \omega_0 n = \frac{1}{2} e^{j\omega_0 n} + \frac{1}{2} e^{-j\omega_0 n}$$

$$\therefore x(n) = \frac{1}{2} e^{j\omega_0 n} u(n) + \frac{1}{2} e^{-j\omega_0 n} u(n)$$

Now we have the known form.

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \qquad ...(1)$$

Comparing Equations (1) and (2) we get.

$$x_1(n) = e^{j\omega_0 n}, \quad \text{here } a_1 = \frac{1}{2} \qquad ...(2)$$

$$\text{and } x_2(n) = e^{-j\omega_0 n}, \quad \text{here } a_2 = \frac{1}{2}$$

Using Linearity property we get,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$$

Here $x_1(n) = e^{j\omega_0 n} u(n) = \left(e^{j\omega_0}\right)^n u(n)$ ...(3)

Recall the standard Z-transform pair,

$$\alpha^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha|$$

Here $\alpha = e^{j\omega_0}$ Thus we get.

$$X_1(Z) = \frac{Z}{Z-e^{j\omega_0}} \qquad \text{ROC is } |Z| > |e^{j\omega_0}|$$

Now $x_2(n) = e^{-j\omega_0 n} u(n) = \left(e^{-j\omega_0}\right)^n u(n)$ ...(4)

Here $\alpha = e^{-j\omega_0}$

$$\therefore X_2(Z) = \frac{Z}{Z-e^{-j\omega_0}} \qquad \text{ROC is } |Z| > |e^{-j\omega_0}| \qquad ...(5)$$

Putting Equations (4) and (5) in Equation (3) we get.

$$X(Z) = a_1 \frac{Z}{Z-e^{j\omega_0}} + a_2 \frac{Z}{Z-e^{-j\omega_0}}$$

But $a_1 = a_2 = \frac{1}{2}$

$$\therefore X(Z) = \frac{1}{2}\left[\frac{Z}{Z-e^{j\omega_0}} + \frac{Z}{Z-e^{-j\omega_0}}\right] \qquad ...(6)$$

ROC :

For $X_1(Z)$, ROC is $|Z| > |e^{j\omega_0}|$

We have $e^{j\theta} = \cos\theta + j\sin\theta$

$$\therefore e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$$

Now $|e^{j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$

Thus ROC is $|Z| > 1$

Similarly for $X_2(Z)$, ROC is $|Z| > |e^{-j\omega_0 n}|$

We have, $e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$

Thus $|e^{-j\omega_0}| = \sqrt{\cos^2\omega_0 + \sin^2\omega_0} = 1$

Thus ROC is $|Z| > 1$

So the combined ROC is $|Z| > 1$.

This is shown in Fig. P. 8.3.5.

This is a standard Z-transform pair.

We can further simplify Equation (6) as follows.

We have $e^{j\omega_0} = \cos\omega_0 + j\sin\omega_0$ and $e^{-j\omega_0} = \cos\omega_0 - j\sin\omega_0$

Putting these values in Equation (6) we get,

$$X(Z) = \frac{1}{2}\left[\frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)} + \frac{Z}{Z-(\cos\omega_0 - j\sin\omega_0)}\right] \quad ...(7)$$

Consider the first term inside the bracket,

$$\frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)} = \frac{Z}{Z-\cos\omega_0 - j\sin\omega_0}$$

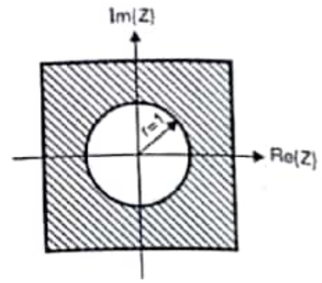
Rationalizing the equation,

$$= \frac{Z(Z-\cos\omega_0 + j\sin\omega_0)}{(Z-\cos\omega_0 - j\sin\omega_0)(Z-\cos\omega_0 + j\sin\omega_0)}$$

$$= \frac{Z(Z-\cos w_0 + j\sin w_0)}{Z^2 - Z\cos w_0 + jZ\sin w_0 - Z\cos w_0 + \cos^2 w_0 - j\sin w_0\cos w_0 - jZ\sin w_0 + j\sin w_0\cos w_0 - j^2\sin^2 w_0}$$

$$= \frac{Z(Z-\cos w_0 + j\sin w_0)}{Z^2 - 2Z\cos w_0 + 1} \quad ...(8)$$

...[Here $j^2 = -1$ and $\sin^2\omega_0 + \cos^2\omega_0 = 1$]



(H-1015) **Fig. P. 8.3.5 : ROC of $x(n) = (\cos\omega_0 n)\,u(n)$**

Now consider second term,

$$\frac{Z}{Z-(\cos\omega_0 - j\sin\omega_0)} = \frac{Z}{Z-\cos\omega_0 + j\sin\omega_0}$$

Rationalizing the equation.

$$= \frac{Z(Z-\cos\omega_0 - j\sin\omega_0)}{(Z-\cos\omega_0 + j\sin\omega_0)(Z-\cos\omega_0 - j\sin\omega_0)} = \frac{Z(Z-\cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} \quad ...(9)$$

Putting Equations (8) and (9) in Equation (7) we get.

$$X(Z) = \frac{1}{2}\left[\frac{Z(Z-\cos\omega_0 + j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1} + \frac{Z(Z-\cos\omega_0 - j\sin\omega_0)}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore X(Z) = \frac{1}{2}\left[\frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0 + Z^2 - Z\cos\omega_0 - jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore X(Z) = \frac{1}{2}\left[\frac{2Z^2 - 2Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}\right]$$

$$\therefore X(Z) = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

Therefore the standard Z-transform identity is.

$$(\cos\omega_0 n)\,u(n) \xleftrightarrow{Z} \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}, \quad \text{ROC is } |Z| > 1$$

Ex. 8.3.6 : Determine Z-transform of :

$$x(n) = \sin\omega_0 n\, u(n)$$

Soln. : We can obtain Z-transform of $\sin\omega_0 n\, u(n)$ similar to the last Example using Euler's identify. This Example can also be solved using another method, which is more simple.

We have standard Z-transform pair.

$$\alpha^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-\alpha} \qquad \text{ROC is } |Z| > |\alpha| \quad ...(1)$$

Let $\alpha = e^{j\omega_0 n}$. Thus we can write,

$$Z\{e^{j\omega_0 n} u(n)\} = \frac{Z}{Z-e^{j\omega_0}} \qquad \text{ROC}: |Z| > |e^{j\omega_0}| \quad ...(2)$$

But we have, $e^{j\omega_0 n} = \cos\omega_0 n + j\sin\omega_0 n$

Putting this value in Equation (3) we get,

$$Z\{\cos\omega_0 n + j\sin\omega_0 n\} = \frac{Z}{Z-(\cos\omega_0 + j\sin\omega_0)}$$

$$= \frac{Z}{Z-\cos\omega_0 - j\sin\omega_0}$$

$$= \frac{Z(Z-\cos\omega_0 + j\sin\omega_0)}{(Z-\cos\omega_0 - j\sin\omega_0)(Z-\cos\omega_0 + j\sin\omega_0)}$$

$$\therefore Z\{\cos\omega_0 n + j\sin\omega_0 n\} = \frac{Z^2 - Z\cos\omega_0 + jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore Z\{\cos\omega_0 n\} + Z\{j\sin\omega_0 n\}$$

$$= \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} + j\frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \quad ...(3)$$

Comparing imaginary part we get,

$$Z\{j\sin\omega_0 n\} = \frac{jZ\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1}$$

$$\therefore Z\{\sin\omega_0 n\} = \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \quad \text{ROC is } |Z| > |e^{j\omega_0}|$$

Similarly comparing real part we can obtain Z-transform of $\cos\omega_0 n$. Now ROC is $|Z| > |e^{j\omega_0}|$. But we know that $|e^{j\omega_0}| = 1$. Thus ROC is $|Z| > 1$. This ROC is same as shown in Fig. P. 8.3.5. This is a standard Z-transform pair.

$$\therefore \sin\omega_0 n\, u(n) \xleftrightarrow{Z} \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \quad \text{ROC}: |Z| > 1$$

8.3.2 Time Shifting :

University Question **MU - Dec. 05**

Q. 1 State and prove time shifting property of z transform. **(Dec. 05, 5 Marks)**

Statement : If $x(n) \xleftrightarrow{Z} X(Z)$

then $x(n-k) \xleftrightarrow{Z} Z^{-k}X(Z)$

The ROC of $Z^{-k}X(Z)$ is same as that of $X(Z)$ except for $Z = 0$ if $k > 0$ and $Z = \infty$ if $k < \infty$.

Proof : According to definition of Z-transform,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \quad ...(8.3.4)$$

Then $Z\{x(n-k)\}$ can be written as,

$$\therefore Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)Z^{-n} \quad ...(8.3.5)$$

Now Z^{-n} can be written as $Z^{-n} = Z^{-(n-k)} \cdot Z^{-k}$. Thus Equation (8.3.5) becomes,

$$Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)Z^{-(n-k)} \cdot Z^{-k}$$

Since the limits of summation are in terms of 'n' we can take Z^{-k} outside the summation sign.

$$\therefore Z\{x(n-k)\} = Z^{-k}\sum_{n=-\infty}^{\infty} x(n-k)Z^{-(n-k)} \quad ...(8.3.6)$$

Now put $n - k = m$ on R.H.S. The limits will change as follows :

at $n = -\infty$, $-\infty - k = m \Rightarrow m = -\infty$

at $n = +\infty$, $\infty - k = m \Rightarrow m = \infty$

$$\therefore Z\{x(n-k)\} = Z^{-k}\sum_{m=-\infty}^{\infty} x(m)Z^{-m} \quad ...(8.3.7)$$

Compare Equation (8.3.7) with Equation (8.3.4),

$$\therefore Z\{x(n-k)\} = Z^{-k}X(Z)$$

$$\therefore x(n-k) \xleftrightarrow{Z} Z^{-k}X(Z)$$

Similarly it can be shown that,

$$x(n+k) \xleftrightarrow{Z} Z^{+k}X(Z)$$

Note : Here $x(n-k)$ indicates that the sequence is shifted in time domain (delayed). Thus shifting the sequence in time domain by $(-k)$ samples corresponds to multiplication by Z^{-k} in the frequency domain.

8.3.2.1 Solved Examples on Time Shifting Property :

Ex. 8.3.7 : Find the Z-transform of :

$$x(n) = \delta(n-k)$$

Soln. : We know that $\delta(n)$ is unit impulse and $Z\{\delta(n)\} = 1$ that means $\delta(n) \xrightarrow{Z} 1$

According to time shifting property we have.

$$x(n-k) \xrightarrow{Z} Z^{-k}X(Z)$$

$$\therefore \delta(n-k) \xrightarrow{Z} Z^{-k} \cdot 1$$

ROC : Entire Z-plane except $Z = 0$.

Ex. 8.3.8 : Find the Z-transform of :

$$x(n) = \delta(n+2)$$

Soln. : We have,

$$\delta(n) \xrightarrow{Z} 1$$

Using time shifting property we can write.

$$x(n+k) \xrightarrow{Z} Z^{+k}X(Z)$$

$$\therefore \delta(n+2) \xrightarrow{Z} Z^2$$

ROC : Entire Z-plane except $Z = \infty$.

Ex. 8.3.9 : Determine the z - transform and sketch R.O.C :

1. $x_1[n] = \left[\frac{1}{3}\right]^n ; n \geq 0$
2. $x_2[n] = x_1[n+4]$

Dec. 11, 8 Marks

Soln. :

1. **Given :** $x_1[n] = \left[\frac{1}{3}\right]^n ; n \geq 0$

According to the definition of z transform,

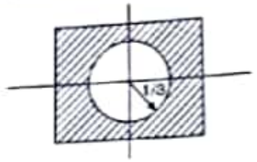
$$X_1(z) = \sum_{n=0}^{\infty} x_1(n)z^{-n}$$

$$= \sum_{n=0}^{\infty}\left[\frac{1}{3}\right]^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{3}\cdot z^{-1}\right)^n$$

$$\therefore X_1(z) = \frac{1}{1-\frac{1}{3}z^{-1}}, \left|\frac{1}{3}\cdot z^{-1}\right| < 1$$

$$\therefore\ X_1(z) = \frac{z^5}{z-\frac{1}{3}},\ |z| > \frac{1}{3} \qquad \text{...Ans.}$$

The ROC is as shown in Fig. P. 8.3.9.



(H-1221) **Fig. P. 8.3.9**

2. **Given :** $x_2[n] = x_1[n+4]$

Using time shifting property.

$X_2(z) = z^4 \cdot X_1(z)$: ROC remains same

$$\therefore\ X_2(z) = z^4 \cdot \frac{z}{z-\frac{1}{3}},\ |z| > \frac{1}{3}$$

$$\therefore\ X_2(z) = \frac{z^5}{z-\frac{1}{3}},\ |z| > \frac{1}{3} \qquad \text{...Ans.}$$

Ex. 8.3.10 : Find z transform of $\left\{\cos\left\{\frac{n\pi}{4}+\alpha\right\}\right\}$ $n \geq 0$.

Soln. : We have standard z transform pair.

$$\cos\omega_0 n\, u(n) = \frac{z^2 - z\cos\omega_0}{z^2 - 2z\cos\omega_0 + 1}$$

Consider the term $\cos\left(n\frac{\pi}{4}\right)$. Here $\omega_0 = \frac{\pi}{4}$

$$\therefore\ \cos\left(n\frac{\pi}{4}\right) = \frac{z^2 - z\cos\left(\frac{\pi}{4}\right)}{z^2 - 2z\cos\left(\frac{\pi}{4}\right) + 1}$$

$$= \frac{z^2 - 0.707\,z}{z^2 - 1.414\,z + 1}$$

According to time shifting property.

$$x(n+k) = z^k X(z)$$

$$\therefore\ \cos\left(\frac{n\pi}{4}+\alpha\right) = z^n\left[\frac{z^2 - 0.707\,z}{z^2 - 1.414\,z + 1}\right]$$

Ex. 8.3.11 : Find y (Z) for the following causal systems

1. $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = \delta(n-2)$
2. $y(n) = u(n) + 2u(n-1) - 4u(n-2) + u(n-3)$.

Soln. :

1. Given equation is,

$$y(n) = \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = \delta(n-2) \qquad \text{...(1)}$$

We have $\delta(n) \xleftrightarrow{Z} 1$ and using time shifting property,

$$\delta(n-2) \leftrightarrow Z^{-2} \cdot 1$$

Taking Z transform of Equation (1) we get,

$$Y(Z) - \frac{3}{4}Z^{-1}Y(Z) + \frac{1}{8}Z^{-2}Y(Z) = Z^{-2}$$

$$\therefore\ Y(Z)\left[1 - \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}\right] = Z^{-2}$$

$$\therefore\ Y(Z) = \frac{Z^{-2}}{1 - \frac{3}{4}Z^{-2} + \frac{1}{8}Z^{-2}}$$

$$\therefore\ Y(Z) = \frac{1}{Z^2 - \frac{3}{4}Z + \frac{1}{8}}$$

2. Given equation is,

$$y(n) = u(n) + 2u(n-1) - 4u(n-2) + u(n-3) \qquad \text{...(2)}$$

We have $u(n) \xleftrightarrow{Z} \frac{Z}{Z-1}$ and using time shifting property

$$u(n-k) \xleftrightarrow{Z} Z^{-k} \cdot \frac{Z}{Z-1}$$

Taking Z transform of Equation (2) we get,

$$Y(Z) = \frac{Z}{Z-1} + 2 \cdot Z^{-1} \cdot \frac{Z}{Z-1} - 4Z^{-2} \cdot \frac{Z}{Z-1} + Z^{-3} \cdot \frac{Z}{Z-1}$$

$$\therefore\ Y(Z) = \frac{Z}{Z-1} + \frac{2}{Z-1} - 4\frac{Z^{-1}}{Z-2} + \frac{Z^{-2}}{Z-1}$$

Ex. 8.3.12 : Calculate Z transform of the following signals :

$$x[n] = u[n-6] - u[n-10]$$

Dec. 2014, 5 Marks

Soln. :

$x[n] = u[n-6] - u[n-10]$:

Given : $x[n] = u[n-6] - u[n-10]$

We have, $Z\{u(n)\} = \frac{Z}{Z-1}$, $|Z| > 1$

Using shifting property,

$$Z\{u(n-6)\} = Z^{-6} \cdot \frac{Z}{Z-1} = \frac{Z^{-5}}{Z-1}$$

and $Z\{u(n-10)\} = Z^{-10} \cdot \frac{Z}{Z-1} = \frac{Z^{-9}}{Z-1}$

$$\therefore\ X(Z) = \frac{Z^{-5}}{Z-1} - \frac{Z^{-9}}{Z-1}$$

$$\therefore\ X(Z) = \frac{Z^{-5} - Z^{-9}}{Z-1}$$

8.3.3 Scaling in the Z-domain (Multiplication by Exponential Sequence) :

Statement : If $x(n) \xleftrightarrow{Z} X(Z)$

ROC : $r_1 < |Z| < r_2$

then $a^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right)$

ROC : $|a|r_1 < |Z| < |a|r_2$

For any constant 'a' real or complex.

Proof : According to definition of Z-transform.

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n} \qquad \text{...(8.3.8)}$$

$$\therefore Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) a^n Z^{-n} = \sum_{n=-\infty}^{\infty} x(n)(a^{-1}Z)^{-n}$$

$$\therefore Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{Z}{a}\right)^{-n} \qquad \text{...(8.3.9)}$$

Comparing Equations (8.3.8) and (8.3.9),

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right)$$

$$\therefore\ a^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right)$$

ROC : ROC of X (Z) is $r_1 < |Z| < r_2$

To obtain ROC of $X\left(\frac{Z}{a}\right)$, replace 'Z' by $\frac{Z}{a}$.

$$\therefore\ \text{ROC of } X\left(\frac{Z}{a}\right) : r_1 < \left|\frac{Z}{a}\right| < r_2$$

$$\therefore\ \text{ROC of } X\left(\frac{Z}{a}\right) : |a|r_1 < |Z| < |a|r_2$$

Note :
1. Scaling in the time domain corresponds to shrinking or expanding the Z-plane.
2. By taking 'a' as complex variable one can change scale of radius and ω in the Z-plane.

8.3.3.1 Solved Examples on Scaling Property :

Ex. 8.3.13 : Obtain Z-transform of $x(n) = a^n u(n)$ using scaling property.

Soln. : Here u (n) is a unit step. Its Z- transform is,

$$Z\{u(n)\} = \frac{Z}{Z-1}$$

ROC : $|Z| > 1$ ...(1)

According to scaling property we have,

$$a^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right) \qquad \text{...(2)}$$

Let $x(n) = u(n)$. Thus $Z\{a^n x(n)\} = Z\{a^n u(n)\}$ is obtained by replacing Z by $\frac{Z}{a}$ in Equation (1).

$$\therefore Z\{a^n u(n)\} = \frac{Z/a}{Z/a - 1} \quad \text{ROC} \left|\frac{Z}{a}\right| > 1$$

$$\therefore Z\{a^n u(n)\} = \frac{Z}{Z-a} \quad \text{ROC} : |Z| > |a|$$

We have already obtained Z-transform of this sequence. Thus using scaling property, same result is obtained.

Ex. 8.3.14 : Obtain Z-transform of

$$x(n) = a^n \cos\omega_0 n\, u(n).$$

Soln. : We have already obtained the Z-transform of $\cos\omega_0 n\, u(n)$.

Recall that result,

$$Z\{\cos\omega_0 n\, u(n)\} = \frac{Z^2 - Z\cos\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \qquad \text{...(1)}$$

ROC : $|Z| > 1$

According to scaling property we have,

$$a^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right) \qquad \text{...(2)}$$

Remember that whenever we use scaling property; we will have to replace Z by $\frac{Z}{a}$ in the equation of X (Z). Thus replacing Z by $\frac{Z}{a}$ in Equation (1) we get,

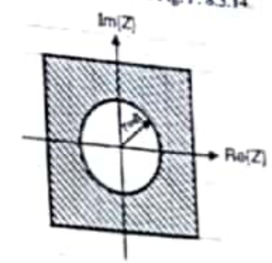
$$Z\{a^n \cos\omega_0 n\} = \frac{\left(\frac{Z}{a}\right)^2 - \left(\frac{Z}{a}\right)\cos\omega_0}{\left(\frac{Z}{a}\right)^2 - 2\left(\frac{Z}{a}\right)\cos\omega_0 + 1}$$

ROC : $\left|\frac{Z}{a}\right| > 1$ ...(3)

Multiplying and dividing numerator and denominator by a^2 we get,

$$Z\{a^n \cos\omega_0 n\} = \frac{Z^2 - aZ\cos\omega_0}{Z^2 - 2aZ\cos\omega_0 + a^2} \quad \text{ROC} : |Z| > |a|$$

Here ROC is $|Z| > |a|$. That means it is exterior part of circle having radius r = a. This is shown in Fig. P. 8.3.14.



(H-1017) **Fig. P. 8.3.14 : ROC of ($a^n \cos\omega_0 n$) u (n)**

Ex. 8.3.15 : Find Z transform of x (n) and draw its ROC.

$$x(n) = \left[(0.5)^n \sin\frac{n\pi}{4}\right] u(n)$$

Soln. : The given function is,

$$x(n) = \left[(0.5)^n \sin\frac{n\pi}{4}\right] u(n) = (0.5)^n \sin\frac{n\pi}{4}\, u(n) \qquad \text{...(1)}$$

We have standard Z transform pair.

$$\sin\omega_0 n\, u(n) \xleftrightarrow{Z} \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \quad \text{ROC} : |Z| > 1 \qquad \text{...(2)}$$

Consider the term $\sin\frac{n\pi}{4}\, u(n)$. Here, let $\omega_0 = \frac{\pi}{4}$. Thus Equation (2) becomes,

$$\sin\frac{n\pi}{4}\, u(n) \xleftrightarrow{Z} \frac{Z\sin\frac{\pi}{4}}{Z^2 - 2Z\cos\frac{\pi}{4} + 1}, \text{ROC} : |Z| > 1$$

$$\therefore\ Z\left\{\sin\frac{\pi}{4} n\, u(n)\right\} = \frac{0.707\,Z}{Z^2 - 1.414\,Z + 1}, \text{ROC} : |Z| > 1 \qquad \text{...(3)}$$

According to the scaling property we have,

$$a^n u(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right)$$

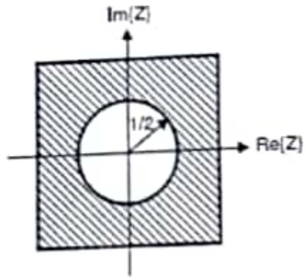
Here a = 0.5. Thus Z transform of given function, x (n) is obtained by replacing Z by $\frac{Z}{0.5}$ that means by 2Z in Equation (3) we get,

$$\therefore Z\left\{\left[(0.5)^n \sin\frac{\pi n}{4}\right]u(n)\right\} = \frac{0.707(2Z)}{(2Z)^2 - 1.414(2Z)+1}.$$

ROC : $|2Z| > 1$

$$\therefore X(Z) = \frac{1.414Z}{4Z^2 - 2.828 + 1}. \quad ROC : |Z| > \frac{1}{2}$$

Thus ROC is exterior part of circle having radius $\frac{1}{2}$ as shown in Fig. P. 8.3.15.



(II-1018) Fig. P. 8.3.15

Ex. 8.3.16 : Obtain Z-transform of $x(n) = a^n \sin\omega_0 n\, u(n)$.

Soln. : We have already obtained Z-transform of $\sin\omega_0 n$. Recall that result.

$$Z\{\sin\omega_0 n\, u(n)\} = \frac{Z\sin\omega_0}{Z^2 - 2Z\cos\omega_0 + 1} \quad ...(1)$$

ROC : $|Z| > 1$

Here we have.

$x(n) = a^n \sin\omega_0 n\, u(n)$. Thus according to scaling property, $X(Z)$ can be obtained by replacing Z by $\frac{Z}{a}$ in Equation (1).

$$\therefore Z\{a^n \sin\omega_0 n\, u(n)\} = \frac{\left(\frac{Z}{a}\right)\sin\omega_0}{\left(\frac{Z}{a}\right)^2 - 2\left(\frac{Z}{a}\right)\cos\omega_0 + 1}$$

ROC : $\left|\frac{Z}{a}\right| > 1$.

Dividing numerator and denominator by a^2 we get,

$$Z\{a^n \sin\omega_0 n\, u(n)\} = \frac{aZ\sin\omega_0}{Z^2 - 2aZ\cos\omega_0 + a^2} \quad ROC : |Z| > |a|$$

Ex. 8.3.17 : Find Z-transform and sketch the ROC $x(n) = (-1)^n 2^{-n} u(n)$

Soln. : Given $x(n) = (-1)^n 2^{-n} u(n)$

$$\therefore x(n) = (-1)^n \frac{1}{(2)^n} u(n)$$

$$\therefore x(n) = \left(-\frac{1}{2}\right)^n u(n) \quad ...(1)$$

Here $u(n)$ is unit step and its Z-transform is given by,

$$Z\{u(n)\} = \frac{Z}{Z-1} \quad ROC : |Z| > 1 \quad ...(2)$$

According to scaling property, we can write,

$$Z\{a^n u(n)\} = \frac{Z/a}{Z/a - 1} : ROC \left|\frac{Z}{a}\right| > 1 \quad ...(3)$$

Observe Equation (1). Here $a = -1/2$. Putting this value in Equation (3) we get,

$$Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{Z/(-1/2)}{\left(\frac{Z}{-1/2}\right) - 1} :$$

$$ROC\,|Z| > \left|\left(-\frac{1}{2}\right)\right| = \frac{-2Z}{-2Z-1} : ROC\,|Z| > \left|-\frac{1}{2}\right|$$

$$\therefore Z\left\{\left(-\frac{1}{2}\right)^n u(n)\right\} = \frac{2Z}{2Z+1} : ROC\,|Z| > 1/2$$

Thus ROC is exterior part of circle having radius r = 1/2.

Ex. 8.3.18 : Determine Z-transform including ROC of the following,

$$x(n) = \left(\frac{1}{2}\right)^n \{u(n) - u(n-10)\}$$

May 11, 8 Marks

Soln. : The given expression can be written as,

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n-10) \quad ...(1)$$

According to linearity property we have,

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z) \quad ...(2)$$

Recall the standard Z-transform pair,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad ROC : |Z| > |\alpha| \quad ...(3)$$

Consider first term of Equation (1).

Let, $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$

Using Equation (3) we can write,

$$X_1(Z) = Z\left\{\left(\frac{1}{2}\right)^n u(n)\right\}$$

$$= \frac{Z}{Z - 1/2} = \frac{2Z}{2Z-1} \quad ROC : |Z| > \frac{1}{2} \quad ...(4)$$

Now consider second term of Equation (1),

Let, $x_2(n) = \left(\frac{1}{2}\right)^n u(n-10)$ ...(5)

First we will obtain Z-transform of $u(n-10)$. Recall the time shifting property,

$$x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} \cdot X(Z)$$

Now we have, $u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}$; ROC : $|Z| > 1$

$$\therefore Z\{u(n-10)\} = Z^{-10}\left(\frac{Z}{Z-1}\right) : ROC\,|Z| > 1$$

$$\therefore Z\{u(n-10)\} = \frac{Z^{-9}}{Z-1} \quad ...(6)$$

According to scaling property we have,

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right)$$

That means we have to replace Z by $\frac{Z}{a}$.

In this case, $a = \frac{1}{2}$. Thus applying scaling property to Equation (6) we can write,

$$Z\left\{\left(\frac{1}{2}\right)^n u(n-10)\right\} = \frac{\left[\frac{Z}{(1/2)}\right]^{-9}}{\frac{Z}{1/2} - 1}$$

$$= \frac{(Z^{-9})\left(\frac{1}{1/2}\right)^{-9}}{\frac{Z}{1/2} - 1} ; \quad ROC : \left|\frac{Z}{1/2}\right| > 1$$

$$= \frac{Z^{-9} \cdot \left(\frac{1}{2}\right)^{+9}}{2Z-1} ; \quad ROC : |Z| > \frac{1}{2}$$

$$X_2(Z) = Z\left\{\left(\frac{1}{2}\right)^n u(n-10)\right\} = \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)}$$

ROC : $|Z| > \frac{1}{2}$ ...(7)

Putting Equations (4) and (7) in Equation (2) we get,

$$X(Z) = \frac{2Z}{2Z-1} - \frac{\left(\frac{1}{2}\right)^9}{Z^9(2Z-1)} ; \quad ROC : |Z| > \frac{1}{2}$$

$$\therefore X(Z) = \frac{2Z \times Z^9 - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)} ; \quad ROC : |Z| > \frac{1}{2}$$

$$\therefore X(Z) = \frac{2Z^{10} - \left(\frac{1}{2}\right)^9}{Z^9(2Z-1)} ; \quad ROC : |Z| > \frac{1}{2}$$

Thus ROC is exterior part of circle having radius r = 1/2.

Ex. 8.3.19 : Determine Z transform and draw ROC of the following signal $x(n) = (2)^{n+2} u(n-1)$. Is the signal causal ?

Soln. : The given signal is,

$$x(n) = (2)^{n+2} u(n-1)$$

$$\therefore x(n) = 2^n \cdot 2^2 u(n-1)$$

$$\therefore x(n) = 4 \cdot 2^n u(n-1) \quad ...(1)$$

First we will obtain Z transform of $u(n-1)$. We have,

$$u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-1}, ROC : |Z| > 1$$

According to shifting property we have,

$$u(n-1) \overset{Z}{\longleftrightarrow} Z^{-1}\left[\frac{Z}{Z-1}\right], ROC : |Z| > 1$$

$$\therefore Z\{u(n-1)\} = \frac{1}{Z-1}, ROC : |Z| > 1 \quad ...(2)$$

Now according to the scaling property, we have,

$$Z\{a^n u(n)\} = X\left(\frac{Z}{a}\right)$$

Here a = 2. Thus applying scaling property to Equation (2).

$$Z\{2^n u(n-1)\} = \frac{1}{\frac{Z}{2} - 1}, \quad ROC : \left|\frac{Z}{2}\right| > 1$$

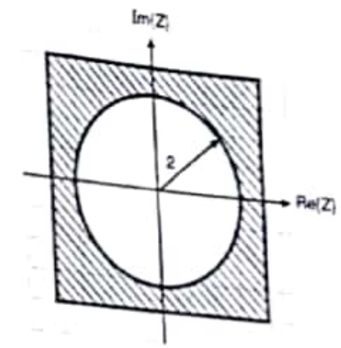
$$\therefore Z\{2^n u(n-1)\} = \frac{2}{Z-2}. \quad ROC : |Z| > 2$$

Thus for Equation (1) we can write,

$$Z\{4 \cdot 2^n u(n-1)\} = 4 \cdot \frac{2}{Z-2}. \quad ROC : |Z| > 2$$

$$\therefore X(Z) = \frac{8}{Z-2}. \quad ROC : |Z| > 2$$

Here ROC is exterior part of circle having radius 2, as shown in Fig. P. 8.3.19. Since ROC is exterior part of circle, the given sequence is causal.



(II-1020) Fig. P. 8.3.19

8.3.4 Time Reversal Property :

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$

ROC : $r_1 < |Z| < r_2$

Then $x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$

ROC : $\frac{1}{r_2} < |Z| < \frac{1}{r_1}$

Proof : According to definition of Z-transform we have,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$\therefore Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n} \quad ...(8.3.10)$$

Put $l = -n$: the limits will change as follows :

when $n = -\infty$, $l = \infty$ and when $n = \infty$, $l = -\infty$.

$$\therefore Z\{x(-n)\} = \sum_{l=\infty}^{-\infty} x(l) Z^{l}$$

$$\therefore Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l)(Z^{-1})^{-l} \quad ...(8.3.11)$$

Comparing this equation with the definition of Z-transform we get,

$$Z\{x(-n)\} = X(Z^{-1}) \quad ...(8.3.12)$$

Note : Here $x(-n)$ is the folded version of $x(n)$. So $x(-n)$ is the time reversed signal. Thus the folding of signal in time domain is equivalent to replacing Z by Z^{-1} in the Z-domain. Replacing Z by Z^{-1} is called as inversion. Hence folding in time domain is equivalent to inversion in the Z-domain.

Comment on ROC :

ROC of $x(n)$ is inverse of that of $x(-n)$. This means that, if Z_0 belongs to the ROC of $x(n)$ then $1/Z_0$ is in the ROC for $x(-n)$.

Given ROC is $r_1 < |Z| < r_2$. Thus ROC of $X(Z^{-1})$ is obtained by replacing Z by Z^{-1}.

Thus ROC of Z^{-1} : $r_1 < |Z^{-1}| < r_2$

$$\therefore \quad r_1 < \left|\frac{1}{Z}\right| < r_2$$

$$\therefore \quad |Z| < \frac{1}{r_1} \text{ and } |Z| > \frac{1}{r_2}$$

$$\therefore \quad \frac{1}{r_2} < |Z| < \frac{1}{r_1}$$

8.3.4.1 Solved Examples on Time Reversal Property :

Ex. 8.3.20 : Obtain the Z-transform of signal $x(n) = u(-n)$.

Soln. :

We have the Z-transform of unit step.

$$Z\{u(n)\} = \frac{Z}{Z-1} \quad ROC : |Z| > 1 \quad ...(1)$$

According to the time reversal property,

$$Z\{x(-n)\} = X(Z^{-1})$$

Thus Z-transform of $x(-n)$ is obtained by replacing Z by Z^{-1} in Equation (1).

$$\therefore \quad Z\{u(-n)\} = \frac{Z^{-1}}{Z^{-1}-1} \quad ROC : |Z^{-1}| > 1$$

Multiplying numerator and denominator by Z we get,

$$Z\{u(-n)\} = \frac{1}{1-Z} \quad ROC : |Z^{-1}| > 1$$

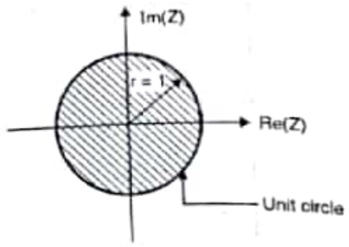
Here ROC is $|Z^{-1}| > 1$

$$\therefore \quad \left|\frac{1}{Z}\right| > 1 \text{ that means } |Z| < 1.$$

This is the standard Z-transform pair,

$$\therefore \quad u(-n) \overset{Z}{\longleftrightarrow} \frac{1}{1-Z} \quad ROC : |Z| < 1$$

Thus ROC is interior part of circle having radius r = 1. ***The circle or radius = 1 is also called as unit circle.*** Thus ROC is interior part of unit circle as shown in Fig. P. 8.3.20.



(H-1021) **Fig. P. 8.3.20 : ROC of u (– n)**

Ex. 8.3.21 : Determine Z-transform and ROC of :

$$x(n) = \left(\frac{1}{2}\right)^n u(-n)$$

Soln. : We have $Z\{u(-n)\} = \frac{1}{1-Z}$

$$ROC : |Z| < 1$$

Here the given signal is $\left(\frac{1}{2}\right)^n u(-n)$. According to scaling property we have,

$$Z\{a^n x(n)\} = X\left(\frac{Z}{a}\right).$$

That means we will have to replace Z by $\frac{Z}{a}$.

In this example $a = \frac{1}{2}$.

$$\text{Thus. } Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-\frac{Z}{1/2}}$$

$$ROC : \left|\frac{Z}{1/2}\right| < 1$$

$$\therefore \quad Z\left\{\left(\frac{1}{2}\right)^n u(-n)\right\} = \frac{1}{1-2Z} \quad ROC : |Z| < \frac{1}{2}$$

Thus ROC is interior part of circle having radius r = 1/2. This ROC is similar to Fig. P. 8.3.12. Only difference is here, r = 1/2.

Ex. 8.3.22 : Find Z-transform of following signals and comment on the results :

1. $x(n) = a^n u(n-1)$
2. $x(n) = a^n u(-n-1)$
3. $x(n) = -(a^n) u(n-1)$
4. $x(n) = -(a^n) u(-n-1)$

Dec. 10, 5 Marks

Soln. :

1. $x(n) = a^n u(n-1)$

Here $u(n-1)$ is delayed unit step. First we will obtain $Z\{u(n-1)\}$.

Now we have,

$$Z\{u(n)\} = \frac{Z}{Z-1} \quad ROC : |Z| > 1$$

Applying time shifting property we can write.

$$Z\{u(n-1)\} = Z^{-1} \times \frac{Z}{Z-1} = \frac{1}{Z-1} \quad ROC : |Z| > 1$$

Now we have to obtain $Z\{a^n u(n-1)\}$.

Applying scaling property we get,

$$Z\{a^n u(n-1)\} = \frac{1}{\frac{Z}{a}-1} \quad ROC : \left|\frac{Z}{a}\right| > 1$$

$$\therefore \quad Z\{a^n u(n-1)\} = \frac{a}{Z-a} \quad ROC : |Z| > a \quad ...(1)$$

Thus ROC is exterior part of circle having radius, r = a.

2. $x(n) = a^n u(-n-1)$

First we will obtain $Z\{u(-n-1)\}$. Here $u(-n)$ is folded unit step. The Z-transform of folded unit step is given by,

$$Z\{u(-n)\} = \frac{1}{1-Z} \quad ROC : |Z| < 1$$

Now recall time shifting property, it is

If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ then $x(n-k) \overset{Z}{\longleftrightarrow} Z^{-k} X(Z)$.

Here we have folded sequence. For folded sequence this property can be written as,

$$x(-n-k) \overset{Z}{\longleftrightarrow} Z^{+k} X(Z)$$

This is because folded sequence is mirror image of original sequence.

Thus we can write,

$$Z\{u(-n-1)\} = Z^{+1} \cdot Z\{u(-n)\} = Z^{1} \cdot \frac{1}{1-Z} \quad ROC : |Z| < 1$$

$$Z\{u(-n-1)\} = \frac{Z}{1-Z} \quad ROC : |Z| < 1$$

Now we have to obtain $Z\{a^n u(-n-1)\}$. Using scaling property we get.

$$Z\{a^n u(-n-1)\} = \frac{Z/a}{1-Z/a} \quad ROC : \left|\frac{Z}{a}\right| < 1$$

$$Z\{a^n u(-n-1)\} = \frac{Z}{a-Z} \quad ROC : |Z| < |a| \quad ...(2)$$

Thus ROC is interior part of circle having radius r = a.

3. $x(n) = -a^n u(n-1)$

From Equation (1) we have,

$$Z\{a^n u(n-1)\} = \frac{a}{Z-a}$$

$$ROC : |Z| > |a|$$

$$\text{Thus } Z\{-a^n u(n-1)\} = -\frac{a}{Z-a}$$

$$ROC : |Z| > |a|$$

$$Z\{-an\, u(n-1)\} = \frac{a}{a-Z} \quad ROC : |Z| > |a| \quad ...(3)$$

4. $x(n) = -(a)^n u(-n-1)$

From Equation (2) we have,

$$Z\{a^n u(-n-1)\} = \frac{Z}{a-Z}$$

$$ROC : |Z| < |a|$$

$$\text{Thus } Z\{-a^n u(-n-1)\} = -\frac{Z}{a-Z}$$

$$ROC : |Z| < |a|$$

$$Z\{-a^n u(-n-1)\} = \frac{Z}{Z-a} \quad ROC : |Z| < |a| \quad ...(4)$$

Note : Compare given Equations (1) and (3) as well as (2) and (4). Here magnitude of sequence is changed in time domain. From the result of Z-transform, Equations (1) and (3), and Equations (2) and (4), we can say that the magnitude in Z-domain is also reversed. Thus changing the magnitude in time domain, changes the magnitude in Z-domain also; while the ROC remains same.

Ex. 8.3.23 : Find the Z - transform of the following discrete-time signals and find ROC for each :

1. $x(n) = \left(\frac{-1}{5}\right)^n u(n) + 5\left(\frac{1}{2}\right)^{-n} u(-n-1)$
2. $x(n) = u(n-2)$

May 13, 10 Marks

Soln. :

1. $x(n) = \left(\frac{-1}{5}\right)^n u(n) + 5\left(\frac{1}{2}\right)^{-n} u(-n-1)$

Let $x(n) = x_1(n) + x_2(n)$

Here $x_1(n) = \left(\frac{-1}{5}\right)^n u(n)$

$$\text{We have, } \alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha}, |Z| > |\alpha|$$

$$\therefore X_1(Z) = \frac{Z}{Z+\frac{1}{5}}, |Z| > \frac{1}{5} \quad ...(1)$$

Now

$$x_2(n) = 5\left(\frac{1}{2}\right)^{-n} u(-n-1)$$

$$\therefore x_2(n) = 5 \cdot (2)^n u(-n-1)$$

We have,

$$Z\{u(-n)\} = \frac{1}{1-Z}, |Z| < 1$$

$$\therefore Z\{u(-n-1)\} = \frac{Z}{1-Z}, |Z| < 1$$

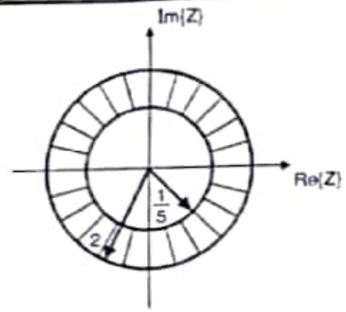
Using scaling property,

$$X_2(Z) = 5 \cdot \frac{Z/2}{1-Z/2}, \left|\frac{Z}{2}\right| < 1$$

$$\therefore X_2(Z) = \frac{5Z}{2-Z}, |Z| < 2$$

$$\therefore X(Z) = \frac{Z}{Z+\frac{1}{5}} + \frac{5Z}{2-Z}$$

The combined ROC is, $\frac{1}{5} < |Z| < 2$



(H-1202) **Fig. P. 8.3.23**

It is shown in Fig. P. 8.3.23.

2. $x(n) = u(n-2)$

We have,

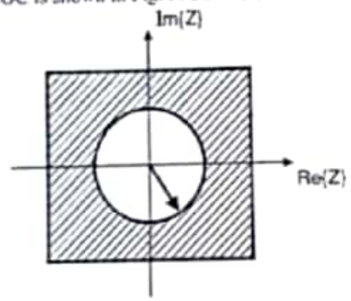
$$Z\{u(n)\} = \frac{Z}{Z-1}, |Z|>1$$

Using time shifting property,

$$Z\{u(n-2)\} = Z^{-2}\frac{Z}{Z-1}, |Z|>1$$

$$\therefore X(Z) = \frac{Z^{-1}}{Z-1}, |Z|>1$$

This ROC is shown in Fig. P. 8.3.23(a).



(H-1203) **Fig. P. 8.3.23(a)**

8.3.5 Differentiation in Z domain :

MU - May 04, Dec. 04

University Question

Q. 1 State and prove differentiation of Z transform. **(May 04, Dec. 04, 5 Marks)**

Statement : If $x(n) \xleftrightarrow{Z} X(Z)$ then $n\,x(n) \xleftrightarrow{Z} -Z\frac{dX(Z)}{dZ}$

Proof : According to the definition of Z-transform.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad ...(8.3.13)$$

Differentiate both sides with respect to Z.

$$\frac{dX(Z)}{dZ} = \frac{d}{dZ}\left[\sum_{n=-\infty}^{\infty} x(n) Z^{-n}\right]$$

We can transfer d/dZ inside the summation sign,

$$\therefore \frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} \frac{d}{dZ}[x(n) Z^{-n}]$$

$$= \sum_{n=-\infty}^{\infty} (-n) x(n) Z^{-n-1}$$

But Z^{-n-1} can be written as $Z^{-n} \cdot Z^{-1}$

$$\therefore \frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} (-n) x(n) Z^{-n} \cdot Z^{-1}$$

Since the limits of summation are in terms of n ; we can take Z^{-1} outside the summation.

$$\therefore \frac{dX(Z)}{dZ} = Z^{-1} \sum_{n=-\infty}^{\infty} (-n) x(n) Z^{-n}$$

Taking negative sign outside the summation,

$$\frac{dX(Z)}{dZ} = -Z^{-1} \sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n}$$

$$\therefore \frac{dX(Z)}{dZ} = -\frac{1}{Z} \sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n}$$

$$\therefore -Z\frac{dX(Z)}{dZ} = \sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n} \quad ...(8.3.14)$$

Comparing R.H.S. with Equation (8.3.13), we can say that $\sum_{n=-\infty}^{\infty} [n x(n)] Z^{-n}$ is the Z – transform of $n\,x(n)$. Thus Equation (8.3.14) becomes,

$$-Z\frac{dX(Z)}{dZ} = Z\{n x(n)\}$$

$$n x(n) \xleftrightarrow{Z} -Z\frac{dX(Z)}{dZ}$$

Hence proved. Note that both transforms have the same ROC.

Note : Multiplying the sequence in time domain by 'n' is equivalent to multiplying the derivative of its Z-transform, by – Z in the 'Z' domain.

8.3.5.1 Solved Examples of Differential Property :

Ex. 8.3.24 : Using differentiation property obtain the Z-transform of unit ramp sequence.

Soln. : We know that unit ramp sequence is given by,

$$x(n) = n u(n) \quad ...(1)$$

According to differentiation property we have,

$$Z\{n x(n)\} = -Z\frac{dX(Z)}{dZ} \quad ...(2)$$

In this case we can write,

$$Z\{n u(n)\} = -Z\frac{d}{dZ}[Z(u(n))] \quad ...(3)$$

We have $Z\{u(n)\} = \frac{Z}{Z-1}$ ROC : $|Z|>1$

$$\therefore Z\{n u(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z-1}\right]$$

$$\therefore Z\{n u(n)\} = -Z\frac{d}{dZ}[(Z)\cdot(Z-1)^{-1}]$$

$$= -Z\left[Z\cdot\frac{d}{dZ}[(Z-1)^{-1}] + (Z-1)^{-1}\cdot\frac{d}{dZ}Z\right]$$

$$= -Z[Z(-1)(Z-1)^{-2} + (Z-1)^{-1}\cdot 1]$$

$$= -Z\left[\frac{-Z}{(Z-1)^2} + \frac{1}{(Z-1)}\right] = -Z\left[\frac{-Z+Z-1}{(Z-1)^2}\right]$$

$$= -Z\left[\frac{-1}{(Z-1)^2}\right]$$

$$\therefore Z\{n u(n)\} = \frac{Z}{(Z-1)^2} \quad \text{ROC} : |Z|>1$$

This is the standard Z-transform pair. Earlier we had obtained the Z-transform of $n\,u(n)$ without using differentiation property.

Ex. 8.3.25 : Obtain the Z-transform of signal, $x(n) = n a^n u(n)$.

Soln. :

Let $x_1(n) = a^n u(n)$

$$\therefore x(n) = n x_1(n)$$

According to differentiation property,

$$n x_1(n) \xleftrightarrow{Z} -Z\frac{d}{dZ} X_1(Z)$$

In this case we can write, ...(1)

$$Z\{n a^n u(n)\} = -Z\frac{d}{dZ}\{a^n u(n)\} \quad ...(2)$$

Here the Z-transform of $a^n u(n)$ is given by,

$$Z\{a^n u(n)\} = \frac{Z}{Z-a} \quad \text{ROC} : |Z|>|a|$$

Putting this value in Equation (2) we get,

$$Z\{n a^n u(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z-a}\right]$$

$$\therefore Z\{n a^n u(n)\} = -Z\frac{d}{dZ}[Z\cdot(Z-a)^{-1}]$$

$$= -Z\left\{Z\cdot\frac{d}{dZ}[(Z-a)^{-1}] + (Z-a)^{-1}\cdot\frac{d}{dZ}Z\right\}$$

$$= -Z[Z\cdot(Z-a)^{-2}\cdot(-1) + (Z-a)^{-1}\cdot 1]$$

$$= -Z\left[\frac{-Z}{(Z-a)^2} + \frac{1}{(Z-a)}\right] = -Z\left[\frac{-Z+Z-a}{(Z-a)^2}\right]$$

$$= -Z\left[\frac{-a}{(Z-a)^2}\right]$$

$$\therefore Z\{n a^n u(n)\} = \frac{aZ}{(Z-a)^2} \quad \text{ROC} : |Z|>|a| \quad ...(3)$$

This is also standard Z-transform pair.

$$\therefore n a^n u(n) \xleftrightarrow{Z} \frac{aZ}{(Z-a)^2} \quad \text{ROC} : |Z|>|a|$$

Note : From this equation we can obtain Z-transform of unit ramp sequence, $n\,u(n)$. By putting a = 1 in Equation (3) we get,

$$Z\{n u(n)\} = \frac{Z}{(Z-1)^2} \quad \text{ROC} : |Z|>1$$

Ex. 8.3.26 : Find z transform of :

1. $x[n] = n^2 u[n]$
2. $x[n] = 2^n u[n-2]$ **May 13, 5 Marks**

Soln. :

1. **Given :** $x(n) = n^2 u(n) = n \cdot n u(n)$

We have Z transform of unit ramp,

$$Z\{n u(n)\} = \frac{Z}{(Z-1)^2}, |Z|>1$$

According to differentiation property,

$$Z\{n \cdot x(n)\} = -Z\frac{d}{dZ}X(Z)$$

$$\therefore Z\{n^2 u(n)\} = -Z\frac{d}{dZ}\left[\frac{Z}{(Z-1)^2}\right]$$

$$= -Z\left\{\frac{(Z-1)^2\cdot 1 - Z\cdot 2(Z-1)\cdot 1}{(Z-1)^4}\right\}$$

$$= -Z\left\{\frac{Z^2 - 2Z + 1 - 2Z^2 + 2Z}{(Z-1)^4}\right\} = -Z\left\{\frac{-Z^2+1}{(Z-1)^4}\right\}$$

$$\therefore Z\{n^2 u(n)\} = +Z\left\{\frac{Z^2-1}{(Z-1)^4}\right\} = Z\left\{\frac{(Z-1)(Z+1)}{(Z-1)^4}\right\}$$

$$\therefore Z\{n^2 u(n)\} = \frac{Z^2+Z}{(Z-1)^3}, |Z|>1$$

2. **Given :** $x(n) = 2^n u(n-2)$

We have $Z\{u(n)\} = \frac{Z}{Z-1}, |Z|>1$

According to time shifting property,

$$Z\{x(n-k)\} = Z^{-k}\cdot X(Z)$$

$$\therefore Z\{u(n-2)\} = Z^{-2}\cdot\frac{Z}{Z-1}, |Z|>1$$

$$\therefore Z\{u(n-2)\} = \frac{Z^{-1}}{Z-1}, |Z|>1$$

$$\therefore Z\{u(n-2)\} = \frac{1}{Z^2-Z}, |Z|>1$$

Now according to scaling property,

$$a^n x(n) \longleftrightarrow X\left(\frac{Z}{a}\right)$$

$$\therefore 2^n u(n-2) \longleftrightarrow \frac{1}{\left(\frac{Z}{2}\right)^2 - \frac{Z}{2}}, \left|\frac{Z}{2}\right|>1$$

$$\therefore 2^n u(n-2) \longleftrightarrow \frac{1}{\frac{Z^2}{4} - \frac{Z}{2}}, |Z|>2$$

$$\therefore 2^n u(n-2) \longleftrightarrow \frac{4}{Z^2-2Z}, |Z|>2$$

Ex. 8.3.27 : Find the ZT along with its ROC of $x(n) = (n+1)\,u(n)$ **May 13, 3 Marks**

Soln. :

Given,

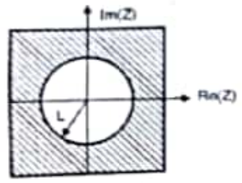
$$x(n) = (n+1)\,u(n) = n u(n) + u(n)$$

We have, $z\{n u(n)\} = \frac{z}{(z-1)^2}, |z|>1$

And $z\{u(n)\} = \frac{z}{z-1}, |z|>1$

$$\therefore x(z) = \frac{z}{(z-1)^2} + \frac{z}{z-1}, |z|>1$$

This ROC is shown in Fig. P. 8.3.27.



(B-1522) Fig. P.8.3.27

8.3.6 Convolution of Two Sequences :

MU - May 10

University Question

Q. 1 State and explain convolution property of Z-transform. **(May 10, 5 Marks)**

Statement : If $x_1(n) \xleftrightarrow{Z} X_1(Z)$ and $x_2(n) \xleftrightarrow{Z} X_2(Z)$

Then $x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$

and ROC is atleast the intersection of ROC of $X_1(Z)$ and $X_2(Z)$.

Proof : According to the definition of convolution, the convolution of $x_1(n)$ and $x_2(n)$ can be written as,

$$x(n) = x_1(n) * x_2(n) \quad \text{...(8.3.15)}$$

$$\therefore \quad x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \quad \text{...(8.3.16)}$$

Taking Z-transform of $x(n)$,

$$X(Z) = Z\left\{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right\} \quad \text{...(8.3.17)}$$

According to the definition of Z-transform we have,

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} \quad \text{...(8.3.18)}$$

In Equation (8.3.17) take the bracket term as $x(n)$. Thus Equation (8.3.18) becomes,

$$\therefore \quad Z\{x(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] \cdot Z^{-n}$$

Rearranging the summation terms,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-n}\right] \quad \text{...(8.3.19)}$$

Now Z^{-n} can be written as,

$$Z^{-n} = Z^{-n} \times Z^{k} \times Z^{-k} = Z^{-(n-k)} \times Z^{-k}$$

Putting this value in Equation (8.3.19) we get,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)} \cdot Z^{-k}\right]$$

In the second summation; the limits of summation are in terms of 'n'. So we can take Z^{-k} out of the summation sign.

$$\therefore \quad Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot Z^{-k}\right]\left[\sum_{k=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)}\right]$$

In the second summation put $n - k = m$. The limits will change as follows :

when $n = -\infty \Rightarrow -\infty - k = m \quad \therefore m = -\infty$

and when $n = +\infty \Rightarrow \infty - k = m \quad \therefore m = \infty$

$$\therefore \quad Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \times Z^{-k}\right]\left[\sum_{m=-\infty}^{\infty} x_2(m)\, Z^{-m}\right] \quad \text{...(8.3.20)}$$

Compare each bracket of R.H.S. with the definition of Z– transform.

$$\therefore \quad Z\{x(n)\} = X_1(Z) \cdot X_2(Z) \quad \text{...(8.3.21)}$$

But here $\quad x(n) = x_1(n) * x_2(n)$

$$\therefore \quad Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$$

ROC is atleast the intersection of $X_1(Z)$ and $X_2(Z)$.

> **Note :** Convolution of two sequences in time domain is equivalent to multiplication of its Z-transforms in Z domain.

Solved Examples :

Ex. 8.3.28 : Find the linear convolution of $x_1(n)$ and $x_2(n)$ using Z-transform

$x_1(n) = \{1, 2, \underset{\uparrow}{3}, 3, 4\}$ and $x_2(n) = \{1, \underset{\uparrow}{2}, 0, 2, 1\}$.

Soln. :

According to the property of linear convolution in Z-domain we have,

$$Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z) \quad \text{...(1)}$$

Step 1 : We have $x_1(n) = \{1, 2, \underset{\uparrow}{3}, 4\}$

According to the definition of Z-transform,

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n)\, Z^{-n}$$

Here limits of summation are from $n = -2$ to $n = 1$.

$$\therefore X_1(Z) = \sum_{n=-2}^{1} x_1(n)\, Z^{-n}$$

$$= x_1(-2)Z^2 + x_1(-1)Z + x_1(0)Z^0 + x_1(1)Z^{-1}$$

$$\therefore X_1(Z) = 1Z^2 + 2Z + 3 + 4Z^{-1} \quad \text{...(2)}$$

Let us determine the ROC of $X_1(Z)$.

We have $\quad X_1(Z) = Z^2 + 2Z + 3 + \frac{4}{Z}$

ROC : 1. Putting $Z = 0$ we get,

$$X_1(Z) = 0 + 0 + 3 + \frac{4}{0} = 0 + 0 + 3 + \infty = \infty$$

Thus $\quad Z = 0$ is not allowed.

2. Putting $Z = \infty$ we get,

$$X_1(Z) = \infty + \infty + 3 + \frac{4}{\infty} = \infty + \infty + 3 + 0 = \infty$$

Thus $Z = \infty$ is not allowed.

Therefore ROC is entire Z plane except $Z = 0$ and $Z = \infty$

Step 2 : Now we will obtain Z-transform of $x_2(n)$.

We have $x_2(n) = \{1, 2, \underset{\uparrow}{0}, 2, 1\}$

Using definition of Z-transform,

$$X_2(Z) = \sum_{n=-\infty}^{\infty} x_2(n)\, Z^{-n}$$

Here the range of n is $n = -2$ to $n = +2$

$$\therefore \quad X_2(Z) = \sum_{n=-2}^{2} x_2(n)\, Z^{-n}$$

$$\therefore X_2(Z) = x_2(-2)Z^2 + x_2(-1)Z^1 + x_2(0)Z^0 + x_2(1)Z^{-1} + x_2(2)Z^{-2}$$

$$\therefore \quad X_2(Z) = 1Z^2 + 2Z + 0Z^0 + 2Z^{-1} + 1Z^{-2}$$

$$\therefore \quad X_2(Z) = Z^2 + 2Z + 2Z^{-1} + Z^{-2}$$

Let us determine the ROC of $X_2(Z)$ ...(3)

We have $X_2(Z) = Z^2 + 2Z + \frac{2}{Z} + \frac{1}{Z^2}$

ROC :

1. Putting $Z = 0$ we get,

$$X_2(Z) = 0 + 0 + \frac{2}{0} + \frac{1}{0} = 0 + 0 + \infty + \infty = \infty.$$

Thus $Z = 0$ is not allowed.

2. Putting $Z = \infty$ we get,

$$X_2(Z) = \infty + \infty + \frac{2}{\infty} + \frac{1}{\infty} = \infty + \infty + 0 + 0 = \infty.$$

Thus $Z = \infty$ is not allowed.

Therefore the ROC is entire Z plane except $Z = 0$ and $Z = \infty$.

Step 3 : Now we have,

$$X(Z) = X_1(Z) \cdot X_2(Z)$$

$$\therefore \quad X(Z) = (Z^2 + 2Z + 3 + 4Z^{-1}) \cdot (Z^2 + 2Z + 2Z^{-1} + Z^{-2})$$

$$\therefore X(Z) = Z^4 + 2Z^3 + 2Z + 1 + 2Z^3 + 4Z^2 + 4 + 2Z^{-1} + 3Z^2 + 6Z + 6Z^{-1} + 3Z^{-2} + 4Z + 8 + 8Z^{-2} + 4Z^{-3}$$

$$\therefore X(Z) = Z^4 + 4Z^3 + 7Z^2 + 12Z + 13 + 8Z^{-1} + 11Z^{-2} + 4Z^{-3} \quad \text{...(4)}$$

Step 4 : According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)\, Z^{-n}$$

From Equation (4), we get the range of n. It is from $n = -4$ to $n = 3$.

$$\therefore \quad X(Z) = \sum_{n=-4}^{3} x(n)\, Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(-4)Z^4 + x(-3)Z^3 + x(-2)Z^2 + x(-1)Z + x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} \quad \text{...(5)}$$

Comparing Equations (4) and (5),

$x(-4) = 1$, $\quad x(-3) = 4$,
$x(-2) = 7$, $\quad x(-1) = 12$,
$x(0) = 13$, $\quad x(1) = 8$,
$x(2) = 11$, $\quad x(3) = 4$.

Thus we can write,

$$x(n) = \{x(-4), x(-3), x(-2), x(-1), \underset{\uparrow}{x(0)}, x(1), x(2), x(3)\}$$

$$\therefore \quad x(n) = \{1, 4, 7, 12, \underset{\uparrow}{13}, 8, 11, 4\}$$

Ex. 8.3.29 : Calculate Z transform of the following signals :

$$x[n] = n\left(-\frac{1}{4}\right)^n u[n] * \left(-\frac{1}{6}\right)^n u[-n]$$

Soln. : **Dec. 2014, 5 Marks**

Given : $\quad x[n] = n\left(-\frac{1}{4}\right)^n u[n] * \left(-\frac{1}{6}\right)^n u[-n]$

Convolution is equivalent to multiplication in Z domain.

$$\therefore \quad x(Z) = Z\left\{n\left(-\frac{1}{4}\right)^n u(n)\right\} \cdot Z\left\{\left(-\frac{1}{6}\right)^n u(-n)\right\}$$

Now we have, $n\, a^n u(n) \longleftrightarrow \frac{aZ}{(Z-a)^2}, \quad |Z| > |a|$

$$\therefore \quad Z\left\{n\left(-\frac{1}{4}\right)^n u(n)\right\} = \frac{\frac{1}{4}Z}{\left(Z - \frac{1}{4}\right)^2}, \quad |Z| > \frac{1}{4}$$

Consider second term,

We have, $\quad Z\{u(n)\} = \frac{Z}{Z-1}, \quad |Z| > 1$

Using time reversal property,

$$Z\{u(-n)\} = \frac{Z^{-1}}{Z^{-1} - 1}, \quad |Z| < 1$$

$$= \frac{1}{1 - Z}, \quad |Z| < 1$$

Using scaling property,

$$Z\left\{\left(-\frac{1}{6}\right)^n u(-n)\right\} = \frac{1}{1 - \frac{Z}{\left(-\frac{1}{6}\right)}}, \quad \left|\frac{Z}{\left(-\frac{1}{6}\right)}\right| < 1$$

$$= \frac{6}{6 + Z}, \quad |Z| < \frac{1}{6}$$

$$\therefore \quad x(Z) = \frac{\frac{1}{4}Z}{(Z - 1/4)^2} \cdot \frac{6}{(6 + Z)}$$

Ex. 8.3.30 : Convolve $x[n] = \left(\frac{1}{3}\right)^n u[n]$ with $h[n] = \left(\frac{1}{2}\right)^n u[n]$ using convolution sum formula and verify your answer using Z transform. **Dec. 2014, 10 Marks**

Soln. :

Given : $\quad x(n) = \left(\frac{1}{3}\right)^n u(n)$ and $h(n) = \left(\frac{1}{2}\right)^n u(n)$

Using convolution :

According to definition of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

Here $x(k) = \left(\frac{1}{3}\right)^k u(k)$, $h(n-k) = \left(\frac{1}{2}\right)^{n-k} u(n-k)$

$$= \left(\frac{1}{2}\right)^n \cdot 2^k \cdot u(-k+n)$$

$$\therefore\ y(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u(k) \cdot \left(\frac{1}{2}\right)^n \cdot 2^k \cdot u(-k+n)$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{\infty} \left(\frac{2}{3}\right)^k u(k) \cdot u(-k+n)$$

Due to $u(k) \cdot u(-k+n)$, the range of summation becomes $k = 0$ to n.

$$\therefore\ y(n) = \left(\frac{1}{2}\right)^n \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

$$\therefore\ y(n) = \left(\frac{1}{2}\right)^n \left[\frac{\left(\frac{2}{3}\right)^0 - \left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1-(2/3)^{n+1}}{\frac{1}{3}}\right]$$

$$= 3\left(\frac{1}{2}\right)^n \left[1-\left(\frac{2}{3}\right)^{n+1}\right]$$

$$\therefore\ y(n) = 3\left(\frac{1}{2}\right)^n - 3\left(\frac{1}{2}\right)^n \cdot \left(\frac{2}{3}\right)^n \cdot \left(\frac{2}{3}\right)$$

$$= \left[3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n\right]$$

Using Z transform :

We have, $x(Z) = \frac{Z}{Z-\frac{1}{3}}$ and $H(Z) = \frac{Z}{Z-\frac{1}{2}}$

Now convolution becomes multiplication in Z domain.

$$\therefore\ Y(Z) = X(Z) \cdot H(Z) = \frac{Z}{Z-\frac{1}{3}} \cdot \frac{Z}{Z-\frac{1}{2}}$$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{Z}{\left(Z-\frac{1}{3}\right)\left(Z-\frac{1}{2}\right)}$$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{A_1}{Z-\frac{1}{3}} + \frac{A_2}{Z-\frac{1}{2}}$$

Here $A_1 = \frac{\frac{1}{3}}{\frac{1}{3}-\frac{1}{2}} = \frac{\frac{1}{3}}{-\frac{1}{6}} = -2$

and $A_2 = \frac{\frac{1}{2}}{\frac{1}{2}-\frac{1}{3}} = \frac{\frac{1}{2}}{\frac{1}{6}} = 3$

$$\therefore\ \frac{y(Z)}{Z} = \frac{-2}{Z-\frac{1}{3}} + \frac{3}{Z-\frac{1}{2}}$$

$$\therefore\ y(Z) = \frac{-2Z}{Z-\frac{1}{3}} + \frac{3Z}{Z-\frac{1}{2}}$$

Taking IZT we get,

$$y(n) = -2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n).$$

Ex. 8.3.31 : A DT signal is given by the following expression. Find its Z transform.

$$x[n] = n\left(-\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^{-n} u[-n]$$

May 2016, 10 Marks

Soln. :

Given : $x(n) = n\left(-\frac{1}{2}\right)^n u(n) * \left(\frac{1}{4}\right)^{-n} u(-n) = x_1(n) * x_2(n)$

In Z domain convolution becomes multiplication

$\therefore X(Z) = X_1(Z) \cdot X_2(Z)$

Here $x_1(n) = n\left(-\frac{1}{2}\right)^n u(n)$

We have, $a^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-a}, |Z| > |a|$

$$\therefore\ \left(-\frac{1}{2}\right)^n u(n) \longleftrightarrow \frac{Z}{Z+\frac{1}{2}}, |Z| > \frac{1}{2}$$

Using differentiation property,

$$n\{n\,x(n)\} \longleftrightarrow -Z\frac{d}{dz} X(Z)$$

$$\therefore\ X_1(Z) = Z\left\{n\left(-\frac{1}{2}\right)^n u(n)\right\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z+\frac{1}{2}}\right]$$

$$= -Z\left[\frac{\left(Z+\frac{1}{2}\right)\cdot 1 - Z(1)}{\left(Z+\frac{1}{2}\right)^2}\right]$$

$$= \frac{-Z\left(Z+\frac{1}{2}\right)+Z^2}{\left(Z+\frac{1}{2}\right)^2} = \frac{-Z^2-\frac{1}{2}Z+Z^2}{\left(Z+\frac{1}{2}\right)^2}$$

$$\therefore\ X_1(Z) = \frac{-\frac{1}{2}Z}{\left(Z+\frac{1}{2}\right)^2}, |Z| > \frac{1}{2} \qquad ...(1)$$

Now $x_2(n) = \left(\frac{1}{4}\right)^{-n} u(-n) = (4)^n u(-n)$

Using time reversal property,

$$u(-n) \longleftrightarrow \frac{1}{1-Z}, |Z| < 1$$

Using scaling property,

$$X_2(\tau) = Z\{(4)^n u(-n)\} = \frac{1}{1-\frac{Z}{4}}, \left|\frac{Z}{4}\right| < 1$$

$$\therefore\ X_2(\tau) = \frac{4}{4-Z}, |Z| < 4 \qquad ...(2)$$

Thus from Equations (1) and (2),

$$X(z) = \frac{-\frac{1}{2}Z}{\left(Z+\frac{1}{2}\right)^2} \cdot \frac{4}{(4-Z)}, \frac{1}{2} < |Z| < 4$$

Ex. 8.3.32 : State and prove with appropriate mathematical derivation, 'convolution in time domain' property and 'time reversal' property of Z transform. Also comment on importance of these properties in the field of communication and signal processing.

Dec. 2016, 10 Marks

Soln. :

Convolution of Two Sequences :

Statement : If $x_1(n) \xleftrightarrow{Z} X_1(Z)$ and $x_2(n) \xleftrightarrow{Z} X_2(Z)$

Then $x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$

and ROC is atleast the intersection of ROC of $X_1(Z)$ and $X_2(Z)$.

Proof : According to the definition of convolution, the convolution of $x_1(n)$ and $x_2(n)$ can be written as,

$$x(n) = x_1(n) * x_2(n) \qquad ...(1)$$

$$\therefore\ x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \qquad ...(2)$$

Taking Z-transform of $x(n)$,

$$X(Z) = Z\left\{\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right\} \qquad ...(3)$$

According to the definition of Z-transform we have,

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} \qquad ...(4)$$

In Equation (3) take the bracket term as $x(n)$. Thus Equation (4) becomes,

$$\therefore\ Z\{x(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] \cdot Z^{-n}$$

Rearranging the summation terms,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-n}\right] \qquad ...(5)$$

Now Z^{-n} can be written as,

$$Z^{-n} = Z^{-n} \times Z^k \times Z^{-k} = Z^{-(n-k)} \times Z^{-k}$$

Putting this value in Equation (5) we get,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)} \cdot Z^{-k}\right]$$

In the second summation; the limits of summation are in terms of 'n'. So we can take Z^{-k} out of the summation sign.

$$\therefore\ Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot Z^{-k}\right]\left[\sum_{k=-\infty}^{\infty} x_2(n-k)\, Z^{-(n-k)}\right]$$

In the second summation put $n - k = m$. The limits will change as follows :

when $n = -\infty \Rightarrow -\infty - k = m \quad \therefore m = -\infty$

and when $n = +\infty \Rightarrow \infty - k = m \quad \therefore m = \infty$

$$\therefore\ Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \times Z^{-k}\right]\left[\sum_{m=-\infty}^{\infty} x_2(m)\, Z^{-m}\right] \qquad ...(6)$$

Compare each bracket of R.H.S. with the definition of Z-transform.

$$\therefore\ Z\{x(n)\} = X_1(Z) \cdot X_2(Z)$$

But here $x(n) = x_1(n) * x_2(n) \qquad ...(7)$

$$\therefore\ Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$$

ROC is atleast the intersection of $X_1(Z)$ and $X_2(Z)$.

Time Reversal Property :

Statement : If $x(n) \xleftrightarrow{Z} X(Z)$ ROC : $r_1 < |Z| < r_2$

Then $x(-n) \xleftrightarrow{Z} X(Z^{-1})$ ROC : $\frac{1}{r_2} < |Z| < \frac{1}{r_1}$

Proof : According to definition of Z-transform we have,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$\therefore\ Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n} \qquad ...(8)$$

Put $l = -n$; the limits will change as follows :

when $n = -\infty$, $l = \infty$ and when $n = \infty$, $l = -\infty$.

$$\therefore\ Z\{x(-n)\} = \sum_{l=\infty}^{-\infty} x(l) Z^{l}$$

$$\therefore\ Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l)(Z^{-1})^{-l} \qquad ...(9)$$

Comparing this equation with the definition of Z-transform we get,

$$Z\{x(-n)\} = X(Z^{-1}) \qquad ...(10)$$

Comment on ROC : ROC of $x(n)$ is inverse of that of $x(-n)$. This means that, if Z_0 belongs to the ROC of $x(n)$ then $1/Z_0$ is in the ROC for $x(-n)$.

Given ROC is $r_1 < |Z| < r_2$. Thus ROC of $X(Z^{-1})$ is obtained by replacing Z by Z^{-1}.

Thus ROC of Z^{-1} : $r_1 < |Z^{-1}| < r_2$

Using convolution :

According to definition of convolution,

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Here $x(k) = \left(\frac{1}{3}\right)^k u(k)$, $h(n-k) = \left(\frac{1}{2}\right)^{n-k} u(n-k)$

$$= \left(\frac{1}{2}\right)^n \cdot 2^k \cdot u(-k+n)$$

$$\therefore\ y(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u(k) \cdot \left(\frac{1}{2}\right)^n \cdot 2^k \cdot u(-k+n)$$

$$= \left(\frac{1}{2}\right)^n \sum_{k=-\infty}^{\infty} \left(\frac{2}{3}\right)^k u(k) \cdot u(-k+n)$$

Due to $u(k) \cdot u(-k+n)$, the range of summation becomes $k = 0$ to n.

$$\therefore\ y(n) = \left(\frac{1}{2}\right)^n \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k$$

$$\therefore\ y(n) = \left(\frac{1}{2}\right)^n \left[\frac{\left(\frac{2}{3}\right)^0 - \left(\frac{2}{3}\right)^{n+1}}{1-\frac{2}{3}}\right]$$

$$= \left(\frac{1}{2}\right)^n \left[\frac{1-(2/3)^{n+1}}{\frac{1}{3}}\right]$$

$$= 3\left(\frac{1}{2}\right)^n \left[1-\left(\frac{2}{3}\right)^{n+1}\right]$$

$$\therefore\ y(n) = 3\left(\frac{1}{2}\right)^n - 3\left(\frac{1}{2}\right)^n \cdot \left(\frac{2}{3}\right)^n \cdot \left(\frac{2}{3}\right)$$

$$= \left[3\left(\frac{1}{2}\right)^n - 2\left(\frac{1}{3}\right)^n\right]$$

Using Z transform :

We have, $x(Z) = \dfrac{Z}{Z-\frac{1}{3}}$ and $H(Z) = \dfrac{Z}{Z-\frac{1}{2}}$

Now convolution becomes multiplication in Z domain.

$$\therefore\ Y(Z) = X(Z) \cdot H(Z) = \frac{Z}{Z-\frac{1}{3}} \cdot \frac{Z}{Z-\frac{1}{2}}$$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{Z}{\left(Z-\frac{1}{3}\right)\left(Z-\frac{1}{2}\right)}$$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{A_1}{Z-\frac{1}{3}} + \frac{A_2}{Z-\frac{1}{2}}$$

Here $A_1 = \dfrac{\frac{1}{3}}{\frac{1}{3}-\frac{1}{2}} = \dfrac{\frac{1}{3}}{-\frac{1}{6}} = -2$

and $A_2 = \dfrac{\frac{1}{2}}{\frac{1}{2}-\frac{1}{3}} = \dfrac{\frac{1}{2}}{\frac{1}{6}} = 3$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{-2}{Z-\frac{1}{3}} + \frac{3}{Z-\frac{1}{2}}$$

$$\therefore\ Y(Z) = \frac{-2Z}{Z-\frac{1}{3}} + \frac{3Z}{Z-\frac{1}{2}}$$

Taking IZT we get,

$$y(n) = -2\left(\frac{1}{3}\right)^n u(n) + 3\left(\frac{1}{2}\right)^n u(n).$$

Ex. 8.3.31 : A DT signal is given by the following expression. Find its Z transform.

$$x[n] = n\left(-\frac{1}{2}\right)^n u[n] * \left(\frac{1}{4}\right)^{-n} u[-n]$$

May 2016, 10 Marks

Soln. :

Given : $x(n) = n\left(-\frac{1}{2}\right)^n u(n) * \left(\frac{1}{4}\right)^{-n} u(-n) = x_1(n) * x_2(n)$

In Z domain convolution becomes multiplication

$\therefore X(Z) = X_1(Z) \cdot X_2(Z)$

Here $x_1(n) = n\left(-\frac{1}{2}\right)^n u(n)$

We have, $a^n u(n) \xleftrightarrow{Z} \dfrac{Z}{Z-a}, |Z| > |a|$

$$\therefore\ \left(-\frac{1}{2}\right)^n u(n) \longleftrightarrow \frac{Z}{Z+\frac{1}{2}}, |Z| > \frac{1}{2}$$

Using differentiation property,

$$n\{n\,x(n)\} \longleftrightarrow -Z\frac{d}{dz}X(Z)$$

$$\therefore X_1(Z) = Z\left\{n\left(-\frac{1}{2}\right)^n u(n)\right\} = -Z\frac{d}{dZ}\left[\frac{Z}{Z+\frac{1}{2}}\right]$$

$$= -Z\left[\frac{\left(Z+\frac{1}{2}\right)\cdot 1 - Z(1)}{\left(Z+\frac{1}{2}\right)^2}\right]$$

$$= \frac{-Z\left(Z+\frac{1}{2}\right)+Z^2}{\left(Z+\frac{1}{2}\right)^2} = \frac{-Z^2-\frac{1}{2}Z+Z^2}{\left(Z+\frac{1}{2}\right)^2}$$

$$\therefore X_1(Z) = \frac{-\frac{1}{2}Z}{\left(Z+\frac{1}{2}\right)^2}, |Z| > \frac{1}{2} \qquad ...(1)$$

Now $x_2(n) = \left(\frac{1}{4}\right)^{-n} u(-n) = (4)^n u(-n)$

Using time reversal property,

$$u(-n) \longleftrightarrow \frac{1}{1-Z}, |Z| < 1$$

Using scaling property,

$$X_2(\tau) = Z\{(4)^n u(-n)\} = \frac{1}{1-\frac{Z}{4}}, \left|\frac{Z}{4}\right| < 1$$

$$\therefore X_2(\tau) = \frac{4}{4-Z}, |Z| < 4$$

Thus from Equations (1) and (2), ...(2)

$$X(z) = \frac{-\frac{1}{2}Z}{\left(Z+\frac{1}{2}\right)^2} \cdot \frac{4}{(4-Z)}, \frac{1}{2} < |Z| < 4$$

Ex. 8.3.32 : State and prove with appropriate mathematical derivation, 'convolution in time domain' property and 'time reversal' property of Z transform. Also comment on importance of these properties in the field of communication and signal processing.

Dec. 2016, 10 Marks

Soln. :

Convolution of Two Sequences :

Statement : If $x_1(n) \xleftrightarrow{Z} X_1(Z)$ and $x_2(n) \xleftrightarrow{Z} X_2(Z)$

Then $x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$

and ROC is atleast the intersection of ROC of $X_1(Z)$ and $X_2(Z)$.

Proof : According to the definition of convolution, the convolution of $x_1(n)$ and $x_2(n)$ can be written as,

$$x(n) = x_1(n) * x_2(n) \qquad ...(1)$$

$$\therefore\ x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k) \qquad ...(2)$$

Taking Z-transform of $x(n)$,

$$X(Z) = Z\left\{\sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)\right\} \qquad ...(3)$$

According to the definition of Z-transform we have,

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n} \qquad ...(4)$$

In Equation (3) take the bracket term as $x(n)$. Thus Equation (4) becomes,

$$\therefore\ Z\{x(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)\right] \cdot Z^{-n}$$

Rearranging the summation terms,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\,Z^{-n}\right] \qquad ...(5)$$

Now Z^{-n} can be written as,

$$Z^{-n} = Z^{-n} \times Z^k \times Z^{-k} = Z^{-(n-k)} \times Z^{-k}$$

Putting this value in Equation (5) we get,

$$Z\{x(n)\} = \sum_{k=-\infty}^{\infty} x_1(k)\left[\sum_{n=-\infty}^{\infty} x_2(n-k)\,Z^{-(n-k)} \cdot Z^{-k}\right]$$

In the second summation; the limits of summation are in terms of 'n'. So we can take Z^{-k} out of the summation sign.

$$\therefore\ Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot Z^{-k}\right]\left[\sum_{k=-\infty}^{\infty} x_2(n-k)\,Z^{-(n-k)}\right]$$

In the second summation put $n - k = m$. The limits will change as follows :

when $n = -\infty \Rightarrow -\infty - k = m \quad \therefore m = -\infty$

and when $n = +\infty \Rightarrow \infty - k = m \quad \therefore m = \infty$

$$\therefore\ Z\{x(n)\} = \left[\sum_{k=-\infty}^{\infty} x_1(k) \times Z^{-k}\right]\left[\sum_{m=-\infty}^{\infty} x_2(m)\,Z^{-m}\right] \qquad ...(6)$$

Compare each bracket of R.H.S. with the definition of Z-transform.

$$\therefore\ Z\{x(n)\} = X_1(Z) \cdot X_2(Z)$$

But here $x(n) = x_1(n) * x_2(n)$ ...(7)

$$\therefore Z\{x_1(n) * x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$$

ROC is atleast the intersection of $X_1(Z)$ and $X_2(Z)$.

Time Reversal Property :

Statement : If $x(n) \xleftrightarrow{Z} X(Z)$ ROC : $r_1 < |Z| < r_2$

Then $x(-n) \xleftrightarrow{Z} X(Z^{-1})$ ROC : $\frac{1}{r_2} < |Z| < \frac{1}{r_1}$

Proof : According to definition of Z-transform we have,

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n)\,Z^{-n}$$

$$\therefore\ Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\,Z^{-n} \qquad ...(8)$$

Put $l = -n$; the limits will change as follows :

when $n = -\infty$, $l = \infty$ and when $n = \infty$, $l = -\infty$.

$$\therefore\ Z\{x(-n)\} = \sum_{l=\infty}^{-\infty} x(l)\,Z^{l}$$

$$\therefore\ Z\{x(-n)\} = \sum_{l=-\infty}^{\infty} x(l)\,(Z^{-1})^{-l} \qquad ...(9)$$

Comparing this equation with the definition of Z-transform we get,

$$Z\{x(-n)\} = X(Z^{-1}) \qquad ...(10)$$

Comment on ROC : ROC of $x(n)$ is inverse of that of $x(-n)$. This means that, if Z_0 belongs to the ROC of $x(n)$ then $1/Z_0$ is in the ROC for $x(-n)$.

Given ROC is $r_1 < |Z| < r_2$. Thus ROC of $X(Z^{-1})$ is obtained by replacing Z by Z^{-1}.

Thus ROC of Z^{-1} : $r_1 < |Z^{-1}| < r_2$

$$\therefore\ r_1 < \left|\frac{1}{2}\right| < r_2$$

$$\therefore\ |Z| < \frac{1}{r_1} \text{ and } |Z| > \frac{1}{r_2}$$

$$\therefore\ \frac{1}{r_2} < |Z| < \frac{1}{r_1}$$

8.3.7 Initial Value Theorem : MU - May 08, Dec. 09

University Question

Q. 1 State and prove initial and final value theorem in Z-transform.

(May 06, 8 Marks, Dec. 09, 5 Marks)

Statement : If $x(n)$ is a causal sequence then its initial value is given by,

$$x(0) = \lim_{Z \to \infty} X(Z)$$

Proof : According to the definition of Z-transform.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{...(8.3.22)}$$

But if $x(n)$ is a causal sequence then limits of summation will be from $n = 0$ to $n = \infty$.

$$\therefore\ X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \text{...(8.3.23)}$$

Expanding the summation we get,

$$X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots$$

Applying limits as $Z \to \infty$ we get.

$$\lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1) Z^{-1} + \lim_{Z \to \infty} x(2) Z^{-2} + \ldots\ldots \text{(as } Z^0 = 1)$$

$$\therefore\ \lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} x(0) + \lim_{Z \to \infty} x(1) \cdot \frac{1}{Z} + \lim_{Z \to \infty} x(2) \frac{1}{Z^2} + \ldots \quad \text{...(8.3.24)}$$

Now as $Z \to \infty$ the terms $\frac{1}{Z} = \frac{1}{Z^2}$ etc. will be $\frac{1}{\infty}$ which is zero.

Thus Equation (8.3.24) becomes,

$$\lim_{Z \to \infty} X(Z) = x(0) \quad \text{...(8.3.25)}$$

But $x(0)$ is called as initial value of $x(n)$.

$$\therefore\ \text{Initial value} = x(0) = \lim_{Z \to \infty} X(Z) \text{ Hence proved.}$$

8.3.8 Final Value Theorem : MU - May 06, Dec. 09

University Question

Q. 1 State and prove initial and final value theorem in Z-transform.

(May 06, 8 Marks, Dec. 09, 5 Marks)

Statement : If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$ then,

$$x(\infty) = \lim_{Z \to 1} [(Z-1)X(Z)]$$

Proof : According to the definition of Z-transform for causal sequence we have,

$$Z\{x(n)\} = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \text{...(8.3.26)}$$

Consider the term $x(n) - x(n-1)$, using Equation (8.3.26), Z-transform of this term can be expressed as,

$$Z\{x(n) - x(n-1)\} = \sum_{n=0}^{\infty} [x(n) - x(n-1)] Z^{-n}$$

$$\therefore\ Z\{x(n)\} - Z\{x(n-1)\} = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-n} \quad \text{...(8.3.27)}$$

Now we have $Z\{x(n)\} = X(Z)$ and

According to shifting property, $Z\{x(n-1)\} = Z^{-1} X(Z)$

$$\therefore\ X(Z) - Z^{-1}X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-n}$$

$$\therefore\ X(Z)(1 - Z^{-1}) = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-n} \quad \text{...(8.3.28)}$$

Now consider the second summation. Here Z^{-n} can be written as.

$$Z^{-n} = Z^{-n} \cdot Z^1 \cdot Z^{-1} = Z^{-(n-1)} \cdot Z^{-1}$$

$$\therefore\ X(Z)(1 - Z^{-1}) = \sum_{n=0}^{\infty} x(n) Z^{-n} - \sum_{n=0}^{\infty} x(n-1) Z^{-(n-1)} Z^{-1}$$

$$\therefore\ X(Z)(1 - Z^{-1}) = \sum_{n=0}^{\infty} x(n) Z^{-n} - Z^{-1} \sum_{n=0}^{\infty} x(n-1) Z^{-(n-1)}$$

Taking limit as $Z \to 1$ on both sides,

$$\lim_{Z \to 1} X(Z)(1 - Z^{-1}) = \lim_{Z \to 1} \sum_{n=0}^{\infty} x(n) Z^{-n} - \lim_{Z \to 1} Z^{-1} \cdot \sum_{n=0}^{\infty} x(n-1) Z^{-(n-1)}$$

Remember that the first summation is $Z\{x(n)\}$ and second summation is $Z\{x(n-1)\}$. Now expanding the summations we can write,

$$\therefore \lim_{Z \to 1} X(Z)(1 - Z^{-1}) = \lim_{Z \to 1} \left[\begin{array}{l} (X(0) - Z^{-1}X(-1)) \\ + (X(1) - Z^{-1}X(0)) \\ + (X(2) - Z^{-1}X(1)) + \ldots \\ + [X(\infty) - Z^{-1}X(\infty - 1) \end{array} \right] \quad \text{...(8.3.29)}$$

Here as $\lim_{Z \to 1}$ we get $\lim_{Z \to 1} Z^{-1} = \lim_{Z \to 1} \frac{1}{Z} = 1$ and $X(-1) = 0$ because $x(n) = 0$ for $n = -1$ since it is a causal sequence.

$$\lim_{Z \to 1} X(Z)(1 - Z^{-1}) = [X(0) - 0] + [X(1) - X(0)] + \ldots\ldots + [X(\infty) - X(\infty - 1)] = X(\infty)$$

... (Here we are adding the difference between two terms and this addition is upto ∞).

$$X(\infty) = \lim_{Z \to 1} X(Z)(1 - Z^{-1})$$

Hence proved.

Solved Examples :

Ex. 8.3.33 : Find initial and final values of $x(n)$ if Z transform is, $X(Z) = \frac{2}{Z^2 + \frac{1}{6}Z - \frac{1}{6}}$

Soln. : According to initial value theorem,

$$x(0) = \lim_{Z \to \infty} X(Z) = \lim_{Z \to \infty} \frac{2}{Z^2 + \frac{1}{6}Z - \frac{1}{6}}$$

$$\therefore\ x(0) = 0$$

According final value theorem,

$$x(\infty) = \lim_{Z \to 1} (Z-1) \cdot X(Z) = \lim_{Z \to 1} (Z-1) \cdot \frac{2}{Z^2 + \frac{1}{6}Z - \frac{1}{6}}$$

We will obtain roots of denominator term.

$$Z^2 + \frac{1}{6}Z - \frac{1}{6} \Rightarrow \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{\frac{-1}{6} \pm \sqrt{\frac{1}{36} + \frac{4}{6}}}{2}$$

$$\therefore\ \text{roots} = \frac{-\frac{1}{6} \pm \sqrt{\frac{1+24}{36}}}{2} = \frac{-\frac{1}{6} \pm \frac{5}{6}}{2} = \frac{1}{3}, -\frac{1}{2}$$

$$\therefore\ x(\infty) = \lim_{Z \to 1} (Z-1) \cdot \frac{2}{\left(Z - \frac{1}{3}\right)\left(Z + \frac{1}{2}\right)}$$

$$\therefore\ x(\infty) = 0$$

Ex. 8.3.34 : Determine the value of signal $x(n)$ at $n = 0$ and $n = \infty$ if, $X(Z) = \frac{2Z^2 + 0.25}{(Z + 0.25)(Z - 1)}$

Soln. : According to initial value theorem,

$$x(0) = \lim_{Z \to \infty} X(Z)\ x(0) = \lim_{Z \to \infty} \frac{2Z^2 + 0.25}{(Z + 0.25)(Z - 1)}$$

Multiplying numerator and denominator by Z^{-2};

$$X(0) = \lim_{Z \to \infty} \frac{2 + 0.25Z^{-2}}{Z^{-1}(Z + 0.25) \cdot Z^{-1}(Z - 1)} = \lim_{Z \to \infty} \frac{2 + \frac{0.25}{Z^2}}{\left(1 + \frac{0.25}{Z}\right)\left(1 - \frac{1}{Z}\right)}$$

$$\therefore\ x(0) = \frac{2}{1} = 2$$

According to final value theorem,

$$x(\infty) = \lim_{Z \to 1} (Z-1)\ X(Z)$$

$$\therefore\ x(\infty) = \lim_{Z \to 1} (Z-1) \cdot \frac{2Z^2 + 0.25}{(Z + 0.25)(Z - 1)} = \lim_{Z \to 1} \frac{2Z^2 + 0.25}{Z + 0.25} = \frac{2 + 0.25}{1 + 0.25}$$

$$\therefore\ x(\infty) = 1.8$$

Ex. 8.3.35 : Find the z transform and region of convergence for the following sequence.

$$X(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$$

Apply initial value theorem and check the z transform whether it is correct or not.

Soln. :

Given : $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - \left(\frac{1}{2}\right)^n u(n)$

We have,

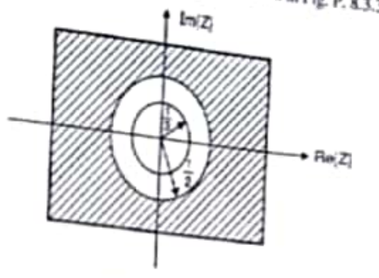
$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z - \alpha}, \quad |z| > |\alpha|$$

$$\therefore\ \left(\frac{1}{3}\right)^n u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z - \frac{1}{3}}, |z| > \frac{1}{3}$$

and $$\left(\frac{1}{2}\right)^n u(n) \overset{Z}{\longleftrightarrow} \frac{z}{z - \frac{1}{2}}, |z| > \frac{1}{2}$$

$$\therefore\ X(z) = 7 \cdot \frac{z}{z - \frac{1}{3}} - 6 \cdot \frac{z}{z - \frac{1}{2}}$$

The combined ROL is $|z| > \frac{1}{2}$. It is shown in Fig. P. 8.3.35.



(B-1323) **Fig. P. 8.3.35**

Now we will apply initial value theorem. According to initial value theorem.

$$x(0) = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{7z}{z - \frac{1}{3}} - \frac{6z}{z - \frac{1}{2}}$$

Dividing numerator and denominator by z we get,

$$x(0) = \lim_{z \to \infty} \frac{7}{1 - \frac{1}{3z}} - \frac{6}{1 - \frac{1}{2z}} = 7 - 6 = 1$$

Now consider given equation,

$$x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$$

Putting $n = 0$ we get,

$$x(0) = 7\left(\frac{1}{3}\right)^0 u(0) - 6\left(\frac{1}{2}\right)^0 u(0) = 7 - 6 = 1$$

Thus calculated z transfer is correct.

Ex. 8.3.36 : Find the initial value $X(0)$ and final value $X(\infty)$ of following :

$$X(z) = \frac{1}{1 + 2z^{-1} - 3z^{-2}}$$

Dec. 11, 4 Marks

Soln. :

Given : $X(z) = \dfrac{1}{1 + 2z^{-1} - 3z^{-2}}$

$$\therefore \; X(z) = \frac{z^2}{z^2 + 2z - 3}$$

Initial value is given by,

$$X(0) = \lim_{z \to \infty} X(z) = \lim_{z \to \infty} \frac{z^2}{z^2 + 2z - 3} = \lim_{z \to \infty} \frac{1}{1 + \frac{2}{z} - \frac{3}{z^2}}$$

$$\therefore \; X(0) = 1$$

Final value is given by,

$$X(\infty) = \lim_{z \to 1} (z - 1) \cdot X(z)$$

$$= \lim_{z \to 1} (z - 1) \frac{z^2}{z^2 - 2z - 3}$$

$$= \lim_{z \to 1} (z - 1) \cdot \frac{z^2}{(z - 3)(z + 1)}$$

$$\therefore \; X(\infty) = 0$$

8.4 Summary of the Properties of Z-Transform :

Table 8.4.1 shows the summary of Z-transform properties.

Table 8.4.1

	Property	Equation Time domain Z-domain	ROC
1.	Linearity	$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} a_1 X_1(Z) + a_2 X_2(Z)$	Intersection of the ROC of $X_1(Z)$ and $X_2(Z)$
2.	Time shifting	$x(n-k) \xleftrightarrow{Z} Z^{-k} X(Z)$	Same as ROC of $X(Z)$ Except $Z = 0$ if $K > 0$ and $Z = \infty$ if $K < 0$
3.	Scaling in Z domain	$a^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right)$	$\lvert a \rvert r_1 < \lvert Z \rvert < \lvert a \rvert r_2$
4.	Time reversal	$x(-n) \xleftrightarrow{Z} X(Z^{-1})$	$\frac{1}{r_2} < \lvert Z \rvert < \frac{1}{r_1}$
5.	Differentiation in Z domain	$nx(n) \xleftrightarrow{Z} -Z \frac{d}{dZ} X(Z)$	Same as ROC of $X(Z)$
6.	Convolution	$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(2) \cdot X_2(Z)$	ROC is atleast intersection of ROC of $X_1(Z)$ and $X_2(Z)$
7.	Correlation	$r_{x_1 x_2}(l) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z^{-1})$	ROC is intersection of ROC of X_1 (Z) and $X_2(Z)$
8.	Multiplication	$X_1(n) \cdot x_2(n) \xleftrightarrow{Z} \frac{1}{2\pi j} \oint_c X_1(v) X_2\left(\frac{Z}{v}\right) v^{-1} dv$	ROC is at least $r_{1l}\, r_{2l} \lvert Z \rvert < r_{1u}\, r_{2u}$
9.	Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n) x_2^*(n) \xleftrightarrow{Z} \frac{1}{2\pi j} \oint_c X_1(v) X_2^*\left(\frac{1}{v}\right) v^{-1} dv$	–
10.	Initial value theorem	Causal $x(n)$ $x(0) = \lim_{Z \to \infty} X(Z)$	–
11.	Final value theorem	Causal $x(n)$: $x(\infty) = \lim_{Z \to 1} X(Z)$	–

8.5 Inverse Z Transform (IZT) :

In the last sections we have studied how to obtain Z-transform of given discrete time sequence $x(n)$. Now in this section we will discuss how to obtain sequence $x(n)$ from given $X(Z)$. This procedure of obtaining $x(n)$ from its Z-transform $X(Z)$ is called as inverse Z-transform (IZT).

Methods for obtaining IZT :

The different methods for obtaining IZT from $X(Z)$ are as follows :

1. Power series expansion.
2. Partial fraction expansion.
3. Residue method.

8.5.1 Power Series Expansion Method :

This method is also called as ***long division method or direct division method***. Generally $X(Z)$ is expressed as power series in 'Z'.

$$\therefore \; X(Z) = a_0 + a_1 Z^{-1} + a_2 Z^{-2} + \ldots\ldots + a_n Z^{-n} \qquad \ldots(8.5.1)$$

Equation (8.5.1) can be easily obtained by dividing numerator by the denominator of $X(Z)$. Now according to the definition of 'Z' transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(8.5.2)$$

If the sequence is causal then the limits of 'n' will be from $n = 0$ to $n = \infty$.

$$\therefore \; X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \qquad \ldots(8.5.3)$$

Expanding Equation (8.5.3) we get,

$$X(Z) = x(0) Z^0 + x(1) Z^{-1} + x(2) Z^{-2} + \ldots\ldots + x(n) Z^{-n}$$

But $Z^0 = 1$.

$$\therefore \; X(Z) = x(0) + x(1) Z^{-1} + x(2) Z^{-2} + \ldots\ldots + x(n) Z^{-n} \qquad \ldots(8.5.4)$$

Now by comparing Equations (8.5.1) and (8.5.4) we can write,

$$x(0) = a_0$$
$$x(1) = a_1$$
$$x(2) = a_2$$
$$\vdots$$
$$x(n) = a_n$$

Thus the general expression of discrete time causal sequence $x(n)$ is,

$$x(n) = a_n \quad n \geq 0 \qquad \ldots(8.5.5)$$

Rules for power stries expansion :

1. In case of anti-causal sequence carryout the long division by writing the polynomials in the reverse order that means starting with the most negative term on the left.
2. In case of causal sequence, carryout the long division without changing order of the polynomial. That means starting with the most positive term on the left.

8.5.1.1 Solved Examples on IZT using Power Series Expansion :

Ex. 8.5.1 : Using power series method for determining Inverse Z transform (IZT) of

$$X(Z) = \frac{1}{1 - 1.5Z^{-1} + 0.5Z^{-2}}$$

for

1. ROC $|Z| > 1$
2. ROC $|Z| < 0.5$.

Soln. :

$$X(Z) = \frac{1}{1 - 1.5Z^{-1} + 0.5Z^{-2}}$$

1. ROC $|Z| > 1$

That means it is causal sequence. So most positive term should be at the beginning. Here given term is in the proper form. The long division is performed as follows.

$$1 - 5Z^{-1} + 0.5Z^{-2} \overline{\big)\, 1} \quad 1 + 1.5Z^{-1} + 1.75Z^{-2} + 1.875Z^{-3} + \ldots$$

$$\begin{array}{l}
1 - 1.5Z^{-1} + 0.5Z^{-2} \\
- \quad + \quad - \\
\hline
1.5Z^{-1} - 0.5Z^{-2} \\
1.5Z^{-1} - 2.25Z^{-2} + 0.75Z^{-3} \\
- \quad + \quad - \\
\hline
1.75Z^{-2} - 0.75Z^{-3} \\
1.75Z^{-2} - 2.625Z^{-3} + 0.875Z^{-4} \\
- \quad + \quad - \\
\hline
1.875Z^{-3} - 0.875Z^{-4}
\end{array}$$

$$\therefore \; X(Z) = 1 + 1.5Z^{-1} + 1.75Z^{-2} + 1.875Z^{-3} \ldots \qquad \ldots(1)$$

According to the definition of Z transform for causal sequence, $X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$

Expanding the summation we get,

$$X(Z) = x(0) + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3} + \ldots \qquad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$x(0) = 1$, $x(1) = 1.5$, $x(2) = 1.75$, $x(3) = 1.875$, ...

Now,

$x(n) = \{x(0), x(1), x(2), x(3) \ldots\}$

$\therefore \; x(n) = \{1, 1.5, 1.75, 1.875, \ldots\}$

2. ROC $|Z| < 0.5$

It represents non-causal sequence. For non-causal sequence; the denominator should have most negative of Z at the beginning. So given equation should be written as,

$$X(Z) = \frac{1}{0.5Z^{-2} - 1.5Z^{-1} + 1}$$

Now we will perform the long division as follows :

$$
\begin{array}{r|l}
 & 2Z^2 + 6Z^3 + 14Z^4 + 30Z^5 \\
0.5z^{-2} - 1.5z^{-1} + 1 & 1 \\
 & 1 - 3Z + 2Z^2 \\
 & - \quad + \quad - \\
 & 0 + 3Z - 2Z^2 \\
 & 3Z - 9Z^2 + 6Z^3 \\
 & - \quad + \quad - \\
 & 7Z^2 - 6Z^3 \\
 & 7Z^2 - 21Z^3 + 14Z^4 \\
 & - \quad + \quad - \\
 & 15Z^3 - 14Z^4 \\
 & 15Z^3 - 45Z^4 + 30Z^5 \\
 & - \quad + \quad - \\
 & 31Z^4 - 30Z^5
\end{array}
$$

Thus X (Z) can be approximately written as,

$$X(Z) = 2Z^2 + 6Z^3 + 14Z^4 + 30Z^5 + \ldots\ldots\ldots \quad \ldots(3)$$

Now according to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

For anti-causal sequence; the limits of summation will be from $n = -1$ to $n = -\infty$.

$$\therefore X(Z) = \sum_{n=-1}^{\infty} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(-1)Z^1 + x(-2)Z^2 + x(-3)Z^3 + x(-4)Z^4 + x(-5)Z^5 + \ldots\ldots \quad \ldots(4)$$

Comparing Equations (3) and (4) we have,

$x(0) = 0$ $\quad$ $x(-3) = 6$

$x(-1) = 0$ $\quad$ $x(-4) = 14$

$x(-2) = 2$ $\quad$ $x(-5) = 30 \ldots\ldots$

Thus sequence x (n) is,

$$x(n) = \{\ldots\ldots x(-5), x(-4), x(-3), x(-2), x(-1), \underset{\uparrow}{x(0)}\}$$

$$\therefore \quad x(n) = \{\ldots\ldots 30, 14, 6, 2, 0, \underset{\uparrow}{0}\}$$

Ex. 8.5.2 : Find inverse Z of

$$X(Z) = \frac{1 - \frac{1}{2}Z^{-1}}{1 - \frac{1}{4}Z^{-2}} \quad |Z| > 1/2$$

Soln. : The given ROC is $|Z| > 1/2$. That means it is causal sequence. Now for the causal sequence the denominator polynomial should have the maximum power of Z on its left. The given expression has the denominator in the proper form. So we don't have to rearrange the expression.

Now we will perform the long division as follows :

$$
\begin{array}{r|l}
 & 1 - \frac{1}{2}Z^1 + \frac{1}{4}Z^2 - \frac{1}{8}Z^3 \ldots\ldots \\
1 - \frac{1}{4}Z^2 & 1 - \frac{1}{2}Z^{-1} \\
 & 1 - \frac{1}{4}Z^{-2} \\
 & - \quad + \\
 & -\frac{1}{2}Z^1 + \frac{1}{4}Z^2 \\
 & -\frac{1}{2}Z^{-1} + \frac{1}{8}Z^{-3} \\
 & + \quad - \\
 & \frac{1}{4}Z^2 - \frac{1}{8}Z^{-3} \\
 & \frac{1}{4}Z^2 - \frac{1}{16}Z^{-4} \\
 & - \quad + \\
 & -\frac{1}{8}Z^3 + \frac{1}{16}Z^{-4} \\
 & -\frac{1}{8}Z^{-3} + \frac{1}{32}Z^5 \\
 & + \quad - \\
 & \frac{1}{16}Z^{-4} - \frac{1}{32}Z^5 \ldots\ldots
\end{array}
$$

Thus the expression X (Z) can be approximately written as,

$$X(Z) = 1 - \frac{1}{2}Z^{-1} + \frac{1}{4}Z^{-2} - \frac{1}{8}Z^{-3} + \quad \ldots(1)$$

According to the definition of Z-transform we have,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

The given sequence is a causal sequence. For causal sequence the limits of summation will be from $n = 0$ to $n = \infty$.

$$\therefore X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

Expanding the summation we get,

$$X(Z) = x(0)Z^0 + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots\ldots$$

$$\therefore X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots\ldots \quad (2)$$

Comparing Equations (1) and (2) we get,

$x(0) = 1$ $\quad$ $x(2) = \frac{1}{4}$

$x(1) = -\frac{1}{2}$ $\quad$ $x(3) = -\frac{1}{8}$

Thus the sequence x (n) is written as,

$$x(n) = \{x(0), x(1), x(2), x(3) \ldots\ldots\}$$

$$\therefore x(n) = \left\{1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8} \ldots\ldots\ldots\right\} \quad \ldots(3)$$

This is the required sequence x(n). Now Equation (3) can also be written as,

$$x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

Ex. 8.5.3 : Find the inverse Z of : $X(Z) = \frac{Z}{Z-1}$ $\quad |Z| > 1$.

Soln. :

Since the given ROC is $|Z| > 1$; it is causal sequence. We know that for the causal sequence the denominator polynomial should have maximum power of Z on its left. Thus the given expression is in the proper form.

> **Note :** We can also convert $X(Z) = \frac{Z}{Z-1}$ into negative powers of Z by multiplying and dividing by Z^{-1}. Thus $X(Z) = \frac{1}{1 - Z^{-1}}$ and we can solve using the same method.

Now performing the long division we get,

$$
\begin{array}{r|l}
 & 1 - Z^{-1} + Z^{-2} + Z^{-3} + \ldots\ldots \\
Z - 1 & 1 \\
 & Z - 1 \\
 & - \quad + \\
 & 1 \\
 & 1 - Z^{-1} \\
 & - \quad + \\
 & Z^{-1} \\
 & Z^{-1} - Z^{-2} \\
 & - \quad + \\
 & Z^{-2} \\
 & Z^{-2} - Z^{-3} \\
 & - \quad + \\
 & Z^{-3}
\end{array}
$$

Thus X (Z) can be approximately written as,

$$X(Z) = 1 + Z^{-1} + Z^{-2} + Z^{-3} + \ldots\ldots \quad \ldots(1)$$

Now the Z-transform of causal sequence is given by,

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$$

Expanding the summation,

$$X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + \ldots \quad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$$x(n) = \{x(0), x(1), x(2), x(3) \ldots\ldots\}$$

$$\therefore x(n) = \{\underset{\uparrow}{1}, 1, 1, 1, \ldots\ldots\ldots\}$$

$$\therefore \quad x(n) = u(n)$$

$$\therefore Z^{-1}\left\{\frac{Z}{Z-1}\right\} = u(n)$$

This is standard Z-transform pair.

Ex. 8.5.4 : Obtain inverse Z of : $X(Z) = \frac{Z}{Z-a}$ if $|Z| < |a|$.

Soln. : Given ROC is $|Z| < |a|$. So the sequence is anti-causal. That means the numerator polynomial should have minimum power of Z on its left. Thus we can write X (Z) as :

$$X(Z) = \frac{Z}{-a + Z} \quad \text{if } |Z| < |a| \quad \ldots(1)$$

Now we will perform the long division as follows :

$$
\begin{array}{r|l}
 & -a^{-1}Z - a^{-2}Z^2 - a^{-3}Z^3 \ldots \\
-a + Z & Z \\
 & Z - a^{-1}Z^2 \\
 & - \quad + \\
 & a^{-1}Z^2 \\
 & a^{-1}Z^2 - a^{-2}Z^3 \\
 & - \quad + \\
 & a^{-2}Z^3 \\
 & a^{-2}Z^3 - a^{-3}Z^4 \\
 & - \quad + \\
 & a^{-3}Z^4
\end{array}
$$

Thus X (Z) can be approximately written as,

$$X(Z) = -a^{-1}Z^1 - a^{-2}Z^2 - a^{-3}Z^3 - \ldots\ldots\ldots \quad \ldots(2)$$

According to definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

For anti-causal sequence, the range of n is from $n = -1$ to $n = -\infty$

$$\therefore \quad X(Z) = \sum_{n=-1}^{-\infty} x(n) Z^{-n}$$

Expanding the summation,

$$X(Z) = x(-1)Z^1 + x(-2)Z^2 + x(-3)Z^3 + \ldots \quad \ldots(3)$$

Comparing Equations (2) and (3) we get,

$x(-1) = -a^{-1}$

$x(-2) = -a^{-2}$

$x(-3) = -a^{-3} \ldots\ldots$

Thus sequence x(n) can be written as,

$$x(n) = \{\ldots\ldots -a^{-3}, -a^{-2}, -a^{-1}, \underset{\uparrow}{0}\} \quad \ldots(4)$$

Equation (4) can be written as,

$$x(n) = -a^n u(-n-1)$$

This is the standard Z-transform pair.

Ex. 8.5.5 : Find inverse Z-transform :

$$X(Z) = \log(1 + aZ^{-1}), \quad |Z| > |a|$$

Soln. :

The given expression is,

$$X(Z) = \log(1 + aZ^{-1}), \quad |Z| > |a| \quad \ldots(1)$$

Now the power series expansion for $\log(1+x)$ with $|x|<1$ is given by,

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots\ldots\ldots$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n \qquad \ldots(2)$$

Let $x = aZ^{-1}$ Thus Equation (2) becomes,

$$X(Z) = \log(1 + aZ^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} [aZ^{-1}]^n$$

$$\therefore \quad X(Z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^n \cdot Z^{-n} \qquad \ldots(3)$$

Now according to the definition of Z-transform,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(4)$$

But the given sequence is causal, since $|Z| > |a|$. And we will assume that the sequence is present from $n = 1$ to $n = \infty$. Thus changing the limits of summation in Equation (4) we get,

$$X(Z) = \sum_{n=1}^{\infty} x(n) Z^{-n} \qquad \ldots(5)$$

Now comparing Equations (3) and (5), we get,

$$x(n) = \frac{(-1)^{n+1}}{n} a^n \quad \text{if } n \geq 1$$

This is the required IZT.

Ex. 8.5.6 : Find inverse z transform of

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - z^{-1} + 0.356\, z^{-2}}$$

Soln. :

We have to use long division method.

$$1 - z^{-1} + 0.356z^{-2} \;\big)\; 1 + 2z^{-1} + z^{-2} \;\big(\; 1 + 3z^{-1} + 3.644z^{-2} + 2.57z^{-3}$$

$$\begin{array}{l}
1 - z^{-1} + 0.356z^{-2} \\
\underline{- \quad + \quad -} \\
\quad 3z^{-1} + 0.644z^{-2} \\
\quad 3z^{-1} - 3z^{-2} + 1.068z^{-3} \\
\quad \underline{- \quad + \quad -} \\
\qquad 3.644z^{-2} - 1.068z^{-3} \\
\qquad 3.644z^{-2} - 3.694z^{-3} + 1.297z^{-4} \\
\qquad \underline{- \quad + \quad -} \\
\qquad\qquad 2.57z^{-3} - 1.297z^{-4}
\end{array}$$

$$\therefore \quad X(z) = 1 + 3z^{-1} + 3.644z^{-2} + 2.57z^{-3} + \ldots \qquad \ldots(1)$$

Using definition of z, $\quad X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$\therefore \quad X(z) = x(0) + x(1) z^{-1} + X(2) z^{-2} + x(3) z^{-3} + \ldots\ldots(2)$$

Comparing Equations (1) and (2) we get,

$x(0) = 1,\ x(1) = 3, \quad x(2) = 3.644, \quad x(3) = 2.57 \ldots$

$$\therefore \quad x(n) = \{1, 3, 3.6444, 2.57, \ldots\}$$

Ex. 8.5.7 : Obtain the inverse Z-transform of :

$$X(z) = \frac{1 - az^{-1}}{z^{-1} - a}, \ |z| > |1/a|$$

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Soln. :

$$\frac{1 - az^{-1}}{z^{-1} - a} = \frac{z^2}{z^2}\left(\frac{1 - az^{-1}}{z^{-1} - a}\right)$$

$$= \frac{z^2 - az}{z - az^2}$$

$$\frac{X(z)}{z} = \frac{-a + z}{z - az^2}$$

$$= \frac{-a + z}{z(1 - az)}$$

$$= \frac{A_1}{2} + \frac{A_2}{1 - az}$$

$$A_1 = z \frac{-a + z}{z(1 - az)} \Bigg|_{z=0}$$

$$A_1 = -a$$

$$A_2 = \frac{-a + z}{z(1 - az)}(1 - az) \Bigg|_{z = \frac{1}{a}}$$

$$= \frac{-a + \frac{1}{a}}{\frac{1}{a}} = -a^2 + 1$$

$$\frac{X(z)}{z} = \frac{-a}{z} + \frac{1 - a^2}{1 - az}$$

$$X(z) = \frac{-az}{z} + \frac{(1 - a^2) z}{1 - az}$$

$$= -a - \frac{(1 - a^2) z}{(az - 1)}$$

$$= -a - \left(\frac{1}{a}\right)(1 - a^2) \frac{z}{\left(z - \frac{1}{a}\right)}$$

$$X(z) = -a - \left(\frac{1 - a^2}{a}\right)\left(\frac{z}{z - \frac{1}{a}}\right)$$

$$X(t) = -a - \left(\frac{1 - a^2}{a}\right)\left(\frac{1}{a}\right)^n u(n)$$

Ex. 8.5.8 : Find a casual sequence where z transform is given by $X(z) = \sin\left(\frac{1}{z}\right)$

Soln. :

Given : $\quad X(z) = \sin\left(\frac{1}{z}\right) = \sin(z^{-1})$

We have a series expansion formula,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n (z^{-1})^{2n+1}}{(2n+1)!}$$

$$\therefore \quad x(z) = \sin(z^{-1}) = \sum_{n=0}^{\infty} \frac{(-1)^n (z^{-1})^{2n+1}}{(2n+1)!}$$

By expanding the summation term we get,

$$X(z) = \frac{z^{-1}}{1} - \frac{z^{-3}}{6} + \frac{z^{-5}}{120} - \frac{z^{-7}}{5040} + \ldots \qquad \ldots(1)$$

Using definition of z, $\quad X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$\therefore X(z) = x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4} + x(5)z^{-5} + x(6)z^{-6} + x(7)z^{-7} + \ldots \qquad \ldots(2)$$

Comparing Equations (1) and (2) we get,

$x(0) = 0, \quad x(1) = 1, \quad x(2) = 0, \quad x(3) = -\frac{1}{6}, \quad x(4) = 0,$

$x(5) = \frac{1}{120}, \quad x(6) = 0, \quad x(7) = -\frac{1}{5040} \ldots$

$$\therefore \quad x(n) = \left\{0, 1, 0, -\frac{1}{6}, 0, -\frac{1}{5040} \ldots\right\}$$

Limitations of power series expansion method :

If 'n' is large then the long division will become tedious. The method is used only when the resulting pattern is simple enough or if we want the values of only first few samples.

8.5.2 Partial Fraction Expansion (P.F.E.) Method :

This is possible for the Z transforms which are rational in nature, that means they are expressed as the ratio of two polynomials.

That means

$$X(Z) = \frac{N(Z)}{D(Z)}$$

$$= \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + \ldots\ldots + b_M Z^{-M}}{a_0 + a_1 Z^{-1} + a_2 Z^{-2} + \ldots a_N Z^{-N}} \qquad \ldots(8.5.6)$$

Here $N(Z)$ = Numerator polynomial

$D(Z)$ = Denominator polynomial

$b_0, b_1 \ldots b_M$ = Coefficients of numerator

$a_0, a_1 \ldots a_N$ = Coefficients of denominator

M = Degree of numerator

N = Degree of denominator.

Steps to follow in Partial Fraction Expansion :

1. Check whether the given function is in proper form or not. The function is said to be in proper form when following conditions are satisfied.
   (a) The coefficient a_0 in Equation (8.5.6) should be equal to 1 ($a_0 = 1$). If $a_0 \neq 1$ then the polynomials are adjusted accordingly.
   (b) In Equation (8.5.6) $a_n \neq 1$ and the degree of numerator should be less than the degree of denominator ($M < N$). If this condition is not satisfied then the long division is carried out to make $M < N$.

   Thus the first step is to obtain the function in the proper form.
2. Multiply the numerator and denominator by Z^N. That means convert the function in terms of positive powers of Z.
3. Obtain the equation $\frac{X(Z)}{Z}$.
4. Factorise the denominator and obtain the roots. Then the denominator will be in the form $(Z - P_1)(Z - P_2) \ldots\ldots (Z - P_N)$. Here $P_1, P_2 \ldots\ldots P_N$ are called as poles.
5. Write down the equation in partial fraction expansion form as follows.

$$\frac{X(Z)}{Z} = \frac{A_1}{Z - P_1} + \frac{A_2}{Z - P_2} + \ldots\ldots + \frac{A_N}{Z - P_N} \qquad \ldots(8.5.7)$$

Here $A_1, A_2, \ldots A_N$ are coefficients. The coefficient A_K is calculated as,

$$A_K = (Z - P_K) \cdot \frac{X(Z)}{Z}\Bigg|_{Z = P_K} \qquad \ldots(8.5.8)$$

Equation (8.7.8) indicates that the calculation $(Z - P_K) \cdot \frac{X(Z)}{Z}$ is performed by putting $Z = P_K$. Here $P_K = P_1, P_2, \ldots\ldots$

6. By calculating values of $A_1, A_2, \ldots A_N$, transfer 'Z' to the R.H.S. of Equation (8.5.7). Now we standard Z transform pairs to obtain inverse Z transform.

$$a^n u(n) \xleftrightarrow{Z} \frac{Z}{Z - a}, \text{ROC } |Z| > |a| \qquad \ldots(8.5.9)$$

$$\text{and } -a^n u(-n-1) \xleftrightarrow{Z} \frac{Z}{Z - a}, \text{ ROC } |Z| < |a| \qquad \ldots(8.5.10)$$

Thus depending on given ROC and by using Equations (8.5.9) and (8.5.10), the sequence $x(n)$ is obtained as follows :

$$x(n) = IZT\left\{\frac{Z}{Z - P_K}\right\} = (P_K)^n u(n) \quad \text{if ROC} : |Z| > |P_K|$$

That means causal sequence $\qquad \ldots(8.5.11)$

$$\text{and } x(n) = IZT\left\{\frac{Z}{Z - P_K}\right\} = -(P_K)^n u(-n-1)$$

if ROC : $|Z| < |P_K|$

That means anticausal sequence. $\qquad \ldots(8.5.12)$

8.5.2.1 Solved Examples using P.F.E. :

While solving the numericals we will consider different cases depending on the nature of poles.

Case I : When there are simple poles :

Ex. 8.5.9 : Find inverse Z of the following :

$$X(Z) = \frac{1 - 1/2\, Z^{-1}}{1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}}, \ |Z| > \frac{1}{2}$$

Soln. :

Step 1 : We will check whether the given function is in proper form or not.

Here $a_0 = 1$, M = Degree of numerator = 1,

N = Degree of denominator = 2

$\therefore M < N$ and $a_0 = 1$

Hence the given function is in proper form.

Note : While checking the degree of numerator and denominator, do not consider the sign.

Step 2 : We will convert the given function in terms of positive powers of Z. The given function is,

$$X(Z) = \frac{1 - \frac{1}{2}Z^{-1}}{1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}} \qquad \ldots(1)$$

To obtain positive powers of Z: multiply numerators and denominator by Z^2.

$$\therefore X(Z) = \frac{Z^2[1 - 1/2 Z^{-1}]}{Z^2\left[1 + \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}\right]}$$

$$\therefore X(Z) = \frac{Z^2 - \frac{1}{2}Z}{Z^2 + \frac{3}{4}Z + \frac{1}{8}} \quad ...(2)$$

Step 3 : We will convert Equation (2) in the form of $\frac{X(Z)}{Z}$.

To do this, take Z common from the numerator.

$$\therefore X(Z) = \frac{Z\left(Z - \frac{1}{2}\right)}{Z^2 + \frac{3}{4}Z + \frac{1}{8}}$$

$$\therefore \frac{X(Z)}{Z} = \frac{Z - 1/2}{Z^2 + \frac{3}{4}Z + \frac{1}{8}} \quad ...(3)$$

Step 4 : We will obtain the roots (poles) of the denominator. To obtain poles P_1 and P_2 use following equation.

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad ...(4)$$

The equation of denominator is, $Z^2 + \frac{3}{4}Z + \frac{1}{8}$

Thus here $a = 1$, $b = \frac{3}{4}$ and $c = \frac{1}{8}$

Putting these values in Equation (4) we get,

$$P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - 4\left(1 \times \frac{1}{8}\right)}}{(2 \times 1)} = \frac{-\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}}{2}$$

$$\therefore P_1, P_2 = \frac{-\frac{3}{4} \pm \sqrt{\frac{9-8}{16}}}{2} = \frac{-\frac{3}{4} \pm \frac{1}{4}}{2} = -\frac{3}{8} \pm \frac{1}{8}$$

$$\therefore P_1 = -\frac{3}{8} - \frac{1}{8} = -\frac{4}{8} = -\frac{1}{2}$$

and $$P_2 = -\frac{3}{8} + \frac{1}{8} = \frac{-3+1}{8} = -\frac{1}{4}$$

Step 5 : We will write down Equation (3) in the partial fraction expansion form.

$$\therefore \frac{X(Z)}{Z} = \frac{Z - \frac{1}{2}}{(Z - P_1)(Z - P_2)} = \frac{Z - \frac{1}{2}}{\left(Z + \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)} \quad ...(5)$$

Thus in terms of coefficients A_1 and A_2 we can write,

$$\frac{X(Z)}{Z} = \frac{A_1}{\left(Z + \frac{1}{2}\right)} + \frac{A_2}{\left(Z + \frac{1}{4}\right)} \quad ...(6)$$

Step 6 : We will obtain values of A_1 and A_2. The general expression to calculate A_K is,

$$A_K = (Z - P_K)\frac{X(Z)}{Z}\bigg|_{Z = P_K}$$

$$\therefore A_1 = (Z - P_1)\frac{X(Z)}{Z}\bigg|_{Z = P_1}$$

Here $P_1 = -1/2$

$$\therefore A_1 = \left(Z + \frac{1}{2}\right) \cdot \frac{X(Z)}{Z}\bigg|_{Z = -\frac{1}{2}}$$

Putting the value of $\frac{X(Z)}{Z}$ from Equation (5) we get,

$$A_1 = \left(Z + \frac{1}{2}\right) \cdot \frac{Z - \frac{1}{2}}{\left(Z + \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)}\bigg|_{Z = -\frac{1}{2}}$$

$$\therefore A_1 = \frac{Z - \frac{1}{2}}{Z + \frac{1}{4}}\bigg|_{Z = -\frac{1}{2}} = \frac{-\frac{1}{2} - \frac{1}{2}}{-\frac{1}{2} + \frac{1}{4}} = \frac{\frac{-1}{-2+1}}{4} = \frac{-1}{-\frac{1}{4}} = 4$$

Similarly, $$A_2 = (Z - P_2)\frac{X(Z)}{Z}\bigg|_{Z = P_2}$$

Here $P_2 = -\frac{1}{4}$

$$\therefore A_2 = \left(Z + \frac{1}{4}\right) \cdot \frac{Z - \frac{1}{2}}{\left(Z + \frac{1}{2}\right)\left(Z + \frac{1}{4}\right)}\bigg|_{Z = -\frac{1}{4}}$$

$$A_2 = \frac{Z - \frac{1}{2}}{Z + \frac{1}{2}}\bigg|_{Z = -\frac{1}{4}} = \frac{-\frac{1}{4} - \frac{1}{2}}{-\frac{1}{4} + \frac{1}{2}} = \frac{-\frac{3}{4}}{1/4} = -3$$

Thus $A_1 = 4$ and $A_2 = -3$.

Step 7 : Putting values of A_1 and A_2 in Equation (6) we get,

$$\frac{X(Z)}{Z} = \frac{4}{Z + \frac{1}{2}} - \frac{3}{Z + \frac{1}{4}}$$

Now we will obtain equation of X (Z) by transferring Z to the R.H.S.

$$\therefore X(Z) = \frac{4Z}{Z + \frac{1}{2}} - \frac{3Z}{Z + \frac{1}{4}} \quad ...(7)$$

Step 8 : The first term of Equation (7) is $4 \cdot \frac{Z}{Z + \frac{1}{2}}$. This term is causal when $|Z| > \frac{1}{2}$.

> **Note :** The standard form is $\frac{Z}{Z + a}$. It is causal when $|Z| > |a|$.

Now the given ROC is $|Z| > \frac{1}{2}$. So first term is causal.

Now recall the standard equation,

$$IZT\left\{\frac{Z}{Z - P_K}\right\} = (P_K)^n u(n) \text{ when } |Z| > |P_K| \quad ...(8)$$

Here $P_K = P_1 = -\frac{1}{2}$

$$\therefore IZT\left\{\frac{Z}{Z + \frac{1}{2}}\right\} = \left(-\frac{1}{2}\right)^n u(n) \quad ...(9)$$

Now consider second term. It is $\frac{Z}{Z + \frac{1}{4}}$. This term is causal when $|Z| > \frac{1}{4}$. Given ROC is $|Z| > \frac{1}{2}$. That means by default it is $> \frac{1}{4}$. Thus the second term is also causal. Thus using Equation (8).

$$IZT\left\{\frac{Z}{Z + \frac{1}{4}}\right\} = \left(\frac{-1}{4}\right)^n u(n)$$

$$\ldots \text{ since } P_2 = -\frac{1}{4} \ldots(10)$$

Thus from Equations (9) and (10) we can write IZT of given function as,

$$x(n) = 4\left(-\frac{1}{2}\right)^n u(n) - 3\left(-\frac{1}{4}\right)^n u(n)$$

Ex. 8.5.10 : Obtain inverse Z of the following :

$$X(Z) = \frac{1 - \frac{1}{2}Z^{-1}}{1 - \frac{1}{4}Z^{-2}}, \quad |Z| > \frac{1}{2}$$

Soln. :

Step 1 : First we will check whether given function is in the proper form or not.

Here $a_0 = 1$, $M = 1$ and $N = 2$.

Since $M < N$; the function is in the proper form.

Step 2 : The given function is,

$$X(Z) = \frac{1 - \frac{1}{2}Z^{-1}}{1 - \frac{1}{4}}Z^{-2} \quad ...(1)$$

Multiply numerator and denominator by Z^2.

$$\therefore X(Z) = \frac{Z^2 - \frac{1}{2}Z}{Z^2 - \frac{1}{4}} \quad ...(2)$$

Step 3 : We will obtain the equation of $\frac{X(Z)}{Z}$ as follows :

$$X(Z) = \frac{Z\left(Z - \frac{1}{2}\right)}{Z^2 - \frac{1}{4}}$$

$$\therefore \frac{X(Z)}{Z} = \frac{\left(Z - \frac{1}{2}\right)}{Z^2 - \frac{1}{4}} \quad ...(3)$$

Step 4 : We will obtain the roots of denominator $Z^2 - \frac{1}{4}$.

Here $Z^2 - \frac{1}{4} = Z^2 - \left(\frac{1}{2}\right)^2$

$\therefore$ Roots are $\left(Z - \frac{1}{2}\right)$ and $\left(Z + \frac{1}{2}\right)$

Thus the poles are,

$$\left(Z - \frac{1}{2}\right) = (Z - P_1) \Rightarrow P_1 = \frac{1}{2}$$

and $$\left(Z + \frac{1}{2}\right) = (Z - P_2) \Rightarrow P_2 = -\frac{1}{2}$$

Step 5 : Thus Equation (3) can be written as,

$$\frac{X(Z)}{Z} = \frac{Z - \frac{1}{2}}{\left(Z - \frac{1}{2}\right)\left(Z + \frac{1}{2}\right)} = \frac{1}{Z + \frac{1}{2}} \quad ...(4)$$

Step 6 : The partial fraction expansion form is,

$\frac{X(Z)}{Z} = \frac{A_1}{Z - P_1}$. Thus $A_1 = 1$.

We do not have to calculate this value.

Step 7 : Now from Equation (4),

$$X(Z) = \frac{Z}{Z + \frac{1}{2}} \quad ...(5)$$

This function is causal if $|Z| > \frac{1}{2}$. The given ROC is $|Z| > \frac{1}{2}$.

Thus given function is causal. Now we will use the standard formula of causal sequence.

$$IZT\left\{\frac{Z}{Z - P_K}\right\} = (P_K)^n u(n)$$

$$\therefore x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

Ex. 8.5.11 : Determine the inverse Z-transform for the function :

$$X[Z] = \frac{Z^2 + Z}{Z^2 - 2Z + 1} \text{ ROC } |Z| > 1$$

May 2014, 5 Marks

Soln. :

Given,

$$X(Z) = \frac{Z^2 + Z}{Z^2 - 2Z + 1}$$

$$\therefore \frac{X(Z)}{Z} = \frac{Z + 1}{(Z - 1)^2} = \frac{A_1}{(Z - 1)} + \frac{A_2}{(Z - 1)^2}$$

Here, $$A_1 = \frac{d}{dZ}\left[(Z - 1)^2 \cdot \frac{Z + 1}{(Z - 1)^2}\right]\bigg|_{Z = 1}$$

$$= \frac{d}{dZ}[Z + 1]\bigg|_{Z = 1}$$

$$\therefore A_1 = 1$$

and $A_2 = (Z-1)^2 \cdot \frac{Z^2+Z}{(Z-1)^2}\Big|_{Z=1}$

$\therefore A_2 = 1+1 = 2$

$\therefore \frac{X(Z)}{Z} = \frac{1}{Z-1} + \frac{2}{(Z-1)^2}$

$\therefore X(Z) = \frac{Z}{Z-1} + \frac{2Z}{(Z-1)^2}$

We have standard Z transform pairs.

$\alpha^n u(n) \leftrightarrow \frac{Z}{Z-\alpha}$

and $n a^n u(n) \leftrightarrow \frac{aZ}{(Z-a)^2}$

$\therefore x(n) = (1)^n u(n) + 2n(1)^n u(n)$

Ex. 8.5.12 : Obtain inverse z - transform of the following X(z) **May 11, 8 Marks**

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})}, \text{ ROC } |z| > 1$$

Soln. :

Given : $X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})}$ ROC, $|z| \geq 1$

To convert into positive powers of z; multiply and divide by z^2.

$$\therefore X(z) = \frac{z^2}{(z+1)(z-1)}$$

$$\therefore \frac{X(z)}{z} = \frac{z}{(z+1)(z-1)}$$

In Partial Fraction Expansion form we can write,

$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{z-1} \quad ...(1)$$

Here $A_1 = (z+1) \cdot \frac{z}{(z+1)(z-1)}\Big|_{z=-1}$

$\therefore A_1 = \frac{-1}{-1-1} = \frac{1}{2}$

and $A_2 = (z-1) \cdot \frac{z}{(z+1)(z-1)}\Big|_{z=1}$

$\therefore A_2 = \frac{1}{1+1} = \frac{1}{2}$

Putting these values in Equation (1) we get,

$\therefore X(z) = \frac{1}{2} \cdot \frac{z}{z+1} + \frac{1}{2} + \frac{z}{z-1}$

Taking Inverse Z-Transform we get,

$x(n) = \frac{1}{2}(-1)^n u(n) + \frac{1}{2}(1)^n u(n)$ ...Ans.

Ex. 8.5.13 : Determine the causal signal x (n) if its, z-transform X (z) is given by,

$$X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$$

Soln. :

Given : $X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}}$

Converting X (z) into positive powers of z

$\therefore X(z) = \frac{z^2+3z}{z^2+3z+2}$

$\therefore \frac{X(z)}{z} = \frac{z+3}{z^2+3z+2}$

$\therefore \frac{X(z)}{z} = \frac{z+3}{(z+1)(z+2)}$

In terms of partial fraction expansion we can write,

$$\frac{X(z)}{z} = \frac{A_1}{z+1} + \frac{A_2}{z+2}$$

Now, $A_1 = (z+1)\frac{(z+3)}{(z+1)(z+2)}\Big|_{z=-1}$

$\therefore A_1 = \frac{-1+3}{-1+2} = 2$

and $A_2 = (z+2)\frac{(z+3)}{(z+1)(z+2)}\Big|_{z=-2}$

$\therefore A_2 = \frac{-2+3}{-2+1} = -1$

$\therefore \frac{X(z)}{z} = \frac{2}{z+1} + \frac{1}{z+2}$

$\therefore X(z) = \frac{2z}{z+1} + \frac{z}{z+2}$

Since the system is causal; its inverse z can be written as,

$x(n) = 2(1)^n u(n) - (2)^n u(n)$

Ex. 8.5.14 : Determine inverse Z transform of :

$$X(Z) = \frac{3}{(1-Z^{-1})(1+Z^{-1})(1-0.5Z^{-1})(1-0.2Z^{-1})}$$

May 2016, 6 Marks

Soln. :

Given : $X(Z) = \frac{3}{(1-Z^{-1})(1+Z^{-1})(1-0.5Z^{-1})(1-0.2Z^{-1})}$

Converting into positive powers of Z,

$$X(Z) = \frac{3Z^4}{(Z-1)(Z+1)(Z-0.5)(Z-0.2)}$$

$$\therefore \frac{X(Z)}{Z} = \frac{3Z^3}{(Z-1)(Z+1)(Z-0.5)(Z-0.2)}$$

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z+1} + \frac{A_3}{Z-0.5} + \frac{A_4}{Z-0.2}$$

Here $A_1 = \frac{3Z^3}{(Z+1)(Z-0.5)(Z-0.2)}\Big|_{Z=1}$

$= \frac{3}{(1+1)(1-0.5)(1-0.2)} = 3.75$

$A_2 = \frac{3Z^3}{(Z-1)(Z-0.5)(Z-0.2)}\Big|_{Z=-1}$

$= \frac{-3}{(-1-1)(-1-0.5)(-1-0.2)} = 0.833$

$A_3 = \frac{3Z^3}{(Z+1)(Z-1)(Z-0.2)}\Big|_{Z=0.5}$

$= \frac{3(0.5)^3}{(0.5+1)(0.5-1)(0.5-0.2)} = -1.67$

and $A_4 = \frac{3Z^3}{(Z+1)(Z-1)(Z-0.5)}\Big|_{Z=0.2}$

$= \frac{3(0.2)^3}{(0.2+1)(0.2-1)(0.2-0.5)} = 0.0833$

$$\therefore \frac{X(Z)}{Z} = \frac{3.75}{Z-1} + \frac{0.833}{Z+1} - \frac{1.67}{Z-0.5} + \frac{0.0833}{Z-0.2}$$

$$\therefore X(Z) = 3.75\frac{Z}{Z-1} + 0.833\frac{Z}{Z+1} - 1.67\frac{Z}{Z-0.5} + 0.0833\frac{Z}{Z-0.2}$$

Taking IZT,

$x(n) = 3.75(1)^n u(n) + 0.0833(-1)^n u(n) - 1.67(0.5)^n u(n) + 0.0833(0.2)^n u(n)$

Ex. 8.5.15 : Determine all possible sequences x (n) associated with Z transform

$$X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(3-Z^{-1})}$$

Soln. : The given function is,

$$X(Z) = \frac{5Z^{-1}}{(1-2Z^{-1})(3-Z^{-1})}$$

$$= \frac{5Z^{-1}}{3-Z^{-1}-6Z^{-1}-2Z^{-2}}$$

$$\therefore X(Z) = \frac{5Z^{-1}}{3-7Z^{-1}-2Z^{-2}} \quad ...(1)$$

Step 1 : Here M = 1 and N = 2 ; M < N but $a_0 = 3$. We want $a_0 = 1$.

Thus to convert the given function into proper form; multiply numerator and denominator by $\frac{1}{3}$.

$$\therefore X(Z) = \frac{\frac{5}{3}Z^{-1}}{1-\frac{7}{3}Z^{-1}-\frac{2}{3}Z^{-2}}$$

Step 2 : Multiply numerator and denominator by Z^2

$$\therefore X(Z) = \frac{\frac{5}{3}Z}{Z^2-\frac{7}{3}Z-\frac{2}{3}}$$

Step 3 : The equation of $\frac{X(Z)}{Z}$ is,

$$\frac{X(Z)}{Z} = \frac{\frac{5}{3}}{Z^2-\frac{7}{3}Z-\frac{2}{3}} \quad ...(2)$$

Step 4 : We will obtain roots of denominator

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{\frac{7}{3} \pm \sqrt{\frac{49}{9}-\frac{8}{3}}}{2} = \frac{7}{6} \pm \frac{5}{6}$$

$\therefore P_1 = \frac{7}{6} + \frac{5}{6} = 2$ and $P_2 = \frac{7}{6} - \frac{5}{6} = \frac{1}{3}$.

Step 5 : Thus Equation (2) can be written as,

$$\therefore \frac{X(Z)}{Z} = \frac{\frac{5}{3}}{(Z-2)\left(Z-\frac{1}{3}\right)} \quad ...(3)$$

In the form of partial fraction expansion we can write,

$$\therefore \frac{X(Z)}{Z} = \frac{A_1}{Z-2} + \frac{A_2}{Z-\frac{1}{3}} \quad ...(4)$$

Step 6 : Now, $A_1 = (Z-P_1) \cdot \frac{X(Z)}{Z}\Big|_{Z=P_1}$

$\therefore A_1 = (Z-2)\frac{5/3}{(Z-2)\left(Z-\frac{1}{3}\right)}\Big|_{Z=2}$

$= \frac{5/3}{2-1/3} = \frac{5/3}{5/3} = 1$

and $A_2 = \left(Z-\frac{1}{3}\right) \cdot \frac{5/3}{(Z-2)\left(Z-\frac{1}{3}\right)}\Big|_{Z=\frac{1}{3}}$

$= \frac{5/3}{1/3-2} = \frac{5/3}{-5/3} = -1$

Step 7 : Putting values of A_1 and A_2 in Equation (4) we get,

$$\frac{X(Z)}{Z} = \frac{1}{Z-2} - \frac{1}{Z-\frac{1}{3}}$$

$$\therefore X(Z) = \frac{Z}{Z-2} - \frac{Z}{Z-1/3} \quad ...(5)$$

Step 8 : To obtain all possible sequences we will consider the following ROC.

1. When ROC is | Z | > 2

The first term of Equation (5) will be causal if | Z | > 2 and the second term will be causal if | Z | > 1 / 3. Now if ROC is | Z | > 2 then by default it is greater than 1/3. Thus for this ROC both the terms are causal.

$\therefore x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\}$

$= (P_K)^n u(n)$ ...(6)

$\therefore x(n) = (2)^n u(n) - (1/3)^n u(n)$

2. When ROC is $2 > |Z| > \frac{1}{3}$

For the first term to be causal, the required ROC is | Z | > 2. But the given ROC is 2 > | Z | that means | Z | < 2. Thus first term is anticausal. We will use the standard formula of IZT for anticausal sequence

$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\}$

$= -(P_K)^n u(-n-1)$ ...(7)

Thus IZT of first term is,

$x_1(n) = -(2)^n u(-n-1)$

Now for the second term to be causal the required ROC is | Z | > 1/3) and the given ROC is $|Z| > \frac{1}{3}$. So second term is causal. Thus using Equation (6) IZT of second term is,

$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$

Thus IZT of total function is,

$$x(n) = x_1(n) + x_2(n)$$

$$\therefore \quad x(n) = -(2)^n u(-n-1) + \left(\frac{1}{3}\right)^n u(n)$$

3. When ROC is $|Z| < \frac{1}{3}$

We know that the required ROC for first and second term to be causal are $|Z| > 2$ and $|Z| > \frac{1}{3}$ respectively. But the given ROC is $|Z| < \frac{1}{3}$. Thus both the terms will be anticausal. Now using Equation (7),

$$x(n) = -(2)^n u(-n-1) - \left(\frac{1}{3}\right)^n u(-n-1)$$

Ex. 8.5.16 : Using partial fraction expansion method find inverse Z transform of following system given by,

$$X(Z) = \frac{Z(Z^2 - 4Z + 5)}{(Z-1)(Z-2)(Z-3)}$$

for ROC being $|Z| > 3$, $2 < |Z| < 3$, $|Z| < 2$.

Soln. :

The given function is,

$$X(Z) = \frac{Z(Z^2 - 4Z + 5)}{(Z-1)(Z-2)(Z-3)} \quad \ldots(1)$$

Step 1 :

$$\frac{X(Z)}{Z} = \frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)} \quad \ldots(2)$$

Equation (2) is in the proper form. The roots of denominator are already given, so the poles are $P_1 = 1$, $P_2 = 2$ and $P_3 = 3$.

Step 2 : In the P.F.E. form Equation (2) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-2} + \frac{A_3}{Z-3} \quad \ldots(3)$$

We will obtain values of A_1, A_2 and A_3 as follows :

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\bigg|_{Z=1}$$

$$\therefore \quad A_1 = (Z-1)\cdot\frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)}\bigg|_{Z=1}$$

$$\therefore \quad A_1 = \frac{1 - 4 + 5}{(1-2)(1-3)}$$

$$= \frac{2}{(-1)(-2)}$$

$$\therefore \quad A_1 = +1$$

$$A_2 = (Z - P_2)\cdot\frac{X(Z)}{Z}\bigg|_{Z=P_2}$$

$$\therefore \quad A_2 = (Z-2)\cdot\frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)}\bigg|_{Z=2}$$

$$\therefore \quad A_2 = \frac{4 - 8 + 5}{(2-1)(2-3)} = \frac{1}{(1)(-1)}$$

$$\therefore \quad A_2 = -1$$

And $$A_3 = (Z - P_3)\frac{X(Z)}{Z}\bigg|_{Z=P_3}$$

$$\therefore \quad A_3 = (Z-3)\cdot\frac{Z^2 - 4Z + 5}{(Z-1)(Z-2)(Z-3)}\bigg|_{Z=3}$$

$$\therefore \quad A_3 = \frac{9 - 12 + 5}{(3-1)(3-2)} = \frac{2}{(2)(1)}$$

$$\therefore \quad A_3 = 1$$

Step 3 : Putting values of A_1, A_2 and A_3 in Equation (3) we get,

$$\therefore \quad \frac{X(Z)}{Z} = \frac{1}{Z-1} - \frac{1}{Z-2} + \frac{1}{Z-3}$$

$$\therefore \quad X(Z) = \frac{Z}{Z-1} - \frac{Z}{Z-2} + \frac{Z}{Z-3} \quad \ldots(4)$$

Step 4 : We will obtain $x(n)$ for different ROC conditions as follows :

1. For ROC : $|Z| > 3$

Since ROC is $|Z| > 3$ then all three terms in Equation (4) are causal.

$$\therefore \quad x(n) = (1)^n u(n) - (2)^n u(n) + (3)^n u(n)$$

2. For ROC : $2 < |Z| < 3$

That means $|Z| > 2$ and $|Z| < 3$

For this ROC; first term is causal, second term is causal and the third term is anticausal.

$$\therefore x(n) = (1)^n u(n) - (2)^n u(n) - (3)^n u(-n-1)$$

3. For ROC : $|Z| < 2$

In this case all terms of Equation (4), will be anti-causal.

$$\therefore x(n) = -(1)^n u(-n-1) + (2)^n u(-n-1) - (3)^n u(-n-1)$$

Ex. 8.5.17 : Determine inverse Z-transform of

$$X(Z) = \frac{1}{1 - 1.5Z^{-1} + 0.5Z^{-2}}$$

if 1. ROC : $|Z| > 1$
2. ROC : $|Z| < 0.5$
3. ROC : $0.5 < |Z| < 1.0$

Soln. :

Step 1 : Here $a_0 = 1$, $M = 0$, $N = 2$

Thus the given function is in the proper form.

Step 2 : To convert $X(Z)$ into positive powers of Z; multiply numerator and denominator by Z^2.

$$\therefore \quad X(Z) = \frac{Z^2}{Z^2 - 1.5Z + 0.5} \quad \ldots(1)$$

Step 3 : We will obtain the expression $\frac{X(Z)}{Z}$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{Z}{Z^2 - 1.5Z + 0.5} \quad \ldots(2)$$

Step 4 : We will obtain the roots of denominator

$$Z^2 - 1.5Z + 0.5$$

Here $a = 1$, $b = -1.5$ and $c = 0.5$

$$\therefore \quad P_1, P_2 = \frac{1.5 \pm \sqrt{(1.5)^2 - 4 \times 0.5}}{2}$$

$$\therefore \quad P_1 = 1 \text{ and } P_2 = 0.5$$

Step 5 : In PFE form Equation (2) can be written as,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z-0.5} \quad \ldots(3)$$

Step 6 : We will calculate the values of A_1 and A_2 as follows :

$$A_1 = (Z - P_1)\frac{X(Z)}{Z}\bigg|_{Z=P_1}$$

$$\therefore \quad A_1 = (Z-1)\cdot\frac{Z}{(Z-1)(Z-0.5)}\bigg|_{Z=1}$$

$$\therefore \quad A_1 = \frac{1}{1-0.5} = \frac{1}{0.5}$$

$$\therefore A_1 = 2$$

and $$A_2 = (Z - P_2)\frac{X(Z)}{Z}\bigg|_{Z=P_2}$$

$$\therefore \quad A_2 = (Z-0.5)\frac{Z}{(Z-1)(Z-0.5)}\bigg|_{Z=0.5}$$

$$\therefore \quad A_2 = \frac{0.5}{0.5-1} = \frac{0.5}{-0.5}$$

$$\therefore A_2 = -1$$

Step 7 : Putting values of A_1 and A_2 in Equation (3) we get,

$$\frac{X(Z)}{Z} = \frac{2}{Z-1} - \frac{1}{Z-0.5}$$

$$\therefore \quad X(Z) = \frac{2Z}{Z-1} - \frac{Z}{Z-0.5} \quad \ldots(4)$$

Step 8 : We will obtain $x(n)$ according to given ROC conditions.

1. ROC : $|Z| > 1$

It means that $|Z| > 0.5$ also. So both the terms of Equation (4) are causal. For the causal sequence we have,

$$x(n) = IZT\left\{\frac{Z}{Z - P_k}\right\} = (P_k)^n u(n)$$

$$\therefore x(n) = 2(1)^n u(n) - (0.5)^n u(n)$$

2. ROC : $|Z| < 0.5$

For this ROC both the terms of Equation (4) become anticausal. And for anticausal sequence we have,

$$x(n) = IZT\left\{\frac{Z}{Z - P_k}\right\} = -(P_k)^n u(-n-1)$$

$$\therefore x(n) = 2[-(1)^n u(-n-1)] - [(0.5)^n u(-n-1)]$$

3. ROC : $0.5 < |Z| < 1$

For this ROC ; the first term is anticausal and second term is causal.

$$\therefore x(n) = 2[-(1)^n u(-n-1)] - (0.5)^n u(n)$$

Ex. 8.5.18 : Find all possible inverse z transform of the following :

$$x(z) = \frac{z + 0.3}{(z + 0.2)(z + 0.5)}$$

Soln. :

$$\frac{X(z)}{z} = \frac{z + 0.3}{(z + 0.2)(z + 0.5)}$$

In partial fraction expansion form we can write,

$$\frac{X(z)}{z} = \frac{A_1}{z} + \frac{A_2}{z + 0.2} + \frac{A_1}{z + 0.5}$$

$$A_1 = z\cdot\frac{z + 0.3}{z(z + 0.2)(z + 0.5)}\bigg|_{z=0} = \frac{0.3}{(0.2)(0.5)} = 3$$

$$A_2 = (z + 0.2)\cdot\frac{z + 0.3}{z(z + 0.2)(z + 0.5)}\bigg|_{z=-0.2}$$

$$\therefore \quad A_2 = \frac{-0.2 + 0.3}{(-0.2)(-0.2 + 0.5)} = -1.67$$

and $$A_3 = (z + 0.5)\cdot\frac{z + 0.3}{z(z + 0.2)(z + 0.5)}\bigg|_{z=-0.5}$$

$$\therefore \quad A_3 = \frac{-0.5 + 0.3}{(-0.5)(-0.5 + 0.3)}$$

$$\therefore \quad A_3 = \frac{-0.2}{(-0.5)(-0.3)} = -1.33$$

$$\therefore \quad \frac{X(z)}{z} = \frac{3}{z} - 1.67\frac{1}{z + 0.2} - 1.33\cdot\frac{1}{z + 0.5}$$

$$\therefore \quad X(z) = 3 - 1.67\frac{z}{z + 0.2} - 1.33\frac{z}{z + 0.5}$$

The different possible ROCs are,

1. $|z| > 0.5$

$$x(n) = 3\delta(n) - 1.67(-0.2)^n u(n) - 1.33(-0.5)^n u(n)$$

2. $|z| < 0.2$

$$x(n) = 3\delta(n) + 1.67(-0.2)^n u(-n-1) + 1.33(-0.5)^n u(-n-1)$$

3. $0.5 > |z| > 0.2$

$$\therefore \quad x(n) = 3\delta(n) - 1.67(-0.2)^n u(n) + 1.33(-0.5)^n u(-n-1)$$

Ex. 8.5.19 : Determine IZT of :

$$X(Z) = \frac{4Z^{-2} + 3Z^{-1} + 2}{\frac{1}{2}Z^{-2} - \frac{3}{2}Z^{-1} + 1}$$; For causal sequence

Soln. :

Step 1 : Here $a_0 = 1$, $M = 2$, $N = 2$

Since $M = 2$ (M not less than N); the given function is not in the proper form. To obtain it in the proper form we will perform the long division.

$$\begin{array}{r|l} & 8 \\ \hline \frac{1}{2}Z^{-2} - \frac{3}{2}Z^{-1} + 1 & 4Z^{-2} + 3Z^{-1} + 2 \\ & 4Z^{-2} - 12Z^{-1} + 8 \\ & - \quad + \quad - \\ \hline & 15Z^{-1} - 6 \end{array}$$

Thus we can write, $$X(Z) = Q + \frac{R}{D}$$

$$\therefore \quad X(Z) = 8 + \frac{15Z^{-1} - 6}{\frac{1}{2}Z^{-2} - \frac{3}{2}Z^{-1} + 1} \quad \ldots(1)$$

Now consider, $$X_1(Z) = \frac{15Z^{-1} - 6}{\frac{1}{2}Z^{-2} - \frac{3}{2}Z^{-1} + 1} \quad \ldots(2)$$

First we will obtain inverse Z of $X_1(Z)$. Here $M = 1$ and $N = 2$. Thus $X_1(Z)$ is in the proper form.

Step 2 : Multiply numerator and denominator by Z^2.

$$\therefore \quad X_1(Z) = \frac{15Z - 6Z^2}{\frac{1}{2} - \frac{3}{2}Z + Z^2} = \frac{-6Z^2 + 15Z}{Z^2 - \frac{3}{2}Z + \frac{1}{2}}$$

Step 3 :

$$\therefore \frac{X_1(Z)}{Z} = \frac{-6Z+15}{Z^2-\frac{3}{2}Z+\frac{1}{2}} \quad ...(3)$$

Step 4 : Now we will obtain roots of $Z^2-\frac{3}{2}Z+\frac{1}{2}$

Here $a = 1, b = -\frac{3}{2}$ and $c = \frac{1}{2}$

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{\frac{3}{2} \pm \sqrt{\frac{9}{4}-2}}{2} = \frac{\frac{3}{2} \pm \sqrt{\frac{1}{4}}}{2} = \frac{\frac{3}{2} \pm \frac{1}{2}}{2}$$

$$\therefore P_1 . P_2 = \frac{3}{4} \pm \frac{1}{4}$$

$$\therefore P_1 = \frac{3}{4}+\frac{1}{4} = 1 \text{ and } P_2 = \frac{3}{4}-\frac{1}{4} = \frac{1}{2}$$

Step 5 : Thus Equation (3) can be written as.

$$\text{Thus } \frac{X_1(Z)}{Z} = \frac{-6Z+15}{(Z-1)\left(Z-\frac{1}{2}\right)} \quad ...(4)$$

In the form of partial fraction expansion we can write.

$$\frac{X_1(Z)}{Z} = \frac{A_1}{(Z-P_1)} + \frac{A_2}{(Z-P_2)} \quad ...(5)$$

Step 6 :

$$\text{Now, } A_1 = (Z-P_1)\frac{X_1(Z)}{Z}\bigg|_{Z=P_1}$$

$$\therefore A_1 = \frac{(Z-1)(-6Z+15)}{(Z-1)\left(Z-\frac{1}{2}\right)}\bigg|_{Z=1/2}$$

$$= \frac{-6+15}{1-1/2} = \frac{9}{1/2} = 18$$

$$\text{and } A_2 = (Z-P_2)\frac{X_1(Z)}{Z}\bigg|_{Z=P_2}$$

$$\therefore A_2 = \frac{(Z-1/2)(-6Z+15)}{(Z-1)(Z-1/2)}\bigg|_{Z=1/2}$$

$$= \frac{-6\left(\frac{1}{2}\right)+15}{\frac{1}{2}-1} = \frac{12}{-1/2} = -24$$

Step 7 :

$$\text{Thus } \frac{X_1(Z)}{Z} = \frac{18}{(Z-1)} - \frac{24}{(Z-1/2)}$$

$$\therefore X_1(Z) = \frac{18Z}{Z-1} - \frac{24Z}{(Z-1/2)} \quad ...(6)$$

Step 8 : Now given sequence is causal. Thus

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

$\therefore$ Thus IZT of Equation (6) is.

$$x_1(n) = 18(1)^n u(n) - 24\left(\frac{1}{2}\right)^n u(n) \quad ...(7)$$

Now the first term in Equation (1) is 8. We have standard Z transform pair $\delta(n) \xleftrightarrow{Z} 1$

That means $IZT\{1\} = \delta(n)$

$\therefore IZT\{8\} = 8\,\delta(n)$

Thus the total IZT of given function is,

$$x(n) = 8\,\delta(n) + 18(1)^n u(n) - 24\left(\frac{1}{2}\right)^n u(n)$$

Ex. 8.5.20 : Obtain inverse Z-transform (IZT) using PFE where

$X(Z) = \frac{1}{(Z-1)(Z-3)}$ Comment on ROC's of $X(Z)$.

Soln. : The given function is,

$$X(Z) = \frac{1}{(Z-1)(Z-3)}$$

$$\therefore X(Z) = \frac{1}{Z^2-4Z+3} \quad ...(1)$$

Step 1 : Here $a_0 = 1$, $M = 0$ and $N = 2$

Since $M < N$ the function is in the proper form.

Step 2 : The given function $X(Z)$ is already in terms of positive powers of Z.

Step 3 : To obtain $\frac{X(Z)}{Z}$ we will multiply both sides of Equation (1) by $\frac{1}{Z}$.

$$\therefore \frac{X(Z)}{Z} = \frac{1}{Z(Z^2-4Z+3)}$$

$$\therefore \frac{X(Z)}{Z} = \frac{1}{Z(Z-1)(Z-3)} \quad ...(2)$$

Step 4 : Equation (2) can be written as.

$$\frac{X(Z)}{Z} = \frac{1}{(Z-0)(Z-1)(Z-3)} \quad ...(3)$$

Thus poles are, $P_1 = 0$, $P_2 = 1$ and $P_2 = 3$

Step 5 : In the partial fraction expansion form we can write,

$$\frac{X(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{(Z-1)} + \frac{A_3}{(Z-3)} \quad ...(4)$$

Step 6 : We will calculate values of A_1, A_2 and A_3.

$$A_1 = (Z-P_1)\frac{X(Z)}{Z}\bigg|_{Z=P_1}$$

$$= (Z-0) \times \frac{1}{(Z-0)(Z-1)(Z-3)}\bigg|_{Z=0}$$

$$= \frac{1}{(-1)(-3)} = \frac{1}{3} A_2 = (Z-P_2)\frac{X(Z)}{Z}\bigg|_{Z=P_2}$$

$$\therefore A_2 = (Z-1)\frac{1}{Z(Z-1)(Z-3)}\bigg|_{Z=1}$$

$$= \frac{1}{(Z)(Z-3)}\bigg|_{Z=1} = \frac{1}{1(1-3)} = -\frac{1}{2}$$

$$\text{and } A_3 = (Z-P_3)\frac{X(Z)}{Z}\bigg|_{Z=P_3}$$

$$\therefore A_3 = (Z-3)\frac{1}{Z(Z-1)(Z-3)}\bigg|_{Z=3}$$

$$= \frac{1}{Z(Z-1)}\bigg|_{Z=3} = \frac{1}{3(3-1)} = \frac{1}{6}$$

Step 7 : Putting A_1, A_2 and A_3 in Equation (4) we get,

$$\frac{X(Z)}{Z} = \frac{1/3}{Z} - \frac{1/2}{(Z-1)} + \frac{1/6}{(Z-3)}$$

$$\therefore X(Z) = \frac{1}{3} - \frac{1}{2}\cdot\frac{Z}{Z-1} + \frac{1}{6}\frac{Z}{Z-3} \quad ...(5)$$

Step 8 : **Comments on ROC :**

The first term in equation is constant. So for this term the ROC is entire Z-plane. For the second term to be causal ROC should be $|Z| > 1$ and for the third term to be causal ROC should be $|Z| > 3$. Thus for the entire function to be causal ROC is $|Z| > 1$.

Assuming sequence is causal; IZT of Equation (5) is,

$$x(n) = \frac{1}{3}\delta(n) - \frac{1}{2}(1)^n u(n) + \frac{1}{6}(3)^n u(n)$$

Case II : When there are repeated poles :

When there are repeated poles then, for obtaining A_K we have to use another method. Let us solve one example.

Ex. 8.5.21 : Determine IZT of the following :

$$X(Z) = \frac{3Z^{-3}}{\left(1-\frac{1}{4}Z^{-1}\right)^2}; \; x(n) \text{ left handed system.}$$

Soln. : Observe the denominator; there are repeated poles. Now first we will convert given function into positive powers of Z.

Step 1 : Multiply numerator and denominator by Z^3.

$$\therefore X(Z) = \frac{Z^3 \times 3Z^{-3}}{Z^3\left(1-\frac{1}{4}Z^{-1}\right)^2}$$

$$\therefore X(Z) = \frac{3}{Z \times Z^2\left(1-\frac{1}{4}Z^{-1}\right)^2}$$

Remember that when we transfer Z^2 inside the square bracket, it becomes Z.

$$\therefore X(Z) = \frac{3}{Z(Z \times 1 - Z \times 4Z^{-1})^2}$$

$$\therefore X(Z) = \frac{3}{Z\left(Z-\frac{1}{4}\right)^2} \quad ...(1)$$

Step 2 : Now we will obtain $\frac{X(Z)}{Z}$

$$\therefore \frac{X(Z)}{Z} = \frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2} \quad ...(2)$$

Step 3 : Since there are repeated roots, partial fraction expansion form of Equation (2) can be written as :

$$\frac{X(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z^2} + \frac{A_3}{\left(Z-\frac{1}{4}\right)} + \frac{A_4}{\left(Z-\frac{1}{4}\right)^2} \quad ...(3)$$

Step 4 : In Equation (3), Z^2 and $\left(Z-\frac{1}{4}\right)^2$ are original terms. So value of A_2 and A_4 is obtained by using normal method. While Z and $\left(Z-\frac{1}{4}\right)$ are extra terms. Thus value of A_1 and A_3 is obtained as follows :

$$A_1 = \frac{d}{dZ}\left[Z^2\frac{X(Z)}{Z}\right]\bigg|_{Z=0}$$

$$\therefore A_1 = \frac{d}{dZ}\left[Z^2 \times \frac{3}{Z(Z-1/4)^2}\right]\bigg|_{Z=0}$$

$$\therefore A_1 = \frac{d}{dZ}\left[\frac{3}{\left(Z-\frac{1}{4}\right)^2}\right]\bigg|_{Z=0}$$

$$A_1 = \left[3(-2)\left(Z-\frac{1}{4}\right)^{-3}\right]\bigg|_{Z=0} = 384$$

$$\text{and } A_3 = \frac{d}{dZ}\left[\left(Z-\frac{1}{4}\right)^2\frac{X(Z)}{Z}\right]\bigg|_{Z=\frac{1}{4}}$$

$$= \frac{d}{dZ}\left[\left(Z-\frac{1}{4}\right)^2 \times \frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2}\right]\bigg|_{Z=\frac{1}{4}}$$

$$\therefore A_3 = \frac{d}{dZ}\left[\frac{3}{Z^2}\right]\bigg|_{Z=\frac{1}{4}} = \left[3 \times (-2)Z^{-3}\right]\bigg|_{Z=\frac{1}{4}}$$

$$\therefore A_3 = \left[3(-2)\left(\frac{1}{4}\right)^{-3}\right] = -384$$

$$\text{Now } A_2 = Z^2\frac{X(Z)}{Z}\bigg|_{Z=0} = Z^2\left[\frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2}\right]\bigg|_{Z=0}$$

$$\therefore A_2 = \frac{3}{\left(Z-\frac{1}{4}\right)^2}\bigg|_{Z=0} = 48$$

And

$$A_4 = \left(Z-\frac{1}{4}\right)^2\frac{X(Z)}{Z}\bigg|_{Z=\frac{1}{4}} = \left(Z-\frac{1}{4}\right)^2\frac{3}{Z^2\left(Z-\frac{1}{4}\right)^2}\bigg|_{Z=\frac{1}{4}}$$

$$\therefore A_4 = \frac{3}{(1/4)^2} = 48$$

Step 5 : Now putting values of A_1, A_2, A_3 and A_4 in Equation (3).

$$\frac{X(Z)}{Z} = \frac{384}{Z} + \frac{48}{Z^2} - \frac{384}{Z-\frac{1}{4}} + \frac{48}{\left(Z-\frac{1}{4}\right)^2}$$

$$\therefore X(Z) = 384 + \frac{48}{Z} - \frac{384Z}{Z-\frac{1}{4}} + \frac{48Z}{\left(Z-\frac{1}{4}\right)^2} \quad ...(4)$$

Step 6 : The given sequence is left-handed that means anticausal. Now we will calculate IZT of each term separately.

1. $IZT\{384\} = 384\, IZT\{1\}$

We know that $\delta(n) \xrightarrow{Z} 1 \therefore IZT\{1\} = \delta(n)$

$\therefore \quad IZT\{384\} = 384\,\delta(n)$ ...(5)

2. $IZT\left\{\frac{48}{Z}\right\} = 48\, IZT\left\{\frac{1}{Z}\right\} = 48\, IZT\{Z^{-1}\}$

Now we have $\delta(n-1) \xrightarrow{Z} Z^{-1}$

$\therefore \quad IZT\{Z^{-1}\} = \delta(n-1)$

$\therefore \quad IZT\frac{48}{Z} = 48\,\delta(n-1)$ ...(6)

3. $IZT\left\{\frac{-384Z}{Z-\frac{1}{4}}\right\} = -384\, IZT\left\{\frac{Z}{Z-\frac{1}{4}}\right\}$

Since the sequence is anticausal we have.

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = -(P_K)^n u(-n-1)$$

Here $P_K = \frac{1}{4}$

$\therefore \quad IZT\left\{\frac{-384Z}{Z-\frac{1}{4}}\right\} = +384\left(\frac{1}{4}\right)^n u(-n-1)$ ...(7)

4. $IZT\left\{\frac{48Z}{\left(Z-\frac{1}{4}\right)^2}\right\} = 48\, IZT\left\{\frac{Z}{\left(Z-\frac{1}{4}\right)^2}\right\}$

Now we have standard Z-transform pair,

$$-n a^n u(-n-1) = \frac{aZ}{(Z-a)^2} \quad \text{if } |a| < 1$$

Here $a = 1/4$

$$\therefore IZT\left\{\frac{48Z}{\left(Z-\frac{1}{4}\right)^2}\right\} = 48 \times 4\, IZT\left\{\frac{\frac{1}{4}Z}{\left(Z-\frac{1}{4}\right)^2}\right\}$$

$$= -192\, n\left(\frac{1}{4}\right)^n u(-n-1) \quad ...(8)$$

Thus IZT of total function is.

$$x(n) = 384\,\delta(n) + 48\,\delta(n-1) + 384\left(\frac{1}{4}\right)^n u(-n-1) - 192\, n\left(\frac{1}{4}\right)^n u(-n-1)$$

Ex. 8.5.22 : Determine inverse Z transform using PFE where

$$X(Z) = \frac{Z^3}{(Z+1)(Z-1)^2}.$$

Soln. :

Step 1 : Given :

$$\therefore \quad X(Z) = \frac{Z^3}{(Z+1)(Z-1)^2}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{Z^2}{(Z+1)(Z-1)^2}$$

Step 2 : This is the case of repeated roots. So in P.F.F form we can write.

$$\frac{X(Z)}{Z} = \frac{A_1}{(Z+1)} + \frac{A_2}{(Z-1)} + \frac{A_3}{(Z-1)^2}$$

Here $A_1 = (Z+1)\frac{Z^2}{(z+1)(z-1)^2}\Big|_{Z=-1}$

$\therefore \quad A_1 = \frac{(-1)^2}{(-1-1)^2} \qquad \therefore A_1 = \frac{1}{4}$

$$A_3 = (Z-1)^2 \cdot \frac{Z^2}{(Z+1)(Z-1)^2}\Big|_{Z=1}$$

$\therefore \quad A_3 = \frac{1}{(1+1)} \qquad \therefore A_3 = \frac{1}{2}$

and $A_2 = \frac{d}{dZ}\left[(Z-1)^2 \cdot \frac{Z^2}{(Z+1)(Z-1)^2}\right]\Big|_{Z=1}$

$$= \frac{d}{dZ}\left[\frac{Z^2}{(Z+1)}\right]\Big|_{Z=1}$$

$$= \left[\frac{(Z+1)\cdot 2Z - Z^2 \cdot 1}{(Z+1)^2}\right]\Big|_{Z=1}$$

$$= \frac{(1+1)\cdot 2 - 1}{(1+1)^2} = \frac{4-1}{4} \qquad \therefore A_2 = \frac{3}{4}$$

Step 3 : Putting the values of A_1, A_2 and A_3 in the equation of $\frac{X(Z)}{Z}$.

$$\frac{X(Z)}{Z} = \frac{1/4}{(Z+1)} + \frac{3/4}{(Z-1)} + \frac{\frac{1}{2}}{(Z-1)^2}$$

$$\therefore \quad X(Z) = \frac{1}{4}\cdot\frac{Z}{Z+1} + \frac{3}{4}\cdot\frac{Z}{Z-1} + \frac{1}{2}\cdot\frac{Z}{(Z-1)^2}$$

Step 4 : We have standard Z transform pairs,

$$\alpha^n u(n) \leftrightarrow \frac{Z}{Z-\alpha}, \ |Z| > |\alpha|$$

and $n a^n u(n) \leftrightarrow \frac{az}{(Z-a)^2}, \ |Z| > |a|.$

Thus inverse Z can be written as,

$$x(n) = \frac{1}{4}\cdot(-1)^n u(n) + \frac{3}{4}(1)^n u(n) + \frac{1}{2}\cdot n(1)^n u(n)$$

Ex. 8.5.23 : Find IZT of the following

$$X(Z) = \frac{2Z^2 + 3Z}{Z^2 + Z + 1}; \text{ if } x(n) \text{ is causal}$$

Soln. :

The given function is already in terms of positive powers of Z.

Step 1 : First we will obtain $\frac{X(Z)}{Z}$

$$\therefore X(Z) = \frac{Z(2Z+3)}{Z^2+Z+1} \therefore \frac{X(Z)}{Z} = \frac{2Z+3}{Z^2+Z+1} \quad ...(1)$$

Step 2 : Now we will obtain the roots of denominator.

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}$$

$$\therefore \quad P_1 = -\frac{1}{2} + j\frac{\sqrt{3}}{2} \text{ and } P_2 = -\frac{1}{2} - j\frac{\sqrt{3}}{2}$$

...[Hence $\sqrt{(-1)} = j$]

Step 3 : Thus Equation (1) can be written as :

$$\frac{X(Z)}{Z} = \frac{2Z+3}{(Z-P_1)(Z-P_2)}$$

$$= \frac{2Z+3}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} \quad ...(2)$$

Step 4 : In the partial expansion form we can write,

$$\frac{X(Z)}{Z} = \frac{A_1}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)} + \frac{A_2}{\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} \quad ...(3)$$

Now $A_1 = (Z-P_1)\frac{X(Z)}{Z}\Big|_{Z=P_1}$

$$\therefore A_1 = \frac{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)(2Z+3)}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}\Bigg|_{Z=-\frac{1}{2}+j\frac{\sqrt{3}}{2}}$$

$$\therefore A_1 = \frac{-1+j\sqrt{3}+3}{-\frac{1}{2}+j\frac{\sqrt{3}}{2}+\frac{1}{2}+j\frac{\sqrt{3}}{2}} = \frac{2+j\sqrt{3}}{j\sqrt{3}}$$

Similarly, $A_2 = (Z-P_2)\frac{X(Z)}{Z}\Big|_{Z=P_2}$

$$= \frac{\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)(2Z+3)}{\left(Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}\Bigg|_{Z=-\frac{1}{2}-j\frac{\sqrt{3}}{2}}$$

$$\therefore A_2 = \frac{-1-j\sqrt{3}+3}{-\frac{1}{2}-j\frac{\sqrt{3}}{2}+\frac{1}{2}-j\frac{\sqrt{3}}{2}} = \frac{2-j\sqrt{3}}{-j\sqrt{3}}$$

Step 5 : Putting values of A_1 and A_2 in Equation (3) we get,

$$\frac{X(Z)}{Z} = \frac{\frac{(2+j\sqrt{3})}{j\sqrt{3}}}{Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}} + \frac{\frac{(2-j\sqrt{3})}{-j\sqrt{3}}}{Z+\frac{1}{2}+j\frac{\sqrt{3}}{2}}$$

$$\therefore \quad \frac{X(Z)}{Z} = \frac{1}{j\sqrt{3}}\left[\frac{2+j\sqrt{3}}{Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}} - \frac{2-j\sqrt{3}\,Z}{Z+\frac{1}{2}-j\frac{\sqrt{3}}{2}}\right]$$

Transferring Z to R.H.S. we get,

$$\therefore \quad X(Z) = \frac{1}{j\sqrt{3}}\left[\frac{(2+j\sqrt{3}\,Z)}{Z-\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)} - \frac{(2-j\sqrt{3})Z}{Z-\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)}\right] \quad ...(4)$$

Step 6 : The given sequence is causal; so we can use the formula,

$$x(n) = IZT\left\{\frac{Z}{Z-P_K}\right\} = (P_K)^n u(n)$$

$$\therefore x(n) = \frac{1}{N\sqrt{3}}\left[(2+j\sqrt{3})\cdot\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^n u(n) - (2-j\sqrt{3})\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^n u(n)\right]$$

$$\therefore x(n) = \left[\left(\frac{2+j\sqrt{3}}{N\sqrt{3}}\right)\cdot\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)^n u(n) + \left(\frac{-2+j\sqrt{3}}{N\sqrt{3}}\right)\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)^n u(n)\right]$$

Converting from rectangular to polar form we get. ...(5)

$$x(n) = \left[\left(\frac{2.64\angle 40.89^\circ}{1.73\angle 90^\circ}\right)(1\angle 120^\circ)^n u(n) + \left(\frac{2.64\angle 139.11^\circ}{1.73\angle 90^\circ}\right)(1\angle -120^\circ)^n u(n)\right]$$

$$\therefore x(n) = [(1.52\angle -49.11^\circ)(1\angle 120^\circ)^n u(n) + (1.52\angle 49.11^\circ)(1\angle -120^\circ)^n u(n)]$$

Now the term Magnitude $\angle$ Angle° can be written as Magnitude $\cdot e^{j\,Angle}$

$$\therefore x(n) = [(1.52 e^{-j49.11^\circ})(1 e^{-j120^\circ})^n u(n) + (1.52 e^{j49.11^\circ})(1 e^{j120^\circ})^n u(n)]$$

$$x(n) = [(1.52 e^{-j49.11^\circ})(1^n \cdot e^{-j120n})u(n) + (1.52 e^{j49.11^\circ})(1^n \cdot e^{j120n})u(n)]$$

$$\therefore x(n) = 1.52(1)^n[e^{-j49.11^\circ}\cdot e^{-j120n} + e^{j49.11^\circ}\cdot e^{j120n}]u(n)$$

$$x(n) = 1.52(1)^n[e^{-j(49.11^\circ+120n)} + e^{j(49.11^\circ+120n)}]u(n)$$

Now we have $e^{j\theta} + e^{-j\theta} = \cos 2\theta$

$$\therefore \quad x(n) = 1.52(1)^n[\cos 2(49.11^\circ - 120n)]u(n)$$

$$\therefore \quad x(n) = 1.52[\cos 2(49.11^\circ - 120n)]u(n)$$

Ex. 8.5.24 : Find the inverse Z transform of the following function.

Dec. 2016.10 Marks

$$X(Z) = \frac{1}{1-0.8Z^{-1}+0.12Z^{-2}}$$

for the following ROCs

(a) $|Z| > 0.6$ (b) $|Z| < 0.2$

(c) $0.2 < |Z| < 0.6$

Soln. :

Given, $X(Z) = \frac{1}{1-0.8Z^{-1}+0.12Z^{-2}}$

$$\therefore X(Z) = \frac{Z^2}{Z^2-0.8Z+0.12}$$

$$\therefore \frac{X(Z)}{Z} = \frac{Z}{(Z-0.6)(Z-0.2)}$$

$$\therefore \frac{X(Z)}{Z} = \frac{A_1}{Z-0.6} + \frac{A_2}{Z-0.2}$$

Here $A_1 = \frac{0.6}{0.6-0.2} = 1.5$ and $A_2 = \frac{0.2}{0.2-0.6} = 0.5$

$$\therefore \frac{X(Z)}{Z} = \frac{1.5}{Z-0.6} - \frac{0.5}{Z-0.2}$$

(i) $|Z| > 0.6$

$$x(n) = 1.5(0.6)^n u(n) - 0.5(0.2)^n u(n)$$

(ii) $|Z| > 0.2$

$$x(n) = -1.5(0.6)^n u(-n-1) + 0.5(0.2)^n u(-n-1)$$

(iii) $0.2 < |Z| < 0.6$

$$x(n) = -1.5(0.6)^n u(-n-1) - 0.5(0.2)^n u(n)$$

8.5.2.2 Numericals on Calculations of Convolution using Z-transform :

Consider two sequences $x_1(n)$ and $x_2(n)$. According to convolution property of Z transform

$$x_1(n) * x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z).$$

Use following steps to obtain convolution using Z transform.

Step 1 : Calculate Z transform of given sequences $x_1(n)$ and $x_2(n)$. It is $X_1(Z)$ and $X_2(Z)$.

Step 2 : Perform multiplication of $X_1(Z)$ and $X_2(Z)$. Denote the result by $Y(Z)$

$\therefore\ Y(Z) = X_1(Z) \cdot X_2(Z)$.

This gives convolution in Z domain.

Step 3 : Obtain IZT of Y (Z) that means y (n). This gives result of convolution in discrete domain.

Ex. 8.5.25 : Find the convolution of the signal using Z-transform.

$x_1(n) = a^n u(n) \qquad x_2(n) = u(n)$

[Note : a < 1]

Soln. :

Given :

$x_1(n) = a^n u(n), a < 1 \qquad x_2(n) = u(n)$

Step 1 : Calculation of $X_1(Z)$ and $X_2(Z)$:

$$\therefore\ X_1(Z) = \frac{Z}{Z-a} \qquad X_2(Z) = \frac{Z}{Z-1}$$

Step 2 : $Y(Z) = X_1(Z) \cdot X_2(Z)$

$$\therefore\ Y(Z) = \frac{Z}{Z-a} \cdot \frac{Z}{Z-1} \qquad Y(Z) = \frac{Z^2}{(Z-a)(Z-1)}$$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{Z}{(Z-a)(Z-1)}$$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{A}{(Z-a)} + \frac{B}{(Z-1)} \qquad \text{...(1)}$$

Where $A = (Z-a)\dfrac{Y(Z)}{Z}\Big|_{Z=a}$

$$\therefore\ A = (Z-a)\frac{z}{(Z-a)(Z-1)}\Big|_{Z=a}$$

$$\therefore\ A = \frac{Z}{Z-1}\Big|_{Z=a} \qquad \therefore\ A = \frac{a}{a-1} \qquad \text{...(2)}$$

Similarly, $B = (Z-1)\dfrac{Y(Z)}{Z}\Big|_{Z=1}$

$$\therefore\ B = (Z-1)\frac{Z}{(Z-1)(Z-a)}\Big|_{Z=1}$$

$$\therefore\ B = \frac{Z}{(Z-a)}\Big|_{Z=1} \qquad \therefore\ B = \frac{1}{1-a} \qquad \text{...(3)}$$

Putting values of A and B in Equation (1).

$$\frac{Y(Z)}{Z} = \frac{a/a-1}{Z-a} + \frac{1/1-a}{Z-1}$$

$$\therefore\ Y(Z) = \left(\frac{a}{a-1}\right)\frac{Z}{Z-a} + \left(\frac{1}{1-a}\right)\cdot\frac{Z}{Z-1}$$

Taking inverse Z transform

$$y(n) = \left(\frac{a}{a-1}\right)a^n u(n) + \left(\frac{1}{1-a}\right)u(n)$$

$$\therefore\ x_1(n) * x_2(n) = \left(\frac{a}{a-1}\right)a^n u(n) + \left(\frac{1}{1-a}\right)u(n)$$

Ex. 8.5.26 : Find convolution of the signal using 'z' transform

$x_1(n) = (0.25)^n u(n-1)$

$x_2(n) = [1 + 0.5^n]\, u[n]$

Dec. 11, 8 Marks

Soln. :

Step 1 : Calculation of $x_1(z)$ and $x_2(z)$:

$$x_1(n) = (0.25)^n u(n-1)$$

$$x_2(n) = (1 + 0.5^n)\, u(n)$$

According to property of linear convolution

$$Z[x_1 + (n) * x_2(n)] = X_1(Z) \cdot X_2(Z) \qquad \text{...(1)}$$

$\therefore$ Z transform of $x_1(n) = X_1(Z)$

$$x_1(n) = (0.25)^n u(n-1)$$

as $u(n) \xleftrightarrow{Z} \dfrac{Z}{Z-1}$

Using time shift property

$$\therefore\ u(n-1) \xleftrightarrow{Z} Z^{-1}\left[\frac{Z}{Z-1}\right]$$

$$\therefore\ u(n-1) \xleftrightarrow{Z} \frac{1}{Z-1}$$

Using scaling property

$$Z[(0.25)^n u(n-1)] = \frac{1}{\frac{Z}{0.25} - 1}$$

$$\therefore\ Z[(0.25)^n u(n-1)] = \frac{1}{4Z-1}$$

and

$$x_2(n) = [1 + (0.5)^n]\, u(n)$$

$$\therefore\ x_2(n) = u(n) + (0.5)^n u(n)$$

$$u(n) \xleftrightarrow{Z} \frac{Z}{Z-1}$$

Using scaling property

$$(0.5)^n u(n) \xleftrightarrow{Z} \frac{\frac{Z}{0.5}}{\frac{Z}{0.5} - 1}$$

$$\therefore\ (0.5)^n u(n) \xleftrightarrow{Z} \frac{2Z}{2Z-1}$$

$$\therefore\ X_2(Z) = \frac{Z}{Z-1} + \frac{2Z}{2Z-1}$$

Step 2 : $Y(Z) = X_1(Z) \cdot X_2(Z)$

$$Y(Z) = X_1(Z) \cdot X_2(Z) = Z\{x_1(n) * x_2(n)\}$$

$$\therefore\ Y(Z) = \frac{1}{4Z-1}\left[\frac{Z}{Z-1} + \frac{2Z}{2Z-1}\right]$$

$$\therefore\ Y(Z) = \frac{1}{4Z-1}\left[\frac{Z(2Z-1) + 2Z(Z-1)}{(Z-1)(2Z-1)}\right]$$

$$Y(Z) = \frac{4Z^2 - 3Z}{(4Z-1)(2Z-1)(Z-1)} \qquad \text{...(2)}$$

Step 3 : To obtain y (n), perform partial fraction expansion. To adjust denominators, Equation (2) can be written as,

$$Y(Z) = \frac{\left(\frac{1}{4}\times\frac{1}{2}\right)(4Z^2 - 3Z)}{\frac{1}{4}(4Z-1)(Z-1)\cdot\frac{1}{2}(2Z-1)}$$

$$\therefore\ Y(Z) = \frac{\frac{1}{8}(4Z^2 - 3Z)}{\left(Z-\frac{1}{4}\right)(Z-1)\left(Z-\frac{1}{2}\right)}$$

$$\therefore\ Y(Z) = \frac{\frac{1}{2}Z^2 - \frac{3}{8}Z}{\left(Z-\frac{1}{4}\right)(Z-1)\left(Z-\frac{1}{2}\right)}$$

$$\therefore\ \frac{Y(Z)}{Z} = \frac{\frac{1}{2}Z - \frac{3}{8}}{\left(Z-\frac{1}{4}\right)(Z-1)\left(Z-\frac{1}{2}\right)} \qquad \text{...(3)}$$

This equation can be written as

$$\frac{Y(Z)}{Z} = \frac{A_1}{\left(Z-\frac{1}{4}\right)} + \frac{A_2}{Z-\frac{1}{2}} + \frac{A_3}{(Z-1)} \qquad \text{...(4)}$$

To calculate A_1, A_2, A_3

$$A_1 = \left(Z-\frac{1}{4}\right)\frac{X(Z)}{2}\Big|_{Z=\frac{1}{4}}$$

$$\therefore\ A_1 = \frac{\left(Z-\frac{1}{4}\right)\left(\frac{1}{2}Z-\frac{3}{8}\right)}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)}\Bigg|_{Z=\frac{1}{4}}$$

$$\therefore\ A_1 = \frac{\left(\frac{1}{2}\times\frac{1}{4}\right)-\frac{3}{8}}{\left(\frac{1}{4}-\frac{1}{2}\right)\left(\frac{1}{4}-1\right)}$$

$$= \frac{\frac{1}{8}-\frac{3}{8}}{\left(-\frac{1}{4}\right)\left(-\frac{3}{4}\right)} = \frac{-2/8}{3/16}$$

$$\therefore\ A_1 = -4/3$$

and $A_2 = \left(Z-\dfrac{1}{2}\right)\dfrac{X(Z)}{Z}\Big|_{Z=\frac{1}{2}}$

$$A_2 = \frac{\left(Z-\frac{1}{2}\right)\left(\frac{1}{2}Z-\frac{-3}{8}\right)}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)}\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore\ A_2 = \frac{\left(\frac{1}{4}-\frac{-3}{8}\right)}{\left(\frac{1}{2}-\frac{1}{4}\right)\left(\frac{1}{2}-1\right)}$$

$$= \frac{-1/8}{\left(\frac{1}{4}\right)\left(-\frac{1}{2}\right)} = \frac{-1/8}{-1/8} = 1$$

and $A_3 = \dfrac{(Z-1)X(Z)}{Z}\Big|_{Z=1}$

$$A_3 = \frac{(Z-1)\left(\frac{1}{2}Z-\frac{3}{8}\right)}{\left(Z-\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)(Z-1)}\Bigg|_{Z=1}$$

$$= \frac{\left(\frac{1}{2}\times\frac{3}{8}\right)}{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{2}\right)}$$

$$= \frac{\frac{1}{8}}{\left(\frac{3}{4}\right)\left(\frac{1}{2}\right)} = \frac{1/8}{3/8} = 1/3$$

Putting these values in Equation (4),

$$\frac{Y(Z)}{Z} = \frac{-4/3}{Z-\frac{1}{4}} + \frac{1}{Z-\frac{1}{2}} + \frac{1/3}{Z-1}$$

$$\therefore\ Y(Z) = -\frac{4}{3}\left[\frac{Z}{Z-4}\right] + \left[\frac{Z}{Z-\frac{1}{2}}\right] + \frac{1}{3}\left[\frac{Z}{Z-1}\right] \qquad \text{...(5)}$$

Combined ROC is $|Z| > 1$

and $IZT \cdot \left[\dfrac{Z}{Z-\alpha}\right] = \alpha^n u(n)$

After taking inverse Z transform Equation (5) becomes

$$x(n) = \frac{-4}{3}\left(\frac{1}{4}\right)^n u(n) + \left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}(1)^n u(n)$$

Ex. 8.5.27 : Find the response of the time invariant system with impulse response $h[n] = \{1, 2, 1, -1\}$ to an input signal $x[n] = \{1, 2, 3, 1\}$ using convolution as well as using Z transform. Verify your answers.

Dec. 2016, 10 Marks

Soln. : Using convolution

We will obtain convolution using tabular method

$\therefore\ y(n) = \{1, 4, 8, 8, 3, -2, -1\}$

Using Z-transform

Given, $x(n) = \{1, 2, 3, 1\}$

$$\text{Now } X(Z) = \sum_{n=0}^{3} x(n) Z^{-n}$$

$$\therefore\ X(Z) = x(0) + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3}$$

$$\therefore\ X(Z) = 1 + 2Z^{-1} + 3Z^{-2} + Z^{-3}$$

Now $h(n) = \{1, 2, 1, -1\}$

$$\therefore\ H(Z) = 1 + 2Z^{-1} + Z^{-2} - Z^{-3}$$

Let $Y(Z) = X(Z) \cdot H(Z)$

	1	2	3	1
1	1	2	3	1
2	2	4	6	2
1	1	2	3	1
-1	-1	-2	-3	-1

$$\therefore\ Y(Z) = [1 + 2Z^{-1} + 3Z^{-2} + Z^{-3}]\,[1 + 2Z^{-1} + Z^{-2} - Z^{-3}]$$
$$= 1 + 2Z^{-1} + Z^{-2} - Z^{-3} + 2Z^{-1} + 4Z^{-2} + 2Z^{-3} - 2Z^{-4} + 3Z^{-2} + 6Z^{-3} + 3Z^{-4} - 3Z^{-5} + Z^{-3} + 2Z^{-4} + Z^{-5} - Z^{-6}$$
$$Y(Z) = 1 + 4Z^{-1} + 8Z^{-2} + 8Z^{-3} + 3Z^{-4} - 2Z^{-5} - Z^{-6} \quad ...(1)$$

Using definition of Z-transform

$$Y(Z) = \sum_{n=0}^{\infty} y(n)\,Z^{-n} = y(0) + y(1)\,Z^{-1} + y(2)\,Z^{-2} + ... \quad ...(2)$$

Comparing Equations (1) and (2),

$y(n) = [1, 4, 8, 8, 3, -2, -1]$

Thus using both methods, answer is same.

8.5.2.3 Calculation of Inverse Z using Convolution Theorem :

The following steps are used to calculate inverse z using convolution theorem.

Step 1 :	Split the given sequence X (z) into two parts $X_1(z)$ and $X_2(z)$.
Step 2 :	Using standard formulae calculate IZT of $X_1(z)$ and $X_2(z)$.
Step 3 :	Let $Y(Z) = X_1(z) \cdot X_2(z)$. Thus using equation of convolutions calculate Y (n).

Ex. 8.5.28 : Find the inverse z-transform of $X(z) = \frac{z}{(z-1)^2}$ using convolution theorem.

Soln. :

Given :

$$X(z) = \frac{z}{(z-1)^2}$$

$$\therefore\ X(z) = \frac{z}{z-1}\,\frac{1}{z-1}$$

Let $X_1(z) = \frac{z}{z-1}$ and $X_2(z) = \frac{1}{z-1} = z^{-1} \cdot \frac{1}{z-1}$

That means $X(z) = X_1(z) \cdot X_2(z)$

Taking IZT of $X_1(z)$ and $X_2(z)$ we get.

$x_1(n) = u(n)$ and $X_2(n) = u(n-1)$

Now, $x_1(n) * x_2(n) \overset{z}{\leftrightarrow} X_1(z) \cdot X_2(z)$

Using equation of convolution,

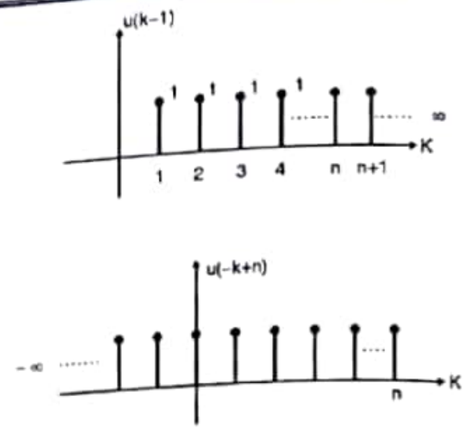
$$y(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_2(k) \cdot x_1(n-k)$$

Here $x_2(k) = u(k-1)$

$x_1(k) = u(k) \quad \therefore\ x_1(n-k) = u(-k+n)$

$$\therefore\ y(n) = \sum_{k=-\infty}^{\infty} u(k-1) \cdot u(-k+n)$$

These two signals are shown in Fig. P. 8.5.28.



(II-1524) Fig. P. 8.5.28

This overlapping is in the range k = 1 to n

$$\therefore\ y(n) = \sum_{k=1}^{n} 1 = \sum_{k=1}^{n} (1)^k$$

We have standard summation formula.

$$\sum_{n=N_1}^{N_2} A^n = N_2 - N_1 + 1 \text{ if } A = 1$$

Here $N_2 = n,\ N_1 = 1$

$\therefore\ y(n) = n - 1 + 1 = n$

$\therefore\ x_1(n) * x_2(n) = n\,u(n)$

Ex. 8.5.29 : Find inverse Z transform of :

1. $X(Z) = \frac{Z^2}{\left(Z - \frac{1}{4}\right)^2}$, ROC : $|Z| > \frac{1}{4}$

2. $X(Z) = \frac{1}{1 - 3Z^{-1} + 2Z^{-2}}$

using convolution method.

Soln. :

1. **Given :** $X(Z) = \frac{Z^2}{\left(Z - \frac{1}{4}\right)^2}$

According to the convolution property,

$$x_1(n) * x_2(n) = X_1(Z) \cdot X_2(Z) \quad ...(1)$$

Let $X_1(Z) = \frac{Z}{Z - \frac{1}{4}}$ and $X_2(Z) = \frac{Z}{Z - \frac{1}{4}}$

$$\therefore\ X(Z) = X_1(Z) \cdot X_2(Z) \quad ...(2)$$

We have standard Z transform formula

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z - \alpha},\ |Z| > |\alpha|$$

$$\therefore\ x_1(n) = \left(\frac{1}{4}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{4}\right)^n u(n)$$

Now $x(n) = x_1(n) * x_2(n)$

According to the definition of convolution,

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)$$

Here $x_1(k) = \left(\frac{1}{4}\right)^k \cdot u(k)$ and $x_2(n-k)\left(\frac{1}{4}\right)^{n-k} \cdot u(n-k)$

$$\therefore\ x(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k \cdot u(k) \cdot \left(\frac{1}{4}\right)^{n-k} \cdot u(n-k)$$

$$\therefore x(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{4}\right)^k \cdot \left(\frac{1}{4}\right)^n \cdot \left(\frac{1}{4}\right)^{-k} \cdot u(k) \cdot u(-k+n)$$

$$\therefore x(n) = \left(\frac{1}{4}\right)^n \sum_{k=-\infty}^{\infty} u(k)\,u(-k+n)$$

Here u (k) is unit step; existing from k = 0 to ∞ and u (− k + n) indicates folded and delayed unit step which exists from − ∞ to n. That means u (k) · u (− k + n) represents unit step in the range k = 0 to n.

$$\therefore\ x(n) = \left(\frac{1}{4}\right)^n \cdot \sum_{k=0}^{n} 1$$

$$\therefore\ x(n) = \left(\frac{1}{4}\right)^n \cdot \sum_{k=0}^{n} (k)^0$$

Now $\sum_{k=0}^{n} k^0 = n + 1$

$$\therefore\ x(n) = \left(\frac{1}{4}\right)^n \cdot (n+1) \cdot u(n)$$

2. **Given :**

$$X(Z) = \frac{1}{1 - 3Z^{-1} + 2Z^{-2}}$$

$$\therefore\ X(Z) = \frac{Z^2}{Z^2 - 3Z + 2} = \frac{Z^2}{(Z-1)(Z-2)}$$

Let $X_1(Z) = \frac{Z}{Z-1}$ and $X_2(Z) = \frac{Z}{Z-2}$

Now $x_1(n) * x_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)$

Here $x_1(n) = u(n)$ and $x_2(n) = 2^n u(n)$

Using definition of convolution,

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)$$

Here $x_1(k) = u(k)$ and $x_2(n-k) = 2^{n-k} \cdot u(n-k)$

$$\therefore\ x(n) = \sum_{k=-\infty}^{\infty} u(k) \cdot 2^{n-k} \cdot u(n-k)$$

$$\therefore\ x(n) = \sum_{k=-\infty}^{\infty} 2^n \cdot 2^{-k} \cdot u(k)\,u(-k+n)$$

$$\therefore\ x(n) = 2^n \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \cdot u(-k+n)$$

Now u (k) · u (− k + n) indicates unit step existing from k = 0 to n

$$\therefore\ x(n) = 2^n \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k$$

We have standard summation formula.

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \quad \text{for } a \neq 1$$

Here $a = \frac{1}{2},\ N_1 = 0,\ N_2 = n$

$$\therefore\ x(n) = 2^n \cdot \left[\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}\right]$$

$$\therefore\ x(n) = 2^n \cdot 2\left[1 - \left(\frac{1}{2}\right)^{n+1}\right]$$

$$\therefore\ x(n) = 2^{n+1}\left[1 - \left(\frac{1}{2}\right)^{n+1}\right] \cdot u(n)$$

Ex. 8.5.30 : Find causal sequence x [n] by using convolution theorem of z transform

$$x[z] = \frac{1}{\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 + \frac{1}{4}Z^{-1}\right)}$$

Soln. : Given equation is,

$$X(Z) = \frac{1}{\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 + \frac{1}{4}Z^{-1}\right)} = X_1(Z) \cdot X_2(Z)$$

Let $X_1(Z) = \frac{1}{1 - \frac{1}{2}Z^{-1}} = \frac{Z}{Z - \frac{1}{2}}$

and $X_2(Z) = \frac{1}{1 - \frac{1}{4}Z^{-1}} = \frac{Z}{Z + \frac{1}{4}}$

$$\therefore\ X_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

and $X_2(n) = \left(-\frac{1}{4}\right)^n u(n)$

Now according to convolution theorem,

$$X_1(n) * X_2(n) \overset{Z}{\longleftrightarrow} X_1(Z) \cdot X_2(Z)$$

According to the definition of convolution

$$X_1(n) * X_2(n) = \sum_{k=-\infty}^{\infty} X_1(k)\,X_2(n-k)$$

Here $X_1(k) = \left(\frac{1}{2}\right)^k u(k)$

and $X_2(n-k) = \left(-\frac{1}{4}\right)^{n-k} u(n-k)$

$$= \left(-\frac{1}{4}\right)^{n} \cdot \left(-\frac{1}{4}\right)^{-k} \cdot u(n-k)$$

$$\therefore\ X_2(n-k) = \left(-\frac{1}{4}\right)^{n} \cdot \left(-\frac{1}{4}\right)^{-k} \cdot u(-k+n)$$

$$\therefore\ X_1(n) * X_2(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k} u(k) \cdot \left(-\frac{1}{4}\right)^{n} \cdot \left(-\frac{1}{4}\right)^{-k} u(-k+n)$$

Here u (k) is unit step from k = 0 to + ∞ and u (– k + n) is unit step from – ∞ to + n. So limits of summation will be from 0 to n

$$\therefore\ X_1(n) * X_2(n) = \sum_{k=0}^{n} \left(-\frac{1}{4}\right)^{k} \cdot \left(\frac{1}{2}\right)^{k} \cdot \left(-\frac{1}{4}\right)^{-k}$$

$$= \left(-\frac{1}{4}\right)^{n} \cdot \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k} \cdot \frac{1}{\left(-\frac{1}{4}\right)^{k}}$$

$$= \left(-\frac{1}{4}\right)^{n} \cdot \sum_{k=0}^{n} \left[\frac{1}{2} \cdot (-4)\right]$$

$$= \left(-\frac{1}{4}\right)^{n} \sum_{k=0}^{n} (-2)^{k}$$

We have standard summation formula

$$\sum_{k=N_1}^{N_2} A^{k} = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here A = – 2, $N_1 = 0$ $N_2 = n$

$$\therefore\ X_1(n) * X_2(n) = \left(-\frac{1}{4}\right)^{n} \cdot \left[\frac{(-2)^{0} - (-2)^{n+1}}{1+2}\right]$$

$$\therefore\ X(n) = X_1(n) * X_2(n) = \left(-\frac{1}{4}\right)^{n} \cdot \left[\frac{1-(-2)^{n+1}}{3}\right]$$

8.5.3 IZT using Cauchy's Residue Theorem :

Let X (Z) be the function of complex variable "Z" and 'C' be the closed path in the Z plane. If the derivative $\frac{dX(Z)}{dZ}$ exists on and inside the contour "C" and if X (Z) has no poles at $Z = Z_0$ then,

$$\frac{1}{2\pi j} \oint_C \frac{X(Z)}{Z - Z_0}\, dZ = \begin{cases} x(z_0) & \text{if } Z_0 \text{ is inside 'C'} \\ 0 & \text{if } Z_0 \text{ is outside 'C'} \end{cases} \qquad \ldots(8.5.13)$$

Now if the (K + 1) order derivative of X (Z) exists and if X (Z) has no poles at $Z = Z_0$ then,

$$\frac{1}{2\pi j} \oint_C \frac{X(Z)}{(Z - Z_0)^{K}}\, dz$$

$$= \begin{cases} \frac{1}{(K+1)!} \frac{d^{K-1}}{dZ^{K-1}} X(Z) \Big|_{Z = Z_0} & \text{if } Z_0 \text{ is inside 'C'} \\ 0 & \text{if } Z_0 \text{ is outside 'C'} \end{cases}$$

These values on the R.H.S. are called as residues of the pole at $Z = Z_0$. Here residues are the values of the function at the particular pole.

Now we will discuss how to obtain IZT using residue theorems.

Steps to be followed :

1. Multiply given X (Z) by Z^{n-1}
2. Obtain the residue of $X(Z) \cdot Z^{n-1}$ using the equation,

$$R_{Z=a} = \frac{d^{m-1}}{dZ^{m-1}} \left\{ \frac{(Z-a)^{m}}{(m-1)!} X(Z) \cdot Z^{n-1} \right\} \Big|_{Z=a} \qquad \text{for } m \geq 1$$

$$R_{Z=a} = (Z-a)^{2} X(Z) \cdot Z^{n-1} \Big|_{Z=a} \qquad \text{for } m = 1$$

$$R_{Z=a} = \frac{d}{dZ} \left[(Z-a)^{2} X(Z) \cdot Z^{n-1} \right] \Big|_{Z=a} \qquad \text{for } m = 2$$

3. Inverse Z of X (Z) is obtained by adding the residues in Step 2.

8.5.3.1 Solved Examples using Residue Method :

Ex. 8.5.31 : Find x (n) for the given X (Z)

$$X(Z) = \frac{10Z}{(Z-1)(Z-2)}$$

Soln. : The given equation is,

$$X(Z) = \frac{10Z}{(Z-1)(Z-2)} \qquad \ldots(1)$$

Step 1 : Multiply both sides of Equation (1) by Z^{n-1}.

$$\therefore\ X(Z) \times Z^{n-1} = \frac{10Z^{1} \times Z^{n-1}}{(Z-1)(Z-2)}$$

$$\therefore\ X(Z) \times Z^{n-1} = \frac{10Z^{n}}{(Z-1)(Z-2)} \qquad \ldots(2)$$

Step 2 : Now the residues of $X(Z) \times Z^{n-1}$ are obtained as follows :

Note that the poles of Equation (2) are $P_1 = 1$ and $P_2 = 2$.

Thus residue at Z = 1 is,

$$\therefore\ R_{Z=1} = (Z-1) \times \frac{10Z^{n}}{(Z-1)(Z-2)} \Big|_{Z=1}$$

$$= \frac{10Z^{n}}{Z-2} \Big|_{Z=1} = \frac{10 \times (1)^{n}}{1-2}$$

$$\therefore\ R_{Z=1} = -10(1)^{n} \qquad \ldots(3)$$

Now residue at Z = 2 is,

$$R_{Z=2} = (Z-2) \frac{10Z^{n}}{(Z-1)(Z-2)} \Big|_{Z=2}$$

$$\therefore\ R_{Z=2} = \frac{10Z^{n}}{(Z-1)} \Big|_{Z=2} = \frac{10(2)^{n}}{(2-1)}$$

$$\therefore\ R_{Z=2} = 10(2)^{n} \qquad \ldots(4)$$

Step 3 : The sequence x(n) is obtained by adding Equations (3) and (4).

$$\therefore\ x(n) = -10(1)^{n} + 10(2)^{n} \quad \text{for } n \geq 0$$

Ex. 8.5.32 : Obtain inverse Z-transform (IZT) using residue method where $X(Z) = \frac{1}{(Z-1)(Z-3)}$.

Soln. :

Step 1 : Multiply both sides of given equation by Z^{n-1}

$$\therefore\ X(Z) \times Z^{n-1} = \frac{Z^{n-1}}{(Z-1)(Z-3)} \qquad \ldots(1)$$

Step 2 : The poles are at Z = 1 and Z = 3.

The residue at Z = 1 is,

$$R_{Z=1} = (Z-1) \times \frac{Z^{n-1}}{(Z-1)(Z-3)} \Big|_{Z=1}$$

$$= \frac{Z^{n-1}}{Z-3} \Big|_{Z=1} = \frac{(1)^{n-1}}{(1-3)} = \frac{(1)^{n} \times (1)^{-1}}{-2}$$

$$= \frac{(1)^{n}}{-2}$$

$$\therefore\ R_{Z=1} = -0.5(1)^{n} \qquad \ldots(2)$$

Residue at Z = 3 is,

$$R_{Z=3} = \frac{(Z-3)Z^{n-1}}{(Z-1)(Z-3)} \Big|_{Z=3} = \frac{Z^{n-1}}{(Z-1)} \Big|_{Z=3}$$

$$= \frac{(3)^{n-1}}{(3-1)} = \frac{(3)^{n} \times (3)^{-1}}{2} = 0.17(3)^{n} \qquad \ldots(3)$$

Step 3 : The sequence x (n) is,

$$x(n) = -0.5(1)^{n} + 0.17(3)^{n} \quad n \geq 0$$

Ex. 8.5.33 : Use residue method to obtain x (n) from :

$$X(Z) = \frac{Z(Z+1)}{(Z-1)^{2}}$$

Soln. :

This is the case of repeated poles.

Step 1 : Multiply both sides of given equation by Z^{n-1}

$$\therefore X(Z) Z^{n-1} = \frac{Z(Z+1) \times Z^{n-1}}{(Z-1)^{2}}$$

$$= \frac{Z^{1}(Z+1)Z^{n} \times Z^{-1}}{(Z-1)^{2}} = \frac{(Z+1)Z^{n}}{(Z-1)^{2}}$$

$$\therefore X(Z) \times Z^{n-1} = \frac{Z^{n} + Z^{n+1}}{(Z-1)^{2}} \qquad \ldots(1)$$

Step 2 : Now the given function has second order pole so the corresponding residue is given by,

$$R_{Z=1} = \frac{d}{dZ} \left[(Z-1)^{2} \times \left(\frac{Z^{n} + Z^{n+1}}{(Z-1)^{2}} \right) \right] \Big|_{Z=1}$$

$$= \frac{d}{dZ} \left[Z^{n} + Z^{n+1} \right] \Big|_{Z=1}$$

$$= nZ^{n-1} + (n+1)Z^{n} \Big|_{Z=1}$$

$$\therefore R_{Z=1} = n(1)^{n-1} + (n+1)1^{n} = n(1)^{n}(1)^{-1} + (n+1)(1)^{n} = (2n+1)(1)^{n}$$

Step 3 : Thus the sequence x(n) is,

$$x(n) = (2n+1)(1)^{n} \quad n \geq 0$$

8.6 Relationship between Fourier Transform and Z-Transform :

MU - May 05, May 07, Dec. 07, Dec. 11

University Questions

Q. 1 Obtain the relationship between Discrete Time, Fourier Transform and Z-Transform and show that the two are same on the unit circle. **(May 05, 5 Marks)**

Q. 2 Explain the relationship between Discrete Time Fourier Transform and z-Transform. **(May 07, 6 Marks, Dec. 11, 4 Marks)**

Q. 3 Compare Discrete Time Fourier Transform and Z-transform. **(Dec. 07, 6 Marks)**

The fourier transform of discrete time signal can be expressed as,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \qquad \ldots(8.6.1)$$

While Z-transform of sequence x(n) is given by,

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \qquad \ldots(8.6.2)$$

As 'Z' is a complex variable; it can be represented as,

$$Z = re^{j\omega}$$

Here 'r' is the magnitude of Z, that means $r = |Z|$ and 'ω' is the angle of Z, that means $\omega = \angle Z$

Substitute $Z = re^{j\omega}$ in Equation (8.6.2),

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} x(n) [re^{j\omega}]^{-n} \qquad \ldots(8.6.3)$$

$$\therefore X(Z) = \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n} \qquad \ldots(8.6.4)$$

This Equation (8.6.4) shows that X(Z) is a fourier transform (FT) of a sequence $[x(n) r^{-n}]$. Now if we evaluate Equation (8.6.4) on the unit circle that means $|Z| = 1$ then we an write,

$$r = |Z| \Rightarrow r = 1 \qquad \therefore\ r^{-n} = (1)^{-n} = 1$$

Thus if X (Z) is evaluated on unit circle then Equation (8.6.4) becomes.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \quad \text{...if } Z = e^{-j\omega n} \text{ and } = |Z| = 1 \quad ...(8.6.5)$$

The R.H.S. of Equation (8.6.5) is nothing else but the fourier transform of x(n). Thus fourier transform of ***a signal is basically equal to the Z-transform of a signal evaluated on a unit circle.***

***Frequency* scale on the unit circle :**

We know that $Z = re^{j\omega}$ ...(8.6.6)

Equation (8.6.6) represents a phasor with magnitude 'r' and angle 'ω'. Now on the unit circle $r = |Z| = 1$. The magnitude is expressed as :

$$|Z| = \sqrt{R_e^2\{Z\} + I_m^2\{Z\}} \quad ...(8.6.7)$$

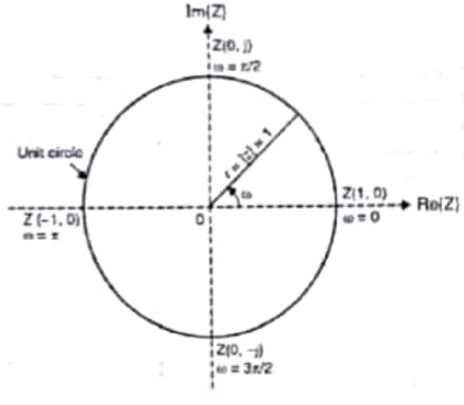
and angle is expressed as, $\angle Z = \tan^{-1}\left[\frac{I_m\{Z\}}{R_e\{Z\}}\right]$

$$\therefore \omega = \tan^{-1}\left[\frac{I_m\{Z\}}{R_e\{Z\}}\right] \quad ...(8.6.8)$$

On the unit circle, 'ω' that means < Z is going to take values from 0 to 2π radians. However because FT = ZT on the unit circle; these values of 'ω' correspond to frequencies. Thus unit circle becomes frequency axis.

The range of frequency 'ω' is from 0 to 2 π or – π to π. Note that 'zero' and '2π' points correspond to the same point on the unit circle. Also π and – π points correspond to the same point (– 1, 0 j) on the unit circle.

If the maximum frequency is π then the upper part of unit circle represents the actually used frequency scale that is 0 to π . This frequency scale on unit circle is shown in Fig. 8.6.1.



(II-1035) **Fig. 8.6.1 : Frequency scale on unit circle**

8.7 Plotting Poles and Zeros of Transfer Function :

An important property of z-transform is that; it can be expressed as a rational function. That means ratio of two polynomials. In this section we will discuss the concept of pole zero related to rational function.

8.7.1 Pole Zero Plot :

Consider an LTI system with input x (n) and output y (n). In terms of Z transform; we can define system transfer function [H (Z)] as the ratio of output and input.

Thus $$H(Z) = \frac{Y(Z)}{X(Z)}$$

We have discussed that the Z-transform can be expressed as the ratio of two polynomials. It is called as rational Z-transform. Thus,

$$X(Z) = \frac{N(Z)}{D(Z)} = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2} + \ldots\ldots + b_M Z^{-M}}{a_0 + a_1 Z^{-1} + a_2 Z^{-2} + \ldots\ldots + a_N Z^{-N}} \quad ...(8.7.1)$$

1. Poles are the values of Z for which X (Z) is ∞.
2. Zeros are the values of Z for which X (Z) is 0.

Thus the poles are not present in the ROC of X (Z) but the zeros can be obtained in the ROC. In terms of summation, Equation (8.7.1) can be written as,

$$X(Z) = \frac{\sum_{K=0}^{M} b_K Z^{-K}}{\sum_{K=0}^{N} a_K Z^{-K}} \quad ...(8.7.2)$$

Rearranging Equation (8.7.1) we get,

$$X(Z) = \frac{Z^{-M}\left[b_0 Z^M + b_1 Z^{M-1} + b_2 Z^{M-2} + \ldots + b_M\right]}{Z^{-N}\left[a_0 Z^N + a_1 Z^{N-1} + a_2 Z^{N-2} + \ldots + a_N\right]} \quad ...(8.7.3)$$

Thus the polynomials are covered into positive powers of Z-Equation (8.7.3) can be further rearranged as,

$$X(Z) = \frac{Z^{-M} b_0\left[Z^M + \frac{b_1}{b_0}Z^{M-1} + \frac{b_2}{b_0}Z^{M-2} + \ldots + \frac{b_M}{b_0}\right]}{Z^{-N} a_0\left[Z^N + \frac{a_1}{a_0}Z^{N-1} + \frac{a_2}{a_0}Z^{N-2} + \ldots + \frac{a_N}{a_0}\right]} \quad ...(8.7.4)$$

Here numerator and denominator are polynomials, so they can be expressed in factored form as :

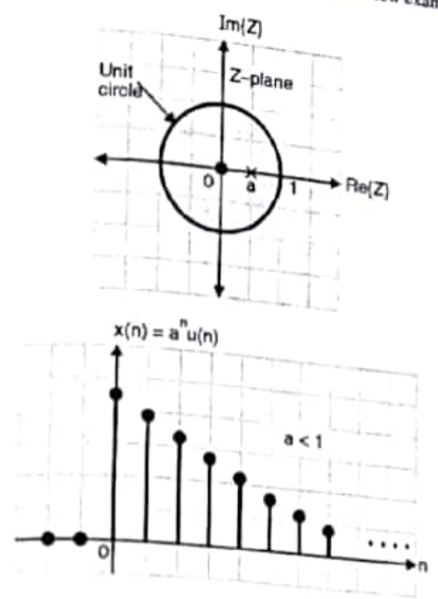
$$X(Z) = \frac{b_0}{a_0} Z^{N-M} \frac{(Z-Z_1)(Z-Z_2)\ldots\ldots(Z-Z_M)}{(Z-P_1)(Z-P_2)\ldots\ldots(Z-P_N)} \quad ...(8.7.5)$$

Here $Z_1, Z_2 \ldots\ldots Z_M$ are zeros and $P_1, P_2 \ldots\ldots P_N$ are the poles. From Equation (8.7.5) we can conclude that.

1. $X(Z) = 0$ for $Z = Z_1, Z_2 \ldots\ldots Z_M$
2. $X(Z) = \infty$ for $Z = P_1, P_2 \ldots\ldots P_N$.

8.7.2 Behaviours of Causal DT Signals :

A causal LSI system is a system present at positive values of Z. All the characteristics of such system depends on pole-zero location with respect to unit circle. We will consider few examples.



(II-1025) **Fig. 8.7.1(a) : Pole-zero plot of a^n u (n)**

1. Let, $x(n) = a^n u(n)$

If a < 1, we will get decaying exponential. Its Z-transform is,

$$X(Z) = \frac{Z}{Z-a} = \frac{(Z-0)}{Z-a} \quad ...(8.7.6)$$

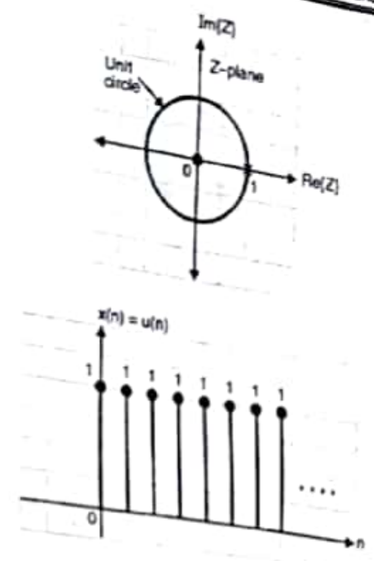
Thus zero is at Z = 0 (origin) and poles is at Z = a. It is shown in Fig. 8.7.1(a).

Here zero is marked by '0' and pole is marked by 'x'. As shown in Fig. 8.7.1(a); the pole is inside unit circle.

2. Let x (n) = u (n)

Then $$X(Z) = \frac{Z}{Z-1} = \frac{(Z-0)}{(Z-1)} \quad ...(8.7.7)$$

Thus zero is at Z = 0 (origin) and pole is at Z = 1 (on the unit circle.) It is shown in, Fig. 8.7.1(b).

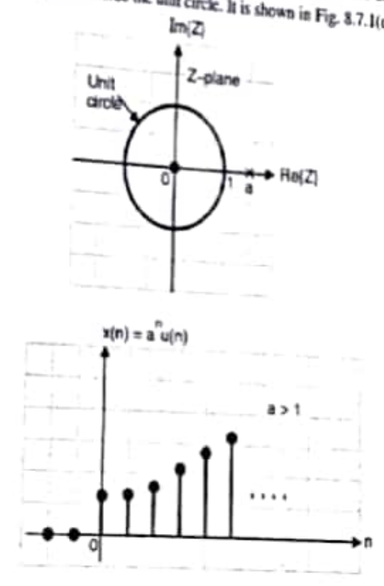


(II-1026) **Fig. 8.7.1(b) : Pole-zero plot of u(n)**

3. Let $x(n) = a^n u(n)$ and a > 1 then we will get rising exponential.

Now $$X(Z) = \frac{Z}{Z-a} = \frac{Z-0}{Z-a} \quad ...(8.7.8)$$

Thus zero is at '0' (origin) and pole is at 'a'. But in this case a > 1. So pole is outside the unit circle. It is shown in Fig. 8.7.1(c).

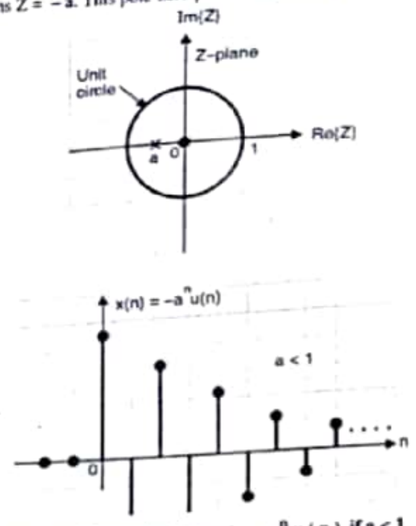


(II-1027) **Fig. 8.7.1(c)**

4. Let $x(n) = -a^n u(n)$ and $a < 1$

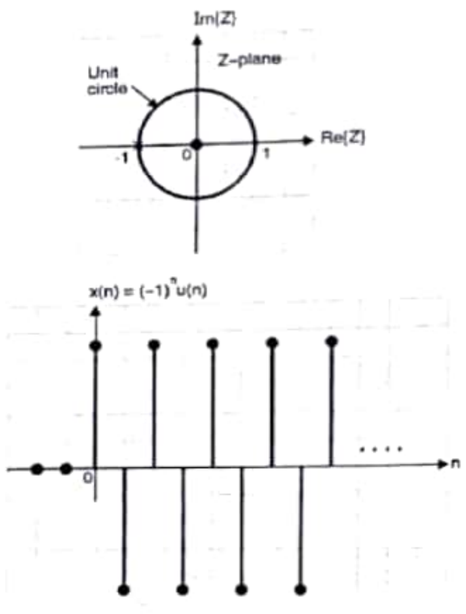
$$\text{then } X(Z) = -\frac{Z}{Z-a} = \frac{Z}{-Z+a} \quad \text{...(8.7.9)}$$

Thus zero is at Z = '0' (origin) and pole is at $-Z + a = 0$ that means $Z = -a$. This pole-zero plot is shown in Fig. 8.7.1(d).



(H-1028) Fig. 8.7.1(d) : $x(n) = -a^n u(n)$, if $a < 1$

5. Let $x(n) = (-1)^n u(n)$, we have standard Z-transform pair.



(H-1029) Fig. 8.7.1(e) : $x(n) = (-1)^n u(n)$

$$a^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-a}$$

Let $a = -1$

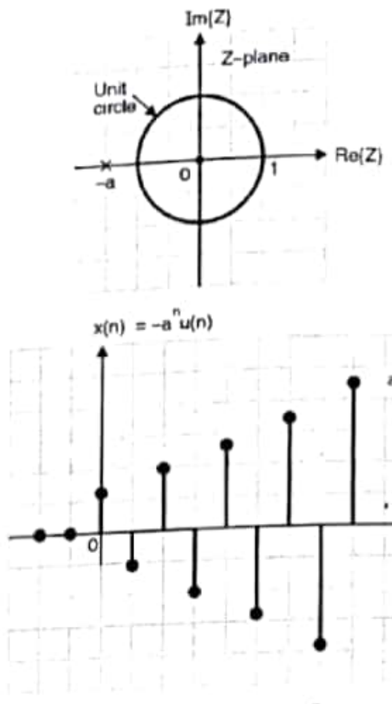
$$\therefore (-1)^n u(n) \xleftrightarrow{Z} \frac{Z}{Z+1} \quad \text{...(8.7.10)}$$

Thus zero is at Z = '0' (origin) and pole is at $Z + 1 = 0$ that means $Z = -1$. This pole zero plot is shown in Fig. 8.7.1(e).

6. Let $x(n) = -a^n u(n)$ and $a > 1$.

$$\therefore X(Z) = -\frac{Z}{Z-a} = \frac{Z}{-Z+a} \quad \text{...(8.7.11)}$$

Thus zero is at Z = 0 (origin) and pole is at $-Z + a = 0$ that means at $Z = -a$. But in this case $a > 1$ that means $-a > -1$. This pole zero plot is shown in Fig. 8.7.1(f).



(H-1030) Fig. P. 8.7.1(f) : $x(n) = -a^n u(n)$ if $a > 1$

From this discussion we can conclude the following points :

1. If the pole is inside unit circle then the signal decays [Fig. 8.7.1(a)].
2. If the pole is on the unit circle then the signal has fixed amplitude [Fig. 8.7.1(b)].
3. If the pole is outside the unit circle then the amplitude of signal is increasing [Fig. 8.7.1(c)]
4. If there is negative pole then the signal alters in sign as shown in Figs. 8.7.1(d), 8.7.1(e) and 8.7.1(f).

Also for the causal signals if the poles are outside the unit circle [Figs. 8.7.1(c) and 8.7.1(f)] then the signal is unstable. Because we know that for stability; ROC should include unit circle. The position of zeros also affect the behaviour of causal signal, but not as strong as the poles.

8.7.3 Solved Examples on Pole Zero Plot :

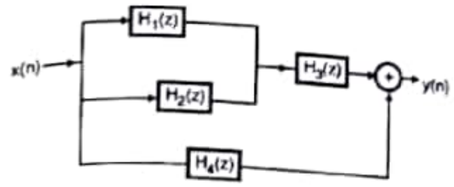
Ex. 8.7.1 : A DT system has transfer function

$H_1(Z)$ has pole at 0.6 and zero at 0.

$H_2(Z)$ zero at -2 and pole at 0.

$H_3(Z) = Z^{-1}$ and $H_4(Z) = 2$.

Find overall impulse response.



(H-1049) Fig. P. 8.7.1

Soln. : As shown in Fig. P. 8.7.1 $H_1(Z)$ and $H_2(Z)$ are in parallel and this parallel combination is in series with $H_3(Z)$. While $H_4(Z)$ is parallel to this combination. Thus overall transfer function is.

$$H(Z) = [[H_1(Z) + H_2(Z)] \cdot H_3(Z)] + H_4(Z) \quad \text{...(1)}$$

$H_1(Z)$ has pole at 0.6 and zero at 0.

$$\therefore H_1(Z) = \frac{Z}{Z-0.6} \quad \text{...(2)}$$

$H_2(Z)$ has pole at origin and zero at -2.

$$\therefore H_2(Z) = \frac{Z+2}{Z} \quad \text{...(3)}$$

$H_3(Z) = Z^{-1}$ and $H_4(Z) = 2$.

Putting these values in Equation (1).

$$H(Z) = \left[\left(\frac{Z}{Z-0.6} + \frac{Z+2}{Z}\right) \cdot Z^{-1}\right] + 2$$

$$\therefore H(Z) = \frac{1}{Z-0.6} + \frac{1+2Z^{-1}}{Z} + 2$$

$$\therefore H(Z) = Z^{-1}\left(\frac{Z}{Z-0.6}\right) + \frac{1}{Z} + \frac{2}{Z^2} + 2$$

$$\therefore H(Z) = Z^{-1}\left(\frac{Z}{Z-0.6}\right) + Z^{-1} + Z^{-2} \cdot 2 + 2$$

We have standard Z-transform pairs,

$$(\alpha)^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-\alpha} \text{ and } \delta(n) \xleftrightarrow{Z} 1$$

$$\therefore h(n) = (0.6)^{n-1} \cdot u(n-1) + \delta(n-1) + 2\delta(n-2) + 2\delta(n)$$

Ex. 8.7.2 : Two LTI systems in cascade have impulse response $h_1[n]$ and $h_2[n]$.

$h_1[n] = (0.9)^n u[n] - 0.5(0.9)^{n-1} u[n-1]$

$h_2[n] = (0.5)^n u[n] - (0.5)^{n-1} u[n-1]$

Find the equivalent response h [n] of the system. **May 2014, 10 Marks**

Soln. :

Given $h_1(n) = (0.9)^n u(n) - 0.5(0.9)^{n-1} u(n-1)$

We have standard Z transform pair,

$$\alpha^n u(n) \leftrightarrow \frac{Z}{Z-\alpha}$$

and according to time shifting property.

$x(n-k) \leftrightarrow Z^{-k} X(Z)$

$$\therefore H_1(Z) = \frac{Z}{Z-0.9} - 0.5Z^{-1}\frac{Z}{Z-0.9}$$

$$= \frac{Z}{Z-0.9} - \frac{0.5}{Z-0.9}$$

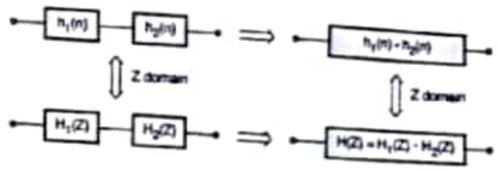
$$\therefore H_1(Z) = \frac{Z-0.5}{Z-0.9}$$

Now, $h_2(n) = (0.5)^n u(n) - (0.5)^{n-1} u(n-1)$

$$\therefore H_2(Z) = \frac{Z}{Z-0.5} - Z^{-1}\frac{Z}{Z-0.5} = \frac{Z}{Z-0.5} - \frac{1}{Z-0.5}$$

$$\therefore H_2(Z) = \frac{Z-1}{Z-0.5}$$

If two systems are in cascade then its equivalent response is obtained by taking convolution of individual response. In Z domain, convolution becomes multiplication. It is shown in Fig. P. 8.7.2.



(H-1527) Fig. P. 8.7.2

Since. $H(Z) = H_1(Z) \cdot H_2(Z)$

$$\therefore H(Z) = \frac{Z-0.5}{Z-0.9} \cdot \frac{Z-1}{Z-0.5} = \frac{Z-1}{Z-0.9}$$

$$\therefore H(Z) = \frac{Z}{Z-0.9} - \frac{1}{Z-0.9} = \frac{Z}{Z-0.9} - Z^{-1} \cdot \frac{Z}{Z-0.9}$$

Taking IZT we get,

$$h(n) = (0.9)^n u(n) - (0.9)^{n-1} u(n-1)$$

Ex. 8.7.3 : Find the Z transform of

$x_1(n) = e^{-(n/40)} u(n)$

$x_2(n) = e^{-(n/40)} \sin\left(\frac{2\pi n}{8}\right) u(n)$ Draw the pole zero diagram for $X_1(Z)$ and $X_2(Z)$

Soln. :

Calculation of $X_1(Z)$:

Given : $x_1(n) = e^{-(n/40)} u(n)$

$$\therefore x_1(n) = (e^{-1/40})^n u(n)$$

$$\therefore x_1(n) = (0.975)^n u(n)$$

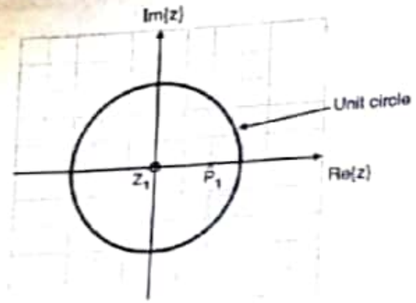
We have standard formula.

$$\alpha^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-\alpha}, |Z| > |\alpha|$$

$$\therefore X_1(Z) = \frac{Z}{Z-0.975}, |Z| > 0.975$$

Zero is at $Z_1 = 0$; pole is at $P_1 = 0.975$

The pole - zero plot is shown in Fig. P. 8.7.3(a).



(H-1068) **Fig. P. 8.7.3(a)**

Calculation of X_2 (Z)

Given, $x_2(n) = e^{-(n/40)} \sin\left(\frac{2\pi n}{8}\right) u(n)$

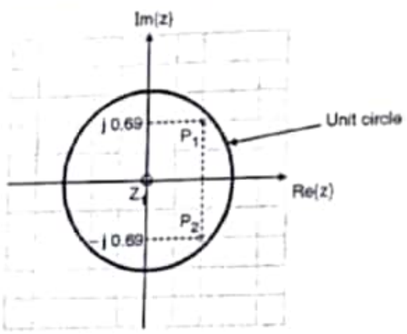
$\therefore \quad x_2(n) = (e^{-1/40})^n \sin\left(\frac{2\pi n}{8}\right) u(n)$

$\therefore \quad x_2(n) = (0.975)^n \sin\left(\frac{\pi}{4} n\right) u(n)$

We have standard formula.

$$a^n \sin \omega_0 n\, u(n) \xleftrightarrow{Z} \frac{a Z \sin \omega_0}{Z^2 - 2a Z \cos \omega_0 + a^2}$$

Here $a = 0.975$, $\omega_0 = \frac{\pi}{4}$



(H-1034) **Fig. P. 8.7.3(b)**

$$\therefore \quad X_2(Z) = \frac{0.975 Z \sin\left(\frac{\pi}{4}\right)}{Z^2 - (2 \times 0.975) Z \cos\left(\frac{\pi}{4}\right) + (0.975)^2}$$

$$\therefore \quad X_2(Z) = \frac{0.689 Z}{Z^2 - 1.38 Z + 0.95}$$

$$\therefore \quad X_2(Z) = \frac{0.689 Z}{(Z - 0.69 - j\,0.69)(Z - 0.69 + j\,0.69)}, \quad |Z| > 0.975$$

Here zero is at

$Z_1 = 0$

Poles are at $P_1 = 0.69 + j\,0.69$, $P_2 = 0.69 - j\,0.69$

Ex. 8.7.4 : Show pole-zero diagram for a system having transfer function $H(Z) = 1 + Z^{-4}$.

Soln. : Given transfer function is.

$$H(Z) = 1 + Z^{-4} \qquad \ldots(1)$$

To convert it into positive powers of Z; we will divide and multiply Equation (1) by Z^4.

$$\therefore \quad H(Z) = \frac{Z^4 (1 + Z^{-4})}{Z^4}$$

$$\therefore \quad H(Z) = \frac{Z^4 + 1}{Z^4} \qquad \ldots(2)$$

We know that the numerator term represents zeros and denominator term represents poles of a system.

Zeros : Thus zeros are at, $Z^4 + 1 = 0$

That means $\quad Z^4 = -1 \qquad \ldots(3)$

We can write $\quad -1 = e^{j\pi}$. Because according to Euler's identity,

$$e^{j\pi} = \cos \pi + j \sin \pi;$$

where $-\cos \pi = -1$ and $\sin \pi = 0 \quad \therefore e^{j\pi} = -1$

$$\therefore \quad Z^4 = e^{j\pi} \qquad \ldots(4)$$

Now consider the term $e^{j\pi k}$. We can write.

$$e^{j\pi k} = \cos 2\pi k + j \sin 2\pi k$$

Let 'k' be any integer. Then $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$

$$\therefore \quad e^{j2\pi k} = 1$$

So we can multiply R.H.S. of Equation (4) by $e^{j2\pi k}$

$$\therefore Z^4 = e^{j\pi} \cdot e^{j1\pi k} \qquad \therefore Z^4 = e^{j(2\pi k \times \pi)}$$

$$\therefore Z = \left[e^{j(2\pi k + \pi)}\right]^{\frac{1}{4}} \quad \therefore \quad Z = e^{j\left(\frac{2\pi k + \pi}{4}\right)} \qquad \ldots(5)$$

Thus general expression for zeros can be written as,

$$Z_k = e^{j\left(\frac{2\pi k + \pi}{4}\right)} \qquad \ldots(6)$$

Here k is an integer. Since the power of Z in Equation (4) is '4'; the range of k is from $k = 0$ to $k = 3$.

$$\therefore \quad \text{For } k = 0 \quad \Rightarrow Z_0 = e^{j\frac{\pi}{4}}$$

We have general expression of Z,

$$Z = r e^{j\omega}$$

Where $\quad r$ = Magnitude of Z and ω = angle

$\therefore Z_0 = e^{j\frac{\pi}{4}}$ respresents magnitude 1 and angle $\frac{\pi}{4}$

$$\text{For } k = 1 \Rightarrow Z_1 = e^{j\left(\frac{2\pi + \pi}{4}\right)} = e^{j\left(\frac{3\pi}{4}\right)}$$

That means magnitude 1 and angle $\frac{3\pi}{4}$

$$\text{For } k = 2 \Rightarrow Z_2 = e^{j\left(\frac{4\pi + \pi}{4}\right)} = e^{j\left(\frac{5\pi}{4}\right)}$$

That means magnitude 1 and angle $\frac{5\pi}{4}$.

$$\text{For } k = 3 \Rightarrow Z_3 = e^{j\left(\frac{6\pi + \pi}{4}\right)} = e^{j\left(\frac{7\pi}{4}\right)}$$

That means magnitude 1 and angle $\frac{7\pi}{4}$.

Table P. 8.7.4 shows the magnitude and angles of zeros.

Table P. 8.7.4

Zeros	Magnitude	Angle
Z_0	1	$\frac{\pi}{4}$
Z_1	1	$\frac{3\pi}{4}$
Z_2	1	$\frac{5\pi}{4}$
Z_3	1	$\frac{7\pi}{4}$

Note : We can also plot zeros using $e^{j\theta} = \cos\theta + j \sin\theta$.

For example $Z_0 = e^{j\frac{\pi}{4}} = \cos\frac{\pi}{4} + j \sin\frac{\pi}{4}$

$= 0.707 + j\,0.707$.

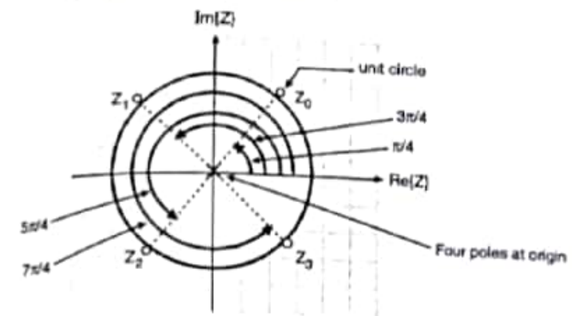
Thus Z_0 is marked at point 0.707 and j 0.707.

Poles : Poles are obtained by equating denominator term to zero.

$$\therefore \quad Z^4 = 0$$

Thus there are four poles at origin.

The pole zero plot is as shown in Fig. P. 8.7.4.



(H-1031) **Fig. P. 8.7.4**

8.8 System Function H (Z) :

The linear convolution of two sequences x(n) and h(n) can be expressed as,

$$y(n) = x(n) * h(n) \qquad \ldots(8.8.1)$$

If we take Z-transform of Equation (8.8.1) then the convolution becomes multiplication. Thus,

$$Y(Z) = X(Z) \cdot H(Z) \qquad \ldots(8.8.2)$$

$$\therefore \quad H(Z) = \frac{Y(Z)}{X(Z)} \qquad \ldots(8.8.3)$$

Here H(Z) is known as system transfer function. Note that H(Z) is the Z-transform of h(n); where h(n) is an impulse response of the system. Thus the system is characterized by H(Z) in the Z domain. H(Z) can be obtained from its difference equation. The LSI system has a rational system function.

8.9 Solved Examples on Transfer Function H (Z) :

Ex. 8.9.1 : Find the difference equation satisfying the input and output of an LTI system with transfer function :

$$H(Z) = \frac{(1 + Z^{-1})^2}{\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 + \frac{3}{4}Z^{-1}\right)}$$

Soln. :

The different notations used for representation of LTI systems as follows :

$x(n)$ = Input to the system and $Z\{x(n)\} = X(Z)$

$h(n)$ = Impulse response of system

and $Z\{h(n)\} = H(Z)$

and $y(n)$ = Output of the system and $Z\{y(n)\} = Y(Z)$.

The given equation is.

$$H(Z) = \frac{(1 + Z^{-1})^2}{\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 + \frac{3}{4}Z^{-1}\right)} \qquad \ldots(1)$$

$$\text{But } H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = \frac{(1 + Z^{-1})^2}{\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 + \frac{3}{4}Z^{-1}\right)}$$

$$\therefore Y(Z)\left[\left(1 - \frac{1}{2}Z^{-1}\right)\left(1 + \frac{3}{4}Z^{-1}\right)\right] = X(Z)(1 + Z^{-1})^2$$

$$\therefore Y(Z)\left[1 + \frac{3}{4}Z^{-1} - \frac{1}{2}Z^{-1} - \frac{3}{8}Z^{-2}\right] = X(Z)\left[1 + 2Z^{-1} + Z^{-2}\right]$$

$$\therefore Y(Z) + \frac{3}{4}Z^{-1}Y(Z) - \frac{1}{2}Z^{-1}Y(Z) - \frac{3}{8}(Z)^{-2}Y(Z) = X(Z) + 2Z^{-1}X(Z) + Z^{-2}X(Z) \qquad \ldots(2)$$

Taking inverse Z-transform of Equation (2) we get,

$$y(n) + \frac{3}{4}y(n-1) - \frac{1}{2}y(n-1) - \frac{3}{8}y(n-2) = x(n) + 2x(n-1) + x(n-2)$$

$$\therefore y(n) - \frac{1}{4}y(n-1) - \frac{3}{8}y(n-2) = x(n) + 2x(n-1) + x(n-2)$$

$$y(n) = \frac{1}{4}y(n-1) + \frac{3}{8}y(n-2) + x(n) + 2x(n-1) + x(n-2)$$

This is the difference equation for the given transfer function.

Ex. 8.9.2 : A system is described by the following difference equation. Find out its transfer function H(Z).

$y[n] = 34\, y[n-1] - 18\, y[n-2] + x[n] + 12\, x[n-1]$.

Dec. 2014, 10 Marks

Soln. :

Given :

Given : $y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{2} x(n-1)$

Take Z transform of both sides.

$$Y(Z) = \frac{3}{4} Z^{-1} y(Z) - \frac{1}{8} Z^{-2} y(Z) + x(Z) + \frac{1}{2} Z^{-1} x(Z)$$

$$\therefore\ y(Z) - \frac{3}{4} Z^{-1} y(Z) + \frac{1}{8} Z^{-2} y(Z) = x(Z) + \frac{1}{2} Z^{-1} x(Z)$$

$$\therefore\ y(Z)\left[1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right] = x(Z)\left[1 + \frac{1}{2} Z^{-1}\right]$$

The transfer function is given by,

$$H(Z) = \frac{y(Z)}{x(Z)}$$

$$\therefore\ H(Z) = \frac{1 + \frac{1}{2} Z^{-1}}{1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}}$$

Ex. 8.9.3 : Consider an LTI system which is stable and for which H (Z), the Z transform of impulse response is given by,

$$H(Z) = \frac{3 - 7Z^{-1} + 5Z^{-2}}{1 - 2.5Z^{-1} + Z^{-2}}$$

Suppose x(n), the input to the system is unit step. Determine the output.

Soln. :

Given : $$H(Z) = \frac{3 - 7Z^{-1} + 5Z^{-2}}{1 - 2.5Z^{-1} + Z^{-2}} \quad \ldots(1)$$

Given input is $x(n) = u(n)$; unit step. We have,

$$Z\{x(n)\} = X(Z) = Z\{u(n)\} = \frac{Z}{Z-1} \quad \ldots(2)$$

Now, $$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore\ Y(Z) = H(Z) \cdot X(Z) \quad \ldots(3)$$

Putting Equations (1) and (2) in Equation (3) we get,

$$Y(Z) = \left[\frac{3 - 7Z^{-1} + 5Z^{-2}}{1 - 2.5Z^{-1} + Z^{-2}}\right] \times \left[\frac{Z}{Z-1}\right] \quad \ldots(4)$$

We will convert first bracket that means H (Z) in terms of positive powers of Z. So multiplying numerator and denominator of H (Z) by Z^2 we get,

$$Y(Z) = \frac{(3Z^2 - 7Z + 5)}{(Z^2 - 2.5Z + 1)} \times \frac{Z}{(Z-1)} \quad \ldots(5)$$

We have to calculate output y (n). So we will take IZT of Y (Z) using partial fraction expansion. From Equation (5) we get,

$$\frac{Y(Z)}{Z} = \frac{3Z^2 - 7Z + 5}{(Z^2 - 2.5Z + 1)(Z-1)} \quad \ldots(6)$$

Since M < N; the equation is in proper form. To obtain the poles we will find out roots of denominator. Consider the equation $Z^2 - 2.5Z + 1$.

Here $a = 1$, $b = -2.5$ and $c = 1$.

Now $$P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore\ P_1, P_2 = \frac{2.5 \pm \sqrt{6.25 - 4}}{2} = \frac{2.5 \pm \sqrt{2.25}}{2}$$

$$\therefore\ P_1 = 2 \text{ and } P_2 = \frac{1}{2}$$

Now the second bracket in denominator is (Z – 1). So we can directly write, $P_3 = 1$.

Putting these values in Equation (6) we get,

$$\frac{Y(Z)}{Z} = \frac{3Z^2 - 7Z + 5}{(Z-2)\left(Z - \frac{1}{2}\right)(Z-1)}$$

In the partial fraction expansion form we can write,

$$\frac{Y(Z)}{Z} = \frac{A_1}{(Z-2)} + \frac{A_2}{\left(Z - \frac{1}{2}\right)} + \frac{A_3}{(Z-1)} \quad \ldots(7)$$

We will calculate the values of A_1, A_2 and A_3.

$$A_1 = (Z - P_1)\frac{Y(Z)}{Z}\bigg|_{Z=P_1}$$

$$= (Z-2)\left[\frac{3Z^2 - 7Z + 5}{(Z-2)\left(Z - \frac{1}{2}\right)(Z-1)}\right]\bigg|_{Z=2}$$

$$\therefore\ A_1 = \frac{3(2)^2 - 7(2) + 5}{\left(2 - \frac{1}{2}\right)(2-1)} \Rightarrow A_1 = 2.$$

$$A_2 = (Z - P_2)\frac{Y(Z)}{Z}\bigg|_{Z=P_2}$$

$$= \left(Z - \frac{1}{2}\right)\left[\frac{3Z^2 - 7Z + 5}{(Z-2)\left(Z - \frac{1}{2}\right)(Z-1)}\right]\bigg|_{Z=1/2}$$

$$\therefore\ A_2 = \frac{3\left(\frac{1}{2}\right)^2 - 7\left(\frac{1}{2}\right) + 5}{\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} - 1\right)} \Rightarrow A_2 = 3$$

and $$A_3 = (Z - P_3)\frac{Y(Z)}{Z}\bigg|_{Z=P_3}$$

$$= (Z-1)\left[\frac{3Z^2 - 7Z + 5}{(Z-2)\left(Z - \frac{1}{2}\right)(Z-1)}\right]\bigg|_{Z=1}$$

$$\therefore\ A_3 = \frac{3(1) - 7(1) + 5}{(1-2)\left(1 - \frac{1}{2}\right)} \Rightarrow A_3 = -2.$$

Putting these values in Equation (7) we get,

$$\frac{Y(Z)}{Z} = \frac{2}{Z-2} + \frac{3}{Z - \frac{1}{2}} - \frac{2}{Z-1}$$

$$\therefore\ Y(Z) = \frac{2Z}{Z-2} + \frac{3Z}{Z - \frac{1}{2}} - \frac{2Z}{Z-1} \quad \ldots(8)$$

In the question it is mentioned that the system is stable. So its ROC must include unit circle. The poles are at $\frac{1}{2}$, 1 and 2. Thus to include the unit circle; for these poles the ROC is,

ROC

$$\frac{1}{2} < |Z| < 2 \quad \ldots(9)$$

Then the first term of Equation (8) becomes anticausal. Because for this term to be causal, required ROC is $|Z| > 2$.

$$\therefore\ IZT\left\{\frac{2Z}{Z-2}\right\} = 2\, IZT\left\{\frac{Z}{Z-2}\right\}$$

$$= -2(2)^n u(-n-1)$$

The second term is causal. Because the required ROC for this term is $|Z| > \frac{1}{2}$; so Equation (9) is satisfied.

$$\therefore\ IZT\left\{\frac{3Z}{Z - \frac{1}{2}}\right\} = 3\, IZT\left\{\frac{Z}{Z - \frac{1}{2}}\right\} = 3\left(\frac{1}{2}\right)^n u(n)$$

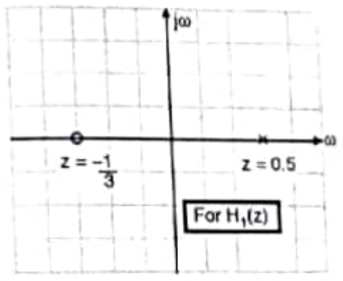
The third term is also causal. For this term to be causal the required ROC is $|Z| > 1$. This satisfies Equation (9)

$$\therefore\ IZT\left\{\frac{2Z}{Z-1}\right\} = 2\, IZT\left\{\frac{Z}{Z-1}\right\} = 2(1)^n u(n)$$

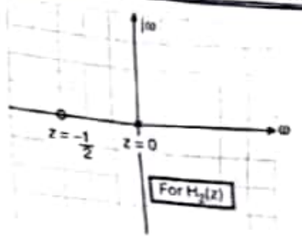
Thus equation of output can be written as,

$$y(n) = -2(2)^n u(-n-1) + 3\left(\frac{1}{2}\right)^n u(n) + 2(1)^n u(n)$$

Ex. 8.9.4 : A causal DT system has transfer function H (Z) such that $H(Z) = H_1(Z) \cdot H_2(Z)$. The pole-zero diagram is as follows :



(II-1530) **Fig. P. 8.9.4(a)**



(II-1530) **Fig. P. 8.9.4(b)**

1. Find transfer function of total system.
2. Find difference equation of system.
3. Find response of system to input $x(n) = \left(-\frac{1}{2}\right)^n u(n)$

Soln. :

1. $H_1(Z)$ has pole at Z = 0.5 and zero at $Z = -\frac{1}{3}$.

We have,

$$H_1(Z) = \frac{Z - Z_1}{Z - P_1}$$

$$\therefore\ H_1(Z) = \frac{Z + 1/3}{Z - 0.5} \quad \ldots(1)$$

$H_2(Z)$ has zero at Z = – 1 / 2 and pole at Z = 0.

$$\therefore\ H_2(Z) = \frac{Z - Z_1}{Z - P_1}$$

$$\therefore\ H_2(Z) = \frac{Z + 1/2}{Z} \quad \ldots(2)$$

Given $H(Z) = H_1(Z) \cdot H_2(Z)$

$$\therefore\ H(Z) = \frac{Z + \frac{1}{3}}{Z - 0.5} \cdot \frac{Z + \frac{1}{2}}{Z} \quad \ldots(3)$$

$$\therefore\ H(Z) = \frac{Z^2 + \frac{1}{2}Z + \frac{1}{3}Z + \frac{1}{6}}{Z(Z - 0.5)}$$

$$\therefore\ H(Z) = \frac{Z^2 + \frac{5}{6}Z + \frac{1}{6}}{Z\left(Z - \frac{1}{2}\right)}$$

This equation gives transfer function of total system.

2. We have

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore\ \frac{Y(Z)}{X(Z)} = \frac{Z^2 + \frac{5}{6}Z + \frac{1}{6}}{Z\left(Z - \frac{1}{2}\right)} \quad \ldots(4)$$

To convert R.H.S. into negative powers of Z, we will multiply numerator and denominator by Z^{-2}.

$$\therefore\ \frac{Y(Z)}{X(Z)} = \frac{1 + \frac{5}{6}Z^{-1} + \frac{1}{6}Z^{-2}}{Z^{-1}\left(Z - \frac{1}{2}\right)} = \frac{1 + \frac{5}{6}Z^{-1} + \frac{1}{6}Z^{-2}}{1 - \frac{1}{2}Z^{-1}}$$

$$\therefore Y(Z)\left[1-\frac{1}{2}Z^{-1}\right] = X(Z)\left[1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}\right]$$

$$\therefore Y(Z)-\frac{1}{2}Z^{-1}Y(Z) = X(Z)+\frac{5}{6}Z^{-1}X(Z)+\frac{1}{6}Z^{-2}X(Z)$$

Taking IZT of both sides we get,

$$y(n)-\frac{1}{2}y(n-1) = x(n)+\frac{5}{6}x(n-1)+\frac{1}{6}x(n-2) \quad ...(5)$$

This is the difference equation of a system.

3. Given input is,

$$x(n) = \left(-\frac{1}{2}\right)^n u(n)$$

We have standard Z-transform pair.

$$a^n u(n) \xleftrightarrow{Z} \frac{Z}{Z-a}$$

Here $a = -\frac{1}{2}$ $\quad \therefore X(Z) = \frac{Z}{Z+\frac{1}{2}}$ ...(6)

Now $H(Z) = \frac{Y(Z)}{X(Z)}$

$$\therefore Y(Z) = H(Z)\cdot X(Z) \quad ...(7)$$

Putting Equations (3) and (6) in Equation (7)

$$Y(Z) = \left[\frac{Z+\frac{1}{3}}{Z-0.5}\cdot\frac{Z+\frac{1}{2}}{Z}\right]\cdot\frac{Z}{Z+\frac{1}{2}}$$

$$\therefore Y(Z) = \frac{Z+\frac{1}{3}}{Z-0.5} \quad \therefore Y(Z) = \frac{Z+\frac{1}{3}}{Z-\frac{1}{2}}$$

We will obtain IZT using P.F.E.

$$\frac{Y(Z)}{Z} = \frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)} \quad ...(8)$$

In PFE form Equation (8) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z}+\frac{A_2}{Z-\frac{1}{2}} \quad ...(9)$$

$$A_1 = (Z-0)\cdot\frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)}\Bigg|_{Z=0} = \frac{1/3}{-1/2} = -\frac{2}{3}$$

$$\text{and } A_2 = \left(Z-\frac{1}{2}\right)\cdot\frac{Z+\frac{1}{3}}{Z\left(Z-\frac{1}{2}\right)}\Bigg|_{Z=\frac{1}{2}}$$

$$\therefore A_2 = \frac{\frac{1}{2}+\frac{1}{3}}{\frac{1}{2}} = \frac{5}{6}\times 2 = \frac{5}{3}$$

Thus Equation (9) becomes,

$$\frac{Y(Z)}{Z} = \frac{-2/3}{Z}+\frac{5/3}{Z-\frac{1}{2}}$$

$$\therefore Y(Z) = -\frac{2}{3}+\frac{5}{3}\cdot\frac{Z}{Z-\frac{1}{2}} \quad ...(10)$$

IZT of Equation (10) can be written as,

$$y(n) = -\frac{2}{3}\delta(n)+\frac{5}{3}\left(\frac{1}{2}\right)^n u(n)$$

This is the response of system.

Ex. 8.9.5 : Find the impulse response for a causal system $y[n] - y[n-1] = x[n] + x[n-1]$. Determine the response of the system for inputs.

1. $x[n] = u[n]$ and 2. $x[n] = 2^{-n}u[n]$.

Soln. :

Given : $y(n) - y(n-1) = x(n) + x(n-1)$

Taking Z transform of both sides

$$Y(Z) - Z^{-1}Y(Z) = X(Z) + Z^{-1}X(Z)$$

$$\therefore Y(Z)[1-Z^{-1}] = X(Z)[1+Z^{-1}]$$

Now $$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1+Z^{-1}}{1-Z^{-1}} \quad ...(1)$$

$$\therefore H(Z) = \frac{Z+1}{Z-1} = \frac{Z}{Z-1}+\frac{1}{Z-1}$$

$$= \frac{Z}{Z-1}-\frac{1}{1-Z}$$

We have standard Z transform pair,

$$\alpha^n u(n) \longleftrightarrow \frac{Z}{Z-\alpha} \text{ and } u(-n) \longleftrightarrow \frac{1}{1-Z}$$

$$\therefore h(n) = u(n) - u(-n)$$

1. Here $x(n) = u(n)$

$$\therefore X(Z) = \frac{Z}{Z-1}$$

Now $Y(Z) = H(Z)\cdot X(Z)$

$$\therefore Y(Z) = \frac{Z+1}{Z-1}\cdot\frac{Z}{Z-1}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{Z+1}{(Z-1)2}$$

Using P.F.E we can write

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z-1}+\frac{A_2}{(Z-1)^2}$$

$$A_1 = \frac{d}{dZ}\left[(Z-1)^2\cdot\frac{Z+1}{(Z-1)^2}\right]\Bigg|_{Z=1}$$

$$\therefore A_1 = 1$$

and $$A_1 = \left[(Z-1)^2\cdot\frac{Z+1}{(Z-1)^2}\right]\Bigg|_{Z=1}$$

$$\therefore A_2 = 2$$

$$\therefore \frac{Y(Z)}{Z} = \frac{1}{Z-1}+\frac{2}{(Z-1)^2}$$

$$\therefore Y(Z) = \frac{Z}{Z-1}+2\frac{Z}{(Z-1)^2}$$

Taking IZT we get,

$$y(n) = u(n) + 2nu(n)$$

2. Given

$$x(n) = 2^{-n}u(n) = \left(\frac{1}{2}\right)^n u(n)$$

$$\therefore X(Z) = \frac{Z}{Z-\frac{1}{2}}$$

Now $Y(Z) = H(Z)\cdot X(Z)$

$$\therefore Y(Z) = \frac{Z+1}{Z-1}\cdot\frac{Z}{Z-\frac{1}{2}}$$

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z-1}+\frac{A_2}{Z-\frac{1}{2}}$$

Here $$A_1 = (Z-1)\cdot\frac{Z+1}{(Z-1)\left(Z-\frac{1}{2}\right)}\Bigg|_{Z=1}$$

$$\therefore A_1 = \frac{2}{1-\frac{1}{2}} = \frac{2}{\frac{1}{2}} \quad \therefore A_1 = 4$$

and $$A_2 = \left(Z-\frac{1}{2}\right)\cdot\frac{Z+1}{(Z-1)\left(Z-\frac{1}{2}\right)}\Bigg|_{Z=\frac{1}{2}}$$

$$= \frac{\frac{1}{2}+1}{\frac{1}{2}-1} = \frac{\frac{3}{2}}{-\frac{1}{2}}$$

$$\therefore A_2 = -3$$

Ex. 8.9.6 : Determine H(Z) and draw a pole zero plot for a system $y(n) + \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$

Soln. :

Given difference equation is,

$$y(n)+\frac{3}{8}y(n-1)+\frac{1}{8}y(n-2) = x(n)+x(n-1)$$

Taking Z transform of both sides we get,

$$Y(Z)+\frac{3}{8}Z^{-1}Y(Z)+\frac{1}{8}Z^{-2}Y(Z) = X(Z)+Z^{-1}X(Z)$$

$$\therefore Y(Z)\left[1+\frac{3}{8}Z^{-1}+\frac{1}{8}Z^{-2}\right] = X(Z)[1+Z^{-1}]$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1+Z^{-1}}{1+\frac{3}{8}Z^{-1}+\frac{1}{8}Z^{-2}}$$

Converting into positive powers of Z we get,

$$H(Z) = \frac{Z^2+Z}{Z^2+\frac{3}{8}Z+\frac{1}{8}}$$

$$\therefore H(Z) = \frac{Z(Z+1)}{(Z+0.1875-0.3j)(Z+0.1875+0.3j)}$$

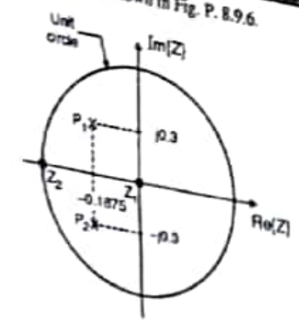
Thus zeros are at,

$$Z_1 = 0, \quad Z_2 = 1$$

And poles are at,

$$P_1 = -0.1875+j0.3, \; P_2 = -0.1875-j0.3$$

The pole zero plot is shown in Fig. P. 8.9.6.



(M-15211) Fig. P. 8.9.6

Ex. 8.9.7 : Determine impulse response of a system $y(n) = 2x(n) - x(n-1) - 3y(n-1) - 2y(n-2)$

Soln. :

Given difference equation is,

$$y(n) = 2x(n) - x(n-1) - 3y(n-1) - 2y(n-2)$$

Taking Z transform of both sides we get,

$$Y(Z) = 2X(Z) - Z^{-1}X(Z) - 3Z^{-1}Y(Z) - 2Z^{-2}Y(Z)$$

$$\therefore Y(Z)+3Z^{-1}Y(Z)+2Z^{-2}Y(Z) = 2X(Z) - Z^{-1}X(Z)$$

$$\therefore Y(Z)[1+3Z^{-1}+2Z^{-2}] = X(Z)[2-Z^{-1}]$$

Now $$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{2-Z^{-1}}{1+3Z^{-1}+2Z^{-2}}$$

We will obtain impulse response h(n) by performing partial fraction expansion as follows :

$$H(Z) = \frac{Z^2-Z}{Z^2+3Z+2}$$

$$\therefore \frac{H(Z)}{Z} = \frac{Z^{-1}}{(Z+2)(Z+1)}$$

In P.F.E. form we can write,

$$\frac{H[Z]}{Z} = \frac{A_1}{Z+2}+\frac{A}{Z+1}$$

Here $$A_1 = (Z+2)\cdot\frac{Z-1}{(Z+2)(Z+1)}\Bigg|_{Z=-2}$$

$$\therefore A_1 = \frac{-3}{-2+1} \quad \therefore A_1 = 3$$

And $$A_2 = (Z+1)\cdot\frac{Z-1}{(Z+2)(Z+1)}\Bigg|_{Z=-1}$$

$$\therefore A_2 = \frac{-1-1}{-1+2}$$

$$\therefore A_2 = -2$$

$$\therefore \frac{H(Z)}{Z} = \frac{3}{Z+2}-\frac{2}{Z+1}$$

$$\therefore H(Z) = 3\frac{Z}{Z+2}-2\frac{Z}{Z+1}$$

Taking IZT we get,

$$h(n) = 3(-2)^n u(n) - 2(-1)^n u(n)$$

Ex. 8.9.8 : Determine an impulse response of a system described as,

$$y(n) = x(n) + 0.6\,y(n-1) - 0.08\,y(n-2)$$

Soln. : Given difference equation is,

$$y(n) = x(n) + 0.6y(n-1) - 0.08y(n-2)$$

Taking Z transform of both sides we get,

$$Y(Z) = X(Z) + 0.6Z^{-1}Y(Z) - 0.08Z^{-2}Y(Z)$$

$$\therefore Y(Z) - 0.6Z^{-1}Y(Z) + 0.08Z^{-2}Y(Z) = X(Z)$$

$$\therefore Y(Z)[1 - 0.6Z^{-1} + 0.08Z^{-2}] = X(Z)$$

Now, $$H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1}{1 - 0.6Z^{-1} + 0.08Z^{-2}}$$

We will obtain h(n) by using PFE as follows :

$$H(Z) = \frac{Z^2}{Z^2 - 0.6Z + 0.08}$$

$$\therefore \frac{H(Z)}{Z} = \frac{Z}{(Z-0.4)(Z-0.2)}$$

$$\therefore \frac{H(Z)}{Z} = \frac{A_1}{Z-0.4} + \frac{A_2}{Z-0.2}$$

Here, $$A_1 = (Z-0.4)\cdot\frac{Z}{(Z-0.4)(Z-0.2)}\Bigg|_{Z=0.4}$$

$$\therefore A_1 = \frac{0.4}{0.4-0.2}$$

$$\therefore A_1 = 2$$

And $$A_2 = (Z-0.2)\cdot\frac{Z}{(Z-0.4)(Z-0.2)}\Bigg|_{Z=0.2}$$

$$\therefore A_2 = \frac{0.2}{0.2-0.4} \quad \therefore A_2 = -1$$

$$\therefore \frac{H(Z)}{Z} = \frac{2}{Z-0.4} - \frac{1}{Z-0.2}$$

$$\therefore H(Z) = 2\frac{Z}{Z-0.4} - \frac{Z}{Z-0.2}$$

Taking IZT we get,

$$h(n) = 2(0.4)^n u(n) - (0.2)^n u(n)$$

Ex. 8.9.9 : Obtain system function H(Z) for

$$y(n) + \frac{1}{2}y(n-1) = x(n) - x(n-1)$$

Determine the poles and zeros and draw a pole zero plot.

Soln. :

Given difference equation is,

$$y(n) + \frac{1}{2}y(n-1) = x(n) - x(n-1)$$

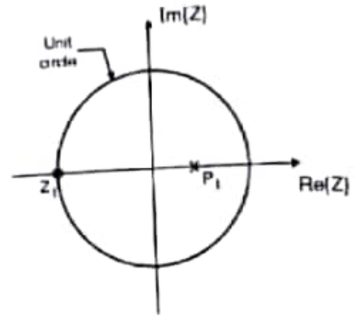
Taking Z transform of both sides.

$$Y(Z) + \frac{1}{2}Z^{-1}Y(Z) = X(Z) - Z^{-1}X(Z)$$

$$\therefore Y(Z)\left[1 + \frac{1}{2}Z^{-1}\right] = X(Z)[1 - Z^{-1}]$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1 - Z^{-1}}{1 + \frac{1}{2}Z^{-1}}$$

$$\therefore H(Z) = \frac{Z-1}{Z+\frac{1}{2}}$$



(H-1532) **Fig. P. 8.9.9**

Thus zero is at $Z_1 = 1$ and pole is at $P_1 = -\frac{1}{2}$.

The pole zero plot is shown in Fig. P. 8.9.9

8.10 Analysis of Causality and Stability of DT System :

We know that the condition for causality in terms of h(n) is,

$$h(n) = 0 \quad n < 0 \qquad \ldots(8.10.1)$$

Here the condition $n < 0$ indicates that value of $h(n)$ for negative 'n' is zero. That h(n) is causal sequence. In Z-transforms we know that if a sequence is causal then the ROC is exterior part of circle having radius 'r'. Now the system transfer function is,

$$H(Z) = Z\{h(n)\} \qquad \ldots(8.10.2)$$

Thus the condition for causality is that the ROC of system transfer function should be exterior part of circle having radius 'r' and $r < \infty$.

We have already studied the condition for stability. It is,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \qquad \ldots(8.10.3)$$

Now according to the definition of Z-transform; $Z\{h(n)\}$ can be written as,

$$Z\{h(n)\} = H(Z) = \sum_{n=-\infty}^{\infty} h(n)Z^{-n} \qquad \ldots(8.10.4)$$

Taking absolute value on both the sides,

$$|H(Z)| = \left|\sum_{n=-\infty}^{\infty} h(n)Z^{-n}\right| \qquad \ldots(8.10.5)$$

Now the absolute value on a sum is always less than the sum of absolute value.

$$\therefore |H(Z)| < \sum_{n=-\infty}^{\infty} |h(n)||Z^{-n}| \qquad \ldots(8.10.6)$$

If we evaluate the Z-transform on the unit circle that means $|Z^{-n}| = |Z| = 1$ then Equation (8.10.6) becomes,

$$|H(Z)| < \sum_{n=-\infty}^{\infty} |h(n)| \qquad \ldots(8.10.7)$$

According to Equation (8.10.3), if the system is BIBO stable then $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$. Thus the condition of stability is Z-domain is.

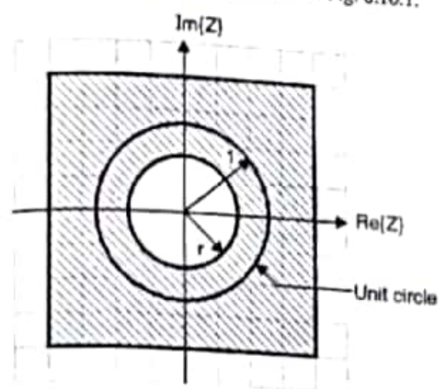
$$|H(Z)| < \infty \text{ when evaluated on unit circle.} \qquad \ldots(8.10.8)$$

That means the system transfer function should be finite if evaluated on unit circle.

Equation (8.10.6) gives the condition of stability in Z domain. ***This condition requires that, unit circle must be present in the ROC of H(Z).*** Otherwise we cannot find H(Z) on unit circle at all.

Combined Condition

1. For a causal LSI system we know that ROC is exterior part of circle; that mean $|Z| > |r|$
2. For the stable LSI system, ROC of H(Z) must include unit circle. Here unit circle is a circle having radius 1. Now consider we have a causal system having ROC as exterior part of circle with radius 'r'. Then for this system to be stable; it should include unit circle as shown in Fig. 8.10.1.



(H-1050) **Fig. 8.10.1 : ROC including unit circle**

From Fig. 8.10.1 we can say that,

$$r < 1$$

Thus the combined condition for causal and stable LSI system is,

$$|Z| > r < 1 \qquad \ldots(8.10.9)$$

We know that :

1. ROC of H(Z) does not contain any poles.
2. ROC of stable and causal LSI system includes the unit circle.

Thus we can state that ***all the poles of H(Z) of a causal and stable system are inside the unit circle.***

8.11 Solved Examples on Stability and Causality :

Ex. 8.11.1 : Consider the system

$$H(Z) = \frac{1 - 2Z^{-1} + 2Z^{-2} - Z^{-3}}{(1 - Z^{-1})(1 - 0.5Z^{-1})(1 - 0.2Z^{-1})}$$

with ROC : $0.5 < |Z| < 1$.

1. Sketch the pole-zero pattern. Is the system stable ?
2. Determine the impulse response of system.

Soln. :

1. **Pole-zero pattern :** The given function is,

$$H(Z) = \frac{1 - 2Z^{-1} + 2Z^{-2} - Z^{-3}}{(1 - Z^{-1})(1 - 0.5Z^{-1})(1 - 0.2Z^{-1})} \qquad \ldots(1)$$

To convert it into positive powers of Z we will multiply numerator and denominator by Z^3.

$$\therefore H(Z) = \frac{Z^3(1 - 2Z^{-1} + 2Z^{-2} - Z^{-3})}{Z(1 - Z^{-1})Z(1 - 0.5Z^{-1})Z(1 - 0.2Z^{-1})}$$

$$\therefore H(Z) = \frac{Z^3 - 2Z^2 + 2Z - 1}{(Z-1)(Z-0.5)(Z-0.2)} \qquad \ldots(2)$$

To obtain zeros, consider the numerator polynomial. We will obtain the roots of $Z^3 - 2Z^2 + 2Z - 1$. Let us perform synthetic division.

$1 - 2 + 2 - 1 = 0$. Thus $(Z - 1)$ exists. Now dividing $Z^3 - 2Z^2 + 2Z - 1$ by $(Z - 1)$ we get,

$$Z^3 - 2Z^2 + 2Z - 1 = (Z-1)(Z^2 - Z + 1) \qquad \ldots(3)$$

Now we will obtain roots of $Z^2 - Z + 1$

$$Z^2 - Z + 1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm j\frac{\sqrt{3}}{2}$$

$$\therefore Z^2 - Z + 1 = \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

Thus Equation (1) becomes,

$$Z^3 - 2Z^2 + 2Z - 1 = (Z-1)\left(Z - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(Z - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \qquad \ldots(4)$$

Putting Equation (4) in Equation (2) we get,

$$H(Z) = \frac{(Z-1)\left(Z - \frac{1}{2} - j\frac{\sqrt{3}}{2}\right)\left(Z - \frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}{(Z-1)(Z-0.5)(Z-0.2)}$$

$$\therefore H(Z) = \frac{\left(Z - \frac{1}{2} - \frac{j\sqrt{3}}{2}\right)\left(Z - \frac{1}{2} + \frac{j\sqrt{3}}{2}\right)}{(Z-0.5)(Z-0.2)} \qquad \ldots(5)$$

We have the standard equation of $H(Z)$ in terms of poles and zeros,

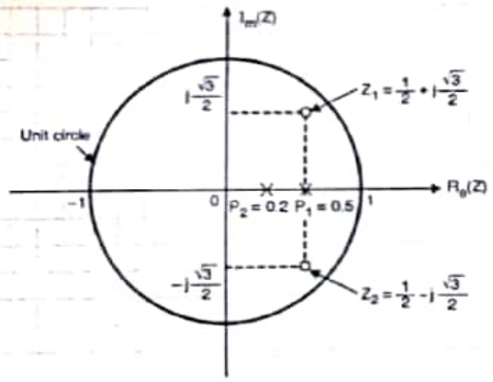
$$H(Z) = \frac{(Z - Z_1)(Z - Z_2)}{(Z - P_1)(Z - P_2)} \qquad \ldots(6)$$

Comparing Equations (5) and (6) we can write,

$$\text{Zeros} \Rightarrow Z_1 = \frac{1}{2} + j\frac{\sqrt{3}}{2} \text{ and } Z_2 = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

$$\text{and Poles} \Rightarrow P_1 = 0.5 \text{ and } P_2 = 0.2.$$

The pole-zero plot is shown in Fig. P. 8.11.1.



(B-1051) **Fig. P. 8.11.1 : Pole-zero plot**

2. Impulse response :

Impulse response is denoted by h (n). It is obtained by taking IZT of H (Z).

We have, $$H(Z) = \frac{\left(Z-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(Z-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}{(Z-0.5)(Z-0.2)}$$

$$\therefore H(Z) = \frac{Z^2-Z+1}{Z^2-0.7Z+0.1} \quad ...(7)$$

Here M = N; we want M < N. Thus multiplying and dividing by Z.

$$H(Z) = \frac{Z(Z^2-Z+1)}{Z(Z^2-0.7Z+0.1)}$$

$$\therefore \frac{H(Z)}{Z} = \frac{Z^2-Z+1}{Z(Z^2-0.7Z+0.1)} \quad ...(8)$$

Now the degree of numerator (M) is less than the degree of denominator (N). So the function is in proper form.

$$\therefore \frac{H(Z)}{Z} = \frac{Z^2-Z+1}{Z(Z-0.5)(Z-0.2)} \quad ...(9)$$

Here poles are $P_1 = 0$, $P_2 = 0.5$ and $P_3 = 0.2$.

We will write Equation (9) in partial fraction expansion form.

$$\therefore \frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-0.5} + \frac{A_3}{Z-0.2} \quad ...(10)$$

Now $$A_1 = (Z-P_1)\frac{H(Z)}{Z}\bigg|_{Z=P_1}$$

$$= \frac{Z(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\bigg|_{Z=0}$$

$$\therefore A_1 = \frac{0+0+1}{(0-0.5)(0-0.2)} = 10$$

$$A_2 = (Z-P_2)\frac{H(Z)}{Z}\bigg|_{Z=P_2}$$

$$= \frac{(Z-0.5)(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\bigg|_{Z=0.5}$$

$$\therefore A_2 = \frac{Z^2-Z+1}{Z(Z-0.2)}\bigg|_{Z=0.5} = \frac{(0.5)^2-0.5+1}{0.5(0.5-0.2)} = 5$$

and $$A_3 = (Z-P_3)\frac{H(Z)}{Z}\bigg|_{Z=P_3}$$

$$= \frac{(Z-0.2)(Z^2-Z+1)}{Z(Z-0.5)(Z-0.2)}\bigg|_{Z=0.2}$$

$$\therefore A_3 = \frac{Z^2-Z+1}{Z(Z-0.5)}\bigg|_{Z=0.2}$$

$$= \frac{(0.2)^2-0.2+1}{0.2(0.2-0.5)} = -14$$

Putting these values in Equation (10) we get,

$$\frac{H(Z)}{Z} = \frac{10}{Z} + \frac{5}{Z-0.5} + \frac{(-14)}{Z-0.2}$$

$$\therefore H(Z) = 10 + \frac{5Z}{Z-0.5} - \frac{14Z}{Z-0.2} \quad ...(11)$$

The given ROC is $0.5 < |Z| < 1$. So all the terms are causal.

$$\therefore IZT\{10\} = 10\,IZT\{1\} = 10\,\delta(n)$$

$$IZT\left\{\frac{5Z}{Z-0.5}\right\} = 5\,IZT\left\{\frac{Z}{Z-0.5}\right\} = 5(0.5)^n u(n)$$

and $$IZT\left\{\frac{-14Z}{Z-0.2}\right\} = -14\,IZT\left\{\frac{Z}{Z-0.2}\right\}$$

$$= -14(0.2)^n u(n).$$

$$\therefore h(n) = 10\,\delta(n) + 5(0.5)^n u(n) - 14(0.2)^n u(n)$$

Ex. 8.11.2 : Calculate Z transform of the following signals :

1. $x[n] = n\left(-\frac{1}{4}\right)^n u[n] * \left(-\frac{1}{6}\right)^{-n} u[-n]$
2. $x[n] = u[n-6] - u[n-10]$

Dec. 2014, 10 Marks

Soln. :

1. $x[n] = n\left(-\frac{1}{4}\right)^n u[n] * \left(-\frac{1}{6}\right)^{-n} u[-n]$:

Convolution is equivalent to multiplication in Z domain.

$$\therefore x(Z) = Z\left\{n\left(-\frac{1}{4}\right)^n u(n)\right\} \cdot Z\left\{\left(-\frac{1}{6}\right)^n u(-n)\right\}$$

Now we have, $n a^n u(n) \longleftrightarrow \frac{aZ}{(Z-a)^2}$, $|Z| > |a|$

$$\therefore Z\left\{n\left(-\frac{1}{4}\right)^n u(n)\right\} = \frac{\frac{1}{4}Z}{\left(Z-\frac{1}{4}\right)^2}, \quad |Z| > \frac{1}{4}$$

Consider second term.

We have, $Z\{u(n)\} = \frac{Z}{Z-1}$, $|Z| > 1$

Using time reversal property,

$$Z\{u(-n)\} = \frac{Z^{-1}}{Z^{-1}-1}, \quad |Z| < 1$$

$$= \frac{1}{1-Z}, \quad |Z| < 1$$

Using scaling property,

$$Z\left\{\left(-\frac{1}{6}\right)^n u(-n)\right\} = \frac{1}{1-\frac{Z}{\left(-\frac{1}{6}\right)}}, \quad \left|\frac{Z}{\left(-\frac{1}{6}\right)}\right| < 1$$

$$= \frac{6}{6+Z}, \quad |Z| < \frac{1}{6}$$

$$\therefore x(Z) = \frac{\frac{1}{4}Z}{(Z-1/4)^2} \cdot \frac{6}{(6+Z)}$$

2. $x[n] = u[n-6] - u[n-10]$:

Given : $x[n] = u[n-6] - u[n-10]$

We have, $Z\{u(n)\} = \frac{Z}{Z-1}$, $|Z| > 1$

Using shifting property,

$$Z\{u(n-6)\} = Z^{-6} \cdot \frac{Z}{Z-1} = \frac{Z^{-5}}{Z-1}$$

and $$Z\{u(n-10)\} = Z^{-10} \cdot \frac{Z}{Z-1} = \frac{Z^{-9}}{Z-1}$$

$$\therefore X(Z) = \frac{Z^{-5}}{Z-1} - \frac{Z^{-9}}{Z-1}$$

$$\therefore X(Z) = \frac{Z^{-5}-Z^{-9}}{Z-1}$$

Ex. 8.11.3 : A causal LTI system has transfer function,

$$H(Z) = \frac{(1-0.5Z^{-1})(1-Z^{-1})}{(1+0.2Z^{-1})(1+0.8Z^{-1})(1-0.8Z^{-1})}$$

1. Give ROC condition.
2. Show pole zero diagram of system.
3. Find impulse response of system.
4. Comment on system stability.

Soln. :

1. The given transfer function is,

$$H(Z) = \frac{(1-0.5Z^{-1})(1-Z^{-1})}{(1+0.2Z^{-1})(1+0.8Z^{-1})(1-0.8Z^{-1})} \quad ...(1)$$

To convert it into positive powers of 'Z' we will multiply numerator and denominator by Z^3.

$$\therefore H(Z) = \frac{Z \cdot Z(1-0.5Z^{-1}) \cdot Z(1-Z^{-1})}{Z(1+0.2Z^{-1}) \cdot Z(1+0.8Z^{-1}) \cdot Z(1-0.8Z^{-1})}$$

$$\therefore H(Z) = \frac{Z(Z-0.5) \cdot (Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)} \quad ...(2)$$

Here, the poles are at – 0.2, – 0.8 and + 0.8.

The given system is causal. So ROC should be greater than the poles having larger value.

$\therefore$ Required ROC is $|Z| > 0.8$.

2. Similar to Equation (2); we can write generalised expression in terms of poles and zeros as,

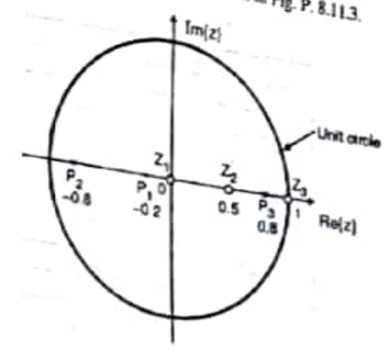
$$H(Z) = \frac{(Z-Z_1)(Z-Z_2)(Z-Z_3)}{(Z-P_1)(Z-P_2)(Z-P_3)} \quad ...(3)$$

Comparing Equations (2) and (3);

Zeros $\Rightarrow Z_1 = 0$, $Z_2 = 0.5$ and $Z_3 = 1$.

Poles $\Rightarrow P_1 = -0.2$, $P_2 = -0.8$ and $P_3 = 0.8$

The pole-zero diagram is as shown in Fig. P. 8.11.3.



(B-1052) **Fig. P. 8.11.3**

3. To calculate impulse response h (n), we have to obtain IZT of H (Z). From Equation (2) we get,

$$\frac{H(Z)}{Z} = \frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)} \quad ...(4)$$

In P.F.E. form Equation (4) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z+0.2} + \frac{A_2}{Z+0.8} + \frac{A_3}{Z-0.8} \quad ...(5)$$

The values of A_1, A_2 and A_3 are calculated as follows :

$$A_1 = (Z-P_1)\frac{H(Z)}{Z}\bigg|_{Z=P_1}$$

$$\therefore A_1 = (Z+0.2) \cdot \frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)}\bigg|_{Z=-0.2}$$

$$\therefore A_1 = \frac{(-0.2-0.5)(-0.2-1)}{(-0.2+0.8)(-0.2-0.8)} = \frac{(-0.7)(-1.2)}{(0.6)(-1)}$$

$$\therefore A_1 = -1.4$$

$$\therefore A_2 = (Z-P_2)\frac{H(Z)}{Z}\bigg|_{Z=P_2}$$

$$\therefore A_2 = (Z+0.8)\frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)}\bigg|_{Z=-0.8}$$

$$\therefore A_2 = \frac{(-0.8-0.5)(-0.8-1)}{(-0.8+0.2)(-0.8-0.8)} = \frac{(-1.3)(-1.8)}{(-0.6)(-1.6)}$$

$$\therefore A_2 = 2.44$$

$$A_3 = (Z-P_3)\frac{H(Z)}{Z}\bigg|_{Z=P_3}$$

$$\therefore A_3 = (Z-0.8) \cdot \frac{(Z-0.5)(Z-1)}{(Z+0.2)(Z+0.8)(Z-0.8)}\bigg|_{Z=0.8}$$

$$\therefore A_3 = \frac{(0.8-0.5)(0.8-1)}{(0.8+0.2)(0.8+0.8)} = \frac{(0.3)(-0.2)}{1(1.6)}$$

$$\therefore A_3 = -0.0375$$

Putting these values in Equation (5) we get,

$$\frac{H(Z)}{Z} = \frac{-1.4}{Z+0.2} + \frac{2.44}{Z+0.8} - \frac{0.0375}{Z-0.8}$$

$$\therefore H(Z) = \frac{-1.4Z}{Z+0.2} + \frac{2.44Z}{Z+0.8} - \frac{0.0375Z}{Z-0.8} \quad ...(6)$$

Since the given function is causal; its IZT can be written as,

$$h(n) = -1.4(-0.2)^n u(n) + 2.44(-0.8)^n u(n) - 0.0375(0.8)^n u(n)$$

4. Since all the poles lie inside the unit circle; the given system is stable.

Ex. 8.11.4 : A causal LTI system is descrited by $y[n] = \frac{3}{4} y[n-1] - \frac{1}{8} y[n-2] + x[n]$.

Where y [n] response of the system and x [n] is excitation to the system.

1. Determine impulse response of the system.
2. Determine step response of the system.
3. Plot pole-zero pattern and state whether system is stable.

May 2014, 10 Marks

Soln. :

1. Impulse response is output of system when applied input is $x(n) = \delta(n)$ and it is denoted by h (n). Thus replacing y (n) by h (n) and x (n) by $\delta(n)$ in given equation we get,

$$h(n) = \frac{3}{4} h(n-1) - \frac{1}{8} h(n-2) + \delta(n)$$

Taking Z transform of both sides,

$$H(Z) = \frac{3}{4} Z^{-1} H(Z) - \frac{1}{8} Z^{-2} H(Z) + 1$$

$$\therefore H(Z) = -\frac{3}{4} Z^{-1} H(Z) + \frac{1}{8} Z^{-2} H(Z) = 1$$

$$\therefore H(Z)\left(1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right) = 1$$

$$\therefore H(Z) = \frac{1}{1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}}$$

To obtain h (n), we will perform P.F.E. as follows :

$$H(Z) = \frac{Z^2}{Z^2 - \frac{3}{4} Z + \frac{1}{8}}$$

$$\therefore \frac{H(Z)}{Z} = \frac{Z}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}$$

$$\therefore \frac{H(Z)}{Z} = \frac{A_1}{Z - \frac{1}{2}} + \frac{A_2}{Z - \frac{1}{4}}$$

Here, $A_1 = \left(Z - \frac{1}{2}\right) \left.\frac{Z}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}\right|_{Z = \frac{1}{2}}$

$$\therefore A_1 = \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

and $A_2 = \left(Z - \frac{1}{4}\right) \left.\frac{Z}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}\right|_{Z = \frac{1}{4}}$

$$\therefore A_2 = \frac{\frac{1}{4}}{\frac{1}{4} - \frac{1}{2}} = \frac{\frac{1}{4}}{-\frac{1}{4}} = -1$$

$$\therefore \frac{H(Z)}{Z} = \frac{2}{Z - \frac{1}{2}} - \frac{1}{Z - \frac{1}{4}} \quad \therefore H(Z) = 2\frac{Z}{Z - \frac{1}{2}} - \frac{Z}{Z - \frac{1}{4}}$$

Taking IZT we get,

$$h(n) = 2\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

This is impulse response of system.

2. Step response of the system, means output of system when applied input is unit step. Thus putting x (n) = u (n) in given equation we get,

$$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + u(n)$$

Taking Z transform of both sides we get,

$$Y(Z) = \frac{3}{4} Z^{-1} Y(Z) - \frac{1}{8} Z^{-2} + \frac{Z}{Z-1}$$

$$\therefore Y(Z) - \frac{3}{4} Z^{-1} Y(Z) + \frac{1}{8} Z^{-2} = \frac{Z}{Z-1}$$

$$\therefore Y(Z)\left[1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right] = \frac{Z}{Z-1}$$

$$\therefore Y(Z) = \frac{Z}{Z-1} \cdot \frac{1}{1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{1}{Z-1} \cdot \frac{Z^2}{Z - \frac{3}{4} Z + \frac{1}{8}}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{Z^2}{(Z-1)\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z - \frac{1}{2}} + \frac{A_3}{Z - \frac{1}{4}}$$

$$A_1 = (Z-1) \cdot \left.\frac{Z^2}{(Z-1)\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}\right|_{Z=1}$$

$$\therefore A_1 = \left.\frac{Z^2}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}\right|_{Z=1}$$

$$= \frac{1}{\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)} = \frac{1}{\left(\frac{1}{2}\right)\left(\frac{3}{4}\right)} = \frac{1}{\frac{3}{8}} = \frac{8}{3}$$

$$A_2 = \left(Z - \frac{1}{2}\right) \left.\frac{Z^2}{(Z-1)\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}\right|_{Z = \frac{1}{2}}$$

$$= \frac{\left(\frac{1}{2}\right)^2}{\left(\frac{1}{2} - 1\right)\left(\frac{1}{2} - \frac{1}{4}\right)} = \frac{\frac{1}{4}}{\left(-\frac{1}{2}\right)\left(\frac{1}{4}\right)} = -2$$

and $A_3 = \left(Z - \frac{1}{4}\right) \left.\frac{Z^2}{(Z-1)\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}\right|_{Z = \frac{1}{4}}$

$$\therefore A_3 = \frac{\left(\frac{1}{4}\right)^2}{\left(\frac{1}{4} - 1\right)\left(\frac{1}{4} - \frac{1}{2}\right)} = \frac{\frac{1}{16}}{\left(-\frac{3}{4}\right)\left(-\frac{1}{4}\right)} = \frac{1}{3}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{\frac{8}{3}}{Z-1} - \frac{2}{Z - \frac{1}{2}} + \frac{\frac{1}{3}}{Z - \frac{1}{4}}$$

$$\therefore Y(Z) = \frac{8}{3}\frac{Z}{Z-1} - \frac{2Z}{Z - \frac{1}{2}} + \frac{\frac{1}{3}Z}{Z - \frac{1}{4}}$$

Taking IZT we get,

$$y(n) = \frac{8}{3}(1)^n u(n) - 2\left(\frac{1}{2}\right)^n u(n) + \frac{1}{3}\left(\frac{1}{4}\right)^n u(n)$$

This is the step response of system.

3. Given difference equation is,

$$y(n) = \frac{3}{4} Y(n-1) - \frac{1}{8} Y(n-2) + x(n)$$

Taking Z transform of both sides,

$$Y(Z) = \frac{3}{4} Z^{-1} y(Z) - \frac{1}{8} Z^{-2} y(Z) + X(Z)$$

$$\therefore Y(Z) - \frac{3}{4} Z^{-1} Y(Z) + \frac{1}{8} Z^{-2} Y(Z) = X(Z)$$

$$\therefore Y(Z)\left[1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right] = X(Z)$$

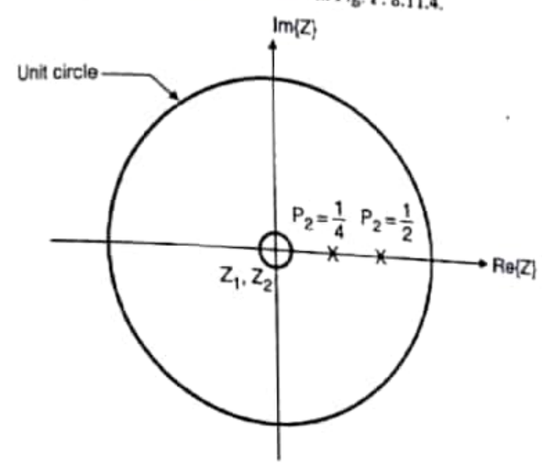
Now $H(Z) = \frac{Y(Z)}{X(Z)}$

$$\therefore H(Z) = \frac{1}{1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}} = \frac{Z^2}{Z^2 - \frac{3}{4} Z + \frac{1}{8}}$$

$$\therefore H(Z) = \frac{(Z-0)^2}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{4}\right)}$$

Thus zeros are at $Z_1 = Z_2 = 0$ and poles are at $P_1 = \frac{1}{2}$, $P_2 = \frac{1}{4}$

The pole zero diagram is shown in Fig. P. 8.11.4.



(H-1528) Fig. P. 8.11.4

Since both poles are inside unit circle, it is stable system.

Ex. 8.11.5 : Using the Z transform method, solve the difference equation :

$$y[n] - 4y[n-1] + 4y[n-2] = x[n] - x[n-1]$$

When $y(-1) = y(-2) = 0$

May 2015, 10 Marks

Soln. :

Since initial conditions are zero, we will use bilateral z transform.

Given :

$$y[n] - 4y[n-1] + 4y[n-2] = x[n] - x[n-1]$$

Calculation of zero input response :

Input is zero. Thus x (n) = 0 $\quad \therefore X(Z) = 0$

Taking Z transform of given equation.

$$Y(Z) - 4Z^{-1} Y(Z) + 4Z^{-2} Y(Z) = 0$$

$$\therefore Y(Z)[1 - 4Z^{-1} + 4Z^{-2}] = 0$$

$$\therefore Y(Z) = 0$$

$$\therefore Y(n) = 0$$

Thus if input is zero ; output is also zero because initial conditions are zero.

Calculation of zero state response :

Input is not given. So consider that input is unit step.

$$\therefore x(n) = u(n) \text{ and } X(Z) = \frac{Z}{Z-1}$$

Taking Z transform of given equation.

$$\therefore Y(Z) - 4Z^{-1} Y(Z) + 4Z^{-2} Y(Z) = X(Z) - Z^{-1} X(Z)$$

$$\therefore Y(Z)[1 - 4Z^{-1} + 4Z^{-2}] = X(Z)[1 - Z^{-1}]$$

$$\therefore Y(Z)[1 - 4Z^{-1} + 4Z^{-2}] = \frac{Z}{Z-1} \cdot [1 - Z^{-1}]$$

$$\therefore Y(Z) = \frac{Z}{Z-1} \cdot \frac{1 - Z^{-1}}{1 - 4Z^{-1} + 4Z^{-2}}$$

$$= \frac{Z}{Z-1} \cdot \frac{Z^2 - Z}{Z^2 - 4Z + 4}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{Z^2 - Z}{(Z-1)(Z-2)^2}$$

$$= \frac{Z(Z-1)}{(Z-1)(Z-2)^2} = \frac{Z}{(Z-2)^2}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{A}{Z-2} + \frac{B}{(Z-2)^2}$$

Here $A = \frac{d}{dZ}\left[(Z-2)^2 \cdot \frac{Z}{(Z-2)^2}\right]\Big|_{Z=2} = 1$

$$B = (Z-2)^2 \cdot \left.\frac{Z}{(Z-2)^2}\right|_{Z=2}$$

$$\therefore B = 2$$

$$\therefore \frac{Y(Z)}{Z} = \frac{1}{Z-2} + \frac{2}{(Z-2)^2}$$

$$\therefore Y(Z) = \frac{Z}{Z-2} + \frac{2Z}{(Z-2)^2}$$

Taking inverse Z transform,

$$y(n) = (2)^n u(n) + n(2)^n u(n)$$

Ex. 8.11.6 : A LTI system has the following transfer function.

$$H(Z) = \frac{Z}{\left(Z-\frac{1}{4}\right)\left(Z+\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)}$$

1. Give all possible ROC conditions
2. Show pole-zero diagram of a system
3. Find impulse response of system
4. Comment on the system stability and causality for all possible ROCs.

May 2016, 10 Marks

Soln. :

Given : $H(Z) = \dfrac{Z}{\left(Z-\frac{1}{4}\right)\left(Z+\frac{1}{4}\right)\left(Z-\frac{1}{2}\right)}$

The poles are at $P_1 = \frac{1}{4}$, $P_2 = -\frac{1}{4}$ and $P_3 = \frac{1}{2}$

The three possible ROC are,

1. $|Z| > \frac{1}{2}$ 2. $|Z| < \frac{1}{4}$ 3. $\frac{1}{4} < |Z| < \frac{1}{2}$

1. **For $|Z| > \frac{1}{2}$:**

- Here ROC is exterior portion of all poles, so system is causal.
- Since all poles are inside unit circle, it is stable system.

Now, $\dfrac{H(Z)}{Z} = \dfrac{A}{Z-\frac{1}{4}} + \dfrac{B}{Z+\frac{1}{4}} + \dfrac{C}{Z-\frac{1}{2}}$

Here $A = \dfrac{1/4}{\left(\frac{1}{4}+\frac{1}{4}\right)\left(\frac{1}{4}-\frac{1}{2}\right)} = -2$

$B = \dfrac{-1/4}{\left(-\frac{1}{4}-\frac{1}{4}\right)\left(-\frac{1}{4}-\frac{1}{2}\right)} = -0.67$

$C = \dfrac{1/2}{\left(\frac{1}{2}-\frac{1}{4}\right)\left(\frac{1}{2}+\frac{1}{4}\right)} = 2.67$

$$\therefore \frac{H(Z)}{Z} = \frac{-2}{Z-\frac{1}{4}} - \frac{0.67}{Z+\frac{1}{4}} + \frac{2.67}{Z-\frac{1}{2}}$$

$$\therefore H(Z) = \frac{-2Z}{Z-\frac{1}{4}} - \frac{0.67Z}{Z+\frac{1}{4}} + \frac{2.67Z}{Z-\frac{1}{2}} \quad \ldots(1)$$

Thus for $|Z| > \frac{1}{2}$, IZT is,

$$h(n) = -2\left(\frac{1}{4}\right)^n u(n) - 0.67\left(-\frac{1}{4}\right)^n u(n) + 2.67\left(\frac{1}{2}\right)^n u(n)$$

2. **ROC $|Z| < \frac{1}{4}$:**

$$h(n) = 2\left(\frac{1}{4}\right)^n u(-n-1) + 0.67\left(-\frac{1}{4}\right)^n u(-n-1) - 2.67\left(\frac{1}{2}\right)^n u(-n-1)$$

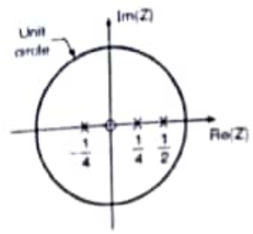
3. **ROC $\frac{1}{4} < |Z| < \frac{1}{2}$:**

$$h(n) = -2\left(\frac{1}{4}\right)^n u(n) - 0.67\left(-\frac{1}{4}\right)^n u(n) - 2.67\left(\frac{1}{2}\right)^n u(-n-1)$$

The zeros are at $Z_1 = 0$

Poles are at $P_1 = \frac{1}{4}$, $P_2 = -\frac{1}{4}$ and $P_3 = \frac{1}{2}$

Pole zero diagram is shown in Fig. P. 8.1.16.



(B-1655) **Fig. P. 8.1.16**

Ex. 8.11.7 : Find h(n) for a causal system $y(n) - y(n-5) = x(n) + x(n-1)$. Determine the response of the system for inputs $x(n) = 2^{-n} u(n)$.

Soln. :

Given : $y(n) - y(n-5) = x(n) + x(n-1)$

Taking Z transform of both sides,

$$Y(Z) - Z^{-5} Y(Z) = X(Z) + Z^{-1} X(Z)$$

$$\therefore Y(Z)\left[1 - Z^{-5}\right] = X(Z)\left[1 + Z^{-1}\right]$$

$$\text{Now } H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore H(Z) = \frac{1+Z^{-1}}{1-Z^{-5}} \quad \ldots(1)$$

Calculation of impulse response :

Impulse response is obtained by performing inverse Z transform of H (Z) as follows.

We will use power series expansion method.

The given sequence is causal. The most positive term of denominator should be at the beginning. So given equation is in proper fori

$$\begin{array}{r|l} & 1 + 1Z^{-1} + Z^{-5} + Z^{-6} + \ldots\ldots \\ \hline 1 - Z^{-5} & 1 + Z^{-1} \\ & 1 - Z^{-5} \\ & + \\ \hline & Z^{-1} + Z^{-5} \\ & Z^{-1} - Z^{-6} \\ & - \quad + \\ \hline & Z^{-5} + Z^{-6} \\ & Z^{-5} - Z^{-26} \\ & - \quad + \\ \hline & Z^{-6} + Z^{-26} \\ & Z^{-6} - Z^{-30} \\ & - \quad + \\ \hline & \ldots \end{array}$$

$$H(Z) = 1 + Z^{-1} + Z^{-5} + Z^{-6} + \ldots \quad \ldots(2)$$

Now for causal sequence, $H(Z) = \sum_{n=0}^{\infty} h(n) Z^{-n}$

Expanding the summation we get,

$$H(Z) = h(0) + h(1) Z^{-1} + h(2) Z^{-2} + \ldots \quad \ldots(3)$$

Comparing Equations (2) and (3) we get,

$$h(0) = 1, h(1) = 1, h(2) = 0, h(3) = 0, h(4) = 0, h(5) = 1, h(6) = 1 \ldots$$

$$\therefore h(n) = \{\underset{\uparrow}{1}, 1, 0, 0, 0, 1, 1 \ldots\}$$

This is the impulse response of system.

Calculation of response to input $x(n) = 2^{-n} u(n)$

Given $x(n) = 2^{-n} u(n) = \left(\frac{1}{2}\right)^n u(n)$

$$\therefore X(Z) = \frac{Z}{Z-\frac{1}{2}} \quad \ldots(4)$$

Now $H(Z) = \dfrac{Y(Z)}{X(Z)}$

$$\therefore Y(Z) = H(Z) \cdot X(Z)$$

Using Equations (1) and (4) we get,

$$Y(Z) = \frac{1+Z^{-1}}{1-Z^{-5}} \cdot \frac{Z}{Z-\frac{1}{2}}$$

$$\therefore Y(Z) = \frac{Z+1}{Z-\frac{1}{2}-Z^{-4}-\frac{1}{2}Z^{-5}}$$

To convert equation into negative powers of Z, multiply numerator and denominator by Z^{-1}.

$$\therefore Y(Z) = \frac{1+Z^{-1}}{1-\frac{1}{2}-Z^{-1}-Z^{-5}-Z^{-6}}$$

Now we will perform long division as follows

$$\begin{array}{r|l} & 1 + \frac{3}{2}Z^{-1} + \frac{5}{4}Z^{-2} \ldots\ldots \\ \hline 1 - \frac{1}{2}Z^{-1} - Z^{-5} - Z^{-6} & 1 + Z^{-1} \\ & 1 - \frac{1}{2}Z^{-1} - Z^{-5} - Z^{-6} \\ & - \quad + \quad + \quad + \\ \hline & \frac{3}{2}Z^{-1} + Z^{-5} + Z^{-6} \\ & \frac{3}{2}Z^{-1} - \frac{3}{4}Z^{-2} - \frac{3}{2}Z^{-5} - \frac{3}{2}Z^{-6} \\ & - \quad + \quad + \quad + \\ \hline & \frac{5}{4}Z^{-2} + \frac{3}{2}Z^{-5} - \frac{3}{2}Z^{-6} \end{array}$$

$$\therefore Y(Z) = 1 + \frac{3}{2}Z^{-1} + \frac{5}{4}Z^{-2} + \ldots \quad \ldots(5)$$

Now for causal sequence.

$$Y(Z) = \sum_{n=0}^{\infty} y(n) Z^{-n}$$

$$\therefore Y(Z) = y(0) + y(1) Z^{-1} + y(2) Z^{-2} + \ldots \quad \ldots(6)$$

Comparing Equations (5) and (6) we get,

$$y(0) = 1, \quad y(1) = \frac{3}{2}, \quad y(2) = \frac{5}{4}$$

$$\therefore y(n) = \left\{1, \frac{3}{2}, \frac{5}{4} \ldots\right\}$$

Ex. 8.11.8 : When input to causal LTI system is,

$$x(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{4}{3} 2^n u(-n-1)$$

Z-transform of output is, $Y(Z)$

$$= \frac{1+Z^{-1}}{(1-Z^{-1})\left(1+\frac{1}{2}Z^{-1}\right)(1-2Z^{-1})}$$

(a) Find Z-transform of $x(n)$.
(b) What is region of convergence of $Y(Z)$.
(c) Find impulse response of system.
(d) Is the system stable.

Soln. :

(a) Given : $x(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n) - \frac{4}{3} 2^n u(-n-1)$ ...(1)

Let $x_1(n) = -\frac{1}{3}\left(\frac{1}{2}\right)^n u(n)$

and $x_2(n) = -\frac{4}{3} 2^n u(-n-1)$

First term of Equation (1) is causal; since it is multiplied by U (n). Use standard Z-transform pair.

$$\alpha^n u(n) \xrightarrow{Z} \frac{Z}{Z-\alpha}, \text{ ROC} : |Z| > |\alpha|$$

$$\therefore X_1(Z) = -\frac{1}{3}\left[\frac{Z}{Z-\frac{1}{2}}\right], \text{ROC} : |Z| > \frac{1}{2}$$

Second term is anticausal, since it is multiplied by $u(-n-1)$. Use standard Z-transform pair.

$$-\alpha^n u(-n-1) \xrightarrow{Z} \frac{Z}{Z-\alpha}, \text{ ROC} : |Z| < |\alpha|$$

$$\therefore X_2(Z) = +\frac{4}{3} \cdot \frac{Z}{Z-2}, \text{ROC} : |Z| < 2$$

Now $X(Z) = X_1(Z) + X_2(Z)$

$$X(Z) = -\frac{1}{3}\left[\frac{Z}{Z-\frac{1}{2}}\right] + \frac{4}{3}\left[\frac{Z}{Z-2}\right] \quad \ldots(2)$$

(b) Given $Y(Z) = \dfrac{1+Z^{-1}}{(1+Z^{-1})\left(1+\frac{1}{2}Z^{-1}\right) \cdot (1-2Z^{-1})}$

Multiply numerator and denominator by Z^3

$$Y(Z) = \frac{Z^3(1+Z^{-1})}{Z(1-Z^{-1})\left(1+\frac{1}{2}Z^{-1}\right)(1-2Z^{-1})}$$

$$\therefore Y(Z) = \frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)} \quad \ldots(3)$$

Given LTI system is causal; thus **ROC is $|Z| > 2$.**

(c) We have,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right] \cdot \left[\frac{1}{-\frac{1}{3}\left(\frac{Z}{Z-\frac{1}{2}}\right)+\frac{4}{3}\left(\frac{Z}{Z-2}\right)}\right]$$

We will simplify second bracket as follows :

$$\therefore H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{3}{\frac{-Z}{Z-\frac{1}{2}}+\frac{4Z}{Z-2}}\right]$$

$$\therefore H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{3}{\frac{-Z^2+2Z+4Z^2-2Z}{\left(Z-\frac{1}{2}\right)(Z-2)}}\right]$$

$$\therefore H(Z) = \left[\frac{Z^3+Z^2}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{3\left(Z-\frac{1}{2}\right)(Z-2)}{3Z^2}\right]$$

$$\therefore H(Z) = \left[\frac{Z^2(Z+1)}{(Z-1)\left(Z+\frac{1}{2}\right)(Z-2)}\right]\left[\frac{\left(Z-\frac{1}{2}\right)(Z-2)}{Z^2}\right]$$

$$\therefore H(Z) = \frac{(Z+1)\left(Z-\frac{1}{2}\right)}{(Z-1)\left(Z+\frac{1}{2}\right)}$$

$$\therefore \frac{H(Z)}{Z} = \frac{(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)} \quad \ldots(4)$$

In P.F.E. form we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z}+\frac{A_2}{Z-1}+\frac{A_3}{Z+\frac{1}{2}} \quad \ldots(5)$$

The values of A_1, A_2 and A_3 are calculated as follows :

$$A_1 = Z \cdot \frac{H(Z)}{Z}\Big|_{Z=0}$$

$$\therefore A_1 = \frac{Z(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)}\Bigg|_{Z=0}$$

$$\therefore A_1 = \frac{(1)\left(-\frac{1}{2}\right)}{(-1)\left(\frac{1}{2}\right)} = 1$$

$$A_2 = \frac{(Z-1)\cdot(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)}\Bigg|_{Z=1}$$

$$= \frac{(1+1)\left(1-\frac{1}{2}\right)}{1\left(1+\frac{1}{2}\right)} = \frac{1}{3/2} = \frac{2}{3}$$

$$A_3 = \frac{\left(Z+\frac{1}{2}\right)\cdot(Z+1)\left(Z-\frac{1}{2}\right)}{Z(Z-1)\left(Z+\frac{1}{2}\right)}\Bigg|_{Z=\frac{-1}{2}}$$

$$\therefore A_3 = \frac{\left(-\frac{1}{2}+1\right)\left(-\frac{1}{2}-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}$$

$$= \frac{\left(\frac{1}{2}\right)(-1)}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)} = -\frac{2}{3}$$

Putting these values in Equation (5) we get,

$$\frac{H(Z)}{Z} = \frac{1}{Z}+\frac{2/3}{Z-1}-\frac{2/3}{Z+\frac{1}{2}}$$

$$\therefore H(Z) = 1+\frac{2}{3}\cdot\frac{Z}{Z-1}-\frac{2}{3}\cdot\frac{Z}{Z+\frac{1}{2}}$$

Since the given system is causal; its IZT can be written as,

$$h(n) = \delta(n)+\frac{2}{3}\cdot(1)^n u(n)-\frac{2}{3}\cdot\left(-\frac{1}{2}\right)^n u(n)$$

(d) ROC for this total system is $|Z| > 1$. **So the system is stable.**

Ex. 8.11.9 : Using the Z transform, solve the difference equation and find out impulse response.

$$y[n]-2y[n-1]+y[n-2] = x[n]+3x[n-3]$$

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Soln. :

Given : $y(n)-2y(n-1)+y(n-2) = x(n)+3x(n-3)$.

Taking Z transform of both sides,

$$Y(Z)-2Z^{-1}Y(Z)+Z^{-2}Y(Z) = X(Z)+3Z^{-1}X(Z)$$

$$\therefore Y(Z)[1-2Z^{-1}+Z^{-2}] = X(Z)[1+3Z^{-1}]$$

$$\text{Now, } H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore H(Z) = \frac{1+3Z^{-1}}{1-2Z^{-1}+Z^{-2}}$$

$$\therefore H(Z) = \frac{Z^2+3Z}{Z^2-2Z+1}$$

$$\therefore \frac{H(Z)}{Z} = \frac{Z+3}{Z^2-2Z+1} = \frac{Z+3}{(Z-1)^2}$$

In P.F.E. we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z-1}+\frac{A_2}{(Z-1)^2}$$

$$\text{Here } A_1 = \frac{d}{dZ}[Z+3]\Big|_{Z=1} = 1$$

$$\text{and } A_2 = Z+3\Big|_{Z=1} = 4$$

$$\therefore \frac{H(Z)}{Z} = \frac{1}{Z-1}+\frac{4}{(Z-1)^2}$$

$$\therefore H(Z) = \frac{Z}{Z-1}+\frac{4Z}{(Z-1)^2}$$

Taking inverse Z transform

$$h(n) = (1)^n u(n)+4n(1)^n u(n)$$

This is the impulse response of system.

Ex. 8.11.10 : Consider a causal LSI system function :

$$H(Z) = \frac{1-a^{-1}Z^{-1}}{1-aZ^{-1}} \text{ where 'a' is real.}$$

Determine the value of 'a' for which the system is stable. Show graphically for $0 < a < 1$; pole-zero plot and the ROC.

Soln. : The given function is,

$$H(Z) = \frac{1-a^{-1}Z^{-1}}{1-aZ^{-1}} \quad \ldots(1)$$

To convert it into positive powers of Z; multiply numerator and denominator by Z.

$$\therefore H(Z) = \frac{Z(1-a^{-1}Z^{-1})}{Z(1-aZ^{-1})} = \frac{Z-a^{-1}}{Z-a}$$

$$\therefore H(Z) = \frac{\left(Z-\frac{1}{a}\right)}{(Z-a)} \quad \ldots(2)$$

Now we have the standard equation,

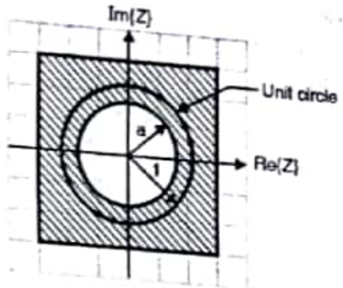
$$H(Z) = \frac{Z-Z_1}{Z-P_1} \quad \ldots(3)$$

Comparing Equations (2) and (3).

Zero is at, $Z_1 = \frac{1}{a}$ and pole is at $P_1 = a$.

Now in the question it is mentioned that, the system is causal. We know that for the causal system ROC is exterior part of circle. Thus here ROC is $|Z| > |a|$.

We want this system to be stable also. We know that for the system to be stable ROC should include unit circle. Thus to satisfy this condition; ***value of 'a' should be less than 1***. So for a causal and stable system; we can write the ROC as,



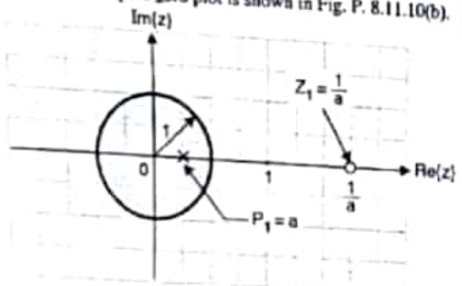
(H-1053) **Fig. P. 8.11.10(a) : ROC : $|Z| > |a| < 1$**

$\therefore$ ROC : $|Z| > |a|$ and $|a| < 1$

ROC : $|Z| > |a| < 1$.

This ROC is shown in Fig. P. 8.11.10(a).

Zero is at $Z_1 = \frac{1}{a}$ and pole is at $P_1 = a$. Here the value of 'a' is less than 1. The pole zero plot is shown in Fig. P. 8.11.10(b).



(H-1054) **Fig. P. 8.11.10(b) : Pole-zero plot**

Ex. 8.11.11 : The transfer function of a causal LTI system is,

$$H(Z) = \frac{1-Z^{-1}}{1+\frac{3}{4}Z^{-1}}$$

1. Find the impulse response of the system
2. Find the output of the system to the input

$$x(n) = \left(\frac{1}{3}\right)^n u(n)+u(-n-1)$$

3. Is the system stable ?

Soln. :

1. Calculation of impulse response h (n)

Given : $$H(Z) = \frac{1-Z^{-1}}{1+\frac{3}{4}Z^{-1}} \quad \ldots(1)$$

We will obtain h (n) by taking IZT of H (Z) using partial fraction expansion. Multiplying numerator and denominator by Z we get,

$$H(Z) = \frac{Z-1}{Z+\frac{3}{4}}$$

$$\frac{H(Z)}{Z} = \frac{Z-1}{Z\left(Z+\frac{3}{4}\right)} \quad \ldots(2)$$

The poles are at $P_1 = 0$ and $P_2 = -\frac{3}{4}$. We can write Equation (2) in the partial fraction expansion form as.

$$\frac{H(Z)}{Z} = \frac{A_1}{(Z-0)} + \frac{A_2}{\left(Z+\frac{3}{4}\right)} \quad ...(3)$$

Here $A_1 = (Z-P_1)\left.\frac{H(Z)}{Z}\right|_{Z=P_1}$

$$= \left.\frac{(Z-0)(Z-1)}{Z\left(Z+\frac{3}{4}\right)}\right|_{Z=0}$$

$$\therefore A_1 = \frac{(0-1)}{(0+3/4)} \Rightarrow A_1 = -\frac{4}{3}$$

and $A_2 = (Z-P_2)\left.\frac{H(Z)}{Z}\right|_{Z=P_1}$

$$= \left.\frac{\left(Z+\frac{3}{4}\right)(Z-1)}{Z(Z+3/4)}\right|_{Z=-\frac{3}{4}}$$

$$\therefore A_2 = \left.\frac{Z-1}{Z}\right|_{Z=-\frac{3}{4}} = \frac{-\frac{3}{4}-1}{-\frac{3}{4}} = \frac{-\frac{7}{4}}{-\frac{3}{4}}$$

$$\therefore A_2 = \frac{7}{3}$$

Putting values of A_1 and A_2 in Equation (3) we get

$$\frac{H(Z)}{Z} = -\frac{4/3}{Z} + \frac{7/3}{Z+3/4}$$

$$\therefore H(Z) = \frac{-4/3Z}{Z} + \frac{7/3Z}{Z+3/4}$$

$$\therefore H(Z) = -4/3 + \frac{7/3Z}{Z+3/4} \quad ...(4)$$

The system is causal. So we can obtain h (n) as follows :

$$h(n) = IZT\left\{\left(-\frac{4}{3}\right)\right\} + \frac{7}{3}\ IZT\left\{\frac{Z}{Z+3/4}\right\}$$

$$\therefore h(n) = -\frac{4}{3}\delta(n) + \frac{7}{3}\left(-\frac{3}{4}\right)^n u(n) \quad ...(5)$$

2. Calculation of output y (n)

Given :

$$x(n) = \left(\frac{1}{3}\right)^n u(n) + u(-n-1) \quad ...(6)$$

Taking Z-transform we get,

$$X(Z) = \frac{Z}{Z-\frac{1}{3}} - \frac{Z}{Z-1}$$

$$... \left[\text{since } Z\{u(-n-1)\} = \frac{-Z}{Z-1}\right] \quad ...(7)$$

And given H (Z) is, $H(Z) = \frac{Z-1}{Z+\frac{3}{4}}$ ...(8)

Now $H(Z) = \frac{Y(Z)}{X(Z)}$

$$\therefore Y(Z) = H(Z) \times X(Z)$$

$$\therefore Y(Z) = \left(\frac{Z-1}{Z+\frac{3}{4}}\right)\left(\frac{Z}{Z-\frac{1}{3}} - \frac{Z}{Z-1}\right)$$

$$\therefore Y(Z) = \frac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)} - \frac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)(Z-1)}$$

$$\therefore Y(Z) = \frac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)} - \frac{Z}{Z+\frac{3}{4}} \quad ...(9)$$

Consider, $Y(Z) = \frac{Z(Z-1)}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)}$

and $Y_2(Z) = \frac{Z}{Z+\frac{3}{4}}$

$$\therefore \frac{Y_1(Z)}{Z} = \frac{Z-1}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)}$$

$$\therefore \frac{Y_1(Z)}{Z} = \frac{A_1}{\left(Z+\frac{3}{4}\right)} + \frac{A_2}{\left(Z-\frac{1}{3}\right)} \quad ...(10)$$

The poles are at $P_1 = -\frac{1}{3}$ and $P_2 = \frac{1}{3}$

Here $A_1 = (Z-P_1)\left.\frac{Y_1(Z)}{Z}\right|_{Z=P_1}$

$$\therefore A_1 = \left(Z+\frac{3}{4}\right)\left[\frac{(Z-1)}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)}\right]\Bigg|_{Z=\frac{3}{4}}$$

$$\therefore A_1 = \left[\frac{Z-1}{Z-\frac{1}{3}}\right]\Bigg|_{Z=-\frac{3}{4}} = \frac{-\frac{3}{4}-1}{-\frac{3}{4}-\frac{1}{3}} = \frac{-\frac{7}{4}}{\frac{-9-4}{12}}$$

$$\therefore A_1 = \frac{-7}{4} \times \left(\frac{-12}{13}\right) = +\frac{21}{13}$$

and $A_2 = (Z-P_2)\left.\frac{Y_1(Z)}{Z}\right|_{Z=P_2}$

$$= \left(Z-\frac{1}{3}\right)\left[\frac{Z-1}{\left(Z+\frac{3}{4}\right)\left(Z-\frac{1}{3}\right)}\right]\Bigg|_{Z=\frac{1}{3}}$$

$$\therefore A_2 = \left[\frac{Z-1}{Z+\frac{3}{4}}\right]\Bigg|_{Z=\frac{1}{3}} = \frac{\frac{1}{3}-1}{\frac{1}{3}+\frac{3}{4}} = \frac{-\frac{2}{3}}{\frac{4+9}{12}} = \frac{-2}{3} \times \frac{12}{13}$$

$$\therefore A_2 = \frac{-8}{13}$$

Putting A_1 and A_2 in Equation (10) we get,

$$\frac{Y_1(Z)}{Z} = \frac{\frac{21}{13}}{\left(Z+\frac{3}{4}\right)} - \frac{\frac{8}{13}}{\left(Z-\frac{1}{3}\right)}$$

$$\therefore Y_1(Z) = \frac{\left(\frac{21}{13}\right)Z}{Z+\frac{3}{4}} - \frac{\left(\frac{8}{13}\right)Z}{Z-\frac{1}{3}}$$

Given system is causal.

$$\therefore y(n) = \frac{21}{13}\left(\frac{-3}{4}\right)^n u(n) - \frac{8}{13}\left(\frac{1}{3}\right)^n u(n) \quad ...(11)$$

Now consider

$$Y_2(Z) = \frac{Z}{Z+\frac{3}{4}}$$

$$\therefore y_2(n) = IZT\left\{\frac{Z}{Z+\frac{3}{4}}\right\} = \left(-\frac{3}{4}\right)^n u(n) \quad ...(12)$$

Thus total output $y(n) = y_1(n) + y_2(n)$

$$\therefore y(n) = \frac{21}{13}\left(-\frac{3}{4}\right)^n u(n) - \frac{8}{13}\left(\frac{1}{3}\right)^n u(n) + \left(-\frac{3}{4}\right)^n u(n)$$

3. Deciding the stability of system

The given transfer function is,

$$H(Z) = \frac{Z-1}{Z+\frac{3}{4}} \quad ...(13)$$

It has one pole at $P_1 = -\frac{3}{4}$. It is clear that the pole lies inside the unit circle. when the pole is inside the unit circle then the system is stable. So this is a stable system.

Ex. 8.11.12 : Plot the pole zero pattern and determine which of the system is stable.

1. $y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1)$
2. $y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2)$

Soln. :

1. Given,

$$y(n) = y(n-1) - 0.5y(n-2) + x(n) + x(n-1) \quad ...(1)$$

Taking IZT we get,

$$Y(Z) = Z^{-1}Y(Z) - 0.5Z^{-2}Y(Z) + X(Z) + Z^{-1}X(Z)$$

$$\therefore Y(Z) - Z^{-1}Y(Z) + 0.5Z^{-2}Y(Z) = X(Z)[1+Z^{-1}]$$

$$\therefore Y(Z)[1 - Z^{-1} + 0.5Z^{-2}] = X(Z)[1+Z^{-1}]$$

$$\therefore \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1+Z^{-1}}{1-Z^{-1}+0.5Z^{-2}}$$

Multiplying numerator and denominator by Z^2 we get,

$$H(Z) = \frac{Z^2+Z}{Z^2-Z+0.5} \quad \therefore H(Z) = \frac{Z(Z+1)}{Z^2-Z+0.5} \quad ...(2)$$

We will obtain the roots of denominator,

$$P_1, P_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm \sqrt{-1}}{2}$$

$$\therefore P_1, P_2 = \frac{1 \pm j\sqrt{1}}{2} = \frac{1}{2} \pm j\frac{1}{2}$$

Thus poles are at,

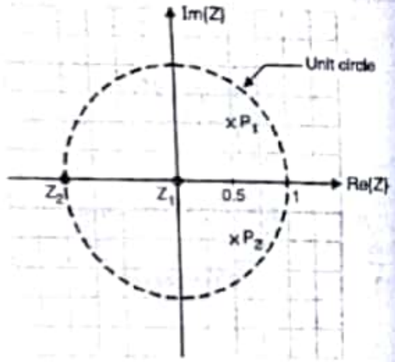
$\therefore P_1 = 0.5 + j0.5$ and $P_2 = 0.5 - j0.5$

Thus Equation (2) becomes,

$$H(Z) = \frac{(Z-0)(Z+1)}{(Z-0.5-j0.5)(Z-0.5+j0.5)}$$

Here zeros are at $Z_1 = 0$ and $Z_2 = -1$

The pole zero plot is as shown in Fig. P. 8.11.12(a).



(II-1857) **Fig. P. 8.11.12(a) : Pole zero plot**

Since both the poles are inside the unit circle; the system is stable.

2. Given,

$$y(n) = 0.7y(n-1) - 0.1y(n-2) + 2x(n) - x(n-2) \quad ...(3)$$

Taking IZT we get,

$$Y(Z) = 0.7Z^{-1}Y(Z) - 0.1Z^{-2}Y(Z) + 2X(Z) - Z^{-2}X(Z)$$

$$\therefore Y(Z) - 0.7Z^{-1}Y(Z) + 0.1Z^{-2}Y(Z) = 2X(Z) - Z^{-2}X(Z)$$

$$\therefore Y(Z)[1 - 0.7Z^{-1} + 0.1Z^{-2}] = X(Z)[2 - Z^{-2}]$$

$$\frac{Y(Z)}{X(Z)} = H(Z) = \frac{2-Z^{-2}}{1-0.7Z^{-1}+0.1Z^{-2}}$$

Multiplying numerator and denominator by Z^2 we get,

$$H(Z) = \frac{2Z^2-1}{Z^2-0.7Z+0.1} \quad ...(4)$$

We will obtain roots of numerator and denominator,

$$Z_1, Z_2 = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\therefore Z_1, Z_2 = \frac{0 \pm \sqrt{0+2}}{2} = \pm\frac{\sqrt{2}}{2} = \pm 0.707$$

$$\therefore Z_1 = 0.707 \text{ and } Z_2 = -0.707$$

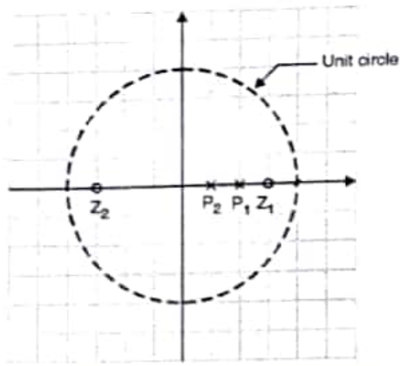
Now, $P_1, P_2 = \frac{0.7 \pm \sqrt{0.49 - 0.4}}{2} = \frac{0.7 \pm 0.3}{2}$

$\therefore P_1 = 0.5$ and $P_2 = 0.2$

Putting these values in Equation (4) we get,

$$H(Z) = \frac{(Z - 0.707)(Z + 0.707)}{(Z - 0.5)(Z - 0.2)}$$

The pole zero plot is shown in Fig. P. 8.11.12(b).



(II-1058) **Fig. P. 8.11.12(b) : Pole-zero plot**

Since both the poles lie inside the unit circle; the system stable.

Ex. 8.11.13 : Check the BIBO stability of

$$T.F\ H(z) = \frac{z - 2}{z(z - 0.8)}$$

Soln. :

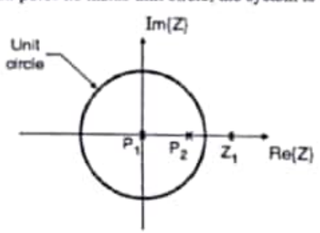
Given $H(Z) = \frac{Z - 2}{Z(Z - 0.8)}$

Thus zero is at $Z_1 = 2$ and poles are at,

$P_1 = 0$ and $P_2 = 0.8$

The pole zero plot is shown in Fig. P. 8.11.13.

Since all poles lie inside unit circle; the system is stable.



(II-1195) **Fig. P. 8.11.13**

Ex. 8.11.14 : A causal system has difference equation given by, $y(n) = 0.5\,y(n-1) - 0.25\,y(n-2) + x(n)$ What is the ROC of transfer function of system.

Soln. : The given equation is,

$$y(n) = 0.5\,y(n-1) - 0.25\,y(n-2) + x(n) \quad ...(1)$$

Taking Z-transform of both sides,

$$\therefore Y(Z) = 0.5 Z^{-1} Y(Z) - 0.25 Z^{-2} Y(Z) + X(Z)$$

$$\therefore Y(Z) - 0.5 Z^{-1} Y(Z) + 0.25 Z^{-2} Y(Z) = X(Z)$$

$$\therefore Y(Z)[1 - 0.5 Z^{-1} + 0.25 Z^{-2}] = X(Z)$$

$$\therefore \frac{Y(Z)}{X(Z)} = \frac{1}{1 - 0.5 Z^{-1} + 0.25 Z^{-2}} \quad ...(2)$$

But $\frac{Y(Z)}{X(Z)} = H(Z)$

$$\therefore H(Z) = \frac{1}{1 - 0.5 Z^{-1} + 0.25 Z^{-2}} \quad ...(3)$$

To convert Equation (3) into positive powers of Z; multiply numerator and denominator by .

$$\therefore H(Z) = \frac{Z^2}{Z^2 - 0.5 Z + 0.25} \quad ...(4)$$

We will obtain the poles of a system by calculating roots of denominator term.

Here $a = 1, b = -0.5, c = 0.25$

$$\therefore P_1, P_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore P_1, P_2 = \frac{0.5 \pm \sqrt{(0.5)^2 - 4 \times 0.25}}{2} = \frac{0.5 \pm \sqrt{-0.75}}{2}$$

$\therefore P_1 = 0.25 + j\,0.433$ and $P_2 = 0.25 - j\,0.433$

Now Equation (4) can be written as,

$$H(Z) = \frac{Z^2}{(Z - P_1)(Z - P_2)}$$

$$\therefore H(Z) = \frac{Z^2}{(Z - 0.25 - j\,0.433)(Z - 0.25 + j\,0.433)} \quad ...(5)$$

The given system is causal and **for causal system ROC should be $|Z| > |P_1|$ or $|P_2|$.**

Here $|P_1|$ or $|P_2| = \sqrt{(-0.25)^2 + (0.433)^2} = 0.499 \approx 0.5$

Thus ROC is $|Z| > 0.5$

Ex. 8.11.15 : Determine the system function, unit sample response and pole zero plot of the system described by the difference equation,

$$y(n) - \frac{1}{2} y(n-1) = 2x(n)$$

May 11, May 12, 10 Marks

Soln. :

Calculation of H(z)

Given difference equation is,

$$y(n) - \frac{1}{2} y(n-1) = 2x(n)$$

Taking z transform of both sides,

$$y(z) - \frac{1}{2} z^{-1} y(z) = 2x(z)$$

$$\therefore y(z)\left[1 - \frac{1}{2} z^{-1}\right] = 2x(z)$$

$$\therefore H(z) = \frac{y(z)}{x(z)} = \frac{2}{1 - \frac{1}{2} z^{-1}}$$

$$\therefore H(z) = \frac{2z}{z - \frac{1}{2}} \quad ...(1)$$

This is the system transfer function.

Calculation of h(n)

h(n) is obtained by taking IZT of H(z). Using standard z transform formula,

$$\alpha^n u(n) \leftrightarrow \frac{z}{z - \alpha}$$

$$\therefore IZT\left\{\frac{2z}{z - \frac{1}{2}}\right\} = 2\left(\frac{1}{2}\right)^n u(n)$$

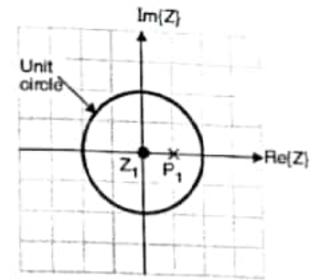
$$\therefore h(n) = 2\left(\frac{1}{2}\right)^n u(n)$$

Pole zero diagram

We have, $H(z) = \frac{2z}{z - \frac{1}{2}}$

Here zero is at $z_1 = 0$ and pole is at $p_1 = \frac{1}{2}$

The pole zero diagram is shown in Fig. P. 8.11.15.



(II-1533) **Fig. P. 8.11.15**

8.12 Unilateral (One-Sided) Z-transform :

Definition :

Unilateral or one sided Z-transform of DT signal x (n) is defined by,

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad ...(8.12.1)$$

For unilateral Z-transform, some-times the notation Z^+ is also used. Basically one sided Z-transform is used for causal system.

Characteristics

1. **ROC :** for unilateral Z-transform [$X^+(Z)$]; is always exterior part of the circle. Thus for unilateral Z-transform; it is not necessary to refer ROC.
2. Z-transform is unique for causal signal.
3. It does not contain information for $n < 0$.

8.12.1 Shifting Property of Unilateral Z-Transform :

The properties of unilateral Z-transform are same as two sided Z-transform properties except shifting property. Under shifting we will consider two cases as follows :

CASE - I : Time delay

Statement : If $x(n) \xleftrightarrow{Z} X(Z)$

Then $x(n-k) \xrightarrow{Z} Z^{-k}\left[X(Z) + \sum_{n=1}^{k} x(-n) Z^{n}\right]$ for $k > 0$

Proof : According to the definition of one-sided Z-transform we have,

$$Z\{x(n)\} = X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad ...(8.12.2)$$

$$\therefore Z\{x(n-k)\} = \sum_{n=0}^{\infty} x(n-k) Z^{-n} \quad ...(8.12.3)$$

Put $n - k = l$ $\therefore n = k + l$ then limits will change as follows :

When $n = 0 \Rightarrow 0 - k = l$ $\therefore l = -k$

When $n = \infty \Rightarrow \infty - k = l$ $\therefore l = \infty$

Thus Equation (8.12.3) becomes,

$$Z\{x(n-k)\} = \sum_{l=-k}^{\infty} x(l) Z^{-(k+l)}$$

$$= \sum_{l=-k}^{\infty} x(l) Z^{-k} \cdot Z^{-l}$$

$$\therefore Z\{x(n-k)\} = Z^{-k} \sum_{l=-k}^{\infty} x(l) Z^{-l} \quad ...(8.12.4)$$

The summation in Equation (8.12.4) can be splitted into two parts as follows :

$$Z\{x(n-k)\} = Z^{-k}\left[\sum_{l=0}^{\infty} x(l) Z^{-l} + \sum_{l=-k}^{-1} x(l) Z^{-l}\right] \quad ...(8.12.5)$$

The first summation term at R.H.S. indicates X (Z).

$$\therefore Z\{x(n-k)\} = Z^{-k} X(Z) + Z^{-k} \sum_{l=-k}^{-1} x(l) Z^{-l} \quad ...(8.12.6)$$

To make the limits of summation positive put $l = -m$ $\therefore m = -l$.

$$\therefore Z\{x(n-k)\} = Z^{-k} X(Z) + Z^{-k} \sum_{m=k}^{l} x(-m) Z^{m}$$

$$\therefore Z\{x(n-k)\} = Z^{-k}\left[X(Z) + \sum_{m=k}^{1} x(-m) Z^{m}\right] \quad ...(8.12.7)$$

In Equation (8.12.7), m is a variable. Instead of m we can put another variable n.

$$\therefore Z\{x(n-k)\} = Z^{-k}\left[X(Z) + \sum_{n=k}^{1} x(-n) Z^{n}\right] \quad ...(8.12.8)$$

CASE - II : Time advance :

Statement : If $x(n) \xrightarrow{Z} X(Z)$

then $x(n+k) \xleftrightarrow{Z} Z^k\left[X(Z) - \sum_{n=0}^{k-1} x(n)Z^{-n}\right]$, $k > 0$ ...(8.12.9)

Proof : According to the definition of one-sided Z-transform.

$$Z\{x(n)\} = X(Z) = \sum_{n=0}^{\infty} x(n)Z^{-n}$$

$$\therefore Z\{x(n+k)\} = \sum_{n=0}^{\infty} x(n+k)Z^{-n} \quad ...(8.12.10)$$

Put $n + k = l$ $\therefore$ $n = l - k$. The limits of summation will change as follows :

When $n = 0 \Rightarrow 0 + k = l \therefore l = k$

When $n = \infty \Rightarrow \infty + k = l \therefore l = \infty$

Thus Equation (8.12.10) becomes.

$$Z\{x(n+k)\} = \sum_{l=k}^{\infty} x(l)Z^{-(l-k)} = \sum_{l=k}^{\infty} x(l)Z^{-l}\cdot Z^k$$

$$\therefore Z\{x(n+k)\} = Z^k\left[\sum_{l=k}^{\infty} x(l)Z^{-l}\right] \quad ...(8.12.11)$$

In Equation (8.12.11), the limits of summation are from $l = k$ to $l = \infty$.

These limits can be written as follows.

$$\sum_{l=k}^{\infty} = \sum_{l=0}^{\infty} - \sum_{l=0}^{k-1}$$

Thus Equation (8.12.11) becomes.

$$Z\{x(n+k)\} = Z^k\left[\sum_{l=0}^{\infty} x(l)Z^{-l} - \sum_{l=0}^{k-1} x(l)Z^{-l}\right]$$

$$\therefore Z\{x(n+k)\} = Z^k\sum_{l=0}^{\infty} x(l)Z^{-l} - Z^k\sum_{l=0}^{k-1} x(l)Z^{-l} \quad ...(8.12.12)$$

The first summation in Equation (8.12.12) indicates $X(Z)$

$$\therefore Z\{x(n+k)\} = Z^k X(Z) - Z^k\sum_{l=0}^{k-1} x(l)Z^{-l}$$

$$\therefore Z\{x(n+k)\} = Z^k\left[X(Z) - \sum_{l=0}^{k-1} x(l)Z^{-l}\right] \quad ...(8.12.13)$$

Replacing the variable 'l' by 'n' we get,

$$Z\{x(n+k)\} = Z^k\left[X(Z) - \sum_{l=0}^{k-1} x(n)Z^{-n}\right] \quad ...(8.12.14)$$

8.12.2 Solution of Difference Equation using Unilateral Z Transform (Impulse and Step Response) :

We have derived shifting property of unilateral Z-transform. Using this property, we can solve the difference equations.

CASE - I : Time delay :

We have, $Z\{x(n-k)\} = Z^{-k}\left[X(Z) + \sum_{n=1}^{k} x(-n)Z^{n}\right]$

Here k is constant which represents the amount by which the signal is delayed. Let k = 1.

$$\therefore Z\{x(n-1)\} = Z^{-1}[X(Z) + x(-1)Z^1]$$

$$\therefore Z\{x(n-1)\} = Z^{-1}X(Z) + x(-1).$$

Similarly we can obtain results for other values of k. For the ready reference; time delay property, in the expanded form can be written for k = 1, 2, 3 and 4 as follows :

$$x(n-1) \xleftrightarrow{Z} Z^{-1}X(Z) + x(-1)$$

$$x(n-2) \xleftrightarrow{Z} Z^{-2}X(Z) + Z^{-1}x(-1) + x(-2)$$

$$x(n-3) \xleftrightarrow{Z} Z^{-3}X(Z) + Z^{-2}x(-1) + Z^{-1}x(-2) + x(-3)$$

$$x(n-4) \xleftrightarrow{Z} Z^{-4}X(Z) + Z^{-3}x(-1) + Z^{-2}x(-2) + Z^{-1}x(-3) + x(-4)$$

CASE - II : Time advance :

Similarly, we can write, time advance property in the expanded form as follows :

$$x(n+1) \xleftrightarrow{Z} Z^{+1}X(Z) - Z^1x(0)$$

$$x(n+2) \xleftrightarrow{Z} Z^2X(Z) - Z^2x(0) - Z^1x(1)$$

$$x(n+3) \xleftrightarrow{Z} Z^3X(Z) - Z^3x(0) - Z^2x(1) - Z^1x(2)$$

$$x(n+4) \xleftrightarrow{Z} Z^4X(Z) - Z^4x(0) - Z^3x(1) - Z^2x(2) - Z^1x(3)$$

Ex. 8.12.1 : A LTI is described by equation,

$$2y(n) + 3y(n-1) + y(n-2) = u(n) + u(n-1) - u(n-2).$$

Find response of system when initial conditions are given by $y(-1) = 2$ and $y(-2) = -1$ and when unit step is applied at the input.

Soln. : Given difference equation is,

$$2y(n) + 3y(n-1) + y(n-2) = u(n) + u(n-1) - u(n-2) \quad ...(1)$$

Now according to shifting property of unilateral Z-transform;

$$x(n-K) = Z^{-K}\left[X(Z) + \sum_{n=1}^{K} x(-n)Z^{n}\right] \quad ...(2)$$

Applying this property to Equation (1) we get,

$$2Y(Z) + 3Z^{-1}[Y(Z) + y(-1)\cdot Z] + Z^{-2}\left[Y(Z) + \sum_{n=1}^{2} y(-n)Z^{n}\right]$$

$$= U(Z) + Z^{-1}[U(Z) + u(-1)Z] - Z^{-2}\left[U(Z) + \sum_{n=1}^{2} u(-n)Z^{n}\right]$$

$$\therefore 2Y(Z) + 3Z^{-1}[Y(Z) + y(-1)\cdot Z] + Z^{-2}[Y(Z) + y(-1)Z^1 + y(-2)Z^2]$$
$$= U(Z) + Z^{-1}[U(Z) + u(-1)Z] - Z^{-2}[U(Z) + u(-1)Z + u(-2)Z^2] \quad ...(3)$$

Given initial conditions are,

$$y(-1) = 2, \quad y(-2) = -1$$

Since unit step is applied at the input,

$$u(-1) = u(-2) = 0.$$

Putting these values in Equation (3) we get,

$$2Y(Z) + 3Z^{-1}[Y(Z) + 2Z] + Z^{-2}[Y(Z) + 2Z - Z^2] = U(Z) + Z^{-1}U(Z) - Z^{-2}U(Z)$$

$$\therefore 2Y(Z) + 3Z^{-1}Y(Z) + 6 + Z^{-2}Y(Z) + 2Z^{-1} - 1 = U(Z)[1 + Z^{-1} - Z^{-2}]$$

$$\therefore Y(Z)[2 + 3Z^{-1} + Z^{-2}] + 5 + 2Z^{-1} = U(Z)[1 + Z^{-1} - Z^{-2}]$$

$$\therefore Y(Z)[2 + 3Z^{-1} + Z^{-2}] = -5 - 2Z^{-1} + U(Z)[1 + Z^{-1} - Z^{-2}]$$

$$\therefore Y(Z) = \frac{-5 - 2Z^{-1}}{2 + 3Z^{-1} + Z^{-2}} + U(Z)\frac{1 + Z^{-1} - Z^{-2}}{2 + 3Z^{-1} + Z^{-2}} \quad ...(4)$$

Now we have, $U(Z) = Z\{u(n)\} = \frac{Z}{Z-1}$.

Putting this value in Equation (4) and converting all terms into positive powers of Z we get,

$$Y(Z) = \frac{-5Z^2 - 2Z}{2Z^2 + 3Z + 1} + \frac{Z}{Z-1}\,\frac{(Z^2 + Z - 1)}{(2Z^2 + 3Z + 1)}$$

$$\therefore Y(Z) = \frac{(Z-1)(-5Z^2 - 2Z) + Z(Z^2 + Z - 1)}{(Z-1)(2Z^2 + 3Z + 1)}$$

$$\therefore Y(Z) = \frac{-5Z^3 - 2Z^2 + 5Z^2 + 2Z + Z^3 + Z^2 - Z}{(Z-1)(2Z^2 + 3Z + 1)}$$

$$\therefore Y(Z) = \frac{-4Z^3 + 4Z^2 + Z}{(Z-1)(2Z^2 + 3Z + 1)}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{-4Z^2 + 4Z + 1}{(Z-1)(2Z^2 + 3Z + 1)}$$

To adjust second term in the denominator divide numerator and denominator by 2;

$$\therefore \frac{Y(Z)}{Z} = \frac{-2Z^2 + 2Z + \frac{1}{2}}{(Z-1)\left(Z^2 + \frac{3}{2}Z + \frac{1}{2}\right)} \quad ...(5)$$

Consider the term $Z^2 + \frac{3}{2}Z + \frac{1}{2}$. We will obtain the roots as follows :

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} - 4\left(\frac{1}{2}\right)}}{2}$$

$$= \frac{-\frac{3}{2} \pm \sqrt{\frac{9}{4} - 2}}{2} = \frac{-\frac{3}{2} \pm \frac{1}{2}}{2} = -\frac{1}{2}, -1$$

$$\therefore \frac{Y(Z)}{Z} = \frac{-2Z^2 + 2Z + 0.5}{(Z-1)(Z+0.5)(Z+1)} \quad ...(6)$$

In P.F.E. form we can write,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z-1} + \frac{A_2}{Z+0.5} + \frac{A_3}{Z+1} \quad ...(7)$$

Here $A_1 = (Z-1)\left[\frac{-2Z^2 + 2Z + 0.5}{(Z-1)(Z+0.5)(Z+1)}\right]\Bigg|_{Z=1}$

$$\therefore A_1 = \frac{-2 + 2 + 0.5}{(1 + 0.5)(1 + 1)} = \frac{0.5}{(1.5)(2)} = 0.17$$

$$A_2 = (Z + 0.5)\left[\frac{-2Z^2 + 2Z + 0.5}{(Z-1)(Z+0.5)(Z+1)}\right]\Bigg|_{Z=-0.5}$$

$$\therefore A_2 = \frac{-2(-0.5)^2 + 2(-0.5) + 0.5}{(-0.5-1)(-0.5+1)} = \frac{-0.5 - 1 + 0.5}{(-1.5)(0.5)} = 1.33$$

and $A_3 = (Z+1)\left[\frac{-2Z^2 + 2Z + 0.5}{(Z-1)(Z+0.5)(Z+1)}\right]\Bigg|_{Z=-1}$

$$\therefore A_3 = \frac{-2(-1)^2 + 2(-1) + 0.5}{(-1-1)(-1+0.5)} = \frac{-2 - 2 + 0.5}{(-2)(-0.5)} = -3.5$$

Putting these values in Equation (7) we get,

$$\frac{Y(Z)}{Z} = \frac{0.17}{Z-1} + \frac{1.33}{Z+0.5} - \frac{3.5}{Z+1}$$

$$\therefore Y(Z) = \frac{0.17Z}{Z-1} + \frac{1.33Z}{Z+0.5} - \frac{3.5Z}{Z+1}$$

Taking IZT we get,

$$y(n) = 0.17(1)^n u(n) + 1.33(-0.5)^n u(n) - 3.5(-1)^n u(n)$$

This is the response of system.

Ex. 8.12.2 : An LTI system is described by the following equation. Determine the transfer function and impulse response of the system. Sketch the poles and zeros of the z-plane.

$$y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$$

Dec. 2016. 10 Marks

Soln. :

Given, $y(n) - 4y(n-1) + 4y(n-2) = x(n-1)$

Taking Z-transform of both sides,

$$Y(Z) - 4Z^{-1}Y(Z) + 4Z^{-2}Y(Z) = Z^{-1}X(Z)$$

$$\therefore Y(Z)[1 - 4Z^{-1} + 4Z^{-2}] = Z^{-1}X(Z)$$

$$\text{Now } H(Z) = \frac{Y(Z)}{X(Z)}$$

$$\therefore H(Z) = \frac{Z^{-1}}{1 - 4Z^{-1} + 4Z^{-2}} \quad ...(1)$$

This is the transfer function of system.

Converting Equation (1) into positive power of Z.

$$H(Z) = \frac{1}{Z^2 - 4Z + 4} \quad ...(2)$$

$$\frac{H(Z)}{Z} = \frac{1}{Z^2 - 4Z + 4} = \frac{1}{(Z-2)^2}$$

$$\therefore H(Z) = \frac{1}{2} \cdot \frac{2Z}{(Z-2)^2}$$

We have $Z\{n a^n u(n)\} = \frac{aZ}{(Z-a)^2}$

$$\therefore h(n) = \frac{1}{2} \cdot n (2)^n u(n) \quad ...(3)$$

This is impulse response of system.

We have, $H(Z) = \frac{Z}{Z^2 - 4Z + 4} = \frac{Z-0}{(Z-2)^2}$

Thus zero is at origin and there are two poles at '2'.

The pole zero diagram is shown in Fig. P. 8.12.2.

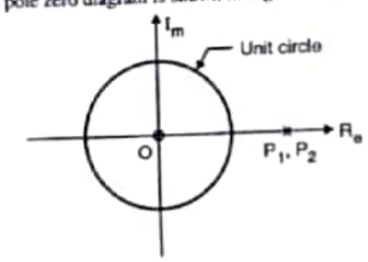


Fig. P. 8.12.2

Ex. 8.12.3 : A DT system is represented as,

$y(n) = 0.5\, y(n-1) - 0.06\, y(n-2) + x(n)$ with $y(-1) = y(-2) = 1$ as initial condition.

Find the response of system in a closed form for an input $x(n) = (0.2)^n u(n)$.

Soln. : Considering initial conditions, using shifting property of unilateral Z-transform we have,

$$x(n-K) \xleftrightarrow{Z} Z^{-K}\left[X(Z) + \sum_{n=1}^{K} x(-n) Z^n\right] \quad ...(1)$$

Given difference is,

$$y(n) = 0.5\, y(n-1) - 0.06\, y(n-2) + x(n)$$

$$Y(Z) = 0.5 Z^{-1}[Y(Z) + y(-1) Z^1] - 0.06 Z^{-2}\left[Y(Z) + \sum_{n=1}^{2} y(-n) Z^n\right] + X(Z)$$

$$\therefore Y(Z) = 0.5 Z^{-1}[Y(Z) + y(-1) Z^1] - 0.06 Z^{-2}[Y(Z) + y(-1) Z + y(-2) Z^2] + X(Z) \quad ...(2)$$

Given initial conditions are,

$$y(-1) = y(-2) = 1.$$

Putting these values in Equation (2) we get,

$$Y(Z) = 0.5 Z^{-1}[Y(Z) + Z^{+1}] - 0.06 Z^{-2}[Y(Z) + Z + Z^2] + X(Z)$$

$$\therefore Y(Z) = 0.5 Z^{-1} Y(Z) + 0.5 - 0.06 Z^{-2} Y(Z) - 0.06 Z^{-1} - 0.06 + X(Z)$$

$$\therefore Y(Z) - 0.5 Z^{-1} Y(Z) + 0.06 Z^{-2} Y(Z) = 0.44 - 0.06 Z^{-1} + X(Z) \quad ...(3)$$

Given input is $x(n) = (0.2)^n u(n)$

$$\therefore X(Z) = \frac{Z}{Z - 0.2}$$

Putting this value in Equation (3) we get,

$$Y(Z) - 0.5 Z^{-1} Y(Z) + 0.06 Z^{-2} Y(Z) = 0.44 - 0.06 Z^{-1} + \frac{Z}{Z-0.2}$$

$$\therefore Y(Z)\left[1 - 0.5 Z^{-1} + 0.06 Z^{-2}\right] = 0.44 - 0.06 Z^{-1} + \frac{Z}{Z-0.2}$$

$$\therefore Y(Z) = \frac{0.44}{1 - 0.5 Z^{-1} + 0.06 Z^{-2}} - \frac{0.06 Z^{-1}}{1 - 0.5 Z^{-1} + 0.06 Z^{-2}} + \left(\frac{Z}{Z-0.2}\right)\left(\frac{1}{1 - 0.5 Z^{-1} + 0.06 Z^{-2}}\right)$$

$$\therefore Y(Z) = \frac{0.44 Z^2}{Z^2 - 0.5Z + 0.06} - \frac{0.06Z}{Z^2 - 0.5Z + 0.06} + \left(\frac{Z}{Z-0.2}\right)\left(\frac{Z^2}{Z^2 - 0.5Z + 0.06}\right) \quad ...(4)$$

Let $Y(Z) = Y_1(Z) - Y_2(Z) + Y_3(Z)$ ...(5)

We will obtain IZT of each term separately.

Here $Y_1(Z) = \frac{0.44 Z^2}{Z^2 - 0.5Z + 0.06}$

The roots of denominator term are as follows :

$$\text{Roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0.5 \pm \sqrt{(0.5)^2 - 4(0.06)}}{2} = \frac{0.5 \pm 0.1}{2}$$

$$\therefore P_1, P_2 = 0.3, 0.2.$$

$$\therefore \frac{Y_1(Z)}{Z} = \frac{0.44Z}{(Z-0.3)(Z-0.2)} \quad ...(6)$$

In PFE form Equation (6) can be written as,

$$\frac{Y_1(Z)}{Z} = \frac{A_1}{Z-0.3} + \frac{A_2}{Z-0.2} \quad ...(7)$$

Here $A_1 = (Z-0.3)\left.\frac{Y_1(Z)}{Z}\right|_{Z=0.3}$

$$\therefore A_1 = (Z-0.3) \cdot \left.\frac{0.44Z}{(Z-0.3)(Z-0.2)}\right|_{Z=0.3}$$

$$\therefore A_1 = \frac{0.44(0.3)}{(0.3-0.2)} = 1.32$$

and $A_2 = (Z-0.2)\left.\frac{Y_1(Z)}{Z}\right|_{Z=0.2}$

$$\therefore A_2 = (Z-0.2) \cdot \left.\frac{0.44Z}{(Z-0.3)(Z-0.2)}\right|_{Z=0.2}$$

$$\therefore A_2 = \frac{0.44(0.2)}{(0.2-0.3)} = -0.88$$

$$\therefore \frac{Y_1(Z)}{Z} = \frac{1.32}{Z-0.3} - \frac{0.88}{Z-0.2}$$

$$\therefore Y_1(Z) = \frac{1.32Z}{Z-0.3} - \frac{0.88Z}{Z-0.2} \quad ...(8)$$

Taking IZT of Equation (8) we get,

$$y_1(n) = 1.32(0.3)^n u(n) - 0.88(0.2)^n u(n) \quad ...(9)$$

Consider the term $Y_2(Z)$.

$$Y_2(Z) = \frac{0.06Z}{Z^2 - 0.5Z + 0.06}$$

$$\therefore \frac{Y_2(Z)}{Z} = \frac{0.06}{(Z-0.3)(Z-0.2)} \quad ...(10)$$

In PFE form we can write,

$$\frac{Y_2(Z)}{Z} = \frac{A_1}{Z-0.3} + \frac{A_2}{Z-0.2}$$

Here $A_1 = (Z-0.3)\left.\frac{Y_2(Z)}{Z}\right|_{Z=0.3}$

$$\therefore A_1 = \left.\frac{(Z-0.3)(0.06)}{(Z-0.3)(Z-0.2)}\right|_{Z=0.3}$$

$$\therefore A_1 = \frac{0.06}{0.3-0.2} = 0.6$$

and $A_2 = (Z-0.2)\left.\frac{Y_2(Z)}{Z}\right|_{Z=0.2}$

$$\therefore A_2 = (Z-0.2)\left.\frac{(0.06)}{(Z-0.3)(Z-0.2)}\right|_{Z=0.2}$$

$$\therefore A_2 = \frac{0.06}{0.2-0.3} = -0.6$$

$$\therefore \frac{Y_2(Z)}{Z} = \frac{0.6}{Z-0.3} - \frac{0.6}{Z-0.2}$$

$$\therefore Y_2(Z) = \frac{0.6Z}{Z-0.3} - \frac{0.6Z}{Z-0.2}$$

Taking IZT we get,

$$y_2(n) = 0.6(0.3)^n u(n) - 0.6(0.2)^n u(n) \quad ...(11)$$

Now consider the term $Y_3(Z)$.

$$Y_3(Z) = \left(\frac{Z}{Z-0.2}\right)\left(\frac{Z^2}{Z^2 - 0.5Z + 0.06}\right)$$

$$\therefore \frac{Y_3(Z)}{Z} = \frac{Z^2}{(Z-0.2)(Z-0.3)(Z-0.2)}$$

$$\therefore \frac{Y_3(Z)}{Z} = \frac{Z^2}{(Z-0.2)^2(Z-0.3)} \quad ...(12)$$

This is the case of repeated poles. In P.F.E. we can write,

$$\frac{Y_3(Z)}{Z} = \frac{A_1}{Z-0.3} + \frac{A_2}{Z-0.2} + \frac{A_3}{(Z-0.2)^2} \quad ...(13)$$

Here $A_1 = (Z-0.3)\left.\frac{Y_3(Z)}{Z}\right|_{Z=0.3}$

$$\therefore A_1 = (Z-0.3) \cdot \left.\frac{Z^2}{(Z-0.2)^2(Z-0.3)}\right|_{Z=0.3}$$

$$\therefore A_1 = \frac{(0.3)^2}{(0.3-0.2)^2} = 9$$

$$A_2 = \frac{d}{dZ}\left[(Z-0.2)^2 \frac{Y_3(Z)}{Z}\right]\Bigg|_{Z=0.2}$$

$$\therefore A_2 = \frac{d}{dZ}\left[(Z-0.2)^2 \cdot \frac{Z^2}{(Z-0.2)^2(Z-0.3)}\right]\Bigg|_{Z=0.2}$$

$$\therefore A_2 = \frac{d}{dZ}\left[Z^2 \cdot (Z-0.3)^{-1}\right]\Big|_{Z=0.2}$$

$$A_2 = \left[Z^2 \cdot (-1)(Z-0.3)^{-2} + (Z-0.3)^{-1} \cdot 2Z\right]\Big|_{Z=0.2}$$

$$\therefore A_2 = \left[\frac{-Z^2}{(Z-0.3)^2} + \frac{2Z}{Z-0.3}\right]\Bigg|_{Z=0.2}$$

$$\therefore A_2 = \frac{-(0.2)^2}{(0.2-0.3)^2} + \frac{2(0.2)}{(0.2-0.3)} = -4 - 4 = -8$$

and $A_3 = (Z-0.2)^2 \left.\frac{Y_3(Z)}{Z}\right|_{Z=0.2}$

$$\therefore A_3 = (Z-0.2)^2 \cdot \left.\frac{Z^2}{(Z-0.2)^2(Z-0.3)}\right|_{Z=0.2}$$

$$\therefore A_3 = \frac{(0.2)^2}{(0.2-0.3)} = -0.4$$

Putting these values in Equation (13),

$$\frac{Y_3(Z)}{Z} = \frac{9}{Z-0.3} - \frac{8}{Z-0.2} - \frac{0.4}{(Z-0.2)^2}$$

$$\therefore Y_3(Z) = \frac{9Z}{Z-0.3} - \frac{8Z}{Z-0.2} - \frac{0.4Z}{(Z-0.2)^2}$$

For the last term we have to use the Z-transform pair

$$n a^n u(n) \xleftrightarrow{Z} \frac{aZ}{(Z-a)^2}$$

$$\therefore Y_3(Z) = \frac{9Z}{Z-0.3} - \frac{8Z}{Z-0.2} - \frac{4}{0.2} \cdot \frac{(0.2Z)}{(Z-0.2)^2}$$

Taking IZT,

$$y_3(n) = 9(0.3)^n u(n) - 8(0.2)^n u(n) - 2n(0.2)^n u(n) \quad ...(14)$$

Now final output is,

$$y(n) = y_1(n) + y_2(n) + y_3(n)$$

Ex. 8.12.4 : A system is described by difference equation

$$y(n) = \frac{1}{2} y(n-1) + x(n)$$

Input given to system is $\left(\frac{1}{3}\right)^n u(n)$ and initial condition is $y(-1) = 1$.

Determine :

1. Zero input response
2. Zero state response
3. Total response.

Soln. :

1. Zero input response is the response of system when input is zero. That means this is the response due to initial conditions only. When initial conditions are present, we have to apply the shifting property of unilateral Z-transform. It is,

$$x(n-K) \xleftrightarrow{Z} Z^{-K}\left[X(Z) + \sum_{n=1}^{K} x(-n) Z^{n}\right] \quad ...(1)$$

Given difference equation is,

$$y(n) = \frac{1}{2} y(n-1) + x(n) \quad ...(2)$$

Applying Equation (1) to Equation (2) we get,

$$Y(Z) = \frac{1}{2} Z^{-1}[Y(Z) + y(-1) Z^{1}] + X(Z)$$

But initial condition is $y(-1) = 1$

$$Y(Z) = \frac{1}{2} Z^{-1}[Y(Z) + Z] + X(Z) \quad ...(3)$$

2. For zero input response, x (n) should be zero. That means X (Z) is zero. Thus Equation (3) becomes,

$$Y(Z) = \frac{1}{2} Z^{-1}[Y(Z) + Z]$$

$$\therefore Y(Z) = \frac{1}{2} Z^{-1} Y(Z) + \frac{1}{2}$$

$$\therefore Y(Z) - \frac{1}{2} Z^{-1} Y(Z) = \frac{1}{2}$$

$$\therefore Y(Z)\left[1 - \frac{1}{2} Z^{-1}\right] = \frac{1}{2}$$

$$\therefore Y(Z) = \frac{1}{2} \cdot \frac{1}{1 - \frac{1}{2} Z^{-1}} = \frac{1}{2} \cdot \frac{1}{Z - \frac{1}{2}} \quad ...(4)$$

IZT of Equation (4) can be written as,

$$y(n) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n} u(n)$$

This is the zero input response of system.

3. Zero state response is the response of system when, initial conditions are zero and input is $x(n) = \left(\frac{1}{3}\right)^{n} u(n)$.

The given difference equation is,

$$y(n) = \frac{1}{2} y(n-1) + x(n)$$

Taking Z-transform without considering initial conditions,

$$Y(Z) = \frac{1}{2} Z^{-1} Y(Z) + X(Z) \quad ...(5)$$

$$\text{But } X(Z) = Z\left\{\left(\frac{1}{3}\right)^{n} u(n)\right\} = \frac{Z}{Z - \frac{1}{3}}$$

$$\therefore Y(Z) = \frac{1}{2} Z^{-1} Y(Z) + \frac{Z}{Z - \frac{1}{3}}$$

$$\therefore Y(Z) - \frac{1}{2} Z^{-1} Y(Z) = \frac{Z}{Z - \frac{1}{3}}$$

$$\therefore Y(Z)\left[1 - \frac{1}{2} Z^{-1}\right] = \frac{Z}{Z - \frac{1}{3}}$$

$$\therefore Y(Z) = \frac{1}{1 - \frac{1}{2} Z^{-1}} \cdot \frac{Z}{Z - \frac{1}{3}}$$

$$\therefore Y(Z) = \frac{Z}{Z - \frac{1}{2}} \cdot \frac{Z}{Z - \frac{1}{3}} = \frac{Z^2}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{3}\right)}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{Z}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{3}\right)} \quad ...(6)$$

In PFE form Equation (6) can be written as,

$$\frac{Y(Z)}{Z} = \frac{A_1}{Z - \frac{1}{2}} + \frac{A_2}{Z - \frac{1}{3}} \quad ...(7)$$

The values of A_1 and A_2 are calculated as follows :

$$A_1 = \left(Z - \frac{1}{2}\right) \cdot \frac{Y(Z)}{Z}\Bigg|_{Z = \frac{1}{2}}$$

$$\therefore A_1 = \left(Z - \frac{1}{2}\right) \cdot \frac{Z}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{3}\right)}\Bigg|_{Z = \frac{1}{2}}$$

$$\therefore A_1 = \frac{\frac{1}{2}}{\frac{1}{2} - \frac{1}{3}} = \frac{\frac{1}{2}}{\frac{1}{6}} = 3$$

$$\text{and } A_2 = \left(Z - \frac{1}{3}\right) \cdot \frac{Y(Z)}{Z}\Bigg|_{Z = \frac{1}{3}}$$

$$\therefore A_2 = \left(Z - \frac{1}{3}\right) \cdot \frac{Z}{\left(Z - \frac{1}{2}\right)\left(Z - \frac{1}{3}\right)}\Bigg|_{Z = \frac{1}{3}}$$

$$\therefore A_2 = \frac{1/3}{\frac{1}{3} - \frac{1}{2}} = -2$$

Putting these values in Equation (7) we get,

$$\frac{Y(Z)}{Z} = \frac{3}{Z - \frac{1}{2}} - \frac{2}{Z - \frac{1}{3}}$$

$$\therefore Y(Z) = \frac{3Z}{Z - \frac{1}{2}} - \frac{2Z}{Z - \frac{1}{3}} \quad ...(8)$$

IZT of Equation (8) can be written as,

$$y(n) = 3 \cdot \left(\frac{1}{2}\right)^{n} u(n) - 2\left(\frac{1}{3}\right)^{n} u(n)$$

This is zero state response of system.

4. Total response is addition of zero input response (ZIR) and zero state response (ZSR).

$$\therefore y(n) = ZIR + ZSR$$

$$\therefore y(n) = \frac{1}{2} \cdot \left(\frac{1}{2}\right)^{n} u(n) + 3\left(\frac{1}{2}\right)^{n} u(n) - 2\left(\frac{1}{3}\right)^{n} u(n)$$

This is the total response of system.

Ex. 8.12.5 : Solve the difference equation
$x(n-z) - 9x(n-1) + 18x(n) = 0$
with IC's $x(-1) = 1$ and $x(-2) = 9$

May 13. 10 Marks

Soln. :

Given difference equation is,

$$x(n-2) - 9x(n-1) + 18x(n) = 0$$

Taking unilateral z transform of both sides we get,

$$\bar{z}^{2}\left[x(z) + \sum_{n=1}^{2} x(-n) z^{n}\right] - 9\bar{z}^{1}\left[x(z) + \sum_{n=1}^{1} x(-n) z^{n}\right] + 18x(z) = 0$$

$$\therefore \bar{z}^{2}[x(z) + x(-1) z + x(-2) z^{2}] - 9\bar{z}^{1}[x(z) + x(-1) z] + 18x(z) = 0$$

Now $x(-1) = 1$ and $x(-2) = 9$

$$\therefore \bar{z}^{2}[x(z) + z + 9z^{2}] - 9\bar{z}^{1}[x(z) + z] + 18x(z) = 0$$

$$\therefore \bar{z}^{2} x(z) + \bar{z}^{1} + 9 - 9\bar{z}^{1} x(z) - 9 + 18x(z) = 0$$

$$\therefore \bar{z}^{2} x(z) - 8\bar{z}^{1} + 18x(z) = 0$$

$$\therefore x(z)\left[\bar{z}^{2} + 18\right] = 8\bar{z}^{1}$$

$$\therefore x(z) = \frac{8\bar{z}^{1}}{\bar{z}^{2} + 18}$$

Multiplying and dividing by z^2 we get,

$$x(z) = \frac{8z}{1 + 18z^{2}}$$

$$\therefore \frac{x(z)}{z} = \frac{8z}{1 + 18z^{2}} = \frac{8}{18} \cdot \frac{1}{z^{2} + \frac{1}{18}}$$

$$\therefore \frac{x(z)}{z} = \frac{8}{18} \cdot \frac{1}{(z - 0.2357j)(z + 0.2357j)}$$

In P.F.E. we can write,

$$\frac{x(z)}{z} = \frac{A_1}{(z - 0.2357j)} + \frac{A_2}{(z + 0.2357j)}$$

Here, $A_1 = (z - 0.2357j) \cdot$

$$\left[\frac{8}{18} \cdot \frac{1}{(z - 0.2357j)(z + 0.2357j)}\right]\Bigg|_{z = 0.2357j}$$

$$\therefore A_1 = \frac{8}{18}\left[\frac{1}{0.2357j + 0.2357j}\right] = -0.9428j$$

$$\text{and } A_2 = (z + 0.2357j)\left[\frac{8}{18} \cdot \frac{1}{(z - 0.2357j)(z + 0.2357j)}\right]\Bigg|_{z = -0.2357j}$$

$$\therefore A_2 = \frac{8}{18}\left[\frac{1}{-0.2357j - 0.2357j}\right] = 0.9428j$$

$$\therefore \frac{x(z)}{z} = \frac{-0.9428j}{z - 0.2357j} + \frac{0.9428j}{z + 0.2357j}$$

$$\therefore x(z) = -0.9428j \frac{z}{z - 0.2357j} + 0.9428j \frac{z}{z + 0.2357j}$$

Taking IZT we get,

$$x(n) = -0.9428j \cdot (0.2357j)^{n} u(n) + 0.9428j(-0.2357j)^{n}$$

Converting into polar form we get,

$$x(n) = 0.9428\angle -90^{\circ}(0.2357\angle 90^{\circ})^{n} u(n) + 0.9428\angle 90(0.2357\angle -90^{\circ})^{n} u(n)$$

The term, magnitude $\angle$ Angle° can be written as, magnitude $e^{j\,\text{Angle}}$

$$\therefore x(n) = (0.9428 e^{-90j})(0.2357 e^{90j})^{n} u(n) + 0.9428 e^{90j}(0.2357 e^{-90j})^{n} u(n)$$

$$\therefore x(n) = 0.9428[e^{-90j} \cdot (0.2357)^{n} \cdot e^{90jn} \cdot u(n) + e^{90j}(0.2357)^{n} e^{-90jn} u(n)]$$

$$\therefore x(n) = 0.9428(0.2357)^{n}[e^{90j(n-1)} + e^{-90j(n-1)}] \cdot u(n)$$

Now we have, $e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$$\therefore x(n) = 0.9428(0.2357)^{n} \cdot 2\cos[90(n-1)] \cdot u(n)$$

$$\therefore x(n) = 1.8856(0.2357)^{n}\cos[90(n-1)]\, u(n)$$

Ex. 8.12.6 : Using the Z transform method, solve the difference equation :
$y[n] - 4y[n-1] + 4y[n-2] = x[n] - x[n-1]$
When $y(-1) = y(-2) = 0$

May 2015. 10 Marks

Soln. :

Since initial conditions are zero, we will use bilateral z transform.

Given :

$$y[n] - 4y[n-1] + 4y[n-2] = x[n] - x[n-1]$$

Calculation of zero input response :

Input is zero. Thus $x(n) = 0$ $\therefore X(Z) = 0$

Taking Z transform of given equation,

$$Y(Z) - 4Z^{-1} Y(Z) + 4Z^{-2} Y(Z) = 0$$

$$\therefore Y(Z)[1 - 4Z^{-1} + 4Z^{-2}] = 0$$

$$\therefore Y(Z) = 0$$

$$\therefore Y(n) = 0$$

Thus if input is zero ; output is also zero because initial conditions are zero.

Calculation of zero state response :

Input is not given. So consider that input is unit step.

$$\therefore x(n) = u(n) \text{ and } X(Z) = \frac{Z}{Z - 1}$$

Taking Z transform of given equation,

$$\therefore Y(Z) - 4Z^{-1} Y(Z) + 4Z^{-2} Y(Z) = X(Z) - Z^{-1} X(Z)$$

$$\therefore Y(Z)[1 - 4Z^{-1} + 4Z^{-2}] = X(Z)[1 - Z^{-1}]$$

$$\therefore Y(Z)[1 - 4Z^{-1} + 4Z^{-2}] = \frac{Z}{Z - 1} \cdot [1 - Z^{-1}]$$

$$\therefore Y(Z) = \frac{Z}{Z - 1} \cdot \frac{1 - Z^{-1}}{1 - 4Z^{-1} + 4Z^{-2}} = \frac{Z}{Z - 1} \cdot \frac{Z^{2} - Z}{Z^{2} - 4Z + 4}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{Z^{2} - Z}{(Z - 1)(Z - 2)^{2}}$$

$$= \frac{Z(Z - 1)}{(Z - 1)(Z - 2)^{2}} = \frac{Z}{(Z - 2)^{2}}$$

$$\therefore \frac{Y(Z)}{Z} = \frac{A}{Z - 2} + \frac{B}{(Z - 2)^{2}}$$

$$\text{Here } A = \frac{d}{dZ}\left[(Z - 2)^{2} \cdot \frac{Z}{(Z - 2)^{2}}\right]\Bigg|_{Z=2} = 1$$

$$B = (Z - 2)^{2} \cdot \frac{Z}{(Z - 2)^{2}}\Bigg|_{Z=2}$$

$$\therefore B = 2$$

$$\therefore \frac{Y(Z)}{Z} = \frac{1}{Z - 2} + \frac{2}{(Z - 2)^{2}}$$

$$\therefore Y(Z) = \frac{Z}{Z - 2} + \frac{2Z}{(Z - 2)^{2}}$$

Taking inverse Z transform,

$$y(n) = (2)^{n} u(n) + n(2)^{n} u(n)$$

8.13 Numericals Based on Calculation of Frequency Response :

Ex. 8.13.1 : Find the magnitude response and impulse response of a system described by the difference equation.

$$y(n)-\frac{1}{2}y(n-1)=x(n)-\frac{1}{4}x(n-1).$$

Soln. :

Calculation of impulse response : The given difference equation is,

$$y(n)-\frac{1}{2}y(n-1)=x(n)-\frac{1}{4}x(n-1) \qquad ...(1)$$

Taking Z-transform

$$Y(Z)-\frac{1}{2}Z^{-1}Y(Z)=X(Z)-\frac{1}{4}Z^{-1}X(Z)$$

$$\therefore Y(Z)\left[1-\frac{1}{2}Z^{-1}\right]=X(Z)\left[1-\frac{1}{4}Z^{-1}\right]$$

$$\therefore \frac{Y(Z)}{X(Z)}=H(Z)=\frac{1-\frac{1}{4}Z^{-1}}{1-\frac{1}{2}Z^{-1}} \qquad ...(2)$$

We will obtain impulse response by taking IZT of Equation (2). This is simple equation; so just by rearranging Equation (2) we can obtain IZT as follows :

$$H(Z)=\frac{1}{1-\frac{1}{2}Z^{-1}}-\frac{\frac{1}{4}Z^{-1}}{1-\frac{1}{2}Z^{-1}}$$

$$\therefore H(Z)=\frac{Z}{Z-\frac{1}{2}}-\frac{1}{4}Z^{-1}\cdot\frac{Z}{Z-\frac{1}{2}} \qquad ...(3)$$

Taking IZT of Equation (3),

$$h(n)=\left(\frac{1}{2}\right)^{n}u(n)-\frac{1}{4}\left(\frac{1}{2}\right)^{n-1}u(n-1)$$

Calculation of magnitude response

The frequency response is obtained by putting $Z=e^{j\omega}$ in the equation of $H(Z)$. The equation of $H(Z)$ is,

$$H(Z)=\frac{1-\frac{1}{4}Z^{-1}}{1-\frac{1}{2}Z^{-1}} \quad \therefore H(\omega)=\frac{1-\frac{1}{4}e^{-j\omega}}{1-\frac{1}{2}e^{-j\omega}} \qquad ...(4)$$

We will obtain different values of $H(\omega)$ by putting values of 'ω' in Equation (4) as follows :

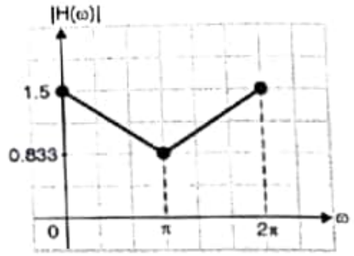
$$\text{At } \omega=0 \Rightarrow H(\omega)=\frac{1-\frac{1}{4}e^{0}}{1-\frac{1}{2}e^{0}}=\frac{1-\frac{1}{4}}{1-\frac{1}{2}}=1.5$$

$$\text{At } \omega=\pi \Rightarrow H(\omega)=\frac{1-\frac{1}{4}e^{-j\pi}}{1-\frac{1}{2}e^{-j\pi}}$$

$$=\frac{1-\frac{1}{4}(\cos\pi-j\sin\pi)}{1-\frac{1}{2}(\cos\pi-j\sin\pi)}=0.833$$

$$\text{At } \omega=2\pi \Rightarrow H(\omega)=\frac{1-\frac{1}{4}e^{-j2\pi}}{1-\frac{1}{2}e^{-j2\pi}}$$

$$=\frac{1-\frac{1}{4}(\cos 2\pi-j\sin 2\pi)}{1-\frac{1}{2}(\cos 2\pi-j\sin 2\pi)}=1.5$$



(II-1038) **Fig. P. 8.13.1 : Magnitude response**

The magnitude response is a plot of ω vs. $H(\omega)$. It is shown in Fig. P. 8.13.1.

Ex. 8.13.2 : LTI system is described by

$h(n)=(0.9)^{n}u(n)$. Plot magnitude response of system using analytical method.

Soln. :

Given $\quad h(n)=(0.9)^{n}u(n)$

We have standard Z-transform pair

$$a^{n}u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-a} \text{ ROC}: |Z|>|a|$$

Here a = 0.9 $\quad \therefore H(Z)=\dfrac{Z}{Z-0.9}$

$$\therefore H(Z)=\frac{1}{1-0.9Z^{-1}} \qquad ...(1)$$

$H(\omega)$ is obtained by putting $Z=e^{j\omega}$ in Equation (1)

$$\therefore H(\omega)=\frac{1}{1-0.9e^{-j\omega}}$$

$$\therefore H(\omega)=\frac{1}{1-0.9(\cos\omega-j\sin\omega)}$$

$$\therefore H(\omega)=\frac{1}{1-0.9\cos\omega-j\,0.9\sin\omega}$$

$$\therefore |H(\omega)|=\frac{1}{\sqrt{(1-0.9\cos\omega)^2+(0.9\sin\omega)^2}}$$

$$\therefore |H(\omega)|=\frac{1}{\sqrt{1-1.8\cos\omega+0.81\cos^2\omega+0.81\sin^2\omega}}$$

$$\therefore |H(\omega)|=\frac{1}{\sqrt{1-1.8\cos\omega+0.81(\cos^2\omega+\sin^2\omega)}}$$

$$\therefore |H(\omega)|=\frac{1}{\sqrt{1-1.8\cos\omega+0.81}} \qquad ...(2)$$

For $\omega=-\pi$ $\quad |H(\omega)|=\dfrac{1}{\sqrt{1-1.8\cos(-\pi)+0.81}}=0.52$

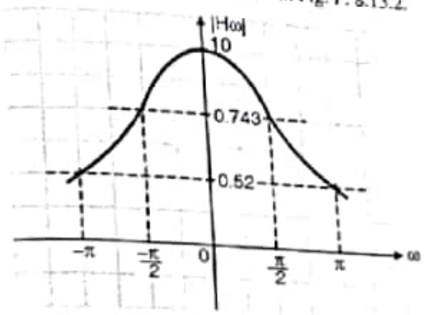
For $\omega=-\dfrac{\pi}{2}$, $\quad |H(\omega)|=\dfrac{1}{\sqrt{1-1.8\cos\left(\frac{-\pi}{2}\right)+0.81}}=0.743$

For $\omega=0$, $\quad |H(\omega)|=\dfrac{1}{\sqrt{1-1.8\cos(0)+0.81}}=10$

For $\omega=\dfrac{\pi}{2}$, $\quad |H(\omega)|=\dfrac{1}{\sqrt{1-1.8\cos\left(\frac{\pi}{2}\right)+0.81}}=0.743$

For $\omega=\pi$, $\quad H(\omega)=\dfrac{1}{\sqrt{1-1.8\cos(\pi)+0.81}}=0.52.$

The magnitude response is as shown in Fig. P. 8.13.2.



(II-1039) **Fig. P. 8.13.2**

Ex. 8.13.3 : Find magnitude and phase response of given system at frequency $\omega=0$ and $\omega=\pi$

$$H(Z)=\frac{1}{1-0.5Z^{-1}}.$$

Soln. : The magnitude response is obtained by putting $Z=e^{+j\omega}$

$$\therefore H(e^{j\omega})=\frac{1}{1-0.5e^{-j\omega}} \qquad ...(1)$$

But $\quad e^{-j\omega}=\cos\omega-j\sin\omega$

$$\therefore H(e^{j\omega})=\frac{1}{1-0.5(\cos\omega-j\sin\omega)}$$

$$=\frac{1}{1-0.5\cos\omega+0.5j\sin\omega} \qquad ...(2)$$

$$\therefore |H(e^{j\omega})|=\frac{1}{\sqrt{(1-0.5\cos\omega)^2+(0.5\sin\omega)^2}}$$

$$\therefore H(e^{j\omega})|=\frac{1}{\sqrt{1-\cos\omega+(0.5)^2\cos^2\omega+(0.5)^2\sin^2\omega}}$$

$$=\frac{1}{\sqrt{1-\cos\omega+(0.5)^2\cdot(\sin^2\omega+\cos^2\omega)}}$$

$$|H(e^{j\omega})|=\frac{1}{\sqrt{1+0.25-\cos\omega}} \qquad ...(3)$$

$$\text{At } \omega=\pi \in |H(e^{j\omega})|=\frac{1}{\sqrt{1+0.25-\cos\pi}}=0.66.$$

From Equation (2), the phase response is given by,

$$\angle H(e^{j\omega})=\tan^{-1}0-\tan^{-1}\frac{0.5\sin\omega}{1-0.5\cos\omega}$$

$$\text{At } \omega=0 \Rightarrow \angle H(e^{j\omega})$$

$$=\tan^{-1}0-\tan^{-1}\frac{0.5\sin 0}{1-0.5\cos 0}=0$$

$$\text{At } \omega=\pi \Rightarrow \angle H(e^{j\omega})$$

$$=\tan^{-1}0-\tan^{-1}\frac{0.5\sin\pi}{1-0.5\cos\pi}=0.$$

Ex. 8.13.4 : A system has unit sample response h (n) given by

$$h(n)=-\frac{1}{4}\delta(n+1)+\frac{1}{2}\delta(n)-\frac{1}{4}\delta(n-1)$$

1. Is the system BIBO stable ?
2. Is the filter causal ?
3. Compute the frequency response.

Soln. :

1. Given : $h(n)=-\frac{1}{4}\delta(n+1)+\frac{1}{2}\delta(n)-\frac{1}{4}\delta(n-1)$

Taking Z transform we get,

$$H(Z)=-\frac{1}{4}Z+\frac{1}{2}-\frac{1}{4}Z^{-1}=-\frac{1}{4}Z+\frac{1}{2}-\frac{1}{4Z}$$

$$\therefore H(Z)=\frac{-Z^2+2Z-1}{4Z}=\frac{-1(Z^2-2Z+1)}{4Z}$$

$$\therefore H(Z)=-\frac{1}{4}\cdot\frac{(Z-1)(Z-1)}{(Z-0)}$$

The pole is at origin. So it lies inside unit circle. Thus it is BIBO stable system.

2. Given : $\quad h(n)=-\frac{1}{4}\delta(n+1)+\frac{1}{2}\delta(n)-\frac{1}{4}\delta(n-1)$

$$\therefore h(n)=\left\{-\frac{1}{4},\underset{\uparrow}{\frac{1}{2}},-\frac{1}{4}\right\}$$

This is anticausal sequence; so filter is not causal.

3. We have, $\quad H(Z)=\dfrac{-Z^2+2Z-1}{4Z}$

Converting into negative powers of Z we get,

$$H(Z)=\frac{-1+2Z^{-1}-Z^{-2}}{4Z^{-1}}$$

To obtain frequency response, put $Z=e^{j\omega}$

$$\therefore H(\omega)=\frac{-1+2e^{-j\omega}-e^{-2j\omega}}{4e^{-j\omega}}$$

$$\text{At } \omega=0 \Rightarrow H(\omega)=\frac{-1+2-1}{4}=0$$

$$\text{At } \omega=\pi \Rightarrow$$

$$H(\omega)=\frac{-1+2e^{-j\pi}-e^{-2j\pi}}{4e^{-j\pi}}=\frac{-1+2(-1+j0)-(1+j0)}{4(-1+j0)}$$

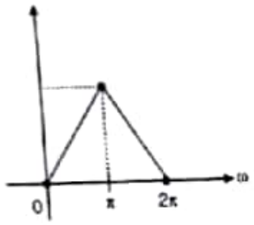
$$\therefore |H(\omega)|=\frac{-1-2-1}{-4}=1$$

$$\text{At } \omega=2\pi \Rightarrow H(\omega)=\frac{-1+2e^{-j2\pi}-e^{-j4\pi}}{4e^{-j2\pi}}$$

$$=\frac{-1+2(1+j0)-(1+j0)}{4(1+j0)}$$

$$\therefore |H(\omega)|=\frac{-1+2-1}{4}=0$$

It is shown in Fig. P. 8.13.4.



(II-1534) **Fig. P. 8.13.4**

8.14 Frequency Response from Pole-Zero Plot using Simple Geometric Construction :

To sketch the frequency response functions of a system from pole-zero plot, geometric construction is very simplest method.

Principle :

Frequency response corresponds to the system function evaluated on the unit circle in the Z-plane and that the complex value of each pole and zero factor can be represented by a vector in the Z-plane from the pole or zero to a point on the unit circle.

Procedure :

1. Find poles and zeros of a given system function and draw pole-zero plot with a unit circle on Z-plane.
2. Draw vectors from all zeros and poles to the same point on unit circle at,

   $Z = re^{j\omega} = e^{j\omega}\ (r = 1)$.

   Vectors from zeros to the unit circle are called zero vectors and vectors from poles to the unit circle are called pole vectors.
3. Find the length of zero vectors and pole vectors. Magnitude response is the product of the lengths of the zero vectors divided by the product of the lengths of pole vectors. With fixed r, sketch magnitude response for different values of θ and ω.
4. Find angle of each zero and pole vector with respect to positive real axis in anticlockwise direction.

   Phase function is the difference between the sum of angles of zero vectors and sum of angles of the pole vectors.

   With fixed r, sketch phase response for different values of θ and ω.

Examples on Geometric Construct :

Ex. 8.14.1 : The difference equation of the system is given as, $y(n) = 2x(n) + x(n-1)$ Find out magnitude and phase response using geometric construction.

Soln. : Given difference equation of the system is,

$$y(n) = 2x(n) + x(n-1)$$

Taking Z-transform on both sides,

$$Y(Z) = 2X(Z) + Z^{-1}X(Z)$$

$$\therefore Y(Z) = X(Z)[2 + Z^{-1}]$$

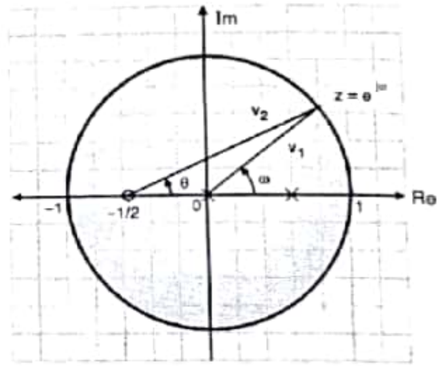
$$\therefore \frac{Y(Z)}{X(Z)} = \left[2 + \frac{1}{Z}\right] = 2\left[1 + \frac{1}{2Z}\right]$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)} = 2\left[\frac{2Z+1}{2Z}\right] \quad \ldots(1)$$

$$= 2\left[\frac{Z + \frac{1}{2}}{Z}\right]$$

where, H (Z) is transfer function.

Step 1 : From Equation (1) :



(H-1535) **Fig. P. 8.14.1**

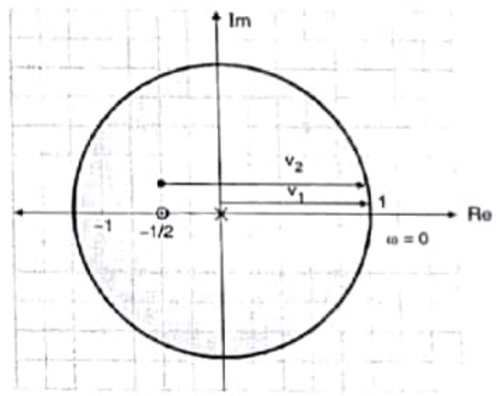
Zero is at $Z = -1/2 \Rightarrow Z_1$ and pole is at $Z = 0 \Rightarrow P_1$

The pole–zero plot is as shown in Fig. P. 8.14.1.

Step 2 : The pole vector and zero vector are shown in Fig. P. 8.14.1(a).

As H (Z) is a function of 'ω', let us consider various values of 'ω' as follows :

Case I : When ω = 0, the pole vector and zero vector are as shown in Fig. P. 8.14.1(a).



(H-1536) **Fig. P. 8.14.1(a)**

$$\therefore |V_1| = 1 \quad \text{and} \quad |V_2| = \left|1 + \frac{1}{2}\right| = \left|\frac{3}{2}\right| = \frac{3}{2}$$

$$\text{and } \angle V_1 = 0 \quad \text{and} \quad \angle V_2 = 0$$

Case II : When $\omega = \pi/2$, the pole vector and zero vector are as shown in Fig. P. 8.14.1(b).

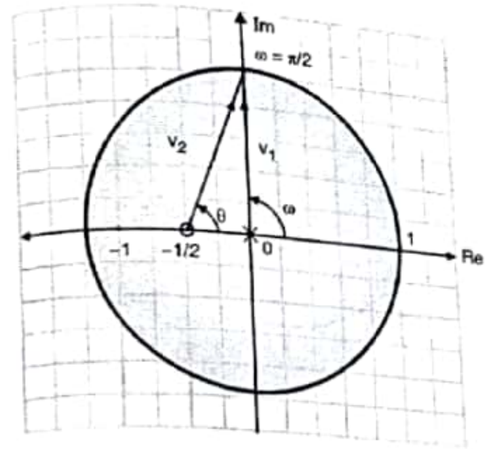
$$\therefore |V_1| = 1,$$

$$|V_2| = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \sqrt{1 + \frac{1}{4}}$$

$$\therefore \quad |V_2| = \sqrt{5}/2$$

$$\angle V_1 = \omega = \pi/2,$$

$$\angle V_2 = \theta = \tan^{-1}\left(\frac{1}{1/2}\right) = 63.43°$$

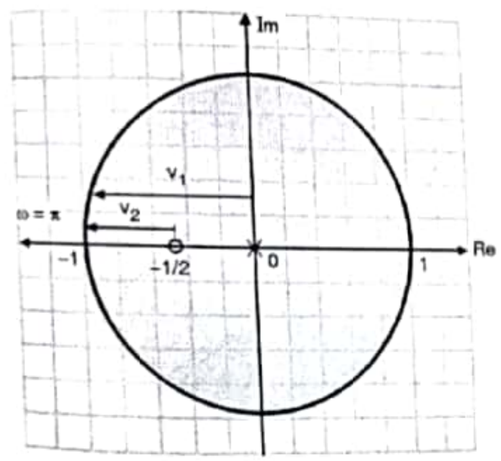


(H-1537) **Fig. P. 8.14.1(b)**

Case III : When ω = π, pole vector and zero vector are as shown in Fig. P. 8.14.1(c).

$$\therefore |V_1| = 1, \qquad |V_2| = \frac{1}{2}$$

$$\angle V_1 = \pi \qquad \text{and } \angle V_2 = \pi$$



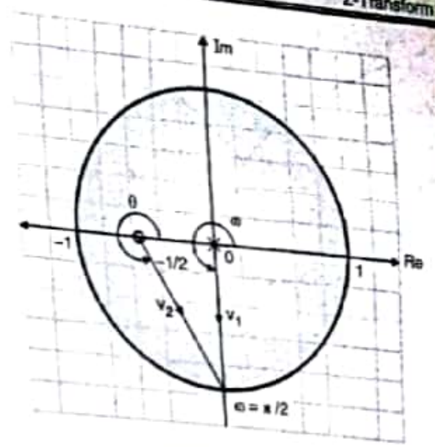
(H-1538) **Fig. P. 8.14.1(c)**

Case IV : When $\omega = 3\pi/2$, pole-zero vectors are as shown in Fig. P. 8.14.1(d)

$$\therefore |V_1| = 1, \quad |V_2| = \left|\sqrt{(1/2)^2 + 1^2}\right| = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$\text{and} \quad \angle V_1 = \omega = \frac{3\pi}{2} \text{ or } \frac{-\pi}{2}$$

$$\angle V_2 = \theta = -\tan^{-1}\left(\frac{1}{1/2}\right) = -63.43°$$



(H-1539) **Fig. P. 8.14.1(d)**

Step 3 : Magnitude response and phase response is given as,

$$\text{At } \omega = 0, |H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{3/2}{1} = 3$$

$$\angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = 0 - 0 = 0$$

$$\text{At } \omega = \frac{\pi}{2}, |H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{\sqrt{5}/2}{1} = \sqrt{5}$$

$$\angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = \theta - \omega = 63.43 - \frac{\pi}{2} = -26.57°$$

$$\text{At } \omega = \pi, |H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{1/2}{1} = 1$$

$$\angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = \theta - \omega = \pi - \pi = 0$$

$$\text{At } \omega = 3\pi/2,$$

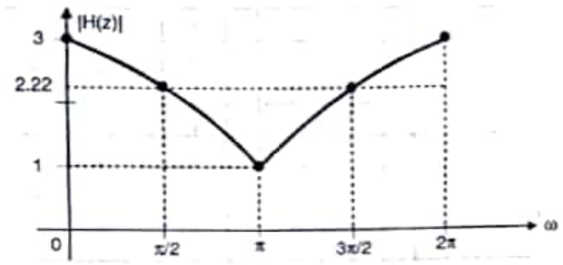
$$|H(e^{j\omega})| = 2\frac{|V_2|}{|V_1|} = 2 \cdot \frac{\sqrt{5}/2}{1} = \sqrt{5}$$

$$\angle H(e^{j\omega}) = \angle V_2 - \angle V_1 = \theta - \omega$$

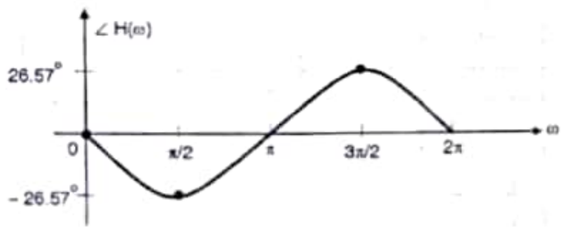
$$= -63.43° - \left(\frac{-\pi}{2}\right) = 26.57°$$

Step 4 : Magnitude and phase response are as shown in Figs. P. 8.14.1(e) and (f) response.

ω	|H(Z)|	∠H(Z)
0	3	0
π/2	$\sqrt{5}$	−26.57°
π	1	0
$\frac{3\pi}{2}$	$\sqrt{5}$	26.57°



(e) Magnitude response



(f) Phase response

(II-1541) Fig. P. 8.14.1

Ex. 8.14.2 : Find and draw magnitude, phase response of the system function

$$H(Z) = \frac{Z+1}{Z(Z-1)}$$

Soln. : The given transfer function is,

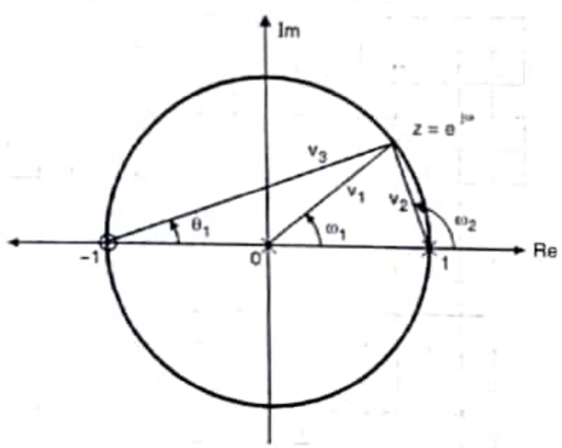
$$H(Z) = \frac{Z+1}{Z(Z-1)}$$

Step 1 : Zero is at $Z_1 = -1$

Poles are at $P_1 = 0$

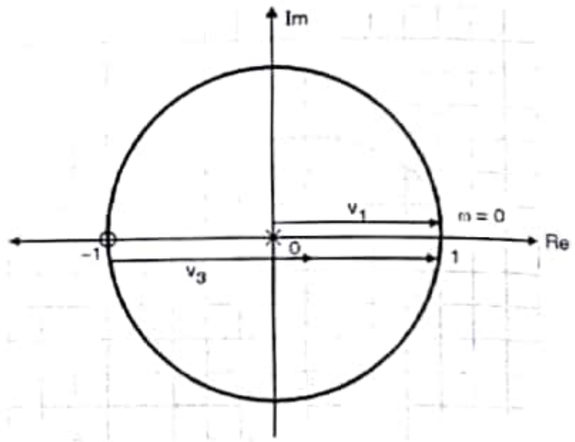
and $P_2 = 1$

The pole-zero plot is as shown in Fig. P. 8.14.2.



(II-1542) Fig. P. 8.14.2

Case I : When $\omega = 0$, pole and zero vectors are as shown in Fig. P. 8.14.2(a).



(II-1543) Fig. P. 8.14.2(a)

$\therefore |V_1| = 1,$

$|V_2| = 0$ and $|V_3| = 1 + 1 = 2$

$\angle V_1 = \omega_1 = 0.$

$\angle V_2 = \omega_2 = \pi$ and $\angle V_3 = \theta_1 = 0$

Case II : When $\omega = \pi / 2$,

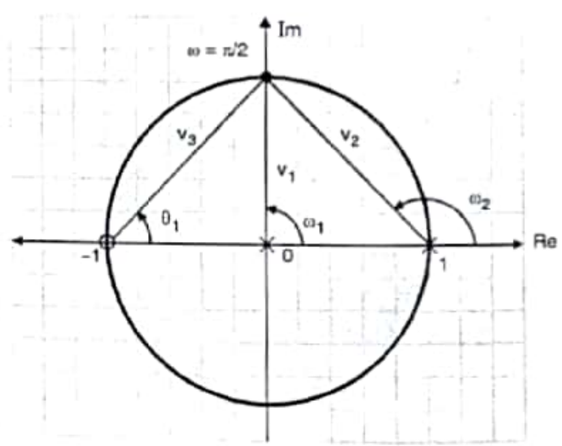
$|V_1| = 1, |V_2| = \sqrt{1^2 + 1^2} = \sqrt{2}.$

$|V_3| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\angle V_1 = \omega_1 = \pi / 2.$

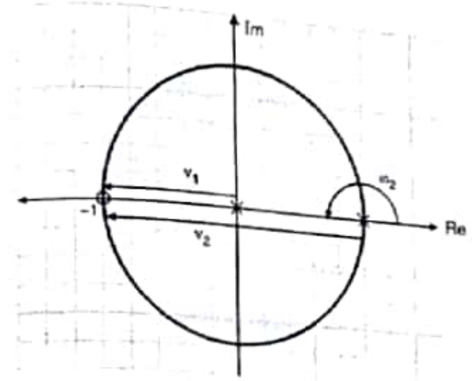
$\angle V_2 = \omega_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4},$

or $\frac{-\pi}{4}$. $\angle V_3 = \theta_1 = \pi / 4$



(II-1544) Fig. P. 8.14.2(b)

Case III : When $\omega = \pi$,



(II-1545) Fig. P. 8.14.2(c)

$|V_1| = 1, |V_2| = 1 + 1 = 2, |V_3| = 0$

$\angle V_1 = \omega_1 = \pi, \angle V_2 = \omega_2 = \pi.$

$\angle V_3 = \theta_1 = \pi / 2$ ...since at $\omega = \pi$,

Complete overlap occurs and the line becomes tangent. Hence θ_3 is $\pi / 2$.

Case IV : When $\omega = 3\pi / 2$,

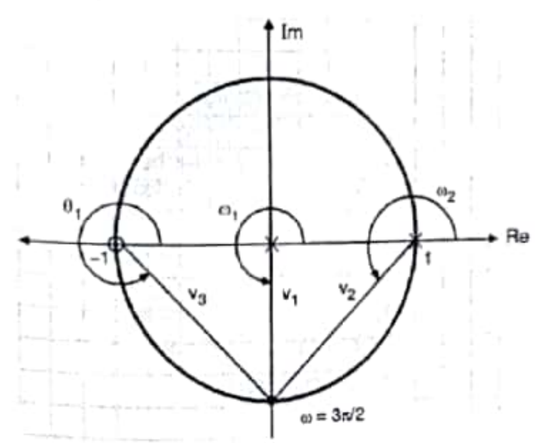
$\therefore |V_1| = 1, |V_2| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$|V_3| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\angle V_1 = \omega_1 = 3\pi / 2$ or $-\pi / 2,$

$\angle V_2 = \omega_2 = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$ or $-\left(\frac{\pi}{2} + \frac{\pi}{4}\right) = \frac{-3\pi}{4}$

$\angle V_3 = \theta_1 = -\pi / 4$



(II-1546) Fig. P. 8.14.2(d)

Step 2 : Magnitude and phase response are given as, at $\omega = 0$,

$$|H(Z)| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{2}{1 \times 0} = \infty$$

$$\angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2)$$
$$= \theta_1 - (\omega_1 + \omega_2)$$
$$= 0 - (0 + 0) = 0$$

At $\omega = \pi / 2$, $|H(Z)| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{\sqrt{2}}{1 \cdot \sqrt{2}}$

$= 1 \angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2) = \theta_1 - (\omega_1 + \omega_2)$

$= \frac{\pi}{4} - \left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 0$

At $\omega = \pi$,

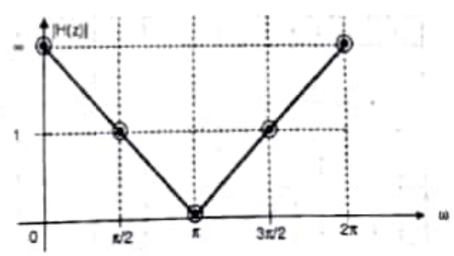
$$|H(Z)| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{0}{1.2} = 0$$

$$\angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2) = \theta_1 - (\omega_1 + \omega_2)$$
$$= \frac{\pi}{2} - (\pi + \pi) = \frac{-3\pi}{2}$$ At $\omega = 3\pi / 2$,

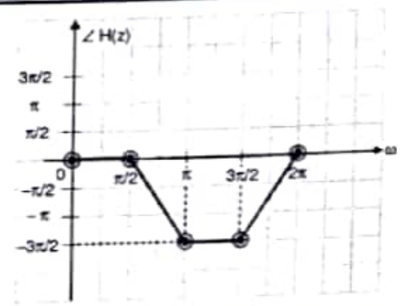
$$|H(Z| = \frac{|V_3|}{|V_1| \cdot |V_2|} = \frac{\sqrt{2}}{1 \cdot \sqrt{2}} = 1$$

$$\angle H(Z) = \angle V_3 - (\angle V_1 + \angle V_2) = \theta_1 - (\omega_1 + \omega_2)$$
$$= -\frac{\pi}{4} - \left(\frac{-\pi}{2} - \frac{3\pi}{4}\right) = \frac{-3\pi}{2}$$

ω	$\lvert H(Z)\rvert$	$\angle H(Z)$
0	∞	0
$\frac{\pi}{2}$	1	0
π	0	$-3\pi/2$
$\frac{3\pi}{2}$	1	$-3\pi/2$
2π	∞	0



(II-1547) Fig. P. 8.14.2(e) : Magnitude Response



(H-1548) **Fig. P. 8.14.2(f) : Phase Response**

8.15 Relationship of Z-transform with Laplace Transform (Mapping between S-plane and Z-plane) :

The poles of analog filters are located at $s = P_k$. Now from Equation (8.15.1) we can say that the poles of digital filter, H (Z) are located at.

$$Z = e^{P_k T_s} \quad ...(8.15.1)$$

This equation indicates that the poles of analog filter at $s = P_k$ are transformed into the poles of digital filter at $Z = e^{P_k T_s}$. Thus the relationship between laplace ('s' domain) and Z domain is given by,

$$Z = e^{s T_s} \quad ...(8.15.2)$$

Here $s = P_k$ and T_s is the sampling time.

Now 's' is the laplace operator and it is expressed as,

$$s = \sigma + j\Omega \quad ...(8.15.3)$$

Here σ = Attenuation factor

and Ω = Analog frequency

We know the 'Z' can be expressed in polar form as,

$$Z = r e^{j\omega} \quad ...(8.15.4)$$

Here 'r' is magnitude and 'ω' is the digital frequency.

Putting Equations (8.15.3) and (8.15.4) in Equation (8.15.2) we get,

$$r e^{j\omega} = e^{(\sigma + j\Omega)Ts}$$

$$\therefore r e^{j\omega} = e^{\sigma T_s} \cdot e^{j\Omega T_s} \quad ...(8.15.5)$$

Separating real and imaginary parts of Equation (8.15.5) we get,

$$r = e^{\sigma T_s} \quad ...(8.15.6)$$

and $$e^{j\omega} = e^{j\Omega T_s}$$

$$\therefore \omega = \Omega T_s \quad ...(8.15.7)$$

Now we will find the relationship between s plane and Z plane. Basically plot in 's'-domain means, σ is plotted on X-axis and jΩ) is plotted on Y- axis. And Z-domain representation means real Z is plotted on X- axis and imaginary Z is plotted on Y-axis.

Now consider Equation (8.15.6), it is

$$r = e^{\sigma T_s}$$

We will discuss the following conditions :

1. If $\sigma < 0$, then r is equal to reciprocal of 'e' raise to some constant. Thus range of r will be 0 to 1.

$$\therefore \sigma < 0 \Rightarrow 0 < r < 1 \quad ...(8.15.8)$$

Now $\sigma < 0$ means negative values of σ. That is L.H.S. of s plane. We know that 'r' is the radius of circle is Z plane. So '$0 < r < 1$' indicates interior part of unit circle. Thus we can conclude that,

L.H.S. of 's' plane is mapped inside the unit circle.

2. If $\sigma = 0$ then $r = e^0 = 1$

$$\therefore \sigma = 0 \Rightarrow r = 1$$

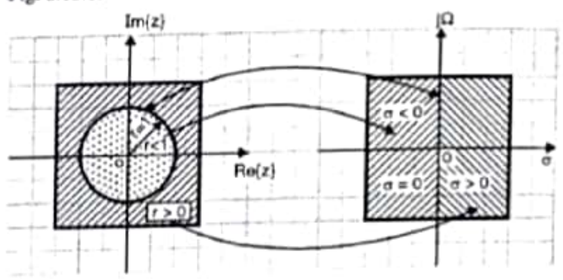
Now $\sigma = 0$ indicates jΩ axis and $r = 1$ indicates unit circle. Thus, **jΩ axis in 's' plane is mapped on the unit circle.**

3. If $\sigma > 0$ then, r is equal to 'e' raise to some constant. That means $r > 1$.

$$\therefore \sigma > 0 \Rightarrow r > 1$$

Now $\sigma > 0$ indicates R.H.S. of 's' plane and '$r > 1$' indicates exterior part of unit circle. Thus, **R.H.S. of 's' plane is mapped outside the unit circle.**

Combining all conditions; this mapping is shown in Fig. 8.15.1.



(H-1549) **Fig. 8.15.1 : Relationship of s plane to Z-plane**

Review Questions

Q. 1 Write short note on : Stability and causaility analysis in Z-domain.

Q. 2 Explain how a time domain behaviour of a DT causal signal can be described by means of a pole zero location of its Z-transform.

Q. 3 What is pole zero plot ? How it can be described for rational ZT ? What are the various features of pole zero plot of rational ZT ?

Q. 4 Write difference equation of system which generates a ramp discrete time signal as an output for $n \geq 0$.

Q. 5 Explain the system classification tests for stability and causality, in time domain and z-domain.

Q. 6 Explain significance of pole-zero plot.

□□□

CHAPTER 9

Module 5

State Space Analysis

Syllabus :

State variable analysis : Introduction to the notion of 'state', systematic procedure for determining state equations, solution of state equations using Laplace transform, definition of exp (A) where A is a matrix, time domain solution of state equations.

Syllabus Topic : Introduction to Notation of State

9.1 Introduction to Notation of State :

Concepts of State, State Variables and State Model

What is meant by state ?

- Fig. 9.1.1 will help us to understand the concept of state and state variables. The voltage across the capacitor is v_o (t). The capacitor gets charged by current i (t).
- We can write the relationship between the current and the capacitor voltage as

$$i(t) = C\frac{dv_o(t)}{dt}$$

$$\frac{dv_o(t)}{dt} = \frac{1}{C} i(t)$$

Integrating both sides we get, $v_o(t) = \frac{1}{C}\int i(t)\,dt + v_{initial}(t)$

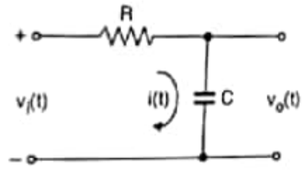


Fig. 9.1.1

- Here $v_{initial}$ (t) is the initial voltage across the capacitor at t = 0. (In the chapter of transfer function, we had assumed this initial voltage across capacitor to be equal to zero).

Apply KVL on Fig. 9.1.1.

We get, $$v_o(t) = R\,i(t) + \frac{1}{C}\int_0^t i(t)\,dt + v_{initial}(t)$$

- Hence the knowledge of the capacitor voltage at t = 0 is needed to determine the new output voltage.
- Hence the initial voltage in a capacitor forms a state variable.

Definition :

State variables are the smallest set of variables which determine the state of a dynamic system.

If n variables x_1 (t), x_2 (t), x_3 (t) .... x_n (t) are needed to completely describe the system behaviour, then once the inputs are given to the system for $t > t_0$ and initial conditions or state t_0 is specified then it is easy to determine the future state of the system completely. Here such n variables x_1 (t), x_2 (t), x_3 (t) .... x_n (t) are a set of state variables. These state variables need not be measurable or observable quantities.

9.1.1 Definition of State and State Variables, State Vectors and State Space :

State :

The state of a dynamic system is the smallest set of variables, called state variables such that the knowledge of these variables at $t = t_0$ together with the inputs for $t \geq t_0$ completely determines the behaviour of the system at $t \geq t_0$.

State variables :

- Smallest set of variables which determine the state of dynamic system.
- For n variables x_1 (t), x_2 (t), x_3 (t) .... x_n (t) are needed to completely describe the behaviour of a dynamic system, together with initial state and inputs then these 'n' variables are known to be set of state variables.
- These state variables need not be physically measurable or observable quantities.

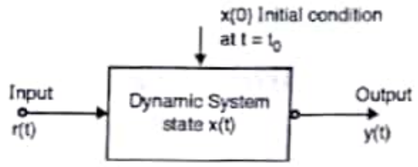


Fig. 9.1.2

State vector :

- If 'n' state variables are needed to completely describe the behaviour of a given system, then these 'n' state variables can be considered to be the n components of a vector x (t). Such a vector is called a **state vector**.
- A state vector is thus a vector which determines uniquely the system state x (t) for any time $t \geq t_0$ once the input u (t) for $t \geq t_0$ is specified.

State space :

- The n dimensional space whose co-ordinate axis consists of the x_1 axis, x_2 axis .... x_n axis is called a **state space.** Any state can be represented by a point in the state space.
- Consider a mechanical system shown in Fig. 9.1.3. Here mass M is acted upon by a force F(t).
- To keep the equations simple, we assume zero friction.

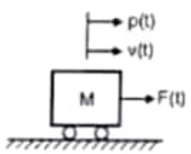


Fig. 9.1.3

Here, M → Mass of the body ; F (t) → Force

p (t) → Displacement ; v (t) → Velocity

(Normally displacement is written as x(t). In this case we represent it as p(t) so as to avoid confusion with $x_1(t)$, $x_2(t)$ which are state variables). We first need to define the state variables.

- Let the state variables be :
  1. Displacement p (t) ; 2. Velocity v (t) ;
- In state space analysis, the state variables are denoted as x_1 (t), x_2 (t).....

$$\therefore \quad x_1(t) = p(t)$$
$$x_2(t) = v(t)$$

From physics, we know the following relationships

$$\frac{d}{dt} p(t) = v(t)$$

and $\frac{d}{dt} v(t) = a(t)$ where a (t) is the acceleration

From Fig. 9.1.3, we know Force = mass × acceleration

$$F = M \cdot a$$

$$\therefore \quad F(t) = M \cdot \frac{d}{dt} v(t) \qquad ...(9.1.1)$$

$$\therefore \quad \frac{d}{dt} v(t) = \frac{F(t)}{M} \qquad ...(9.1.2)$$

We can write Equation (9.1.2) in terms of state variables as

$$\dot{x}_2(t) = \frac{F(t)}{M} \qquad ...(9.1.3)$$

Similarly $$\frac{d}{dt} p(t) = v(t) \qquad ...(9.1.4)$$

We can write Equation (9.1.4) in terms of state variables as

$$\dot{x}_1(t) = x_2(t) \qquad ...(9.1.5)$$

We represent Equation (9.1.3) and Equation (9.1.5) in matrix form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} F(t) \qquad ...(9.1.6)$$

In state variable theory, the input is defined as r (t).

Hence $F(t) = r(t)$

Equation (9.1.6) is modified to give

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} r(t) \qquad ...(9.1.7)$$

Now if the desired output is velocity then

$$y(t) = v(t)$$

In matrix form we get $$y = [0 \quad 1] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad ...(9.1.8)$$

Equation (9.1.7) and Equation (9.1.8) are the state equation and the output equation. They can be written in compressed form as :

$\dot{x} = Ax + r$ → state equation

$y = c' x$ → output equation

where, A → matrix

x → vector

c′ → vector transposer

and y → scalar value

Syllabus Topic : Systematic Procedure for Determining State Equations

9.2 Systematic Procedure for Determining State Equations :

State Variable Representation of Control System :

State Model

General Case :

Consider a system which is multiple input multiple output (MIMO).

Let, the number of inputs - **m**

the number of state variables - n

the number of outputs - **p**

Refer Fig. 9.2.1

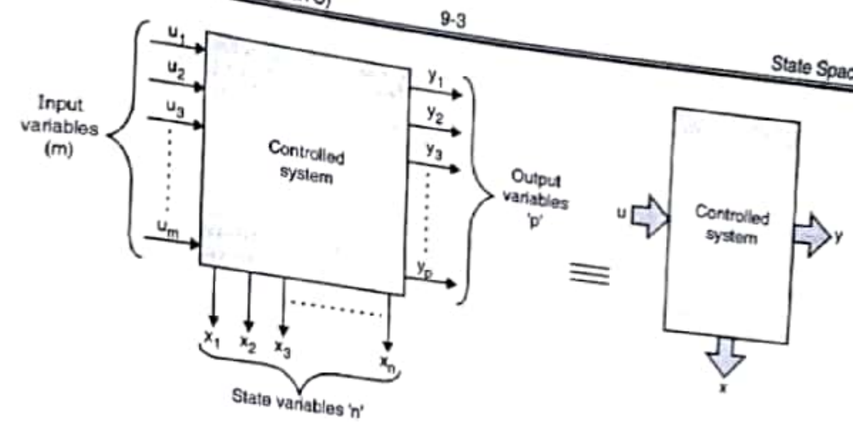


Fig. 9.2.1 : Structure of a general control system

- The n state variables are given by n first order differential equations as follows :

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2, ..., x_n ; u_1, u_2, ..., u_m)$$
$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2, ..., x_n ; u_1, u_2, ..., u_m)$$
$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2, ..., x_n ; u_1, u_2, ..., u_m) \qquad ...(9.2.1)$$

- The n differential Equation (9.2.1) may be written in a vector notation as,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \qquad ...(9.2.2)$$

For a time varying system, t also becomes a parameter,

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \qquad ...(9.2.3)$$

- Similarly the relation between p outputs can be given as,

$$y_1 = g_1(x_1, x_2, ..., x_n ; u_1, u_2, ..., u_m)$$
$$y_2 = g_2(x_1, x_2, ..., x_n ; u_1, u_2, ..., u_m)$$
$$\vdots$$
$$y_p = g_p(x_1, x_2, ..., x_n ; u_1, u_2, ..., u_m) \qquad ...(9.2.4)$$

The equations may be written in vector notation as,

$$y(t) = g(x(t), u(t)) \qquad ...(9.2.5)$$

For a time varying case $$y(t) = g(x(t), u(t), t) \qquad ...(9.2.6)$$

- The vector notation being,

$$\text{State vector, } x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$\text{Input vector, } u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$$

$$\text{Output vector, } y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

$$f(\cdot) = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \\ \vdots \\ f_n(\cdot) \end{bmatrix}$$

$$g(\cdot) = \begin{bmatrix} g_1(\cdot) \\ g_2(\cdot) \\ \vdots \\ g_p(\cdot) \end{bmatrix}$$

Equation (9.2.1) is an expanded form of Equation (9.2.2) and is called the **State equation** for a linear time invariant case and Equation (9.2.3) is the state equation for a linear time varying case.

- The n state variables $x_1, x_2, ... x_n$ from the n dimension space is called **State space.**
- The state vector x for a particular value of $x_1, x_2, ..., x_n$ forms a point, called **State point** in this state space.
- The trajectory traced by a state point from $t = t_0$ to $t = t_1$ is called **State trajectory.**
- Equation (9.2.4) is an expanded form of Equation (9.2.5) and is called **Output equation** for a linear time invariant case and Equation (9.2.6) is the output equation for a linear time varying case.
- The state and output equations together constitute the **State model** of a system.

9.2.1 State Model of Linear Systems :

For a special case of linear time invariant systems Equation (9.2.1) becomes,

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2 + ... + a_{1n} x_n + b_{11} u_1 + b_{12} u_2 + ... + b_{1m} u_m$$
$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2 + ... + a_{2n} x_n + b_{21} u_1 + b_{22} u_2 + ... + b_{2m} u_m$$
$$\vdots$$
$$\dot{x}_n = a_{n1} x_1 + a_{n2} x_2 + ... + a_{nn} x_n + b_{n1} u_1 + b_{n2} u_2 + ... + b_{nm} u_m$$

or

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \qquad ...(9.2.7)$$

i.e. $\dot{x}(t) = Ax(t) + Bu(t)$

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$

= System matrix = $(n \times n)$ matrix.

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nm} \end{bmatrix}$

= Input matrix = $(n \times m)$ matrix

$x(t)$ = State vector $\rightarrow n \times 1$

$u(t)$ = Input vector $\rightarrow m \times 1$

Equation (9.2.7) is the state equation.

Output equation : It is similarly derived as,

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) + \dots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) + \dots + d_{1m}u_m(t)$$

$$y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + \dots + c_{2n}x_n(t) + d_{21}u_1(t) + d_{22}u_2(t) + \dots + d_{2m}u_m(t)$$

$$\vdots$$

$$y_p(t) = c_{p1}x_1(t) + c_{p2}x_2(t) + \dots + c_{pn}x_n(t) + d_{p1}u_1(t) + d_{p2}u_2(t) + \dots + d_{pm}u_m(t) \quad \dots(9.2.8)$$

or

$$y(t) = Cx(t) + Du(t) \quad \dots(9.2.9)$$

where $C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \dots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{p1} & c_{p2} & c_{p3} & \dots & c_{pn} \end{bmatrix}$

= Output matrix $(p \times n)$

$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} & \dots & d_{1m} \\ d_{21} & d_{22} & d_{23} & \dots & d_{2m} \\ \vdots & \vdots & \vdots & & \vdots \\ d_{p1} & d_{p2} & d_{p3} & \dots & d_{pm} \end{bmatrix}$

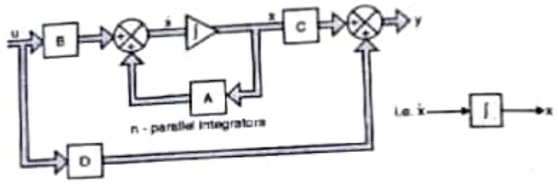
= Transmission matrix $(p \times m)$

Thus the state model for a linear time invariant system is,

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t) \quad \dots(9.2.10)$$

The block diagram for Equation (9.2.10) is as shown in Fig. 9.2.2(a).

Note : The output of integrator for input $\dot{x} = \frac{dx}{dt}$ is $x(t)$. Refer Fig. 9.2.2(b).



(a) A linear MIMO system (b)

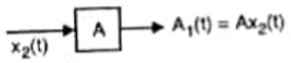
Fig. 9.2.2

9.2.2 State Diagram Representation :

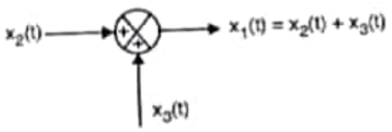
- It is the pictorial representation of the state model derived for a given system.
- It forms a close relationship amongst the state model, differential equations of the system and its solution.
- It is basically a block diagram type approach which is designed from the view of programming of a computer.
- State diagram is a proper interconnection of three units.

(i) Scalars
(ii) Adders
(iii) Integrators.

Scalars are nothing but amplifiers having required gain.



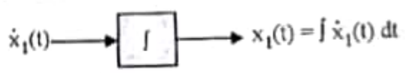
(a) Scalar



(b) Adder

Fig. 9.2.3

- Adders are nothing but summing points. Integrators are the elements which actually integrate the differentiation of state variable to obtain required state variable.



Integrator

Block diagram of MIMO system is drawn in Fig. 9.2.4.

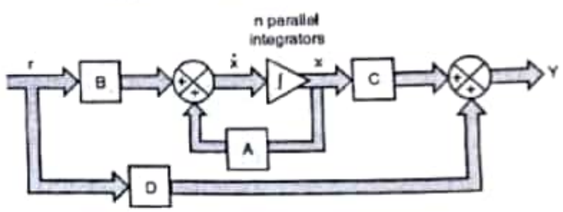


Fig. 9.2.4 : State diagram of MIMO system

Block diagram of SISO system is drawn in Fig. 9.2.5.

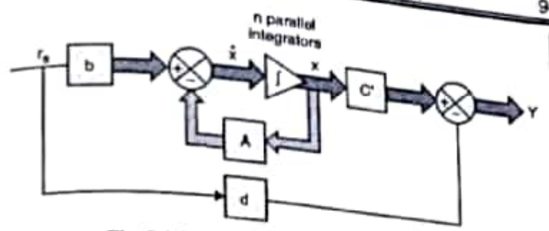


Fig. 9.2.5 : Block diagram of SISO system

9.2.3 Non-uniqueness of the State Variable :

- For a given system we try to build the state equation based on the state variables of the system and formulate a state model. But these state models that are developed are not unique.
- For a given system alternate representations are possible. Consider a system given by,

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

- Let a transformation be, $x = Mz$

where M is a non-singular constant matrix.

$$\therefore x = Mz$$
$$\therefore \dot{x} = A\dot{x} + Bu = AMz + Bu$$

If we substitute in output equation

$$y = Cx + Du = CMz + Du$$

- The 2 equations can also be transformed into equation consisting of variable z.

We have, $\dot{x} = M\dot{z} = AMz + Bu$

$$\therefore \dot{z} = \frac{AMz}{M} + \frac{Bu}{M} = AM^{-1}Mz + BM^{-1}u$$

$$\therefore \text{Let } P = M^{-1}AM,\ Q = M^{-1}B$$
$$\therefore \dot{z} = Pz + Qu$$

Similarly we can write, $y = CMz + Du = Rz + Du$

- These two equations are of a the same system with different variables and hence different state models can be developed. This shows that given state models are not unique.

Explanation with example :

- Consider a simple example of a capacitor as shown in Fig. 9.2.6.
- If we consider e (t) as a state variable, we would have

$$x(t) = e(t)$$

Now, $e(t) = \frac{1}{C}\int i_1(t)\,dt$

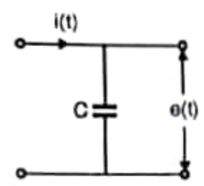


Fig. 9.2.6

$$\therefore \frac{d}{dt}e(t) = \frac{1}{C}i_1(t)$$

$$\therefore \dot{x}(t) = \frac{1}{C}i_1(t)$$

Since inputs are represented as r (t), we get

$$\dot{x}(t) = \frac{1}{C}r(t) \quad \dots(9.2.11)$$

- Now, if we choose charge q (t) as the state variable instead of voltage, we have

$$x(t) = q(t)$$
$$\therefore \dot{x}(t) = \dot{q}(t)$$

We know, $\frac{d}{dt}q(t) = i(t)$

$$\therefore \dot{x}(t) = i(t) \quad \dots(9.2.12)$$

Since inputs are represented as r (t), we get

$$\dot{x}(t) = r(t) \quad \dots(9.2.13)$$

We are analysing the same system, though Equations (9.2.13) and (9.2.11) are different. This points out that the state variable model of a system is not unique.

State variable model of an LTI system is not unique.

9.2.4 Different Representation of State Model :

1. Forming state model by physical variables.
2. Forming state model by phase variables.
3. Canonical state variables.

9.2.5 Forming State Models by Physical Variables :

Physical variables of a system seem to be a natural choice as state variables as these can be currently measured and manipulated.

Ex. 9.2.1 : Obtain the state model of the given circuit.

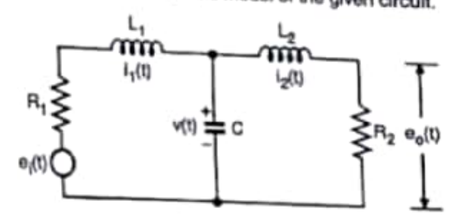


Fig. P. 9.2.1

Soln. : The given circuit has three storage elements viz. two inductors and one capacitor. Here $i_1(t)$ and $i_2(t)$ are the loop currents while v(t) is the voltage across the capacitor. Hence there are three state variable that we use

1. Current passing through inductor $L_1 = i_1(t)$.
2. Current passing through inductor $L_2 = i_2(t)$.
3. Voltage across capacitor v(t).

$$x_1(t) = i_1(t);\quad x_2(t) = i_2(t);\quad x_3(t) = v(t)$$

Here $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the three state variables. We apply KVL on the 1st loop.

$$e_1(t) = R_1 i_1(t) + L_1\frac{di_1(t)}{dt} + v(t)$$

$$\therefore L_1\frac{di_1(t)}{dt} = -R_1 i_1(t) - v(t) + e_1(t)$$

$$\therefore \frac{di_1(t)}{dt} = \frac{1}{L_1}[-R_1 i_1(t) - v(t) + e_i(t)] \quad ...(1)$$

We know $i_1(t) = x_1(t) \therefore \frac{di_1(t)}{dt} = \dot{x}_1(t)$

Also $v(t) = x_3(t)$. Using these in Equation (1) we get

$$\dot{x}_1(t) = \frac{-R_1}{L_1} x_1(t) - \frac{1}{L_1} x_3(t) + \frac{1}{L_1} e_i(t) \quad ...(2)$$

We now apply KVL on the 2nd loop.

$$\therefore v(t) = L_2 \frac{di_2(t)}{dt} + R_2 i_2(t)$$

$$\therefore L_2 \frac{di_2(t)}{dt} = -R_2 i_2(t) - v(t)$$

$$\therefore \frac{di_2(t)}{dt} = \frac{-R_2}{L_2} i_2(t) - \frac{1}{L_2} v(t)$$

This can be written in terms of state variables

$$\therefore \dot{x}_2(t) = \frac{-R_2}{L_2} x_2(t) - \frac{1}{L_2} x_3(t) \quad ...(3)$$

Now finally voltage across the capacitor is

$$v(t) = \frac{1}{C}\int i_1(t)dt - \frac{1}{C}\int i_2(t)\,dt$$

$$v(t) = \frac{1}{C}\int [i_1(t) - i_2(t)]\,dt$$

$$C\,v(t) = \int [i_1(t) - i_2(t)]\,dt$$

$$\therefore C\frac{dv(t)}{dt} = i_1(t) - i_2(t)$$

$$\therefore \frac{dv(t)}{dt} = \frac{1}{C} i_1(t) - \frac{1}{C} i_2(t)$$

We write this in terms of state variable.

$$\therefore \dot{x}_3(t) = \frac{1}{C} x_1(t) - \frac{1}{C} x_2(t) \quad ...(4)$$

The output $e_o(t)$ can be written as

$$e_o(t) = R_2 i_2(t) = R_2 x_2(t) \quad ...(5)$$

We generate the state model using Equations (2), (3) and (4)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} e_i(t)$$

The output Equation (5) can be written in matrix form as shown.

$$e_o(t) = [0 \;\; R_2 \;\; 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

Ex. 9.2.2 : Find state model of following Electrical network.

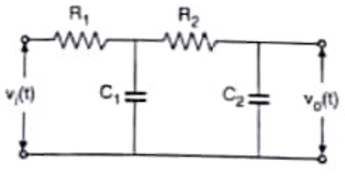


Fig. P. 9.2.2

Soln. :

Assume loop currents as shown in Fig. P. 9.2.2(a).

(a) State space model

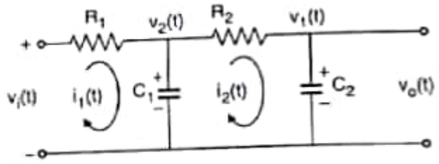


Fig. P. 9.2.2(a)

Applying KVL to the first loop

$$v_i(t) = R_1 i_1(t) + v_2(t)$$

$$\therefore i_1(t) = \frac{v_i(t) - v_2(t)}{R_1} \quad ...(1)$$

Applying kVL to the second loop

$$v_2(t) = R_2 i_2(t) + v_1(t)$$

$$\therefore i_2(t) = \frac{v_2(t) - v_1(t)}{R_2} \quad ...(2)$$

Current through capacitor C_1 is $i_1(t) - i_2(t)$ and through C_2 is $i_2(t)$

$$\therefore i_1(t) - i_2(t) = C_1 \frac{d\,v_2(t)}{dt} \quad ...(3)$$

$$\text{and} \quad i_2(t) = C_2 \frac{d\,v_1(t)}{dt} \quad ...(4)$$

Substituting Equation (1) and Equation (2) in Equation (3) and Equation (4).

$$C_1 \frac{d\,v_2(t)}{dt} = \frac{v_i(t) - v_2(t)}{R_1} - \frac{v_2(t) - v_1(t)}{R_2} \quad ...(6)$$

$$\text{and } C_2 \frac{d\,v_1(t)}{dt} = \frac{v_2(t) - v_1(t)}{R_2} \quad ...(7)$$

Let

$$v_i(t) = v, \quad x_1(t) = v_1(t), \quad x_2(t) = v_2(t)$$

$$\dot{x}_1 = \frac{d\,v_1(t)}{dt}, \qquad \dot{x}_2 = \frac{d\,v_2(t)}{dt}$$

Hence Equations (6) and (7) can be written as,

$$\therefore C_1 \dot{x}_2 = \frac{v - x_2(t)}{R_1} - \frac{x_2(t) - x_1(t)}{R_2} \quad ...(8)$$

$$C_2 \dot{x}_1 = \frac{x_2(t) - x_1(t)}{R_2} \quad ...(9)$$

Rearranging Equations (8) and (9) we obtain

$$\therefore \dot{x}_2 = \frac{v - x_2(t)}{R_1 C_1} - \frac{x_2(t) - x_1(t)}{R_2 C_1} \quad ...(10)$$

$$\text{and} \quad \dot{x}_1 = \frac{x_2(t) - x_1(t)}{R_2 C_2} \quad ...(11)$$

Assume $R_1 C_1 = R_2 C_1 = R_2 C_2 = 1$ in Equations (10) and (11), we obtain

$$\therefore \dot{x}_2 = v - x_2(t) - x_2(t) + x_1(t)$$

$$\therefore \dot{x}_2 = v - 2x_2(t) + x_1(t) \quad ...(12)$$

$$\text{and} \quad \dot{x}_1 = x_2(t) - x_1(t) \quad ...(13)$$

$$\text{Also} \quad y(t) = v_1(t) = x_1(t) \quad ...(14)$$

Using Equations (12), (13) and (14), we obtain the state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

$$\text{and} \quad y(t) = [1 \;\; 0] \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Ex. 9.2.3 : A feedback system is having the closed loop transfer function : $T(s) = \frac{s^2 + 3s + 3}{s^3 + 2s^2 + 3s + 1}$.

Construct a state model for the system.

Soln. :

The given transfer function can be written as,

$$T(s) = \frac{Y(s)}{R(s)} = \frac{X_1(s)}{R(s)} \cdot \frac{Y(s)}{X_1(s)}$$

$$\text{Here let,} \quad \frac{X_1(s)}{R(s)} = \frac{1}{s^3 + 2s^2 + 3s + 1} \quad ...(1)$$

$$\text{and} \quad \frac{Y(s)}{X_1(s)} = s^2 + 3s + 3 \quad ...(2)$$

From Equation (1) we get,

$$s^3 X_1(s) + 2s^2 X_1(s) + 3s X_1(s) = R(s)$$

Taking inverse Laplace transform.

$$\dddot{x}_1(t) + 2\ddot{x}_1(t) + 3\dot{x}_1(t) + x_1(t) = r(t) \quad ...(3)$$

$$\text{Let } \dot{x}_1(t) = x_2(t)$$

$$\therefore \dot{x}_2(t) = x_3(t) = \ddot{x}_1(t)$$

$$\dot{x}_3(t) = \dddot{x}_1(t)$$

Hence Equation (3) becomes,

$$\dot{x}_3(t) + 2x_3(t) + 3x_2(t) + x_1(t) = r(t)$$

$$\dot{x}_3(t) = -x_1(t) - 3x_2(t) - 2x_3(t) + r(t)$$

Thus we have obtained three set of state equations. Writing them in matrix from

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t) \quad ...(4)$$

From Equation (2) we can write,

$$(s^2 + 3s + 3) X_1(s) = Y(s)$$

$$\therefore s^2 X_1(s) + 3s X_1(s) + 3 X_1(s) = Y(s)$$

Taking inverse Laplace,

$$\ddot{x}_1(t) + 3\dot{x}_1(t) + 3x_1(t) = y(t)$$

$$\therefore x_3(t) + 3x_2(t) + 3x_1(t) = y(t)$$

$\therefore$ Output equation in matrix form can be written as.

$$y(t) = [3\;3\;1] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad ...(5)$$

$\therefore$ Equations (4) and (5) represent state model for given system.

Ex. 9.2.4 : Obtain a state space model of the system with transfer function $\frac{Y(s)}{U(s)} = \frac{6}{s^3 + 6s^2 + 11s + 6}$

Soln. :

Given transfer can be written as, $\frac{X_1(s)}{U(s)} \cdot \frac{Y(s)}{X_1(s)}$

$$\text{Here} \quad \frac{X_1(s)}{U(s)} = \frac{1}{s^3 + 6s^2 + 11s + 6} \quad ...(1)$$

$$\text{and} \quad \frac{Y(s)}{X_1(s)} = 6 \quad ...(2)$$

From Equation (1) we get,

$$X_1(s)[s^3 + 6s^2 + 11s + 6] = U(s)$$

$$s^3 X_1(s) + 6s^2 X_1(s) + 11s X_1(s) + 6X(s) = U(s)$$

Taking inverse laplace transform

$$\dddot{x}_1(t) + 6\ddot{x}_1(t) + 11\dot{x}_1(t) + 6x_1(t) = u(t) \quad ...(3)$$

$$\text{Let} \quad \dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = \ddot{x}_1(t) = x_3(t)$$

$$\dot{x}_3(t) = \ddot{x}_2(t) = \dddot{x}_1(t)$$

Therefore Equation (3) becomes,

$$\dot{x}_3(t) + 6x_3(t) + 11x_2(t) + 6x_1(t) = u(t)$$

$$\therefore \dot{x}_3(t) = -6x_3(t) - 11x_2(t) - 6x_1(t) + u(t)$$

Thus we have obtained three sets of state equation writing them in matrix form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \quad ...(4)$$

From Equation (2) we write

$$6X_1(s) = Y(s)$$

$\therefore$ Taking inverse laplace we get,

$$6x_1(t) = y(t)$$

$\therefore$ Output equations in matrix form can be written as,

$$y(t) = [6, 0, 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad ...(5)$$

Equations (4) and (5) represent the state variable model.

Ex. 9.2.5 : Obtain the state model for the system with transfer function $\frac{Y(s)}{U(s)} = \frac{3s + 4}{s^2 + 5s + 6}$

Soln. : The given transfer function can be written in the form

$$T(s) = \frac{X_1(s)}{U(s)} \cdot \frac{Y(s)}{X_1(s)}$$

$$\text{Let} \quad \frac{X_1(s)}{U(s)} = \frac{1}{s^2 + 5s + 2} \quad ...(1)$$

$$\text{and} \quad \frac{Y(s)}{X_1(s)} = 3s + 4 \quad ...(2)$$

From Equation (1) we get

$$X_1(s)\,[s^2 + 5s + 2] = U(s)$$

$$s^2 X_1(s) + 5s\,X_1(s) + 2\,X_1(s) = U(s)$$

Taking inverse laplace transform.

$$\frac{d^2}{dt^2} x_1(t) + 5\frac{d}{dt} x_1(t) + 2x(t) = u(t)$$

$$\ddot{x}_1(t) + 5\,\dot{x}_1(t) + 2x(t) = u(t) \qquad ...(3)$$

Let $\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = x_3(t) = \ddot{x}_1(t)$

$\therefore$ Equation (3) becomes.

$$x_3(t) + 5\,x_2(t) + 2\,x_1(t) = u(t)$$

This can be written as.

$$0\,\dot{x}_3(t) + x_3(t) + 5x_2(t) + 2\,x_1(t) = u(t)$$

$$\therefore 0\,\dot{x}_3(t) = -x_3(t) - 5x_2(t) - 2x_1(t) + u(t)$$

Writing them in matrix form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t) \qquad ...(4)$$

From Equation (2) we get, $\dfrac{Y(s)}{X_1(s)} = 3s + 4$

$$(3s + 4)\,X_1(s) = Y(s)$$

$$3s\,X_1(s) + 4\,X_1(s) = Y(s)$$

Taking inverse laplace we get.

$$3\,\dot{x}_1(t) + 4\,x_1(t) = y(t)$$

We know $\dot{x}_1(t) = x_2(t)$

$$\therefore 3\,x_2(t) + 4\,x_1(t) = y(t)$$

$\therefore$ Output can be written as.

$$y(t) = [4, 3, 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad ...(5)$$

Equations (4) and (5) represent the state variable model.

Ex. 9.2.6 : Obtain a state space representation in controllable and observable canonical form for the system $T(s) = \dfrac{s+3}{s^2 + 3s + 2}$.

Soln. :

$$T(s) = \frac{C(s)}{R(s)} = \frac{s+3}{s^2 + 3s + 2}$$

Let $\dfrac{C(s)}{R(s)} = \dfrac{X_1(s)}{R(s)} \cdot \dfrac{C(s)}{X_1(s)}$

where $\dfrac{X_1(s)}{R(s)} = \dfrac{1}{s^2 + 3s + 2}$ ...(1)

and $\dfrac{C(s)}{X_1(s)} = s + 3$ ...(2)

From Equation (1), we get

$$\frac{X_1(s)}{R(s)} = \frac{1}{s^2 + 3s + 2}$$

$$s^2 X_1(s) + 3s\,X_1(s) + 2\,X_1(s) = R(s)$$

Taking the inverse Laplace transform, we get

$$\ddot{x}_1(t) + 3\,\dot{x}_1(t) + 2x_1(t) = r(t) \qquad ...(3)$$

Let $\dot{x}_1(t) = x_2(t)$ ...(4)

$$\therefore \dot{x}_2(t) = \ddot{x}_1(t) = x_3(t) \qquad ...(5)$$

$\therefore$ Equation (3) can be written as,

$$x_3(t) + 3\,x_2(t) + 2x_1(t) = r(t)$$

i.e., $\dot{x}_3(t) + x_3(t) + 3\,x_2(t) + 2x_1(t) = r(t)$

$$\therefore \dot{x}_3(t) = -x_3(t) - 3\,x_2(t) - 2x_1(t) + r(t) \qquad ...(6)$$

We write Equations (4), (5) and (6) in matrix form.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r(t) \quad ...(A)$$

We now work with Equation (2).

$$\frac{C(s)}{X_1(s)} = s + 3$$

$$\therefore C(s) = sX_1(s) + 3X_1(s)$$

Taking the Laplace transform, we get

$$c(t) = \dot{x}_1(t) + 3x_1(t)$$

we know $\dot{x}_1(t) = x_2(t)$

$$\therefore c(t) = x_2(t) + 3x_1(t) \qquad ...(7)$$

Hence the output Equation (7) can be written as,

$$c(t) = [3 \quad 1 \quad 0] \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \qquad ...(B)$$

Equations (A) and (B) represent the state variable model.

9.2.6 Solution of LTI State Equations :

Solving the State Equation :

The general solution to the state equation is treated for two cases :

(1) Homogeneous case

(2) Non-homogeneous case.

9.2.7 Solution of Homogeneous State Equation :

- In homogeneous case u terms are not taken. Consider a scalar differential equation,

$$\dot{x} = ax \qquad ...(9.2.14)$$

- Assume a solution of x as,

$$x = b_0 + b_1 t + b_2 t^2 + ... + b_k t^k + ... \qquad ...(9.2.15)$$

$$\therefore \dot{x} = b_1 + 2\,b_2 t + 3\,b_3 t^2 + ... \qquad ...(9.2.16)$$

Put this in Equation (9.2.15) LHS $= \dot{x}$

$$RHS = a\,b_0 + a\,b_1 t + a\,b_2 t^2 + ... \qquad ...(9.2.17)$$

For a true solution LHS = RHS

From Equations (9.2.16) and (9.3.17)

$$b_1 = a\,b_0$$

$$b_2 = \frac{1}{2} a\,b_1 = \frac{1}{2} a\,(a\,b_0) = \frac{1}{2} a^2 b_0$$

$$b_3 = \frac{1}{3} a\,b_2 = \frac{1}{3 \times 2} a^3 b_0$$

$$\vdots$$

$$b_k = \frac{1}{k!} a^k b_0$$

- Now, for $t = 0$ $x(0) = b_0$

But we have,

$$x(t) = \left(1 + at + \frac{1}{2!} a^2 t^2 + ... + \frac{1}{k!} a^k t^k ....\right) x(0)$$

$$x(t) = e^{at} \cdot x(0) \qquad ...(9.2.18)$$

- Similarly for vector matrix differential equation,

$$\dot{x} = A\,x \qquad x(t) = e^{At}\,x(0)$$

Now $\dot{x} = A\,x$

Taking Laplace for matrices

$$s\,X(s) - x(0) = AX(s)$$

Take AX (s) on LHS. Also $s\,X(s) = sI\,X(s)$ where I is a $n \times n$ identity matrix of same order as A.

$$\therefore (sI - A)\,X(s) = x(0)$$

$$\therefore X(s) = (sI - A)^{-1}\,x(0)$$

$$\therefore x(t) = L^{-1}(sI - A)^{-1}\,x(0) \qquad ...(9.2.19)$$

Comparing, $e^{At} = L^{-1}(sI - A)^{-1} = \phi(t)$

where $\phi(t)$ is state transition matrix.

Syllabus Topic : Definition of exp (A) where 'A' is a Matrix

9.3 Definition of exp (A), where 'A' is Matrix :

The transition in the state is performed by the term e^{At}; so the term e^{At} is called as state transition matrix, denoted by $\delta(t)$

9.3.1 Properties of State Transition Matrix :

1. $\phi(0) = e^{A0} = I$
2. $\phi(t) = e^{At} = [\phi(-t)]^{-1}$ i.e. $\phi^{-1}(t) = \phi(-t)$
3. $\phi(t_1 + t_2) = e^{A(t_1 + t_2)} = e^{At_1} \cdot e^{At_2} = \phi(t_1)\,\phi(t_2)$
4. $[\phi(t)]^n = \phi(n\,t)$
5. $\phi(t_2 - t_1)\,\phi(t_1 - t_0) = \phi(t_2 - t_0)$

The state transition matrix gives idea about the progress of state from x (0). It gives the free response of the system.

Ex. 9.3.1 : Find the state transition matrix $\phi(t)$:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \text{ Also find } \phi^{-1}(t).$$

Soln. :

Here $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$\phi(t) = L^{-1}[sI - A]^{-1}$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$\therefore [sI - A] = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

To find $[sI - A]^{-1}$

$$(sI - A)^{-1} = \frac{\text{Adj}\,[sI - A]}{\det\,[sI - A]}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$\det\,(sI - A) = \begin{vmatrix} s+3 & 1 \\ -2 & s \end{vmatrix} = s(s+3) - (1)(-2)$$

$$= s^2 + 3s + 2 = (s+2)(s+1)$$

$$\therefore [sI - A]^{-1} = \frac{1}{(s+2)(s+1)} \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+2)(s+1)} & \dfrac{1}{(s+2)(s+1)} \\ \dfrac{-2}{(s+2)(s+1)} & \dfrac{s}{(s+2)(s+1)} \end{bmatrix}$$

$\therefore$ For $L^{-1}[sI - A]^{-1}$ take individual inverse Laplace

$$\frac{s+3}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

$$A = \left.\frac{s+3}{s+1}\right|_{s=-2} = \frac{1}{-1} = -1$$

$$B = \left.\frac{s+3}{s+2}\right|_{s=-1} = \frac{2}{1} = 2$$

$$\therefore L^{-1}\frac{s+3}{(s+2)(s+1)} = L^{-1}\left[\frac{-1}{s+2} + \frac{2}{s+1}\right]$$

$$= 2e^{-t} - e^{-2t}$$

Similarly, $L^{-1}\dfrac{1}{(s+2)(s+1)} = -e^{-2t} + e^{-t} = e^{-t} - e^{-2t}$

$$L^{-1}\frac{-2}{(s+2)(s+1)} = 2e^{-2t} - 2e^{-t}$$

$$L^{-1}\frac{s}{(s+1)(s+2)} = 2e^{-2t} - e^{-t}$$

$$\therefore \phi(t) = L^{-1}[sI - A]^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ 2e^{-2t} - 2e^{-t} & 2e^{-2t} - e^{-t} \end{bmatrix} \quad \text{...Ans.}$$

Ex. 9.3.2 : Obtain state transition matrix for the system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix} x(t)$$

Soln. :

State transition matrix $\phi(t)$ is given by the formula,

$$\phi(t) = L^{-1}[sI - A]^{-1}$$

Let $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore \; sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$\therefore A = \begin{bmatrix} 2 & 3 \\ 0 & -3 \end{bmatrix}$$

$$\therefore [sI - A] = \begin{bmatrix} s-2 & -3 \\ 0 & s+3 \end{bmatrix}$$

We need to find $[sI - A]^{-1}$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{\det\,[sI - A]}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s+3 & 3 \\ 0 & s-2 \end{bmatrix}$$

$$\det\,[sI - A] = (s-2)(s+3) - (-3) \times 0 = (s-2)(s+3)$$

$$\therefore \; [sI - A]^{-1} = \frac{\begin{bmatrix} s+3 & 3 \\ 0 & s-2 \end{bmatrix}}{(s-2)(s+3)}$$

$$\therefore \; [sI - A]^{-1} = \begin{bmatrix} \dfrac{(s+3)}{(s-2)(s+3)} & \dfrac{3}{(s-2)(s+3)} \\ 0 & \dfrac{2}{(s-2)(s+3)} \end{bmatrix}$$

$$\phi(t) = L^{-1}\,[sI - A]^{-1}$$

We take inverse laplace of individual elements.

1. $\dfrac{1}{s-2}$

$$L^{-1}\left[\frac{1}{s-2}\right] = e^{2t}$$

2. $\dfrac{3}{(s-2)(s+3)} = \dfrac{A}{s-2} + \dfrac{B}{s+3}$

$$A = \left.\frac{3}{s+3}\right|_{s=2} \qquad \therefore A = \frac{3}{5}$$

$$B = \left.\frac{3}{s-2}\right|_{s=-3} \qquad \therefore B = \frac{-3}{5}$$

$$\therefore L^{-1}\left[\frac{3}{(s-2)(s+3)}\right] = L^{-1}\left[\frac{\frac{3}{5}}{s-2} \quad \frac{\frac{-3}{5}}{s+3}\right] = \frac{3}{5}e^{2t} - \frac{3}{5}e^{-3t}$$

3. $\dfrac{2}{(s-2)(s+3)} = \dfrac{A}{s-2} + \dfrac{B}{s+3}$

$$A = \left.\frac{2}{s+3}\right|_{s=2} \qquad \therefore A = \frac{2}{5}$$

$$B = \left.\frac{2}{s-2}\right|_{s=-3} \qquad \therefore B = \frac{-2}{5}$$

$$\therefore L^{-1}\left[\frac{2}{(s-2)(s+3)}\right] = L^{-1}\left[\frac{\frac{2}{5}}{s-2} \quad \frac{\frac{-2}{5}}{s+3}\right] = \frac{2}{5}e^{2t} - \frac{3}{5}e^{-3t}$$

$$\therefore \phi(t) = L^{-1}\,[sI - A]^{-1} = \begin{bmatrix} e^{2t} & \frac{3}{5}(e^{2t} - e^{-3t}) \\ 0 & \frac{2}{5}(e^{2t} - e^{-3t}) \end{bmatrix}$$

Ex. 9.3.3 : Obtain state transition matrix for the system.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Soln. :

Let $\quad A = \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix}$

State transition matrix is given by the equation

$$e^{At} = L^{-1}\,[sI - A]^{-1}$$

Now, $\quad sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} s+3 & -1 \\ 0 & s+1 \end{bmatrix}$

To find $[sI - A]^{-1}$

$$\therefore \text{Adj}\,[sI - A] = \begin{bmatrix} s+1 & 1 \\ 0 & s+3 \end{bmatrix} \qquad \text{...(1)}$$

$$\det\,[sI - A] = (s+1)(s+3) \qquad \text{...(2)}$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{\det\,[sI - A]}$$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}$$

Using Equations (1) and (2) we get,

$$\therefore \; [sI - A]^{-1} = \begin{bmatrix} \dfrac{(s+1)}{(s+1)(s+3)} & \dfrac{1}{(s+1)(s+3)} \\ 0 & \dfrac{(s+3)}{(s+1)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{(s+3)} & \dfrac{1}{(s+1)(s+3)} \\ 0 & \dfrac{1}{(s+1)} \end{bmatrix}$$

We now need to take inverse Laplace transforms. We hence find the partial fractions first.

$$[sI - A]^{-1} = \begin{bmatrix} \dfrac{1}{s+3} & \left(\dfrac{0.5}{s+1} - \dfrac{0.5}{s+3}\right) \\ 0 & \dfrac{1}{s+1} \end{bmatrix}$$

Now state transition matrix is

$$e^{At} = L^{-1}\,[sI - A]^{-1}$$

$$= L^{-1}\begin{bmatrix} \dfrac{1}{(s+3)} & \left(\dfrac{0.5}{s+1} - \dfrac{0.5}{s+3}\right) \\ 0 & \dfrac{1}{(s+1)} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-3t} & 0.5\,e^{-t} - 0.5\,e^{-3t} \\ 0 & e^{-t} \end{bmatrix}$$

Ex. 9.3.4 : Obtain state transition matrix if

(i) $\dfrac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$

(ii) $\dfrac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x$

using Laplace transformation.

Soln. :

(i) State transition matrix is given by the equation.

$$\phi(t) = L^{-1}\,[sI - A]^{-1}$$

Here, $\quad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \therefore sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$\therefore \; sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}$$

To find $[sI - A]^{-1}$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{\det\,[sI - A]}$$

Now, $\quad \text{Adj}\,[sI - A] = \begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}$

Now, $\quad \det\,[sI - A] = s(s) - (1 \times -1) = s^2 + 1$

$$\therefore \; [sI - A]^{-1} = \frac{\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}}{s^2 + 1}$$

$$\therefore \; [sI - A]^{-1} = \begin{bmatrix} \dfrac{s}{s^2+1} & \dfrac{1}{s^2+1} \\ \dfrac{-1}{s^2+1} & \dfrac{s}{s^2+1} \end{bmatrix}$$

For $L^{-1}\,[sI - A]^{-1}$, we compute individual inverse Laplace transforms.

(i) $L^{-1}\left[\dfrac{s}{s^2+1}\right]$

This can be written as,

$$L^{-1}\left[\frac{s}{s^2 + 1^2}\right] = \cos t$$

(ii) $L^{-1}\left[\dfrac{1}{s^2+1}\right]$

This can be written as,

$$L^{-1}\left[\frac{1}{s^2 + 1^2}\right] = \sin t$$

(iii) $L^{-1}\left[\dfrac{-1}{s^2+1}\right] = -\sin t$

(iv) $L^{-1}\left[\dfrac{s}{s^2+1}\right] = \cos t$

$$\therefore \phi(t) = L^{-1}\,[sI - A]^{-1} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

(ii) $\dfrac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Here $\quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \therefore sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$$

$$\therefore \; [sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}$$

To find $[sI - A]^{-1}$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{\det\,[sI - A]}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}$$

Now, $\quad \det\,[sI - A] = s(s) - (1 \times 0) = s^2$

$$\therefore \; [sI - A]^{-1} = \frac{\begin{bmatrix} s & 1 \\ 0 & s \end{bmatrix}}{s^2}$$

$$\therefore \; [sI - A]^{-1} = \begin{bmatrix} \dfrac{1}{s} & \dfrac{1}{s^2} \\ 0 & \dfrac{1}{s} \end{bmatrix}$$

For $L^{-1}\,[sI - A]^{-1}$, we compute individual inverse Laplace transforms.

$$L^{-1}\left[\frac{1}{s}\right] = 1$$

$$L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\therefore \phi(t) = L^{-1}\,[sI - A]^{-1} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Ex. 9.3.5 : Obtain the state transition matrix for the system with state equation :

$$[\dot{x}] = \begin{bmatrix} 0 & 1 \\ -8 & -9 \end{bmatrix} [x]$$

using Laplace transformation.

Soln. :

The above example is of the form.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Here $\quad A = \begin{bmatrix} 0 & 1 \\ -8 & -9 \end{bmatrix}$

The state transition matrix is given by the equation.

$$\phi(t) = L^{-1}\,[sI - A]^{-1}$$

Here, $\quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \qquad \therefore sI = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$

$$\therefore \; sI - A = \begin{bmatrix} s & -1 \\ 8 & s+9 \end{bmatrix}$$

To find $[sI - A]^{-1}$

$$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{\det\,[sI - A]}$$

$$\text{Adj}\,[sI - A] = \begin{bmatrix} s+9 & 1 \\ -8 & s \end{bmatrix} \qquad \text{...(1)}$$

$$\det\,[sI - A] = s(s+9) - (-8 \times 1) = s^2 + 9s + 8 \qquad \text{...(2)}$$

$$\therefore \; [sI - A]^{-1} = \frac{\begin{bmatrix} s+9 & 1 \\ -8 & s \end{bmatrix}}{s^2 + 9s + 8}$$

$$\therefore \; [sI - A]^{-1} = \begin{bmatrix} \dfrac{s+9}{s^2+9s+8} & \dfrac{1}{s^2+9s+8} \\ \dfrac{-8}{s^2+9s+8} & \dfrac{s}{s^2+9s+8} \end{bmatrix}$$

$$[sI - A]^{-1} = \begin{bmatrix} \dfrac{s+9}{(s+1)(s+8)} & \dfrac{1}{(s+1)(s+8)} \\ \dfrac{-8}{(s+1)(s+8)} & \dfrac{s}{(s+1)(s+8)} \end{bmatrix}$$

Now, $\quad \phi(t) = L^{-1}\,[sI - A]^{-1}$

For this we compute the Inverse Laplace transform of individual element of the

(i) $\frac{s+9}{(s+1)(s+8)} = \frac{A}{s+1} + \frac{B}{s+9}$

$$A = \left.\frac{(s+9)\cancel{(s+1)}}{\cancel{(s+1)}(s+8)}\right|_{s=-1} = 1.14$$

$$B = \left.\frac{(s+9)\cancel{(s+8)}}{(s+1)\cancel{(s+8)}}\right|_{s=-8} = -0.14$$

$$L^{-1}\left[\frac{s+9}{(s+1)(s+8)}\right] = L^{-1}\left[\frac{1.14}{s+1} - \frac{0.14}{s+8}\right] = 1.14\,e^{-t} - 0.14\,e^{-8t}$$

(ii) $\frac{1}{(s+1)(s+8)} = \frac{A}{s+1} + \frac{B}{s+8}$

$$A = \left.\frac{1\times\cancel{(s+1)}}{\cancel{(s+1)}(s+8)}\right|_{s=-1} = 0.14$$

$$B = \left.\frac{1\times\cancel{(s+8)}}{(s+1)\cancel{(s+8)}}\right|_{s=-8} = -0.14$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s+8)}\right] = L^{-1}\left[\frac{0.14}{s+1} - \frac{0.14}{s+8}\right] = 0.14\,e^{-t} - 0.14\,e^{-8t}$$

(iii) $\frac{-8}{(s+1)(s+8)} = \frac{A}{(s+1)} + \frac{B}{s+8}$

$$A = \left.\frac{-8\times\cancel{(s+1)}}{\cancel{(s+1)}(s+8)}\right|_{s=-1} = -1.14$$

$$B = \left.\frac{-8\times\cancel{(s+8)}}{(s+1)\cancel{(s+8)}}\right|_{s=-8} = 1.14$$

$$\therefore L^{-1}\left[\frac{-8}{(s+1)(s+8)}\right] = L^{-1}\left[\frac{-1.14}{s+1} + \frac{1.14}{s+8}\right] = -1.14\,e^{-t} + 1.14\,e^{-8t}$$

(iv) $\frac{s}{(s+1)(s+8)} = \frac{A}{(s+1)} + \frac{B}{s+8}$

$$A = \left.\frac{s\cancel{(s+1)}}{\cancel{(s+1)}(s+8)}\right|_{s=-1} = -0.142$$

$$B = \left.\frac{s\cancel{(s+8)}}{(s+1)\cancel{(s+8)}}\right|_{s=-8} = 1.14$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s+8)}\right] = L^{-1}\left[\frac{-0.142}{s+1} + \frac{1.14}{s+8}\right] = -0.14\,e^{-t} + 0.14\,e^{-8t}$$

Hence the state transition matrix $\phi(t) = L^{-1}[sI - A]$ is

$$\phi(t) = \begin{bmatrix} 1.14\,e^{-t} - 0.14\,e^{-8t} & 0.14\,e^{-t} - 0.14\,e^{-8t} \\ -1.14\,e^{-t} + 1.14\,e^{-8t} & -0.14\,e^{-t} + 1.14\,e^{-8t} \end{bmatrix}$$

Ex. 9.3.6 : Obtain state transition matrix if :

$$\dot{x} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} x(t).$$

Soln. :

The matrix (sI – A) will be formed as,

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} = \begin{bmatrix} s-0 & 3 \\ -1 & s+4 \end{bmatrix}$$

$$(sI - A)^{-1} = \frac{\text{Adj}(sI - A)}{|sI - A|} = \frac{\begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix}}{\begin{vmatrix} s & 3 \\ -1 & s+4 \end{vmatrix}} = \frac{\begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix}}{s(s+4)+3}$$

$$(sI - A)^{-1} = \frac{\begin{bmatrix} s+4 & -3 \\ 1 & s \end{bmatrix}}{s^2+4s+3} = \begin{bmatrix} \frac{s+4}{s^2+4s+3} & \frac{-3}{s^2+4s+3} \\ \frac{1}{s^2+4s+3} & \frac{s}{s^2+4s+3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+4}{(s+3)(s+1)} & \frac{-3}{(s+3)(s+1)} \\ \frac{1}{(s+3)(s+1)} & \frac{s}{(s+3)(s+1)} \end{bmatrix} \quad \frac{s+4}{(s+3)(s+1)}$$

$$= \frac{-1/2}{s+3} + \frac{3/2}{s+1} \quad \frac{-3}{(s+3)(s+1)} = \frac{3/2}{s+3} + \frac{-3/2}{s+1}$$

$$\frac{1}{(s+3)(s+1)} = \frac{-1/2}{s+3} + \frac{1/2}{s+1} \quad \frac{s}{(s+3)(s+1)}$$

$$= \frac{3/2}{s+3} + \frac{-1/2}{s+1}$$

State transition matrix $\phi(t) = e^{At}$ can be obtained by taking inverse Laplace transform

$$\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}$$

$$= L^{-1}\begin{bmatrix} \frac{-1/2}{s+3} + \frac{3/2}{s+1} & \frac{3/2}{s+3} + \frac{-3/2}{s+1} \\ \frac{-1/2}{s+3} + \frac{1/2}{s+1} & \frac{3/2}{s+3} + \frac{-1/2}{s+1} \end{bmatrix}$$

$$\therefore \phi(t) = \begin{bmatrix} -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t} & \frac{3}{2}e^{-3t} - \frac{3}{2}e^{-t} \\ -\frac{1}{2}e^{-3t} + \frac{1}{2}e^{-t} & \frac{3}{2}e^{-3t} - \frac{1}{2}e^{-t} \end{bmatrix}$$

Syllabus Topic : Time Domain Solution of State Equations

9.4 Time Domain Solution of State Equations :

Solution in Time Domain :

Consider the following linear time invariant state space equations.

$$\dot{x}(t) = Ax(t) + Br(t) \quad \text{...(9.4.1)}$$

$$y(t) = Cx(t) + Dr(t) \quad \text{...(9.4.2)}$$

Multiplying both sides by e^{-At} we get,

$$e^{-At}\dot{x}(t) = e^{-At}Ax(t) + e^{-At}Br(t) \quad \text{...(9.4.3)}$$

$$\therefore e^{-At}\dot{x}(t) - e^{-At}Ax(t) = e^{-At}Br(t) \quad \text{...(9.4.4)}$$

But $e^{-At}\dot{x}(t) - e^{-At}Ax(t) = \frac{d}{dt}[e^{-At}x(t)]$

$\therefore$ Equation (9.4.4) becomes,

$$\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Br(t) \quad \text{...(9.4.5)}$$

Integrating Equation (9.4.5) from 0 to 't' we get

$$[e^{-At}x(t)]_0^t = \int_0^t e^{-A\tau}Br(\tau)\,d\tau$$

$$\therefore e^{-At}x(t) - x(0) = \int_0^t e^{A\tau}Br(\tau)\,d\tau$$

$$e^{-At}x(t) = x(0) + \int_0^t e^{-A\tau}Br(\tau)\,d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{At}e^{-A\tau}Br(\tau)\,d\tau$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Br(\tau)\,d\tau \quad \text{...(9.4.6)}$$

According to these equations the property of e^{At} is to take system from any state x (0) or x $(t - t_0)$ to state at t i.e. x (t). Therefore it is known as the **state transition matrix** and it is denoted as,

$$\phi(t) = e^{At}$$

$$\phi(t_0) = e^{A(t-t_0)}$$

Putting this value of x (t) in output equation we can get value of y (t) of the system.

Syllabus Topic : Solution of State Equations using Laplace Transform

9.5 Solution of State Equations using Laplace Transform :

Solution using Laplace Transform :

- Consider the equation $\dot{x}(t) = Ax(t) + Br(t)$
- Taking Laplace transform of above equation we get,

$$sX(s) - x(0) = AX(s) + BR(s)$$

$$(sI - A)X(s) = x(0) + BR(s)$$

$$X(s) = \underbrace{(sI - A)^{-1}x(0)}_{\text{zero input response}} + \underbrace{(sI - A)^{-1}BR(s)}_{\text{zero state response or forced response.}}$$

- Forced response is due to input r (t). Here $(sI - A)^{-1}$ is responsible for converting initial conditions and input to output. Hence it is called state transition matrix. In time domain state transition matrix is,

$$\phi(t) = L^{-1}[(sI - A)^{-1}]$$

- It is also given by, $\phi(t) = e^{At}$

Now output equation is given as,

$$y(t) = Cx(t) + Dr(t)$$

Taking Laplace Transform $Y(s) = CX(s) + DR(s)$

Substituting X (s) in above equation we get,

$$Y(s) = C[(sI - A)^{-1}x(0) + (sI - A)^{-1}BR(s)] + DR(s)$$

$$= C(sI - A)^{-1}x(0) + [C(sI - A)^{-1}B + D]R(s) \quad \text{...(9.5.1)}$$

From above equation we can obtain transfer function as,

$$Y(s) = C[(sI - A)^{-1}B + D]R(s)$$

...Assuming initial condition as 0

$$\therefore T(s) = \frac{Y(s)}{R(s)} = C(sI - A)^{-1}B + D$$

- This equation we have also obtained earlier. Inverse Laplace transform gives output of the system as well as solution of state equations.

Ex. 9.5.1 : Find the state transition matrix for,

$$A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$$

Soln. : First find A^2, A^3, A^4...

$$A^2 = \begin{bmatrix} -6 & -5 \\ 30 & 19 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 30 & 19 \\ -114 & -65 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} -114 & -65 \\ 390 & 211 \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}t + \begin{bmatrix} -6 & -5 \\ 30 & 19 \end{bmatrix}\frac{t^2}{2} + \begin{bmatrix} 30 & 19 \\ -114 & -65 \end{bmatrix}\frac{t^3}{6} + \ldots$$

$$\therefore e^{At} = \begin{bmatrix} 1 - \frac{6t^2}{2!} + \frac{30t^3}{3!} - \frac{114t^4}{4!} + \ldots & t - \frac{5t^2}{2!} + \frac{19t^3}{3!} - \frac{65t^4}{4!} + \ldots \\ -6t + \frac{30t^2}{2!} - \frac{114t^3}{3!} + \frac{390t^4}{4!} \ldots & 1 - 5t + \frac{19t^2}{2!} - \frac{-65t^3}{3!} + \frac{211t^4}{4!} + \ldots \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

Ex. 9.5.2 : Obtain the response of the states of the following system :

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r(t)$$

Where, r (t) is unit step occurring at t = 0 and $x_0^T = [1\ \ 0]$.

Soln. :

The state transition matrix is given by,

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots$$

$$\therefore e^{At} = \begin{bmatrix} 1 - t + \frac{1}{2}t^2 - \frac{1}{6} + t^3 + \ldots & 0 \\ 0 & 1 - 2t + \frac{4}{2}t^2 - \frac{8}{6}t^3 + \ldots \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix} \quad \text{...(1)}$$

The time response of the system is given by

$$x(t) = e^{At}\left[x_0 + \int_0^t e^{-A\tau} B\tau(r)\,d\tau\right] \quad ...(2)$$

It is given that $r = 1, t \geq 0$ so,

$$e^{-A\tau} Br = \begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{2\tau} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{\tau} \\ e^{2\tau} \end{bmatrix}$$

$$\therefore \int_0^t e^{-A\tau} Br\,d\tau = \begin{bmatrix} e^{+\tau} - 1 \\ \frac{1}{2}(e^{2t} - 1) \end{bmatrix} \quad ...(3)$$

Substituting Equations (1) and (3) in Equation (2) we get,

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} e^{t} - 1 \\ \frac{1}{2}(e^{2t} - 1) \end{bmatrix}\right\}$$

$$\therefore x(t) = \begin{bmatrix} 1 \\ \frac{1}{2}(1 - e^{-2t}) \end{bmatrix}$$

Review Questions

Q. 1 Define the terms :
(i) State (ii) State variables
(iii) State vector (iv) State space
(v) State equation

Q. 2 Obtain the state transition matrix of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Q. 3 Explain the following terms :
(i) State
(ii) State variables
(iii) State equations
(iv) State transition matrix.

Q. 4 Write the derivation of the transfer function from the state equation.

Q. 5 Write short note on advantages and limitations of state variable approach.

Q. 6 Write short note on Salient features of state variable model.

Q. 7 Explain advantages of state variable method over conventional methods for analysis of system.

Q. 8 Write short note on State variable representation.

□□□

CHAPTER 10

Module 5

Realization Structures

Syllabus :

Systems with finite duration and infinite duration, impulse response, recursive and non-recursive discrete time system, realization structures : direct form-I, direct form-II Transpose, cascade, and parallel forms.

10.1 Systems with Finite Duration and Infinite Duration, Impulse Response, Recursive and Nonrecursive Systems

We have already discussed these topics in chapter no. 3.

Syllabus Topic : Realization Structures

10.2 Realization Structures for FIR Systems :

Transfer Function of FIR Filters :

- We know that FIR stands for finite impulse response. The difference equation of FIR system is,

$$y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k) \quad ...(10.2.1)$$

Equation (10.2.1) shows that the system has length M as the limits of summation are from 0 to M – 1. These limits also indicates that the system is causal.

- Taking Z-transform of Equation (10.2.1) we get,

$$Y(Z) = \sum_{k=0}^{M-1} h(k)\,Z^{-k}\,X(Z) \quad ...(10.2.2)$$

Here we have used time shifting property of Z-transform.

$$\therefore Z\{x(n-k)\} = Z^{-k}X(k)$$

- Now the system transfer function is given by,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

Thus from Equation (10.2.2) we get,

$$\boxed{H(Z) = \frac{Y(Z)}{X(Z)} = \sum_{k=0}^{M-1} h(k)\,Z^{-k}} \quad ...(10.2.3)$$

Equation (10.2.3) is called as system transfer function of FIR filter.

10.2.1 Direct Form Structure :

- The direct form realization of FIR filter can be obtained by using the equation of linear convolution. It is,

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k) \quad ...(10.2.4)$$

- If we consider that there are 'M' samples then Equation (10.2.4) becomes,

$$y(n) = \sum_{k=0}^{M-1} h(k)\,x(n-k) \quad ...(10.2.5)$$

Expanding Equation (10.2.5) we get,

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + ... + h(M-1)x(n-M+1) \quad ...(10.2.6)$$

How to draw the structure ?

- Now we will study how to draw structure (block diagram) for this filter. The first term in Equation (10.2.3) is $h(0)x(n)$.
- Here $x(n)$ is input and $h(0)$ is first sample of $h(n)$. So $h(0)$ is constant. That means $h(0)x(n)$ indicates that input $x(n)$ is multiplied by constant $h(0)$. It is represented as shown in Fig. 10.2.1(a).

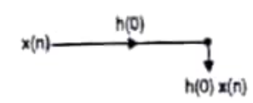


Fig. 10.2.1(a) : $h(0)x(n)$

- The second term in Equation (10.2.3) is $h(1)x(n-1)$. Here $x(n-1)$ indicates delay of input $x(n)$ by 1 sample.
- The term $h(1)x(n-1)$ is represented as shown in Fig. 10.2.1(b).

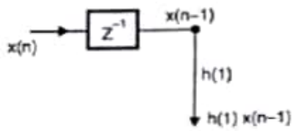


Fig. 10.2.1(b) : h (1) x (n − 1)

- Now, the third term is h (2) x (n − 2). Here x (n − 2) indicates delay of input x (n) by two samples. It is represented as shown in Fig. 10.2.1(c).

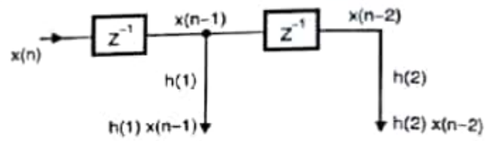


Fig. 10.2.1(c) : h (2) x (n − 2)

- Similarly we can draw the block diagram for remaining terms of Equation (10.2.6). The total structure for Equation (10.2.6) can be drawn by adding the block diagrams for all the terms. This structure is shown in Fig. 10.2.2(a). It is called as direct form realization of FIR system.

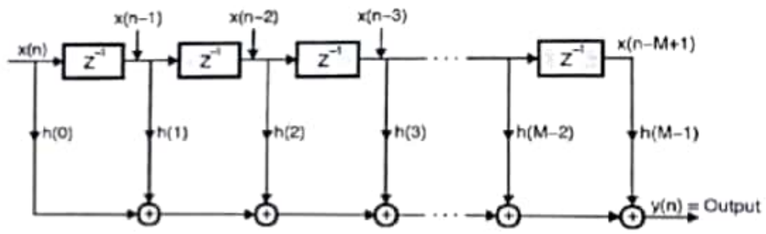


Fig. 10.2.2(a) : Direct form realization of FIR system

Observations :

1. There are 'M − 1' delay blocks. So this is canonic structure.
2. Input signal is delayed 'M − 1' times. So to store this delayed input signal 'M − 1' memory locations are required.
3. This structure has 'M − 1' additions and 'M' multiplications.
4. Equation (10.2.3) shows that present input x (n) and past inputs (delayed inputs) e.g. x (n − 1), x (n − 2), etc. are multiplied by the corresponding sample of h (n) that is h (0), h (1)... etc. Thus output y (n) is weighted linear combination of present input and past inputs.
5. If the structure looks like, Fig. 10.2.2(b) then it is called as pick– off node or tapped line. Now we can draw the same Fig. 10.2.2(a) as shown in Fig. 10.2.2(c). In this case it looks like **tapped delay line**. Therefore the direct form realization is also called as **tapped or transversal delay line filter.**

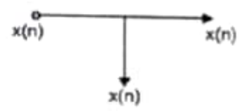


Fig. 10.2.2(b) : Pick–off node

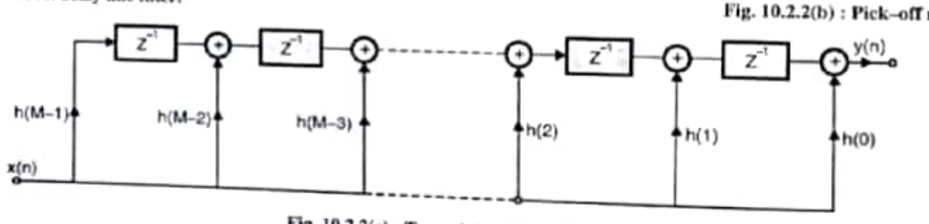


Fig. 10.2.2(c) : Tapped delay line filter

Syllabus Topic : Cascade

10.2.2 Cascade Form Structure :

- Cascade means the number of stages are connected in series. Now we have the system transfer function of the FIR system is given by,

$$H(Z) = \sum_{k=0}^{M-1} h(k) Z^{-k} \qquad \text{...(10.2.7)}$$

- Generally the higher order FIR filter is realized by using a series connection of different FIR sections (cascade connection).

- Here each section is characterized by the second order transfer function. Then due to cascade connection ; the total transfer function [H (Z)] will be multiplication of all second order transfer functions.

$$\therefore H(Z) = H_1(Z) \cdot H_2(Z) \cdot H_3(Z) \ldots H_k(Z) \qquad \text{...(10.2.8)}$$

Here $H_k(Z)$ is the second order transfer function and it is given by,

$$H_k(Z) = h(k_0) + h(k_1) Z^{-1} + h(k_2) Z^{-2} \qquad \text{...(10.2.9)}$$

But we know that, $H_k(Z) = \dfrac{Y_k(Z)}{X_k(Z)}$

$$\therefore \frac{Y_k(Z)}{X_k(Z)} = h(k_0) + h(k_1) Z^{-1} + h(k_2) Z^{-2}$$

$$\therefore Y_k(Z) = h(k_0) X_k(Z) + h(k_1) Z^{-1} X_k(Z) + h(k_2) Z^{-2} X_k(Z) \qquad \text{...(10.2.10)}$$

Taking inverse Z–transform (IZT) of both sides we get,

$$y_k(n) = h(k_0) x_k(n) + h(k_1) x_k(n-1) + h(k_2) x_k(n-2) \qquad \text{...(10.2.11)}$$

From Equation (10.2.11), we can draw the direct form realization of second order section as shown in Fig. 10.2.3(a).

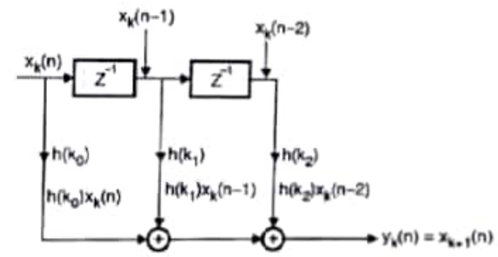


Fig. 10.2.3(a) : Direct–form realization of second order section

- Now the total structure is obtained by cascading all second order sections. The block schematic of cascade structure is shown in Fig. 10.2.3(b).

Fig. 10.2.3(b) : Block diagram of cascade structure

- Thus using Figs. 10.2.3(a) and 10.2.3(b), we can draw the cascade realization of FIR system as shown in Fig. 10.2.3(c).

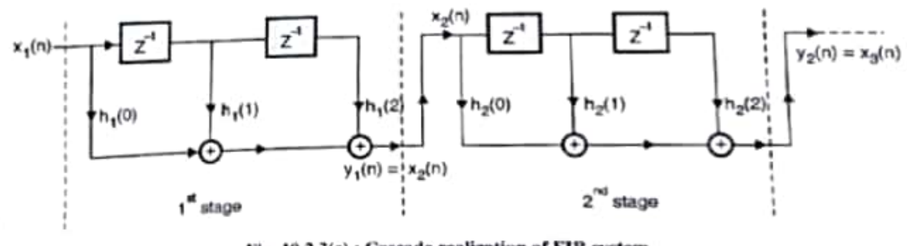


Fig. 10.2.3(c) : Cascade realization of FIR system

Note : This cascade realization is obtained by putting K = 0 in first stage and K = 1 in second stage.

Observations :

1. This cascade form is canonic with respect to delay.
2. It has M − 1 adders and M multipliers for (M − 1)th order FIR transfer function.

10.3 Realization Structures for IIR Systems :

10.3.1 Structure for IIR Filter :

- We know that IIR stands for infinite impulse response. Generally IIR systems are recursive type.
- A recursive system means; feedback connection is present from output side to the input side.
- For the realization of IIR systems present, past, future samples of input and past values of output are required.
- For the realization of IIR systems, following structures are used.

1. Direct form structure
2. Cascade form structure
3. Parallel form structure.

10.3.2 Direct Form Structure for IIR Systems :

Direct form structure is again divided into two types :

(A) Direct form-I structure
(B) Direct form-II structure.

- The general difference equation for discrete time LTI system is given as.

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad ...(10.3.1)$$

- Now let $H_1(Z)$ = Numerator term of Equation (10.3.1)

$$\therefore H_1(Z) = \sum_{k=0}^{M} b_k Z^{-k} \quad ...(10.3.2)$$

And let, $$H_2(Z) = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad ...(10.3.3)$$

Putting Equations (10.3.2) and (10.3.3) in Equation (10.3.1) we get,

$$H(Z) = H_1(Z) \cdot H_2(Z) \quad ...(10.3.4)$$

- This equation shows that $H(Z)$ can be represented as the cascade connection of $H_1(Z)$ and $H_2(Z)$ as shown in Fig. 10.3.1(a).

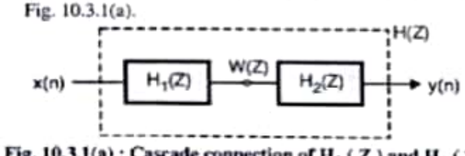


Fig. 10.3.1(a) : Cascade connection of $H_1(Z)$ and $H_2(Z)$

- We know that $H_1(Z)$ is the transfer function of the numerator term. Numerator always contain zeros of the system. So $H_1(Z)$ is called as **all zero system**. Now $H_2(Z)$ is transfer function of denominator. Denominator always contain poles of the system. So $H_2(Z)$ is called as **all pole system**.

Syllabus Topic : Realization Structures : Direct form-I

10.3.3 Direct form-I Structure :

The direct form-I structure is obtained by cascading (connecting in series), the structure for $H_1(Z)$ and $H_2(Z)$. First we will draw the direct form structure for $H_1(Z)$.

Direct form structure for $H_1(Z)$:

- Recall the equation of $H_1(Z)$ (Equation (10.3.2)). It is,

$$H_1(Z) = \sum_{k=0}^{M} b_k Z^{-k} \quad ...(10.3.5)$$

But we know that, $$H(Z) = \frac{\text{Output}[Y(Z)]}{\text{Input}[X(Z)]}$$

- To avoid the confusion we will write,

$$H_1(Z) = \frac{W(Z)}{X(Z)}$$; where $W(Z)$ is output of first stage.

$$\frac{W(Z)}{X(Z)} = \sum_{k=0}^{M} b_k Z^{-k} \quad ...(10.3.6)$$

$$\therefore \frac{W(Z)}{X(Z)} = b_0 Z^0 + b_1 Z^{-1} + b_2 Z^{-2} + ... + b_M Z^{-M}$$

But $Z^0 = 1$

$$\therefore W(Z) = b_0 X(Z) + b_1 Z^{-1} X(Z) + b_2 Z^{-2} X(Z) + ... + b_M Z^{-M} X(Z) \quad ...(10.3.7)$$

- Taking inverse Z-transform (IZT) of Equation (10.3.7) we get,

$$y_1(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + ... + b_M x(n-M) \quad ...(10.3.8)$$

- Here $b_0, b_1, b_2 ... b_M$ are the coefficients. The direct form realization of Equation (10.3.8) is shown in Fig. 10.3.1(b).

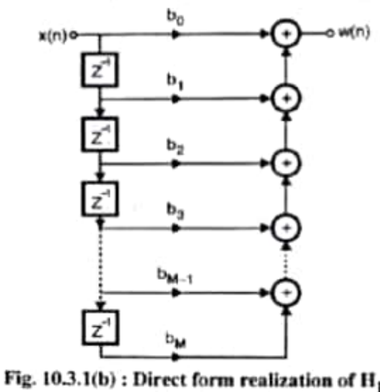


Fig. 10.3.1(b) : Direct form realization of $H_1(Z)$

(All zero system)

Direct form structure for $H_2(Z)$:

- Recall the equation for $H_2(Z)$. It is,

$$H_2(Z) = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad ...(10.3.9)$$

We have, $$H_2(Z) = \frac{\text{Output}[Y(Z)]}{\text{Input}[X(Z)]} \quad ...(10.3.10)$$

- Now $H_2(Z)$ represents second stage of Fig. 10.3.1(a). Input of second stage is the output of first stage.
- Thus input of $H_2(Z)$ is $W(Z)$ and output of $H_2(Z)$ is the output of overall system which is $Y(Z)$.

Thus Equation (10.3.10) becomes,

$$H_2(Z) = \frac{Y(Z)}{W(Z)}$$

Putting this value in Equation (10.3.9) we get,

$$\frac{Y(Z)}{W(Z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}}$$

$$\therefore Y(Z)\left[1 + \sum_{k=1}^{N} a_k Z^{-k}\right] = W(Z)$$

$$\therefore Y(Z) + Y(Z)\left[\sum_{k=1}^{N} a_k Z^{-k}\right] = W(Z)$$

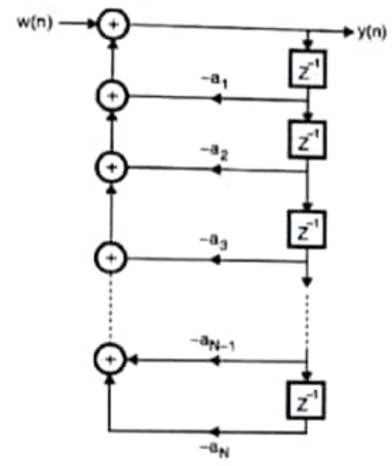


Fig. 10.3.1(c) : Direct form realization of $H_2(Z)$

(All pole system)

$$\therefore Y(Z) = -\left[\sum_{k=1}^{N} a_k Z^{-k}\right] Y(Z) + W(Z) \quad ...(10.3.11)$$

Expanding the summation we get,

$$Y(Z) = -\left[a_1 Z^{-1} + a_2 Z^{-2} + ... + a_N Z^{-N}\right] Y(Z) + W(Z)$$

$$\therefore Y(Z) = -a_1 Z^{-1} Y(Z) - a_2 Z^{-2} Y(Z) ... - a_N Z^{-N} Y(Z) + W(Z) \quad ...(10.3.12)$$

Taking IZT of Equation (10.3.12) we get,

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) ... - a_N y(n-N) + w_n \quad ...(10.3.13)$$

Here $-a_1, -a_2 ... -a_N$ are the coefficients.

- The direct form implementation of Equation (10.3.13) is shown in Fig. 10.3.1(c).
- Here all R.H.S. terms of Equation (10.3.13) except $w(n)$ are the delayed output terms. Thus this is a feedback (recursive) connection. Now **direct form-I** structure is obtained by cascading $H_1(Z)$ and $H_2(Z)$.
- Thus direct form-I realization of IIR system is obtained by connecting Fig. 10.3.1(b) and Fig. 10.3.1(c) in series. It is shown in Fig. 10.3.1(d).

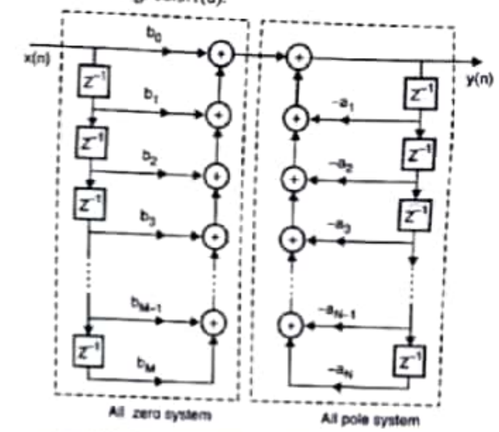


Fig. 10.3.1(d) : Direct form-I realization of IIR system

Computation complexity :

The computational complexity of direct form-I structure is as follows :

1. Number of multiplications = M + N + 1
2. Number of additions = M + N
3. Number of memory locations = M + N + 1.

Syllabus Topic : Realization Structures : Direct form-II

10.3.4 Direct form-II Structure :

- Observe Fig. 10.3.1(d). Here, first block represents all zero system and second block represents all pole system. Thus, in direct form-I structure ; zeros of $H(Z)$ are realized first and the poles of $H(Z)$ are realized second.
- Now we are studying these structures for LTI (linear time invariant) systems. Since the systems are linear we can interchange the positions of $H_1(Z)$ and $H_2(Z)$.
- This will give us the direct form-II structure. Thus in direct form-II structure; poles are realized first and zeros second.

We have, $$H(Z) = H_1(Z) \cdot H_2(Z) \quad ...(10.3.14)$$

- We will interchange the equations of $H_1(Z)$ and $H_2(Z)$.

$$\therefore H_1(Z) = \frac{1}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad ...(10.3.15)$$

and $$H_2(Z) = \sum_{k=0}^{M} b_k Z^{-k} \quad ...(10.3.16)$$

- The cascade connection of $H(Z) = H_1(Z) \cdot H_2(Z)$ is shown in Fig. 10.3.2(a).

Here W (Z) represents the output of first block.

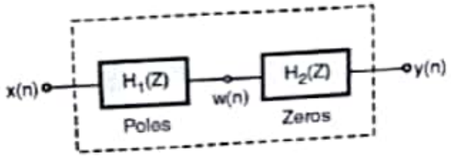


Fig. 10.3.2(a) : Decomposition of direct form-II realization

(1) All pole system :

We can write,

$$H_1(Z) = \frac{\text{Output}}{\text{Input}} = \frac{W(Z)}{X(Z)} \qquad ...(10.3.17)$$

Comparing Equations (10.3.15) and (10.3.16) we get,

$$\frac{W(Z)}{X(Z)} = \frac{1}{1+\sum_{k=1}^{N} a_k Z^{-k}}$$

$$\therefore W(Z)\left[1+\sum_{k=1}^{N} a_k Z^{-k}\right] = X(Z)$$

$$\therefore W(Z)+W(Z)\left[\sum_{k=1}^{N} a_k Z^{-k}\right] = X(Z)$$

$$\therefore W(Z) = X(Z)-W(Z)\left[\sum_{k=1}^{N} a_k Z^{-k}\right]$$

Expanding the summation we get,

$$W(Z) = X(Z)-a_1 Z^{-1}W(Z)-a_2 Z^{-2}W(Z) .... - a_N Z^{-N}W(Z) \qquad ...(10.3.18)$$

Taking IZT of Equation (10.3.18) we get,

$$w(n) = x(n)-a_1 x(n-1)-a_2 x(n-2) ..... - a_N x(n-N) \qquad ...(10.3.19)$$

The direct form implementation of Equation (10.3.19) is shown in Fig. 10.3.2(b).

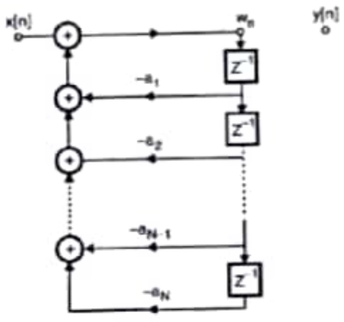


Fig. 10.3.2(b) : Direct form realization of $H_1(Z)$

(All pole system)

(2) All zero system :

- As shown in Fig. 10.3.2(a), the output of second stage is the overall output of the system which is Y (Z). While input to this stage is the output of first stage, which is W (Z).

Thus we can write,

$$H_2(Z) = \frac{\text{Output}}{\text{Input}} = \frac{Y(Z)}{W(Z)} \qquad ...(10.3.20)$$

Comparing Equations (10.3.16) and (10.3.20) we get,

$$\frac{Y(Z)}{W(Z)} = \sum_{k=0}^{M} b_k Z^{-k} \qquad ...(10.3.21)$$

$$\therefore Y(Z) = W(Z)\left[\sum_{k=0}^{M} b_k Z^{-k}\right]$$

Expanding the summation we get,

$$Y(Z) = W(Z)\left[b_0 Z^0 + b_1 Z^{-1} + b_2 Z^{-2} + ... + b_M Z^{-M}\right]$$

Here $Z^0 = 1$; multiplying by W (Z) we get,

$$Y(Z) = b_0 W(Z) + b_1 Z^{-1} W(Z) + b_2 Z^{-2} W(Z) + ... + b_M Z^{-M} W(Z) \qquad ...(10.3.22)$$

Taking IZT of Equation (10.3.22) we get,

$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + ... + b_m w(n-M) \qquad ...(10.3.23)$$

- The direct form implementation of Equation (10.3.23) is shown in Fig. 10.3.2(c).

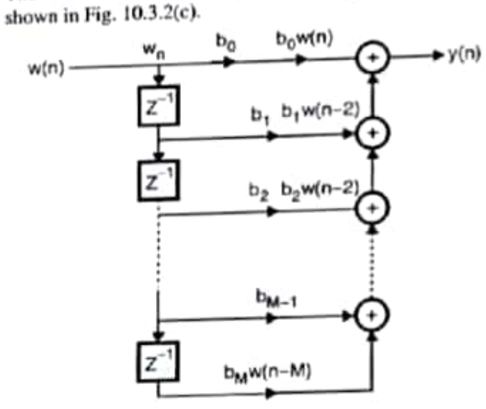


Fig. 10.3.2(c) : Direct form realization of $H_2(Z)$

(All zero system)

- Now **direct form-II** structure is obtained by cascading $H_1(Z)$ and $H_2(Z)$. That means it is obtained by connecting Fig. 10.3.2(b) and Fig. 10.3.2(c) in series. This is shown in Fig. 10.3.2(d).
- In Fig. 10.3.2(d), separate delay elements are used. Generally common delay elements are used in direct form-II structure. By using common delays, direct form II structure is drawn as shown in Fig. 10.3.2(e).
- Note that here we have considered N = M.

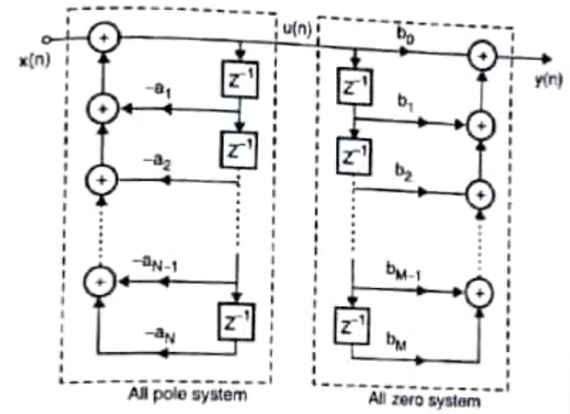


Fig. 10.3.2(d) : Direct form-II realization of IIR system

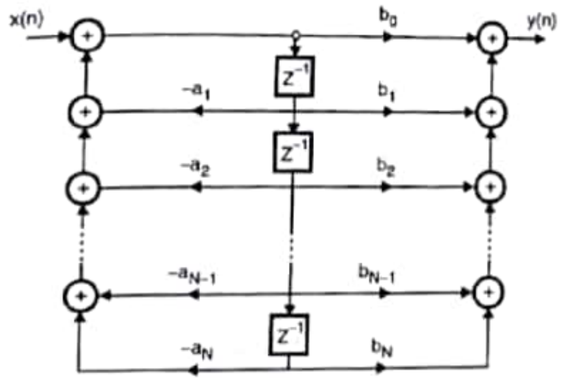


Fig. 10.3.2(e) : Direct form-II realization using common delay elements

Computation complexity :

The computational complexity for direct form-II realization of IIR is as follows :

1. Number of multiplications = M + N + 1.
2. Number of additions = M + N.
3. The number of delay elements are reduced in direct form-II structure, compared to direct form-I structure. That means the memory locations are reduced. The memory locations required for direct form-II structure are { M , N }.

Why these structure are called as direct form structures ?

The direct form-I and direct form-II structures are obtained directly from the corresponding transfer functions without any rearrangements. So these structure is called as direct form structures.

Advantage :

The only advantage of direct form realization is its implementation which is easy. We prefer direct form-II structure compared to direct form-I structure; because less memory locations are required.

Disadvantage :

Both direct form structures are sensitive to the effects of quantization errors in the coefficients. So in the practical applications ; these structures are not preferred.

10.3.5 Cascade Form Structures :

- To obtain the cascade form realization; the numerator and denominator of given transfer function H (Z) is factored into the product of second order terms.
- Then the total transfer function H (Z) is expressed as,

$$H(Z) = H_1(Z)\cdot H_2(Z) ..... H_k(Z) \qquad ...(10.3.24)$$

- Here $H_1(Z)$, $H_2(Z)$ ... $H_k(Z)$ are second order polynomials. Then each subtransfer function (H_1 , H_2 ...etc.) can be realized using direct form-I or direct form-II structures.
- The total transfer function is obtained by connecting all second order subsystems in series as shown in Fig. 10.3.3(a).

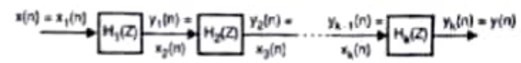


Fig. 10.3.3(a) : Cascade form realization

- Now we have the general difference equation for discrete time LTI system given by,

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1+\sum_{k=1}^{N} a_k Z^{-k}} \qquad ...(10.3.25)$$

- Since $H_1(Z)$, $H_2(Z)$ are second order polynomials ; we can write the second order differential equation for $H_k(Z)$ by putting M = N = 2 in Equation (10.3.25).

$$\therefore H_k(Z) = \frac{\sum_{k=0}^{2} b_k Z^{-k}}{1+\sum_{k=1}^{2} a_k Z^{-k}}$$

$$\therefore H_k(Z) = \frac{b_0 Z^0 + b_1 Z^{-1} + b_2 Z^{-2}}{1+a_1 Z^{-1} + a_2 Z^{-2}}$$

But $Z^0 = 1$

$$\therefore H_k(Z) = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2}}{1+a_1 Z^{-1} + a_2 Z^{-2}} \qquad ...(10.3.26)$$

- We can obtain direct form-II structure of Equation (10.3.26), similar to Fig. 10.3.3(e). Now, Fig. 10.3.3(e) shows the direct form-II realization for N = M ; that means for M stages.
- In this case we have second order equation. So by using Fig. 10.3.3(e) we can draw the direct form structure for Equation (10.3.26) by putting M = N = 2. This structure is shown in Fig. 10.3.4(b).

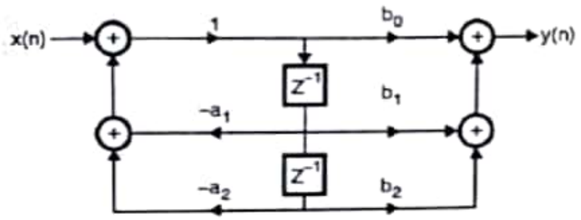


Fig. 10.3.4(b) : Direct form-II realization of second order subsystem

- So all such subsystems should be connected in series to obtain the cascade form realization of IIR system.

Syllabus Topic : Parallel Forms

10.3.6 Parallel Form Structure :

- We have the general difference equation for IIR systems given by,

$$H(Z) = \frac{\sum_{k=0}^{M} b_k Z^{-k}}{1 + \sum_{k=1}^{N} a_k Z^{-k}} \quad \text{...(10.3.27)}$$

- By using partial fraction expansion we can express overall transfer function H (Z) as,

$$H(Z) = C + H_1(Z) + H_2(Z) + \ldots + H_k(Z) \quad \text{...(10.3.28)}$$

- Here 'C' is constant and $H_1(Z)$, $H_2(Z)$... $H_k(Z)$ are second order subsystems. The general block schematic of parallel form realization structure for IIR system is as shown in Fig. 10.3.5.
- Here $H_1(Z)$, $H_2(Z)$...etc. can be realized by using direct form–I or direct form–II structures. Then all these structures are connected in parallel as shown in Fig. 10.3.5 to obtain the parallel form realization for IIR system.

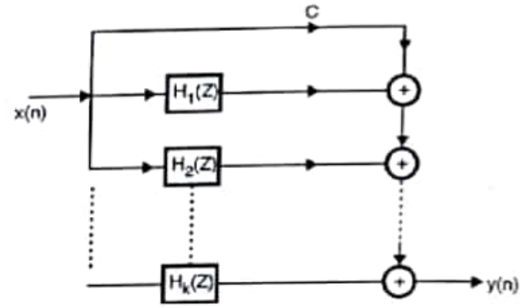


Fig. 10.3.5 : Parallel form realization for IIR system

Applications :

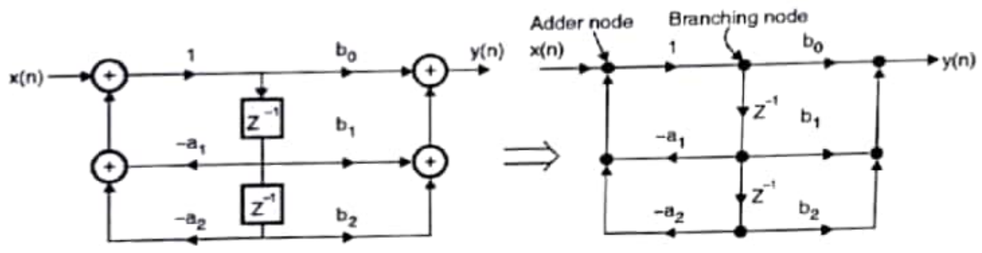
The parallel form realization is generally used for high speed filtering applications. Since this is a parallel connection ; the processing of filtering operation is performed parallely.

10.3.7 Representation of Structures using Signal Flow Graphs :

- Basically a signal flow graph is graphical representation of the block diagram structure. Both the signal flow graph and the block diagram structure provide the same information.
- For example, we will consider the second order subsystem. We know that the transfer function of second order subsystem is given by,

$$H(Z) = \frac{b_0 + b_1 Z^{-1} + b_2 Z^{-2}}{1 + a_1 Z^{-1} + a_2 Z^{-2}} \quad \text{...(10.3.29)}$$

- We have already drawn the direct form-II realization of second order subsystem (Fig. 10.3.2(d)).
- For the reference it is again drawn as shown in Fig. 10.3.6(a) and its signal flow graph representation is shown in Fig. 10.3.6(b).



(a) (b)

Fig. 10.3.6 : Direct form-II realization and its signal flow graph

How to draw a signal flow graph ?

From the given block diagram realization; it is simple to draw the signal flow graph. The procedure is as follows :

1. Replace all adders by adder nodes.

2. Whenever there are different branches ; draw the branching node.
3. Keep the directions of arrows and the corresponding coefficients as it is.
4. Replace every delay element by simple transmittance branch. For that branch, write Z^{-1} to indicate the delay operation.

There is 1:1 correspondence between the signal flow graph and block diagram representation. And the signal flow graph representation is much simple compared to block diagram representation.

Syllabus Topic : Transpose

10.3.8 Transposed Structures :

If two digital filter structures have the same transfer function then they are called as equivalent structures. By using the transpose operation, we can obtain equivalent structure from a given realization structure.

Transposition or flow graph reversal theorem :

If we reverse the directions of all branch transmittances and interchange input and output in the flow graph (or given structure) then the system transfer function remains unchanged. It is called as transposition theorem.

Procedure to obtain transposed structure and transposed flow graph :

1. Reverse all signal flow graph directions.
2. Change branching nodes into adders and vice-versa.
3. Interchange input and output.

The transposed structure for second order subsystem is shown in Fig. 10.3.7(a) and the corresponding transposed flow graph is shown in Fig. 10.3.7(b).

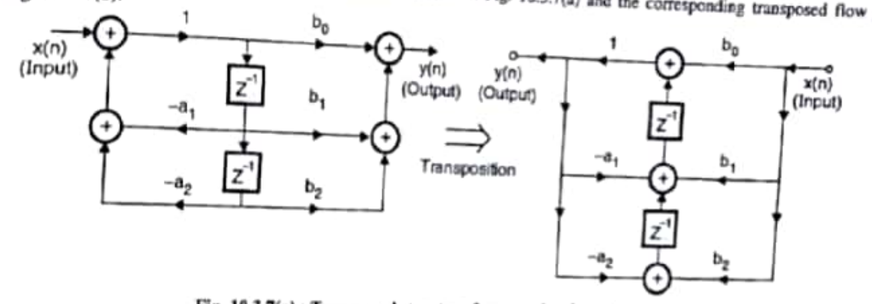


Fig. 10.3.7(a) : Transposed structure for second order subsystem

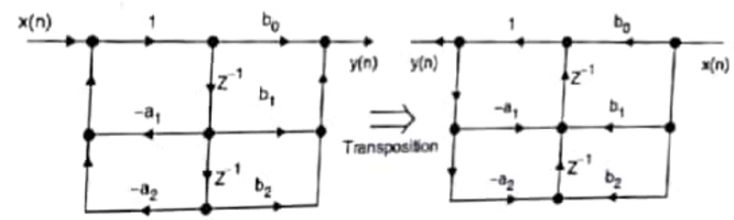


Fig. 10.3.7(b) : Transposed flow graph for second order subsystem

Solved Examples :

Ex. 10.3.1 : Obtain direct form I and II realization of a system described by, $y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) + \frac{1}{2} x(n-1)$.

Soln. : Calculation of H (Z)

Given equation is,

$$y(n) - \frac{3}{4} y(n-1) + \frac{1}{8} y(n-2) = x(n) + \frac{1}{2} x(n-1) \quad \text{...(1)}$$

Taking Z-transform of both sides we get,

$$Y(Z) - \frac{3}{4} Z^{-1} Y(Z) + \frac{1}{8} Z^{-2} Y(Z) = X(Z) + \frac{1}{2} Z^{-1} X(Z)$$

$$\therefore Y(Z)\left[1 - \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}\right] = X(Z)\left[1 + \frac{1}{2}Z^{-1}\right] \quad \text{...(2)}$$

Now the transfer function H (Z) is,

$$H(Z) = \frac{Y(Z)}{X(Z)}$$

Thus from Equation (2) we get,

$$H(Z) = \frac{1 + \frac{1}{2}Z^{-1}}{1 - \frac{3}{4}Z^{-1} + \frac{1}{8}Z^{-2}} \quad \text{...(3)}$$

We will obtain direct form-I and II structures using cascade connection. This cascade connection is shown in Fig. P. 10.3.1(a).

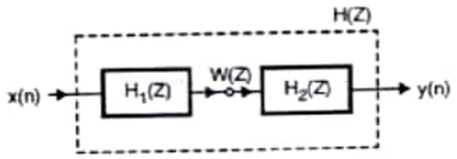


Fig. P. 10.3.1(a) : $H(Z) = H_1(Z) \cdot H_2(Z)$

Direct form-I structure :

We have,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

Let $H_1(Z) = 1 + \frac{1}{2} Z^{-1}$

But $H_1(Z) = \frac{\text{Output}}{\text{Input}} = \frac{W(Z)}{X(Z)}$

$$\therefore \frac{W(Z)}{X(Z)} = 1 + \frac{1}{2} Z^{-1}$$

$$\therefore W(Z) = X(Z)\left[1 + \frac{1}{2} Z^{-1}\right]$$

$$\therefore W(Z) = X(Z) + \frac{1}{2} Z^{-1} X(Z) \quad \text{...(4)}$$

Taking inverse Z-transform (IZT) of Equation (4) we get,

$$w(n) = x(n) + \frac{1}{2} x(n-1) \quad \text{...(5)}$$

The direct form-I realization of Equation (5) is as shown in Fig. P. 10.3.1(b).

Now let, $H_2(Z) = \frac{1}{1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}}$

But $H_2(Z) = \frac{\text{Output}}{\text{Input}} = \frac{Y(Z)}{W(Z)}$

$$\therefore \frac{Y(Z)}{W(Z)} = \frac{1}{1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}}$$

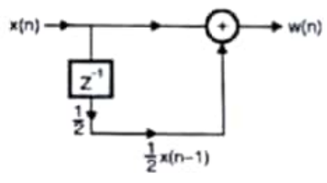


Fig. P. 10.3.1(b) : Direct form-I realization of $H_1(Z)$

$$\therefore Y(Z)\left[1 - \frac{3}{4} Z^{-1} + \frac{1}{8} Z^{-2}\right] = W(Z)$$

$$Y(Z) - \frac{3}{4} Z^{-1} Y(Z) + \frac{1}{8} Z^{-2} Y(Z) = W(Z)$$

$$\therefore Y(Z) = W(Z) + \frac{3}{4} Z^{-1} Y(Z) - \frac{1}{8} Z^{-2} Y(Z) \quad \text{...(6)}$$

Taking IZT of Equation (6),

$$y(n) = w(n) + \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) \quad \text{...(7)}$$

The direct form-I realization of Equation (7) is shown in Fig. P. 10.3.1(c).

Now direct form-I realization of total transfer function $H(Z)$ is obtained by cascading Fig. P. 10.3.1(b) and Fig. P. 10.3.1(c). This is shown in Fig. P. 10.3.1(d).

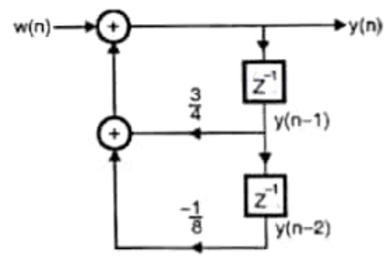


Fig. P. 10.3.1(c) : Direct form-I realization of $H_2(Z)$

Direct form-II realization :

The direct form-II realization is obtained by interchanging the positions of $H_1(Z)$ and $H_2(Z)$. That means by interchanging block-1 and block-2 of Fig. P. 10.3.1(d). It is shown in Fig. P. 10.3.1(e).

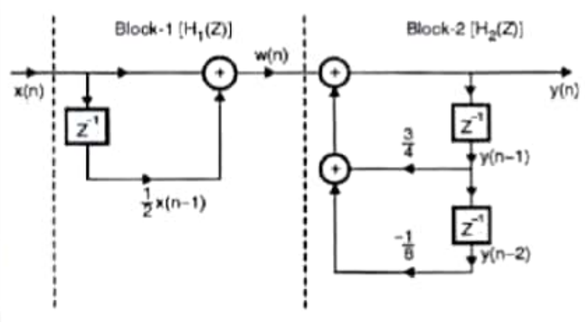


Fig. P. 10.3.1(d) : Direct form-I realization of $H(Z)$

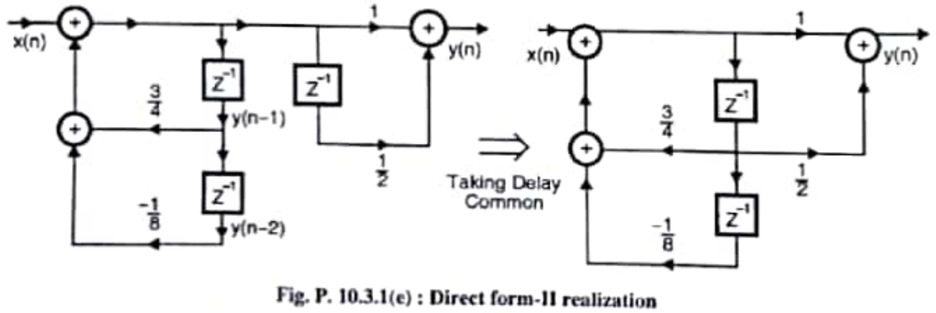


Fig. P. 10.3.1(e) : Direct form-II realization

Ex. 10.3.2 : Draw Direct Form-I IIR Filter structure for :

$$H(Z) = 3 + \frac{4Z}{Z - 0.5} - \frac{2}{Z - 0.25}$$

Soln. :

Given : $H(Z) = 3 + \frac{4Z}{Z - 0.5} - \frac{2}{Z - 0.25}$ ...(1)

Equation (1) can be written as,

$$H(Z) = 3 + \frac{4Z^2 - 3Z + 1}{Z^2 - 0.75Z + 0.125} \quad \text{...(2)}$$

or $$H(Z) = \frac{3[Z^2 - 0.75Z + 0.125] + 4Z^2 - 3Z - 1}{Z^2 - 0.75Z + 0.125}$$

Solving, $$H(Z) = \frac{7Z^2 - 5.25Z - 0.625}{Z^2 - 0.75Z + 0.125}$$

or $$H(Z) = \frac{7 - 5.25Z^{-1} - 0.625Z^{-2}}{1 - 0.75Z^{-1} + 0.125Z^{-2}} \quad \text{...(3)}$$

Now $H(Z) = H_1(Z) \cdot H_2(Z)$

Let $H_1(Z) = \frac{W(Z)}{X(Z)} = 7 - 5.25Z^{-1} - 0.625Z^{-2}$

$\therefore W(Z) = 7X(Z) - 5.25Z^{-1}X(Z)X(Z) - 0.625Z^{-2}X(Z)$

Taking IZT we get,

$\therefore w(n) = 7x(n) - 5.25x(n-1) - 0.625x(n-2)$

The realization of $H_1(Z)$ is shown in Fig. P. 10.3.2.

Now consider $H_2(Z) = \frac{Y(Z)}{W(Z)}$

$$= \frac{1}{1 - 0.75Z^{-1} + 0.125Z^{-2}}$$

$\therefore Y(Z) - 0.75Z^{-1}Y(Z) + 0.125Z^{-2}Y(Z) = W(Z)$

$\therefore Y(Z) = W(Z) + 0.75Z^{-1}Y(Z) - 0.125Z^{-2}Y(Z)$

Taking IZT we get,

$\therefore y(n) = w(n) + 0.75y(n-1) - 0.125y(n-2)$

The realization of $H_2(Z)$ is shown in Fig. P. 10.3.2(a).

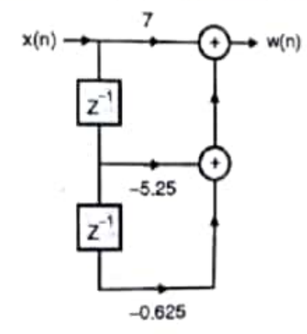


Fig. P. 10.3.2

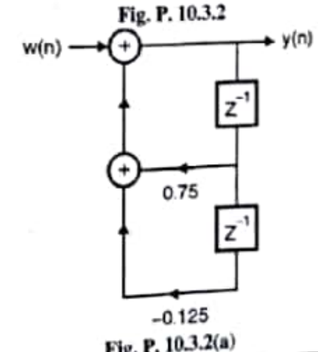


Fig. P. 10.3.2(a)

The direct Form-I realization is obtained by cascading $H_1(Z)$ and $H_2(Z)$. It is shown in Fig. P. 10.3.2(b).

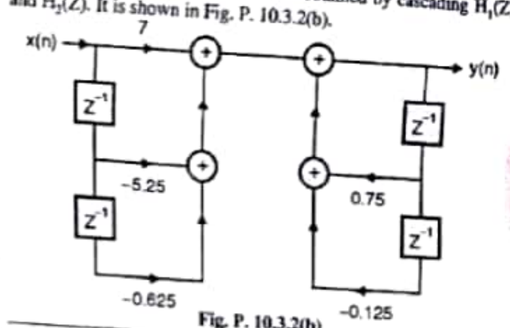


Fig. P. 10.3.2(b)

Ex. 10.3.3 : Realize the system described by following difference equation using direct Form-I structure $y(n) = y(n-1) - \frac{1}{2} y(n-2) + x(n) - x(n-1) + x(n-2)$.

Soln. : First we will obtain transfer function $H(Z)$.

Given equation is,

$$y(n) = y(n-1) - \frac{1}{2} y(n-2) + x(n) - x(n-1) + x(n-2)$$

Taking Z transform on both sides,

$$Y(Z) = Z^{-1}Y(Z) - \frac{1}{2}Z^{-2}Y(Z) + X(Z) - Z^{-1}X(Z) + Z^{-2}X(Z)$$

$$\therefore Y(Z) - Z^{-1}Y(Z) + \frac{1}{2}Z^{-2}Y(Z) = X(Z) - Z^{-1}X(Z) + Z^{-2}X(Z)$$

$$\therefore Y(Z)[1 - Z^{-1} + \frac{1}{2}Z^{-2}] = X(Z)[1 - Z^{-1} + Z^{-2}]$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1 - Z^{-1} + Z^{-2}}{1 - Z^{-1} + \frac{1}{2}Z^{-2}} \quad \text{...(1)}$$

Let $H_1(Z) = \frac{W(Z)}{X(Z)} = 1 - Z^{-1} + Z^{-2}$

$\therefore W(Z) = X(Z) - Z^{-1}X(Z) + Z^{-2}X(Z)$

Taking IZT we get,

$$w(n) = x(n) - x(n-1) + x(n-2) \quad \text{...(2)}$$

This realization is shown in Fig. P. 10.3.3(a).

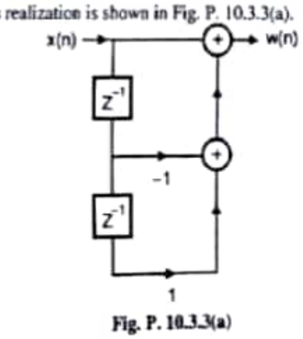


Fig. P. 10.3.3(a)

Now consider, $H_2(Z) = \frac{Y(Z)}{W(Z)} = \frac{1}{1 - Z^{-1} + \frac{1}{2}Z^{-2}}$

$$\therefore Y(Z) - Z^{-1}Y(Z) + \frac{1}{2}Z^{-2}Y(Z) - \frac{1}{2}Z^{-2}Y(Z) = W(Z)$$

$$\therefore \quad Y(Z) = W(Z) + Z^{-1}Y(Z) - \frac{1}{2}Z^{-2}Y(Z)$$

Taking IZT we get,

$$y(n) = w(n) + y(n-1) - \frac{1}{2}y(n-2) \qquad \ldots(3)$$

This realization is shown in Fig. P. 10.3.3(b).

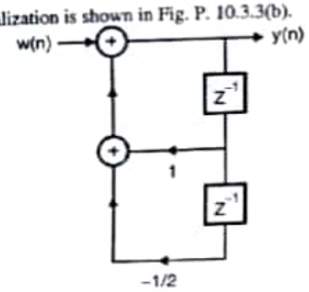


Fig. P. 10.3.3(b)

The direct Form-I realization is obtained by cascading structures of Fig. P. 10.3.3(a) and (b). It is shown in Fig. P. 10.3.3(c).

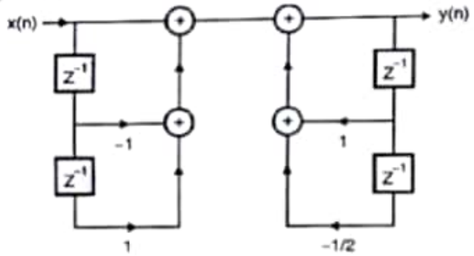


Fig. P. 10.3.3(c)

Ex. 10.3.4 : Consider a causal LTI system with system function,

$$H(Z) = \frac{1 + \frac{1}{5}Z^{-1}}{\left(1 - 0.5Z^{-1} + \frac{1}{3}Z^{-2}\right)(1 + 0.25Z^{-1})}$$

Draw the signal flow graph for implementation of the system using direct form–II.

Soln. : The given transfer function is,

$$H(Z) = \frac{1 + \frac{1}{5}Z^{-1}}{\left(1 - 0.5Z^{-1} + \frac{1}{3}Z^{-2}\right)(1 + 0.25Z^{-1})} \qquad \ldots(1)$$

It can be simplified as follows :

$$H(Z) = \frac{1 + \frac{1}{5}Z^{-1}}{1 - 0.5Z^{-1} + \frac{1}{3}Z^{-2} + 0.25Z^{-1} - 0.125Z^{-2} + 0.0833Z^{-3}}$$

$$\therefore \quad H(Z) = \frac{1 + \frac{1}{5}Z^{-1}}{1 - 0.25Z^{-1} + 0.208Z^{-2} + 0.0833Z^{-3}}$$

For the direct form-II structure, poles are realized first and then the zeros are realized. We can write,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

Here $H_1(Z)$ = All pole system

$$= \frac{1}{1 - 0.25Z^{-1} + 0.208Z^{-2} + 0.0833Z^{-3}}$$

and $H_2(Z)$ = All zero system $= 1 + 0.2Z^{-1}$

consider $H_1(Z)$ terms.

We have $H_1(Z) = \dfrac{\text{Output}}{\text{Input}} = \dfrac{W(Z)}{X(Z)}$

$$\therefore \quad \frac{W(Z)}{X(Z)} = \frac{1}{1 - 0.25Z^{-1} + 0.208Z^{-2} + 0.0833Z^{-3}}$$

$$\therefore \quad W(Z)[1 - 0.25Z^{-1} + 0.208Z^{-2} + 0.0833Z^{-3}] = X(Z)$$

$$\therefore \quad W(Z) - 0.25Z^{-1}W(Z) + 0.208Z^{-2}W(Z) + 0.0833Z^{-3}W(Z) = X(Z)$$

$$\therefore \quad W(Z) = X(Z) + 0.25Z^{-1}W(Z) - 0.208Z^{-2}W(Z) - 0.0833Z^{-3}W(Z) \qquad \ldots(2)$$

Taking IZT of Equation (2) we get,

$$w(n) = x(n) + 0.25\,w(n-1) - 0.208\,w(n-2) - 0.0833\,w(n-3) \qquad \ldots(3)$$

The direct form realization of Equation (3) is shown in Fig. P. 10.3.4(a).

Now consider $H_2(Z)$ that means all zero system.

$$H_2(Z) = 1 + 0.2Z^{-1} \quad \text{But } H_2(Z) = \frac{\text{Output}}{\text{Input}} = \frac{Y(Z)}{W(Z)}$$

$$\therefore \quad \frac{Y(Z)}{W(Z)} = 1 + 0.2Z^{-1}$$

$$\therefore \quad Y(Z) = W(Z) + 0.2Z^{-1}W(Z) \qquad \ldots(4)$$

Taking IZT of Equation (4) we get,

$$y(n) = w(n) + 0.2\,w(n-1) \qquad \ldots(5)$$

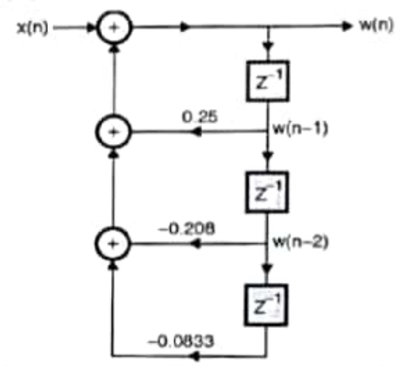


Fig. P. 10.3.4(a) : Direct form realization of $H_1(Z)$

The direct form realization of Equation (5) is shown in Fig. P. 10.3.4(b).

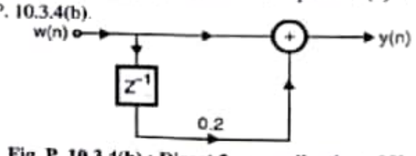


Fig. P. 10.3.4(b) : Direct form realization of $H_2(Z)$

Now direct form-II realization is obtained by connecting Fig. P. 10.3.4(a) and Fig. P. 10.3.4(b) in series (cascade connection). It is shown in Fig. P. 10.3.4(c).

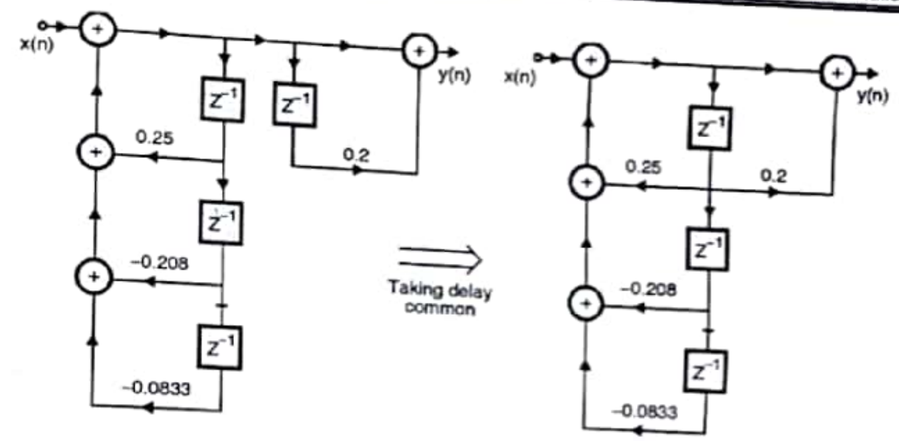


Fig. P. 10.3.4(c) : Direct form-II realization

Note : We can also obtain direct form-II realization by first drawing direct form-I structure. To draw direct form-I structure take $H_1(Z)$ = all zero system. In this example take

$H_1(Z) = 1 + 0.2Z^{-1}$ and take $H_2(Z)$ = all pole system, so here

$$H_2(Z) = \frac{1}{1 - 0.25Z^{-1} + 0.208Z^{-2} + 0.0833Z^{-3}}.$$

Then obtain direct form-I structure.

By interchanging the positions of $H_1(Z)$ and $H_2(Z)$; obtain direct form-II structure.

But this is tedious method. So, whenever it is asked to obtain direct form–II structure the take $H_1(Z)$ = all pole system and $H_2(Z)$ = all zero system. Then cascade $H_1(Z)$ and $H_2(Z)$ to obtain direct form–II structure.

Now the signal flow graph is as shown in Fig. P. 10.3.4(d).

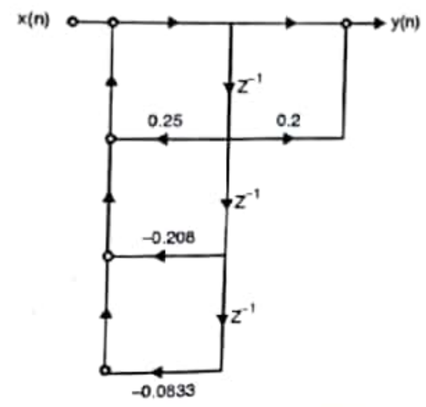


Fig. P. 10.3.4(d) : Signal flow graph

Shortcut method to draw direct Form–II structure :

In Ex. 10.3.1 and Ex. 10.3.2 we have obtained direct form-II structure by considering the cascade connection. That means by using $H(Z) = H_1(Z) \cdot H_2(Z)$. Now we will study the shortcut method to obtain direct form-II structure. By using this shortcut method we will obtain direct form-II structure for Ex. 10.3.2.

In Ex. 10.3.2 the given $H(Z)$ is,

$$H(Z) = \frac{1 + \frac{1}{5}Z^{-1}}{1 - 0.5Z^{-1} + 0.208Z^{-2} + 0.083Z^{-3}}$$

The different steps are as follows :

Step I : Write down three terms namely input $x(n)$, middle term $w(n)$ and output $y(n)$ as shown in Fig. P. 10.3.4(e).

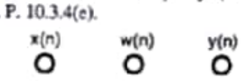


Fig. P. 10.3.4(e) : Step–I

Step II : Check the maximum power of Z inverse present in the numerator or denominator. It represents the maximum delay. For the given $H(Z)$ it is '– 3'. This indicates that total '3' delay boxes are required. Draw three Z^{-1} boxes below $w(n)$ as shown in Fig. P. 10.3.4(f).

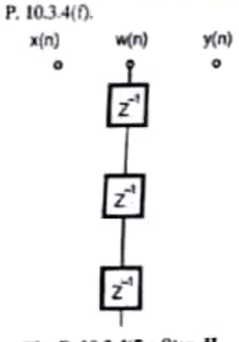


Fig. P. 10.3.4(f) : Step-II

Step III : Now consider the numerator term. Check the constant multiplier value. It is 0.2 and it is multiplied by Z^{-1}. So write this value at the R.H.S. and below first Z^{-1} box as shown in Fig. P. 10.3.4(g).

Then using the adder complete the connection as shown in Fig. P. 10.3.4(g).

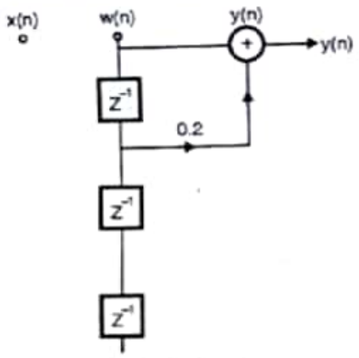


Fig. P. 10.3.4(g) : Step-III

Step IV : Now consider the denominator term. Write down the constant multiplier values at the L.H.S. part of diagram. While writing these values, change the sign. The original constant multipliers are '– 0.5' for Z^{-1}, 0.208 for Z^{-2} and 0.083 for Z^{-3}. So change the signs of these values and write down at the L.H.S. near the corresponding delay box. Then by using adders complete the connection as shown in Fig. P. 10.3.4(h).

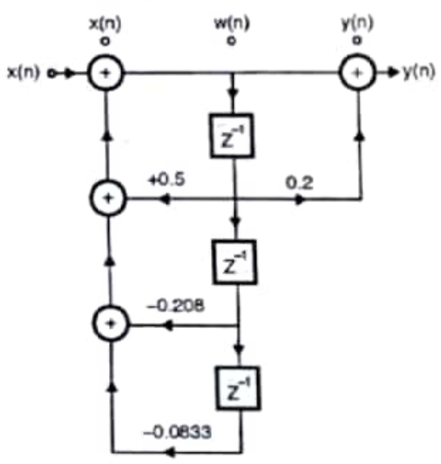


Fig. P. 10.3.4(h) : Step-IV

This diagram is same as Fig. P. 10.3.4(c). So by using this simple method; we can directly draw the direct form-II structure.

Now applying this method we will obtain direct form-II structure for Ex. 10.3.1. In this problem we have obtained the equation of $H(Z)$ as,

$$H(Z) = \frac{1+\frac{1}{2}Z^{-1}}{1-\frac{3}{4}Z^{-1}+\frac{1}{8}Z^{-2}}$$

Here maximum value of Z inverse is '2'. So two delay elements are required. For the numerator the constant multiplier value is $\frac{1}{2}$. While for the denominator term constant multiplier values are $-\frac{3}{4}$ for Z^{-1} and $\frac{1}{8}$ for Z^{-2}. After changing the signs we get $\frac{3}{4}$ for Z^{-1} and $-\frac{1}{8}$ for Z^{-2}. So we can draw direct form-II structure as shown in Fig. P. 10.3.4(i).

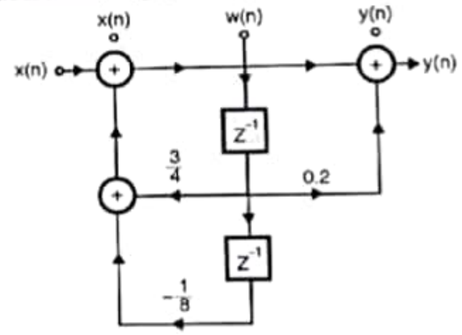


Fig. P. 10.3.4(i) : Direct form-II structure

Ex. 10.3.5 : Develop direct form-II realization of the transfer function.

$$H(Z) = \frac{3+3.6Z^{-1}+0.6Z^{-2}}{1+0.1Z^{-1}-0.2Z^{-2}}$$

Is it advantageous than direct form–I ?

Soln. :

For the direct form-II structure we have to realize the poles first and then zeros.

$\therefore$ Let $H_1(Z)$ =All pole system $= \dfrac{1}{1+0.1Z^{-1}-0.2Z^{-2}}$

Now $H_1(Z) = \dfrac{\text{Output}}{\text{Input}} = \dfrac{W(Z)}{X(Z)}$

$$\therefore \frac{W(Z)}{X(Z)} = \frac{1}{1+0.1Z^{-1}-0.2Z^{-2}}$$

$$\therefore W(Z)[1+0.1Z^{-1}-0.2Z^{-2}] = X(Z)$$

$$\therefore W(Z)+0.1Z^{-1}W(Z)-0.2Z^{-2}W(Z) = X(Z)$$

$$\therefore W(Z) = X(Z)-0.1Z^{-1}W(Z)+0.2Z^{-2}W(Z) \quad ...(1)$$

Taking IZT of Equation (1),

$$w(n) = x(n)-0.1\,w(n-1)+0.2\,w(n-2) \quad ...(2)$$

The direct form implementation of Equation (2) is shown in Fig. P. 10.3.5(a).

Now $H_2(Z)$ = All zero system $= 3+3.6Z^{-1}+0.6Z^{-2}$

But $H_2(Z) = \dfrac{\text{Output}}{\text{Input}} = \dfrac{Y(Z)}{W(Z)}$

$$\therefore \frac{Y(Z)}{W(Z)} = 3+3.6Z^{-1}+0.6Z^{-2}$$

$$\therefore Y(Z) = 3W(Z)+3.6Z^{-1}W(Z)+0.6Z^{-2}W(Z) \quad ...(3)$$

Taking IZT of Equation (3) we get,

$$y(n) = 3w(n)+3.6w(n-1)+0.6w(n-2) \quad ...(4)$$

The direct form implementation of Equation (4) is shown in Fig. P. 10.3.5(b).

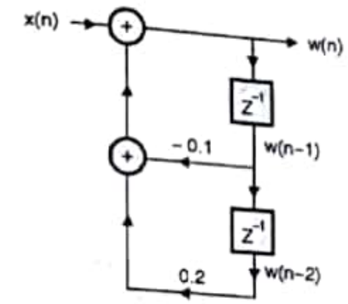


Fig. P. 10.3.5(a) : Direct form realization of $H_1(Z)$

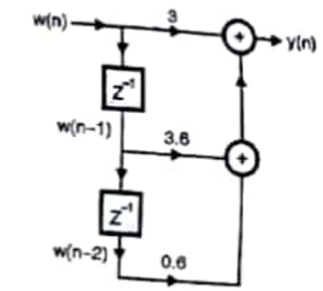


Fig. P. 10.3.5 : (b) Direct form realization of $H_2(Z)$

The direct form-II realization is obtained by cascading $H_1(Z)$ and $H_2(Z)$. It is shown in Fig. P. 10.3.5(c).

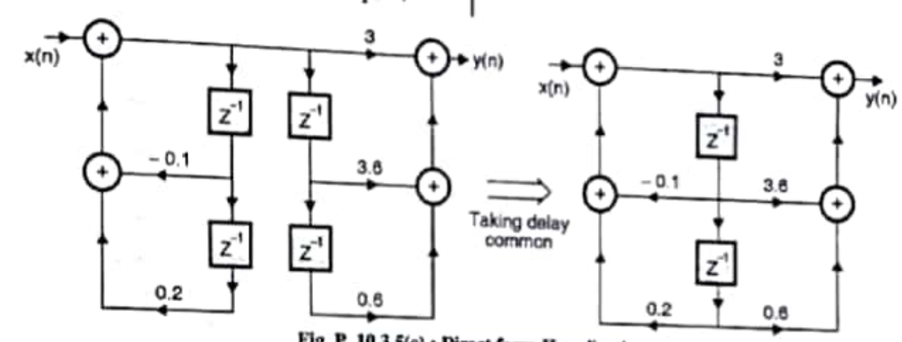


Fig. P. 10.3.5(c) : Direct form-II realization

Now compared to direct form-I structure; nearly half number of delays are required for direct form-II realization. That means, it requires minimum number of storage elements. So it is advantageous than direct form-I structure.

Quick check :

We can check the answer by using shortcut method to obtain direct form-II realization.

Here maximum delays are 2. The coefficients of numerator are 3.6 for Z^{-1} and 0.6 for Z^{-2}. Now changing the signs, we get the coefficients of denominator as – 0.1 for Z^{-1} and + 0.2 for Z^{-2}. Thus direct form-II realization is shown in Fig. P. 10.3.5(d).

Thus Fig. P. 10.3.5(c) and Fig. P. 10.3.5(d) are same.

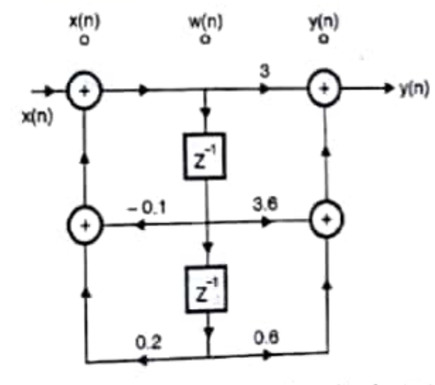


Fig. P. 10.3.5(d) : Direct form-II realization using shortcut method

Ex. 10.3.6 : Obtain direct form-II implementation of LTI system with transfer function.

$$H(Z) = \frac{1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}}{1-\frac{1}{2}Z^{-1}-\frac{1}{2}Z^{-2}}$$

Soln. :

For direct form-II structure. All pole system is realized first and then all zero system is realized.

$$\therefore \quad \text{Let } H_1(Z) = \frac{1}{1-\frac{1}{2}Z^{-1}-\frac{1}{2}Z^{-2}}$$

$$\text{But } H_1(Z) = \frac{W(Z)}{X(Z)}$$

$$\therefore \frac{W(Z)}{X(Z)} = \frac{1}{1-\frac{1}{2}Z^{-1}-\frac{1}{2}Z^{-2}}$$

$$\therefore W(Z)-\frac{1}{2}Z^{-1}W(Z)-\frac{1}{2}Z^{-2}W(Z) = X(Z)$$

$$\therefore W(Z) = X(Z)+\frac{1}{2}Z^{-1}W(Z)+\frac{1}{2}Z^{-2}W(Z) \quad ...(1)$$

Taking IZT of Equation (1),

$$w(n) = x(n)+\frac{1}{2}w(n-1)+\frac{1}{2}w(n-2) \quad ...(2)$$

The direct form realization of Equation (2) is shown in Fig. P. 10.3.6(a).

Now consider all zero system.

$$\therefore H_2(Z) = 1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}$$

$$\text{But } H_2(Z) = \frac{\text{Output}}{\text{Input}} = \frac{Y(Z)}{W(Z)}$$

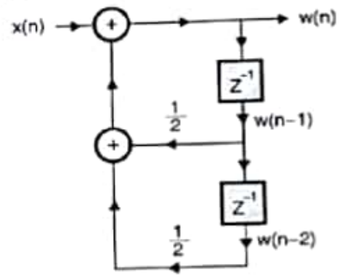
$$\therefore \frac{Y(Z)}{W(Z)} = 1+\frac{5}{6}Z^{-1}+\frac{1}{6}Z^{-2}$$

$$\therefore Y(Z) = W(Z)+\frac{5}{6}Z^{-1}W(Z)+\frac{1}{6}Z^{-2}W(Z) \quad ...(3)$$

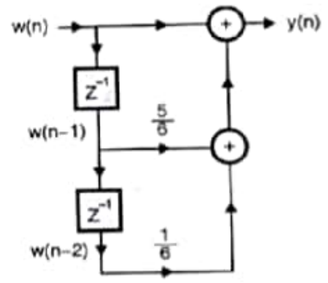
Taking IZT of Equation (3),

$$y(n) = w(n)+\frac{5}{6}w(n-1)+\frac{1}{6}w(n-2) \quad ...(4)$$

The direct form implementation of Equation (4) is shown in Fig. P. 10.3.6(b).



(a) Direct form realization of $H_1(Z)$



(b) Direct form realization of $H_2(Z)$

Fig. P. 10.3.6

Now direct form-II realization is obtained by cascading $H_1(Z)$ and $H_2(Z)$ as shown in Fig. P. 10.3.6(c).

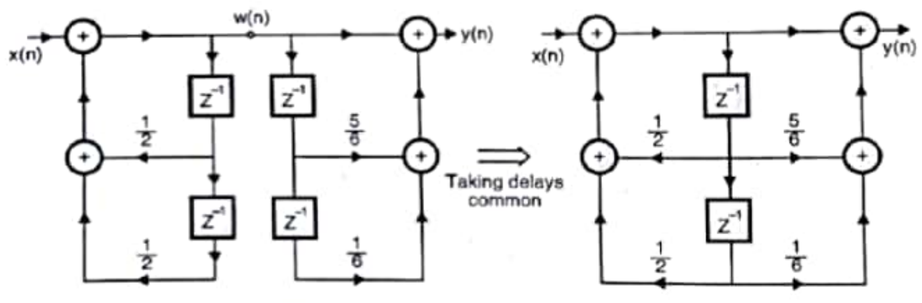


Fig. P. 10.3.6(c) : Direct form-II realization

Note : We can easily check the answer using shortcut method.

Ex. 10.3.7 : A certain discrete time LTI filter has the following data :

Poles are at 0.2 and 0.6.

Zeros are at – 0.4 and origin.

Gain of filter is 5.

Show direct form-II realization and cascade form realization.

Soln. :

$$\text{Given Zeros} \Rightarrow \quad Z_1 = 0 \text{ and } Z_2 = -0.4$$

$$\text{Poles} \Rightarrow \quad P_1 = 0.2 \text{ and } P_2 = 0.6$$

$$\text{Gain} = k = 5$$

The transfer function of system is expressed as,

$$H(Z) = k \cdot \frac{(Z-Z_1)(Z-Z_2)}{(Z-P_1)(Z-P_2)}$$

$$\therefore H(Z) = \frac{5Z(Z+0.4)}{(Z-0.2)(Z-0.6)} \quad ...(1)$$

$$\therefore H(Z) = \frac{5Z^2+2Z}{Z^2-0.8Z+0.12}$$

Converting into negative powers of Z we get,

$$H(Z) = \frac{5+2Z^{-1}}{1-0.8Z^{-1}+0.12Z^{-2}} \quad ...(2)$$

Direct form-II realization :

Using shortcut method, direct form-II realization is shown in Fig. P. 10.3.7(a).

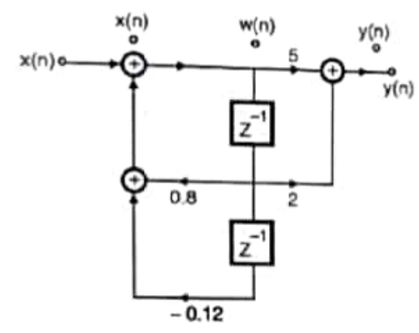


Fig. P. 10.3.7(a)

Cascade form realization :

According to Equation (1),

$$H(Z) = \frac{5Z(Z+0.4)}{(Z-0.2)(Z-0.6)}$$

$$\therefore H(Z) = \frac{5Z}{Z-0.2} \cdot \frac{Z+0.4}{Z-0.6} \quad ...(3)$$

$$\text{Let } H_1(Z) = \frac{5Z}{Z-0.2} = \frac{5}{1-0.2Z^{-1}}$$

$$\text{and } H_2(Z) = \frac{Z+0.4}{Z-0.6} = \frac{1+0.4Z^{-1}}{1-0.6Z^{-1}}$$

Using shortcut method $H_1(Z)$ and $H_2(Z)$ are realized as shown in Fig. P. 10.3.7(b) and (c).

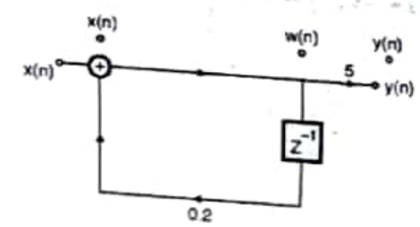


Fig. P. 10.3.7(b)

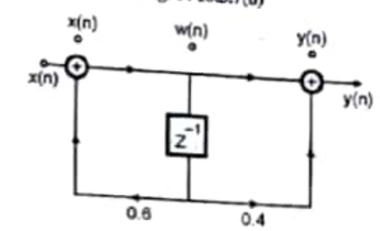


Fig. P. 10.3.7(e)

The cascade realization is shown in Fig. P. 10.3.7(d).

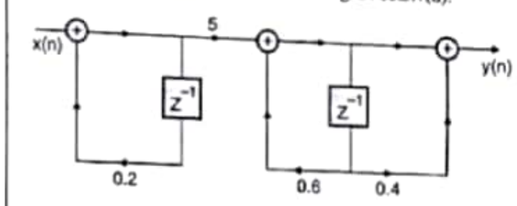


Fig. P. 10.3.7(d)

Ex. 10.3.8 : Determine direct-form-II realization for each of the following LTI system :

(i) $2y(n)+y(n-1)-4y(n-3) = x(n)+3x(n-1)$

(ii) $y(n) = x(n)-x(n-1)+2x(n-2)-3x(n-4)$

Soln. :

(i) Given, $2y(n)+y(n-1)-4y(n-3) = x(n)+3x(n-1)$

Taking Z-transform of both sides.

$$2Y(Z)+Z^{-1}Y(Z)-4Z^{-3}Y(Z) = X(Z)+3Z^{-1}X(Z)$$

$$\therefore Y(Z)[2+Z^{-1}-4Z^{-3}] = X(Z)[1+3Z^{-1}]$$

$$\therefore \frac{Y(Z)}{X(Z)} = H(Z) = \frac{1+3Z^{-1}}{2+Z^{-1}-4Z^{-3}}$$

Using short-cut method, the direct form-II structure is drawn as shown in Fig. P. 10.3.8(a).

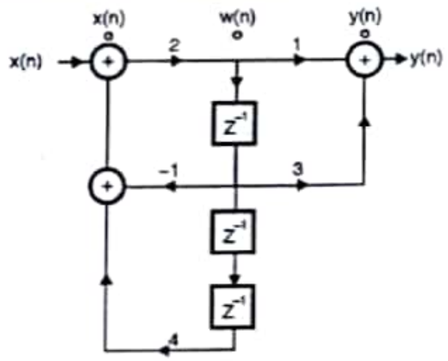


Fig. P. 10.3.8(a)

(ii) We have, $y(n) = x(n) - x(n-1) + 2x(n-2) - 3x(n-4)$

Taking Z-transform of both sides we get,

$$Y(Z) = X(Z) - Z^{-1}X(Z) + 2Z^{-2}X(Z) - 3Z^{-4}X(Z)$$

$$\therefore \quad Y(Z) = X(Z)[1 - Z^{-1} + 2Z^{-2} - 3Z^{-4}]$$

$$\therefore \quad \frac{Y(Z)}{X(Z)} = H(Z) = 1 - Z^{-1} + 2Z^{-2} - 3Z^{-4}$$

Using short-cut method, the direct form-II realization is obtained as shown in Fig. P. 10.3.8(b).

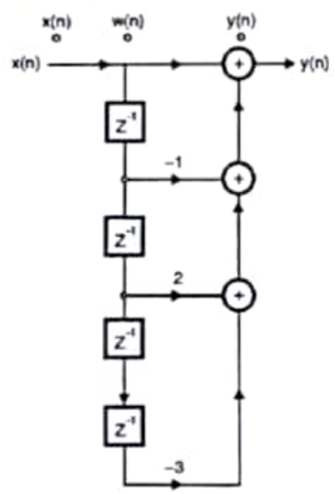


Fig. P. 10.3.8(b)

Ex. 10.3.9 : Determine direct form I and II for the filter given by $y(n) = 2b \cos \omega_0\, y(n-1) - b^2 y(n-2) + x(n) - b \cos \omega_0\, x(n-1)$.

Soln. :

Given :

$$y(n) = 2b \cos \omega_0\, y(n-1) - b^2 y(n-2) + x(n) - b \cos \omega_0\, x(n-1)$$

Taking Z-transform of both sides.

$$Y(Z) = 2b \cos \omega_0 Z^{-1} Y(Z) - b^2 Z^{-2} Y(Z) + X(Z) - b \cos \omega_0 Z^{-1} X(Z)$$

$$Y(Z) - 2b \cos \omega_0 Z^{-1} Y(Z) - b^2 Z^{-2} Y(Z) = X(Z) - b \cos \omega_0 Z^{-1} X(Z)$$

$$\therefore Y(Z)[1 - 2b \cos \omega_0 Z^{-1} + b^2 Z^{-2}] = X(Z)[1 - b \cos \omega_0 Z^{-1}]$$

$$\therefore H(Z) = \frac{Y(Z)}{X(Z)} = \frac{1 - b \cos \omega_0 Z^{-1}}{1 - 2b \cos \omega_0 Z^{-1} + b^2 Z^{-2}}$$

Let $H_1(Z) = \dfrac{W(Z)}{X(Z)} = 1 - b \cos \omega_0 Z^{-1}$

$$\therefore W(Z) = X(Z) - b \cos \omega_0 Z^{-1} X(Z)$$

Taking IZT we get,

$$w(n) = x(n) - b \cos \omega_0\, x(n-1) \qquad \ldots(1)$$

Let $H_2(Z) = \dfrac{Y(Z)}{W(Z)} = \dfrac{1}{1 - 2b \cos \omega_0 Z^{-1} + b^2 Z^{-2}}$

$$\therefore Y(Z) - 2b \cos \omega_0 Z^{-1} Y(Z) - b^2 Z^{-2} Y(Z) = W(Z)$$

$$\therefore Y(Z) = W(Z) + 2b \cos \omega_0 Z^{-1} Y(Z) - b^2 Z^{-2} Y(Z)$$

Taking IZT we get,

$$y(n) = w(n) + 2b \cos \omega_0\, y(n-1) - b^2 y(n-2) \qquad \ldots(2)$$

Using Equations (1) and (2), the direct form I realization is obtained as shown in Fig. P. 10.3.9(a).

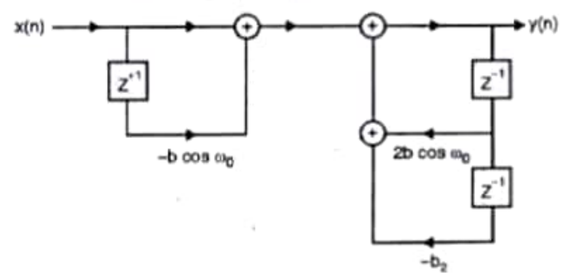


Fig. P. 10.3.9(a)

By interchanging the positions; the direct form-II realization is obtained as shown in Fig. P. 10.3.9(b).

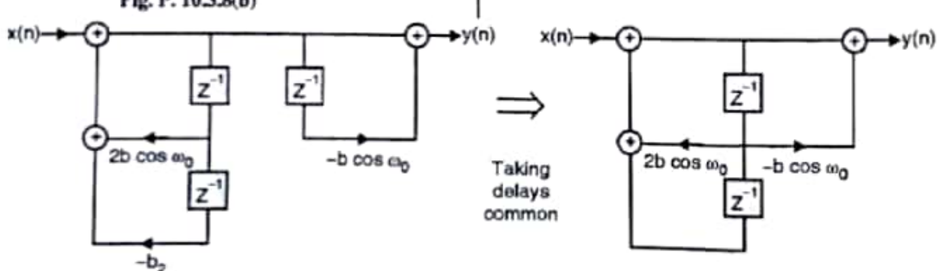


Fig. P. 10.3.9(b)

Ex. 10.3.10 : The transfer function of discrete time causal system is given by,

$$H(Z) = \frac{1 - Z^{-1}}{1 - 0.2Z^{-1} - 0.15Z^{-2}}$$

Draw cascade and parallel realization.

Soln. :

The given transfer function is,

$$H(Z) = \frac{1 - Z^{-1}}{1 - 0.2Z^{-1} - 0.15Z^{-2}} \qquad \ldots(1)$$

Cascade realization :

First we will obtain the roots of denominator, by using the equation,

$$\text{roots} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Here $a = 1$, $b = -0.2$ and $c = -0.15$

$$\therefore \text{roots} = \frac{0.2 \pm \sqrt{0.04 + 0.6}}{2} = \frac{0.2 \pm 0.8}{2} = 0.5, -0.3$$

Thus $(1 - 0.2Z^{-1} - 0.15Z^{-2}) = (1 - 0.5Z^{-1})(1 + 0.3Z^{-1})$

Putting this value in Equation (1) we get,

$$H(Z) = \frac{1 - Z^{-1}}{(1 - 0.5Z^{-1})(1 + 0.3Z^{-1})} \qquad \ldots(2)$$

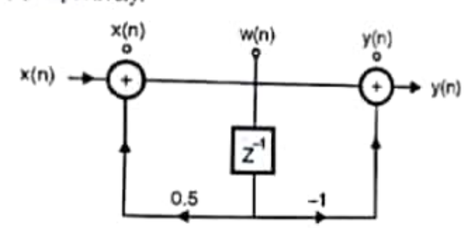
For the cascade realization we have to express $H(Z)$ as,

$$H(Z) = H_1(Z) \cdot H_2(Z)$$

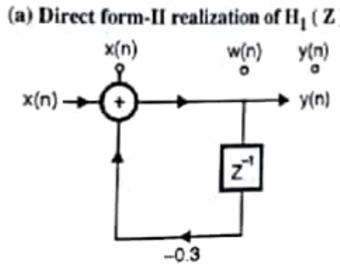
Let $H_1(Z) = \dfrac{1 - Z^{-1}}{1 - 0.5Z^{-1}}$ ...(3)

and let $H_2(Z) = \dfrac{1}{1 + 0.3Z^{-1}}$ ...(4)

Using shortcut method, we can obtain direct form-II structures for $H_1(Z)$ and $H_2(Z)$ as shown in Fig. P. 10.3.10(a) and (b) respectively.



(a) Direct form-II realization of $H_1(Z)$



(b) Direct form-II realization of $H_2(Z)$

Fig. P. 10.3.10

Now for the cascade connection we will have to connect $H_1(Z)$ and $H_2(Z)$ in series. This series connection is shown in Fig. P. 10.3.10(c).

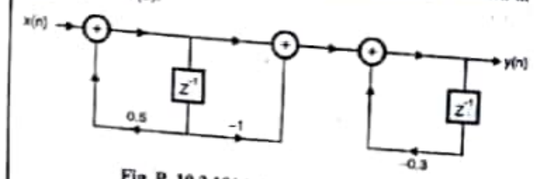


Fig. P. 10.3.10(c) : Cascade realization

Parallel realization :

For the parallel form realization we have to use partial fraction expansion (PFE). It is similar to PFE which we have studied in Z transformation. The given transfer function is,

$$H(Z) = \frac{1 - Z^{-1}}{1 - 0.2Z^{-1} - 0.15Z^{-2}}$$

To convert this equation in terms of positive powers of Z; multiply numerator and denominator by Z^2.

$$\therefore H(Z) = \frac{Z^2(1 - Z^{-1})}{Z^2(1 - 0.2Z^{-1} - 0.15Z^{-2})}$$

$$\therefore H(Z) = \frac{Z^2 - Z}{Z^2 - 0.2Z - 0.15} = \frac{Z(Z-1)}{Z^2 - 0.2Z - 0.15}$$

$$\therefore \frac{H(Z)}{Z} = \frac{Z - 1}{Z^2 - 0.2Z - 0.15} \qquad \ldots(5)$$

Equation (5) is in the proper form. The roots of denominator are 0.5 and – 0.3. That means poles are at, $P_1 = 0.5$ and $P_2 = -0.3$.

$$\therefore \frac{H(Z)}{Z} = \frac{Z - 1}{(Z - 0.5)(Z + 0.3)} \qquad \ldots(6)$$

In the partial fraction expansion form Equation (6) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{(Z - 0.5)} + \frac{A_2}{(Z + 0.3)} \qquad \ldots(7)$$

Now we will calculate the values of A_1 and A_2.

$$A_1 = (Z - p_1)\frac{H(Z)}{Z}\bigg|_{Z = p_1}$$

$$\therefore A_1 = \frac{(Z - 0.5)(Z - 1)}{(Z - 0.5)(Z + 0.3)}\bigg|_{Z = 0.5}$$

$$= \frac{0.5 - 1}{0.5 + 0.3} = -0.625 \qquad \ldots(8)$$

$$\text{Now, } A_2 = (Z - p_2)\frac{H(Z)}{Z}\bigg|_{Z = p_2}$$

$$\therefore A_2 = \frac{(Z + 0.3)(Z - 1)}{(Z - 0.5)(Z + 0.3)}\bigg|_{Z = -0.3}$$

$$= \frac{-0.3 - 1}{-0.3 - 0.5} = 1.625 \qquad \ldots(9)$$

Putting the values of A_1 and A_2 in Equation (7),

$$\frac{H(Z)}{Z} = \frac{-0.625}{Z - 0.5} + \frac{1.625}{Z + 0.3}$$

$$\therefore H(Z) = \frac{-0.625Z}{Z - 0.5} + \frac{1.625Z}{Z + 0.3} \qquad \ldots(10)$$

Remember that, to realize the given transfer function H (Z); it should be in terms of negative powers of Z. So we will convert Equation (10) in terms of negative powers of Z as follows:

$$H(Z) = \frac{Z^{-1}(-0.625\,Z)}{Z^{-1}(Z-0.5)} + \frac{Z^{-1}(1.625\,Z)}{Z^{-1}(Z+0.3)}$$

$$\therefore\ H(Z) = \frac{-0.625}{1-0.5\,Z^{-1}} + \frac{1.625}{1+0.3\,Z^{-1}} \qquad ...(11)$$

The general structure of parallel form realization is as shown in Fig. P. 10.3.10(d). The equation is,

$$H(Z) = H_1(Z) + H_2(Z).$$

Now let $H_1(Z) = \dfrac{-0.625}{1-0.5\,Z^{-1}}$ ...(12)

and $H_2(Z) = \dfrac{1.625}{1+0.3\,Z^{-1}}$ ...(13)

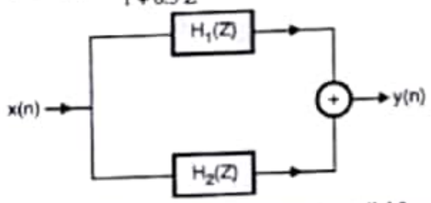
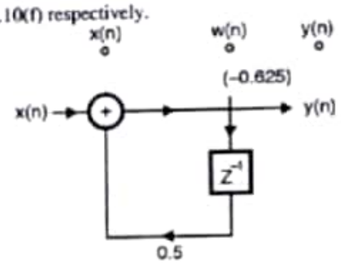
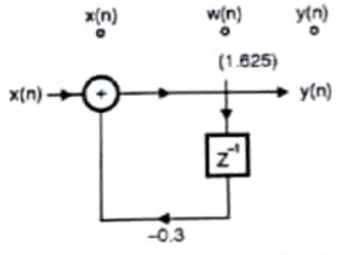


Fig. P. 10.3.10(d) : General structure of parallel form realization

Using shortcut method; the direct form-II realization of $H_1(Z)$ and $H_2(Z)$ is obtained as shown in Fig. P. 10.3.10(e) and Fig. P. 10.3.10(f) respectively.



(e) Direct form-II realization of $H_1(Z)$



(f) Direct form-II realization of $H_2(Z)$

Fig. P. 10.3.10

To obtain the parallel form realization; we have to connect Fig. P. 10.3.10(e) and Fig. P. 10.3.10(f) in parallel as shown in Fig. P. 10.3.10(g).

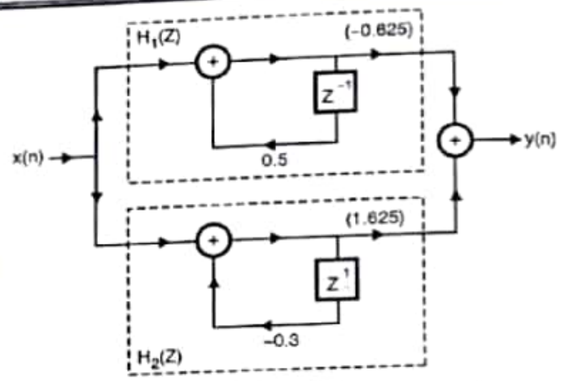


Fig. P. 10.3.10(g) : Parallel form realization

Ex. 10.3.11 : A system has an impulse response, $h(n) = (0.5)^n u(n) + n(0.2)^n u(n)$ show parallel realization of system.

Soln. :

First we will obtain transfer function H (Z).

We have, $h(n) = (0.5)^n u(n) + n(0.2)^n u(n)$ ...(1)

Recall the standard Z-transform pairs,

$$\alpha^n u(n) \overset{Z}{\longleftrightarrow} \frac{Z}{Z-\alpha} \quad \text{and} \quad n\,a^n u(n) \overset{Z}{\longleftrightarrow} \frac{aZ}{(Z-a)^2}$$

Thus Z-transform of Equation (1) can be written as,

$$H(Z) = \frac{Z}{Z-0.5} + \frac{0.2\,Z}{(Z-0.2)^2} \qquad ...(2)$$

Let $H_1(Z) = \dfrac{Z}{Z-0.5} = \dfrac{1}{1-0.5\,Z^{-1}}$

The direct form-II realization of $H_1(Z)$ is shown in Fig. P. 10.3.11(a).

Let $H_2(Z) = \dfrac{0.2\,Z}{(Z-0.2)^2}$

$$Z^2 = \frac{0.2\,Z}{Z^2 - 0.4\,Z + 0.04}$$

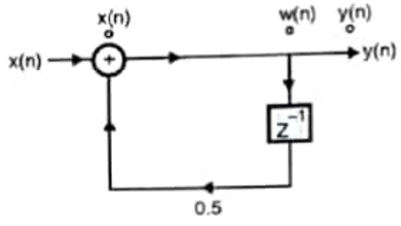


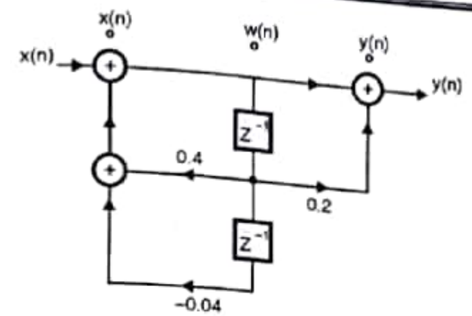
Fig. P. 10.3.11(a)

Multiplying numerator and denominator by Z^{-2} we get,

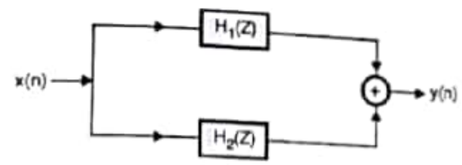
$$H_2(Z) = \frac{0.2\,Z^{-1}}{1-0.4\,Z^{-1}+0.04\,Z^{-2}}$$

The direct form-II realization of $H_2(Z)$ is shown in Fig. P. 10.3.11(b).

The general structure parallel form realization is shown in Fig. P. 10.3.11(c).



(b)



(c)

Fig. P. 10.3.11

Thus using Fig. P. 10.3.11(a) and Fig. P. 10.3.11(b); we can draw the parallel form realization of H (Z) as shown in Fig. P. 10.3.11(d).

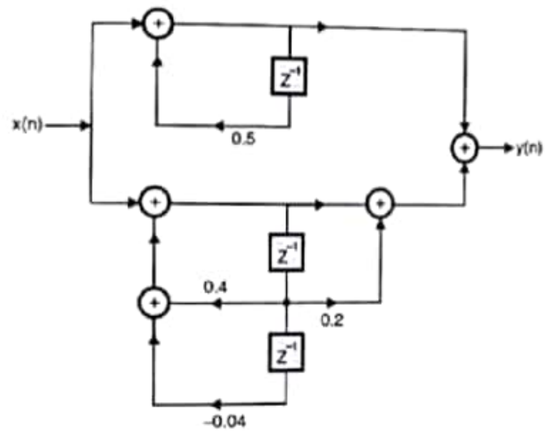


Fig. P. 10.3.11(d)

Ex. 10.3.12 : Develop parallel form realization for the digital filter with transfer function.

$$H(Z) = \frac{1+2\,Z^{-1}+Z^{-2}}{1-0.75\,Z^{-1}+0.125\,Z^{-2}}$$

Using first order systems.

Soln. :

To obtain the parallel form realization we have to use partial fraction expansion.

The given transfer function is,

$$H(Z) = \frac{1+2\,Z^{-1}+Z^{-2}}{1-0.75\,Z^{-1}+0.125\,Z^{-2}} \qquad ...(1)$$

To convert if in terms of positive power of Z; multiply numerator and denominator by Z^2.

$$\therefore\ H(Z) = \frac{Z^2+2Z+1}{Z^2-0.75\,Z+0.125}$$

$$\therefore\ \frac{H(Z)}{Z} = \frac{Z^2+2Z+1}{Z(Z^2-0.75\,Z+0.125)} \qquad ...(2)$$

Now we will obtain roots of $Z^2 - 0.75\,Z + 0.125$

Here $a = 1$, $b = -0.75$ and $c = 0.125$

$$\therefore\ \text{roots} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\therefore\ \text{roots} = \frac{0.75 \pm \sqrt{(-0.75)^2-(4\times 0.125)}}{2}$$

$$= \frac{0.75 \pm 0.25}{2} = 0.5,\ 0.25$$

Thus Equation (2) becomes,

$$\frac{H(Z)}{Z} = \frac{Z^2+2Z+1}{Z(Z-0.5)(Z-0.25)} \qquad ...(3)$$

Thus poles are at $p_1 = 0$, $p_2 = 0.5$ and $p_3 = 0.25$

In the partial fraction expansion form, Equation (3) can be written as,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-0.5} + \frac{A_3}{Z-0.25} \qquad ...(4)$$

Now $A_1 = \dfrac{H(Z)}{Z}\times(Z-P_1)\Big|_{Z=p_2}$

$$= \frac{Z^2+2Z+1}{Z(Z-0.5)(Z-0.25)}\times Z\Big|_{Z=0}$$

$$\therefore\ A_1 = \frac{1}{(-0.5)(-0.25)} = 8 \qquad ...(5)$$

$$A_2 = \frac{H(Z)}{Z}\times(Z-P_2)\Big|_{Z=p_2}$$

$$= \frac{Z^2+2Z+1}{Z(Z-0.5)(Z-0.25)}\times(Z-0.5)\Big|_{Z=0.5}$$

$$\therefore\ A_2 = \frac{Z^2+2Z+1}{Z(Z-0.25)}\Big|_{Z=0.5}$$

$$= \frac{(0.5)^2+2(0.5)+1}{0.5(0.5-0.25)} = 18 \qquad ...(6)$$

and $A_3 = \dfrac{H(Z)}{Z}\times(Z-P_3)\Big|_{Z=p_3}$

$$\therefore\ A_3 = \frac{Z^2+2Z+1}{Z(Z-0.5)\times(Z-0.25)}\times(Z-0.25)\Big|_{Z=0.25}$$

$$\therefore\ A_3 = \frac{(0.25)^2+2(0.25)+1}{0.25(0.25-0.5)} = -25 \qquad ...(7)$$

Putting the values of A_1, A_2 and A_3 in Equation (4) we get,

$$\frac{H(Z)}{Z} = \frac{8}{Z} + \frac{18}{(Z-0.5)} - \frac{25}{(Z-0.25)}$$

$$\therefore\ H(Z) = 8 + \frac{18Z}{Z-0.5} - \frac{25Z}{Z-0.25} \qquad ...(8)$$

To realize H (Z); we want the expression in terms of negative powers of Z.

$$\therefore\ H(Z) = 8 + \frac{Z^{-1}(18Z)}{Z^{-1}(Z-0.5)} - \frac{Z^{-1}(25Z)}{Z^{-1}(Z-0.25)}$$

$$\therefore\ H(Z) = 8 + \frac{18}{1-0.5Z^{-1}} - \frac{25}{1-0.25Z^{-1}} \qquad ...(9)$$

Let $H_1(Z) = 8$

$$H_2(Z) = \frac{18}{1-0.5Z^{-1}}$$

and $$H_3(Z) = \frac{-25}{1-0.25Z^{-1}}$$

Using shortcut method; direct form-II realization of $H_2(Z)$ and $H_3(Z)$ is obtained as shown in Fig. P. 10.3.12(a) and Fig. P. 10.3.12(b) respectively.

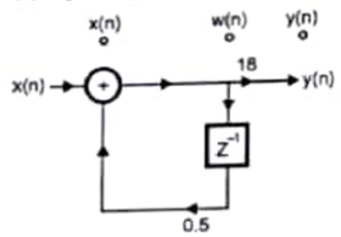


Fig. P. 10.3.12(a) : Direct form-II realization of $H_2(Z)$

Now $H_1(Z) = 8$; which is constant.

So it can be expressed as,

$$H_1(Z) = \frac{Y_1(Z)}{X_1(Z)}$$

$$\therefore\ Y_1(Z) = H_1(Z)\cdot X_1(Z) = 8X_1(Z)$$

Taking IZT of both sides we get,

$$y_1(n) = 8x_1(n) \qquad ...(10)$$

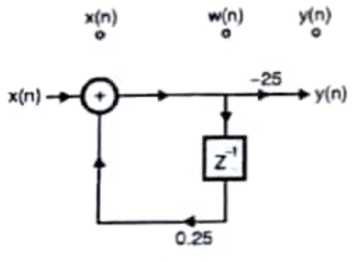


Fig. P. 10.3.12(b) : Direct form-II realization of $H_3(Z)$

Using Fig. P.10.3.12(a), Fig. P.10.3.12(b) and Equation (10); the parallel form realization is drawn as shown in Fig. P. 10.3.12(c).

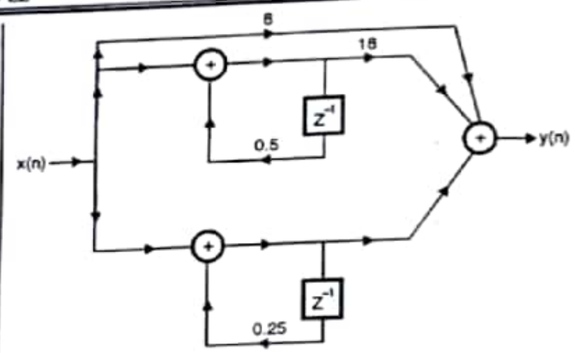


Fig. P. 10.3.12(c) : Parallel form realization

Ex. 10.3.13 : System has transfer function as shown in Fig. P. 10.3.13

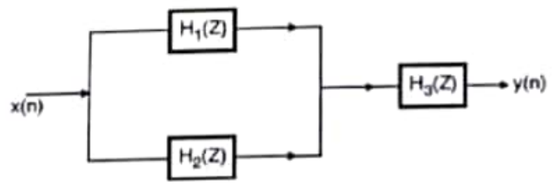


Fig. P. 10.3.13

$H_1(Z)$ and $H_2(Z)$ has poles at + 1 and – 1. $H_1(Z)$ has two zeros at origin, $H_2(Z)$ has only one zero at origin.

The transfer $H_3(Z)$ is given by,

$H_3(Z) = 1 + 1.5Z^{-1} - 1.5Z^{-2} - Z^{-3}$

Show realization diagram of overall transfer function.

Soln. : First we will obtain expressions of $H_1(Z)$ and $H_2(Z)$.

For $H_1(Z)$

Zeros ⇒ $Z_1 = 0$ and $Z_2 = 0$; two zeros at origin.

Poles ⇒ $P_1 = 1$ and $P_2 = -1$.

$$\text{Now } H_1(Z) = \frac{(Z-Z_1)(Z-Z_2)}{(Z-P_1)(Z-P_2)}$$

$$\therefore\ H_1(Z) = \frac{Z^2}{(Z-1)(Z+1)} \qquad ...(1)$$

For $H_2(Z)$

Zeros ⇒ $Z_1 = 0$ Poles ⇒ $P_1 = 1$ and $P_2 = -1$

$$\text{Now } H_2(Z) = \frac{(Z-Z_1)}{(Z-P_1)(Z-P_2)}$$

$$\therefore\ H_2(Z) = \frac{Z}{(Z-1)(Z+1)} \qquad ...(2)$$

Realization of $H_1(Z)$

$$\text{We have } H_1(Z) = \frac{Z^2}{(Z-1)(Z+1)} = \frac{Z^2}{Z^2-1}$$

$$\therefore H_1(Z) = \frac{1}{1-Z^{-2}}$$

Direct form-II realization of $H_1(Z)$ is shown in Fig. P. 10.3.13(a).

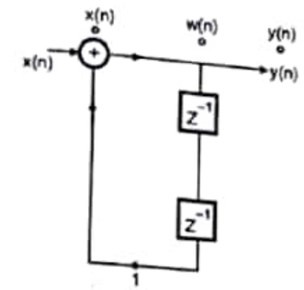


Fig. P. 10.3.13(a)

Realization of $H_2(Z)$:

We have,

$$H_2(Z) = \frac{Z}{(Z-1)(Z+1)} = \frac{Z}{Z^2-1}$$

$$\therefore\ H_2(Z) = \frac{Z^{-1}}{1-Z^{-2}}$$

Direct form-II realization of $H_2(Z)$ is shown in Fig. P. 10.3.13(b).

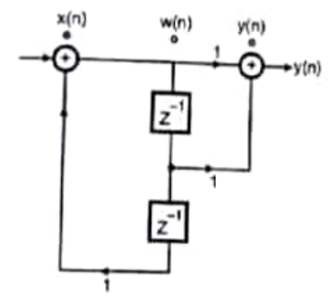


Fig. P. 10.3.13(b)

Realization of $H_3(Z)$:

We have, $H_3(Z) = 1 + 1.5Z^{-1} - 1.5Z^{-2} - Z^{-3}$

Direct form-II realization of $H_3(Z)$ is shown in Fig. P. 10.3.13(c).

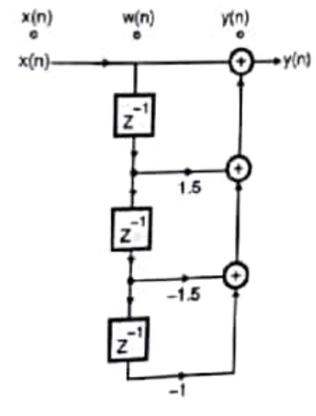


Fig. P. 10.3.13(c)

Realization of overall transfer function :

As shown in Fig. P. 10.3.13 $H_1(Z)$ and $H_2(Z)$ are connected in parallel and this parallel combination is cascaded with $H_3(Z)$. Thus realization of overall transfer function is shown in Fig. P. 10.3.13(d).

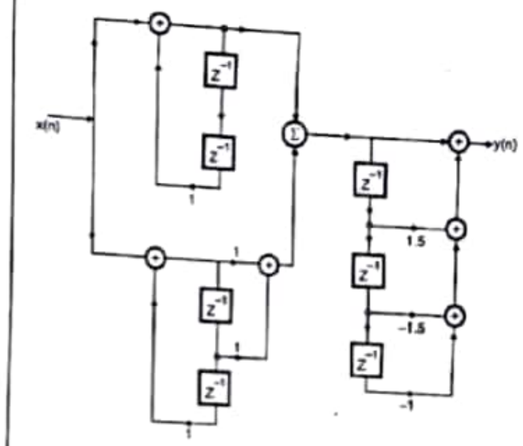


Fig. P. 10.3.13(d)

Ex. 10.3.14 : Using first order sections, obtain cascade realization for

$$H(Z) = \frac{\left(1+\frac{1}{2}Z^{-1}\right)\left(1+\frac{1}{4}Z^{-1}\right)}{\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{1}{4}Z^{-1}\right)\left(1+\frac{1}{8}Z^{-1}\right)}$$

Soln. :

We know that for cascade realization the given transfer function can be expressed as,

$$H(Z) = H_1(Z)\cdot H_2(Z)\cdot H_3(Z)....$$

Thus the given transfer function can be expressed as,

$$H(Z) = \frac{\left(1+\frac{1}{2}Z^{-1}\right)}{1-\frac{1}{2}Z^{-1}}\cdot\frac{1+\frac{1}{4}Z^{-1}}{1-\frac{1}{4}Z^{-1}}\cdot\frac{1}{1-\frac{1}{8}Z^{-1}}$$

$$\text{Here } H_1(Z) = \frac{1+\frac{1}{2}Z^{-1}}{1-\frac{1}{2}Z^{-1}} = \frac{Y_1(Z)}{X_1(Z)}$$

$$Y_1(Z)\left(1-\frac{1}{2}Z^{-1}\right) = X(Z)\left(1+\frac{1}{2}Z^{-1}\right)$$

$$Y_1(Z) - \frac{1}{2}Z^{-1}Y_1(Z) = X(Z) + \frac{1}{2}Z^{-1}X(Z)$$

$$\therefore\ Y_1(Z) = X(Z) + \frac{1}{2}Z^{-1}X(Z) + \frac{1}{2}Z^{-1}Y_1(Z)$$

Taking IZT of both sides,

$$y_1(n) = x(n) + \frac{1}{2}x(n-1) + \frac{1}{2}y_1(n-1) \quad ...(1)$$

It is realized as shown in Fig. P. 10.3.14(a).

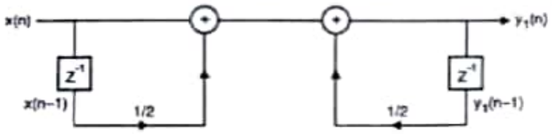


Fig. P. 10.3.14(a)

Its direct form-II realization is as shown in Fig. P. 10.3.14(b).

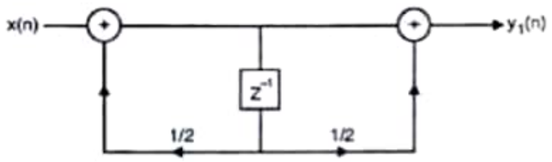


Fig. P. 10.3.14(b) : Direct form-II

$$\text{Now} \quad H_2(z) = \frac{1+\frac{1}{4}Z^{-1}}{1-\frac{1}{4}Z^{-1}} = \frac{Y_2(Z)}{X_2(Z)}$$

$$\therefore \quad Y_2(Z)\left(1-\frac{1}{4}Z^{-1}\right) = X_2(Z)\left(1+\frac{1}{4}Z^{-1}\right)$$

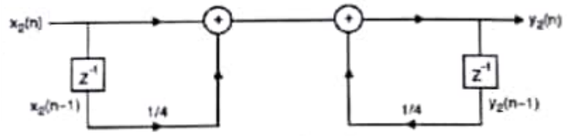
$$Y_2(Z) - \frac{1}{4}Z^{-1}Y_2(Z) = X_2(Z) + \frac{1}{4}Z^{-1}X_2(Z)$$

$$\therefore \quad Y_2(Z) = X_2(Z) + \frac{1}{4}Z^{-1}X_2(Z) + \frac{1}{4}Z^{-1}Y_2(Z)$$

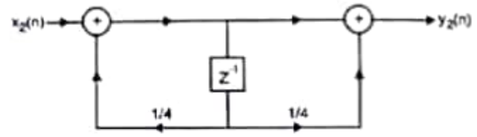
Taking IZT of both sides,

$$y_2(n) = x_2(n) + \frac{1}{4}x_2(n-1) + \frac{1}{4}y_2(n-1) \quad ...(2)$$

Its realization is as shown in Figs. P. 10.3.14(c) and (d).



(c) Direct form-I



(d) Direct form-II

Fig. P. 10.3.14

$$\text{Now} \quad H_3(Z) = \frac{1}{1-\frac{1}{8}Z^{-1}} = \frac{Y(Z)}{X_3(Z)}$$

$$\therefore Y(Z)\left(1-\frac{1}{8}Z^{-1}\right) = X_3(Z)$$

$$\therefore Y(Z) - \frac{1}{8}Z^{-1}Y(Z) = X_3(Z) \therefore Y(Z) = X_3(Z) + \frac{1}{8}Z^{-1}Y_3(Z)$$

Taking IZT of both sides, $y(n) = x_3(n) + \frac{1}{8}y(n-1) \quad ...(3)$

Its realization is as shown in Fig. P. 10.3.14(e).

Now cascade realization is obtained by cascading the direct form- II structures of $H_1(Z)$, $H_2(Z)$ and $H_3(Z)$ as shown in Fig. P. 10.3.14(f).

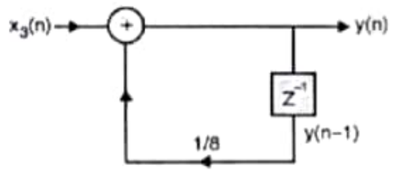


Fig. P. 10.3.14(e)

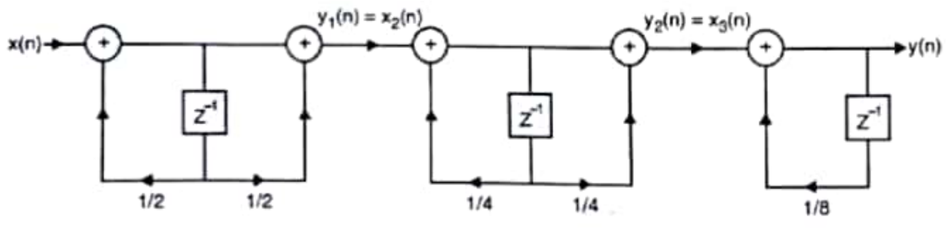


Fig. P. 10.3.14(f)

Ex. 10.3.15 : Determine the parallel realization of the IIR digital filter transfer functions.

(i) $H(Z) = \dfrac{3(2Z^2+5Z+4)}{(2Z+1)(Z+2)}$

(ii) $H(Z) = \dfrac{3Z(5Z-2)}{\left(Z+\frac{1}{2}\right)(3Z-1)}$

Soln. :

(i) Given $H(Z) = \dfrac{3(2Z^2+5Z+4)}{(2Z+1)(Z+2)}$

Multiplying numerator and denominator by $\frac{1}{2}$ we get,

$$H(Z) = \frac{\frac{3}{2}(2Z^2+5Z+4)}{\left(Z+\frac{1}{2}\right)(Z+2)}$$

$$\therefore H(Z) = \frac{3Z^2+\frac{15}{2}Z+12}{\left(Z+\frac{1}{2}\right)(Z+2)}$$

$$\therefore \frac{H(Z)}{Z} = \frac{3Z^2+\frac{15}{2}Z+12}{Z\left(Z+\frac{1}{2}\right)(Z+2)}$$

In the partial fraction expansion form we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z+\frac{1}{2}} + \frac{A_3}{Z+2}$$

$$\text{Here} \quad A_1 = \left.\frac{Z\left(3Z^2+\frac{15}{2}Z+12\right)}{Z\left(Z+\frac{1}{2}\right)(Z+2)}\right|_{Z=0}$$

$$\therefore A_1 = \frac{12}{\frac{1}{2}(2)} = 1$$

$$A_2 = \left.\frac{\left(Z+\frac{1}{2}\right)\cdot\left(3Z^2+\frac{15}{2}Z+12\right)}{Z\left(Z+\frac{1}{2}\right)(Z+2)}\right|_{Z=\frac{1}{2}}$$

$$\therefore A_2 = \frac{3\left(\frac{1}{4}\right)+\left(-\frac{15}{4}\right)+12}{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}+2\right)} = \frac{\frac{3}{3}-\frac{15}{4}+12}{\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right)}$$

$$= \frac{-\frac{12+48}{4}}{-3/4} = \frac{36}{-3} = -12$$

$$\text{and} \quad A_3 = \left.\frac{(Z+2)\cdot\left(3Z^2+\frac{15}{2}Z+12\right)}{Z\left(Z+\frac{1}{2}\right)(Z+2)}\right|_{Z=-2}$$

$$= \frac{(12-15+12)}{(-2)\left(-2+\frac{1}{2}\right)} = \frac{9}{(-2)\left(-\frac{3}{2}\right)}$$

$$\therefore \quad A_3 = \frac{9}{3} = 3$$

$$\therefore \frac{H(Z)}{Z} = \frac{12}{Z} - \frac{12}{Z+\frac{1}{2}} + \frac{3}{Z+2}$$

$$\therefore H(Z) = 12 - \frac{12Z}{Z+\frac{1}{2}} + \frac{3Z}{Z+2}$$

$$\therefore H(Z) = 12 - \frac{12}{1+\frac{1}{2}Z^{-1}} + \frac{3}{1+2Z^{-1}}$$

In the general form we can write,

$$H(Z) = H_1(Z) + H_2(Z) + H_3(Z)$$

Here $H_1(Z) = 12$

$$H_2(Z) = -\frac{12}{1+\frac{1}{2}Z^{-1}}$$

$$\text{and} \quad H_3(Z) = \frac{3}{1+\frac{1}{2}Z^{-1}}$$

The general structure of parallel form realization is shown in Fig. P. 10.3.15(a).

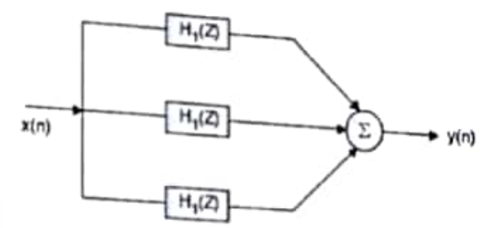


Fig. P. 10.3.15(a)

Using short cut method, it can be realized as shown in Fig. P. 10.3.15(b).

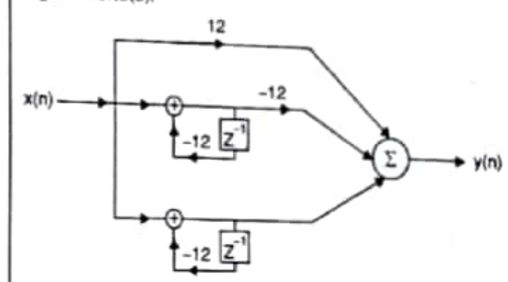


Fig. P. 10.3.15(b)

(ii) $H(Z) = \dfrac{3Z(5Z-2)}{\left(Z+\frac{1}{2}\right)(3Z-1)}$

$$\therefore \frac{H(Z)}{Z} = \frac{3(5Z-2)}{\left(Z+\frac{1}{2}\right)(3Z-1)}$$

$$\frac{H(Z)}{Z} = \frac{15Z-6}{\left(Z+\frac{1}{2}\right)(3Z-1)}$$

Multiplying numerator and denominator by $\frac{1}{3}$ we get,

$$\frac{H(Z)}{Z} = \frac{5Z-2}{\left(Z+\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}$$

In partial fraction expansion from we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{\left(Z+\frac{1}{2}\right)} + \frac{A_2}{\left(Z-\frac{1}{3}\right)}$$

Here $A_1 = \left(Z+\frac{1}{2}\right)\cdot\frac{(5Z-2)}{\left(Z+\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}\Bigg|_{Z=-1/2}$

$$\therefore\ A_1 = \frac{-5/2-2}{-\frac{1}{2}-\frac{1}{3}} = \frac{-\frac{9}{2}}{-\frac{5}{6}} = \frac{27}{5}$$

and $A_2 = \left(Z-\frac{1}{3}\right)\frac{(5Z-2)}{\left(Z+\frac{1}{2}\right)\left(Z-\frac{1}{3}\right)}\Bigg|_{Z=1/3}$

$$\therefore\ A_2 = \frac{5/3-2}{\frac{1}{3}+\frac{1}{2}}$$

$$\therefore\ A_2 = \frac{-\frac{1}{3}}{5/6} = -2/5$$

$$\therefore\ \frac{H(Z)}{Z} = \frac{27/5}{Z+\frac{1}{2}} - \frac{2/5}{Z-\frac{1}{3}}$$

$$\therefore\ H(Z) = \frac{(27/5)Z}{Z+\frac{1}{2}} - \frac{(2/5)Z}{Z-\frac{1}{3}}$$

$$\therefore\ H(Z) = \frac{27/5}{1+\frac{1}{2}Z^{-1}} - \frac{2/5}{1-\frac{1}{3}Z^{-1}}$$

In the general form we can write,

$$H(Z) = H_1(Z) + H_2(Z)$$

Using short cut method it can be realized as shown in Fig. P. 10.3.15(c).

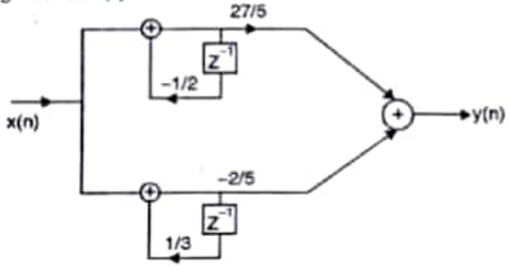


Fig. P. 10.3.15(c)

Ex. 10.3.16 : Fig. P. 10.3.16 shows direct-II form of realization of IIR filter :

(a) Find Transform functions H (Z) of filter.

(b) Find corresponding difference equation.

(c) Realize filter using cascade form using first order module ?

(d) Find impulse response function for filter.

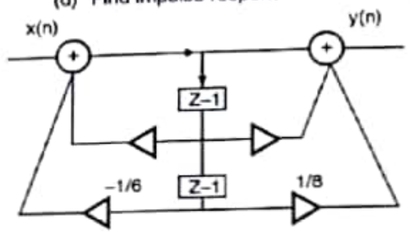


Fig. P. 10.3.16

Soln. :

(a) This is direct from-II structure, using shortcut method. We can write H (z) as,

$$H(Z) = \frac{1+Z^{-1}+\frac{1}{8}Z^{-2}}{1-Z^{-1}+\frac{1}{6}Z^{-2}}$$

(b) $$\frac{Y(Z)}{X(Z)} = \frac{1+Z^{-1}+\frac{1}{8}Z^{-2}}{1-Z^{-1}+\frac{1}{6}Z^{-2}}$$

$$Y(Z) - Z^{-1}Y(Z) + \frac{1}{6}Z^{-2}Y(Z) = Z^{-1}X(Z) + \frac{1}{8}Z^{-2}X(Z) + X(Z)$$

Taking IZT

$$y(n) - y(n-1) + 16\,y(n-2) = x(n) + x(n-1) + \frac{1}{8}x(n-2)$$

$$\therefore\ y(n) = x(n) + x(n-1) + \frac{1}{8}x(n-2) + y(n-1) - \frac{1}{6}y(n-2)$$

This is the difference equation.

(c) We have, $$H(Z) = \frac{1+Z^{-1}+\frac{1}{8}Z^{-2}}{1-Z^{-1}+\frac{1}{6}Z^{-2}}$$

$$\therefore\ H(Z) = \frac{Z^2+Z+\frac{1}{8}}{Z^2-Z+\frac{1}{6}} = \frac{(Z+0.146)(Z+0.853)}{(Z-0.79)(Z-0.21)}$$

Let $H(Z) = H_1(Z)\cdot H_2(Z)$

and $$H_1(Z) = \frac{Z+0.146}{Z-0.79} = \frac{1+0.146Z^{-1}}{1-0.79Z^{-1}}$$

$$H_2(Z) = \frac{Z+0.853}{Z-0.21} = \frac{1+0.853Z^{-1}}{1-0.21Z^{-1}}$$

Using short cut method cascade realization is shown in Fig. P. 10.3.16(a).

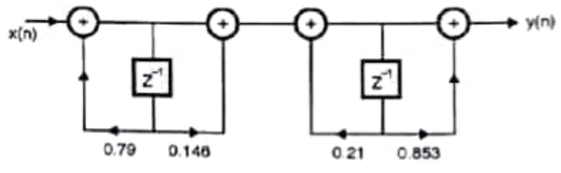


Fig. P. 10.3.16(a)

(d) We have,

$$H(Z) = \frac{Z^2+Z+\frac{1}{8}}{(Z-0.79)(Z-0.21)}$$

$$\therefore\ \frac{H(Z)}{Z} = \frac{Z^2+Z+\frac{1}{8}}{Z(Z-0.79)(Z-0.21)}$$

In P.F.E. from we can write.

$$\frac{H(Z)}{Z} = \frac{A_1}{Z} + \frac{A_2}{Z-0.79} + \frac{A_3}{Z-0.21}$$

$$A_1 = Z\cdot\frac{Z^2+Z+\frac{1}{8}}{Z(Z-0.79)(Z-0.21)}\Bigg|_{Z=0}$$

$$= \frac{\frac{1}{8}}{(-0.79)(-0.21)} = 0.75$$

$$A_2 = (Z-0.79)\cdot\frac{Z^2+Z+\frac{1}{8}}{Z(Z-0.79)(Z-0.21)}\Bigg|_{Z=0.79}$$

$$\therefore\ A_2 = \frac{(0.79)^2+0.79+\frac{1}{8}}{(0.79)(0.79-0.21)} = 3.36$$

$$A_3 = (Z-0.21)\cdot\frac{Z_2+Z+\frac{1}{8}}{Z(Z-0.79)(Z-0.21)}\Bigg|_{Z=0.21}$$

$$= \frac{(0.21)^2+0.21+\frac{1}{8}}{(0.21)(0.21-0.79)} = -3.11$$

$$\therefore\ \frac{H(Z)}{Z} = \frac{0.75}{2} + \frac{3.36}{Z-0.79} - \frac{3.11}{Z-0.21}$$

$$\therefore\ H(Z) = 0.75 + 3.36\frac{Z}{Z-0.79} - 3.11\frac{Z}{Z-0.21}$$

$$\therefore\ h(n) = 0.75\,\delta(n) + 3.36\,(0.79)^n\,w(n) - 3.11\,(0.21)^n\,w(n)$$

Ex. 10.3.17 : Obtain the cascade realization of system characterized by transfer function

$$H(Z) = \frac{2(Z+2)}{Z(Z-0.1)(Z+0.5)(Z+0.4)}$$

Soln. : To convert the equation into negative powers or Z ; multiply and divide by Z^{-4}

$$\therefore\ H(Z) = \frac{2Z^{-4}(Z+2)}{Z^{-4}[Z(Z-0.1)(Z+0.5)(Z+0.4)]}$$

$$= \frac{2Z^{-3}(1+2Z^{-1})}{1(1-0.1Z^{-1})(1+0.5Z^{-1})(1+0.4Z^{-1})}$$

$$\therefore\ H(Z) = \left(\frac{2Z^{-2}}{1-0.1Z^{-1}}\right)\left(\frac{Z^{-1}}{1+0.5Z^{-1}}\right)\left(\frac{1+2Z^{-1}}{1+0.4Z^{-1}}\right)$$

Let $H(Z) = H_1(Z)\cdot H_2(Z)\cdot H_3(Z)$

Using shortcut method, this realization is drawn as follows:

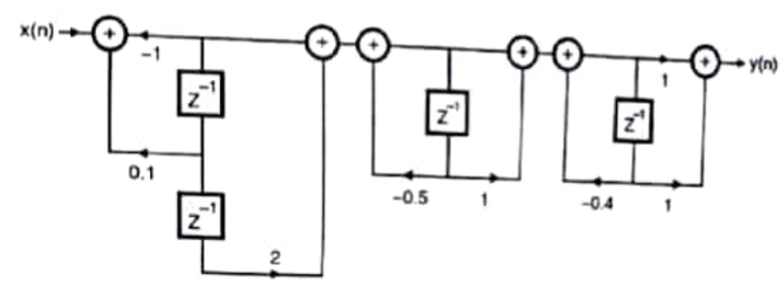


Fig. P. 10.3.17

Ex. 10.3.18 : Determine cascade and parallel realization of given function.

$$H(Z) = \frac{10\left(1-\frac{1}{2}Z^{-1}\right)\left(1-\frac{2}{3}Z^{-1}\right)(1+2Z^{-1})}{\left(1-\frac{3}{4}Z^{-1}\right)\left(1-\frac{1}{8}Z^{-1}\right)\left[1-\left(\frac{1}{2}+\frac{j}{2}\right)Z^{-1}\right]\left[1-\left(\frac{1}{2}-\frac{j}{2}\right)Z^{-1}\right]}$$

Soln. :

Cascade realization : The given function can be expressed as,

$$H(Z) = \frac{10}{1-\frac{3}{4}Z^{-1}}\cdot\frac{\left(1-\frac{1}{2}Z^{-1}\right)}{\left(1-\frac{1}{8}Z^{-1}\right)}\cdot\frac{\left(1-\frac{2}{3}Z^{-1}\right)}{\left[1-\left(\frac{1}{2}+\frac{j}{2}\right)Z^{-1}\right]}\cdot\frac{(1+2Z^{-1})}{\left[1-\left(\frac{1}{2}-\frac{j}{2}\right)Z^{-1}\right]}$$

That means $H(Z) = H_1(Z)\cdot H_2(Z)\cdot H_3(Z)\cdot H_4(Z)$

Using shortcut method, every subtransfer function is realized and connected in cascade as shown in Fig. P. 10.3.18(a).

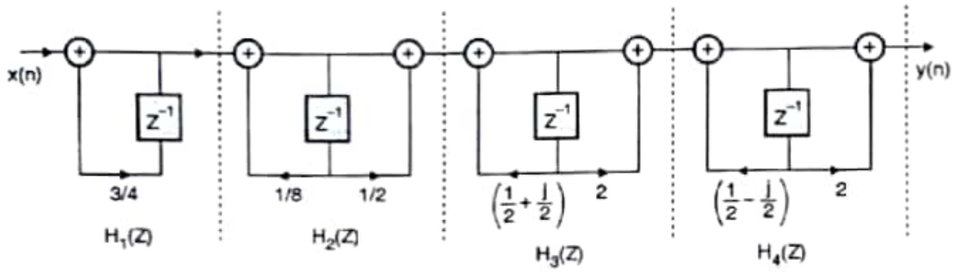


Fig. P. 10.3.18(a)

Parallel realization :

To obtain parallel realization we will perform partial fraction expansion as follow :

To convert given equation into positive power of Z ; multiply numerator and denominator by Z^4. That means multiply every bracket by Z.

$$\therefore \quad H(Z) = \frac{10Z\left(Z-\frac{1}{2}\right)\left(Z-\frac{2}{3}\right)(Z+2)}{\left(Z-\frac{3}{4}\right)\left(Z-\frac{1}{8}\right)\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}$$

$$\therefore \quad \frac{H(Z)}{Z} = \frac{10\left(Z-\frac{1}{2}\right)\left(Z-\frac{2}{3}\right)(Z+2)}{\left(Z-\frac{3}{4}\right)\left(Z-\frac{1}{8}\right)\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}$$

In partial fraction expansion form we can write,

$$\frac{H(Z)}{Z} = \frac{A_1}{Z-\frac{3}{4}} + \frac{A_2}{Z-\frac{1}{8}} + \frac{A_3}{\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]} + \frac{A_4}{\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]} \quad \ldots(1)$$

The coefficients A_1, A_2, A_3, A_4 are calculated as follows :

$$\therefore \quad A_1 = \left(Z-\frac{3}{4}\right)\cdot\frac{H(Z)}{Z}\bigg|_{Z=\frac{3}{4}}$$

$$\therefore A_1 = \left(Z-\frac{3}{4}\right)\cdot\frac{10\left(Z-\frac{1}{2}\right)\left(Z-\frac{2}{3}\right)(Z+2)}{\left(Z-\frac{3}{4}\right)\left(Z-\frac{1}{8}\right)\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}\bigg|_{Z=314}$$

$$\therefore \quad A_1 = \frac{10\left(\frac{3}{4}-\frac{1}{2}\right)\left(\frac{3}{4}-\frac{2}{3}\right)\left(\frac{3}{4}+2\right)}{\left(\frac{3}{4}-\frac{1}{8}\right)\left[\frac{3}{4}-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[\frac{3}{4}-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}$$

$$= \frac{0.5729}{(0.625)(0.3125)}$$

$$\boxed{\therefore \quad A_1 = 2.933}$$

$$A_2 = \left(Z-\frac{1}{8}\right)\frac{10\left(Z-\frac{1}{2}\right)\left(Z-\frac{2}{3}\right)(Z+2)}{\left(Z-\frac{3}{4}\right)\left(Z-\frac{1}{8}\right)\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}\bigg|_{Z=\frac{1}{8}}$$

$$\therefore \quad A_2 = \frac{10\left(\frac{1}{8}-\frac{1}{2}\right)\left(\frac{1}{8}-\frac{2}{3}\right)\left(\frac{1}{8}+2\right)}{\left(\frac{1}{8}-\frac{3}{4}\right)\left[\frac{1}{8}-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[\frac{1}{8}-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}$$

$$= \frac{4.316}{-0.625\,[-0.375-0.5\,j]\,[-0.375+0.5\,j]}$$

$$\boxed{\therefore \quad A_2 = -17.68}$$

$$A_3 = \left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\cdot\frac{10\left(Z-\frac{1}{2}\right)\left(Z-\frac{2}{3}\right)(Z+2)}{\left(Z-\frac{3}{4}\right)\left(Z-\frac{1}{8}\right)\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}\bigg|_{Z=\frac{1}{2}+\frac{j}{2}}$$

$$A_3 = \frac{10\left(\frac{1}{2}+\frac{j}{2}-\frac{1}{2}\right)\left(\frac{1}{2}+\frac{j}{2}-\frac{2}{3}\right)\left(\frac{1}{2}+\frac{j}{2}+2\right)}{\left(\frac{1}{2}+\frac{j}{2}-\frac{3}{4}\right)\left(\frac{1}{2}+\frac{j}{2}-\frac{1}{8}\right)\left[\frac{1}{2}+\frac{j}{2}-\frac{1}{2}+\frac{j}{2}\right]}$$

$$= \frac{10\,(0+0.5j)\,(-0.167+0.5j)\,(2.5+0.5j)}{(-0.25+0.5j)\,(0.375+0.5j)\,(0+j)}$$

$$= \frac{-5.8325-j\,3.3375}{-0.0625-j\,0.34375}$$

$$\therefore \quad \boxed{A_3 = 12.22-j\,14.58}$$

$$A_4 = \frac{\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]10\left(Z-\frac{1}{2}\right)\left(Z-\frac{2}{3}\right)(Z+2)}{\left(Z-\frac{3}{4}\right)\left(Z-\frac{1}{8}\right)\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}\bigg|_{Z=\frac{1}{2}-\frac{j}{2}}$$

$$\therefore \quad A_4 = \frac{10\left(\frac{1}{2}-\frac{j}{2}-\frac{1}{2}\right)\left(\frac{1}{2}-\frac{j}{2}-\frac{2}{3}\right)\left(\frac{1}{2}-\frac{j}{2}+2\right)}{\left(\frac{1}{2}-\frac{j}{2}-\frac{3}{4}\right)\left(\frac{1}{2}-\frac{j}{2}-\frac{1}{8}\right)\left[\frac{1}{2}-\frac{j}{2}-\frac{j}{2}-\frac{j}{2}\right]}$$

$$\therefore \quad A_4 = \frac{10\,(0-0.5j)\,(-0.17-0.5j)\,(2.5-0.5j)}{(-0.25-0.5j)\,(0.375-0.5j)\,(0-j)}$$

$$= \frac{-5.825+j\,3.375}{-0.0625+j\,0.34375}$$

$$\boxed{\therefore \quad A_4 = 12.49+j\,14.67}$$

Putting these values in Equation (1) we get,

$$\frac{H(Z)}{Z} = \frac{2.933}{Z-\frac{3}{4}} - \frac{17.68}{Z-\frac{1}{8}} + \frac{(12.22-j\,14.58)}{\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]} + \frac{(12.49+j\,14.67)}{\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}$$

$$\therefore \quad H(Z) = \frac{2.933Z}{Z-\frac{3}{4}} - \frac{17.68Z}{Z-\frac{1}{8}} + \frac{(12.22-j\,14.58)\,Z}{\left[Z-\left(\frac{1}{2}+\frac{j}{2}\right)\right]} + \frac{(12.49+j\,14.67)\,Z}{\left[Z-\left(\frac{1}{2}-\frac{j}{2}\right)\right]}$$

Converting equation into negative powers of Z,

$$H(Z) = \frac{2.933}{1-\frac{3}{4}Z^{-1}} - \frac{17.68}{1-\frac{1}{8}Z^{-1}} + \frac{(12.22-j\,14.58)}{1-\left(\frac{1}{2}+\frac{j}{2}\right)Z^{-1}} + \frac{(12.49+j\,14.67)}{1-\left(\frac{1}{2}-\frac{j}{2}\right)Z^{-1}}$$

$$\therefore \quad H(Z) = H_1(Z)+H_2(Z)+H_3(Z)+H_4(Z)$$

The parallel realization is shown in Fig. P. 10.3.18(b).

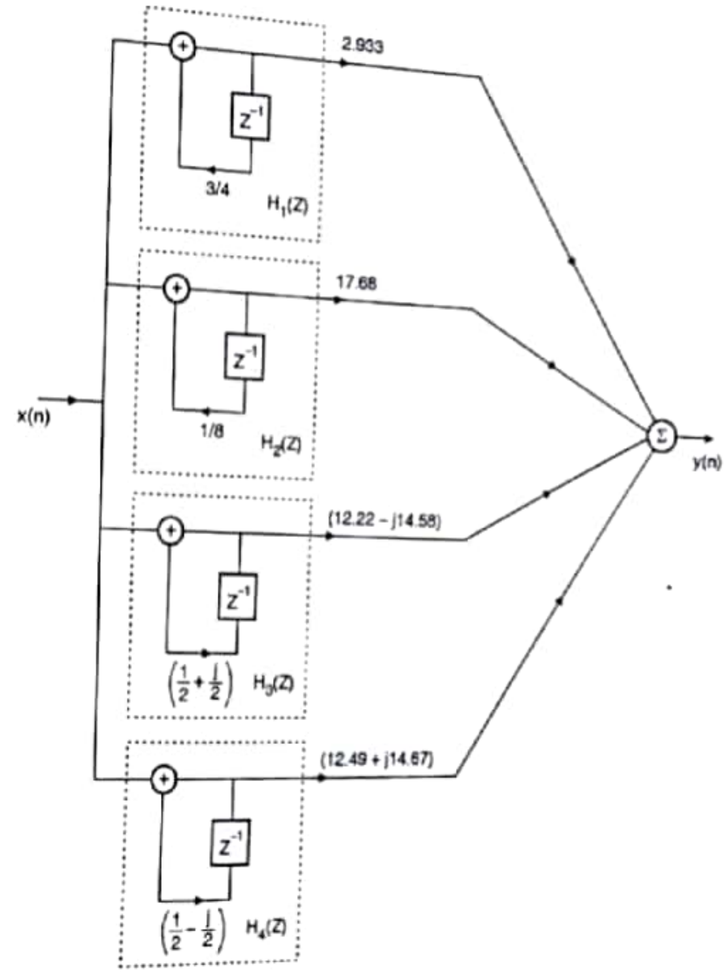


Fig. P. 10.3.18(b)

CHAPTER 11

Module 6

Applications of Signals and Systems

Syllabus :

Signal processing Applications : Speech and Audio Processing, Multimedia (image and video) processing, Underwater acoustic signal processing Biological signal analysis.

Communication and Control System Application : Modulation (Analog and Digital) process, Feedback/Feedforward Control system.

11.1 Speech and Audio Processing :

11.1.1 Acoustic Features of Speech :

- Human speech is generated through vocal cords and the vocal tract.
- The vocal cords, consisting of two bands of tough, elastic tissue, located at the opening of larynx, vibrate when the air from the lungs passes between them producing sound waves which are emitted from the lips and to some extent from the nose, and heard as speech.

Speech sounds are of two types :

1. Voiced sounds :

There are produced by quasi-periodic pulses of air exciting the vocal tract. These pulses, in turn, are generated by forcing air through the epiglottis, with the tension of the vocal cords.

2. Unvoiced sound :

- To produce these sounds, a constriction is formed at some point along the vocal tract, towards the mouth. Air is forced through the constriction at a high speed resulting in turbulent flow. A broad-spectrum noise source is created which excites the vocal tract leading to the emission of a unvoiced sound.
- Human sounds can be characterized in terms of three vital parameters as,
  1. **Pitch :** Related to the frequency of sound (in Hz).
  2. **Loudness :** The physiological perception of the intensity of sound (in dB).
  3. **Quality :** A property possessed by sound by virtue of its harmonic content.

(i) Phonemes :

The smallest units of sound which can be recognized by contrast with their environment are called phonemes.

(ii) Dipones :

The sounds stretching from the middle of one phoneme to the centre of the next is termed as dipones.

(iii) Digitization of speech :

- The acoustic energy of speech is converted into electrical energy by a electroacoustic transducer. First, the speech signal / waveform is passed through a low-pass filter restraining the components below a frequency W Hz.
- It is then sampled at a rate greater than 2W, quantized and converted into binary digits. Although the frequency W is determined by the intended application, it usually lies between 3 and 5 kHz. The sampling rate ≥ 8000 samples per second is adequate for digital representation of telephone bandwidth ~ 4 kHz speech signals. An encoding rate of 16 bits per second is used.
- The digitized signal is now ready for processing and various operations can be carried out on it, as demanded by the problem at hand.

11.1.2 Speech Synthesis (Computer Created Speech) :

Application :

1. Talking games.
2. Text to speech mechanism used by blind people.
3. Medical monitoring systems that gives verbal warnings and directions when some emergency conditions exists.
4. Some controls of space shuttle are operated verbally by pilot.
5. Voice entry systems are used by handicapped people.

11.1.3 Methods for Speech Synthesis :

- For better speech quality, more number of bits should be stored in the memory to represent each word.
- The speed of operation depends on how fast data is send to the synthesiser act.

1. Waveform modification :

- Voiced sounds are produced by vibration of vocal cords as air passes from lungs.
- The frequency of vibration is called as pitch.
- Pitch depends on position of tongue shape of mouth and the position of lips.
- Vowels A, E are examples of voiced sounds.
- Unvoiced sounds are produced by modifying the position of tongue and shape of mouth as consist stream of air comes out from the lungs. Letter s is an example of unvoiced sound.
- Mixture of voiced sound and unvoiced sound is called as nasal sounds or fricatives.

Scanned by CamScanner

Electronic model for human vocal tract

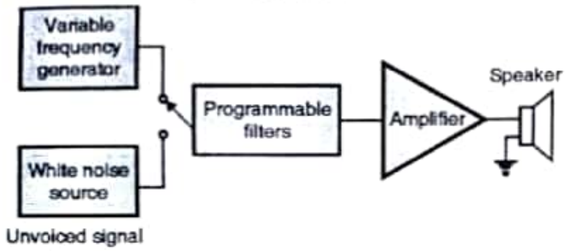


Fig. 11.1.1

- Usually LPC is used as filter. In this case the parameters send by the computer are the coefficients of LPC and the pitch for the pulse source.
- In LPC, current value is predicted as the sum of current input value and the portions of previous input values.
- A high quality LPC synthesis requires 16 K bits/sec.
- IC sp 1000 is used as LPC speech processor, LPC speech recognizer and LPC speech synthesis.
- In learn mode sp 1000 generates LPC coefficients for spoken words. Microcomputer reads these coefficients and stores in memory.
- To operate in recognisation mode, sp 1000 is used to generate the coefficients for unknown words. These coefficients are then compared with known words in memory and then unknown word is identified.
- For synthesis mode, sp 1000 is switched to talk mode. Microcomputer sends the coefficients to it and then sound is generated.

2. Direct digitization speech synthesis :

- This method produces high quality speech because it is actually play back of recorder sound.
- Initially, the word, we want the compare to speak ; is spoken clearly into microphone. Output voltage of microphone is amplified and applied to 12 bit ADC.
- One approach is to store the samples of ADC into memory. When it is required to produce the word than these samples are applied to DAC. But this requires lot of memory space. To reduce this, speech compression techniques are used.
- Sometimes delta modulation techniques are used. In this case only 3 bit or 4 bit code representing how much sample has changed from last sample is stored in memory ; instead of storing complete 12 bit value.

11.1.4 Speech Recognition :

- In the training mode, templates of each of words that the system need to recognize are produced. These templates are stored in the memory.
- To produce a template for a word ; the user speaks the word several times in a microphone connected to system.
- The system then determines several parameters or features of each repetition of words. Then average of all values is taken and template is produced.
- The centre frequency of filters are adjusted to match those of average speaks.
- The output of filter is averaged to produce a signal proportional to energy in each of the frequency bands.
- ZCD is used to give basic frequency information.
- Output of ZCD is converted to volt, so it can be digitized along with the output from averages.
- When the word is spoken, samples of each of the features are taken and digitized at evenly spaced intervals of 10 to 20 ms during the duration of word. These features are stored in memory.
- In training mode ; this set of samples will be averaged with others to form template of the word.
- In recognization mode, this set of feature will be compared with templates stored in memory.
- The best set of matching is considered to be correct word.

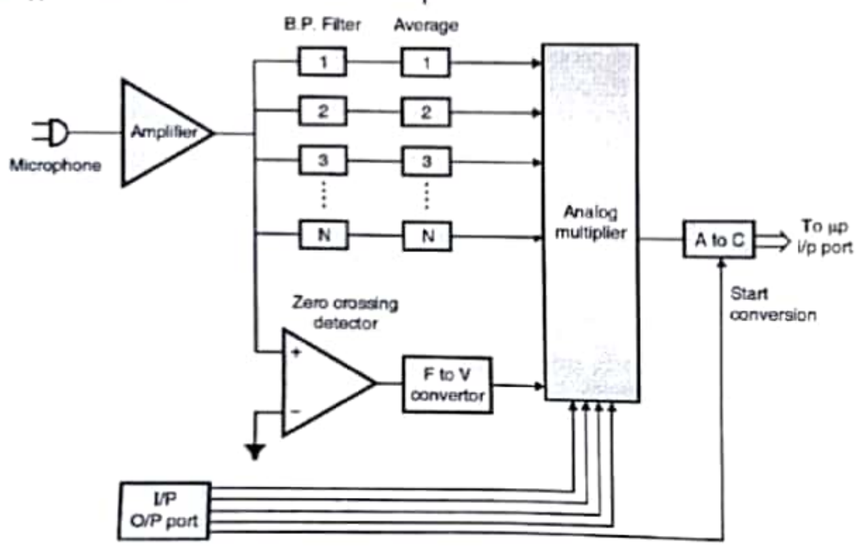


Fig. 11.1.2

Advantages and disadvantages :

1. The performance is sensitive to amount of training data available for creating sound class reference pattern. Generally the more training ; the higher will be performance of system.
2. The reference patters are sensitive to the speaking environment and transmission characteristics of medium used to create the speech. This is because, spectral characters of speech are affected by environmental conditions.
3. No speech specific knowledge is used explicitly in the system; hence method is relatively insensitive to the choice of vocabulary words, task, syntax etc.
4. Computational load is directly proportional to number of patterns being trained or recognized.
5. Because of system is insensitive to sound class, the basic techniques are applicable to wide range of speech sounds including phrases, whole words subwords units.

11.1.5 Voice Response Systems :

- It is designed to respond to a request for information using spoken message.
- It communicates the voice in one direction only i.e. from machine to man.
- A simple example is automated telephone directory system in which improperly dialled telephone numbers are detected, the type of problems is determined (e.g. telephone has been disconnected or a number has been changed etc.). Then a request for an appropriate message is sent to voice response system.
- For such a system the vocabulary entries are generally entirely phrases as well as limited number of isolated words.

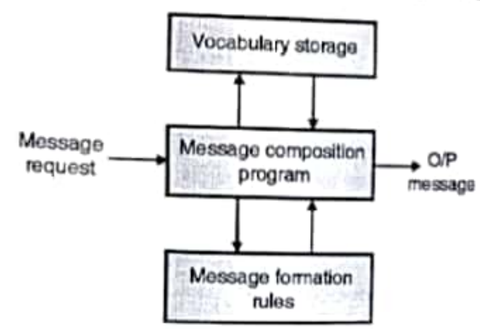


Fig. 11.1.3

- Elements of voice response system are :
  1. Provision for storage of a vocabulatory for voice response system.
  2. Rules for forming messages from elements of vocabulary.
  3. A program for composing voice response messages.
- Input to this system is in the form of message request which may be initiated by another information processing system.
- Output is in the form of speech alterness in response to the message request.

11.1.6 All Digital Voice Response System :

- It is limited vocabulary system in which output message is created by terminating isolated natural speech elements in the vocabulary storage.

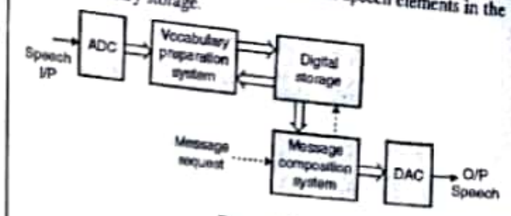


Fig. 11.1.4

- Vocabulary consists of isolated words and phrases which are represented in digital form and stored in digital memory.
- Messages are created by retrieving the required words and phrases from storage and reproducing them in a proper sequence.
- Different considerations in designing are as follows :
  1. A means of representing and storing the basic vocabulary elements must be selected and system designed to permit easy access to each elements of vocabulary.
  2. A means of editing speech recordings to select the desired vocabulary elements must be provided along with a means of recording the vocabulary elements onto the storage medium.
  3. A system for selecting and reproducing vocabulary elements in the prescribed sequence (i.e. Message composition system).

11.1.7 Multiline Digital Voice Response System :

- Vocabulary elements are represented using ADPCM coding at 24 kb/s.
- ADPCM coders and decoders are implemented in hardware.
- The beginning and end points of each word are automatically located by an algorithm which is based upon the computation of short time energy of the ADPCM code words.
- The vocabulary is stored on a fixed head disk for rapid retrieval by message composition program.
- The computer receives message requests through telephone lines or direct digital interfaces to other computers.
- The message composition program locates the required Vocabulary elements and retrieves them from memory.
- The digital representation of desired messages are stored in buffers in RAM of computer.
- These buffers are accessed by ADPCM decoders through DMA channel.
- The system is able to provide voice output simultaneously on many channels.

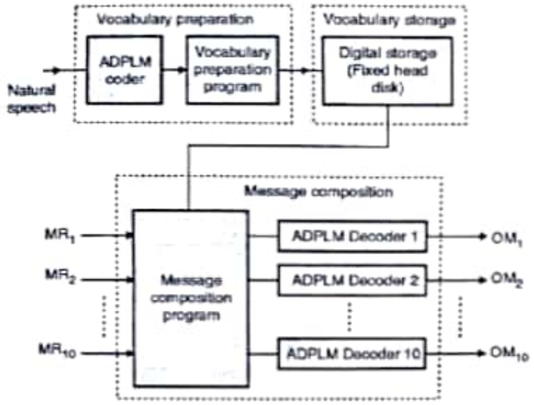


Fig. 11.1.5

11.1.8 Subband Coding of Speech Signal :

- Digital signal is obtained by using A/D converter which may be needed to be stored for future use. In order to increase the efficiency of the system, the digital signal is to be stored over limited bandwidth. This can be achieved by using signal compression techniques. Signal compression is achieved by using subband coding.
- **Example** : In telephone systems, speech signal has bandwidth of 3.4 kHz. For digital transmission, the speech is sampled at 8 kHz by 8 bit A/D converter. This generates a digital signal of 8 kHz × 8 = 64 kbps. If we are able to compress such signal to half of it (i.e. 32 kbps) then, double number of speech signals can be easily accommodated using the same channel bandwidth. By using the non-uniform distribution of energy in frequency bands of speech and audio signal, it is possible to divide them into various subbands.

Subband coding encoder :

Fig. 11.1.6 shows the subband coding encoder. Following steps are involved :

1. The input signal x (n) is given to set of filters which decomposes the input signal into set of narrowband signals occupying contiguous frequency bands.
2. These narrow band signals are down sampled producing subband signals.
3. All the narrow band signals are then compressed by encoder. This reduces the data rate and storage requirement of the signals.
4. These compressed signals are multiplexed and transmitted over the channel. Thus, signal q (n) has reduced data rate as compared to x (n).

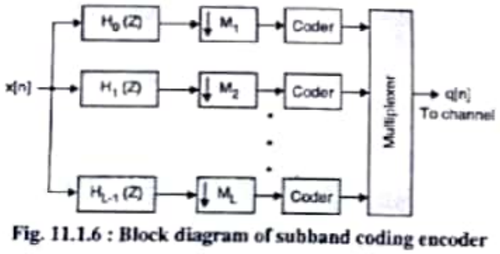


Fig. 11.1.6 : Block diagram of subband coding encoder

Subband coding decoder :

Fig. 11.1.7 shows the subband coding decoder.

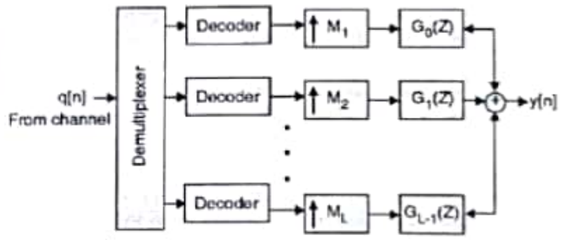


Fig. 11.1.7 : Block diagram of subband coding decoder

Following steps are involved :

1. The received signal q (n) is first demultiplexed into set of subband signals.
2. Then the subband signal is decompressed by decoder.
3. Decoded signal is upsampled by using sampler.
4. The upsampled signal is given to synthesis filter bank $G_0(Z)$ to $G_{L-1}(Z)$.
5. The outputs of synthesis filter bank are combined to produce output signal y (n).

11.1.9 Voice Coders (Vocoders) :

- In our day to day life we listen to different persons speaking the same English language in different styles. And still we can recognize what is being spoken.
- Thus to transmit speech, we need not transmit the precise waveform generated by the speaker.
- Rather we can transmit the "information" from which a waveform can be reconstructed at the receiver which is "similar" to and not "identical" to the waveform generated by the speaker.
- The speech coders or vocoders are used for the digital coding of speech. They are used for voice simulation.
- The advantage of using the vocoders is that they operate at very low bit rates ranging from 1.2 to 2.4 kbits/sec.

11.1.10 Voice Model (Speech Model) :

- A voice model or speech model shown in Fig. 11.1.8 is used to simulate voice. The operation of a voice model is dependent on the fact that speech consists of a sequence of "voiced" and "unvoiced" sounds.
- Therefore we can approximate it by passing a sequence of voiced and unvoiced sounds through a filter.
- The term "voiced sounds" is defined as the sounds which are generated by the vibrations of the vocal cords.
- The "unvoiced" sounds are produced at the time of pronunciation of letters such as "s", "p" or "f".
- These sounds are produced by expelling air through lips and teeth. A speech model is as shown in Fig. 11.1.8.

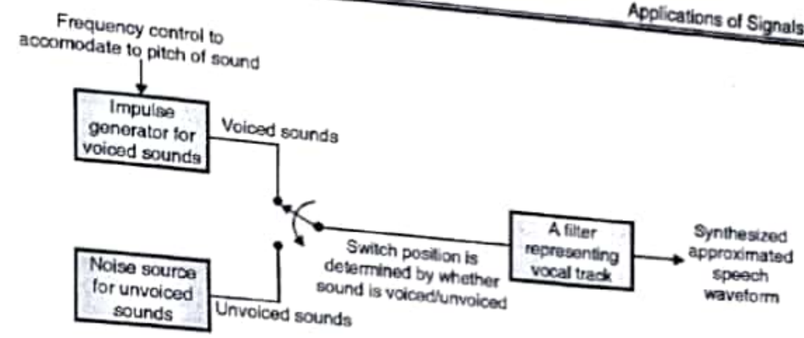


Fig. 11.1.8 : Speech model used in vocoders

- The speech model shown in Fig. 11.1.8 simulates the voiced sounds by an impulse generator whose frequency is equal to the fundamental frequency of the vocal cords.
- The noise source simulates the unvoiced sounds. The switch position determines whether the sound is voiced or unvoiced.
- The selected voiced or unvoiced sound is then passed through a filter, which simulates the effect of the mouth, throat and nasal passage of the speaker on the generated sound.
- The input signal is filtered by this filter in such a way that the required letter is pronounced.
- All the vocoders employ the scheme shown in Fig. 11.1.8 for generation of a synthesized approximation of speech. A voice coder or vocoder consists of voice encoder and voice decoder. Let us discuss their operation one by one.

11.1.11 Voice Encoder :

Fig. 11.1.9(a) shows the block diagram of a voice encoder.

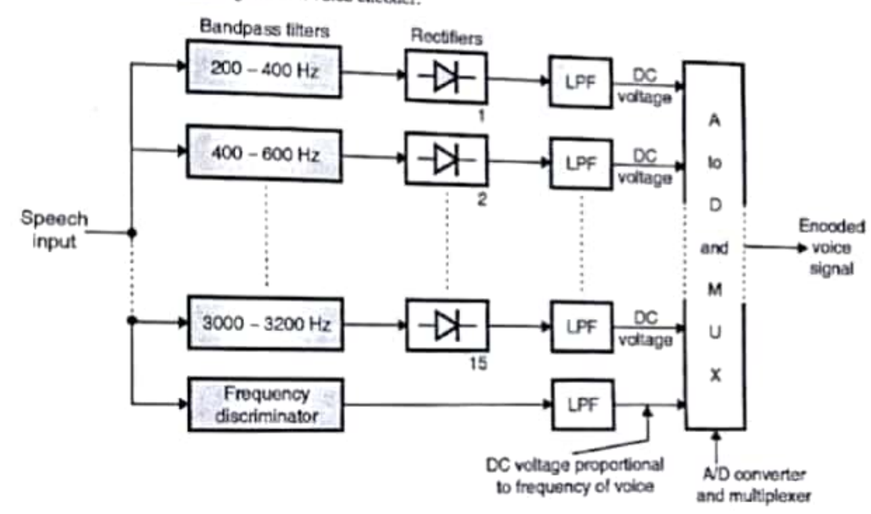


Fig. 11.1.9(a) : Voice encoder

Operation :

Operation of the voice encoder is as follows :

- The speech frequency spectrum (200 Hz – 3200 Hz) is divided into 15 frequency ranges using 15 bandpass filters. Each range is of bandwidth 200 Hz. (200 to 400, 400 to 600 etc.)
- Output of each bandpass filter is rectified and filtered to produce a dc voltage which is proportional to the amplitude of AC signal at filter output.
- The input speech signal is also applied to a frequency discriminator followed by a 20 Hz low pass filter. The output of this filter is a dc voltage which is proportional to the voice frequency. This frequency represents the pitch of the voice.
- The dc voltage at the output of the discriminator-filter combination can also indicate whether the speech is voiced or unvoiced.
- The DC signals available at the outputs of 16 low pass filters are sampled, multiplexed and A/D converted. Thus we get the digital equivalent of the speech signal at the encoder output.

- Then the encoded voice signal transmitted by the voice encoder consists of all the frequency components from 200 to 3200 Hz, information about the "pitch" of the speech signal and an information about whether the speech is voiced or unvoiced.

11.1.12 Voice Decoder :

- Voice decoder first decodes the digital voice signal and then using a speech synthesizer, produces an approximate voice signal at its output.
- Fig. 11.1.9(b) shows the block diagram of a voice decoder.

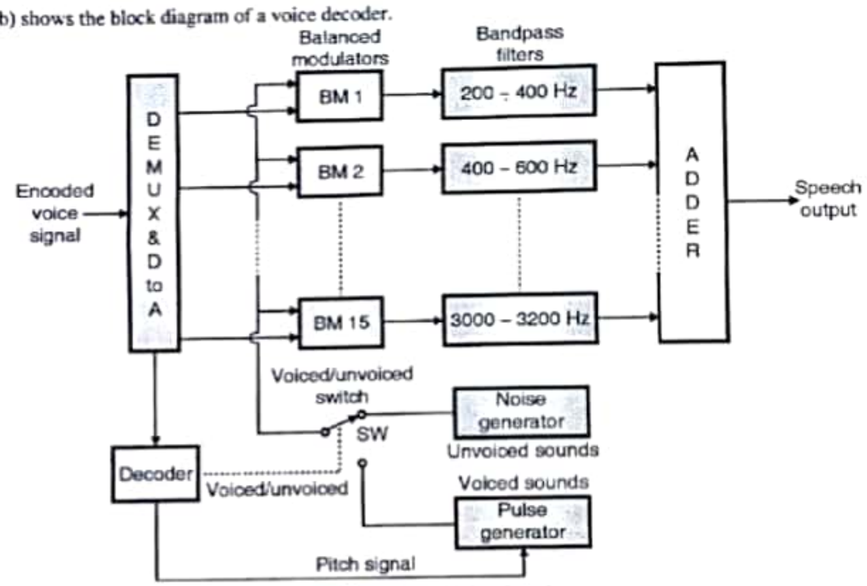


Fig. 11.1.9(b) : Voice decoder

Operation of voice decoders :

Operation of the voice decoder is as follows :

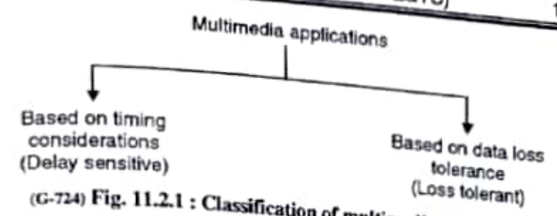
- The received encoded voice signal is demultiplexed and decoded i.e. converted back to analog form by the demultiplexer and D/A converter.
- Corresponding to each rectifier-filter combination at the encoder we have a balanced modulator-filter combination at the receiver. The carrier signal to all these balanced modulators comes from either a noise generator or a pulse generator output depending on the position of switch "SW".
- The position of switch "SW" depends on the decision made by the decoder of whether the received signal is a "voiced" signal or an "unvoiced" signal. When the received signal is "voiced", switch "SW" connects the pulse generator output to the inputs of all the balanced modulators and when it is unvoiced, the switch connects the noise generator at the input of all the balanced modulators.
- The position of switch "SW" is dependent on the decision made by the decoder of whether the received signal is "voiced" or "unvoiced".
- When the received signal is "voiced" the outputs of only certain balanced modulators will be present depending on the frequencies present in the received signal. But if an "unvoiced" signal is received then all the balanced modulator will produce some output.
- The adder will add all the analog signals at its input to produce the voice or speech output.

The only disadvantage of speech transmission using vocoders is that the quality of speech is severally degraded.

11.2 Multimedia (Image and Video) :

11.2.1 Multimedia :

- The meaning of multimedia is two or more media. But in practice when we use the word multimedia, it means the combination of two or more **continuous media.**
- Usually the two media are audio and video.
- Multimedia is a system, which is a combination of following components such as :
  1. Text
  2. Images or Graphics
  3. Animation
  4. Sound
  5. Video
- In multimedia, all these five media are used to represent the information to the users via the computer.
- In practice, many multimedia packages comprise of only two or three medias of the above media, as sometimes the use of all media types may not be appropriate for the information exchange. Typically all computers have inbuilt sound cards. Playing of notes, music is possible.
- We can classify the multimedia applications in two major categories as shown in Fig. 11.2.1.



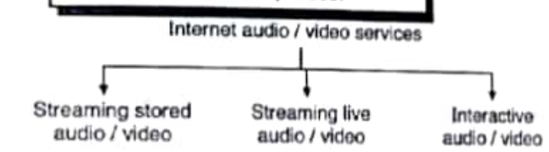
(G-724) **Fig. 11.2.1 : Classification of multimedia applications**

- **Delay sensitive :** These applications are sensitive to timing delays. Many multimedia applications are useless if the sender to receiver delay is more than a few hundred milliseconds.
- **Loss tolerant :** Most of the multimedia applications are loss tolerant. That means they allow the loss of data to some extent without compromising with the quality.
- So the multimedia applications are delay sensitive but loss tolerant, which are different from the elastic applications such as E-mail, FTP and Telnet.

11.2.2 Classification of Audio / Video Services :

Fig. 11.2.2 shows the classification of audio and video services. It shows that audio and video services can be classified into three categories namely :

1. Streaming stored audio / video.
2. Streaming live audio / video.
3. Interactive audio / video.



(O-261) **Fig. 11.2.2 : Classification of internet audio and video services**

Streaming :

Streaming is the process in which the user can listen or watch an audio or video file after the downloading has begun.

1. Streaming stored audio / video :

- In this type of internet service the audio / video files are stored on a server in the compressed form. A user (client) uses Internet to download these stored files, onto his machine.
- This examples of this service are : stored songs, books, educational lectures, movies, TV shows, video clips etc.
- Streamed stored audio / video service is also known as on demand audio / video.

2. Streaming live audio / video :

- In this category the audio / video information is being streamed live on the Internet and the user is listening / watching the broadcast live.
- The examples of this service are : Internet radio and Internet TV (live cricket match broadcast).
- Note that in the live streaming there is no need to download the stored audio / video files from a server.

3. Interactive audio / video :

- This service enables the users to interactively communicate with each other via Internet. The examples of this service are Internet telephony and internet teleconferencing.
- For all these applications the audio and video information has to be in the digital form. Hence it is necessary to first digitize the sound and video information and then it should be compressed to reduce the bandwidth required to transmit it.
- In this chapter we are going to discuss the digitization and compression of the audio and video signals.

11.2.3 Streaming Stored Audio and Video :

- This is the class of applications which allows the clients to request for compressed audio or video files which are stored on the servers.
- Stored audio files may contain educational lectures, songs and other recorded audio stuff.
- Stored video files may contain movies, TV shows, documentaries, cartoons or music video clips.
- This class of applications has three distinguishing factors as follows :

1. Stored media	2. Streaming
3. Continuous playout.	

1. Stored media :

The multimedia content has been prerecorded and stored at the server. So it is possible for the user to make following operations on the stored contents :

1. Pause
2. Rewind
3. Fast forward
4. Index through.

2. Streaming :

- In a streaming stored audio/video applications, a client starts playout of the audio/video a few seconds after receiving the file from the server.
- So the client plays out audio/video from one location in the file when it is receiving later parts of the same file from the server. This technique is called as **streaming.**
- This avoids need to download the entire file before beginning playout.

3. Continuous playout :

- Once the playout of the multimedia content begins, it is necessary to proceed as per the original timing of recording.
- This requirement puts critical delay constraints on data delivery.

11.2.4 Streaming Live Audio and Video :

This is similar to traditional broadcast radio and TV, except for one change that transmission takes place over the Internet.

Internet radio :

- After it became possible to stream audio over the Internet, commercial radio stations started broadcasting their contents over the Internet.
- There are two general approaches followed. In the first one, the programs are prerecorded and stored on disk.
- Listeners can download any of these programs for listening.
- The other approach is to broadcast live over the Internet. With this technique, buffering on the users side is necessary to smooth out jitter.

- The Internet radio should use multicasting with RTP/RTSP protocols.
- The streaming live audio/video is not stored. Hence a client cannot fast forward through the media.
- However local storage is always possible, which makes the other interactive applications possible.
- Distribution of live audio/video to many receivers can be accomplished using the IP multicasting technique or using separate unicast streams.

11.2.5 Real Time Interactive Audio and Video:

- In this class of applications, people are allowed to use audio/video to communicate with each other in real time.
- Real time interactive audio over the internet is called as **Internet phone.** This can provide local, PBX and long distance telephone service at a very low cost.
- This service also facilitates some special services such as :
  1. Web phone integration
  2. Group real time communication
  3. Directory services
  4. Caller filtering
- Real time interactive video is also called as video conferencing. Here the users communicate visually as well as orally.
- The delays should not be longer than 400 msec.

11.2.6 Problems for Multimedia in Today's Internet :

- The protocol used by the Internet is IP. Recall that IP provides the **best effort service** to all the datagrams it carries.
- That means the Internet makes best effort to move each datagram from sender to receiver as quickly as possible.
- But the Internet does not make any promise about end to end delay for an individual packet.
- Same is true for TCP and UDP as they operate on IP.
- As there is no guarantee of delivering the packets in time, it is an extremely challenging problem to develop successful multimedia networking applications for the Internet.
- So multimedia on Internet has not achieved the success that it is expected to get.

11.2.7 Expectations of Multimedia from the Internet :

Different multimedia types have different expectations from the Internet. Some of the important expectations are as follows :

1. Real time characteristics.
2. High bandwidth.
3. Multicasting support.
4. Error characteristics of the medium.
5. Session individuality.
6. Security.
7. Mobility.

11.3 Underwater Acoustic Signal Processing :

- It is basically acoustic communication technique, used to send and receive the messages below water.
- The common method used for such communication is hydrophones.
- The electromagnetic waves are not preferred for such communication, because these waves propagate over short distances.
- Hence acoustic waves are preferred for such communication because acoustic waves can propagate over longer distances.
- The major limitations for underwater communication are as follows
  1. **Multipath propagation**
     The signals generated from transmitter, follows multiple paths because of this, there is a large delay spread.
  2. Available bandwidth is small.
  3. For long range; signals attenuation is strong.
  4. Due to use of acoustic waves; the communication takes place with low data rate.
- The different types of modulation techniques used for acoustic communication are as follows
  1. Phase shift keying
  2. Frequency shift keying
  3. Spread spectrum (Direct sequency or frequency hopped)
  4. Multiple frequency shift keying.
- The vector sensors are used to measure acoustic filed components like wave velocity.
- An internal vector sensor measures the velocity and gradient vector sensor gives approximations of acoustic field.

11.4 Biological Signal Analysis :

11.4.1 Case Study of Real Time DSP Applications to Biomedical Signal Processing :

- Most of the biomedical signals are in audio range and DSP is widely used in many biomedical applications.
- The signals from the body, are captured using electrodes. But the major problem is addition of noise in such signals.
- To remove this noise signals; digital filters are used.

11.4.2 Fetal ECG Monitoring :

- Electrical activity of a heart is called as electrocardiogram (ECG).
- Fetal ECG represents electrical activity of the baby's heart. It is similar to the adult ECG waveform.
- The Fetal ECG (FECG) contains five peaks namely PQRST. This waveform is shown in Fig. 11.4.1.

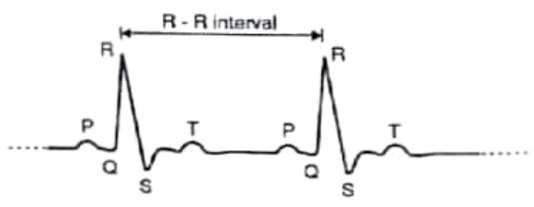


Fig. 11.4.1

- The Fetal Heart Rate (FHR) is obtained from R-to-R interval of the waveform shown in Fig. 11.4.1.
- ECG is having P-wave, QRS complex wave and T wave.
- Instantaneous heart rate is obtained by multiplying time interval between R-to-R (in miliseconds) by 60,000.
- A suitable DSP algorithm is used successive QRS complex waveform and from this, R-to-R interval is calculated. It gives value of corresponding FHR.
- The shape of QRS complex waveform changes from patient to patient.
- The obtained ECG signal is compared with a known standard ECG and then the locations of QRS complex in the obtained ECG can be determined on the basis of similarity.

11.4.3 DSP based Closed Loop Controlled Anaesthesia :

- In case of surgery; anaesthesia is injected to the patient body.
- A proper amount of drug to induce anaesthesia is required to inject in the patient body.
- Extra amount of drug produces side effects, as well as less amount of drug can produce psychological consequences for a long term.
- So it is required to make necessary changes in the dose of anaesthetic drugs and to control the anaesthesia.
- Using DSP, a closed loop system can be designed to monitor the does of anaesthesia and it is an automatic system.
- But, this system requires an accurate measurement of the depth of anaesthesia and it requires a feedback signal to monitor the delivery of dose.
- We will discuss the closed loop system used for EEG (Electroencephalogram).
- EFG represents, electrical activity of the brain and it contains useful information for diagnostic of neurological disorders.
- From the EEG signals; the features like bispectral index Audiotory Evoked Response (AER) are extracted and then the depth of anaesthesia is determined.
- Basically AER signal gives information from consciousness to unconsciousness condition of the patient.
- But these AER signals are usually small and they are mixed up with EEG.
- So a proper system is required to extract these AER signals and then process it.
- A bispectral index is obtained by performing spectrum analysis of EEG.
- It gives information about the changes in frequency components of EEG at different consciousness levels.
- A block diagram of DSP based closed controlled anaesthesia system is shown in Fig. 11.4.2.
- EEG electrodes are placed on the scalp of the patient; which generates EEG signals.
- Using suitable signal processing method; noise contained in EEG is reduced.
- EEG monitor produces EEG index (bispectral index) and it gives the measurement of drug induced in patient body.
- This EEG index is used as a feedback signal and it is compared with the target EEG index.
- The target EEG index is a standard value which is determined by considering many biomedical factors.
- The difference between target EEG index and measured EEG produces an error signal, e (k).
- This signal is applied to PID (Proportional-Integral-Derivative) control; to generate the required control signal, C (k).
- The signal C (k) is applied to pharmacokinetic / pharamaco-dynamic model and this model generates a signal I (k) to determine the rate at which, drug is injected in to the patient body.
- The signal I (k) operates infusion pump which controls the flow rate of drug.

Fig. 11.4.2

11.5 Communication and Control System Applications :

11.5.1 Baseband and Carrier Communication :

- As discussed in chapter 1, the term **baseband** is used to represent the band of frequencies occupied by the signal originated from the source or input transducer.
- In telephony the baseband corresponds to the audio band from 0 to 3.5 kHz, in TV the baseband is from 0 to 4.3 MHz.
- In baseband communication, the baseband signals are transmitted without modulation. The example of baseband communication is telephony.
- Communication that uses modulation to shift the frequency spectrum of the information signal known as **carrier communication.**

11.5.2 Definition of Modulation :

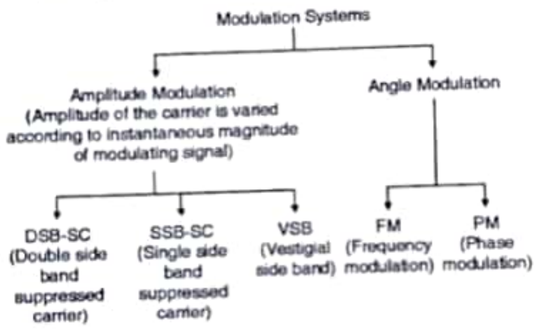
- Modulation is the process by which some characteristics of a carrier is varied in accordance with a modulating wave.
- The three characteristics of the carrier are amplitude, frequency and phase. So one of these characteristics is varied

in proportion with the modulating signal (i.e. message or information signal).

- The message signal is referred to as the modulating signal and the result of modulation process is referred to a **modulated signal.**
- It is important to note that all the information is contained in the "varying characteristics" of the carrier but the carrier itself does not contain any information.

11.5.3 Types of Modulation Systems :

- Depending on which characteristics of the carrier is being changed, the modulation systems are classified as shown in Fig. 11.5.1.



(D-34) **Fig. 11.5.1 : Classification of modulation systems**

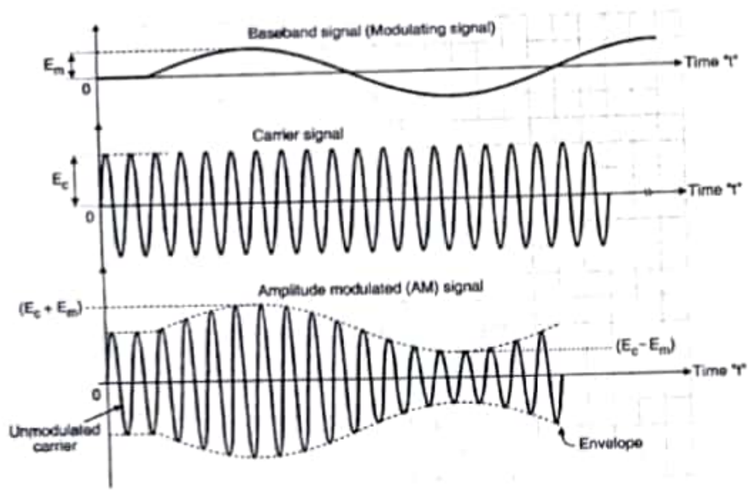
11.5.4 Amplitude Modulation (AM) :

Definition :

- **Amplitude modulation (AM)** or Amplitude Modulation with Full Carrier (AM-FC) is the process of changing the amplitude of a high frequency sinusoidal carrier signal in proportion with the instantaneous value of modulating signal.
- Fig. 11.5.2 shows the amplitude modulated wave when the modulating signal is a sinusoidal signal.

Observations :

1. The frequency of the sinusoidal carrier is much higher than that of the modulating signal.
2. In AM the instantaneous amplitude of the sinusoidal high frequency carrier is changed in proportion to the instantaneous amplitude of the modulating signal. This is the principle of AM.
3. The time domain display of AM signal is as shown in Fig. 11.5.2. This AM signal is transmitted by a transmitter. The information in the AM signal is contained in the amplitude variations of the carrier of the envelope shown by dotted lines in Fig. 11.5.2.
4. Note that the frequency and phase of the carrier remain constant.



(D-12) **Fig. 11.5.2 : AM waveform for sinusoidal modulating signal**

5. AM is used in the applications such as radio transmission, TV transmission etc.

> **Note :** The modulating signal in practice may or may not be purely sinusoidal. Most of the times it will have a complex shape.

11.5.5 Mathematical Representation of an AM Wave :

The mathematical representation of an AM wave can be divided into two parts namely :

1. Time domain description. 2. Frequency domain description.

11.5.6 Time Domain Description :

Assumptions :

- Let the modulating signal be sinusoidal and be represented as.

$$e_m = E_m \cos \omega_m t \quad \text{...(11.5.1)}$$

where, "e_m" is the instantaneous amplitude of the modulating signal, E_m is the peak amplitude, $\omega_m = 2\pi f_m$ and f_m = Frequency of the modulating signal.

- Let the carrier signal also be sinusoidal at a much higher frequency than that of the modulating signal. The instantaneous carrier signal e_c is given by,

$$e_c = E_c \cos \omega_c t \quad \text{...(11.5.2)}$$

where, E_c = Peak carrier amplitude,

f_c = Carrier frequency and $\omega_c = 2\pi f_c$.

- The AM wave is expressed by the following expression.

$$e_{AM} = A \cos (2\pi f_c t) \quad \text{...(11.5.3)}$$

where, A = Envelope of AM wave

- Where A represents the instantaneous value of the envelope. The modulating signal either adds or gets subtracted from the peak carrier amplitude E_c as shown in Fig. 11.5.2. Hence we can represent the instantaneous value of envelope as,

$$A = E_c + e_m = E_c + E_m \cos (2\pi f_m t) \quad \text{...(11.5.4)}$$

- Hence the AM wave is given by,

$$e_{AM} = A \cos (2\pi f_c t)$$
$$= [E_c + E_m \cos (2\pi f_m t)] \cos (2\pi f_c t)$$
$$\therefore \quad e_{AM} = E_c \left[1 + \frac{E_m}{E_c} \cos (2\pi f_m t) \right] \cos (2\pi f_c t)$$

- Let $m = E_m / E_c$ be the modulation index.

$$\therefore \quad e_{AM} = E_c [1 + m \cos (2\pi f_m t)] \cos (2\pi f_c t) \quad \text{...(11.5.5)}$$

This expression represents the time domain representation of an AM signal.

> **Note :** It is not necessary to always consider the cosine waves to obtain the mathematical expression. We can even use the sine waves to obtain the mathematical expression for AM.

11.5.7 Modulation Index or Modulation Factor :

- In AM wave the modulation index (m) is defined as the ratio of amplitudes of the modulating and carrier waves as follows :

$$m = \frac{E_m}{E_c} \quad \text{...(11.5.6)}$$

- When $E_m \le E_c$ the modulation index "m" has values between 0 and 1 and no distortion is introduced in the AM wave. But if $E_m > E_c$ then m is greater than 1. This will distort the shape of AM signal. The distortion is called as "over modulation."
- The modulation index is also called as modulation factor, modulation coefficient or degree of modulation. However if modulation index is expressed as percentage it is called as "percentage modulation."

$$\therefore \ \% \text{ Modulation} = \frac{E_m}{E_c} \times 100 \quad \text{...(11.5.7)}$$

- Note that "m" is a dimensionless quantity.

11.5.8 Frequency Spectrum of the AM Wave (Frequency Domain Description) :

- The frequency spectrum is a graph of amplitude on Y axis versus frequency on X axis. The frequency spectrum of AM wave tells us about which frequency components are present in the AM wave and what are their amplitudes. So consider the equation for AM wave.

$$e_{AM} = (E_c + E_m \cos \omega_m t) \cos \omega_c t$$
$$E_c = \left[1 + \frac{E_m}{E_c} \cos \omega_m t \right] \cos \omega_c t$$

- As per the definition of the modulation index, $m = E_m / E_c$.

$$\therefore \quad e_{AM} = E_c (1 + m \cos \omega_m t) \cos \omega_c t \quad \text{...(11.5.8)}$$

Simplifying we get,

$$e_{AM} = E_c \cos \omega_c t + m E_c \cos \omega_m t \cos \omega_c t \quad \text{...(11.5.9)}$$

- For the second term in the above expression use the following standard identity :

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

- Therefore Equation (11.5.9) gets simplified as follows :

$$e_{AM} = \underbrace{E_c \cos \omega_c t}_{\text{Carrier}} + \underbrace{\frac{mE_c}{2} \cos(\omega_c + \omega_m)t}_{\text{Upper sideband}} + \underbrace{\frac{mE_c}{2} \cos(\omega_c - \omega_m)t}_{\text{Lower sideband}}$$

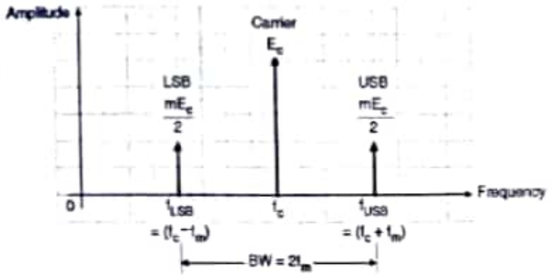
(D-1637) ...(11.5.10)

Observations :

The expression for the AM wave shows that it consists of three terms :

1. First term is nothing else but the unmodulated carrier signal.
2. The second term is a sinusoidal signal at frequency $(f_c + f_m)$. This is called as the upper sideband (USB). Its amplitude is $\frac{m E_c}{2}$.
3. The third term represents a sinusoidal signal at frequency $(f_c - f_m)$. It is called as the lower sideband (LSB). Its amplitude is $\frac{m E_c}{2}$.

- Hence the frequency spectrum of an A.M. wave is as shown in Fig. 11.5.3. Note that it is a single sided spectrum i.e. the spectrum plotted for only the positive values of frequency.



(D-37) **Fig. 11.5.3 : Single sided frequency spectrum of AM wave**

Bandwidth of AM Wave :

The bandwidth of the AM signal is obtained by the subtraction of the highest and the lowest frequency component in the frequency spectrum. Therefore :

$$BW = f_{USB} - f_{LSB} = (f_c + f_m) - (f_c - f_m)$$

$$BW = 2 f_m \quad ...(11.5.11)$$

11.5.9 Spectrum of AM Wave using the Fourier Transform :

- Let the AM wave be denoted by s(t). That means $e_{AM} = s(t)$.
- We know that $s(t) = E_c [1 + m \cos(2\pi f_m t)] \cos(2\pi f_c t)$

$$= E_c \cos(2\pi f_c t) + \frac{m E_c}{2} \cos 2\pi (f_c + f_m) t + \frac{m E_c}{2} \cos 2\pi (f_c - f_m) t \quad ...(11.5.12)$$

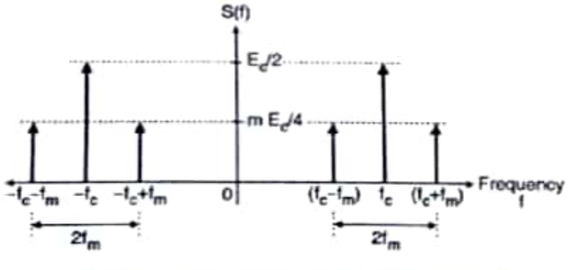
- The Fourier transform of a cosine wave is as follows :

$$\cos(2\pi f_0 t) = \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

- Hence the spectrum of AM wave can be obtained by taking the F.T. of Equation (11.5.12).
- Take the Fourier transform of both sides to get,

$$S(f) = \frac{E_c}{2}[\delta(f + f_c) + \delta(f - f_c)] + \frac{m E_c}{4}[\delta(f + f_c + f_m) + \delta(f - f_c - f_m)] + \frac{m E_c}{4}[\delta(f + f_c - f_m) + \delta(f - f_c + f_m)] \quad ...(11.5.13)$$

- This expression represents the double sided spectrum of AM wave as shown in Fig. 11.5.4.



(D-38) **Fig. 11.5.4 : Double sided spectrum of AM**

Note : The amplitudes of the carrier f_c, the USB $(f_c + f_m)$ and the LSB $(f_c - f_m)$ in the double sided spectrum are exactly half as compared to the corresponding components in the single sided spectrum drawn in Fig. 11.5.3.

11.5.10 Mathematical Treatment as Applied to the General Signals :

- We will obtain the spectrum of an AM wave with a general modulating signal which is not sinusoidal.
- The frequency domain description is same as the frequency spectrum of AM wave.
- It can be obtained by taking the Fourier transform of the time domain expression of the AM wave.
- Let x(t) be the general instantaneous modulating signal and let,

  $c(t) = E_c \cos(2\pi f_c t)$ represent the sinusoidal carrier.

1. The Fourier transforms of x(t) and E_c are as follows :

$$x(t) \xleftrightarrow{F} X(f)$$

$$E_c \cos(2\pi f_c t) \xleftrightarrow{F} \frac{E_c}{2}[\delta(f + f_c) + \delta(f - f_c)]$$

2. The amplitude modulated signal is expressed mathematically in time domain as,

$$s(t) = E_c [1 + m x(t)] \cos 2\pi f_c t \quad ...(11.5.14)$$

To obtain the spectrum of this signal, let us take the Fourier transform of Equation (11.5.14) to get,

$$\therefore \quad S(f) = F[s(t)] = F[E_c \{1 + m x(t)\} \cos 2\pi f_c t] \quad ...(11.5.15)$$

3. Fourier transform of multiplication of two signals in time domain results in convolution of their Fourier transforms. That is,

$$F[x_1(t) x_2(t)] = X_1(f) * X_2(f)$$

Where * represents the convolution.

Applying this rule to Equation (11.5.15) we get,

$$F[s(t)] = F[E_c \cos(2\pi f_c t) + m E_c x(t) \cos(2\pi f_c t)]$$

$$\therefore \quad S(f) = F[E_c \cos(2\pi f_c t)] + m E_c F[x(t) \cos(2\pi f_c t)]$$

$$= \frac{E_c}{2}\{\delta(f + f_c) + \delta(f - f_c)\} + \frac{m E_c}{2}\{X(f) * [\delta(f + f_c) + \delta(f - f_c)]\}$$

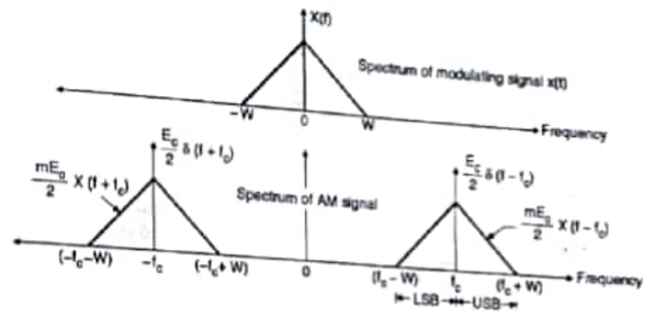
$$\therefore \quad S(f) = \frac{E_c}{2}\{\delta(f + f_c) + \delta(f - f_c)\} + \frac{m E_c}{2}\{X(f) * \delta(f + f_c)\} + \frac{m E_c}{2}\{X(f) * \delta(f - f_c)\}$$

But $X(f) * \delta(f + f_c) = X(f + f_c)$ and $X(f) * \delta(f - f_c) = X(f - f_c)$

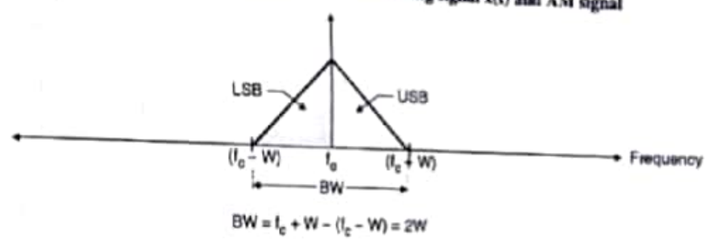
$$\therefore S(f) = \underbrace{\frac{E_c}{2}\delta(f + f_c)}_{(1)} + \underbrace{\frac{E}{2}\delta(f - f_c)}_{(2)} + \underbrace{\frac{mE_c}{2}X(f + f_c)}_{(3)} + \underbrace{\frac{mE_c}{2}X(f - f_c)}_{(4)} \quad ...(11.5.16)$$

(1) → Left shifted weighted delta function
(2) → Right shifted weighted delta function
(3) → X(f) shifted left by f_c
(4) → X(f) shifted right by f_c

(D-49)



(D-50) **Fig. 11.5.5 : Spectrums of modulating signal x(t) and AM signal**



(D-51) **Fig. 11.5.6 : Single sided spectrum of AM wave**

4. We can plot the spectrum of AM wave from Equation (11.5.16) as shown in Fig. 11.5.5. Term (1) and (2) represents the shifted delta functions whereas terms (3) and (4) represents the shifted versions of the spectrum X(f) of modulating signal.
5. The triangular shape of the spectrum X(f) is arbitrarily selected. We can choose any other shape. Fig. 11.5.5 shows the **double sided spectrum of AM wave**, but note that the negative frequencies are imaginary, and used for mathematical convenience. The single sided spectrum of an AM wave is shown in Fig. 11.5.6.

Description of the spectrum :

1. Fig. 11.5.5 shows the spectrum of modulating signal X(f) and the double sided spectrum of the AM wave. Note that the portion of this spectrum which lies on the negative side of frequency axis does not actually exist.
2. Fig. 11.5.6 shows the single sided spectrum of AM wave.
3. **Upper side band :** The portion of the AM wave spectrum extending above the carrier frequency is called as upper sideband or USB. The frequency components corresponding to USB range from f_c to $(f_c + W)$, as shown in Fig. 11.5.6.
4. **Lower side band :** The portion of AM wave spectrum which extends below the carrier frequency f_c is called as the lower sideband LSB. The frequency components corresponding to LSB range from $(f_c - W)$ to f_c, as shown in Fig. 11.5.6.
5. **Highest and Lowest Frequencies :** From Fig. 11.5.6 it is evident that the highest frequency in the spectrum of an AM wave is equal to $(f_c + W)$.

$$\therefore \quad f_{max} = (f_c + W) \quad ...(11.5.17)$$

And the lowest frequency in the spectrum is equal to $(f_c - W)$.

$$\therefore \quad f_{min} = (f_c - W) \quad ...(11.5.18)$$

11.5.11 Transmission Bandwidth (B) :

The transmission bandwidth of an AM wave is defined as the difference between the highest and lowest frequency components present in the spectrum.

∴ Transmission Bandwidth

$$B = f_{max} - f_{min} = (f_c + W) - (f_c - W)$$

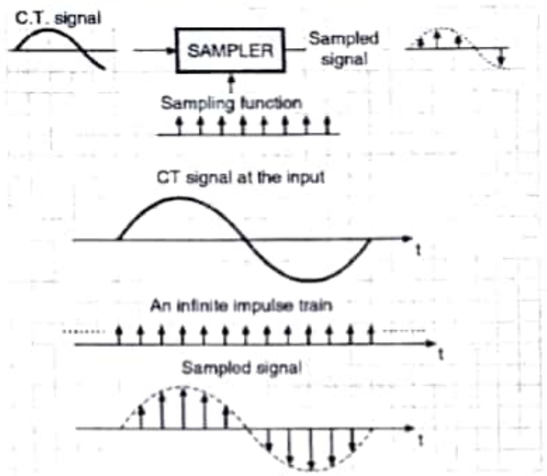
$$B = 2W \text{ Hz} \quad ...(11.5.19)$$

where W = Highest frequency in the spectrum of x(t).

Note : The discussion till now reveals that the spectrum of a standard AM wave consists of the carrier f_c, upper sideband (USB) and lower sideband (LSB).

11.5.12 Application of Digital Communication

- The first step in digital communication systems is the conversion of a continuous time signal into a discrete time signal.
- This conversion is accomplished by means of a process called **sampling**, in which the C.T. signal is multiplied with a train of unit impulses (which acts as a sampling function) to obtain the sampled signal which is a **faithful discrete time representation** of the original C.T. signal.



(H-1665) **Fig. 11.5.7 : Sampling**

- All the signals shown in Fig. 11.5.7 have been defined with the help of the principles learnt in signals and systems.
- How do we confirm that the sampled signal $x_\delta(t)$ is truly a faithful representation of the original signal x(t) ?
- The answer is by plotting the frequency spectrums of the signals x (t) and x_δ (t). We obtain the Fourier transform of these signals to plot their frequency spectrums which we learn as a part of signals and systems syllabus.
- In this way the principles of signals and systems can be used in the sampling system as well as many other digital communication systems in the following areas :
  1. Representation of signals in time domain
  2. Obtaining the spectrums of various signals.
  3. Representation and analysis in frequency domain.

11.6 Feedback and Feedforward Control Systems :

11.6.1 Introduction :

Control systems are an integral part of our lives and play a very important role. From a simple bread toaster to a complex power plant, control system are all around us and are an inseparable part of modern society. Some of the more commonly used control system in our day to day lives are washing machines, elevators, airconditioning units, heaters just to name a few. We also have very advanced control systems such as launching of a satellite in orbit, regulating the generation in a power plant, tracking an enemy plane on radar etc.

All these control system are designed by humans, but we must realize that we are not the only creators of controlled systems. Control systems also occur in nature. Our human body is an excellent example of a very complex control system. In fact our body is made up of a number of control systems viz., respiratory system, digestive system, reproductory system etc.

Hence the principles of control systems that we would study here would be applicable to engineering as well as non engineering fields.

This chapter deals with elementary definitions, classifications, applications and control terminologies.

11.6.2 Important Definitions :

Let us draw a very simplified diagram of a control system.

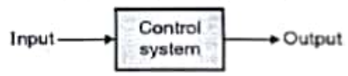


Fig. 11.6.1

1. **Control System :** A control system is an arrangement or a combination of various physical components, also called sub-systems, connected in such a manner so as to attain a certain objective. Thus a system has an input and an output associated with it. A simple example of a system is a light bulb. A bulb has a tungsten filament which is enclosed in a glass case.

   The bulb acts as a system. When this bulb is given an input of 230 V, it starts to glow (output). Hence Fig. 11.6.2 can be used to represent this process.

Fig. 11.6.2

2. **Input :** The excitation or stimulus applied to a system from an external source in order to produce an output is called input signal.
3. **Output :** The actual response obtained from a system is called an output response.
4. **Control :** It means to regulate, direct or command a system so that the desired objective is attained.
5. **Control Action :** Control action is that quantity responsible for activating the system to produce the output.

Thus a control system must have the following :

1. Inputs
2. Outputs
3. An arrangement of subsystems to achieve the input-output combination.

Let us take an example of an elevator. What happens when we get inside an elevator and press a floor button say 7.

Here pressing of a seventh floor button acts as an input to the elevator. The elevator itself is made up of several subsystems. On receiving the input, the motor starts moving at a constant speed and lifts the elevator carriage. The motor slowly stops as the carriage reaches the desired floor. Though this is a very simplistic explanation, it gives us an idea of how the system, the elevator carriage in this case, responds to the floor button press (input) and makes the motor move (output).

There are two control configurations that are used. By configurations we mean how subsystems are connected between the input and the output.

They are :

1. Open loop systems
2. Closed loop systems

11.6.3 Open Loop System :

An open loop system is one which does not automatically correct the change in output by adjusting the input signal. A formal definition would be as follows.

A system in which the control action is totally independent of the output of the system is called as open loop system.

A generalized block diagram of a open loop system is shown in Fig. 11.6.3.

Fig. 11.6.3

In open loop systems, the output remains constant for a given input provided the external conditions are the same. There is no correcting action taking place in an open loop system.

A very common example of a open loop system is the Fan.

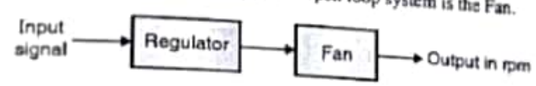


Fig. 11.6.4

In this case, when we switch on the fan, it starts moving at a speed depending on the position of the regulator. It continues to rotate at the same speed without taking into account the surrounding environment. The only way to change the speed of the fan would be to physically adjust the regulator. This is an open loop system.

In this human-system interface, the system (fan in this case) is unable to complete its task (controlling the temperature of the room) without the contribution from a human operator.

Listed below are a few day-to-day examples of open loop systems.

11.6.4 Open Loop Examples :

Normally such systems are calibrated systems. Let us study the following control examples :

Example 1 - Electric Hand Drier :

An electric hand drier sends a gush of hot air when wet hands are taken near it. The input is the wet hands and output is dried hand. The process generates a whirl of hot air. A photocell gives the actuating signal whenever light flow is interrupted (by wet hands) which forms the control action.

This control action has nothing to do with status of the output (how much has the hand dried). Even if we place a dry hand below the machine (photocell), it will generate a whirl of hot air. Hence it is an open loop system.

Example 2 - Automatic Washing Machine

In an automatic washing machine, the output is the clean cloth and the input is the soiled cloth. The soap, water etc. can be taken as inputs too. The control action, is determined by the timer which determines wash and rinse cycles. The control action, has nothing to do with the actual status of 'clean' cloth. If a perfectly clean cloth is put in the machine, it still washes it. Hence it is an open loop system.

Example 3 - Bread Toaster

A toaster simply heats the bread for sometime. This time is determined for a soft or hard toast and is calibrated. This control action has nothing to do with the status of the toast (burnt/untouched/no bread in toaster etc. conditions) and starts and stops after a fixed time. Hence it is an open loop system.

Example 4 - Automatic Coffee/Milk Server

Ready coffee is prepared by fixed proportions of ingredients and is simply released when asked. It assumes coffee cups are placed currently but does not ensure whether the fixed volume is filled in the cup. Hence the system which aims to send a fixed volume of liquid / beverage upon request, is an open loop.

Most open loop systems perform an action based on the time set by the timmer. It does not check if the required objective is met or not.

11.6.5 Advantages and Disadvantages of Open Loop Systems :

Advantages :

1. They are simple in construction and design.
2. They are economic.
3. Easy maintenance.
4. Not much problems of stability (will be discussed in the chapter of stability).

Disadvantages :

1. Inaccurate and unreliable because accuracy is dependent on accuracy of calibration.
2. Inaccurate results are obtained with parameter variations, internal disturbances.
3. To maintain quality and accuracy, recalibration of the controller is necessary from time to time.

11.6.6 Closed Loop System :

A system in which the control action is dependent on the output is called a closed loop system. In closed loop systems, the output is constantly monitored and adjusted to the required value by the controller. A generalized block diagram of a closed loop system is shown in Fig. 11.6.5.

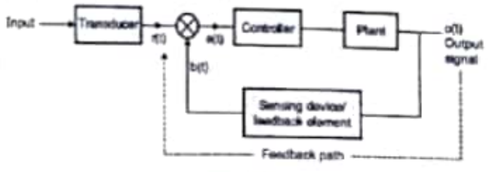


Fig. 11.6.5

As is evident from the block diagram, the output signal is fed back and compared to the reference input. Depending on the difference between the output signal and the reference input, corrective action is taken by the controller to adjust the output.

The general notations used are as follows :

r (t) → reference input

e (t) → error signal

b (t) → feedback signal

c (t) → controlled output signal

The main difference between a closed loop system and an open loop system is that in a closed loop system, there is a constant comparison between the output and the reference input. This is absent in open loop systems.

A common example of a closed loop system is an Air-Conditioning (A.C.) unit.

In an A.C. we adjust the temperature to a particular value (say 24°C).

The A.C. gives out cool air which cools the room, as soon as the temperature of the room drops below 24°C, it shuts off its compressor till the temperature of the room rises above 24°C. It then once again switches on the compressor. This action ensures that the temperature of the room is maintained at a fixed temperature.

The A.C. is constantly checking the temperature of the room and based on the temperature, it switches on or switches off the compressor.

Listed below are a few day-to-day examples of closed loop systems.

11.6.7 Closed Loop Examples :

Now let us discuss some closed loop examples.

Example 1 - Automatic Electric Irons :

An Automatic Electric Iron regulates the temperature of iron such that for a given cloth selection the temperature remains in a specified range. When the temperature falls below the range, heating is started and when it exceeds the range, heating is stopped. Thus it is closed loop with respect to temperature as the control action (which initiates heating or not heating the coil) depends on the output temperature. In this, the thermostat constantly checks the output temperature. This is fed back to a comparator which switches ON or switches OFF the heater. This is the principle of closed loop temperature control. Refer Fig. 11.6.6

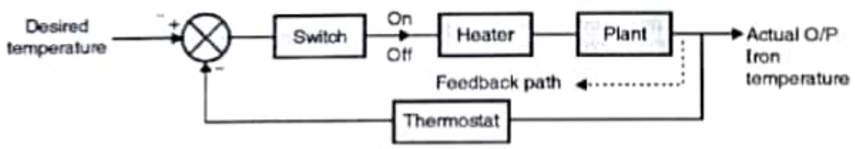


Fig. 11.6.6 : Automatic electric iron

The feedback element could be a RTD, a thermocouple or a thermostat.

Example 2 - Voltage Stabilizers :

Single phase electric supply is usually expected to be 230 V however variations from 200 to 250 V are common, voltages being low in day and high in night. There are many devices that cannot permit such high input variations. Voltages stabilisers accept variable voltage input and give fixed voltage as output. A small discussion will enable the appreciation of control principles involved.

Principle :

When input voltage exceeds normal value (say 250 instead of 230 V), for a 100 turn primary, voltage per turn is $\frac{250}{100} = 2.5$ V. To get 230 V at the output, the switch should be connected below A (towards C side). This makes it a step down transformer. Hence when input is above normal, number of turns of secondary is reduced. Similarly when voltage is say 180 V, the secondary should be such that it is a step-up transformer i.e. tapping towards B side. (Fig. 11.6.7)

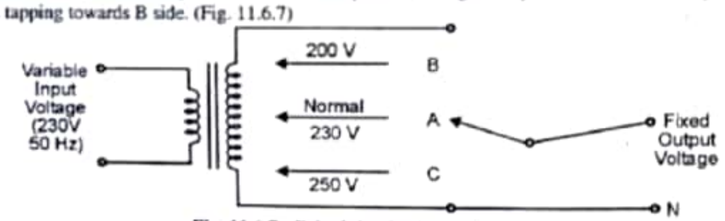


Fig. 11.6.7 : Principle of voltage stabiliser

As fixed taps are used, voltage will be around 230 V but not exactly at 230 V. A simple block diagram is shown in Fig. 11.6.8.

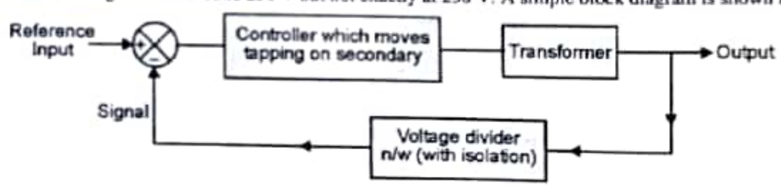


Fig. 11.6.8 : Voltage stabilizer

Instead of using a fixed tap, the entire secondary can be tapped at any point by using a design shown in Fig. 11.6.9

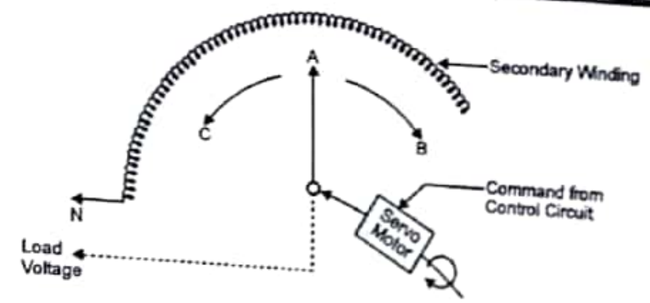


Fig. 11.6.9 : Servo voltage stabilisers

A servo motor drives the shaft (on which the tapping is mounted). Hence voltage is better regulated at 230 V than the fixed tap secondary. Such schemes are available in market as Servo voltage stabilisers. They are usually used as fixed input voltage power supply for computers, photo copy machines, sophisticated CNC machines, etc.

Example 3 - DC Motor Speed Control by Tacho Feedback :

A DC motor speed can be easily controlled by using a Tachometer / Tachogenerator. Refer Fig. 11.6.10. A tachogenerator gives an output voltage proportional to speed. By comparing the output of the tachogenerator with reference speed, the armature field voltage can be made to increase / decrease. This will control the speed of motor.

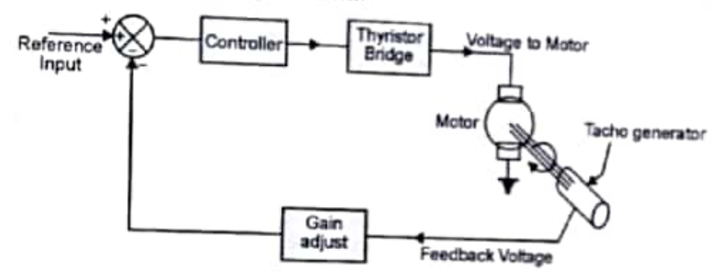


Fig. 11.6.10 : D. C. motor speed control by tacho feedback

Example 4 - Missile Launchers :

When an enemy plane enters the premises it is detected by a radar. Signals are sent to missile launching computers and missile is fired. The radar continuously tracks the plane and directs the released missile such that the error in the position of missile and plane is reduced. Refer Fig. 11.6.11.

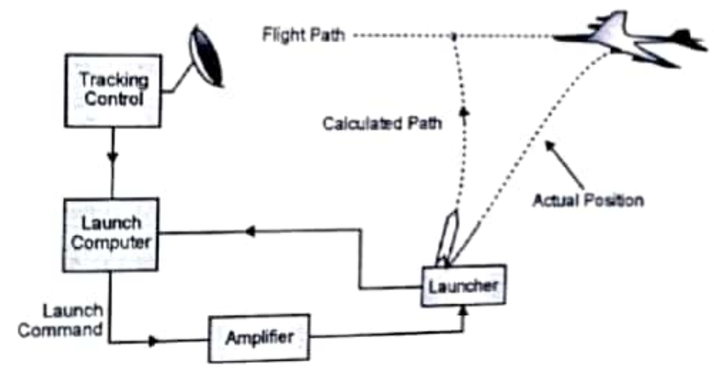


Fig. 11.6.11

Example 5 - Sun Seeker Solar Systems :

To get maximum benefit from solar rays, the photocells array is turned such that it faces the sun always. The actual position of sun is continuously checked and fed back. The position of photocells is changed according to the feedback value. Refer Fig. 11.6.12.

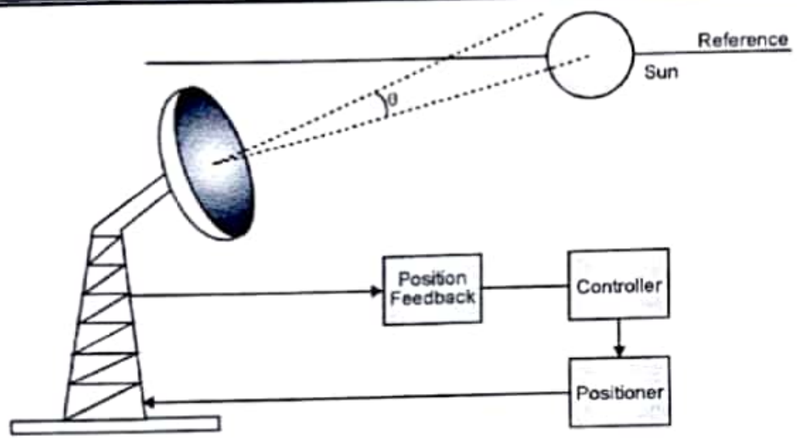


Fig. 11.6.12

11.6.8 Advantages and Disadvantages of Closed Loop System :

Advantages :

1. Accuracy is very high as any error arising is corrected.
2. It senses changes in output due to environmental or parametric changes, internal disturbances etc. and corrects the same.
3. Reduced effect of non-linearties.
4. High Bandwidth.
5. Facilitates Automation.

Disadvantages :

1. Complicated in design and maintenance costlier.
2. System may become unstable.

11.6.9 How Does One Convert an Open Loop System to a Closed Loop System ?

The basic difference between an open loop system and a closed loop system is that the later has a feedback path i.e. the output is fed back to the controller. This is achieved by using a sensor that detects the output, and converts it to a electrical form so that it can be sent back to the controller.

Let us take an example :

Consider a system where a motor needs to move a load upto a particular position. A simple open loop system that needs to be implemented for this purpose is shown using a block diagram in Fig. 11.6.13.

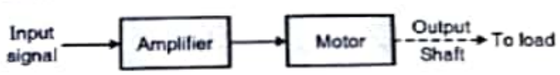


Fig. 11.6.13

In this system, the input signal is fed to an amplifier. After amplification, the amplified signal is given to the motor which moves the output shaft (load).

The motor will continue to move even after the load has reached the required position. To stop the motor, the operator would require to make the input signal equal to zero.

To convert this to a closed loop system, we need to use a position sensor which detects the position of the load. The output of this sensor is then used to control the motor. A simple block diagram would help us understand this.

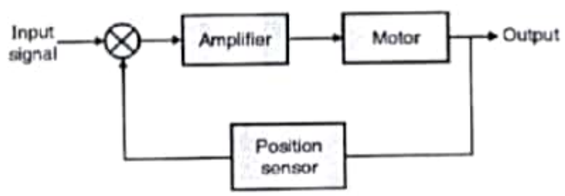


Fig. 11.6.14

Hence to convert a open loop system to a closed loop system we need a feedback path and a feedback element-sensor. All closed loop systems need a sensor-transducer.

11.6.10 Comparison of Open and Closed Loop Systems :

Sr. No.	Open Loop System	Closed Loop System
1.	No feedback. Hence feedback elements absent.	Feedback exists. Hence feedback elements exists.
2.	No error detector.	Error detector is present.
3.	It is inaccurate.	It is accurate.
4.	Highly sensitive to parameter changes.	Less sensitive to parameter changes.
5.	Small bandwidth.	Large bandwidth.
6.	Stable.	May become unstable.
7.	Economical.	Costly.
8.	Examples : Coffee maker, automatic toaster, hand drier.	Examples : Guided missile, Temperature control of oven, Servo voltage stabilisers.

11.6.11 Servomechanisms :

Servomechanisms are closed loop systems where the feedback signals are used to control position, speed, velocity, acceleration etc (position or time derivatives of position).

Servo voltage stabilisers in which position of the secondary tap is controlled using servo motors is an example of servomechanisms.

Since servomechanisms are used to control position, one of the most important components of servomechanisms is the position sensor. The position sensor measures the position of the servomotor and converts it into an electrical signal that the control system can use.

A typical block diagram of position control servomechanism is shown in Fig. 11.6.15.

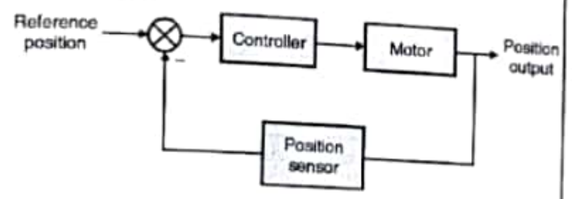


Fig. 11.6.15

A servo control system is one of the most widely used forms of control systems.

11.6.12 Requirements of a Good Control System :

1. **Accuracy :** Accuracy should be very high as any error arising should be corrected. Accuracy can be improved by using feedback element. Because of feedback element, system becomes closed loop system. In closed loop control systems, steady state error tends to zero.

   To increase the accuracy, an error detector should be present in control system.
2. **Sensitivity :** A control system senses changes in output due to environmental or parametric changes, internal disturbances or any other parameters and corrects the same. A good control system should be insensitive to such parameters but very sensitive to the input signals.
3. **Noise :** Noise is an undesired input signal. A good control system should be insensitive to such input signals. A good control system should be able to reduce the effects of noise or undesired inputs.
4. **Stability :** Stability means bounded input and bounded output. In the absence of the inputs, the output should tend to zero as time increases. A good control system response is stable for all variations.
5. **Bandwidth :** Operating frequency range decides the bandwidth of any system. For frequency response of a good control system, bandwidth should be large.
6. **Speed :** A good control system should have high speed. Hence the output of system should be as fast as possible.
7. **Oscillation :** For a good control system, oscillations of output should be constant or sustained.

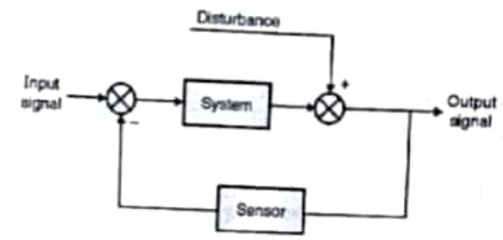
11.6.13 Feed Forward Systems :

We have seen that feedback systems (closed loop systems) perform correction and adjust the output based on the reference input signal.

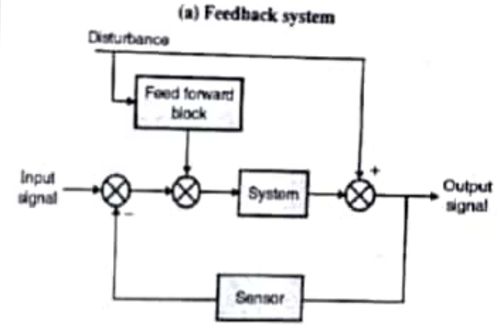
In feed forward systems the output of one stage of the control system is sent to a later stage to improve the overall performance of the control system. It is used whenever there are disturbances present.

Shown is Fig. 11.6.16 are block diagrams : one of a feedback system and another one with feedback as well as feedforward.

In feedback systems, the corrective action starts only after the output has been affected.



(a) Feedback system



(b) Feedback with feed forward

Fig. 11.6.16

In the feedforward systems, the corrective action is initiated without waiting for the effect of disturbance to show up in the output.

Combined feed forward and feedback control can significantly improve the overall performance as compared to a simple feedback control whenever there is a major disturbance that can be measured before it affects the system.

11.6.14 Adaptive Control Systems :

An adaptive Control System is one that continuously and automatically measures the dynamic characteristics (such as the transfer function or state equation) of the plant, compares them with the desired characteristics and uses the difference to vary the adjustable system parameters so that optimal performance can be maintained regardless of environmental changes. It can also be defined as follows, Adaptive Control Systems continuously measure its own performance index, compares it with given performance index and modify parameters such that optimal performance is given under all possible operating conditions.

The block diagram of a Adaptive Control System is shown in Fig. 11.6.17.

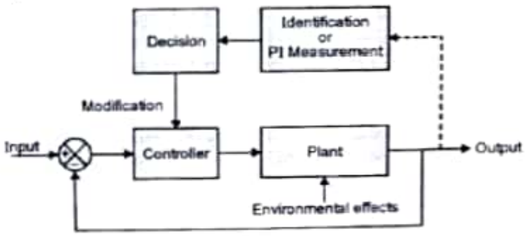


Fig. 11.6.17 : Adaptive control system

Thus an adaptive control system must :

1. Measure its own performance
2. Have some defined performance indices
3. Work in such a way that performance is optimal subject to the available resources and constraints.

Thus for adaptive control system there must be :

1. Identification of dynamic characteristics of plant
2. Decision making based on that identification
3. Modification based on such identification.

The process of identification includes using test signals or stochastic signal analysis. Normally identification should be less time consuming.

Decision making based on identification will measure the present performance and if it is not optimal, it will begin to initiate new changes. After this, decision either controller parameters may be modified or optimal control signal will be generated. Hence it is seen that in adaptive control self organising features exist.

Solved Examples

Ex. 11.6.1 : Draw a block diagram of control system in which door automatically opens when a customer comes near it, and door closes when he goes away from door.

Soln. :

Refer Fig. P. 11.6.1. When a customer is made to come through an entry passage, light falling on door changes. This is sensed using photosensitive cells, which give signals to an electronic circuit. This circuit gives a command to open the door.

As soon as customer moves away from door the light falling on photocells improves, and hence the door closes. It is a open loop system.

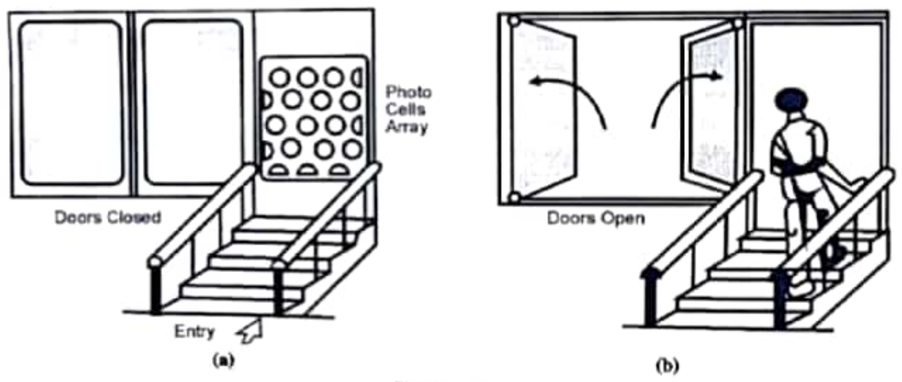


Fig. P. 11.6.1

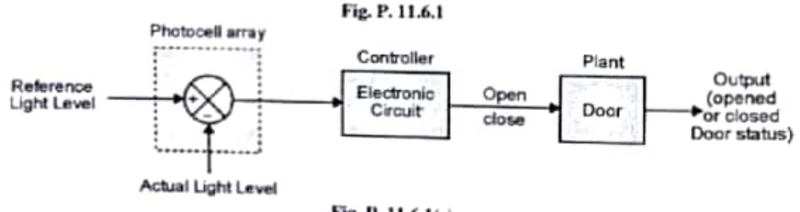


Fig. P. 11.6.1(c)

Ex. 11.6.2 : Consider an industrial oven, which is used for controlling temperature of the chamber of an oven from room temperature plus 5°C to 180°C. The device used for this is a bimetallic strip. Draw suitable block diagram. Indicate various important parameters and different components in same.

Soln. :

This is a closed loop temperature control system. So Fig. 11.6.6 in Example 1 can be used. The desired temperature is a range from room temperature plus 5°C upto max 180°C and is the reference temperature. Refer Fig. P. 11.6.2.

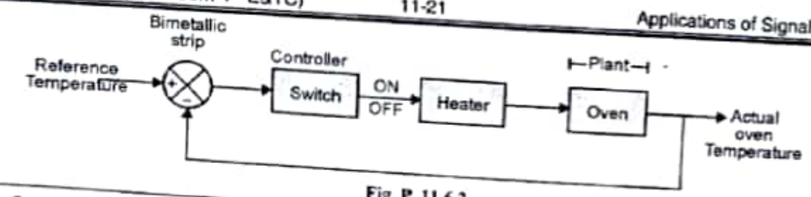


Fig. P. 11.6.2

Ex. 11.6.3 : Consider the system described below and draw the block diagram. State whether it is closed or open loop, with reason. The system is a bottle filling plant on a conveyor belt. Each bottle is to be filled with specified volume. Then a new bottle will replace old bottle. Mention your assumptions specifically.

Soln. :

First let us draw an open loop version. When a specified volume is filled, bottle is moved out. The system does not check whether the bottle is full. Hence some liquid is split down. Control action does not depend on status of output, hence it is a open loop system. Refer Fig. P. 11.6.3.

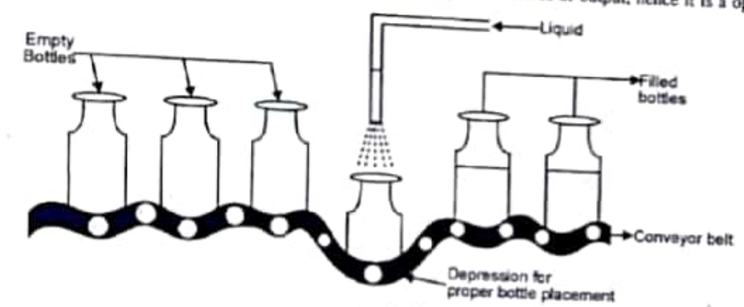


Fig. P. 11.6.3(a)

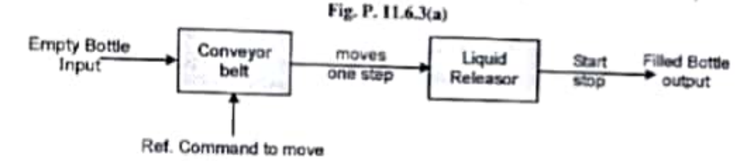


Fig. P. 11.6.3(b) : Open loop block diagram

To make it a closed loop, we need to check the liquid level by using a level marker. Once liquid in bottle touches the level, it changes the conductivity of level marker (Touch Detector Circuit), and we can assume the bottle to be full. Introduction of a touch detector circuit makes the system closed loop. Refer Fig. P. 11.6.3(c) and (d).

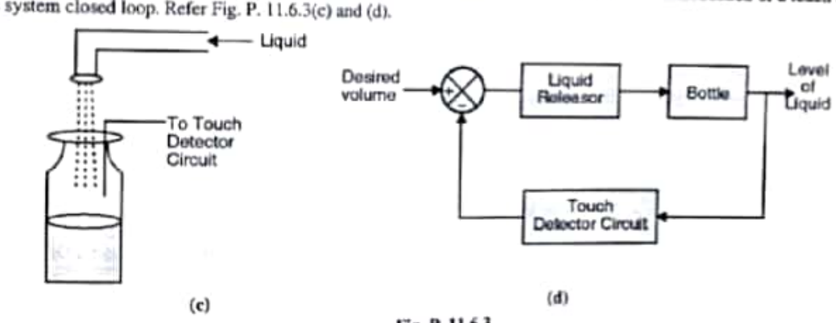


Fig. P. 11.6.3

Ex. 11.6.4 : Identify the input and output of an automatic refrigerator. Is it open loop or closed loop control system ?

Soln. :

A refrigerator is used to keep food in a chilled / cool condition . As per this aim, the input is the unchilled food. The output is the chilled food items. The mechanism or control action is to ensure temperature lies in a certain range. As soon it becomes hotter, thermostat settings make the compressor on and cools the fridge. As soon as temperature falls below a certain level, compressor stops. As this system detects the actual temperature (output) and accordingly monitors the control action, the system is closed loop w.r.t. temperature.

11.6.15 The Design Process :

We shall now discuss an orderly sequence for the design of feedback control systems. We draw a simple block diagram to understand the design process.

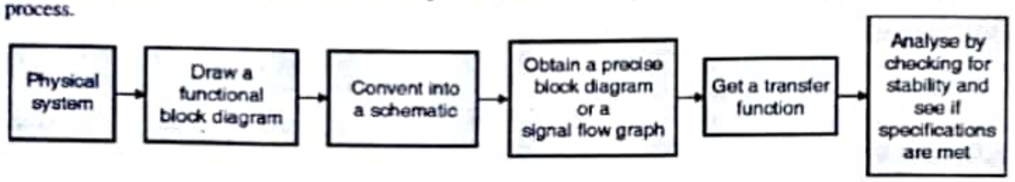


Fig. 11.6.18

Step 1 : We begin by transforming our requirements into a physical system.

Step 2 : The designer converts a qualitative description of the system into a functional block diagram and also shows their interconnections.

Step 3 : A control system can consist of electrical, mechanical and electromechanical components while making the schematic, the design engineer must make approximations about the system and neglect certain phenomena or else the schematic will be to complicated making it difficult to extract a useful mathematical model.

The decision made while preparing the schematic come from the knowledge of the physical system.

Step 4 : Once the schematic is drawn, the designer uses physical laws, for example kirchoffs laws for electrical networks, to model the system mathematically. Hence each such system can now be represented by a set of mathematical equation.

Step 5 : Each of the subsystems can be combined together to get a single mathematical equation known as the transfer function to represent the entire system. This can be done either by block diagram reduction or by signal flow graphs.

Step 6 : The final stage is to analyse the given system. In this the designer checks whether the system is stable or not. If the system is stable, we studies the response of the system to various simulated inputs. If the system is unstable, various methods are worked out to make the system stable.

Review Questions

Q. 1 Can the electrical network of Fig. 1 be deemed as a feedback control system ?

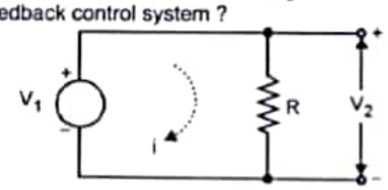


Fig. 1

Q. 2 Identify the input and output of an automatic refrigerator. Is it an open loop or closed loop control system ?

Q. 3 Identify a possible input and possible output for a rotational generator of electricity.

Q. 4 Name three applications for feedback control systems.

Q. 5 Explain various methods of speech synthesis.

Q. 6 With the help of block schematic; explain the working of speech recognition system.

Q. 7 Explain the working of all digital voice response system.

Q. 8 Explain working of voice response system for wiring communication equipments.

Q. 9 With the help of block schematic, explain the working of speakers recognition system.

Q. 10 Explain the subband coding of speech signal as an application of multirate signal processing.

Q. 11 Write short note on real time DSP applications to biomedical signal processing.

Q. 12 How the fetal ECG is monitored using DSP ?

□□□

APPENDIX A

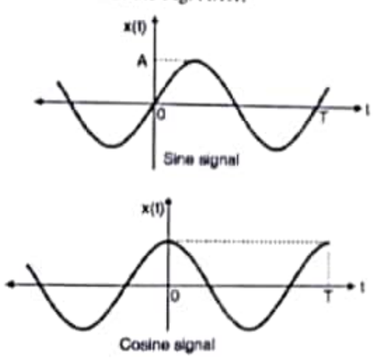
Different Signals from Communication & Bio-medical Field

A.1 Different Signals from Communication Field :

In different analog and digital communication systems, various signals are used. In this section we will discuss some of those signals with their application in various communication systems.

A.1.1 Sinusoidal Signals :

- The CT sinusoidal signals include sine and cosine signals. They are as shown in the Fig. A.1.1.



(II-45) **Fig. A.1.1 : Sinusoidal signals**

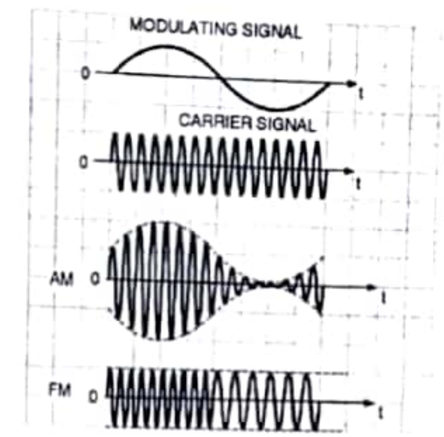
- Mathematically they can be represented as follows :

A sine signal : $x(t) = A \sin \omega t = A \sin (2 \pi f t)$ ...(A.1.1)

A cosine signal : $x(t) = A \cos \omega t = A \cos (2 \pi f t)$ ...(A.1.2)

Application in communication field :

- The sinusoidal signals are used as a **carrier** signal in the analog modulation systems such as Amplitude Modulation (AM), Frequency Modulation (FM) and Phase Modulation (PM), as shown in Fig. A.1.2.



(II-1666) **Fig. A.1.2 : Use of sinusoidal signals in communication**

A.1.2 A Rectangular Signal :

- A rectangular signal of amplitude "A" and a duration T is shown in Fig. A.1.3(a). It is mathematically represented as follows :

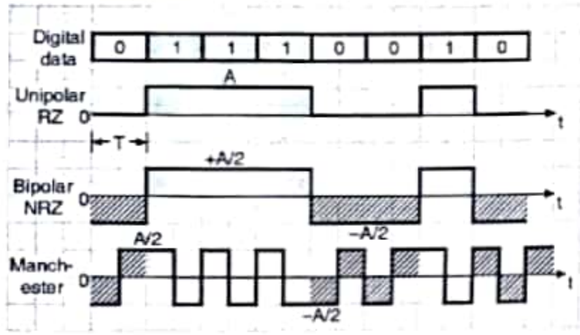
$$A \text{ rect}\left[\frac{t}{T}\right] = A \ldots -T/2 \le t \le T/2$$
$$= 0 \ldots \text{Elsewhere}$$

(II-1667) **Fig. A.1.3(a) : A rectangular pulse**

1. **Application for data representation :**

- The computer data is in the form of binary bits 0 and 1, when this data is to be transferred to another computer or a device like printer, we need to first convert it into a suitable **digital signal** which is in the form of rectangular signal as shown in Fig. A.1.3(b).
- These digital signals are called as **line codes.** The same digital data can be represented with different line codes, as shown.

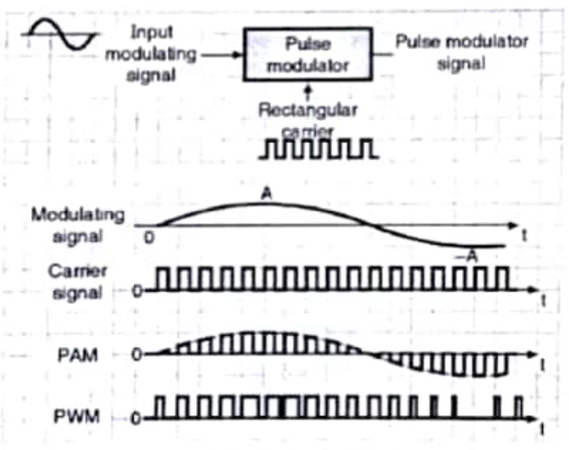
- A suitable line code is selected depending on the need of application.



(H-1668) **Fig. A.1.3(b) : Use of rectangular signal for representing the digital data**

2. Application in analog and digital pulse modulation :

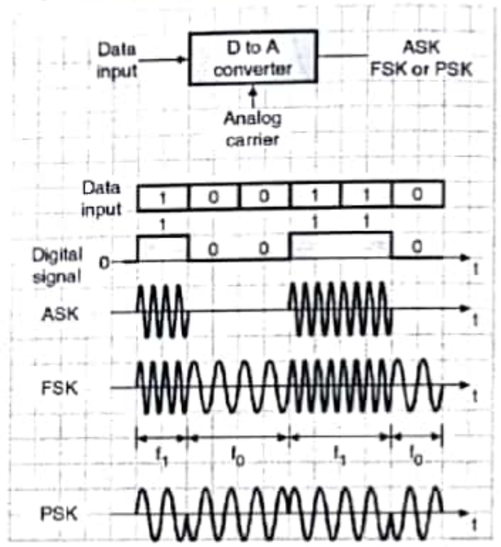
- In the analog pulse modulation techniques such as Pulse Amplitude Modulation (PAM) or pulse width modulation or in the digital pulse modulation technique such as Pulse Code Modulation (PCM), a train OR high frequency **rectangular pulses** is used as **carrier** as shown in Fig. A.1.4.
- The PAM, PWM and PCM waveforms also are in the rectangular signal form as shown.
- The input modulating signal is shown as a sinusoidal signal in Fig. A.1.4 but in reality it can have any arbitrary shape.



(H-1669) **Fig. A.1.4 : Use of rectangular signal in pulse communications**

A.1.3 Application of Sinusoidal and Rectangular Signals :

- The sinusoidal signals are also used in the Continuous Wave (CW) modulation systems such as ASK (Amplitude Shift Keying), PSK (Phase Shift Keying and FSK) (Frequency Shift Keying) as shown in Fig. A.1.5.
- Note that input data stream is represented as a digital signal using the rectangular signal.



(H-1670) **Fig. A.1.5 : Use of sinusoidal and rectangular signals in CW modulation**

A.1.4 Unit Impulse Function δ(t) :

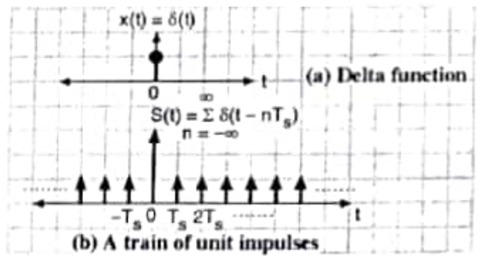
- The unit impulse function δ(t) has been shown in Fig. A.1.6(a). It is mathematically represented as follows :

Delta function : $\delta(t) = 0$ for $t \neq 0$

- An infinite train of unit impulses has been shown in Fig. A.1.6(b) and it is mathematically represented as follows :

$$S(t) = \ldots\delta(t + 2T_s) + \delta(t + T_s) + \delta(t) + \delta(t - T_s) + \ldots$$

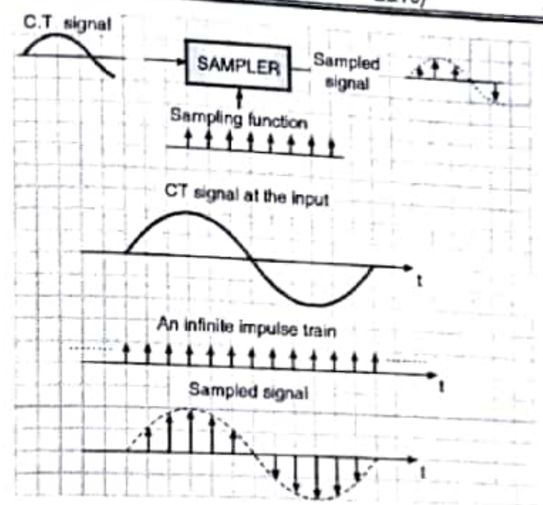
$$\therefore S(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$



(H-1671) **Fig. A.1.6**

Application of delta function :

- The train of unit impulses (Fig. A.1.6(b)) is used as the **sampling function** in the process of sampling as shown in Fig. A.1.6(c).
- Sampling process is used to convert a continuous time signal into its discrete time equivalent signal. It is the first step in any digital communication system.



(H-1665) **Fig. A.1.6(c) : Use of train of unit impulses in sampling**

A.1.5 Other Signals and Their Applications :

- Some other signals are also used in communication applications as mentioned in the Table A.1.1.

Table A.1.1

Sr. No.	Signal	Application
1.	Triangular signal	In the frequency spectrum of AM, or sampled signal the triangular shaped spectrums are used.
2.	Signum function	In order to minimize the intersymb interference (ISI) the digital pulses a shaped to be of signum function. This called as Nyquist shaping.
3.	Unit step signal	It is used to find the unit step response o system. That means a unit step signal applied at the system input and its frequer spectrum is plotted.
4.	Unit impulse signal	It is used for finding the impulse response a system.

A.2 Signals used in the Bio-Medical Field :

- The same signals such as unit step, unit impulse rectangu pulse etc that are used in the communication field are a used in the biomedical field.
- The common applications where these signals are u include :
  1. Electrocardiogram (ECG)
  2. Electroencephalogram (EEG)
  3. Action potential measurement
  4. Electroneurogram (ENG)
- These applications are discussed in chapter 11 of this book.

□□