Chapter 2
Part 2-Stability Analysis

The analysis of whether the given system can reach steady state; passing through the transients successfully is called Stability Analysis of the system.

Ex.1 **Stability of Control Systems**

The stability of a linear closed loop system can be determined from the locations of closed loop poles in the s-plane.

For example : If system has closed loop T.F.

$$\frac{C(s)}{R(s)} = \frac{10}{(s+2)(s+4)}$$

output response for unit step input.

$$R(s) = 1/s$$

$$C(s) = \frac{10}{s(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+4}$$

$$C(s) = 10\left\{\frac{1/8}{s} - \frac{1/4}{s+2} + \frac{1/8}{s+4}\right\} \quad \text{... Finding the partial fractions}$$

$$C(s) = \frac{1.25}{s} - \frac{2.5}{s+2} + \frac{1.25}{s+4}$$

$$c(t) = 1.25 - 2.5\,e^{-2t} + 1.25\,e^{-4t} = C_{ss} + c_t(t)$$

Now as $t \to \infty$ both exponential terms will approach to zero and output will be steady state output.

i.e. as $t \to \infty$, $c_t(t) = 0$

Transient output $= 0$

Such systems are called ***absolutely stable systems.***

Thus if closed loop poles are located in left half, exponential indices in output are negative. And if indices are negative, exponential transient terms will vanish when $t \to \infty$.

Ex.2

$$\frac{C(s)}{R(s)} = \frac{10}{(s-2)(s+4)}$$

Find out unit step response of above system.

$$C(s) = \frac{10}{s(s-2)(s+4)} = \left\{\frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+4}\right\}$$

$$C(s) = -\frac{1.25}{s} + \frac{0.833}{s-2} + \frac{0.416}{s+4}$$

$\therefore$ $$c(t) = -1.25 + 0.833\, e^{+2t} + 0.416\, e^{-4t}$$

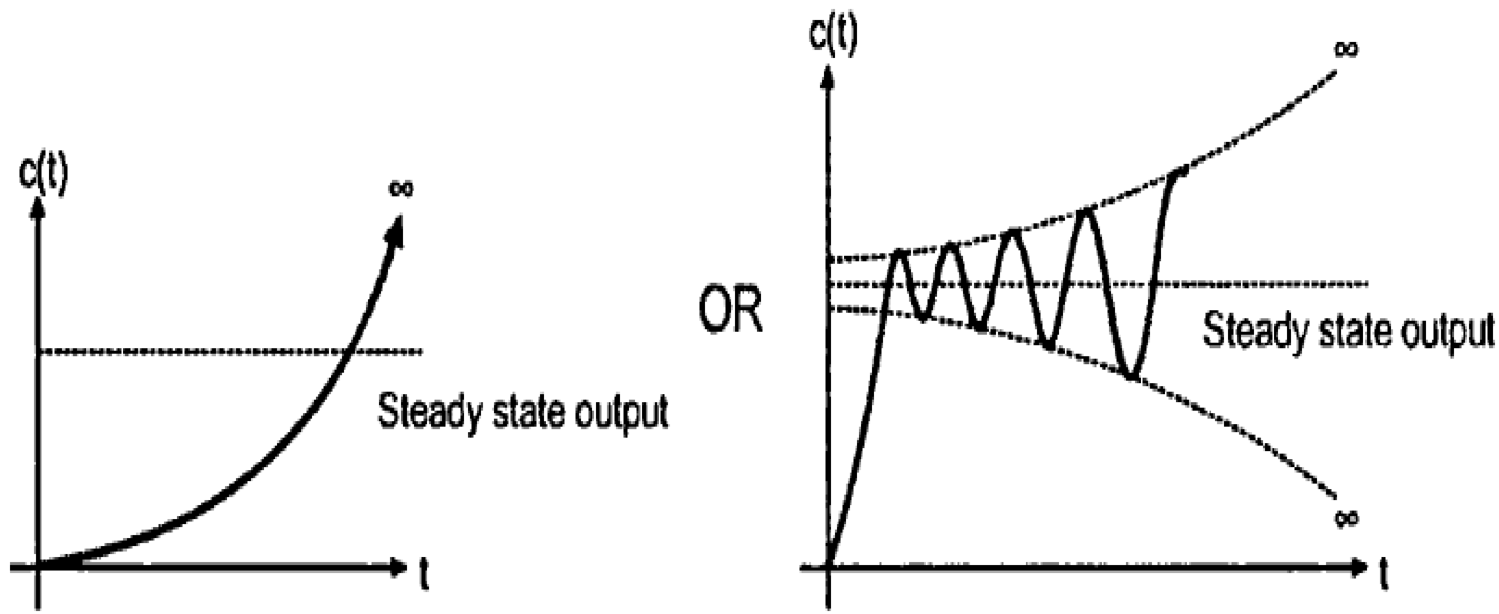
Now due to pole located in right half, there is one exponential term with positive index in transient output.

while $c_{ss}(t) = -1.25$

t	c(t)
0	0
1	+ 4.91
2	+ 44.23
4	+ 2481.88
∞	∞

So it is clear that if any of the closed loop poles lie in right half of s-plane, then it gives the exponential term of positive index and due to that, transient response of increasing amplitude, making system unstable.

In such systems output is uncontrollable and unbounded one. Output response of such systems is as shown in the Fig.



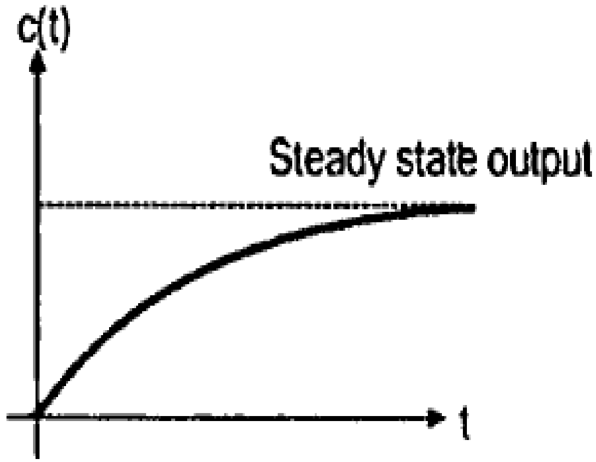
(a) Increasing exponentially

(b) Increasing amplitude oscillations

Uncontrollable response

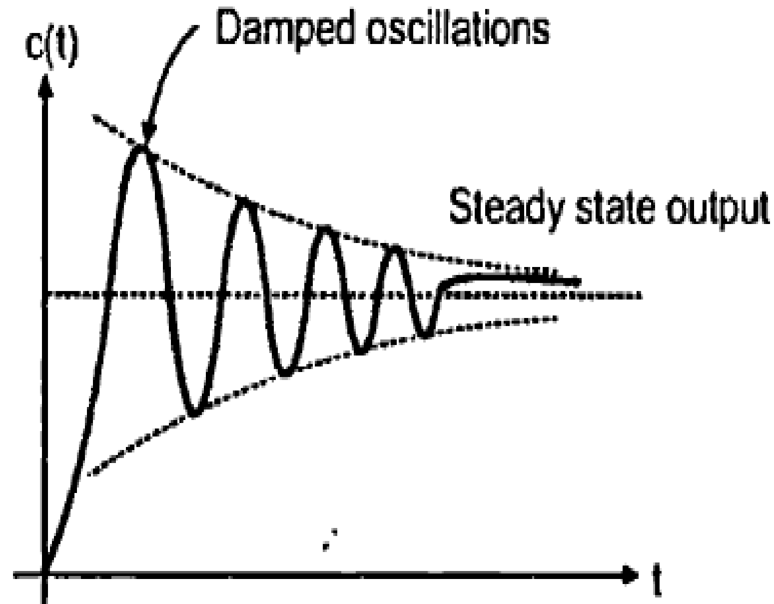
the stability depends on locations of closed loop poles. And the closed loop poles are the roots of the characteristic equation of the system.

So, $\boxed{\text{Closed loop poles = Roots of the characteristic equation}}$



(a)

OR



(b)

Definition of BIBO Stability : *This is Bounded Input Bounded Output stability (BIBO).*

A linear time invariant system is said to be stable if following conditions are satisfied :

i) *When the system is excited by a bounded input, output is also bounded and controllable.*

ii) *In the absence of the input, output must tend to zero irrespective of the initial conditions.*

Unstable System : *A linear time invariant system is said to be unstable if,*

i) *For a bounded input it produces unbounded output.*

ii) *In absence of the input, output may not return to zero. It shows certain output without input.*

Routh-Hurwitz Criterion

This represents a method of determining the location of poles of a characteristic equation with respect to the left half and right half of the s-plane without actually solving the equation.

The T.F. of any linear closed loop system can be represented as,

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + ... + b_m}{a_0 s^n + a_1 s^{n-1} + ... + a_n} = \frac{B(s)}{F(s)}$$

where 'a' and 'b' are constants.

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ... + a_n = 0$$

Thus the roots of the characteristic equation are the closed loop poles of the system which decide the stability of the system.

Necessary Conditions

In order that the above characteristic equation has no root in right of s-plane, it is **necessary but not sufficient** that,

1) All the coefficients of the polynomial have the same sign.
2) None of the coefficient vanishes i.e. all powers of 's' must be present in descending order from 'n' to zero.

These conditions are not sufficient.

Hurwitz's Criterion

The sufficient condition for having all the roots of characteristic equation in left half of s-plane is given by Hurwitz. It is referred as Hurwitz criterion. It states that :

The necessary and sufficient condition to have all roots of characteristic equation in left half of s-plane is that the sub-determinants D_K , K = 1, 2, ...n obtained from Hurwitz's determinant `H' must all be positive.

Method of forming Hurwitz determinant :

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots\cdots & a_{2n-1} \\ a_0 & a_2 & a_4 & \cdots\cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots\cdots & a_{2n-3} \\ 0 & a_0 & a_2 & \cdots\cdots & a_{2n-4} \\ 0 & 0 & a_1 & \cdots\cdots & a_{2n-5} \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ 0 & . & . & \cdots\cdots & a_n \end{vmatrix}$$

$$D_1 = |a_1| \qquad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} \qquad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} \qquad D_K = |H|$$

For the system to be stable, all the above determinants must be positive.

Example *Determine the stability of the given characteristic equation by Hurwitz's method.*

$F(s) = s^3 + s^2 + s^1 + 4 = 0$ is characteristic equation.

$$a_0 = 1,\ a_1 = 1,\ a_2 = 1,\ a_3 = 4,\ n = 3$$

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$$D_1 = |1| = 1$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$$

$$D_3 = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 4 - 16 = -12$$

As D_2 and D_3 are negative, given system is unstable.

Disadvantages of Hurwitz's Method

i) For higher order systems, to solve the determinants of higher order is very complicated and time consuming.

ii) Number of roots located in right half of s-plane for unstable system cannot be judged by this method.

iii) Difficult to predict marginal stability of the system.

Due to these limitations, a new method is suggested by the scientist Routh called Routh's method. It is also called Routh-Hurwitz method.

Routh's Criterion

The necessary and sufficient condition for system to be stable is "All the terms in the first column of Routh's array must have same sign. There should not be any sign change in the first column of Routh's array."

If there are any sign changes existing then,

a) System is unstable.

The number of sign changes equals the number of roots lying in the right half of s-plane.

Examine the stability of given equations using .

Example : $s^3 + 6s^2 + 11s + 6 = 0$

Solution : $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

$$
\begin{array}{c|cc}
s^3 & 1 & 11 \\
s^2 & 6 & 6 \\
s^1 & \dfrac{11 \times 6 - 6}{6} = 10 & 0 \\
s^0 & 6 &
\end{array}
$$

As there is no sign change in first column, system is stable.

Special Cases of Routh's Criterion

Special Case 1

First element of any of the rows of Routh's array is zero and the same remaining row contains at least one non-zero element.

Effect : *The terms in the new row become infinite and Routh's test fails.*

e.g. : $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

s^5	1	3	2	
s^4	2	6	1	
s^3	$\boxed{0}$	1.5	0	Special case 1
s^2	∞	...	...	Routh' array failed

Following two methods are used to remove above said difficulty.

First method : Substitute a small positive number `ε' in place of a zero occurred as a first element in a row. Complete the array with this number `ε'. Then examine the sign change by taking $\lim\limits_{\varepsilon \to 0}$. Consider above example.

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$\frac{6\varepsilon - 3}{\varepsilon}$	1	0
s^1	$\dfrac{\frac{1.5\,(6\varepsilon - 3)}{\varepsilon} - \varepsilon}{\frac{(6\varepsilon - 3)}{\varepsilon}}$	0	
s^0	1		

To examine sign change,

$$\lim_{\varepsilon \to 0}\left(\frac{6\varepsilon - 3}{\varepsilon}\right) = 6 - \lim_{\varepsilon \to 0}\frac{3}{\varepsilon}$$

$$= 6 - \infty$$

$$= -\infty \text{ sign is negative.}$$

$$\lim_{\varepsilon \to 0}\frac{1.5(6\varepsilon - 3) - \varepsilon^2}{6\varepsilon - 3} = \lim_{\varepsilon \to 0}\frac{9\varepsilon - 4.5 - \varepsilon^2}{6\varepsilon - 3}$$

$$= \frac{0 - 4.5 - 0}{0 - 3}$$

$$= +1.5 \text{ sign is positive.}$$

Second method : To solve the above difficulty one more method can be used. In this, replace `s' by `1/z' in original equation. Taking L.C.M., rearrange characteristic equation in descending powers of `z'. Then complete the Routh's array with this new equation in `z' and examine the stability with this array.

Consider $$F(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$$

Put $$s = 1/z$$

$$\therefore \quad \frac{1}{z^5} + \frac{2}{z^4} + \frac{3}{z^3} + \frac{6}{z^2} + \frac{2}{z} + 1 = 0$$

$$z^5 + 2z^4 + 6z^3 + 3z^2 + 2z + 1 = 0$$

z^5	1	6	2
z^4	2	3	1
z^3	4.5	1.5	0
z^2	2.33	1	0
z^1	− 0.429	0	
z^0	1		

As there are two sign changes, **system is unstable.** The result is same.

Special Case 2

All the elements of a row in a Routh's array are zero.

Effect : The terms of the next row cannot be determined and Routh's test fails

$$
\begin{array}{c|ccc}
s^5 & a & b & c \\
s^4 & d & e & f \\
s^3 & \boxed{0 \quad 0 \quad 0} & &
\end{array}
\quad \leftarrow \text{Row of zeros, Special case 2}
$$

This indicates non-availablility of coefficient in that row.

Procedure to Eliminate this Difficulty

i) Form an equation by using the coefficients of a row which is just above the row of zeros. Such an equation is called an **Auxiliary Equation** denoted as A(s). For above case such an equation is,

$$A(s) = ds^4 + es^2 + f$$

The coefficients of any row are corresponding to alternate powers of `s' starting from the power indicated against it.

So 'd' is coefficient corresponding to s^4 so first term is ds^4 of A(s).

Next coefficient 'e' is corresponding to alternate power of 's' from 4 i.e. s^2 hence the term es^2 and so on.

ii) Take the derivative of an auxiliary equation with respect to 's'.

i.e. $$\frac{dA(s)}{ds} = \underline{4d}\, s^3 + \underline{2e}\, s$$

iii) Replace row of zeros by the coefficients of $\frac{dA(s)}{ds}$.

s^5	a	b	c
s^4	d	e	f
s^3	4d	2e	0

iv) Complete the array interms of these new coefficients.

$s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$. *Find the number of roots of this equation with positive real part, zero real part and negative real part.*

Solution :

s^6	1	3	– 16	– 48
s^5	4	0	– 64	0
s^4	3	0	– 48	0
s^3	0	0	0	

$$A(s) = 3s^4 - 48 = 0 \qquad \frac{dA}{ds} = 12s^3$$

s^6	1	3	– 16	– 48
s^5	4	0	– 64	0
s^4	3	0	– 48	0
s^3	12	0	0	0
s^2	[ε] 0	– 48	0	0
s^1	$\frac{576}{\varepsilon}$	0	0	
s^0	– 48			

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

For unity feedback system,

$G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$ *, find range of values of K, marginal value of K and frequency of sustained oscillations.*

Solution : Characteristic equation, $1 + G(s)H(s) = 0$ and $H(s) = 1$

$$\therefore \quad 1+\frac{K}{s(1+0.4s)(1+0.25s)} = 0$$

$$s[1 + 0.65s + 0.1s^2] + K = 0$$

$$\therefore \quad 0.1s^3 + 0.65s^2 + s + K = 0$$

s^3	0.1	1	From s^0, $K > 0$
s^2	0.65	K	From s^1,
s^1	$\frac{0.65-0.1K}{0.65}$	0	$0.65 - 0.1K > 0$ $\therefore\ 0.65 > 0.1\ K$
s^0	K		$\therefore\ 6.5 > K$

$\therefore$ Range of values of K, $0 < K < 6.5$.

The marginal value of 'K' is a value which makes any row other than s^0 as row of zeros.

$$\therefore \quad 0.65 - 0.1\ K_{mar} = 0$$

$$\therefore \quad \boxed{K_{mar} = 6.5}$$

To find frequency, find out roots of auxiliary equation at marginal value of 'K'.

$$A(s) = 0.65s^2 + K = 0\ ;$$

$$\therefore \quad 0.65s^2 + 6.5 = 0 \qquad \because\ K_{mar} = 6.5$$

$$s^2 = -10$$

$$s = \pm j\ 3.162$$

Comparing with $s = \pm j\omega$

ω = Frequency of oscillations

= **3.162 rad/sec.**

Example : $s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$

Solution :

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 15 \\
s^3 & 0 & -12 & 0
\end{array}$$

Replace 0 by small positive number ε.

$$\begin{array}{c|ccc}
s^5 & 1 & 2 & 3 \\
s^4 & 1 & 2 & 15 \\
s^3 & \varepsilon & -12 & 0 \\
s^2 & \dfrac{2\varepsilon + 12}{\varepsilon} & 15 & 0 \\
s^1 & \dfrac{\left(\dfrac{2\varepsilon + 12}{\varepsilon}\right)(-12) - 15\varepsilon}{\dfrac{2\varepsilon + 12}{\varepsilon}} & 0 & 0 \\
s^0 & 15 & &
\end{array}$$

$$\lim_{\varepsilon \to 0} = \frac{2\varepsilon+12}{\varepsilon} = 2+\frac{12}{\varepsilon} = +\infty$$

$$\lim_{\varepsilon \to 0} \frac{\frac{(2\varepsilon+12)}{\varepsilon}(-12)-15\varepsilon}{\frac{(2\varepsilon+12)}{\varepsilon}} = \lim_{\varepsilon \to 0} \frac{-24\varepsilon-144-15\varepsilon^2}{2\varepsilon+12}$$

$$= \frac{0-144-0}{0+12} = -12$$

s^5	1	2	3
s^4	1	2	15
s^3	ε	-12	0
s^2	$+\infty$	15	0
s^1	-12	0	
s^0	15		

There are two sign changes, so **system is unstable.**

Example : $s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$. *Find the number of roots of this equation with positive real part, zero real part and negative real part.*

Solution :

s^6	1	3	– 16	– 48
s^5	4	0	– 64	0
s^4	3	0	– 48	0
s^3	0	0	0	

$$A(s) = 3s^4 - 48 = 0 \qquad \frac{dA}{ds} = 12s^3$$

s^6	1	3	– 16	– 48
s^5	4	0	– 64	0
s^4	3	0	– 48	0
s^3	12	0	0	0
s^2	[ε] 0	– 48	0	0
s^1	$\frac{576}{\varepsilon}$	0	0	
s^0	– 48			

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

$\therefore$ One sign change and system is unstable . Thus there is one root in R.H.S. of s-plane i.e. with positive real part. Now solve A(s) = 0 for the dominant roots.

$$A(s) = 3s^4 - 48 = 0$$

Put $\quad s^2 = y$

$\therefore \quad 3y^2 = 48 \qquad \therefore \quad y^2 = 16, \qquad \therefore \quad y = \pm\sqrt{16} = \pm 4$

$\therefore \quad s^2 = +4 \qquad s^2 = -4$

$\quad s = \pm 2 \qquad s = \pm 2j$

So $s = \pm 2j$ are the two roots on imaginary axis i.e. with zero real part. Root in R.H.S. indicated by a sign change is $s = +2$ as obtained by solving $A(s) = 0$. Total there are 6 roots as $n = 6$.

Roots with positive real part = 1

Roots with zero real part = 2

Roots with negative real part = 6 – 2 – 1 = 3.

· **Example** : *For system $s^4 + 22s^3 + 10s^2 + s + K = 0$, find K_{mar} and ω at K_{mar}.*

Solution :

s^4	1	10	K
s^3	22	1	0
s^2	9.95	K	0
s^1	$\frac{9.95-22K}{9.95}$	0	
s^0	K		

Marginal value of `K' which makes row of s^1 as row of zeros.

$$9.95 - 22\, K_{mar} = 0$$

$$\therefore \quad K_{mar} = \mathbf{0.4524}$$

$$\text{Hence} \quad A(s) = 9.95s^2 + K = 0$$

$$9.95s^2 + 0.4524 = 0$$

$$s^2 = -0.04546$$

$$s = \pm j\, 0.2132$$

Hence frequency of oscillations = **0.2132 rad/sec.**

Example ... *For a system with characteristic equation $F(s) = s^6 + 3s^5 + 4s^4 + 6s^3 + 5s^2 + 3s + 2 = 0$, examine stability.*

Solution :

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	0	0	0	0

Row of zeros

$$A(s) = 2s^4 + 4s^2 + 2 = 0 \quad \text{i.e.} \quad s^4 + 2s^2 + 1 = 0$$

$$\frac{dA(s)}{ds} = 4s^3 + 4s$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	0	0	0	0

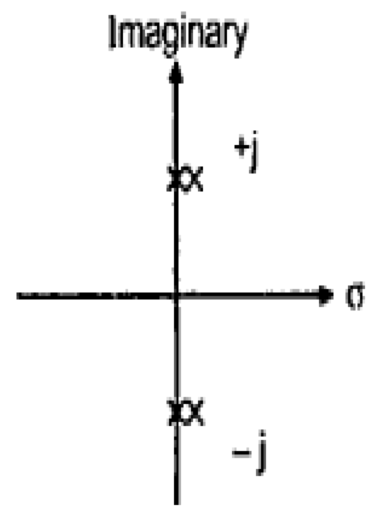
Row of zeros again

$$\therefore \quad A'(s) = 2s^2 + 2 = 0$$

$$\frac{dA'(s)}{ds} = 4s = 0$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	4	0	0	0
s^0	2	0	0	0

No sign change, hence no root is located in R.H.S. of s-plane. As row of zeros occur, system may be marginally stable or unstable. To examine that find the roots of first auxiliary equation.



$A(s) = s^4 + 2s^2 + 1 = 0 \qquad s^2 = \frac{-2 \pm \sqrt{4-4}}{2} = -1$

$s^2 = -1, \qquad s^2 = -1, \qquad s_{1,2} = \pm j, \qquad s_{3,4} = \pm j$

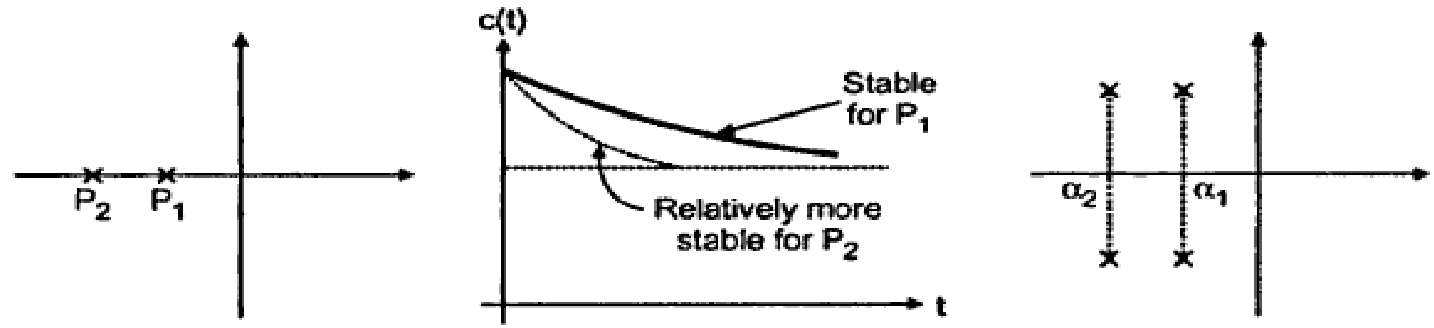
Relative Stability

The system is said to be relatively more stable or unstable on the basis of settling time. System is said to be relatively more stable if settling time for that system is less than that of the other system.

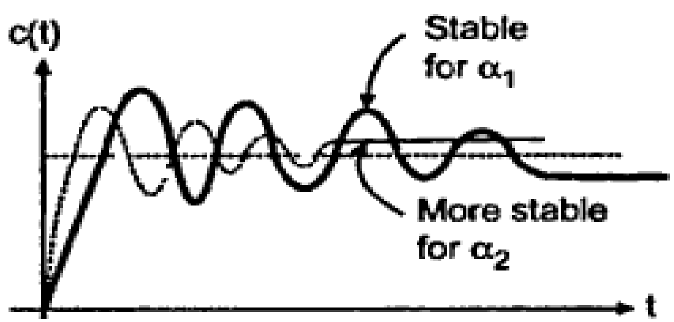
The settling time of the root or pair of complex conjugate roots is inversely proportional to the real part of the roots.

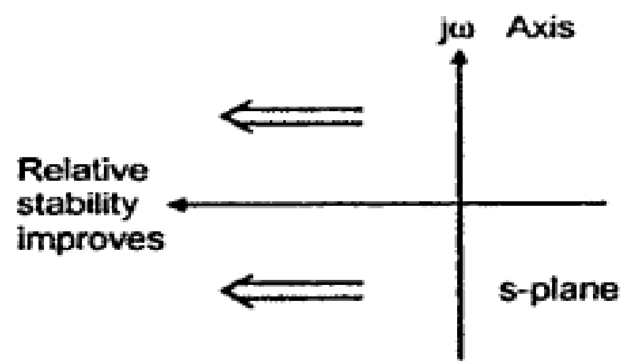
So for the roots located near the jω axis, settling time will be large. As roots or pair of complex conjugate roots moves away from jω – axis i.e. towards left half of s-plane, settling time becomes lesser or smaller and system becomes more and more stable.

So relative stability of the system improves, as the closed loop poles move away from the imaginary axis in left half of s-plane.



(a) (b) (c)



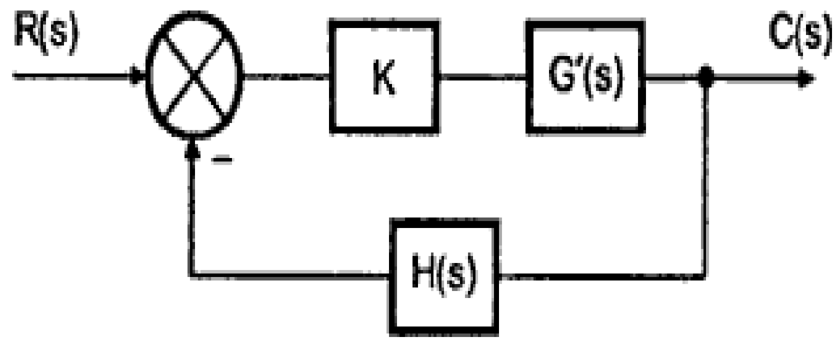


Basic Concepts of Root Locus

In general, the characteristic equation of a closed loop system is given as,

$$1 + G(s)H(s) = 0$$

For root locus, the gain `K' is assumed to be a variable parameter and is a part of forward path of the closed loop system. Consider the system shown in the Fig. 9.1.



$$G(s) = KG'(s)$$

where K = Gain of the amplifier in forward path or also called **System Gain.** The characteristic equation becomes,

$$1 + G(s)H(s) = 0 \quad \text{i.e.} \quad 1 + KG'(s)H(s) = 0$$

which contains `K' as a variable parameter.

The closed loop poles i.e. the roots of the above equation are now dependent on the values of 'K'.

If now gain 'K' is varied from $-\infty$ to $+\infty$ then for each separate value of 'K' we will get separate set of locations of the roots of the characteristic equation. If all such locations are joined, the resulting locus is called **Root Locus.** So we can define root locus as, **the locus of the closed loop poles obtained when system gain 'K' is varied from $-\infty$ to $+\infty$ is called Root Locus.**

Example : *Consider unity feedback system with* $G(s) = \frac{K}{s}$. *Obtain its roots locus.*

Solution : The characteristic equation becomes,

$$1 + G(s)H(s) = 0, \quad H(s) = 1$$

$$\therefore \quad 1 + \frac{K}{s} = 0$$

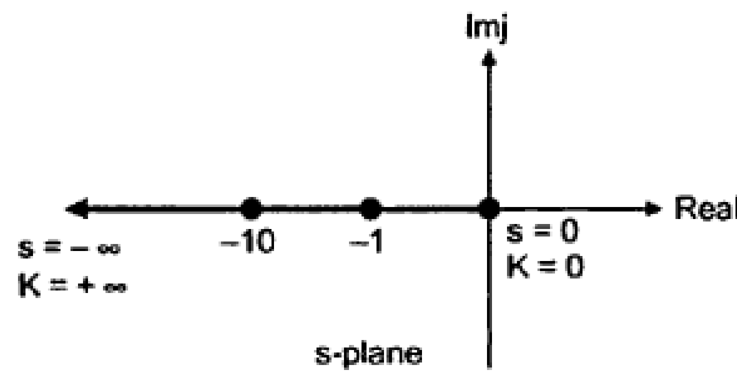
$$\therefore \quad s + K = 0$$

The root of this equation is located at $s = -K$

Now if gain 'K' is varied from 0 to $+\infty$, the location of this root is going to change.

The locus obtained by joining all such locations when K is varied from 0 to $+\infty$ is called Root Locus.

K	s = – K Root location
0	0
1	– 1
10	– 10
⋮	⋮
+ ∞	– ∞



The root locus is nothing but the negative real axis, for this system.

Angle and Magnitude Condition

For a general closed loop system the characteristic equation is,

$$1 + G(s)H(s) = 0$$

i.e. $G(s)H(s) = -1$

As s-plane is complex we can write above equation as,

$$\boxed{G(s)H(s) = -1 + j0}$$

All s-values can be expressed as '$\sigma + j\omega$' i.e. G(s)H(s) term is also complex one.

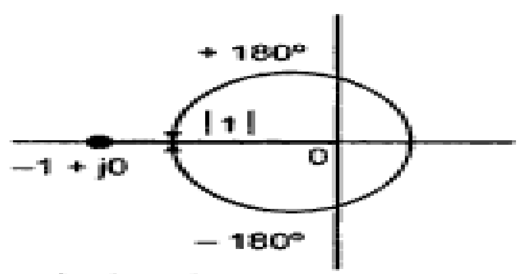
Angle Condition

$$G(s)H(s) = -1 + j0$$

Equating angles of both sides,

$$\angle G(s)H(s) = \pm (2q + 1)\, 180^\circ \qquad q = 0, 1, 2 \,\ldots\ldots$$

$\therefore -1 + j0 = 1 \angle \pm 180^\circ$ but the point $-1 + j0$ is a point on negative real axis which can be traced as magnitude 1 at an angle $\pm 180^\circ, \pm 540^\circ, \pm 900^\circ \ldots\ldots \pm (2q + 1)\, 180^\circ$.



Magnitude = 1
Angle = ± 180°, ± 540°, - - - - - ± (2q + 1) 180°

$\therefore$ Angle condition can be stated as,

$\angle G(s)H(s)|$ for any value of 's' which is the root of equation $[1 + G(s)H(s) = 0]$ is

$$= \pm (2q+1)\, 180^\circ \quad \text{where } q = 0, 1, 2 \,\ldots\ldots$$

$$= \text{Odd multiple of } 180^\circ$$

Example ! : *Consider the system with* $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$. *Find whether* $s = -0.75$ *is on the root locus or not using angle condition.*

Solution : Let us test whether s = – 0.75 is located on the root locus of above system i.e. whether s = – 0.75 is a root of the characteristic equation 1 + G(s)H(s) = 0 or not. Use Angle condition,

$$\angle G(s)H(s)\Big|_{\text{at point } s=-0.75} = \pm(2q+1)\,180^\circ \qquad q = 0, 1, 2, \ldots$$

Substituting s = – 0.75 in all the terms of G(s)H(s),

$$\angle G(s)H(s)\Big|_{\text{at } s=-0.75} = \frac{\angle K + j0}{\angle -0.75 + j0 \cdot \angle\, 1.25 + j0 \cdot \angle\, 3.25 + j0}$$

Converting to polar form and considering angles, **(use calculator to obtain polar form from rectangular form and consider angle.)**

$$= \frac{0^\circ}{180^\circ \cdot 0^\circ \cdot 0^\circ} = -180^\circ$$

That is ∠ G(s)H(s) = – 180° at s = – 0.75 which satisfies angle condition and we can conclude that point s = – 0.75 is on the root locus of the given system.

Let us test, s = – 1 + j4 for its existence on the root locus of the same system,

$$\angle G(s)H(s)\Big|_{\text{at } s=-1+j4} = \frac{\angle K + j0}{\angle -1 + j4 \cdot \angle\, 1 + j4 \cdot \angle\, 3 + j4}$$

$$= \frac{0^\circ}{104.03^\circ \cdot 75.963^\circ \cdot 53.13^\circ}$$

$$= -233.123^\circ$$

$$\angle G(s)H(s)\Big|_{\text{at } s=-1+j4} = -233.123^\circ$$

As this is not satisfying the angle condition, the point (–1 + j4) cannot be on the root locus of the given system.

Magnitude Condition

If magnitudes of both sides of the equation $G(s)H(s) = -1$ are equated then we get a magnitude condition.

$$|G(s)H(s)| = |-1 + j0| = 1$$

So magnitude condition is, $$\left| G(s)H(s) \right|_{\substack{\text{at a point in s - plane} \\ \text{which is on root locus}}} = 1$$

At any point in s-plane, using magnitude condition we can find the value of K. But use of magnitude condition totally depends on the existence of a point on the root locus.

So magnitude condition can be used only when a point in s-plane is confirmed for its existence on the root locus by use of angle condition.

Refer example where $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$ *and* $s = -0.75$ *is confirmed to be on the root locus. Now we are interested in knowing that at what value of K,* $s = -0.75$ *is one of the roots of* $1 + G(s)H(s) = 0$*. Use the magnitude condition.*

Solution :

$$|G(s)H(s)|_{\text{at } s = -0.75} = 1$$

$$\frac{|K|}{|-0.75|\,|1.25|\,|3.25|} = 1$$

$$\therefore \qquad K = 3.0468$$

In this case, $1 + G(s)H(s) = 0$ means $1 + \dfrac{K}{s(s+2)(s+4)} = 0$ i.e.

$s^3 + 6s^2 + 8s + K = 0$ is a cubic equation. But by use of angle and magnitude conditions one after the other we have decided that for $K = 3.0468$, one of the three roots is located at $s = -0.75$. The remaining two roots then can be easily obtained.

Rules for Construction of Root Locus

Rule No. 1 : The root locus is always symmetrical about the real axis. The roots of the characteristic equation are either real or complex conjugates or combination of both. Therefore their locus must be symmetrical about the real axis of the s-plane.

Rule No. 2 : Let G(s)H(s) = Open loop T.F. of the system

P = Number of open loop poles

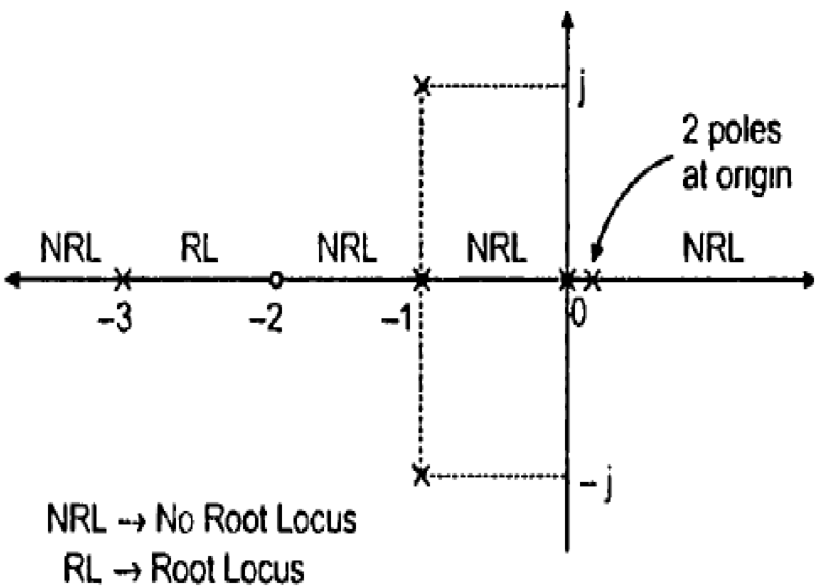
Z = Number of open loop zeros

Case (i) $P > Z$	Case (ii) $Z > P$
Number of branches equal to number of open loop poles $N = P$	Number of branches equal to the number of open loop zeros. $N = Z$
Branches will start from each of the location of open loop pole. Out of 'P' number of branches, 'Z' number of branches will terminate at the locations of open loop zeros. The remaining 'P – Z' branches will approach to infinity e.g. : If $P = 4$ and $Z = 1$ then number of root locus branches = 4, number $P - Z = 3$. 4 branches will start from locations of open loop poles, out of this only one will terminate at the available finite open loop zero location. The remaining $P - Z = 3$ branches will approach to ∞	Branches will terminate at each of the finite location of open loop zero. But out of 'Z' number of branches, 'P' number of branches will start from each of the finite open loop pole locations while remaining $Z - P$ number of branches will originate from infinity and will approach to finite zeros. e.g. : If $P = 1$ and $Z = 4$ then number of separate branches = $Z = 4$, number of $Z - P$ branches = 3. 3 branches will start from infinity while 1 branch will start from location of open loop pole and all 4 branches will terminate at available 4 finite locations of zeros.

Rule No. 3 : A point on the real axis lies on the root locus if the sum of the number of open loop poles and the open loop zeros, on the real axis, to the right hand side of this point is odd.

Example : $G(s)H(s) = \dfrac{K(s+2)}{s^2(s^2+2s+2)(s+3)}$. *Find the sections of real axis which belongs to the root locus.*

Poles are at 0, 0, –1 ± j, –3. [Zero is at –2.]



For positive real axis, there is no pole and zero to right hand side so sum is zero and hence there is no root locus.

For next section between s = 0 to s = –2, to the right hand side sum is 2 which is even hence there is no root locus. Complex conjugate roots should not be considered while using this rule. For next section between s = – 2 and s = –3, to right hand side sum is 3, odd hence full section is part of the root locus. While section to left of s = – 3 there is no root locus.

Rule No. 4 : Generally number of poles are more than number of zeros and in such case `P–Z' branches will approach to infinity. This rule gives us information about how these branches approach to infinity.

The branches which are approaching to infinity, do so along the straight lines called **Asymptotes** of the root locus. Asymptotes are the guidelines for the branches approaching to infinity. Angles of such asymptotes are given by ,

$$\boxed{\theta = \frac{(2q+1)\,180^\circ}{P-Z} \qquad \text{where } q = 0, 1, 2 \ldots\ldots (P - Z - 1)}$$

Asymptotes are always symmetrically located about real axis.

Rule No. 5 : Now only the angles of asymptotes are not sufficient but where the asymptotes are located in s-plane is equally important. Location of asymptotes in s – plane is given by this rule.

All the asymptotes intersect the real axis at a common point known as centroid denoted by σ. The co-ordinates of centroid can be calculated as,

$$\boxed{\sigma = \frac{\sum \text{Real parts of poles of } G(s)H(s) - \sum \text{Real parts of zeros of } G(s)H(s)}{P - Z}}$$

Centroid is always real, it may be located on negative or positive real axis. It may or may not be the part of the root locus.

Example : *For* $G(s)H(s) = \dfrac{K}{(s+1)(s+2+j2)(s+2-j2)}$, *calculate angles of asymptotes and the centroid.*

Solution : P = 3, Z = 0, N = P = 3

$$P - Z = 3 \text{ branches are approaching to infinity.}$$

Poles located at $s = -1, -2 \pm j2$

Angles of asymptotes are given by,

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z} \qquad q = 0, 1, 2$$

Number of asymptotes = Number of branches approaching to infinity.

For $q = 0$, $\theta = \dfrac{180^\circ}{3} = 60^\circ$

$q = 1$, $\theta_1 = \dfrac{(2+1)\,180^\circ}{3} = 180^\circ$

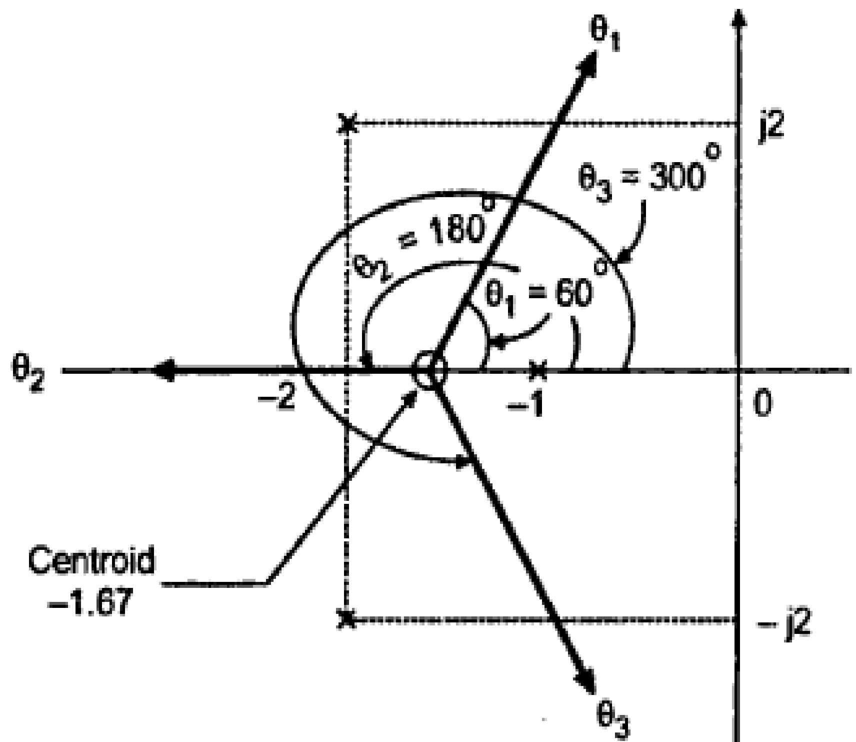
$q = 2$, $\theta_2 = \dfrac{(2\times 2+1)\,180^\circ}{3} = 300^\circ$

All these asymptotes are going to intersect at a common point on real axis called centroid.

$$\sigma = \frac{\sum \text{Real parts of poles} - \sum \text{Real parts of zeros}}{P-Z}$$

$$= \frac{-1-2-2-0}{3} = \frac{-5}{3} = -1.67$$

Centroid and angles of asymptotes are shown in the Fig.



P – Z	Number of asymptotes required	Angles of asymptotes
0	0	--
1	1	$\theta_1 = 180°$
2	2	$\theta_1 = 90°$, $\theta_2 = 270°$
3	3	$\theta_1 = 60°$, $\theta_2 = 180°$, $\theta_3 = 300°$
4	4	$\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$

***Rule No. 6 :* Breakaway Point :**

Breakaway point is a point on the root locus where multiple roots of the characteristic equation occurs, for a particular value of K.

Such a point where two or more roots occur for a particular value of K is called **Breakaway point**. The root locus branches always leave breakaway points at an angle of $\pm\frac{180°}{n}$ where n = number of branches approaching at breakaway point.

As breakaway point indicates values of multiple root, it is **always on the root locus.**

General predictions about existence of breakaway points :

1) *If there are adjacently placed poles on the real axis and the real axis between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed poles.*

Consider example $\quad G(s)H(s) = \dfrac{K}{s(s+2)}$

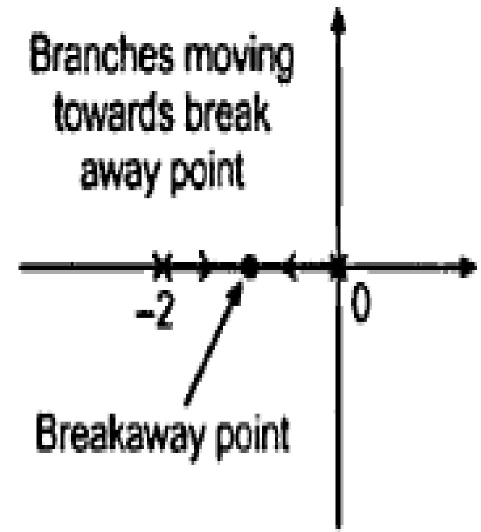
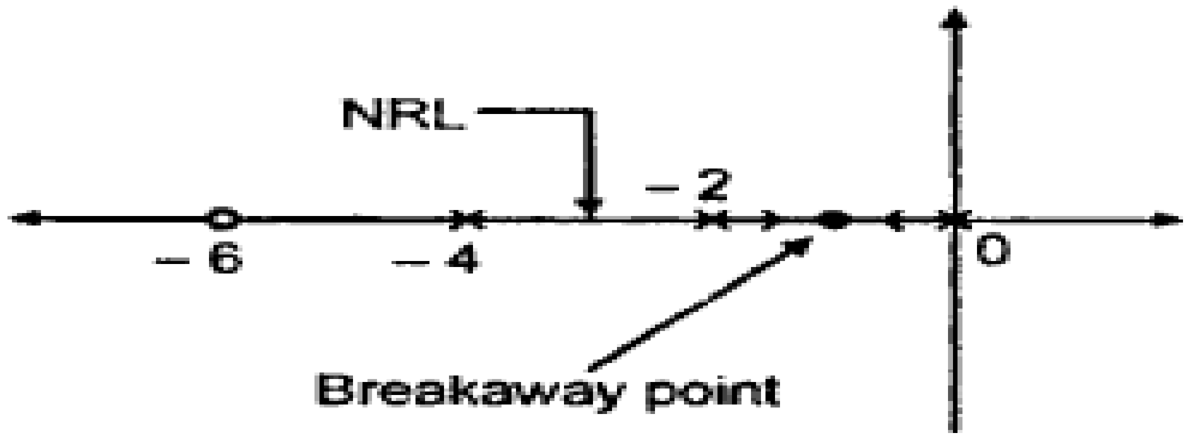


Fig.

s = 0 and s = −2 are adjacently placed poles on real axis and according to rule 3, section between them is a part of the root locus hence there must exist at least one breakaway point in between them.

Example : *For* $G(s)H(s) = \dfrac{K(s+6)}{s(s+2)(s+4)}$ *how many minimum breakaway points exist ?*



branches are approaching towards breakaway point from the poles.

2) *If there are two adjacently placed zeros on real axis and section of real axis in between them is a part of the root locus then there exists minimum one breakaway point in between adjacently placed zeros.*

Example : *For* $G(s)H(s) = \dfrac{K(s+2)(s+4)}{s^2(s+6)}$ *, how many minimum breakaway points exist ?*

Solution : Consider the pole-zero plot as shown in the Fig.

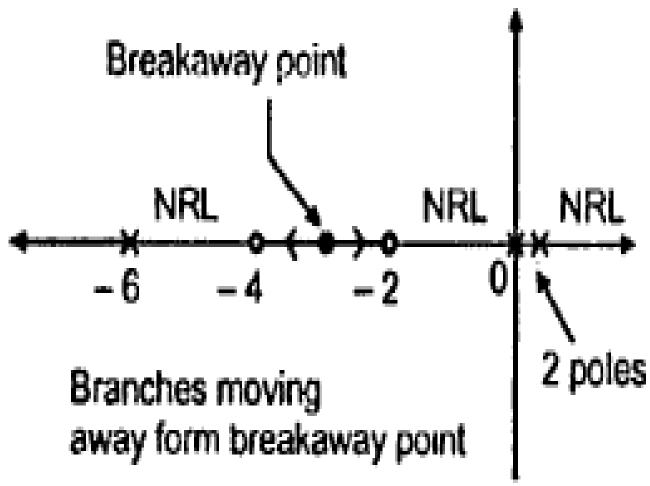


Fig.

3) *If there is a zero on the real axis and to the left of that zero there is no pole or zero existing on the real axis and complete real axis to the left of this zero is a part of the root locus then there exists minimum one breakaway point to the left of that zero.*

Consider open loop transfer function,

$$G(s)H(s) = \frac{K(s+2)(s+4)}{s(s^2+2s+20)}$$

An open loop zero $s = -4$ satisfies all the conditions and hence minimum one breakaway point exists to the left of this zero.

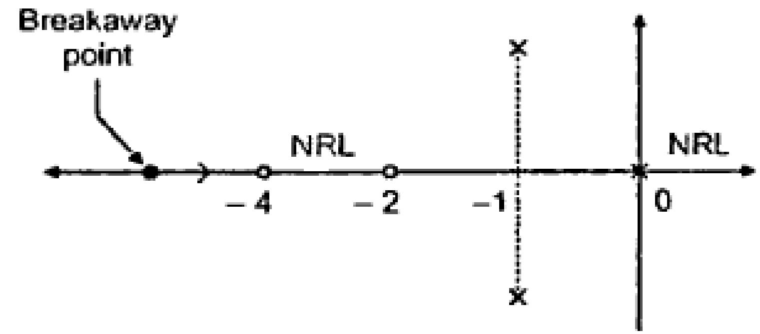


Fig.

Determination of Breakaway Points

Steps to determine the co-ordinates of breakaway points are,

Step 1 : Construct the characteristic equation $1 + G(s)H(s) = 0$ of the system.

Step 2 : From this equation, separate the terms involving 'K' and terms involving 's'. Write the value of K in terms of s.

$$K = f(s)$$

Step 3 : Differentiate above equation w.r.t. 's', equate it to zero.

$$\frac{dK}{ds} = 0$$

Step 4 : Roots of the equation $\frac{dK}{ds} = 0$ gives us the breakaway points.

Example : *For* $G(s)H(s) = \dfrac{K}{s(s+1)(s+4)}$, *determine the co-ordinates of valid breakaway points.*

Solution : Characteristic equation $1 + G(s)H(s) = 0$

Step 1 : $1 + \dfrac{K}{s(s+1)(s+4)} = 0$ i.e. $s^3 + 5s^2 + 4s + K = 0$

Step 2 : $K = -s^3 - 5s^2 - 4s$

Step 3 : $\dfrac{dK}{ds} = -3s^2 - 10s - 4 = 0$

Step 4 : $3s^2 + 10s + 4 = 0$

$\therefore$ Breakaway points $= \dfrac{-10 \pm \sqrt{100 - 4 \times 4 \times 3}}{2 \times 3} = -0.46, -2.86$

Substituting in expression for K

For $s = -0.46$, $K = +0.8793$

For $s = -2.86$, $K = -6.064$

$\therefore$ For $s = -0.46$, K is positive.

i.e. $s = -0.46$ is valid breakaway point for the root locus.

Root locus approaches and leaves breakaway point at an angle $\pm\dfrac{180^0}{n}$.

Here number of branches approaching = 2

$\therefore$ Angle of approaching = $\pm 90^0$ i.e. $\pm\dfrac{\pi}{2}$

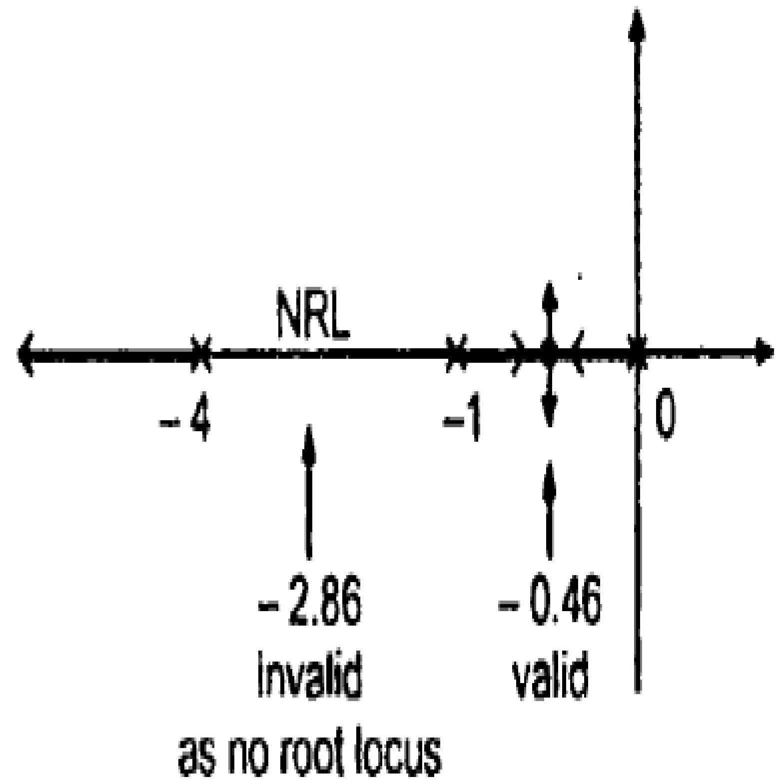


Fig.

Rule No. 7 : Intersection of root locus with imaginary axis. This can be determined by following procedure.

Step 1 :	Consider characteristic equation $1 + G(s)H(s) = 0$ as obtained in Rule 6.
Step 2 :	Construct Routh's array in terms of "K".
Step 3 :	Determine $K_{marginal}$ i.e. value of K which creates one of the rows of Routh's array as row of zeros, except the row of s^0.
Step 4 :	Construct auxiliary equation $A(s) = 0$ by using coefficients of a row which is just above the row of zeros.
Step 5 :	Roots of auxiliary equation $A(s) = 0$ for $K = K_{mar}$ are nothing but the intersection points of the root locus with imaginary axis.

Consider example :

$$G(s)H(s) = \frac{K}{s(s+1)(s+4)}$$

Characteristic equation is given by,

$$1 + G(s)H(s) = 1 + \frac{K}{s(s+1)(s+4)} = 0$$

i.e. $s^3 + 5s^2 + 4s + K = C$

Routh's array,

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 5 & K \\ s^1 & \frac{20-K}{5} & 0 \\ s^0 & K & \end{array}$$

$K_{mar} = 20$ that makes row corresponding to s^1 as row of zeros.

$$\therefore \quad A(s) = 5s^2 + K = 0$$

$$K = K_{mar} = 20$$

$$5s^2 + 20 = 0$$

$$s^2 = -4 \quad \therefore \quad s = \pm j2$$

If K_{mar} is positive, root locus intersects with imaginary axis. But if K_{mar} is negative root locus does not intersect with imaginary axis and lies totally in left half of s-plane.

Rule No. 8 : Angle of departure at complex conjugate poles and angle of arrival at complex conjugate zeros.

Angle of departure at complex pole :

As branch always leaves from an open loop pole, it is advantageous to know at what angle it departs from complex conjugate pole. This angle at which it departs from complex pole is called **angle of departure denoted as ϕ_d.**

$$\phi_d = 180^\circ - \phi \qquad \text{where } \phi = \sum \phi_P - \sum \phi_Z$$

where $\sum \phi_P$ = Contributions by the angles made by remaining open loop poles at the pole at which ϕ_d is to be calculated.

$\sum \phi_Z$ = Contributions by the angles made by the open loop zeros at the pole at which ϕ_d is to be calculated.

To calculate $\Sigma \phi_P$, join all the remaining poles to the complex pole under consideration. Add all the angles subtended by phasors joining poles to pole under consideration. Similarly join all zeros to pole under consideration and adding all angles determine $\Sigma \phi_Z$.

Example : *For* $G(s)H(s) = \dfrac{K(s+2)}{s(s+4)(s^2+2s+2)}$, *calculate angles of departures at complex conjugate poles.*

Solution : $P = 4$, $Z = 1$

Poles are at $s = 0, -4, -1 \pm j$

Zero at $s = -2$.

Draw Pole-Zero plot.

Let us calculate ϕ_d at the pole $s = -1 + j$.

Join all other poles to this pole and measure or calculate the angles $\phi_{P1}, \phi_{P2}, \phi_{P3}$ as shown in the Fig. 9.17.

Join all zeros to this pole and calculate ϕ_{Z1}.

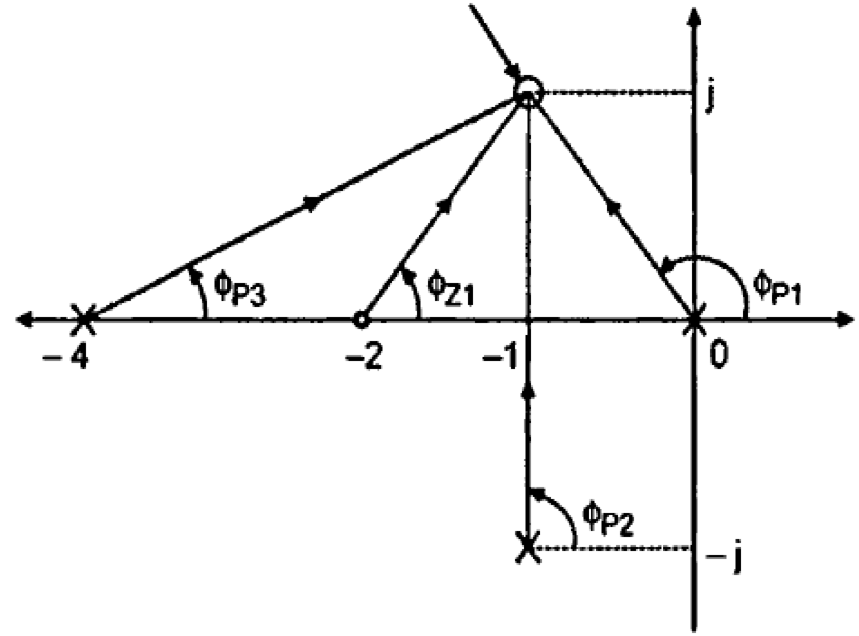


Fig. 9.17

Then , $\quad \sum \phi_P = \phi_{P1} + \phi_{P2} + \phi_{P3}$ while

$$\sum \phi_Z = \phi_{Z1}$$

$$\phi_{P1} = 135°,\quad \phi_{P2} = 90°,\quad \phi_{P3} = 18.43°$$

$$\sum \phi_P = 135° + 90° + 18.43° = 243.43°$$

$$\sum \phi_Z = \phi_{Z1} = 45°$$

$$\phi = \sum \phi_P - \sum \phi_Z = 243.43° - 45° = 198.43°$$

$$\phi_d = 180° - \phi = 180° - 198.43° = -18.43°$$

$\therefore$ Root locus branch leaving this pole will depart tangentially to the line whose angle is given by $\phi_d = -18.43°$ as shown in the Fig. .

For second complex conjugate pole, sign of ϕ_d will be just opposite as root locus is always symmetrical about real axis. So root locus branch departing from $s = -1 - j$ will depart tangentially to the line whose angle is given by $\phi_d = +18.43°$.

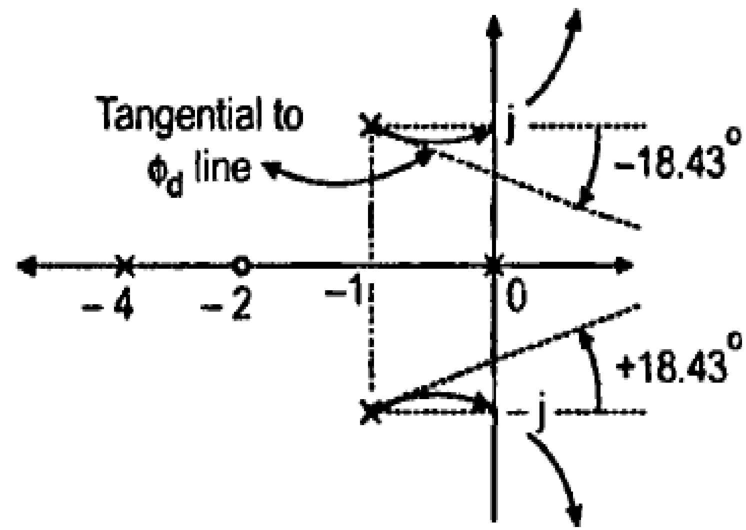


Fig.

Angle of arrival at a complex zero :

Angle of arrival at a complex zero can be calculated by the same method, which is denoted as ϕ_a. The only change to calculate the angle of arrival is,

$$\boxed{\begin{aligned} \phi_a &= 180^\circ + \phi \\ \text{where} \quad \phi &= \Sigma\phi_P - \Sigma\phi_Z \end{aligned}}$$

Such branches will arrive and terminate at the complex zeros running tangentially to the lines whose angles are given by ϕ_a as explained above.

General Steps to Solve the Problem on Root Locus

Step 1 : Get the general information about number of open loop poles, zeros, number of branches etc. from G(s)H(s) .

Step 2 : Draw the pole-zero plot. Identify sections of real axis for the existence of the root locus. And predict minimum number of breakaway points by using general predictions.

Step 3 : Calculate angles of asymptotes.

Step 4 : Determine the centroid. Sketch a separate sketch for step 3 and step 4.

Step 5 : Calculate the breakaway and breakin points. If breakaway points are complex conjugates, then use angle condition to check them for their validity as breakaway points.

Step 6 : Calculate the intersection points of root locus with the imaginary axis.

Step 7 : Calculate the angles of departures or arrivals if applicable.

Step 8 : Combine steps 1 to 7 and draw the final sketch of the root locus.

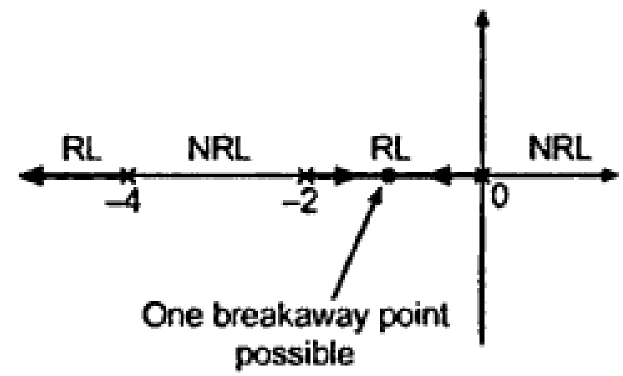
Step 9 : Predict the stability and performance of the given system by using the root locus.

Example : *For a unity feedback system,* $G(s) = \dfrac{K}{s(s+4)(s+2)}$*. Sketch the rough nature of the root locus showing all details on it. Comment on the stability of the system.*

(M.U. : June-92)

Solution : Step 1 : General information from $G(s)H(s) = \dfrac{K}{s(s+2)(s+4)}$

P = 3, Z = 0, number of branches N = P = 3. No finite zero so all P − Z = 3 branches will terminate at infinity. Starting points are locations of open loop poles i.e. 0, − 2, − 4.



Step 2 : Pole-Zero plot and sections of real axis.

Directions of branches away from poles. One breakaway point exists between 0 and − 2 according to general prediction.

Sections of real axis identified as a part of the root locus as to right side sum of poles and zeros is odd for those sections.

Step 3 : Angles of asymptotes.

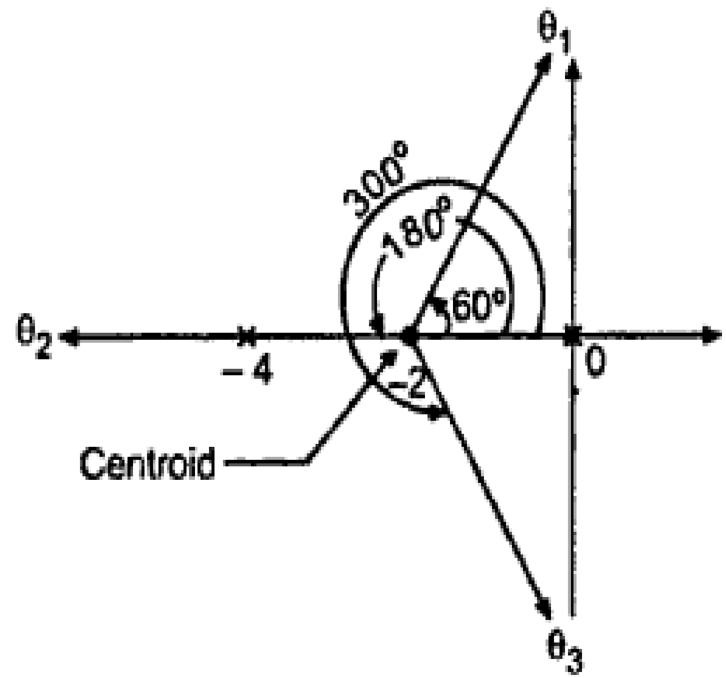
3 branches are approaching to ∞ , 3 asymptotes are required.

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z}, \qquad q = 0, 1, 2$$

$$\therefore\ \theta_1 = \frac{180^\circ}{3} = 60^\circ,\ \theta_2 = \frac{(2+1)\,180^\circ}{3} = 180^\circ,\ \theta_3 = \frac{(2\times 2+1)\,180^\circ}{3} = 300^\circ$$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R.P. of zeros}}{P - Z} = \frac{0-2-4}{3} = -2$$



Branches will approach to ∞ along these lines which are asymptotes.

Step 5 : To find breakaway point (Refer Rule No. 6). Characteristic equation is

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s\,(s+2)\,(s+4)} = 0$$

$$\therefore \qquad s^3 + 6\,s^2 + 8\,s + K = 0$$

$$\therefore \qquad K = -\,s^3 - 6\,s^2 - 8\,s \qquad \text{... (1)}$$

$$\frac{dK}{ds} = -\,3\,s^2 - 12\,s - 8 = 0$$

$$\text{i.e.} \qquad 3\,s^2 + 12\,s + 8 = 0$$

$$\text{Roots i.e. breakaway points} = \frac{-12 \pm \sqrt{144 - 4 \times 3 \times 8}}{2 \times 3} = -\,0.845\,,\,-\,3.15$$

As there is no root locus between – 2 to – 4 , – 3.15 cannot be a breakaway point. It also can be confirmed by calculating `K' for s = – 3.15. It will be negative that confirms s = – 3.15 is not a breakaway point.

For s = – 3.15, K = – 3.079 (Substituting in equation for K)

But as there has to be breakaway point between `0' and `– 2' , s = – 0.845 is vaild breakaway point.

For s = – 0.845 K = + 3.079

As K is positive s = **– 0.845 is vaild breakaway point.**

Step 6 : Intersection point with imaginary axis.

Characteristic equation

$$s^3 + 6s^2 + 8s + K = 0$$

Routh's array

$$\begin{array}{c|cc} s^3 & 1 & 8 \\ s^2 & 6 & K \\ s^1 & \dfrac{48-4}{6} & 0 \\ s^0 & K & \end{array}$$

$$K_{marginal} = 48 \text{ which makes row of } s^1 \text{ as row of zeros.}$$

$$A(s) = 6s^2 + K = 0$$

$$K_{mar} = 48$$

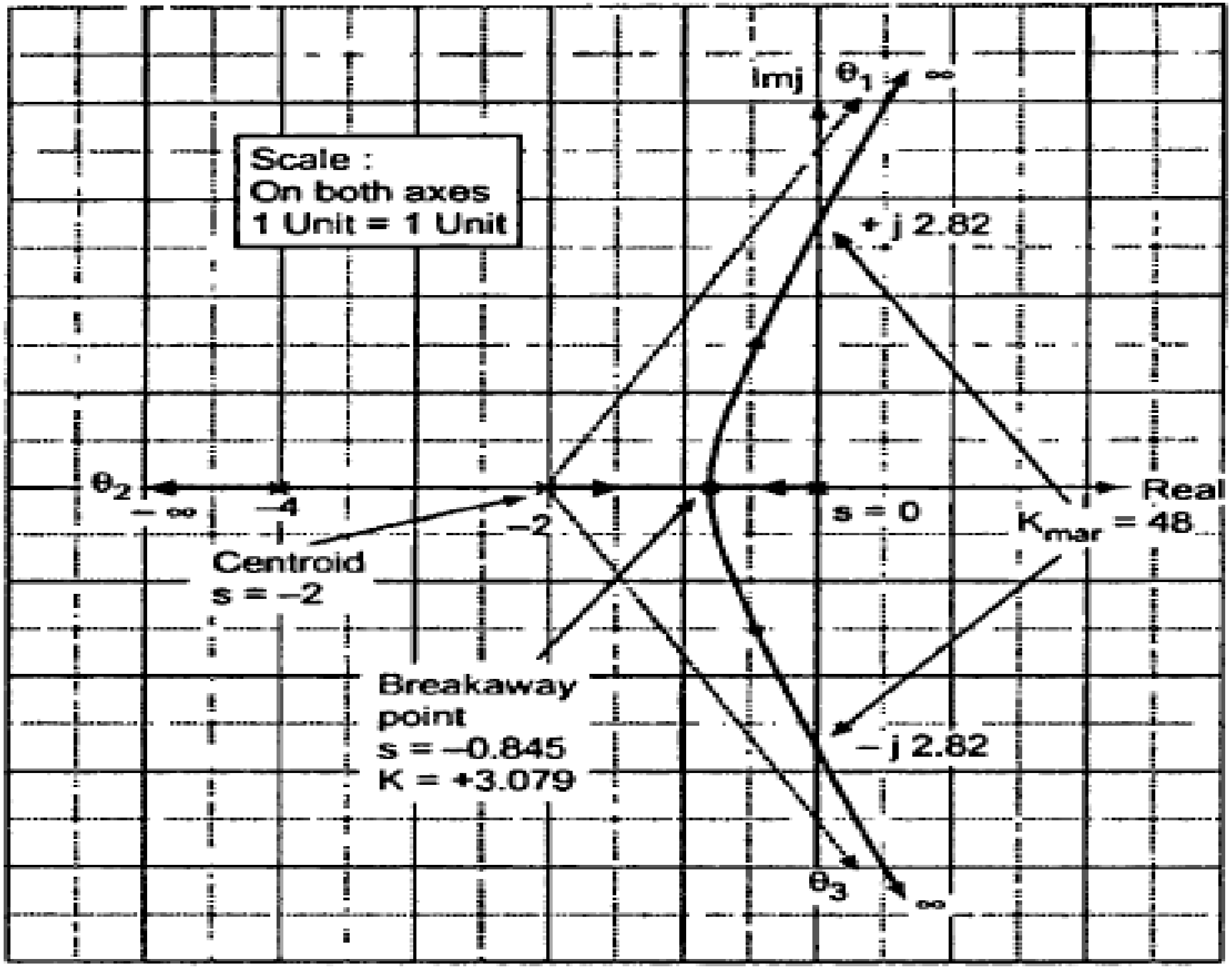
$$\therefore\ 6s^2 + 48 = 0$$

$$s^2 = -8$$

$$\therefore\quad s = \pm j\sqrt{8} = \pm j2.828$$

Intersection of root locus with imaginary axis is at $\pm$ j 2.828 and corresponding value of K_{mar} = 48.

Step 7 : As there are no complex conjugate poles or zeros, no angles of departures or arrivals are required to be calculated.



For $0 < K < 48$, all the roots are in left half of s–plane hence system is absolutely stable. For $K_{mar} = +48$, a pair of dominant roots on imaginary axis with remaining root in left half. So system is marginally stable oscillating at 2.82 rad/sec. For $48 < K < \infty$, dominant roots are located in right half of s-plane hence system is unstable.

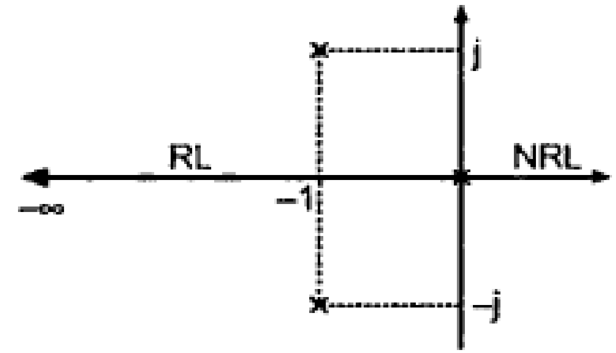
Example : *Sketch the root locus for the system having* $G(s)H(s) = \dfrac{K}{s(s^2+2s+2)}$.

Solution : Step 1 : P = 3, Z = 0, N = P = 3

P – Z = 3 branches approaching to ∞ . Starting points open loop poles,

s = 0, s = – 1 + j , s = –1 – j.

Step 2 : Pole-Zero plot and sections of real axis.



One branch is approaching to – ∞ starting from pole at s = 0. No breakaway point exists according to general predictions.

Step 3 : Angles of asymptotes : 3 branches approaching to ∞, 3 asymptotes required

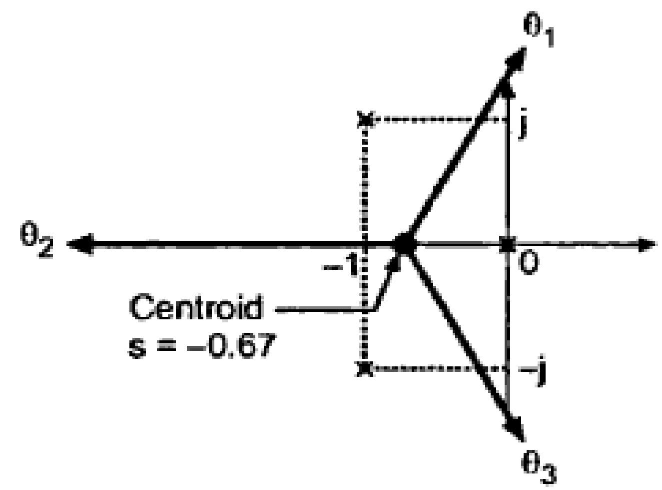
$$\theta = \frac{(2q+1)\,180^\circ}{P-Z}, \quad q = 0, 1, 2.$$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ,\ \theta_2 = \frac{(2+1)\,180^\circ}{3} = 180^\circ, \theta_3 = \frac{(2\times 2+1)\,180^\circ}{3} = 300^\circ$$

Step 4 : Centroid : $\sigma = \dfrac{\sum \text{R. P. of poles} - \sum \text{R. P. of zeros}}{P-Z} = \dfrac{0-1-1-0}{3}$

$$= -\frac{2}{3} = -\mathbf{0.67}$$

One branch approaching to ∞ along θ_2 while remaining two branches starting from $-1+j$ and $-1-j$ will approach to ∞ along θ_1 and θ_3 respectively.



Step 5 : Breakaway point

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s^2+2s+2)} = 0$$

$$s^3 + 2s^2 + 2s + K = 0$$

$$\therefore \qquad K = -s^3 - 2s^2 - 2s$$

$$\frac{dK}{ds} = -3s^2 - 4s - 2 = 0$$

$$\therefore \qquad 3s^2 + 4s + 2 = 0$$

$$\therefore \text{ Breakaway points are } \frac{-4 \pm \sqrt{16-24}}{2 \times 3} = -0.67 \pm j\,0.4714$$

Now instead of substituting $-0.67 \pm j\,0.4714$ in expression for K we can check validity of these points as breakaway points by using angle condition.

The point which satisfies angle condition is on the root locus and point on root locus satisfying $\frac{dK}{ds} = 0$ is nothing but breakaway point.

Let us test $\quad s = -0.67 + j\,0.4714$

$$\angle G(s)H(s) = \pm(2q+1)\,180^\circ, \qquad q = 0, 1, 2 \,.......$$

$$G(s)H(s) = \frac{K}{s(s+1+j)(s+1-j)}$$

at $\quad s = -0.67 + j\,0.4714$

$$\angle G(s)H(s) = \frac{\angle K + j0}{\angle -0.67 + j0.4714 \;\; \angle -0.67 + j0.4714 + 1 + j \;\; \angle -0.67 + j0.4714 + 1 - j}$$

$$= \frac{\angle K + j0}{\angle -0.67 + j0.4714 \;\; \angle 0.33 + 1.47j \;\; \angle 0.33 - j0.53}$$

$$= \frac{0^\circ}{\angle 144.87^\circ \;\; \angle 77.34^\circ \;\; \angle -58.09^\circ} = -164.11^\circ$$

This is not odd multiple of 180°. Hence point is not on the root locus and hence there is no breakaway point existing for this system.

Step 6 : Intersection with imaginary axis.

Characteristic equation : $s^3 + 2s^2 + 2s + K = 0$

Routh's array

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 2 & K \\ s^1 & \frac{4-K}{2} & 0 \\ s^0 & K & \end{array}$$

$K_{mar} = +4$ makes row of $s^1 = 0$

$A(s) = 2s^2 + K = 0$

At $K_{mar} = 4$

$2s^2 + 4 = 0$

$s^2 = -2 \quad \therefore \quad s = \pm j\, 1.414$

Step 7 : Angle of departure : As branch is departing at $-1 + j$ let us calculate angle of departure, at $-1 + j$.

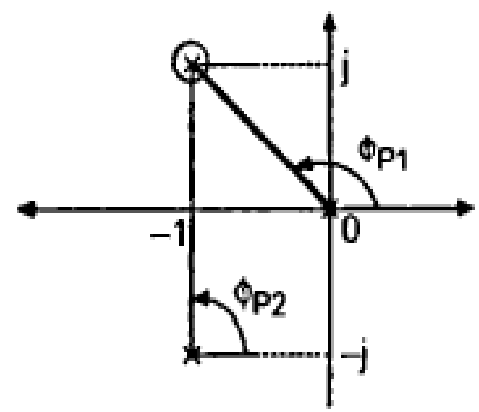
$\phi_{P1} = 135^\circ$, $\phi_{P2} = 90^\circ$

$\Sigma\phi_P = \phi_{P1} + \phi_{P2} = 225^\circ$, $\Sigma\phi_Z = 0$

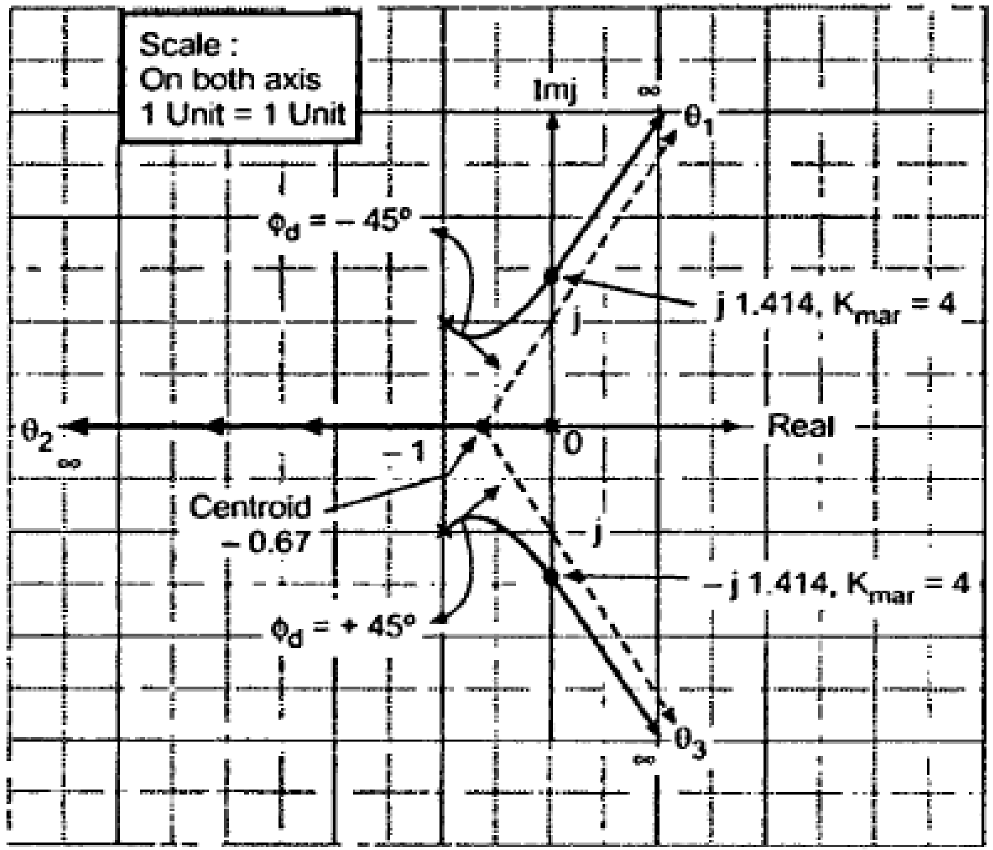
$\therefore \quad \phi = \Sigma\phi_P - \Sigma\phi_Z = 225^\circ$

$\therefore \quad \phi_d = 180^\circ - \phi = 180^\circ - 225^\circ = -45^\circ$

At $-1-j$, $\quad \phi_d = +45^\circ$



Step 8 : Complete Root Locus is :



Step 9 : Comment on stability :

For 0 < K < 4 all roots are in left half of s–plane. System is absolutely stable.

At K = + 4, dominant roots are on imaginary axis, system is marginally stable, oscillating with 1.414 rad/sec.

At K > 4, dominant roots are in right half of s-plane and hence system becomes unstable in nature.

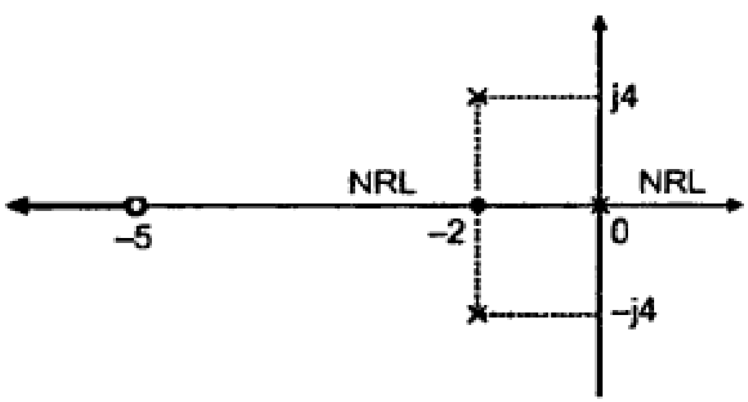
Example : *Sketch the complete root locus for the system having* $G(s)H(s) = \dfrac{K\,(s+5)}{(s^2 + 4\,s + 20)}$.

Solution : Step 1 : Number of poles P = 2, Z = 1 , N = P = Z

One branch has to terminate at finite zero s = – 5 while P – Z = 1 branch has to terminate at ∞. Starting points of branches are,

$$\frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j\,4$$

Step 2 : Pole - Zero plot and sections of real axis are as in following figure.



Step 3 : Angles of asymptotes

One branch approaches to ∞ so one asymptote is required.

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z}, \quad q = 0$$

$$\therefore \qquad \theta_1 = 180^\circ$$

Branch approaches to ∞ along $+180^\circ$ i.e. negative real axis.

Step 4 : Centroid

As there is only one branch approaching to ∞ and one asymptote exists, centroid is not required.

Step 5 : Breakaway point

Characteristic equation : $1 + G(s)H(s) = 0$

$$1 + \frac{K(s+5)}{(s^2+4s+20)} = 0$$

$$\therefore \qquad s^2+4s+20+Ks+5K = 0$$

$$\therefore \qquad s^2+4s+20+K(s+5) = 0$$

$$\therefore \qquad K = \frac{-s^2-4s-20}{(s+5)} \qquad \ldots (1)$$

Now

$$\frac{dK}{ds} = \frac{vu' - uv'}{v^2} = 0$$

$$= (s+5)(-2s-4) - (-s^2-4s-20)(1) = 0$$

$$= -2s^2 - 14s - 20 + s^2 + 4s + 20 = 0$$

i.e. $\qquad -s^2 - 10\,s = 0$

$\therefore \qquad -s\,(s+10) = 0$

s = 0 and s = – 10 are breakaway points. But s = 0 cannot be breakaway point as for s = 0, K = – 4.

For s = – 10 , $\quad K = \dfrac{-100+40-20}{-10+5} = +16$

Hence **s = – 10 is valid breakaway point.**

Step 6 : Intersection with imaginary axis.

Characteristic equation

$$s^2 + 4\,s + 20 + K\,s + 5\,K = 0$$

$$s^2 + s\,(K+4) + (20+5\,K) = 0$$

Routh's array

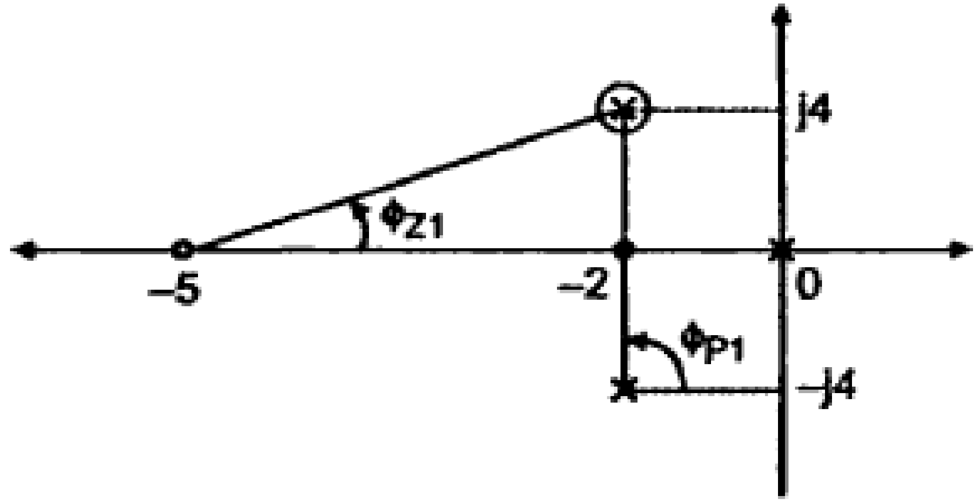
$$\begin{array}{c|ll} s^2 & 1 & 20 + 5\,K \\ s^1 & K + 4 & 0 \\ s^0 & 20 + 5\,K & \end{array}$$

$K_{mar} = \quad -4$ makes s^1 row as row of zeros.

But as it is negative, there is no intersection of root locus with imaginary axis.

Step 7 : Angle of departure

Consider – 2 + j4 join remaining pole and zero to it.



$$\phi_{P1} = 90°, \quad \phi_{Z1} = \tan^{-1}\frac{4}{3} = 53.13°$$

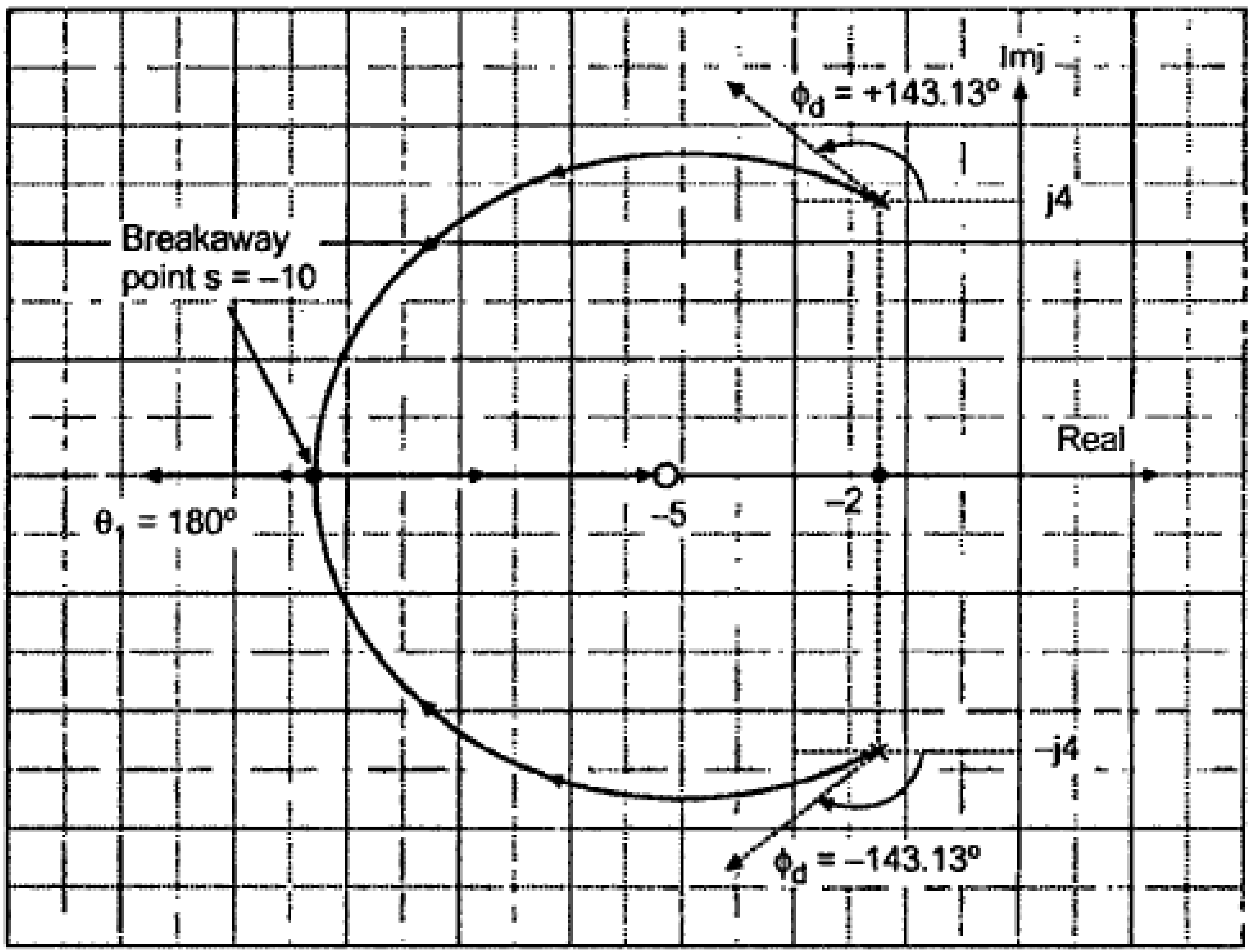
$$\Sigma\phi_{P} = 90°, \quad \Sigma\phi_{Z} = 53.13°$$

$$\therefore \quad \phi = \Sigma\phi_{P} - \Sigma\phi_{Z} = 36.86°$$

$$\therefore \quad \phi_{d} = 180° - \phi = +143.13° \quad \text{at} - 2 + j\,4 \text{ pole}$$

$$\phi_{d} = -143.13° \quad \text{at} - 2 - j\,4 \text{ pole.}$$

Step 8 : Complete Root Locus is using following figure.



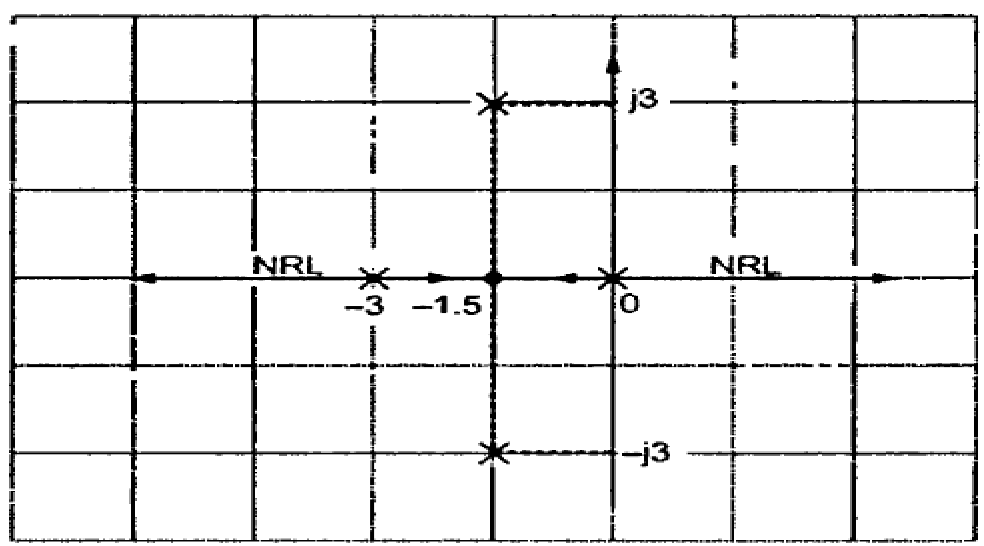
$$G(s)H(s) = \frac{K}{s(s+3)(s^2+3s+11.25)}$$

Solution : Step 1 : P = 4 , Z = 0 , N = 4. All branches approaching to ∞. Starting points s = 0, – 3 and –1.5 ± j3.

Step 2 : Pole - Zero plot is as follows.

Section between 0 and –3 is part of root locus.

Step 3 : Angles of asymptotes



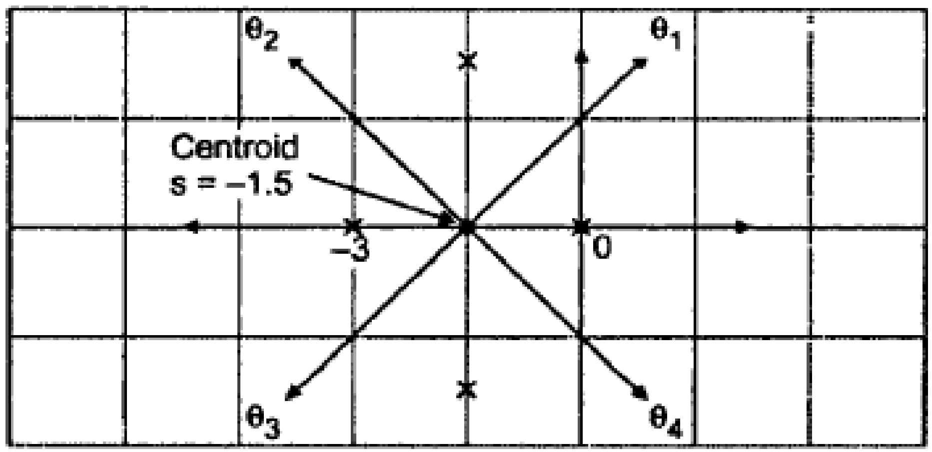
As number of branches approaching to ∞ are 4, angles are $\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$.

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R. P. of poles} - \sum \text{R. P. of zeros}}{P - Z} = \frac{0 - 3 - 1.5 - 1.5}{4}$$

$$= -1.5$$

Step 5 : Breakaway points



Characteristic equation : $1 + G(s)H(s) = 0$

$$\therefore \quad 1 + \frac{K}{s(s+3)(s^2 + 3s + 11.25)} = 0$$

$$\therefore \quad s^4 + 6s^3 + 20.25s^2 + 33.75s + K = 0$$

$$\therefore \quad K = -s^4 - 6s^3 - 20.25s^2 - 33.75s$$

$$\therefore \quad \frac{dK}{ds} = -4s^3 - 18s^2 - 40.5s - 33.75 = 0$$

$$\therefore \; 4s^3 + 18s^2 + 40.5s + 33.75 = 0$$

Test s = –1.5,

– 1.5	4	18	40.5	33.75
		– 6	– 18	–33.75
	4	12	22.5	$\boxed{0}$

$$\therefore \quad \frac{dK}{ds} = (s + 1.5)(4s^2 + 12s + 22.5) = 0$$

$$\therefore \quad s = -1.5 \text{ and } \frac{-12 \pm \sqrt{144 - 4 \times 4 \times 22.5}}{2 \times 4}$$

$\therefore$ Three breakaway points are –1.5 , –1.5 ± j 1.8371

At s = –1.5 , Value of K = + 20.25

So s = –1.5 is valid breakaway point.

But to test s = –1.5 ± j 1.8371 , it is difficult to calculate 'K' so we can use angle condition to test their validity.

$$\angle G(s)H(s) = \pm(2q+1)\,180^\circ$$

$$\left.\frac{\angle K + j0}{\angle s \angle s+3 \angle s+1.5+j3 \angle s+1.5-j3}\right|_{\text{at } s = -1.5 + j1.8371}$$

Note : If a section of the real axis is identified for the existence of root locus and a breakaway point is predicted between that section. Try the midpoint of such section for a root of the equation $\frac{dK}{ds} = 0$, first.

$$= \frac{\angle K + j0}{\angle -1.5 + j\,1.8371\ \angle -1.5 + j\,1.8371 + 3\angle -1.5 + j\,1.8371 + 1.5 + j3\angle -1.5 + j\,1.8371 + 1.5 - j3}$$

$$\therefore \qquad \angle G(s)H(s) = \frac{0^\circ}{\angle -1.5 + j\,1.8371\ \ \angle 1.5 + j\,1.8371\ \ \angle j\,4.8371\ \ \angle -j\,1.1629}$$

$$= \frac{0^\circ}{129.23^\circ\ 50.77^\circ\ 90^\circ\ (-90^\circ)}$$

$$= -180^\circ$$

i.e. it satisfies angle condition so both $-1.5 \pm j1.8371$ are valid breakaway points. To find corresponding 'K' use magnitude condition for same point.

$$\left| G(s)H(s) \right|_{\text{at } s = -1.5 \pm j1.8371} = 1$$

Consider a point $-1.5 + j\,3$

$$\phi_{P1} = 180^\circ - \tan^{-1}\frac{3}{1.5} = 116.56^\circ$$

$$\therefore \quad \phi_{P2} = 90^\circ$$

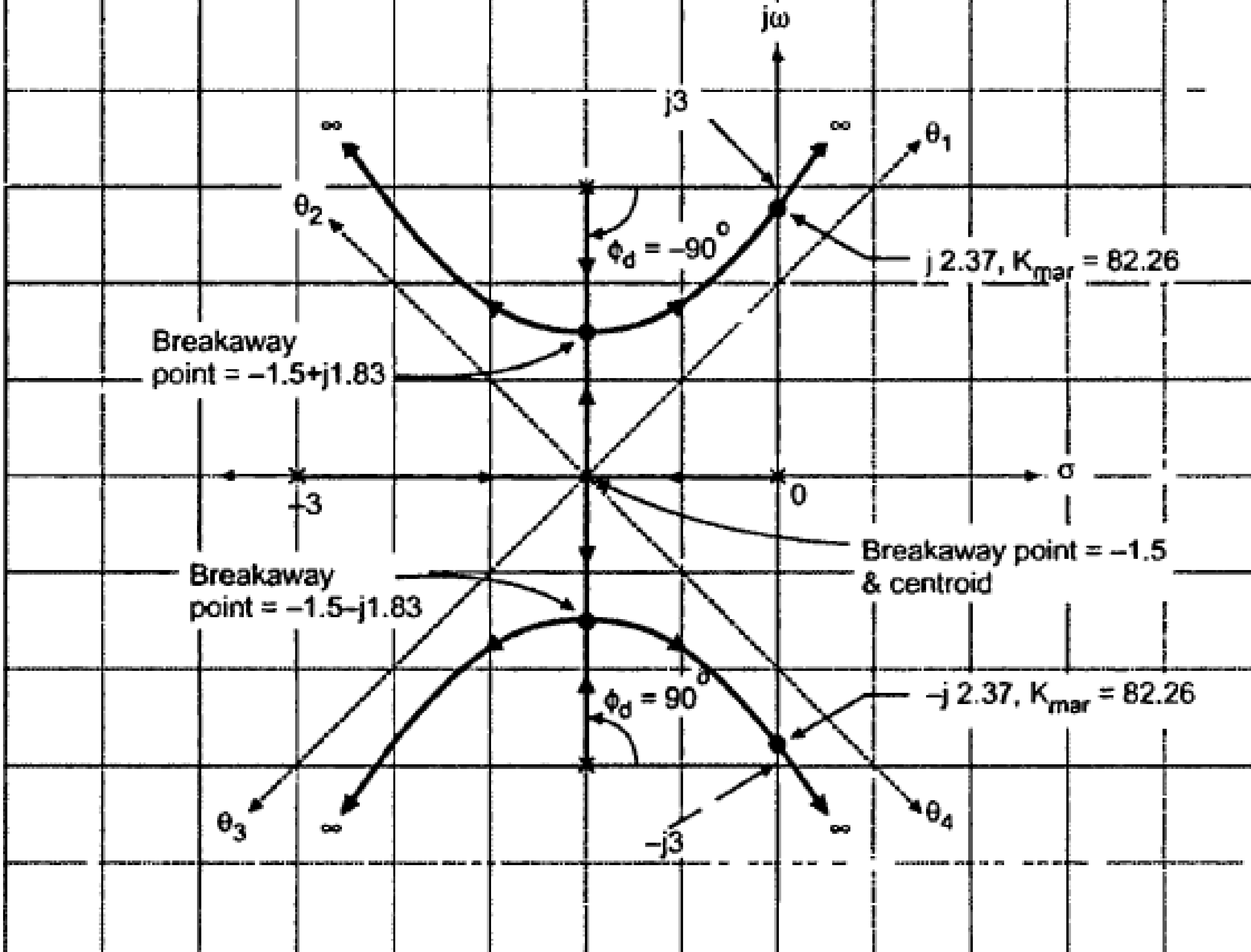
$$\therefore \quad \phi_{P3} = \tan^{-1}\frac{3}{1.5} = 63.43^\circ$$

$$\sum\phi_P = 270^\circ, \quad \sum\phi_Z = 0^\circ$$

$$\phi = \sum\phi_P - \sum\phi_Z = 270^\circ$$

At $-1.5 + j\,3$ $\quad \phi_d = 180^\circ - \phi = -90^\circ$

At $-1.5 + j\,3$ $\quad \phi_d = +90^\circ$



∴ For $0 < K < 82.26$ system is stable.

At $K = 82.26$ system is marginally stable.

$K > 82.26$ system is unstable.

Example *Let us reduce the imaginary part of complex poles such that it is less than distance of s = 0 and − 3 from s = − 1.5.*

$$G(s)H(s) = \frac{K}{s(s+3)(s^2+3s+3)}$$

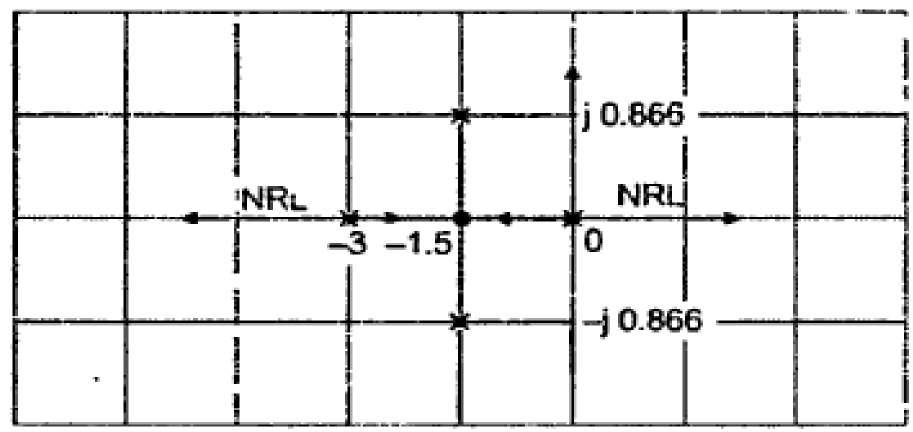
Solution : Step 1 : P = 4, Z = 0, N = 4, All branches approaching to ∞. Starting points $s = 0, -3$ and $\frac{-3 \pm \sqrt{9-12}}{2}$ i.e. $0, -3, -1.5 \pm j\,0.866$.

Step 2 : Pole-Zero plot is as follows.

Minimum one breakaway point exists between 0 and − 3.

Step 3 and 4 : Same as in examples 9.21 and 9.22 as real parts of poles are not changed.

Centroid −1.5 and angles $\theta_1 = 45°$, $\theta_2 = 135°$, $\theta_3 = 225°$, $\theta_4 = 315°$.



Step 5 : Breakaway point

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+3)(s^2+3s+3)} = 0$$

$$s^4 + 6s^3 + 12s^2 + 9s + K = 0$$

$$\therefore \quad K = -s^4 - 6s^3 - 12s^2 - 9s \qquad \text{... (1)}$$

$$\therefore \quad \frac{dK}{ds} = -4s^3 - 18s^2 - 24s - 9 = 0$$

$\therefore \; 4s^3 + 18s^2 + 24s + 9 = 0$

Test $s = -1.5$

-1.5	4	18	24	9
		-6	-18	-9
	4	12	6	$\boxed{0}$

$$\frac{dK}{ds} = (s + 15)(4s^2 + 12s + 6) = 0$$

i.e. $\quad s = -1.5 \text{ and } \dfrac{-12 \pm \sqrt{144 - 16 \times 6}}{2 \times 4}$

$\therefore$ Breakaway points are $= -1.5, -0.633, -2.366$ and all are valid.

For $s = -1.5$, $K = 1.6875$ ... Using equation (1)

$\left.\begin{array}{ll} s = -0.633, & K = 2.25 \\ s = -2.366, & K = 2.25 \end{array}\right\}$ These two occur simultaneously.

Step 6 : Intersection with imaginary axis.

Characteristic equation : $s^4 + 6s^3 + 12s^2 + 9s + K = 0$

Routh's array,

s^4	1	12	K
s^3	6	9	0
s^2	10.5	K	0
s^1	$\dfrac{94.5 - 6K}{10.5}$	0	
s^0	K		

$$\therefore \quad 94.5 - 6K = 0$$

$$K_{mar} = 15.75$$

$$A(s) = 10.5s^2 + K = 0$$

$$\therefore \quad 10.5s^2 + 15.75 = 0$$

$$s^2 = -1.5$$

$$s^2 = -1.5$$

$$\therefore \qquad s = \pm j\,1.224$$

Step 7 : Angle of departure

ϕ_d at $-1.5 + j\,0.866$ is $-90°$. While ϕ_d at $-1.5 - j\,0.866$ is $+90°$.

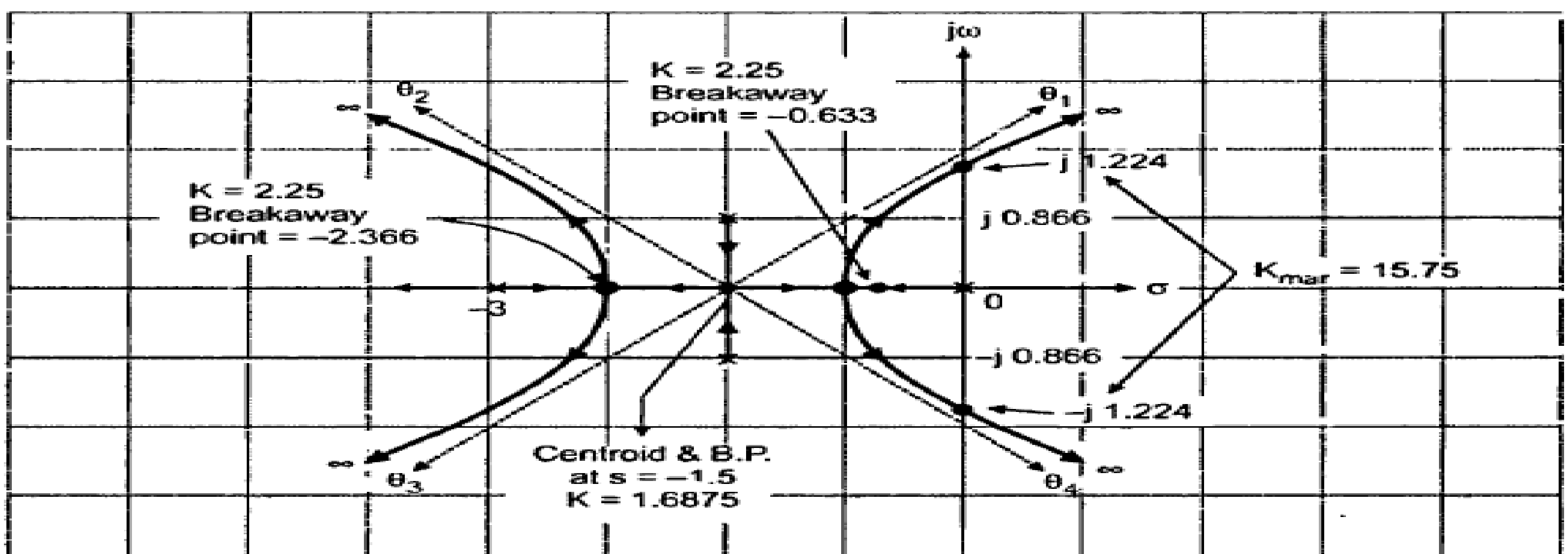
Step 8 : Root locus : In this problem $s = -1.5$ will occur as a breakaway point first as $K = 1.6875$ for this breakaway point. But branches from complex poles will reach at $s = -1.5$ rather than branches from real open loop poles. This is because the complex poles are more closer to $s = -1.5$ than the real poles. The remaining breakaway points will occur later simultaneously. These will exist after the complex poles branches break into real branches at $s = -1.5$.

Step 9 :

For $0 < K < 15.75$ system is stable.

At $K = 15.75$ system is marginally stable.

$K > 15.75$ system is unstable.



HW

Example : $G(s)H(s) = \dfrac{K(s+1)}{s(s-1)(s^2+5s+20)}$

Sketch the complete root locus with approximate indication of breakaway points. Comment on the stability.

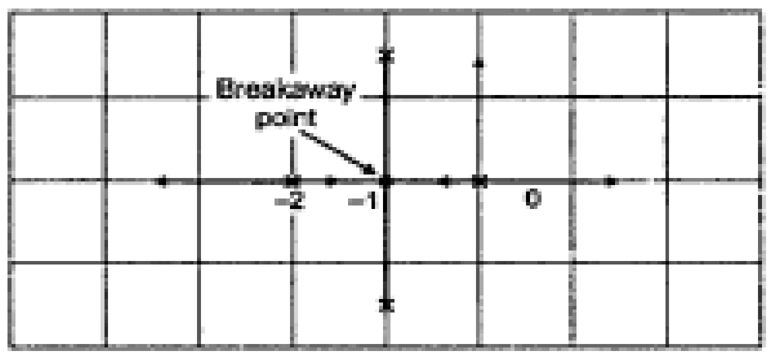
Effect of Addition of Open Loop Poles and Zeros

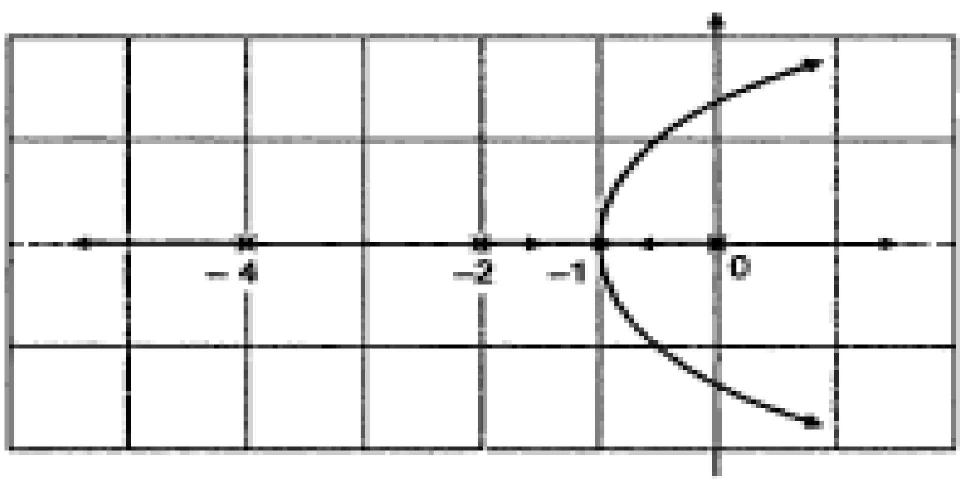
Addition of Pole

In general we can state that adding a pole to the function G(s)H(s) in the left half of the s-plane has the effect of pushing original root locus towards right half of s - plane. This can be proved by following examples.

Consider, $G(s)H(s) = \dfrac{K}{s\,(s+2)}$

Corresponding root locus is shown in the Fig. 9.20.

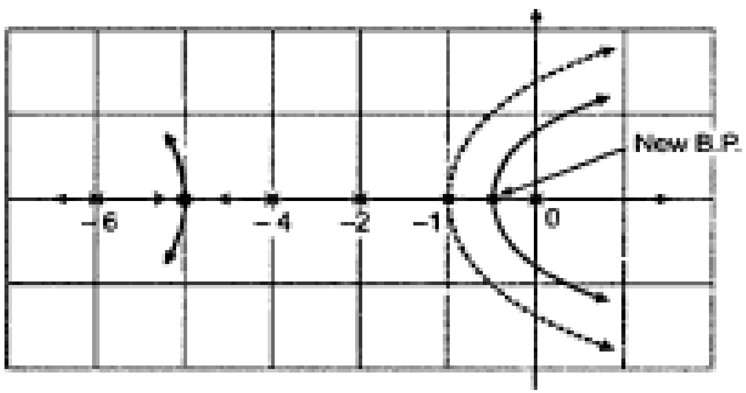




If now one more pole at s = – 6 is added to the system,

$$G(s)H(s) = \frac{K}{s(s+2)(s+4)(s+6)}$$

Breakaway point in section s = 0 and s = –2 gets shifted towards right as compared to previous case. So system stability further gets restricted. This is shown in the Fig. 9.22.



Effects of addition of open loop poles can be summarized as :

1) Root locus shifts towards imaginary axis.
2) System stability relatively decreases.
3) System becomes more oscillatory in nature.
4) Range of operating values of 'K' for stability of the system decreases.