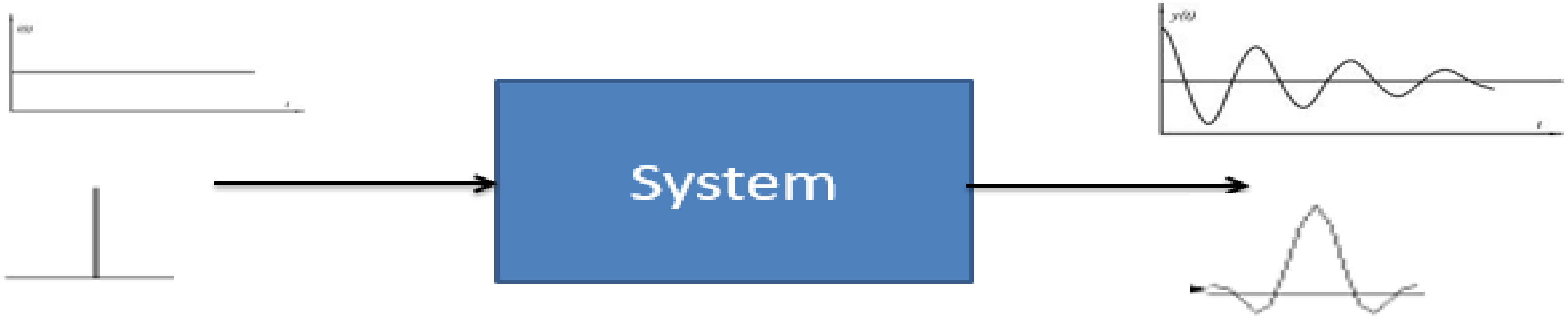
Chapter 2_Time Response Analysis

- Time Response
- Input Supplied System
- Steady State Response and Error
- Time Response specification
- Limitations

Time Response of Control Systems

- Time response of a dynamic system response to an input expressed as a function of time.



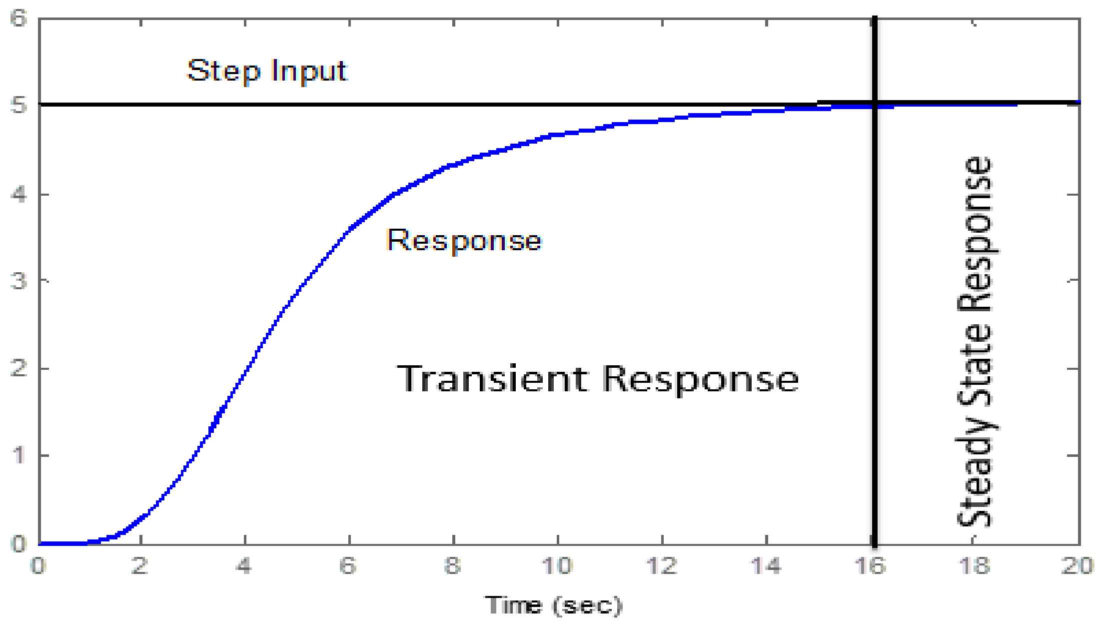
- The time response of any system has two components
    - Transient response
    - Steady-state response.

Time Response of Control Systems

- When the response of the system is changed from equilibrium it takes some time to settle down.

- This is called transient response.

- The response of the system after the transient response is called steady state response.

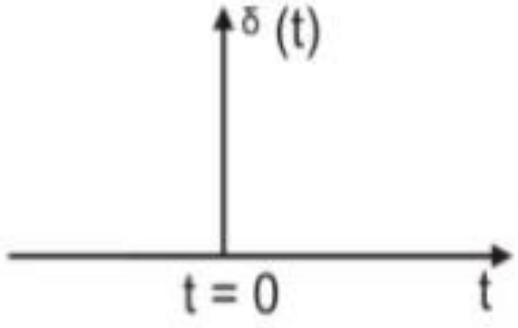


IMPULSE INPUT

- It is sudden change input. An impulse is infinite at t=0 and everywhere else.
- $r(t) = \delta(t) = 1 \quad t = 0$

  $= 0 \quad t \neq 0$

  In laplase domain we have,
- $L[r(t)] = 1$

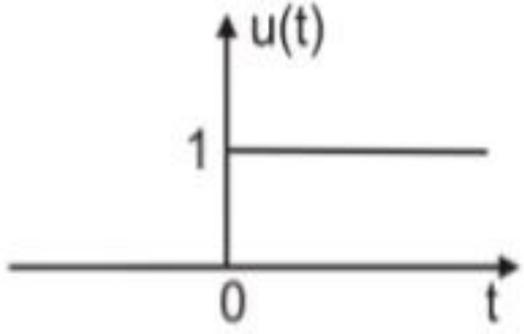


STEP INPUT

- It represents a constant command such as position. Like elevator is a step input.
- $r(t) = u(t) = A \quad t \geq 0$

  $= 0 \quad$ otherwise

  $L[r(t)] = A/s$



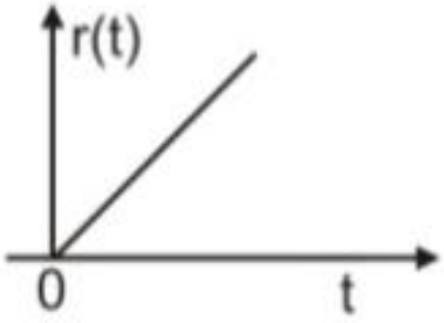
RAMP INPUT

- this represents a linearly increasing input command.
- $r(t) \quad = At \quad t \geq 0$, Aslope

  $= 0 \quad t < 0$

  $L[r(t)] = A/s^2$

  A= 1 then unit ramp



PARABOLIC INPUT

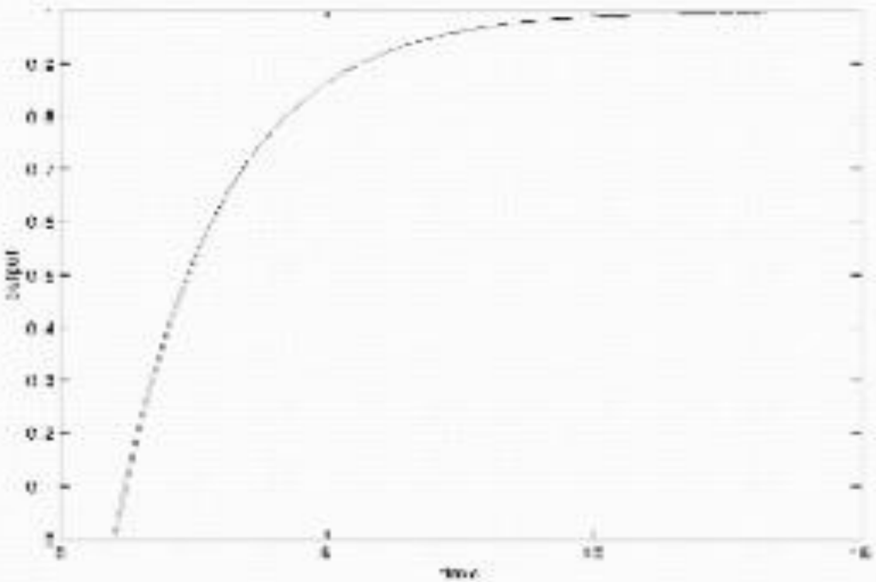
- Rate of change of velocity is acceleration. Acceleration is a parabolic function.
- $r(t) = At^2/2 \qquad t \geq 0$

  $= 0 \qquad t < 0$

  $L[r(t)] = A/s^3$

SINUSOIDAL INPUT

- It input of varying and study the system frequently response.
- $r(t) = A \sin(wt) \qquad t \geq 0$



Classification of Control Systems

- Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs, and so on.

Classification of Control Systems

- Consider the unity-feedback control system with the following open-loop transfer function

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

- It involves the term s^N in the denominator, representing N poles at the origin.

- A system is called type 0, type 1, type 2, ... , if N=0, N=1, N=2, ... , respectively.

Classification of Control Systems

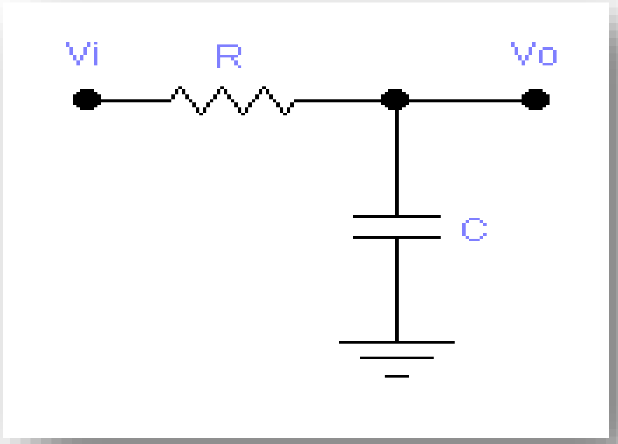
- As the type number is increased, accuracy is improved.
- However, increasing the type number aggravates the stability problem.
- A compromise between steady-state accuracy and relative stability is always necessary.

FIRST-ORDER SYSTEM

- General form:

$$\boxed{G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s+1}}$$

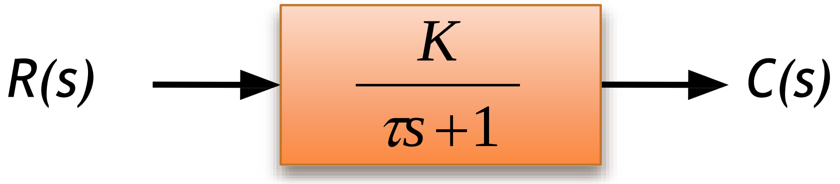
- Problem: Derive the transfer function for the following circuit



$$\boxed{G(s) = \frac{1}{RCs+1}}$$

FIRST-ORDER SYSTEM

- Transient Response: Gradual change of output from initial to the desired condition.
- Block diagram representation:



Where,

K : Gain

τ : Time constant

- By definition itself, the input to the system should be a step function which is given by the following:

$$\boxed{R(s) = \frac{1}{s}}$$

INTRODUCTION

- The first order system has only one pole.

$$\frac{C(s)}{R(s)} = \frac{K}{Ts+1}$$

Where *K* is the D.C gain and *T* is the time constant of the system.

- Time constant is a measure of how quickly a 1st order system responds to a unit step input.

FIRST-ORDER SYSTEM

- General form:

$$\boxed{G(s) = \frac{C(s)}{R(s)} = \frac{K}{\tau s + 1}} \Rightarrow \boxed{C(s) = G(s)R(s)}$$

- Output response:

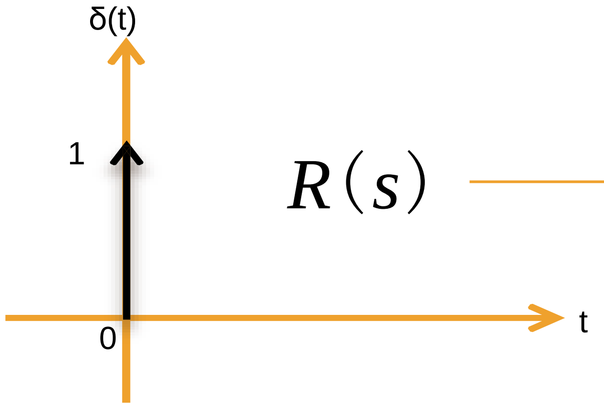
$$C(s) = \left(\frac{1}{s}\right)\left(\frac{K}{\tau s + 1}\right)$$

$$= \frac{A}{s} + \frac{B}{\tau s + 1}$$

$$\boxed{c(t) = A + \frac{B}{\tau} e^{-t/\tau}}$$

IMPULSE RESPONSEOF 1^{ST} ORDER SYSTEM

- Consider the following 1^{st} order system



$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \delta(s) = 1$$

$$C(s) = \frac{K}{Ts+1}$$

IMPULSE RESPONSEOF 1ST ORDER SYSTEM

$$C(s) = \frac{K}{Ts+1}$$

- Re-arrange following equation as

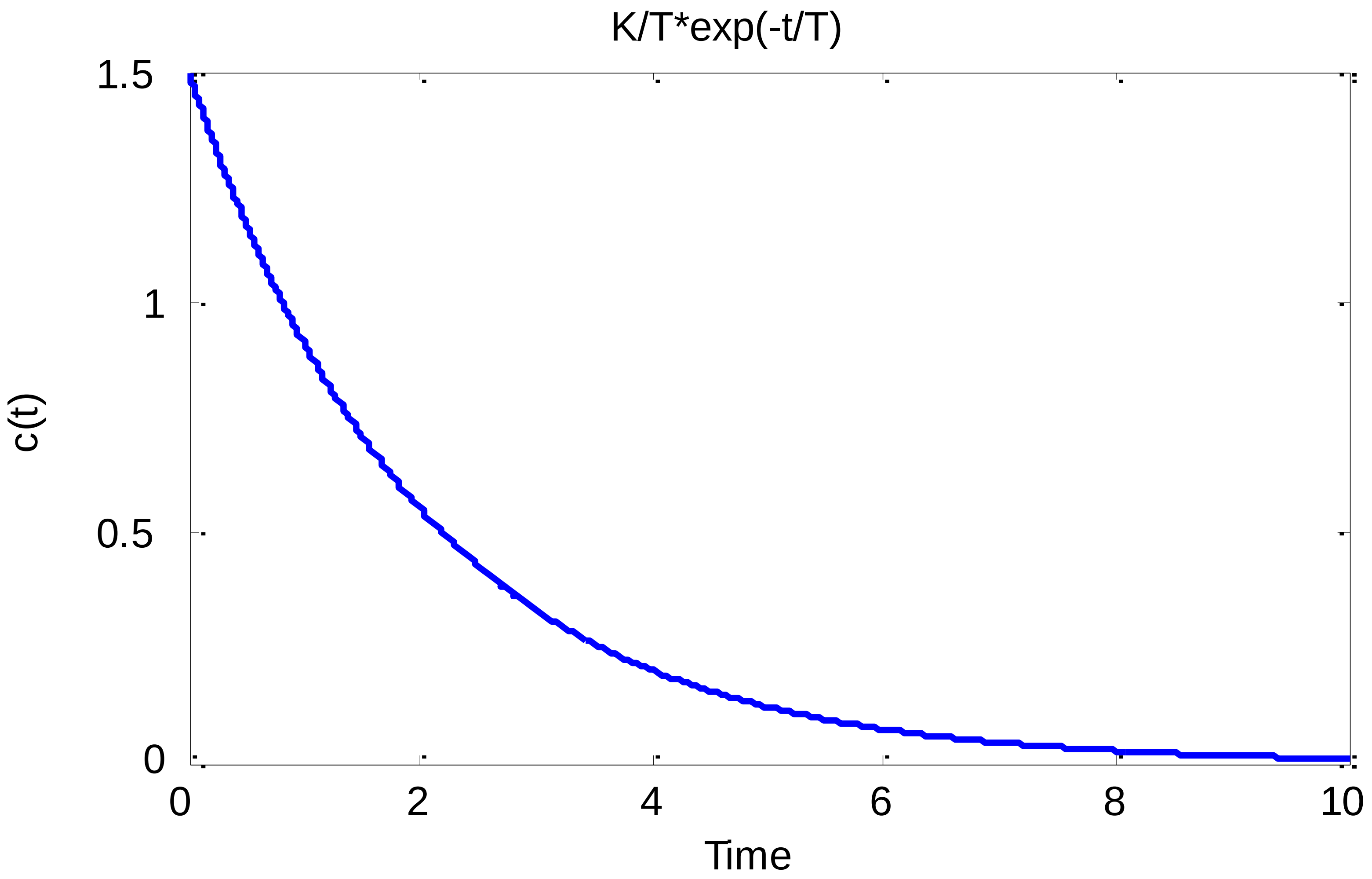
$$C(s) = \frac{K/T}{s+1/T}$$

- In order represent the response of the system in time domain we need to compute inverse Laplace transform of the above equation.

$$c(t) = \frac{K}{T}e^{-t/T}$$

IMPULSE RESPONSE OF 1^{ST} ORDER SYSTEM

- If K=3 and T=2s then $c(t) = \dfrac{K}{T} e^{-t/T}$



STEP RESPONSE OF 1ST ORDERSYSTEM

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = U(s) = \frac{1}{s}$$

$$C(s) = \frac{K}{s(Ts+1)}$$

- In order to find out the inverse Laplace of the above equation, we need to break it into partial fraction expansion

Forced Response **Natural Response**

$$C(s) = \frac{K}{s} - \frac{KT}{Ts+1}$$

STEP RESPONSE OF 1ST ORDER SYSTEM

$$C(s) = K\left(\frac{1}{s} - \frac{T}{Ts+1}\right)$$

- Taking Inverse Laplace of above equation

$$c(t) = K\left(u(t) - e^{-t/T}\right)$$

- Where u(t)=1

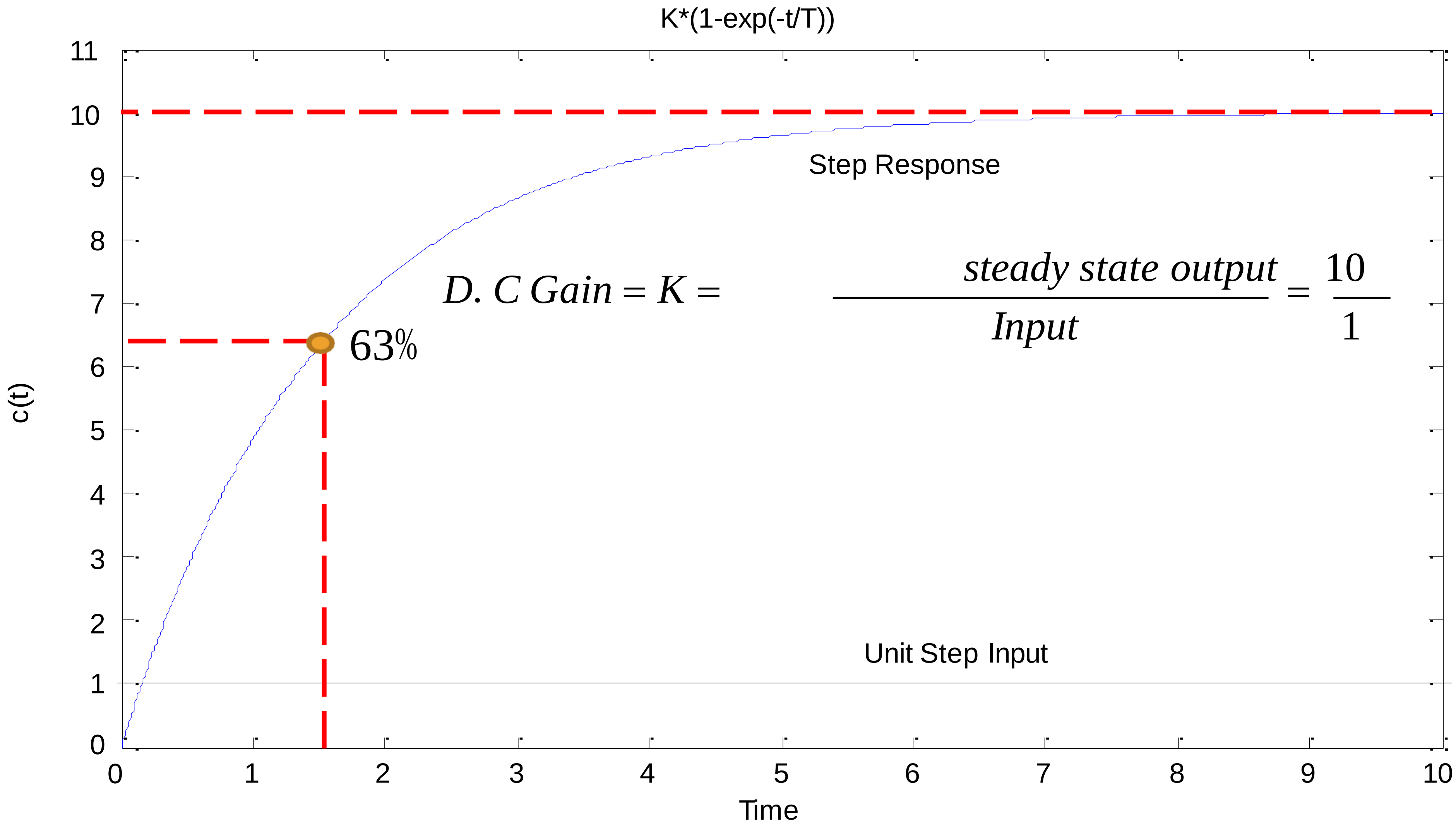
$$c(t) = K\left(1 - e^{-t/T}\right)$$

- When t=T

$$c(t) = K\left(1 - e^{-1}\right) = 0.632K$$

STEP RESPONSE OF 1ST ORDERSYSTEM

- If K=10 and T=1.5s then $c(t) = K\left(1 - e^{-t/T}\right)$

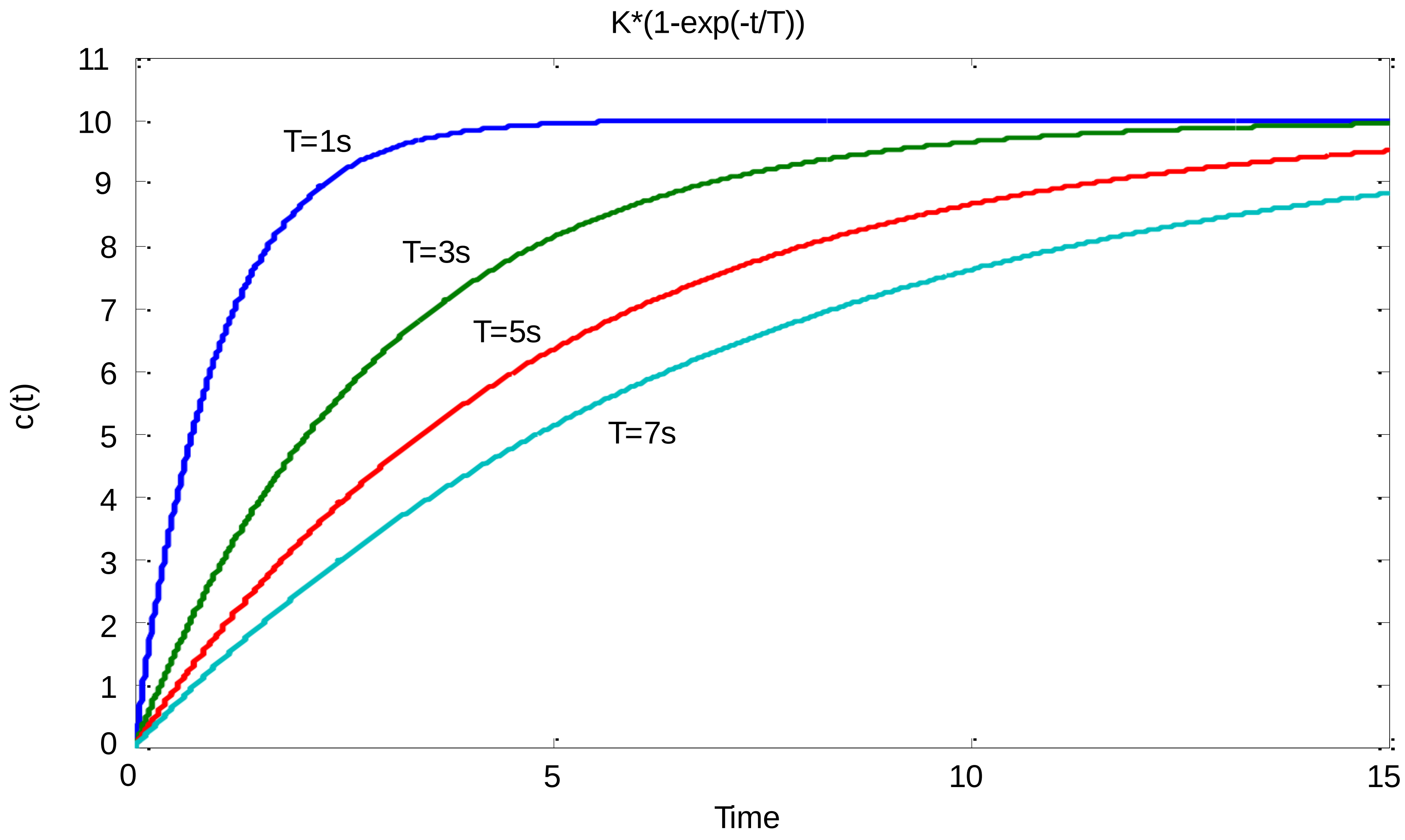


$$D.C\ Gain = K = \frac{steady\ state\ output}{Input} = \frac{10}{1}$$

STEP RESPONSE OF 1ST ORDERSYSTEM

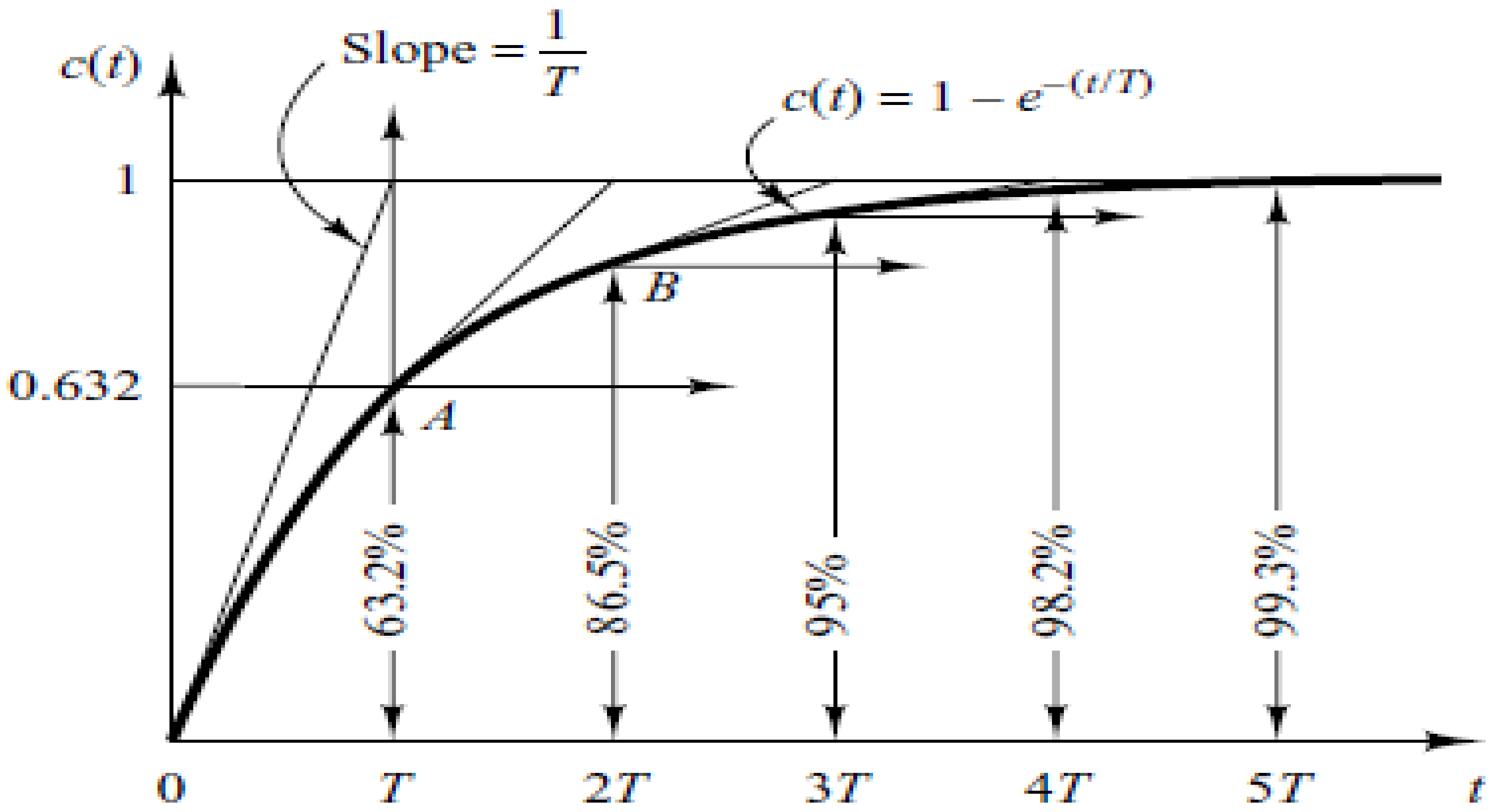
- If K=10 and T=1, 3, 5, 7

$$c(t) = K\left(1 - e^{-t/T}\right)$$



STEP RESPONSE OF 1ST ORDER SYSTEM

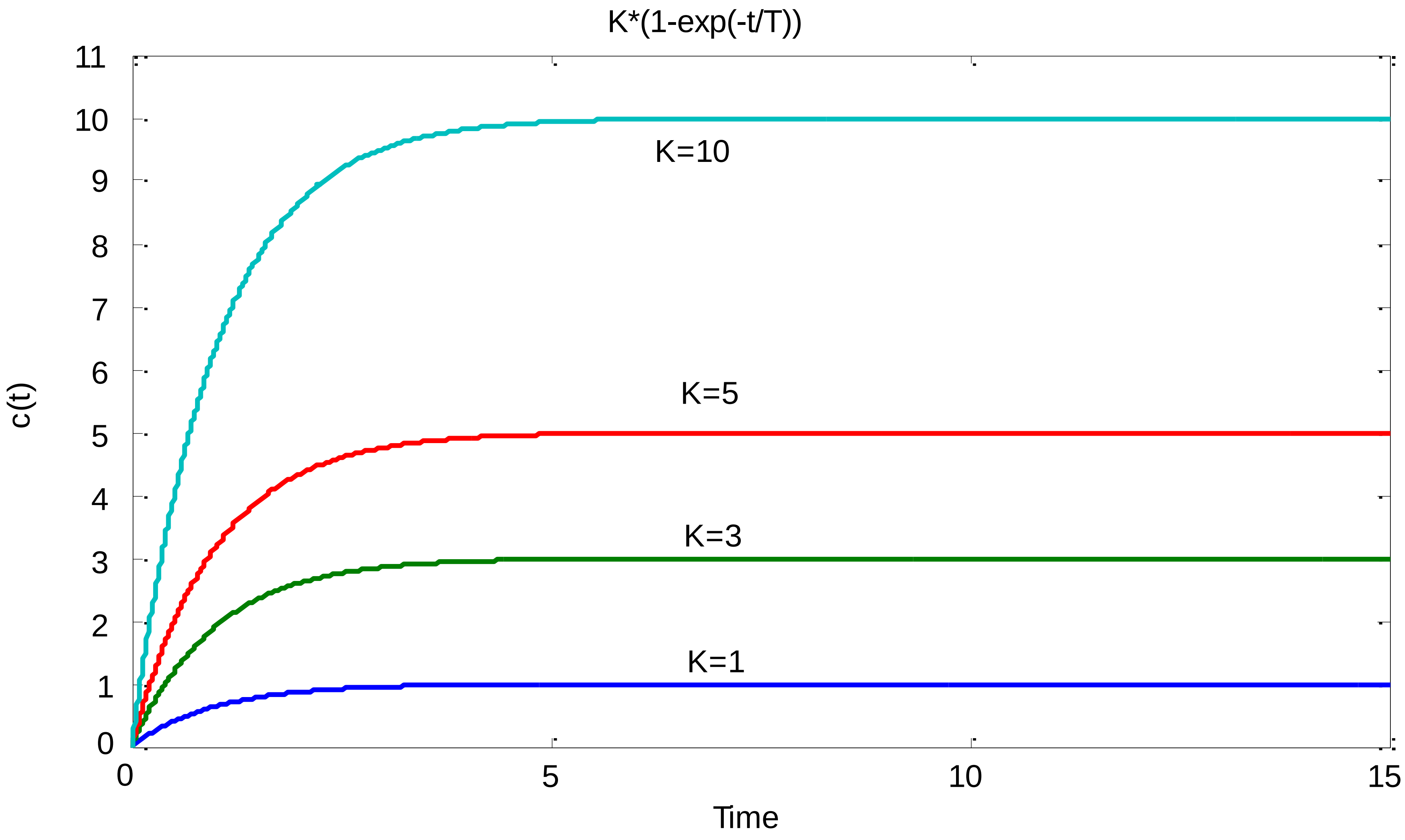
- System takes five time constants to reach its final value.



STEP RESPONSE OF 1^{ST} ORDERSYSTEM

- If K=1, 3, 5, 10 and T=1

$$c(t) = K\left(1 - e^{-t/T}\right)$$



RELATION BETWEEN STEP AND IMPULSE RESPONSE

- The step response of the first order system is

$$c(t) = K\left(1 - e^{-t/T}\right) = K - Ke^{-t/T}$$

- Differentiating c(t) with respect to t yields

$$\frac{dc(t)}{dt} = \frac{d}{dt}\left(K - Ke^{-t/T}\right)$$

$$\frac{dc(t)}{dt} = \frac{K}{T}e^{-t/T}$$

EXAMPLE#1

- The Laplace Transform of Impulse response of a system is actually the transfer function of the system.
- Therefore taking Laplace Transform of the impulse response given by following equation.

$$c(t) = 3e^{-0.5t}$$

$$C(s) = \frac{3}{S+0.5} \times 1 = \frac{3}{S+0.5} \times \delta(s)$$

$$\frac{C(s)}{\delta(s)} = \frac{C(s)}{R(s)} = \frac{3}{S+0.5}$$

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

EXAMPLE#1

- Impulse response of a 1st order system is given below.

$$c(t) = 3e^{-0.5t}$$

- Find out
  - Time constant T=2
  - D.C Gain K=6
  - Transfer Function
  - Step Response
  - Also Draw the Step response on your notebook

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

EXAMPLE#1

- For step response integrate impulse response

$$c(t) = 3e^{-0.5t}$$

$$\int c(t)\,dt = 3\int e^{-0.5t}\,dt \quad c_s(t) = -6e^{-0.5t}$$

$$+C$$

- We can find out C if initial condition is known e.g. $c_s(0)=0$

$$0 = -6e^{-0.5\times 0} + C$$

$$C = 6$$

$$c_s(t) = 6 - 6e^{-0.5t}$$

EXAMPLE#1

- If initial Conditions are not known then partial fraction expansion is a better choice

$$\frac{C(s)}{R(s)} = \frac{6}{2S+1}$$

since $R(s)$ is a step input, $R(s) = \frac{1}{s}$

$$C(s) = \frac{6}{s(2S+1)}$$

$$\frac{6}{s(2S+1)} = \frac{A}{s} + \frac{B}{2s+1}$$

$$\frac{6}{s(2S+1)} = \frac{6}{s} - \frac{6}{s+0.5}$$

$$c(t) = 6 - 6e^{-0.5t}$$

RAMP RESPONSE OF 1ST ORDER SYSTEM

- Consider the following 1st order system

$$R(s) \longrightarrow \boxed{\dfrac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^2}$$

$$C(s) = \frac{K}{s^2(Ts+1)}$$

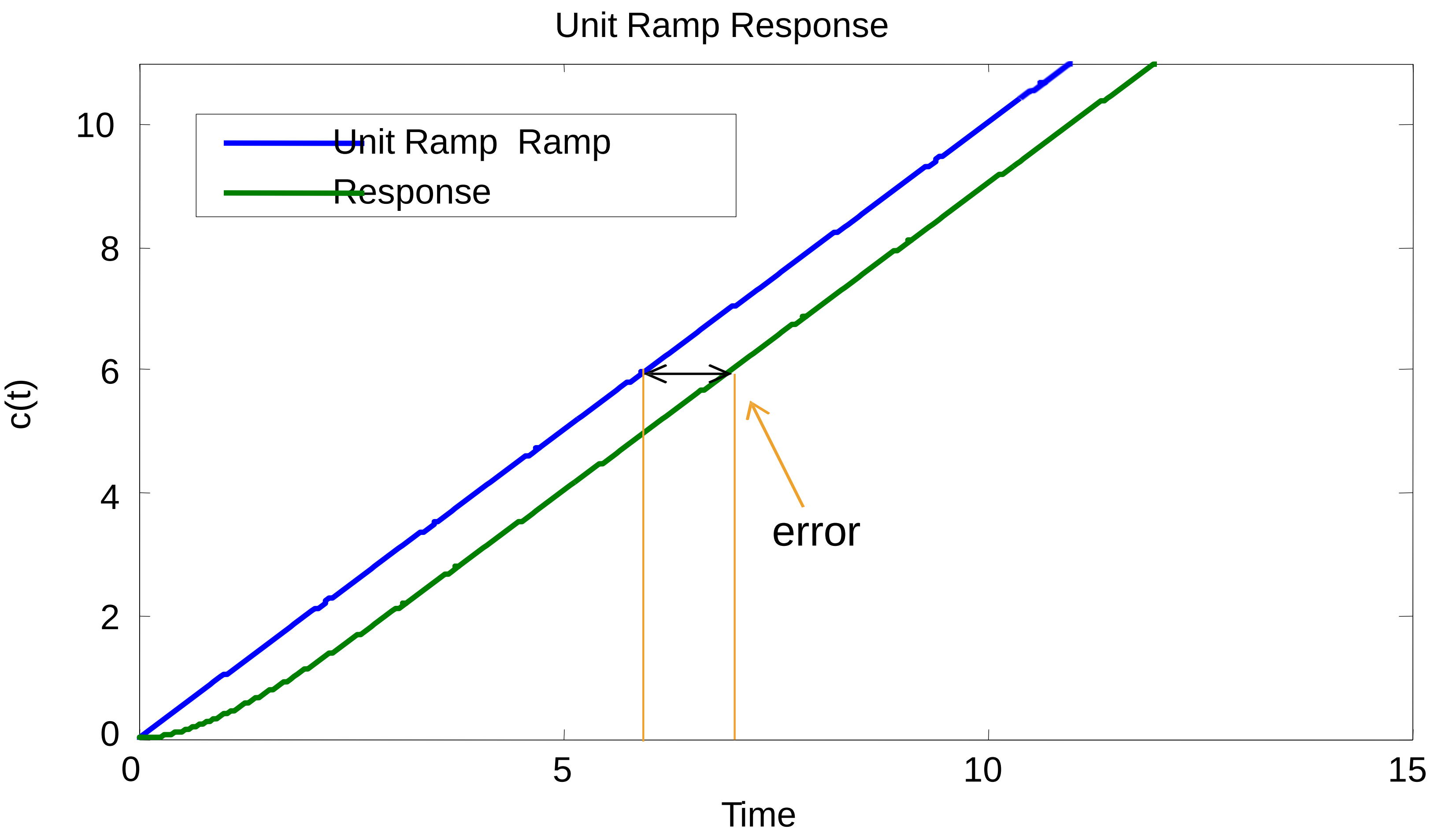
- The ramp response is given as

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$

RAMP RESPONSE OF 1ST ORDER SYSTEM

- If K=1 and T=1

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$

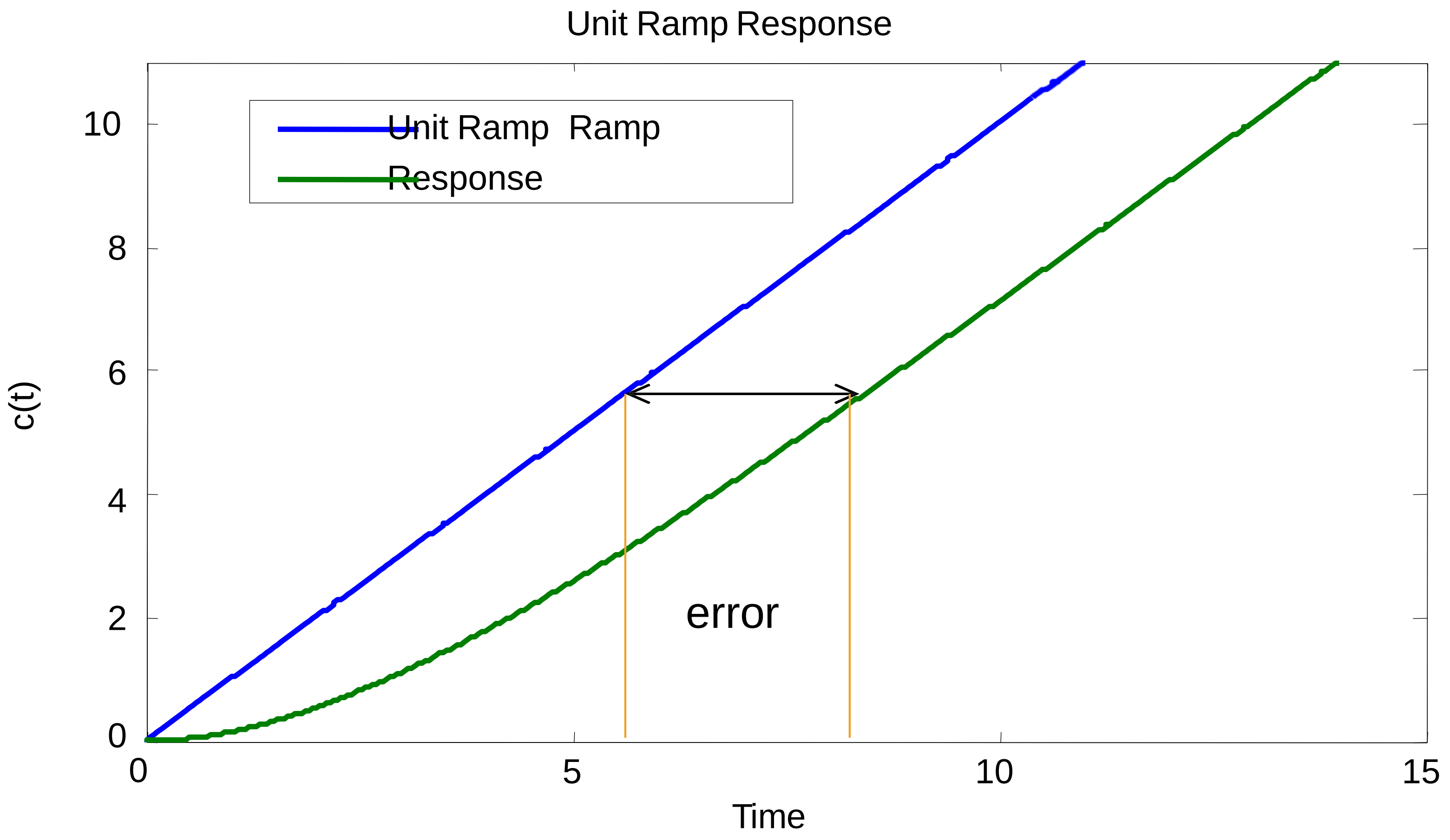


RAMP RESPONSE OF 1ST ORDER SYSTEM

RAMP RESPONSE OF 1ST ORDER SYSTEM

- If K=1 and T=3

$$c(t) = K\left(t - T + Te^{-t/T}\right)$$



PARABOLIC RESPONSE OF 1^{ST} ORDER SYSTEM

- Consider the following 1^{st} order system

$$R(s) \longrightarrow \boxed{\frac{K}{Ts+1}} \longrightarrow C(s)$$

$$R(s) = \frac{1}{s^3}$$

Therefore,

$$C(s) = \frac{K}{s^3(Ts+1)}$$

SECOND-ORDER SYSTEM

- General form:

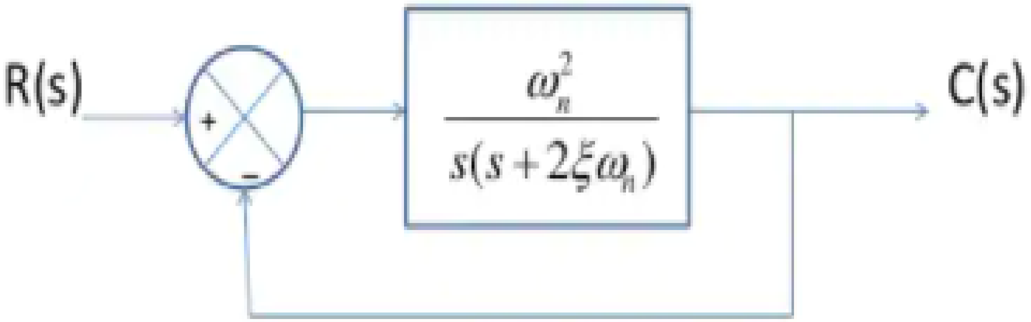
$$G(s) = \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

Where,

K : Gain

ς : Damping ratio

ω_n : Undamped natural frequency



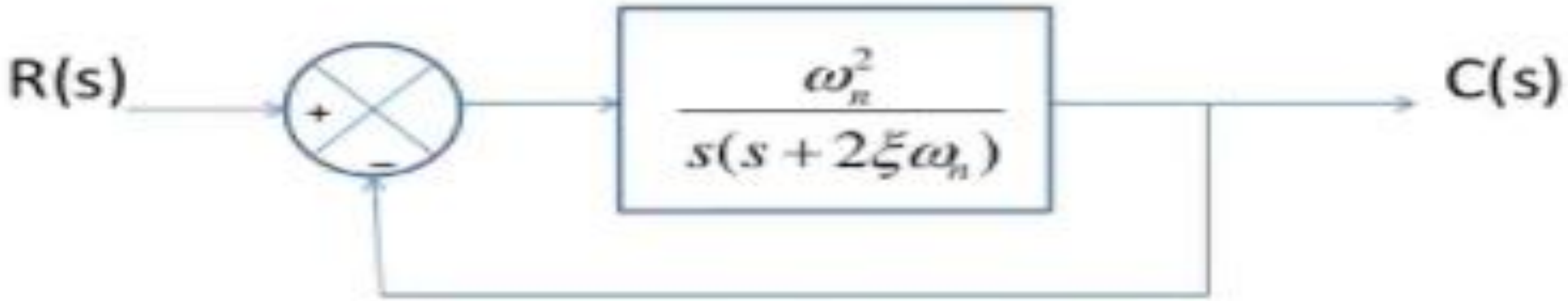
- Roots of denominator:

$$s^2 + 2\varsigma\omega_n s + \omega_n^2 = 0$$

$$\boxed{s_{1,2} = -\varsigma\omega_n \pm \omega_n\sqrt{\varsigma^2 - 1}}$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \phi\right]$$

Where, $\phi = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} \text{-----}(A)$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} R(s)$$

For unit step input $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2} ------(1)$$

Replace $s^2 + 2\xi\omega_n s + \omega_n^2$ by $(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)$

$$C(s) = \frac{1}{s} \cdot \frac{\omega_n^2}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)} -----(2)$$

Break the equation by partial fraction and put $\omega_d^2 = \omega_n^2(1 - \xi^2)$

$$\frac{1}{s} \cdot \frac{\omega_n^2}{(s + \xi\omega_n)^2 + \omega_d^2} = \frac{A}{s} + \frac{B}{(s + \xi\omega_n)^2 + \omega_d^2} ------(3)$$

$$A = 1$$

Multiply equation (3) by $\left[(s+\xi\omega_n)^2+\omega_d^2\right]$ and put

$$s=-\xi\omega_n-j\omega_d$$

$$B=\frac{\omega_n^2}{s}$$

$$B=\frac{\omega_n^2}{-\xi\omega_n-j\omega_d}$$

$$B=\frac{-\omega_n^2(\xi\omega_n-j\omega_d)}{(\xi\omega_n+j\omega_d)(\xi\omega_n-j\omega_d)}$$

$$-j\omega_d=s+\xi\omega_n$$

$$B=-(\xi\omega_n+s+\xi\omega_n)=-(s+2\xi\omega_n)$$

Equation (1) can be written as

$$C(s) = \frac{1}{s} - \frac{s + \xi\omega_n + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}$$

$$C(s) = \frac{1}{s} - \left[\frac{s + \xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2} + \frac{\xi\omega_n}{\omega_d} . \frac{\omega_d}{(s + \xi\omega_n)^2 + \omega_d^2}\right] - - (4)$$

Laplace Inverse of equation (4)

$$c(t) = 1 - \left[e^{-\xi\omega_n t} . \cos\omega_d t + \frac{\xi\omega_n}{\omega_d} e^{-\xi\omega_n t} . \sin\omega_d t\right] - - - (5)$$

Put $\quad \omega_d = \omega_n\sqrt{1 - \xi^2}$

$$c(t) = 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{\xi}{\sqrt{1-\xi^2}}.\sin\omega_d t\right]$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\sqrt{1-\xi^2}.\cos\omega_d t + \xi\sin\omega_d t\right]$$

Put $\sqrt{1-\xi^2} = \sin\phi$

$$\therefore \cos\phi = \xi$$

$$\tan\phi = \frac{\sqrt{1-\xi^2}}{\xi}$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \phi)$$

Put the values of ω_d & ϕ

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2})t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right] --- (6)$$

Error signal for the system

$$e(t) = r(t) - c(t)$$

$$e(t) = \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[(\omega_n\sqrt{1-\xi^2})t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right] --- (7)$$

The steady state value of c(t)

$$e_{ss} = \underset{t\to\infty}{Limit}\, c(t) = 1$$

Therefore at steady state there is no error between input and output.

ω_n = natural frequency of oscillation or undamped natural frequency.

ω_d = damped frequency of oscillation.

$\xi\omega_n$ = damping factor or actual damping or damping coefficient.

For equation (A) two poles (for $0 \le \xi \le 1$) are

$$-\xi\omega_n + j\omega_n\sqrt{1-\xi^2}$$

$$-\xi\omega_n - j\omega_n\sqrt{1-\xi^2}$$

Example:

e.g. :

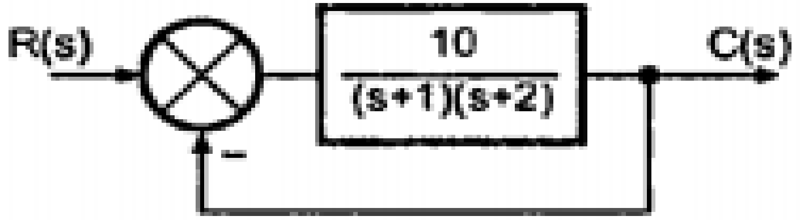


Fig. 7.21

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{\dfrac{10}{(s+1)(s+2)}}{1+\dfrac{10}{(s+1)(s+2)}} = \frac{10}{s^2+3s+12}$$

This C.L.T.F. is not standard as numerator term is not ω_n^2 but denominator always reflects ξ and ω_n. The values can be decided by comparing the denominator with the standard characteristic equation $s^2 + 2\xi\,\omega_n\, s + \omega_n^2 = 0$.

$\therefore \quad \omega_n^2 = 12 \qquad$ i.e. $\omega_n = \sqrt{12}$ rad / sec

While $\quad 2\xi\,\omega_n = 3 \qquad \therefore \; \xi = \dfrac{3}{2\sqrt{12}} = 0.433$

SECOND-ORDER SYSTEM

- Problem: For each of the transfer function, find the values of ζ and ω_n, as well as characterize the nature of the response.

a) $$G(s) = \frac{400}{s^2 + 12s + 400}$$

b) $$G(s) = \frac{900}{s^2 + 90s + 900}$$

c) $$G(s) = \frac{225}{s^2 + 30s + 225}$$

d) $$G(s) = \frac{625}{s^2 + 625}$$

Effect of ξ on Second Order System Performance

Consider input applied to the standard second order system is unit step.

$\therefore$ $$R(s) = 1/s$$

While $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\,\omega_n\,s + \omega_n^2}$$

$\therefore$ $$C(s) = \frac{\omega_n^2}{s\,(s^2 + 2\xi\,\omega_n\,s + \omega_n^2)}$$

Finding the roots of the equation $s^2 + 2\,\xi\,\omega_n\,s + \omega_n^2 = 0$

i.e. $$\frac{-2\xi\,\omega_n \pm \sqrt{4\xi^2\,\omega_n^2 - 4\omega_n^2}}{2}$$

i.e. $$s_{1,2} = -\xi\,\omega_n \pm \omega_n\sqrt{\xi^2 - 1}$$

We can write, $$C(s) = \frac{\omega_n^2}{s\left(s + \xi\,\omega_n + \omega_n\sqrt{\xi^2 - 1}\right)\left(s + \xi\,\omega_n - \omega_n\sqrt{\xi^2 - 1}\right)}$$

Now nature of these roots is dependent on damping ratio ξ. Consider the following cases,

Case 1 : $1 < \xi < \infty$

The roots are,

$$s_{1,2} = -\xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1}$$

i.e. real, unequal and negative, say $-K_1$ and $-K_2$

$$\therefore \qquad C(s) = \frac{\omega_n^2}{s(s+K_1)(s+K_2)} = \frac{A}{s} + \frac{B}{s+K_1} + \frac{C}{s+K_2}$$

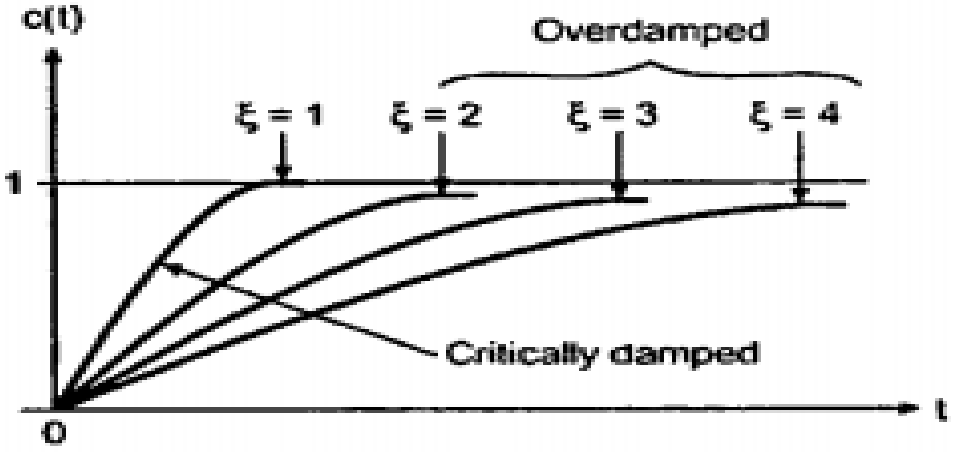
Taking Laplace inverse, c(t) will take the following form,

$$c(t) = C_{ss} + Be^{-K_1 t} + Ce^{-K_2 t},$$

where $\qquad C_{ss}$ = Steady state output = A

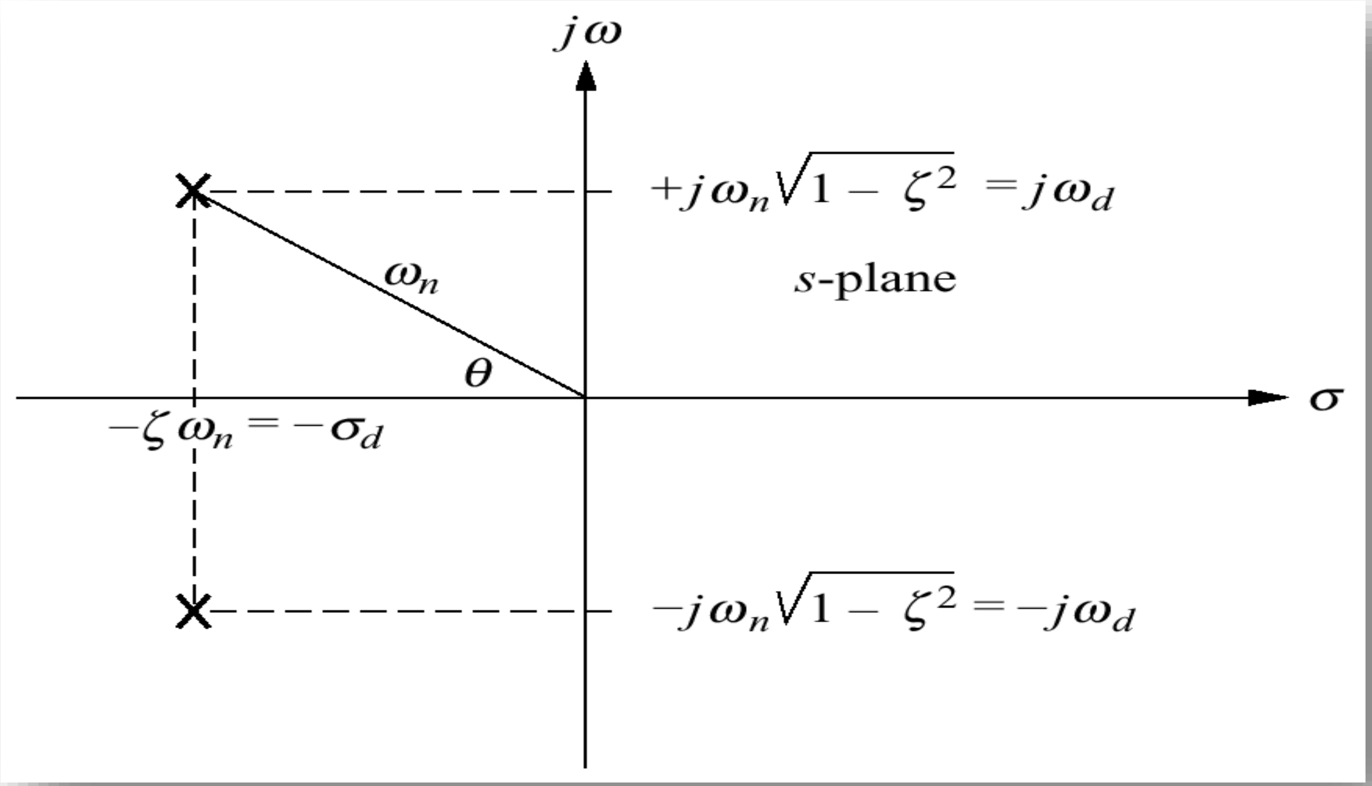
The output is purely exponential. This means damping is so high that there are no oscillations in the output and is purely exponential. Hence such systems are called 'Overdamped'.

Hence nature of response will be as shown in the Fig. 7.22.



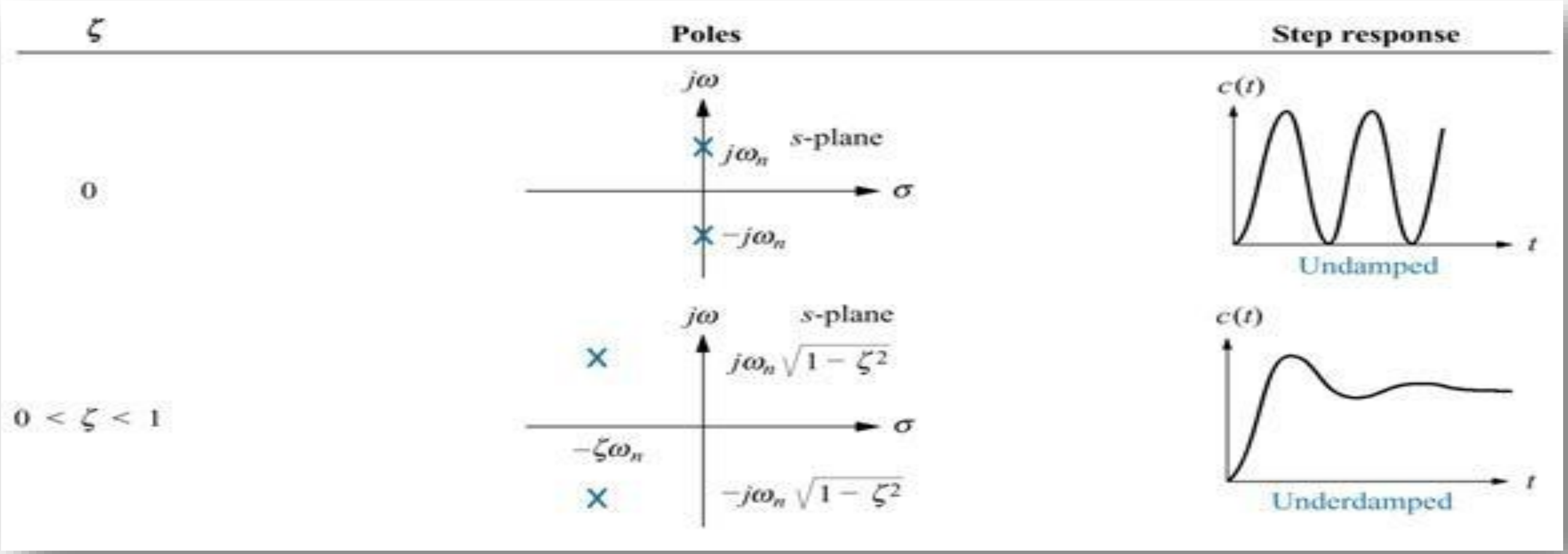
SECOND-ORDER SYSTEM

- Pole plot for the underdamped second-order system



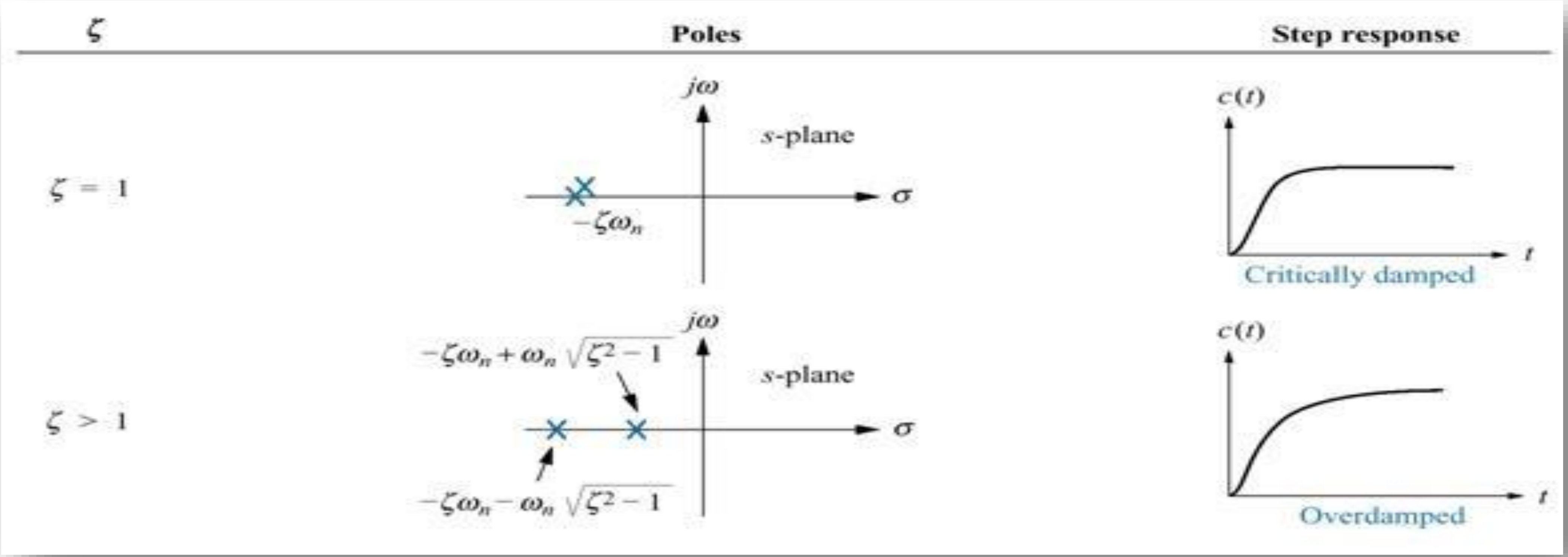
SECOND-ORDER SYSTEM

- Second-order response as a function of damping ratio



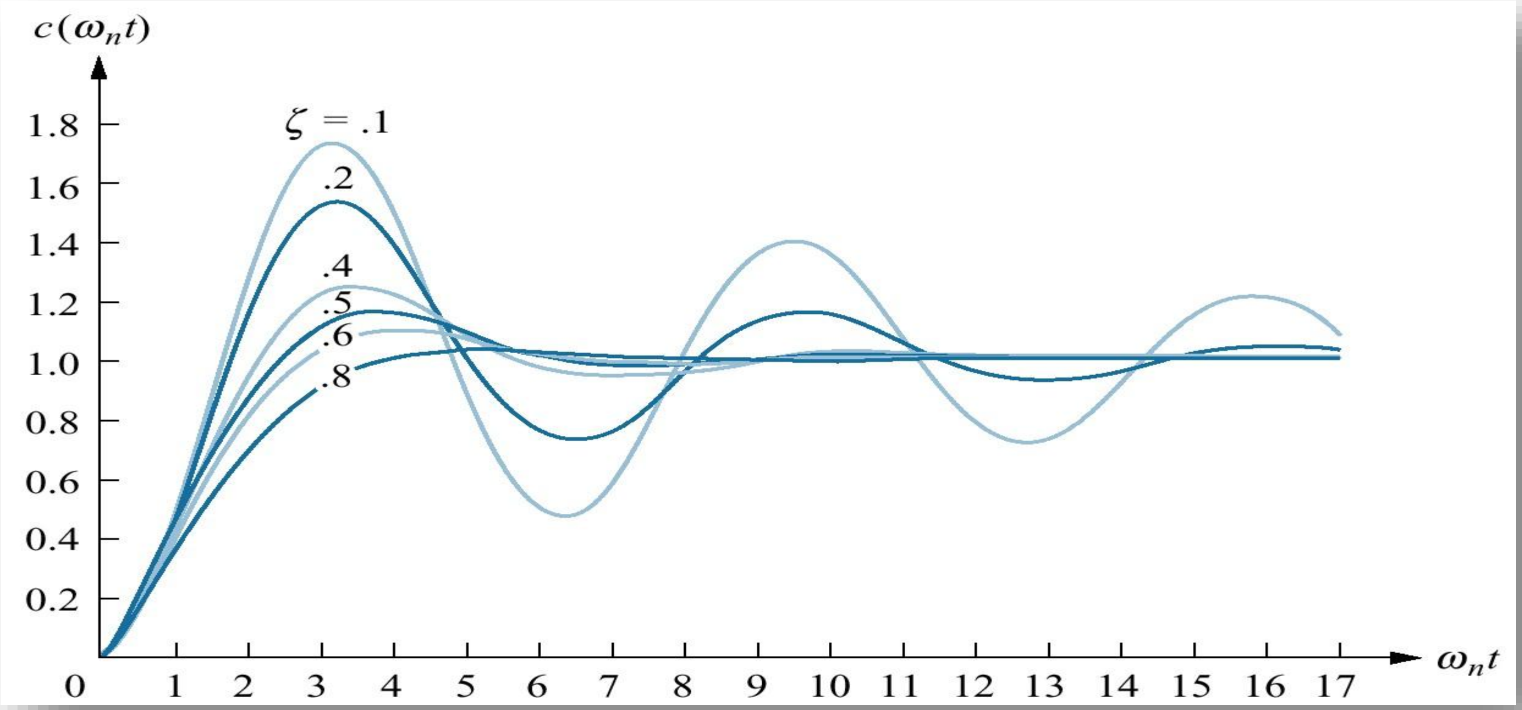
SECOND-ORDER SYSTEM

- Second-order response as a function of damping ratio

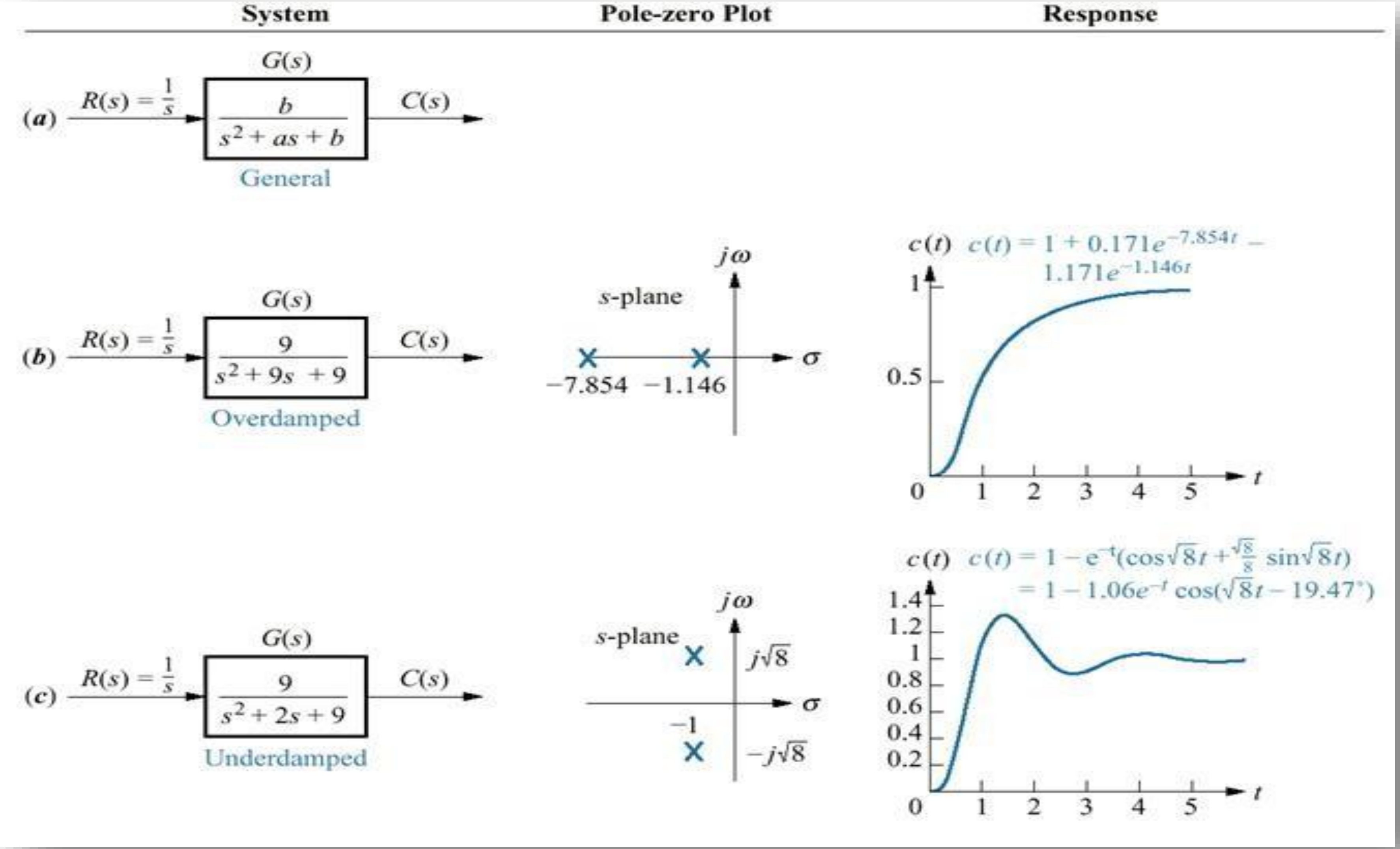


SECOND-ORDER SYSTEM

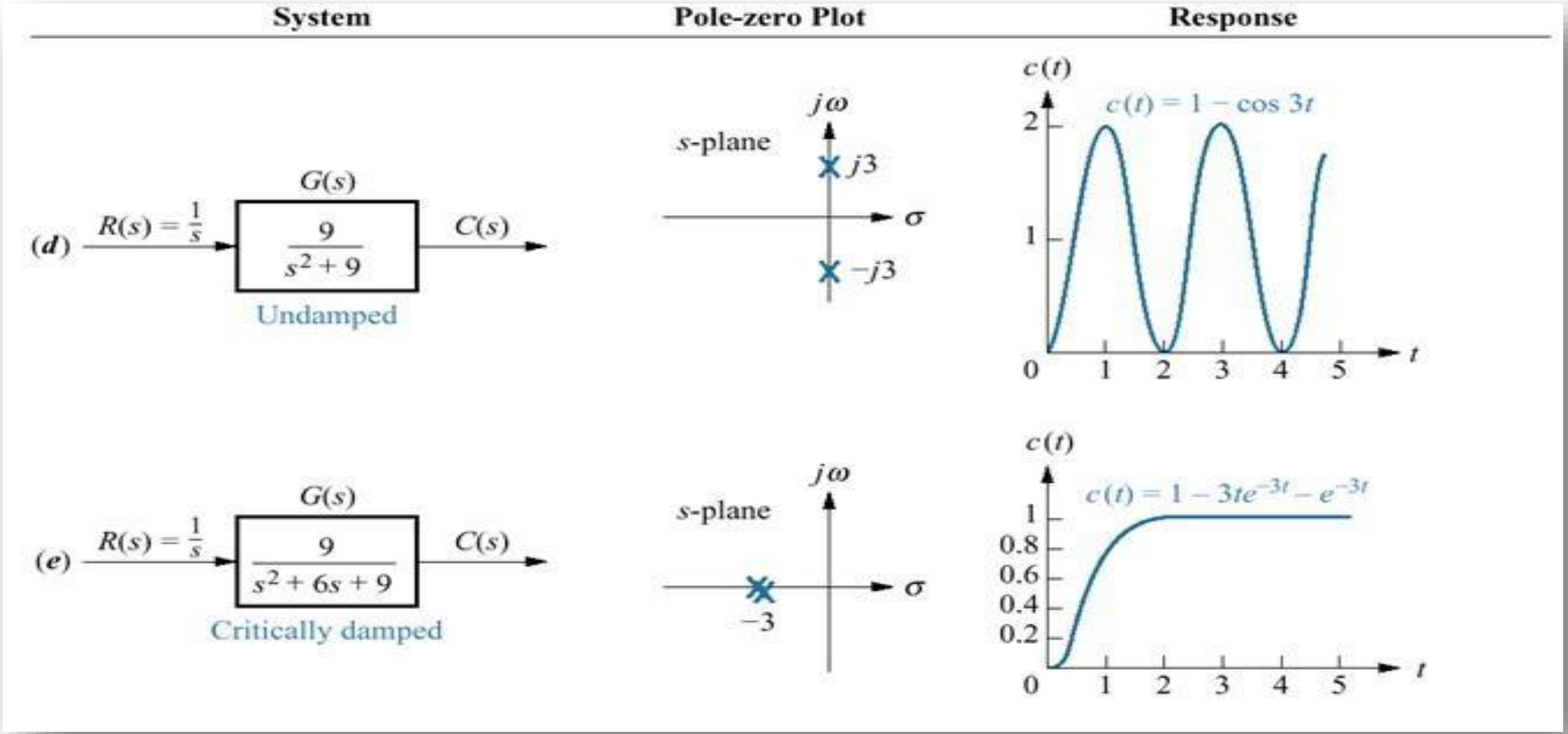
- Second-order underdamped responses for damping ratiovalue



SECOND-ORDER SYSTEM

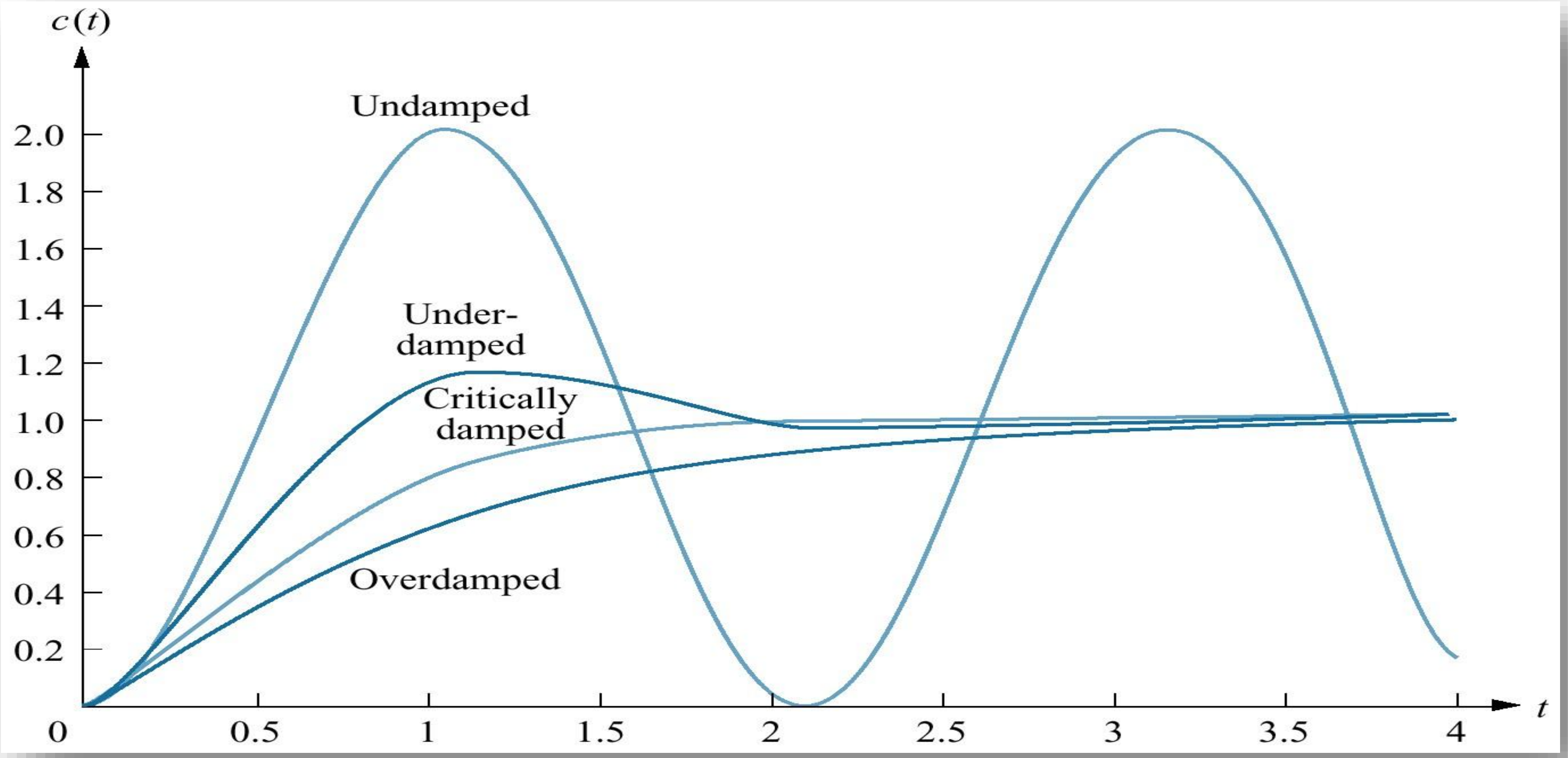


SECOND-ORDER SYSTEM



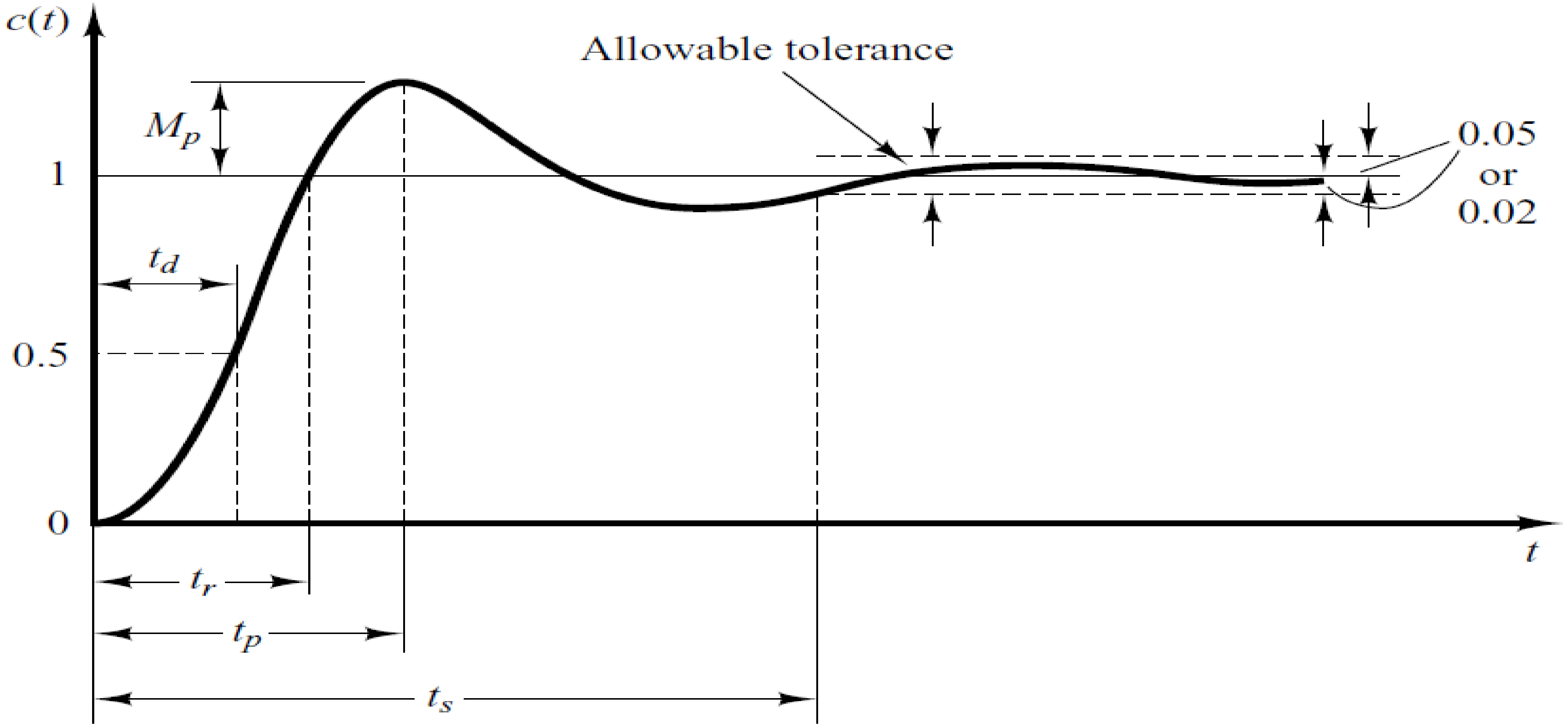
SECOND-ORDER SYSTEM

- Step responses for second-order system damping cases



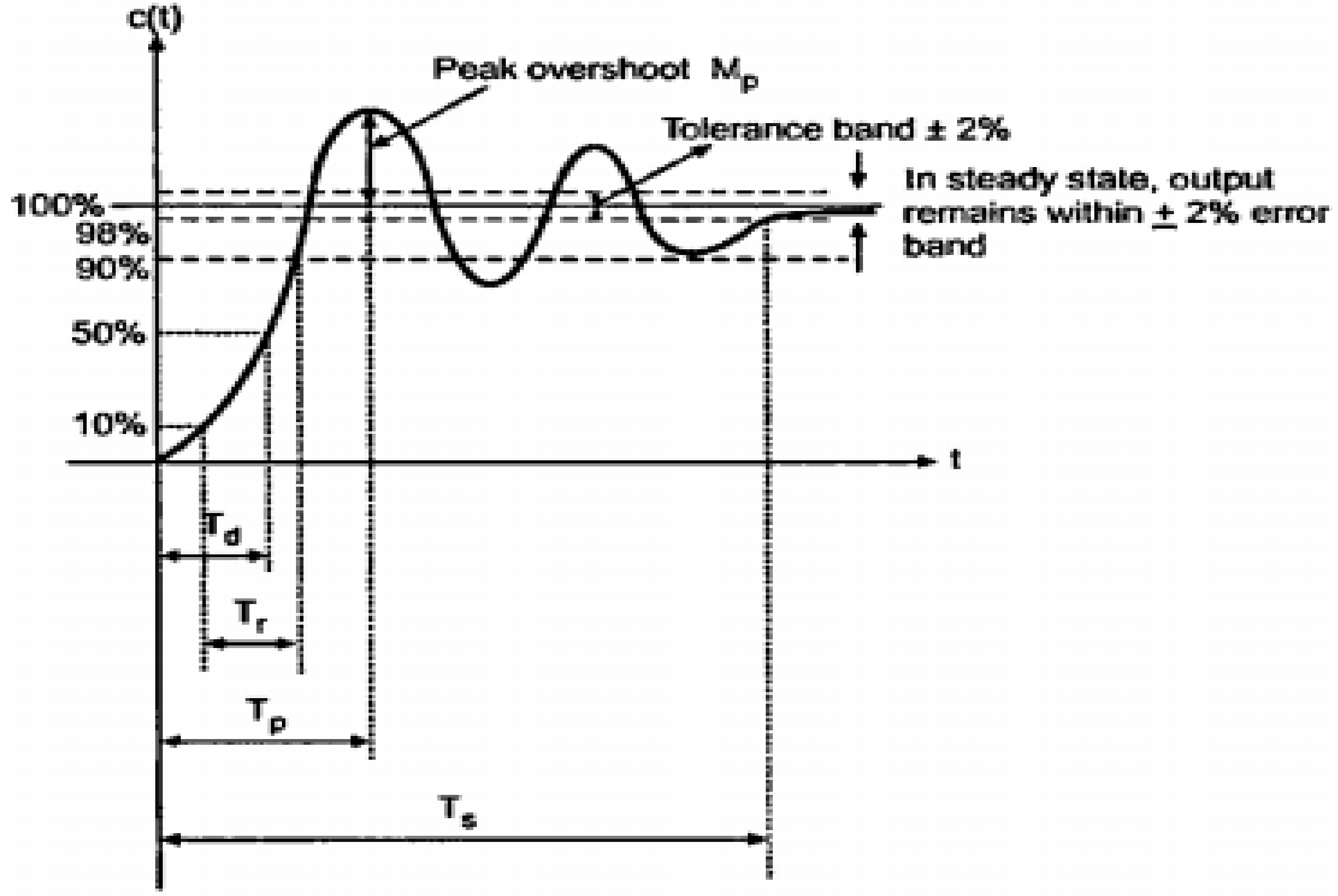
TIME-DOMAIN SPECIFICATION

For $0<\zeta <1$ and $\omega_n > 0$, the 2nd order system's response due to a unit step input looks like



Transient Response Specifications

The actual output behaviour according to the expression derived can be shown as in the Fig.



Transient Response Specifications

1) ***Delay Time T_d** : It is the time required for the response to reach 50% of the final value in the first attempt. It is given by,*

$$\boxed{T_d = \frac{1 + 0.7\,\xi}{\omega_n}}$$

2) ***Rise Time T_r** : It is the time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value for underdamped systems. The rise time is reci ~ocal of the slope of the response at the instant, the response is equal to 50% of the final value. It is given by,*

$$\boxed{T_r = \frac{\pi - \theta}{\omega_d} \text{ sec where } \theta \text{ must be in radians.}}$$

3) ***Peak Time T_p : It is the time required for the response to reach its peak value. It is also defined as the time at which response undergoes the first overshoot which is always peak overshoot.***

$$\boxed{T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}} \text{ sec}}$$

4) ***Peak Overshoot M_p : It is the largest error between reference input and output during the transient period.***

It also can be defined as the amount by which output overshoots its reference steady state value during the first overshoot.

$$M_p = \left\{ c(t) \big|_{t=T_p} \right\} - 1 \qquad 1 \text{ is for unit step input}$$

$$\boxed{\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100}$$

5) ***Settling Time T_s : This is defined as the time required for the response to decrease and stay within specified percentage of its final value (within tolerance band).***

$$\text{Time constant of system} = \frac{1}{\xi\omega_n} = T$$

$$T_s = 4 \times \text{Time constant}$$

Practically the setting time is assumed to be 4 times, the time constant of the system.

$$\boxed{T_s = \frac{4}{\xi\omega_n} \quad \text{..... for a tolerance band of } \pm 2\% \text{ of steady state}}$$

Example 1

A second order system is given by $\frac{C(s)}{R(s)} = \frac{25^2}{s^2 + 6s + 25}$. *Find its rise time, peak time, peak overshoot and settling time if subjected to unit step input. Also calculate expression for its output response.*

Solution : Comparing the denominator of T.F. with the standard form $s^2 + 2\xi\omega_n s + \omega_n^2$

$$\omega_n^2 = 25 \quad \text{and} \quad 2\xi\omega_n = 6$$

$$\omega_n = 5 \qquad \therefore \quad \xi = 0.6$$

$$\theta = \tan^{-1}\left[\frac{\sqrt{1-\xi^2}}{\xi}\right] = 0.9272 \text{ radians}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = 5\sqrt{1-(0.6)^2} = 4 \text{ rad/sec}$$

$$T_r = \frac{\pi-\theta}{\omega_d} = \frac{\pi - 0.9272}{4} = 0.5535 \text{ sec}$$

$$T_P = \frac{\pi}{\omega_d} = \frac{\pi}{4} = 0.785 \text{ sec}$$

$$\% M_P = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100 = 9.48\ \%$$

$$T_s = \frac{4}{\xi\omega_n} = 1.33 \text{ sec}$$

and $$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t + \theta) = 1 - \frac{e^{-3t}}{\sqrt{1-(0.6)^2}}\sin(4t + 0.9272)$$

$\therefore$ $$= 1 - 1.5625\, e^{-3t} \sin(4t + 0.9272)$$