Module - 2

Complex Integration

* Revision:

→ complex number :

$z = x + iy$, where, x, y are Real numbers and $i = \sqrt{-1}$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$, where , $r = \sqrt{x^2 + y^2}$

$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

→ The complex differential :

$dz = dx + i\,dy$

→ A curve in the complex plane :

$z(t) = x(t) + iy(t)$, $a \le t \le b$

→ A complex function :

$f(z) = u(x,y) + iv(x,y)$

→ If $z = x + iy$ and $z_0 = x_0 + iy_0$ then

$|z - z_0| = r$ Represent the equation of circle with centre at z_0 and radius r.

since, $|z - z_0| = r \Rightarrow |z - z_0|^2 = r^2$

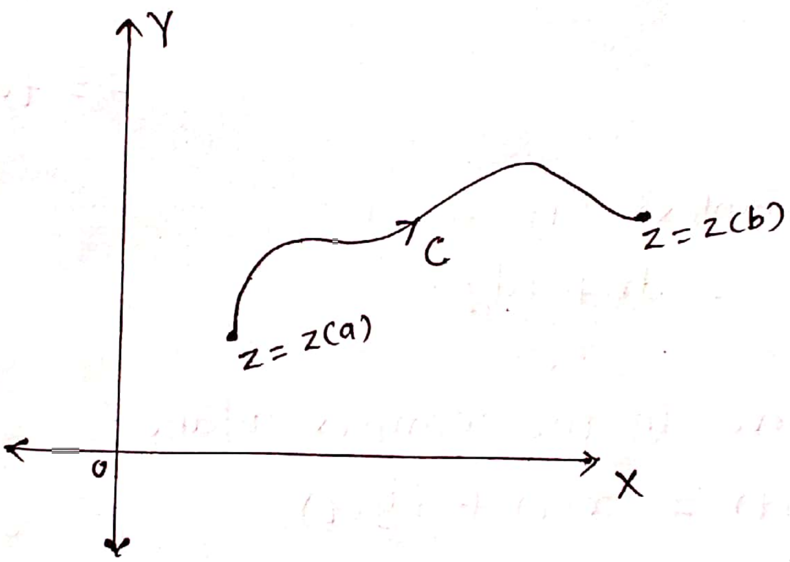
$\Rightarrow |x + iy - (x_0 + iy_0)|^2 = r^2 \Rightarrow |(x - x_0) + i(y - y_0)|^2 = r^2$

$\Rightarrow (x - x_0)^2 + (y - y_0)^2 = r^2$ which is Equation of circle centre at $z_0 = (x_0, y_0)$ and radius r.

I) <u>Line Integral</u> : (Contour Integral / path integral)

let C be the curve from $z = z(a)$ to $z = z(b)$ and the function $f(z)$ be the piecewise contineous on C then the line integral of f over C is defined as

$$\int_C f(z)\,dz = \int_a^b f[z(t)] \cdot z'(t)\,dt, \quad a \leq t \leq b$$



<u>Note that</u> :

① $\int_C [f(z) + g(z)]\,dz = \int_C f(z)\,dz + \int_C g(z)\,dz$

② $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

③ $\int_{C = C_1 + C_2} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$

④ $\int_C k \cdot f(z)\,dz = k \int_C f(z)\,dz$, k - constant

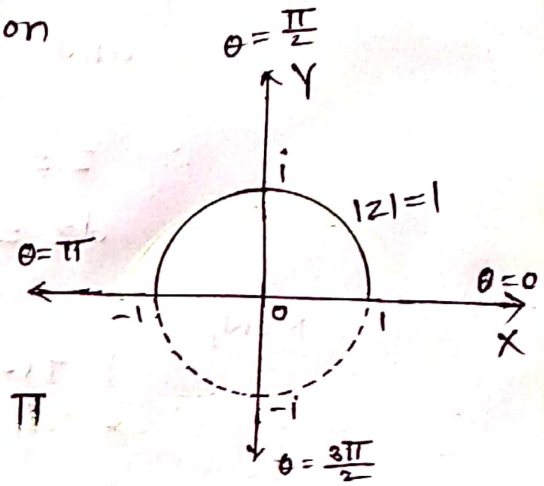
⑤ $\left| \int_C f(z)\,dz \right| \leq \int_C |f(z)| \cdot |dz|$

* Examples on Line Integral :

Example ① Evaluate $\int_C \bar{z} \cdot dz$, where C is the upper half of the circle $|z| = 1$

Solution: Given: $C : |z| = 1$

Clearly, $|z| = 1$ is equation of circle with centre at origin and radius 1



let $z = re^{i\theta}$

since, $r = |z| = 1$, $0 \leq \theta \leq \pi$

$\Rightarrow z = e^{i\theta}$

$\therefore dz = e^{i\theta} \cdot i\, d\theta$ and $\bar{z} = e^{-i\theta}$

Now,
$$\int_C \bar{z}\, dz = \int_{\theta=0}^{\pi} e^{-i\theta} \cdot e^{i\theta} \cdot i\, d\theta$$

$$= \int_0^{\pi} i\, d\theta$$

$$= i\, [\theta]_0^{\pi}$$

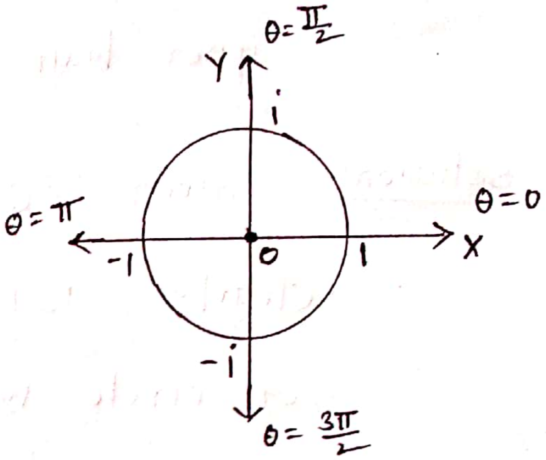
$$= i\, [\pi - 0]$$

$$= \pi i$$

$$\therefore \boxed{\int_C \bar{z}\, dz = \pi i}$$

Example ②. Evaluate $\int_C \log z \, dz$, where C is the unit circle in the z-plane

Solution: Given: $C : |z| = 1$



let $z = re^{i\theta}$

since, $r = |z| = 1$

and $0 \leq \theta \leq 2\pi$

$\therefore \quad z = e^{i\theta}$

$dz = e^{i\theta} \cdot i \cdot d\theta$

Now,

$$\int_C f(z)\,dz = \int_{\theta=0}^{2\pi} \log e^{i\theta} \cdot e^{i\theta} \cdot i \cdot d\theta$$

$$= \int_0^{2\pi} i\theta \cdot e^{i\theta} \cdot i \, d\theta$$

$$= -\int_0^{2\pi} \theta \cdot e^{i\theta} \, d\theta$$

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} - \int \frac{e^{i\theta}}{i} \cdot 1 \cdot d\theta\right]_0^{2\pi}$$

($\because$ using integration by part $\int uv\,dx = u\int v\,dx - \int\left[\frac{\partial u}{\partial x}\int v\,dx\right]dx$)

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} - \frac{e^{i\theta}}{-1}\right]_0^{2\pi}$$

$$= -\left[\theta \cdot \frac{e^{i\theta}}{i} + \frac{e^{i\theta}}{1}\right]_0^{2\pi}$$

$$= -\left[\frac{2\pi e^{2\pi i}}{i} + e^{2\pi i} - 0 - e^{0}\right]$$

$$= -\left[\frac{2\pi}{i} + 1 - 1\right] \qquad (\because e^{2\pi i} = 1)$$

$$= -\frac{2\pi}{i}$$

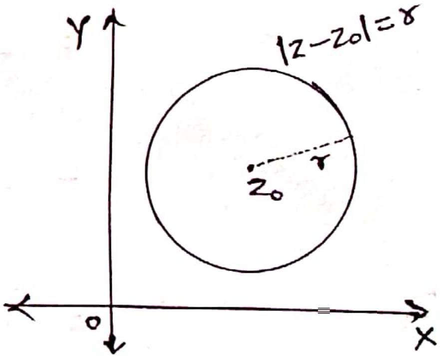
$$= 2\pi i \qquad \left(\because i = \frac{i \cdot i}{i} = -\frac{1}{i}\right)$$

$$\therefore \boxed{\int_C \log z \, dz = 2\pi i}$$

Example 3. Evaluate $\int_C \frac{dz}{(z-z_0)^{n+1}}$,

where, n is an integer and C is the circle $|z-z_0| = r$

<u>Solution</u>: Given: $C : |z-z_0| = r$



Clearly, $|z-z_0| = r$ is the equation of circle with centre at z_0 and radius r

let $z - z_0 = re^{i\theta}$, $\quad 0 \le \theta \le 2\pi$

$$\Rightarrow dz = re^{i\theta} i \, d\theta$$

Now, $$I = \int_C \frac{dz}{(z-z_0)^{n+1}} = \int_0^{2\pi} \frac{r \cdot e^{i\theta} \cdot i \, d\theta}{(re^{i\theta})^{n+1}}$$

$$= \frac{i}{r^n} \int_0^{2\pi} \frac{1}{e^{in\theta}} d\theta$$

$$= \frac{i}{r^n} \int_0^{2\pi} e^{-in\theta} d\theta$$

Case i) if $n=0$ then

$$I = \frac{i}{r^0}\int_0^{2\pi} e^0\, d\theta$$

$$= i\int_0^{2\pi} 1\, d\theta$$

$$= i\,[\theta]_0^{2\pi}$$

$$= i\,[2\pi - 0]$$

$$= 2\pi i$$

Case ii) if $n \neq 0$ then

$$I = \frac{i}{r^n}\int_0^{2\pi} e^{-in\theta}\, d\theta$$

$$= \frac{i}{r^n}\int_0^{2\pi} (\cos n\theta - i\sin n\theta)\, d\theta$$

$$= \frac{1}{r^n}\int_0^{2\pi} (i\cos n\theta + \sin n\theta)\, d\theta$$

$$= \frac{1}{r^n}\left[i\,\frac{\sin n\theta}{n} - \frac{\cos n\theta}{n}\right]_0^{2\pi}$$

$$= \frac{1}{r^n}\left[\frac{i\sin 2n\pi}{n} - \frac{\cos 2n\pi}{n} - \frac{i\sin 0}{n} + \frac{\cos 0}{n}\right]$$

$$= 0$$

Therefore,

$$\int_C \frac{dz}{(z-z_0)^{n+1}} = \begin{cases} 2\pi i & \text{if } n=0 \\ 0 & \text{if } n\neq 0 \end{cases}$$

Home work: Example (4) Evaluate $\int_C \frac{2z+3}{z}\,dz$, where C is

i) the upper half of the circle $|z|=2$

ii) the lower half of the circle $|z|=2$

iii) The whole circle in anticlock-wise direction