Module no 4

Mrs.Manisha Joshi

CHAPTER 3

Laplace Transform

3.1 Introduction

The Laplace transform is used to transform a time signal to complex frequency domain. (The complex frequency domain is also known as Laplace domain or s-domain). This transformation was first proposed by Laplace (in the year 1780) and later adopted for various engineering applications for solving differential equations. Hence this transformation is called ***Laplace transform.***

In signals and systems the Laplace transform is used to transform a time domain system to s-domain. In time domain the equations governing a system will be in the form of differential equations. While transforming the system to s-domain, the differential equations are transformed to simple algebraic equations and so the analysis of systems will be much easier in s-domain.

In this chapter a brief discussion about Laplace transform and its applications for analysis of signals and systems are presented.

Complex Frequency

The ***complex frequency*** is defined as,

Complex frequency, $s = \sigma + j\Omega$

where, σ = ***Neper frequency*** in neper per second
Ω = ***Radian (or Real) frequency*** in radian per second

The complex frequency is involved in the time domain signal of the form Ke^{st}. The signal Ke^{st} can be thought of as an universal signal which represents all types of signals and takes a particular form for various choices of σ and Ω as shown below.

Let, $x(t) = A\,e^{st} = A\,e^{(\sigma + j\Omega)t}$ .....(3.1)

Complex Frequency Plane or s-Plane

The complex frequency is defined as,

$$\text{Complex frequency, } s = \sigma + j\Omega$$

where, σ = Real part of s

Ω = Imaginary part of s

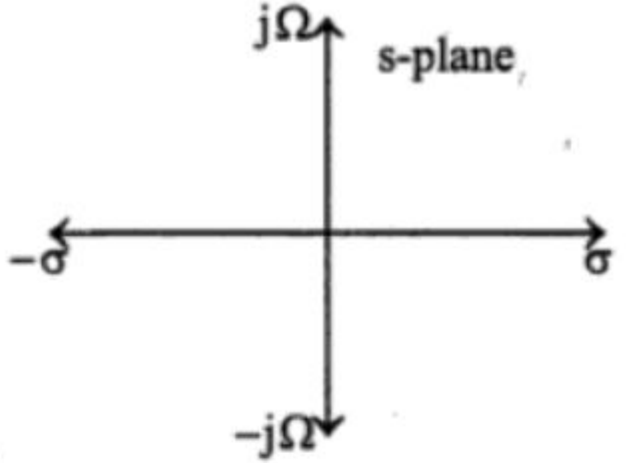


Fig 3.1: Complex frequency plane or s-plane.

The σ and Ω can take values from $-\infty$ to $+\infty$. A two dimensional complex plane with values of σ on horizontal axis and values Ω on vertical axis as shown in fig 3.1 is called ***complex frequency plane*** or ***s-plane.***

The ***s-plane*** is used to represent various critical frequencies (poles and zeros) of signals which are functions of s and to study the path taken by these critical frequencies when some parameters of the signals are varied. This study will be useful to design systems for a desired response.

Definition of Laplace Transform

In order to transform a time domain signal x(t) to s-domain, multiply the signal by e^{-st} and then integrate from $-\infty$ to ∞. The transformed signal is represented as X(s) and the transformation is denoted by the script letter $\mathcal{L}$.

Symbolically the ***Laplace transform*** of x(t) is denoted as,

$$X(s) = \mathcal{L}\{x(t)\}$$

Let x(t) be a continuous time signal defined for all values of t. Let X(s) be Laplace transform of x(t). Now the ***Laplace transform*** of x(t) is defined as,

$$\boxed{\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt} \qquad .....(3.6)$$

If x(t) is defined for $t \geq 0$, (i.e., if x(t) is causal) then,

$$\boxed{\mathcal{L}\{x(t)\} = X(s) = \int_{0}^{+\infty} x(t)\, e^{-st}\, dt} \qquad .....(3.7)$$

The definition of Laplace transform as given by equation (3.6) is called ***Two sided Laplace transform*** or ***Bilateral Laplace Transform*** and the definition of Laplace transform as given by equation (3.7) is called ***One sided Laplace transform*** or ***Unilateral Laplace transform***.

Definition of Inverse Laplace Transform

The s-domain signal X(s) can be transformed to time domain signal x(t) by using inverse Laplace transform.

The ***Inverse Laplace transform*** of X(s) is defined as,

$$\boxed{\mathcal{L}^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\Omega}^{s=\sigma+j\Omega} X(s)\, e^{st}\, ds}$$

The signal x(t) and X(s) are called ***Laplace transform pair*** and can be expressed as,

$$\boxed{x(t) \xrightleftharpoons[\mathcal{L}^{-1}]{\mathcal{L}} X(s)}$$

3.2 Region of Convergence

The Laplace transform of a signal is given by $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$. The values of s for which the integral $\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$ converges is called ***Region Of Convergence (ROC)***. The ROC for the following three types of signals are discussed here.

Case i : Right sided (causal) signal

Case ii : Left sided (anticausal) signal

Case iii : Two sided signal.

***Case i :* Right sided (causal) signal**

Let, $x(t) = e^{-at} u(t)$, where $a > 0$

$= e^{-at}$ for $t \geq 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} e^{-at}\, u(t)\, e^{-st}\, dt$$

$$= \int_{0}^{+\infty} e^{-at}\, e^{-st}\, dt = \int_{0}^{+\infty} e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \frac{e^{-(\sigma + j\Omega + a)\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)} = -\frac{e^{-(\sigma + a)\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

$$\boxed{\begin{array}{c}\text{Put,}\\ s = \sigma + j\Omega\end{array}}$$

$$\therefore \mathcal{L}\{x(t)\} = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a}$$

$$\text{where, } k = \sigma + a = \sigma - (-a)$$

When $\sigma > -a$, $k = \sigma - (-a) = \text{Positive}$, $\therefore e^{-k\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $k = \sigma - (-a) = \text{Negative}$, $\therefore e^{-k\infty} = e^{+\infty} = \infty$

Hence we can say that, X(s) converges, when $\sigma > -a$, and does not converge for $\sigma < -a$.

$\therefore$ Abscissa of convergence, $\sigma_c = -a$.

When $\sigma > -a$, the X(s) is given by,

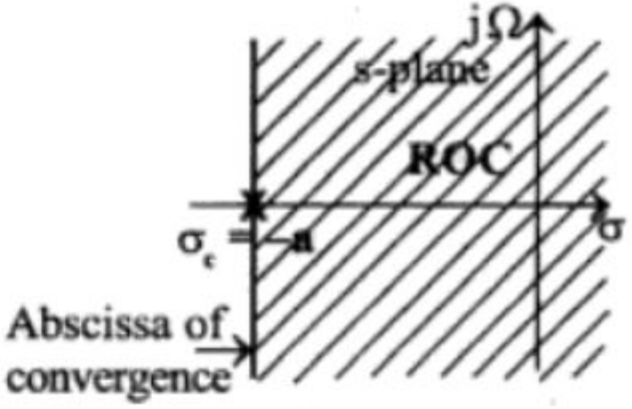


Fig 3.2 : ROC of $x(t) = e^{-at}u(t)$.

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = -\frac{0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} = \frac{1}{s+a}$$

Therefore for a causal signal the ROC includes all points on the s-plane to the right of abscissa of convergence, $\sigma_c = -a$, as shown in fig 3.2.

Case ii : ***Left sided (anticausal) signal***

Let, $x(t) = e^{-bt}u(-t) = e^{-bt}$ for $t \le 0$, where $b > 0$

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} e^{-bt}\, u(-t)\, e^{-st}\, dt = \int_{-\infty}^{0} e^{-bt}\, e^{-st}\, dt$$

$$= \int_{-\infty}^{0} e^{-(s+b)t}\, dt = \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0} = \frac{e^{0}}{-(s+b)} - \frac{e^{(\sigma + j\Omega + b)\infty}}{-(s+b)}$$

Put, $s = \sigma + j\Omega$

$$= -\frac{1}{s+b} + \frac{e^{(\sigma+b)\times\infty} e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{e^{k\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

where, $k = \sigma + b = \sigma - (-b)$

When $\sigma > -b$, $k = \sigma - (-b) =$ Positive, $\therefore e^{k\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $k = \sigma - (-b) =$ Negative, $\therefore e^{k\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when $\sigma < -b$, and does not converge for $\sigma > -b$.

$\therefore$ Abscissa of convergence, $\sigma_c = -b$.

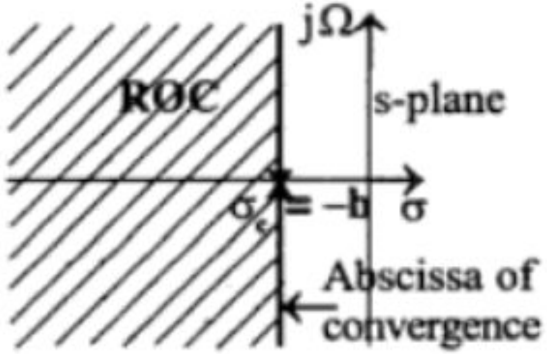


Fig 3.3 : ROC of $x(t) = e^{-bt}u(-t)$.

When $\sigma > -b$, $k = \sigma - (-b) = \text{Positive}$, $\therefore e^{k\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $k = \sigma - (-b) = \text{Negative}$, $\therefore e^{k\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when $\sigma < -b$, and does not converge for $\sigma > -b$.

$\therefore$ Abscissa of convergence, $\sigma_c = -b$.

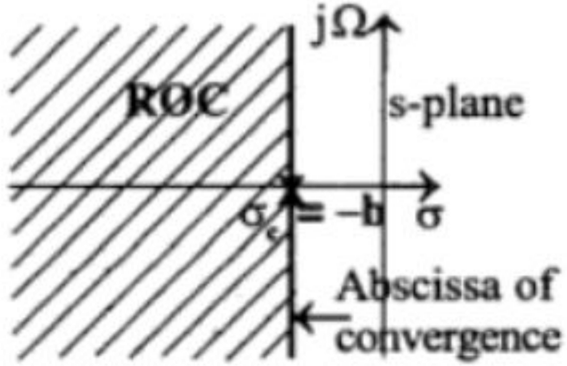


Fig 3.3 : ROC of $x(t) = e^{-bt}u(-t)$.

When $\sigma < -b$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{1}{s+b} + \frac{e^{k\times\infty}\, e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b} + \frac{0 \times e^{j\Omega\times\infty}}{s+b} = -\frac{1}{s+b}$$

Therefore for an anticausal signal the ROC includes all points on the s-plane to the left of abscissa of convergence, $\sigma_c = -b$, as shown in fig 3.3.

Case iii: ***Two sided signal***

Let, $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$, where $a > 0,\ b > 0$, and $a > b$ (i.e., $-a < -b$)

Now, the Laplace transform of x(t) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} \left[e^{-at}\, u(t) + e^{-bt}\, u(-t)\right] e^{-st}\, dt$$

$$= \int_{0}^{+\infty} e^{-at}\, e^{-st}\, dt + \int_{-\infty}^{0} e^{-bt}\, e^{-st}\, dt = \int_{0}^{+\infty} e^{-(s+a)t}\, dt + \int_{-\infty}^{0} e^{-(s+b)t}\, dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty} + \left[\frac{e^{-(s+b)t}}{-(s+b)}\right]_{-\infty}^{0} = \left[\frac{e^{-(\sigma+j\Omega+a)t}}{-(s+a)}\right]_0^{\infty} + \left[\frac{e^{-(\sigma+j\Omega+b)t}}{-(s+b)}\right]_{-\infty}^{0}$$

$$= \frac{e^{-(\sigma+j\Omega+a)\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)} + \frac{e^{0}}{-(s+b)} - \frac{e^{(\sigma+j\Omega+b)\infty}}{-(s+b)}$$

$$\boxed{\text{Put, } s = \sigma + j\Omega}$$

$$= -\frac{e^{-p\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

where, $p = \sigma + a = \sigma - (-a)$ and $q = \sigma + b = \sigma - (-b)$

When $\sigma > -a$, $p = \sigma - (-a) = \text{Positive}$, $\therefore e^{-p\infty} = e^{-\infty} = 0$

When $\sigma < -a$, $p = \sigma - (-a) = \text{Negative}$, $\therefore e^{-p\infty} = e^{+\infty} = \infty$

When $\sigma > -b$, $q = \sigma - (-b) = \text{Positive}$, $\therefore e^{q\infty} = e^{\infty} = \infty$

When $\sigma < -b$, $q = \sigma - (-b) = \text{Negative}$, $\therefore e^{q\infty} = e^{-\infty} = 0$

Hence we can say that, X(s) converges, when σ lies between $-a$ and $-b$ (i.e., $-a < \sigma < -$
does not converge for $\sigma < -a$ and $\sigma > -b$.

$\therefore$ Abscissa of convergences, $\sigma_{c1} = -a$ and $\sigma_{c2} = -b$.

When $-a < \sigma < -b$, the X(s) is given by,

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-p\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

$$= -\frac{0 \times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{0 \times e^{j\Omega\times\infty}}{s+b}$$

$$\mathcal{L}\{x(t)\} = X(s) = -\frac{e^{-p\times\infty}\, e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{e^{q\times\infty}\, e^{j\Omega\times\infty}}{s+b}$$

$$= -\frac{0\times e^{-j\Omega\times\infty}}{s+a} + \frac{1}{s+a} - \frac{1}{s+b} + \frac{0\times e^{j\Omega\times\infty}}{s+b}$$

$$= \frac{1}{s+a} - \frac{1}{s+b}$$

Therefore for a two sided signal the ROC includes all points on the s-plane in the region in between two abscissa of convergences, $\sigma_{c1} = -a$ and $\sigma_{c2} = -b$, as shown in fig 3.4.

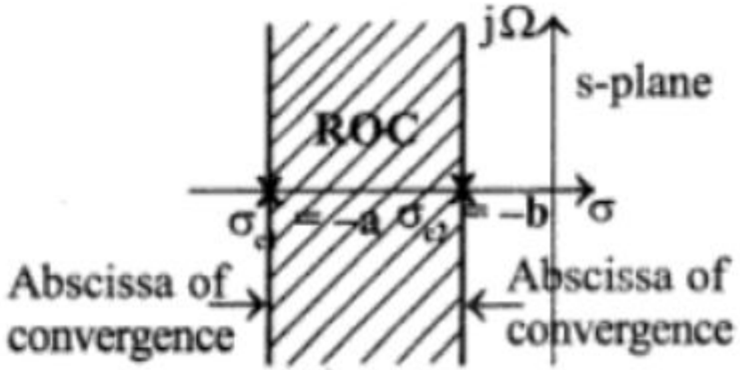


***Fig 3.4 : ROC of** $x(t) = e^{-at}u(t) + e^{-bt}u(-t)$.*

Example 3.1

Determine the Laplace transform of the following continuous time signals and their ROC.

a) $x(t) = A\,u(t)$ b) $x(t) = t\,u(t)$ c) $x(t) = e^{-3t}\,u(t)$ d) $x(t) = e^{-3t}\,u(-t)$ e) $x(t) = e^{-4|t|}$

Solution

a) Given that, $x(t) = A\,u(t) = A \;;\; t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{0}^{\infty} A\,e^{-st}\,dt = A\int_{0}^{\infty} e^{-st}\,dt$$

$\boxed{\text{Put, } s = \sigma + j\Omega}$

$$= A\left[\frac{e^{-st}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma + j\Omega)t}}{-s}\right]_0^{\infty} = A\left[\frac{e^{-(\sigma + j\Omega)\infty}}{-s} - \frac{e^0}{-s}\right] = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$
When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, X(s) converges when $\sigma > 0$.

When $\sigma > 0$, the X(s) is given by,

$$X(s) = A\left[\frac{e^{-\sigma\times\infty}\,e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = A\left[\frac{0 \times e^{-j\Omega\times\infty}}{-s} + \frac{1}{s}\right] = \frac{A}{s}$$

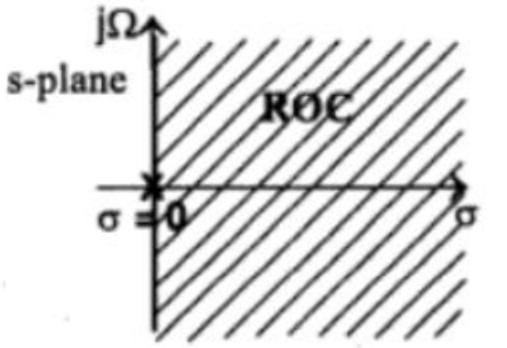


Fig 1 : ROC of $x(t) = A\,u(t)$.

$\therefore\ \mathcal{L}\{A\,u(t)\} = \dfrac{A}{s}$; with ROC as all points is s-plane to the right of line passing through $\sigma = 0$.

(or ROC is right half of s-plane).

b) Given that, $x(t) = t\,u(t) = t\ ;\ t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$\boxed{\int u\,v = u \int v - \int \left[du \int v\right]}$$

$$= \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \int_0^{\infty} 1 \times \frac{e^{-st}}{-s}\, dt = \left[t\,\frac{e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty} = \left[t\,\frac{e^{-(\sigma+j\Omega)t}}{-s}\right]_0^{\infty} - \left[\frac{e^{-(\sigma+j\Omega)t}}{s^2}\right]_0^{\infty}$$

$$\boxed{\begin{array}{c}\text{Put,}\\ s = \sigma + j\Omega\end{array}}$$

$$= \left[\infty \times \frac{e^{-(\sigma+j\Omega)\infty}}{-s} - 0 \times \frac{e^{0}}{-s} - \frac{e^{-(\sigma+j\Omega)\infty}}{s^2} + \frac{e^{0}}{s^2}\right]$$

$$= \left[\infty \times \frac{e^{-\sigma\times\infty}\, e^{-j\Omega\times\infty}}{-s} - 0 - \frac{e^{-\sigma\times\infty}\, e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

When, $\sigma > 0$, (i.e., when σ is positive), $e^{-\sigma\infty} = e^{-\infty} = 0$

When, $\sigma < 0$, (i.e., when σ is negative), $e^{-\sigma\infty} = e^{\infty} = \infty$

Therefore we can say that, $X(s)$ converges when $\sigma > 0$.

When $\sigma > 0$, the $X(s)$ is given by,

$$X(s) = \left[\infty \times \frac{e^{-\sigma\times\infty}\, e^{-j\Omega\times\infty}}{-s} - \frac{e^{-\sigma\times\infty}\, e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right]$$

$$= \left[\infty \times \frac{0 \times e^{-j\Omega\times\infty}}{-s} - \frac{0 \times e^{-j\Omega\times\infty}}{s^2} + \frac{1}{s^2}\right] = \frac{1}{s^2}$$

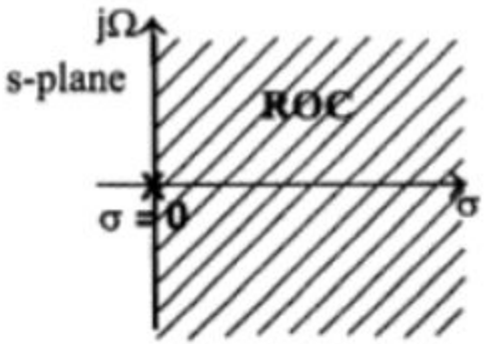


Fig 2 : ROC of x(t) = t u(t).

$$\therefore \mathcal{L}\{t\,u(t)\} = \frac{1}{s^2}\ ;$$ with ROC as all points is s-plane to the right of line passing through $\sigma = 0$.

(or ROC is right half of s-plane).

---

c) Given that, $x(t) = e^{-3t}\,u(t) = e^{-3t}\ ;\ t \geq 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-3t}\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+3)t}\, dt$$

$$= \left[\frac{e^{-(s+3)t}}{-(s+3)}\right]_0^{\infty} = \frac{e^{-(s+3)\infty}}{-(s+3)} - \frac{e^{0}}{-(s+3)} = -\frac{e^{-(\sigma+j\Omega+3)\infty}}{s+3} + \frac{1}{s+3}$$

$$\boxed{\text{Put, } s = \sigma + j\Omega}$$

$$= -\frac{e^{-(\sigma+3)\times\infty}\, e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3}$$

where, $k = \sigma + 3 = \sigma - (-3)$

When, $\sigma > -3$, $k = \sigma - (-3) =$ Positive. $\therefore e^{-k\infty} = e^{-\infty} = 0$

When, $\sigma < -3$, $k = \sigma - (-3) =$ Negative. $\therefore e^{-k\infty} = e^{\infty} = \infty$

Therefore we can say that, $X(s)$ converges when $\sigma > -3$.

When $\sigma > -3$, the $X(s)$ is given by,

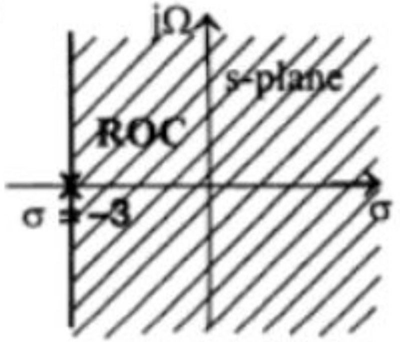


Fig 3 : ROC of $x(t) = e^{-3t}u(t)$.

$$X(s) = -\frac{e^{-k\times\infty}\, e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = -\frac{0 \times e^{-j\Omega\times\infty}}{s+3} + \frac{1}{s+3} = \frac{1}{s+3}$$

$$\therefore \mathcal{L}\left\{e^{-3t}\,u(t)\right\} = \frac{1}{s+3}\ ;$$ with ROC as all points in s-plane to the right of line passing through $\sigma = -3$.

b) Given that, $x(t) = \cos\Omega_0 t\, u(t) = \cos\Omega_0 t \quad ; \quad t \ge 0$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty \cos\Omega_0 t\, e^{-st}\, dt = \int_0^\infty \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}\, e^{-st}\, dt \qquad \boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s - j\Omega_0)t} + e^{-(s + j\Omega_0)t}\right) dt = \frac{1}{2}\left[\frac{e^{-(s - j\Omega_0)t}}{-(s - j\Omega_0)} + \frac{e^{-(s + j\Omega_0)t}}{-(s + j\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s - j\Omega_0)} + \frac{e^{-\infty}}{-(s + j\Omega_0)} - \frac{e^{0}}{-(s - j\Omega_0)} - \frac{e^{0}}{-(s + j\Omega_0)}\right]$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s - j\Omega_0} + \frac{1}{s + j\Omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s + j\Omega_0 + s - j\Omega_0}{(s - j\Omega_0)(s + j\Omega_0)}\right] = \frac{1}{2}\left[\frac{2s}{s^2 + \Omega_0^2}\right] = \frac{s}{s^2 + \Omega_0^2} \qquad \boxed{(a+b)(a-b) = a^2 - b^2} \quad \boxed{j^2 = -1}$$

$$\therefore\ \mathcal{L}\{\cos\Omega_0 t\, u(t)\} = \frac{s}{s^2 + \Omega_0^2}$$

c) **Given that, $x(t) = \cosh\Omega_0 t\ u(t) = \cosh\Omega_0 t \quad ; \quad t \geq 0$**

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)\, e^{-st}\, dt = \int_0^\infty \cosh\Omega_0 t\, e^{-st}\, dt = \int_0^\infty \frac{e^{\Omega_0 t} + e^{-\Omega_0 t}}{2}\, e^{-st}\, dt \qquad \boxed{\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}}$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-(s-\Omega_0)t} + e^{-(s+\Omega_0)t}\right) dt = \frac{1}{2}\left[\frac{e^{-(s-\Omega_0)t}}{-(s-\Omega_0)} + \frac{e^{-(s+\Omega_0)t}}{-(s+\Omega_0)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}}{-(s-\Omega_0)} + \frac{e^{-\infty}}{-(s+\Omega_0)} - \frac{e^{0}}{-(s-\Omega_0)} - \frac{e^{0}}{-(s+\Omega_0)}\right]$$

$$= \frac{1}{2}\left[0 + 0 + \frac{1}{s-\Omega_0} + \frac{1}{s+\Omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+\Omega_0+s-\Omega_0}{(s-\Omega_0)(s+\Omega_0)}\right] = \frac{1}{2}\left[\frac{2s}{s^2-\Omega_0^2}\right] = \frac{s}{s^2-\Omega_0^2} \qquad \boxed{(a+b)(a-b) = a^2 - b^2}$$

$$\therefore \mathcal{L}\{\cosh\Omega_0 t\ u(t)\} = \frac{s}{s^2-\Omega_0^2}$$

Example 3.3

Determine the Laplace transform of the signals shown below.

a)

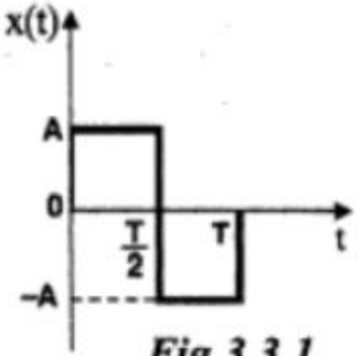


Fig 3.3.1.

b)

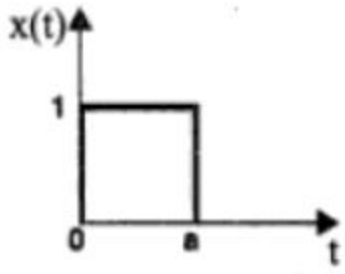


Fig 3.3.2.

c)

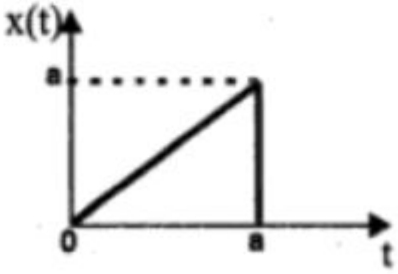


Fig 3.3.3.

d)

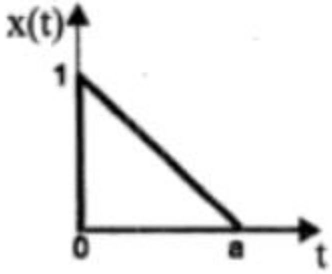


Fig 3.3.4.

Solution

a)

The mathematical equation of the signal shown in fig 3.3.1 is,

$$x(t) = A \quad ;\text{for} \quad 0 < t < T/2$$
$$= -A \quad ;\text{for} \quad T/2 < t < T$$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{T} x(t)\, e^{-st}\, dt$$

$$= \int_{0}^{T/2} A\, e^{-st}\, dt + \int_{T/2}^{T} (-A)\, e^{-st}\, dt = \left[\frac{A\, e^{-st}}{-s}\right]_{0}^{T/2} + \left[\frac{-A\, e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= \left[\frac{A\, e^{\frac{-sT}{2}}}{-s} - \frac{A\, e^{0}}{-s}\right] + \left[\frac{A\, e^{-sT}}{s} - \frac{A\, e^{\frac{-sT}{2}}}{s}\right]$$

$$= -\frac{A\, e^{\frac{-sT}{2}}}{s} + \frac{A}{s} + \frac{A\, e^{-sT}}{s} - \frac{A\, e^{\frac{-sT}{2}}}{s}$$

$$= \frac{A}{s}\left[1 + e^{-sT} - 2\, e^{\frac{-sT}{2}}\right] = \frac{A}{s}\left[1 - e^{\frac{-sT}{2}}\right]^2$$

$$\boxed{(a-b)^2 = a^2 + b^2 - 2ab}$$

b)

The mathematical equation of the signal shown in fig 3.3.2 is,

$$x(t) = 1 \quad ;\text{for } 0 \le t \le a$$
$$= 0 \quad ;\text{for } t > a$$

By definition of Laplace transform,

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt = \int_{0}^{a} 1 \times e^{-st}\, dt = \int_{0}^{a} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^a$$

$$= \frac{e^{-as}}{-s} - \frac{e^{0}}{-s} = -\frac{e^{-as}}{s} + \frac{1}{s} = \frac{1}{s}\left(1 - e^{-as}\right)$$

c)

To Find Mathematical Equation for x(t)

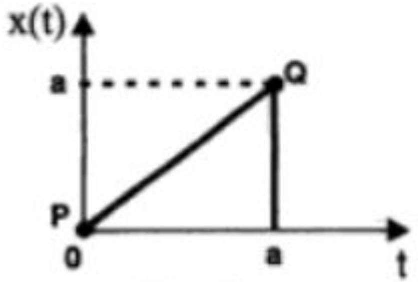


Fig 1.

Consider the equation of straight line, $\dfrac{y - y_1}{y_1 - y_2} = \dfrac{x - x_1}{x_1 - x_2}$

Here, $y = x(t)$, $x = t$.

$\therefore$ The equation of straight line can be written as, $\dfrac{x(t) - x(t_1)}{x(t_1) - x(t_2)} = \dfrac{t - t_1}{t_1 - t_2}$ .....(1)

Consider points P and Q, as shown in fig 1.

Coordinates of point - P = $[t_1, x(t_1)] = [0, 0]$

Coordinates of point - Q = $[t_2, x(t_2)] = [a, a]$

On substituting the coordinates of points - P and Q in equation - (1) we get,

$$\frac{x(t)-0}{0-a} = \frac{t-0}{0-a} \Rightarrow \frac{x(t)}{-a} = \frac{t}{-a} \Rightarrow x(t) = t$$

$$\therefore\ x(t) = t \quad ;\text{for } t = 0 \text{ to } a$$
$$= 0 \quad ;\text{for } t > a$$

To Evaluate Laplace transform of x(t)

$$\mathcal{L}\{x(t)\} = X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = \int_0^a t\, e^{-st}\, dt$$

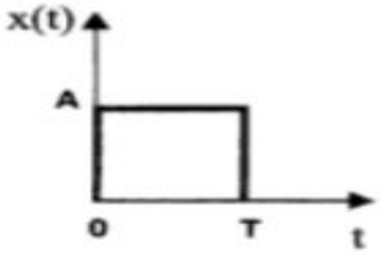
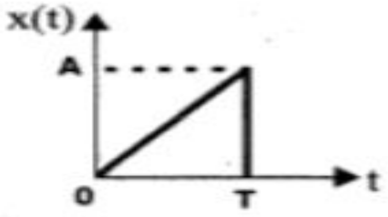
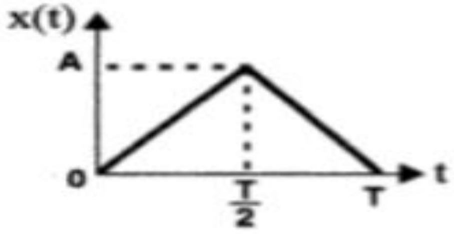
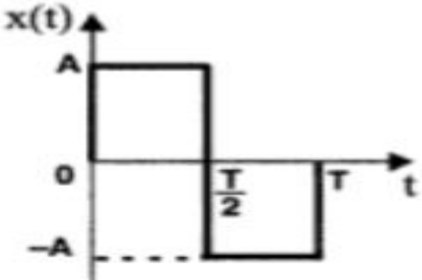
$$= \left[t \times \frac{e^{-st}}{-s} - \int 1 \times \frac{e^{-st}}{-s}\, dt\right]_0^a = \left[-\frac{t\, e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^a$$

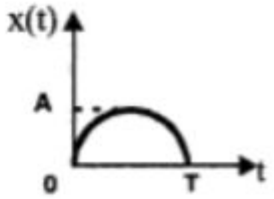
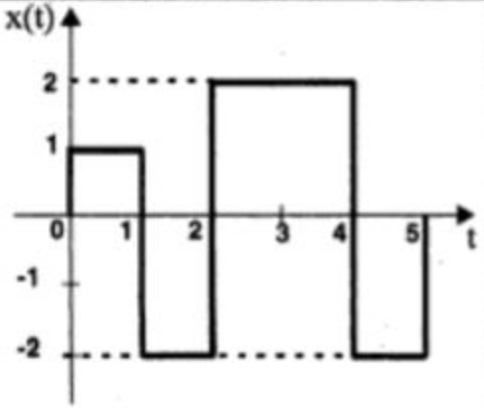
$$= \left[-\frac{a\, e^{-sa}}{s} - \frac{e^{-sa}}{s^2} + 0 + \frac{e^0}{s^2}\right] = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - \frac{a e^{-as}}{s}$$

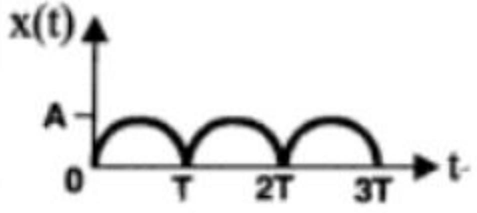
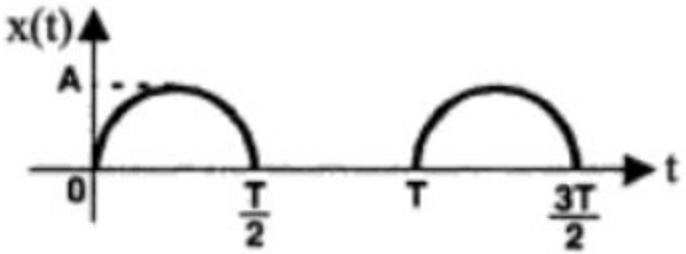
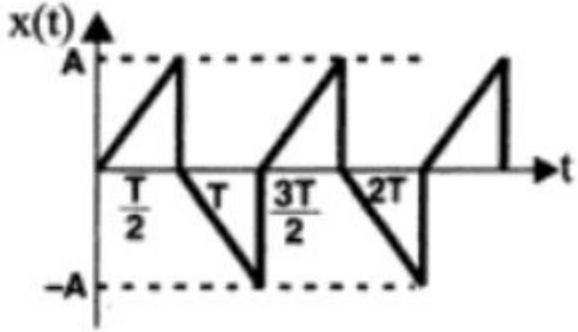
$$= \frac{1}{s^2}\left[1 - e^{-as}(1 + as)\right]$$

$\int uv = u\int v - \int[du \int v]$	
$u = t$	$v = e^{-st}$

Table 3.1 : Laplace Transform of Some Standard Signals

Waveform	x(t)	$X(s) = \mathcal{L}\{x(t)\}$
	$x(t) = A \;;\; 0 < t < T$ $= 0 \;;\; t > T$	$X(s) = \dfrac{A}{s}(1 - e^{-sT})$
	$x(t) = \dfrac{At}{T} \;;\; 0 < t < T$ $= 0 \;;\; t > T$	$X(s) = \dfrac{A}{Ts^2}\left[1 - e^{-sT}(1 + sT)\right]$
	$x(t) = \dfrac{2At}{T} \;;\; 0 < t < \dfrac{T}{2}$ $= 2A - \dfrac{2At}{T} \;;\; \dfrac{T}{2} < t < T$	$X(s) = \dfrac{2A}{Ts^2}\left(1 - e^{\frac{-sT}{2}}\right)^2$
	$x(t) = A \;;\; 0 < t < \dfrac{T}{2}$ $= -A \;;\; \dfrac{T}{2} < t < T$	$X(s) = \dfrac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$

	$x(t) = A \sin t \;;\; 0 < t < T$ $\quad = 0 \;;\; t > T$	$X(s) = \dfrac{A}{s^2 + 1}\left(e^{-sT} + 1\right)$
	$x(t) = 1 \;;\; 0 < t < 1$ $\quad = -2 \;;\; 1 < t < 2$ $\quad = 2 \;;\; 2 < t < 4$ $\quad = -2 \;;\; 4 < t < 5$ $\quad = 0 \;;\; t > 5$	$X(s) = \dfrac{1}{s}\left(1 - 3e^{-s} + 4e^{-2s} - 4e^{-4s} + 2e^{-5s}\right)$

Waveform	x(t)	$X(s) = \mathcal{L}\{x(t)\}$
	$x(t) = A \sin t \; ; \; 0 < t < T$ and $x(t + nT) = x(t)$	$X(s) = \dfrac{A}{s^2 + 1}\left(\dfrac{1 + e^{-sT}}{1 - e^{-sT}}\right)$
	$x(t) = A \sin t \; ; \; 0 < t < \dfrac{T}{2}$ $= 0 \; ; \; \dfrac{T}{2} < t < T$ and $x(t + nT) = x(t)$	$X(s) = \dfrac{A}{\left(s^2 + 1\right)\left(1 - e^{-\frac{sT}{2}}\right)}$
	$x(t) = \dfrac{2At}{T} \; ; \; 0 < t < \dfrac{T}{2}$ $= A - \dfrac{2At}{T} \; ; \; \dfrac{T}{2} < t < T$ and $x(t + nT) = x(t)$	$X(s) = \dfrac{2A\left[1 - \left(1 + \dfrac{Ts}{2}\right)e^{-\frac{sT}{2}}\right]}{Ts^2\left(1 + e^{-\frac{sT}{2}}\right)}$

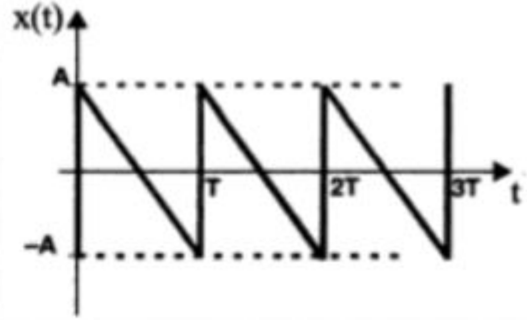
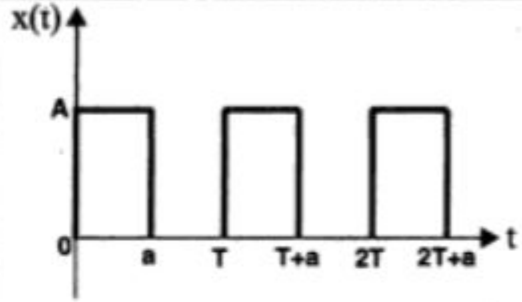
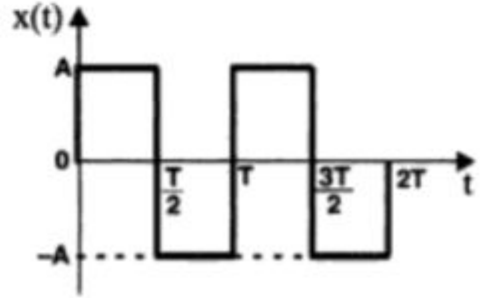
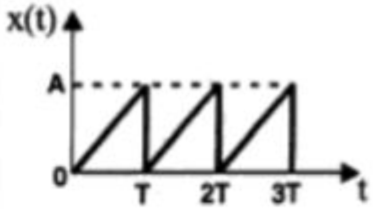
	$x(t) = A - \frac{2At}{T}\ ; 0 < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{2A}{Ts}\left(\frac{T}{2}\ \frac{1 + e^{-sT}}{1 - e^{-sT}} - \frac{1}{s}\right)$
	$x(t) = A \ ;\ 0 < t < a$ $= 0 \ ;\ a < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{A}{s}\ \frac{1 - e^{-as}}{1 - e^{-sT}}$
	$x(t) = A \ ;\ 0 < t < \frac{T}{2}$ $= -A \ ;\ \frac{T}{2} < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{A}{s}\left(\frac{1 - e^{-\frac{sT}{2}}}{1 + e^{-\frac{sT}{2}}}\right)$
	$x(t) = \frac{At}{T} \ ;\ 0 < t < T$ and $x(t + nT) = x(t)$	$X(s) = \frac{A}{Ts^2}\left[\frac{1 - e^{-sT}(1 + sT)}{1 - e^{-sT}}\right]$

Table 3.2 : Some Standard Laplace Transform Pairs

Note : σ = Real part of s

x(t)	X(s)	ROC
$\delta(t)$	1	Entire s-plane
$u(t)$	$\frac{1}{s}$	$\sigma > 0$
$t\,u(t)$	$\frac{1}{s^2}$	$\sigma > 0$
$\frac{t^{n-1}}{(n-1)!}u(t)$ where, $n = 1, 2, 3, ...$	$\frac{1}{s^n}$	$\sigma > 0$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\sigma > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\sigma < -a$
$t^n u(t)$ where, $n = 1, 2, 3 .....$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$t\,e^{-at}u(t)$	$\frac{1}{(s+a)^2}$	$\sigma > -a$

$\frac{1}{(n-1)!} t^{n-1} e^{-at} u(t)$ where, $n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$	$\sigma > -a$
$t^n e^{-at} u(t)$ where, $n = 1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$	$\sigma > -a$
$\sin \Omega_0 t\, u(t)$	$\frac{\Omega_0}{s^2 + \Omega_0^2}$	$\sigma > 0$
$\cos \Omega_0 t\, u(t)$	$\frac{s}{s^2 + \Omega_0^2}$	$\sigma > 0$
$\sinh \Omega_0 t\, u(t)$	$\frac{\Omega_0}{s^2 - \Omega_0^2}$	$\sigma > \Omega_0$
$\cosh \Omega_0 t\, u(t)$	$\frac{s}{s^2 - \Omega_0^2}$	$\sigma > \Omega_0$
$e^{-at} \sin \Omega_0 t\, u(t)$	$\frac{\Omega_0}{(s+a)^2 + \Omega_0^2}$	$\sigma > -a$
$e^{-at} \cos \Omega_0 t\, u(t)$	$\frac{s+a}{(s+a)^2 + \Omega_0^2}$	$\sigma > -a$

3.3 Properties and Theorems of Laplace Transform

The properties and theorems of Laplace transform are listed in table 3.3. The proof of properties and theorems are presented in this section.

1. Amplitude Scaling

In amplitude scaling, if the amplitude (or magnitude) of a time domain signal is multiplied by a constant A, then its Laplace transform is also multiplied by the same constant.

i.e., if $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{A\,x(t)\} = A\,X(s)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt \qquad \text{.....(3.8)}$$

$$\mathcal{L}\{A\,x(t)\} = \int_{-\infty}^{+\infty} A\,x(t)\,e^{-st}\,dt$$

$$= A\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

$$= A\,X(s) \qquad \boxed{\text{Using equation (3.8)}}$$

2. *Linearity*

The linearity property states that, Laplace transform of weighted sum of the two or more signals is equal to similar weighted sum of Laplace transforms of the individual signals.

i.e., if $\mathcal{L}\{x_1(t)\} = X_1(s)$ and $\mathcal{L}\{x_2(t)\} = X_2(s)$, then

$$\mathcal{L}\{a_1\, x_1(t) + a_2\, x_2(t)\} = a_1\, X_1(s) + a_2\, X_2(s)$$

Proof :

By definition of Laplace transform,

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt \qquad .....(3.9)$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt \qquad .....(3.10)$$

$$\mathcal{L}\{a_1\, x_1(t) + a_2\, x_2(t)\} = \int_{-\infty}^{+\infty} \left[a_1\, x_1(t) + a_2\, x_2(t)\right] e^{-st}\, dt$$

$$= a_1 \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt + a_2 \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt$$

$$= a_1\, X_1(s) + a_2\, X_2(s)$$

Using equations (3.9) and (3.10)

3. Time Differentiation

The time differentiation property states that if a causal signal x(t) is piecewise continuous, and Laplace transform of x(t) is X(s) then, Laplace transform of $\frac{d}{dt}x(t)$ is given by sX(s) − x(0).

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0) \quad ; \text{ where, } x(0) \text{ is value of } x(t) \text{ at } t = 0.$$

Proof :

By definition of Laplace transform, the Laplace transform of a causal signal is given by,

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)\, e^{-st}\, dt \qquad .....(3.11)$$

$$\therefore \mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = \int_0^\infty e^{-st}\, \frac{dx(t)}{dt}\, dt$$

$\int u\,v = u\int v - \int\left[du \int v\right]$	
$u = e^{-st}$	$v = \frac{dx(t)}{dt}$

$$= \left[e^{-st}\, x(t)\right]_0^\infty - \int_0^\infty -s\, e^{-st}\, x(t)\, dt$$

$$= e^{-\infty}\, x(\infty) - e^0\, x(0) + s\int_0^\infty x(t)\, e^{-st}\, dt \qquad \boxed{e^{-\infty} = 0 \text{ and } e^0 = 1}$$

$$= s\int_0^\infty x(t)\, e^{-st}\, dt - x(0) = s\,X(s) - x(0) \qquad \boxed{\text{Using equation (3.11)}}$$

4. Time Integration

The time integration property states that, if a causal signal x(t) is continuous and Laplace transform of x(t) is X(s), then the Laplace transform of $\int x(t)\,dt$ is given by, $\frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\int x(t)\,dt\right\} = \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s}$$

Proof :

By definition of Laplace transform, the Laplace transform of a causal signal is given by,

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^\infty x(t)\,e^{-st}\,dt \qquad .....(3.12)$$

$$\therefore\ \mathcal{L}\left\{\int x(t)\,dt\right\} = \int_0^\infty \left[\int x(t)\,dt\right] e^{-st}\,dt$$

$$= \left[\left[\int x(t)\,dt\right] \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty x(t)\,\frac{e^{-st}}{-s}\,dt$$

$\int uv = u\int v - \int\left[du\int v\right]$	
$u = \int x(t)\,dt$	$v = e^{-st}$

$$= \left[\int x(t)\,dt\right]\Big|_{t=\infty} \frac{e^{-\infty}}{-s} - \left[\int x(t)\,dt\right]\Big|_{t=0} \frac{e^{0}}{-s} + \frac{1}{s}\int_0^\infty x(t)\,e^{-st}\,dt$$

$$= \frac{1}{s}\left[\int x(t)\,dt\right]\Big|_{t=0} + \frac{1}{s}\int_0^\infty x(t)\,e^{-st}\,dt \qquad \boxed{e^{-\infty} = 0 \text{ and } e^{0} = 1}$$

$$= \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]\Big|_{t=0}}{s} \qquad \boxed{\text{Using equation (3.12)}}$$

5. *Frequency shifting*

The frequency shifting property of Laplace transform says that,

If, $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{e^{\pm at}\, x(t)\right\} = X(s \mp a) \qquad \left[\text{i.e., } \mathcal{L}\left\{e^{at}\, x(t)\right\} = X(s-a) \text{ and } \mathcal{L}\left\{e^{-at}\, x(t)\right\} = X(s+a)\right]$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.13)$$

$$\therefore \mathcal{L}\left\{e^{\pm at}\, x(t)\right\} = \int_{-\infty}^{+\infty} e^{\pm at}\, x(t)\, e^{-st}\, dt$$

$$= \int_{-\infty}^{+\infty} x(t)\, e^{-(s \mp a)t}\, dt$$

$$= X(s \mp a)$$

The term $\int_{-\infty}^{+\infty} x(t)\, e^{-(s \mp a)t}\, dt$ is similar to the form of definition of Laplace transform (equation(3.13)) except that s is replaced by $(s \mp a)$.

$$\therefore \int_{-\infty}^{+\infty} x(t)\, e^{-(s \mp a)t} dt = X(s \mp a)$$

6. Time shifting

The time shifting property of Laplace transform says that,

If, $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x(t \pm a)\} = e^{\pm as} X(s) \qquad \left[\text{i.e.,} \mathcal{L}\{x(t+a)\} = e^{as} X(s) \text{ and } \mathcal{L}\{x(t-a)\} = e^{-as} X(s)\right]$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.14)$$

Let, $t \pm a = \tau$
$\therefore t = \tau \mp a$
On differentiating
$dt = d\tau$

$$\therefore \mathcal{L}\{x(t \pm a)\} = \int_{-\infty}^{+\infty} x(t \pm a)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-s(\tau \mp a)}\, d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau)\, e^{-s\tau} \times e^{\pm as}\, d\tau = e^{\pm as} \int_{-\infty}^{+\infty} x(\tau)\, e^{-s\tau}\, d\tau$$

Since τ is a dummy variable for integration we can change τ to t.

$$= e^{\pm as} \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = e^{\pm as} X(s)$$

Using equation (3.14)

7. Frequency Differentiation

The frequency differentiation property of Laplace transform says that,

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

On differentiating the above equation with respect to s we get,

$$\frac{d}{ds}X(s) = \frac{d}{ds}\left(\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt\right)$$

$$= \int_{-\infty}^{+\infty} x(t)\left(\frac{d}{ds}e^{-st}\right)dt = \int_{-\infty}^{+\infty} x(t)\left(-t\,e^{-st}\right)dt$$

Interchanging the order of integration and differentiation

$$= \int_{-\infty}^{+\infty} (-t\,x(t))\,e^{-st}\,dt = \mathcal{L}\{-t\,x(t)\} = -\mathcal{L}\{t\,x(t)\}$$

$$\therefore \mathcal{L}\{t\,x(t)\} = -\frac{d}{ds}X(s)$$

8. Frequency Integration

The frequency integration property of Laplace transform says that,

i.e., If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\left\{\frac{1}{t}\,x(t)\right\} = \int_s^{\infty} X(s)\,ds$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

On integrating the above equation with respect to s between limits s to ∞ we get,

$$\int_s^{\infty} X(s)\,ds = \int_s^{\infty}\left[\int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt\right]ds$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\int_s^{\infty} e^{-st}\,ds\right]dt$$

Interchanging the order of integrations.

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-st}}{-t}\right]_s^{\infty} dt = \int_{-\infty}^{+\infty} x(t)\left[\frac{e^{-\infty}}{-t} - \frac{e^{-st}}{-t}\right]dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[0 + \frac{e^{-st}}{t}\right]dt = \int_{-\infty}^{+\infty}\left[\frac{1}{t}\,x(t)\right]e^{-st}\,dt = \mathcal{L}\left\{\frac{1}{t}\,x(t)\right\}$$

9. *Time scaling*

The time scaling property of Laplace transform says that,

If $\mathcal{L}\{x(t)\} = X(s)$, then

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$$

Proof :

By definition of Laplace transform,

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt \qquad .....(3.15)$$

Put, $at = \tau$

$\therefore\ t = \dfrac{\tau}{a}$

On differentiating

$dt = \dfrac{d\tau}{a}$

$$\therefore\ \mathcal{L}\{x(at)\} = \int_{-\infty}^{+\infty} x(at)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} x(\tau)\, e^{-s\left(\frac{\tau}{a}\right)} \frac{d\tau}{a}$$

$$= \frac{1}{a} \int_{-\infty}^{+\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau = \frac{1}{a} X\left(\frac{s}{a}\right)$$

The above transform is applicable for positive values of "a".

If "a" happens to be negative it can be proved that,

$$\mathcal{L}\{x(at)\} = -\frac{1}{a} X\left(\frac{s}{a}\right)$$

Hence in general,

$$\mathcal{L}\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right) \text{ for both positive and negative values of "a"}$$

The term $\int_{-\infty}^{+\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau$ is similar to the form of definition of Laplace transform(equation(3.15)) except that s is replaced by $\left(\frac{s}{a}\right)$.

$$\therefore \int_{-\infty}^{+\infty} x(\tau) e^{-\left(\frac{s}{a}\right)\tau} d\tau = X\left(\frac{s}{a}\right)$$

10. Periodicity

The periodicity property of Laplace transform says that,

If $x(t) = x(t+nT)$, and $x_1(t)$ be one period of $x(t)$, and $\mathcal{L}\{x_1(t)\} = \int_0^T x_1(t)\, e^{-st}\, dt$, then

$$\mathcal{L}\{x(t + nT)\} = \frac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

Proof :

By definition of Laplace transform,

$$\mathcal{L}\{x(t + nT)\} = \int_0^\infty x(t + nT)\, e^{-st}\, dt$$

$$= \int_0^T x_1(t)\, e^{-st}\, dt + \int_T^{2T} x_1(t - T)\, e^{-s(t+T)}\, dt + \int_{2T}^{3T} x_1(t - 2T)\, e^{-s(t+2T)}\, dt + \ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots + \int_{pT}^{(p+1)T} x_1(t - pT)\, e^{-s(t+pT)}\, dt + \ldots\ldots\ldots$$

$$= \sum_{p=0}^{\infty} \int_{pT}^{(p+1)T} x_1(t - pT)\, e^{-s(t+pT)}\, dt$$

$$= \sum_{p=0}^{\infty} \int_{0}^{T} x_1(t)\, e^{-st}\, e^{-psT}\, dt$$

The periodic signal will be identical in every period and so, $x_1(t+pT) = x_1(t)$.

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\sum_{p=0}^{\infty} e^{-psT} \right) dt$$

Interchanging the order of integration and summation

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\sum_{p=0}^{\infty} e^{-sT} \right)^{p} dt$$

Using infinite geometric series sum formula $\sum_{n=0}^{\infty} C^n = \dfrac{1}{1 - C}$

$$= \int_{0}^{T} x_1(t)\, e^{-st} \left(\frac{1}{1 - e^{-sT}} \right) dt$$

$$= \frac{1}{1 - e^{-sT}} \int_{0}^{T} x_1(t)\, e^{-st}\, dt$$

The term $\dfrac{1}{1 - e^{-sT}}$ is independent of **t**

11. Initial Value Theorem

The initial value theorem states that, if x(t) and its derivative are Laplace transformable then,

$$\underset{t \to 0}{Lt}\ x(t) = \underset{s \to \infty}{Lt}\ s\,X(s)$$

i.e., Initial value of signal, $x(0) = \underset{t \to 0}{Lt}\ x(t) = \underset{s \to \infty}{Lt}\ s\,X(s)$

Proof :

We know that, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\,X(s) - x(0)$

On taking limit $s \to \infty$ on both sides of the above equation we get,

$$\underset{s \to \infty}{Lt}\ \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \underset{s \to \infty}{Lt}\ [s\,X(s) - x(0)]$$

By definition of Laplace transform, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt$

$$\underset{s \to \infty}{Lt} \int_0^{\infty} \frac{dx(t)}{dt}\, e^{-st}\, dt = \underset{s \to \infty}{Lt}\ [s\,X(s) - x(0)]$$

Here $\frac{dx(t)}{dt}$ and x(0) are not functions of s

$$\int_0^{\infty} \frac{dx(t)}{dt}\left(\underset{s \to \infty}{Lt}\ e^{-st}\right) dt = \left(\underset{s \to \infty}{Lt}\ s\,X(s)\right) - x(0)$$

$\underset{s \to \infty}{Lt}\ e^{-st} = 0$

$$0 = \underset{s \to \infty}{Lt}\ s\,X(s) - x(0)$$

$$\therefore\ x(0) = \underset{s \to \infty}{Lt}\ s\,X(s)$$

$x(0) = \underset{t \to 0}{Lt}\ x(t)$

$$\therefore\ \underset{t \to 0}{Lt}\ x(t) = \underset{s \to \infty}{Lt}\ s\,X(s)$$

12. Final Value Theorem

The final value theorem states that if x(t) and its derivative are Laplace transformable then $\underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ s\,X(s)$

i.e., Final value of signal, $x(\infty) = \underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ s\,X(s)$

Proof :

We know that, $\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = s\,X(s) - x(0)$

On taking limit $s \to 0$ on both sides of the above equation we get,

$$\underset{s \to 0}{Lt}\ \mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \underset{s \to 0}{Lt}\ [s\,X(s) - x(0)]$$

By definition of Laplace transform

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt$$

$$\underset{s \to 0}{Lt} \int_0^\infty \frac{dx(t)}{dt}\, e^{-st}\, dt = \underset{s \to 0}{Lt}\ [s\,X(s) - x(0)]$$

Here $\frac{dx(t)}{dt}$ and x(0) are not functions of s

$$\int_0^\infty \frac{dx(t)}{dt}\left(\underset{s \to 0}{Lt}\ e^{-st}\right) dt = \left(\underset{s \to 0}{Lt}\ s\,X(s)\right) - x(0)$$

$$\underset{s \to 0}{Lt}\ e^{-st} = 1$$

$$\int_0^\infty \frac{dx(t)}{dt}\, dt = \underset{s \to 0}{Lt}\ s\,X(s) - x(0)$$

$$[x(t)]_0^\infty = \underset{s \to 0}{Lt}\ s\,X(s) - x(0)$$

$$x(\infty) - x(0) = \underset{s \to 0}{Lt}\ s\,X(s) - x(0)$$

$$\therefore x(\infty) = \underset{s \to 0}{Lt}\ s\,X(s)$$

$$\therefore \underset{t \to \infty}{Lt}\ x(t) = \underset{s \to 0}{Lt}\ s\,X(s)$$

$$x(\infty) = \underset{t \to \infty}{Lt}\ x(t)$$

13. Convolution Theorem

The convolution theorem of Laplace transform says that, Laplace transform of convolution of two signals is given by the product of the Laplace transform of the individual signals.

i.e., if $\mathcal{L}\{x_1(t)\} = X_1(s)$ and $\mathcal{L}\{x_2(t)\} = X_2(s)$ then,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\, X_2(s) \qquad .....(3.16)$$

The equation (3.16) is also known as convolution property of Laplace transform.

With reference to chapter-2, section 2.9 we get,

$$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\, x_2(t-\lambda)\, d\lambda \qquad ....(3.17)$$

where, λ is a dummy variable used for integration.

Proof :

Let $x_1(t)$ and $x_2(t)$ be two time domain signals.

By definition of Laplace transform,

$$X_1(s) = \mathcal{L}\{x_1(t)\} = \int_{-\infty}^{+\infty} x_1(t)\, e^{-st}\, dt \qquad \text{.....(3.18)}$$

$$X_2(s) = \mathcal{L}\{x_2(t)\} = \int_{-\infty}^{+\infty} x_2(t)\, e^{-st}\, dt \qquad \text{.....(3.19)}$$

Let $x_3(t)$ be the signal obtained by convolution of $x_1(t)$ and $x_2(t)$. Now from equation (3.17) we get,

$$x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(t-\lambda)\, d\lambda \qquad \text{.....(3.20)}$$

Let, $\mathcal{L}\{x_3(t)\} = X_3(s)$. Now by definition of Laplace transform we can write,

$$X_3(s) = \mathcal{L}\{x_3(t)\} = \int_{-\infty}^{+\infty} x_3(t)\, e^{-st}\, dt \qquad \text{..... (3.21)}$$

On substituing for $x_3(t)$ from equation (3.20) in equation (3.21) we get,

$$X_3(s) = \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} x_1(\lambda)\ x_2(t-\lambda)\ d\lambda \right] e^{-st}\ dt \qquad \text{..... (3.22)}$$

Let, $e^{-st} = e^{s\lambda} \times e^{-s\lambda} \times e^{-st} = e^{-s\lambda} \times e^{-s(t-\lambda)} = e^{-s\lambda} \times e^{-sM}$ ..... (3.23)

where, $M = t - \lambda$ and so, $dM = dt$ .....(3.24)

Using equations (3.23) and (3.24), the equation (3.22) can be written as,

$$X_3(s) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1(\lambda)\ x_2(M)\ e^{-s\lambda}\ e^{-sM}\ d\lambda\ dM$$

$$= \int_{-\infty}^{+\infty} x_1(\lambda)\ e^{-s\lambda}\ d\lambda \times \int_{-\infty}^{+\infty} x_2(M)\ e^{-sM}\ dM \qquad \text{.....(3.25)}$$

In equation (3.25), λ and M are dummy variables used for integration, and so they can be changed to t.

Therefore equation (3.25) can be written as,

$$X_3(s) = \int_{-\infty}^{+\infty} x_1(t)\ e^{-st}\ dt \times \int_{-\infty}^{+\infty} x_2(t)\ e^{-st}\ dt$$

$$= X_1(s)\ X_2(s)$$

Using equations (3.18) and (3.19)

$$\therefore \mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\ X_2(s)$$

Table 3.3 : Properties of Laplace Transform

Note : $\mathcal{L}\{x(t)\} = X(s);\quad \mathcal{L}\{x_1(t)\} = X_1(s);\quad \mathcal{L}\{x_2(t)\} = X_2(s)$

Property	Time domain signal	s-domain signal
Amplitude scaling	$A\,x(t)$	$A\,X(s)$
Linearity	$a_1 x_1(t) \pm a_2 x_2(t)$	$a_1 X_1(s) \pm a_2 X_2(s)$
Time differentiation	$\frac{d}{dt}x(t)$	$s\,X(s) - x(0)$
	$\frac{d^n}{dt^n}x(t)$ where $n = 1, 2, 3$	$s^n X(s) - \sum_{K=1}^{n} s^{n-K} \left.\frac{d^{(K-1)}x(t)}{dt^{K-1}}\right\|_{t=0}$
Time integration	$\int x(t)\,dt$	$\frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]_{t=0}}{s}$
	$\int \text{.....} \int x(t)\,(dt)^n$ where $n = 1, 2, 3$	$\frac{X(s)}{s^n} + \sum_{K=1}^{n} \frac{1}{s^{n-K+1}} \left[\int \text{....} \int x(t)\,(dt)^k\right]_{t=0}$
Frequency shifting	$e^{\pm at}\,x(t)$	$X(s \mp a)$
Time shifting	$x(t \pm \alpha)$	$e^{\pm \alpha s}\,X(s)$

Frequency differentiation	$t\, x(t)$	$-\dfrac{dX(s)}{ds}$
	$t^n\, x(t)$ where $n = 1, 2, 3 \ldots\ldots$	$(-1)^n \dfrac{d^n}{ds^n} X(s)$
Frequency integration	$\dfrac{1}{t}\, x(t)$	$\int_s^{\infty} X(s)\, ds$
Time scaling	$x(at)$	$\dfrac{1}{\lvert a \rvert} X\left(\dfrac{s}{a}\right)$
Periodicity	$x(t + nT)$	$\dfrac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$ where, $x_1(t)$ is one period of $x(t)$.
Initial value theorem	$\underset{t \to 0}{\mathrm{Lt}}\; x(t) = x(0)$	$\underset{s \to \infty}{\mathrm{Lt}}\; s\, X(s)$
Final value theorem	$\underset{t \to \infty}{\mathrm{Lt}}\; x(t) = x(\infty)$	$\underset{s \to 0}{\mathrm{Lt}}\; s\, X(s)$
Convolution theorem	$x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\lambda)\; x_2(t-\lambda)\, d\lambda$	$X_1(s)\, X_2(s)$

Example 3.5

Determine Laplace transform of periodic square wave shown in fig 3.5.1.

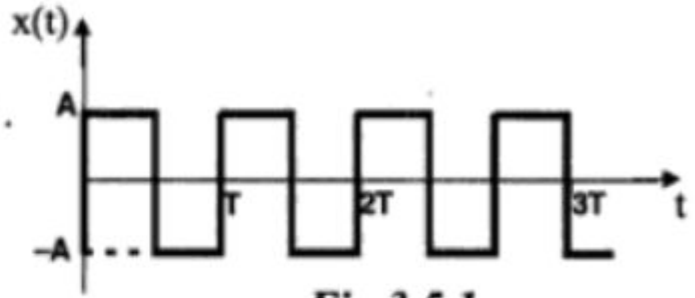


Fig 3.5.1.

Solution

The given waveform satisfy the condition, $x(t+nT) = x(t)$, and so it is periodic.

Let $x_1(t)$ be one period of $x(t)$. The equation for one period of the periodic waveform of fig 3.5.1 is,

$$x_1(t) = A \quad ; \text{for } t = 0 \text{ to } \frac{T}{2}$$

$$= -A \quad ; \text{for } t = \frac{T}{2} \text{ to } T$$

From periodicity property of Laplace transform,

If $X(s) = \mathcal{L}\{x(t)\}$, and if $x(t) = x(t + nT)$ then, $X(s) = \dfrac{1}{1 - e^{-sT}} \displaystyle\int_0^T x_1(t)\, e^{-st}\, dt$, where $x_1(t)$ is one period of $x(t)$.

$$\therefore \mathcal{L}\{x(t)\} = X(s) = \frac{1}{1 - e^{-sT}} \int_0^T x_1(t)\, e^{-st}\, dt$$

Using the result of example 3.3(a), the above equation can be written as,

From example 3.3(a) we get,

$$\int_0^T x_1(t)\, e^{-st}\, dt = \frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$$

$$X(s) = \frac{1}{1 - e^{-sT}} \left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

If $X(s) = \mathcal{L}\{x(t)\}$, and if $x(t) = x(t + nT)$ then, $X(s) = \dfrac{1}{1 - e^{-sT}} \int\limits_0^T x_1(t)\, e^{-st}\, dt$, where $x_1(t)$ is one period of $x(t)$.

$$\therefore \mathcal{L}\{x(t)\} = X(s) = \frac{1}{1 - e^{-sT}} \int\limits_0^T x_1(t)\, e^{-st}\, dt$$

Using the result of example 3.3(a), the above equation can be written as,

From example 3.3(a) we get,

$$\int\limits_0^T x_1(t)\, e^{-st}\, dt = \frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2$$

$$X(s) = \frac{1}{1 - e^{-sT}}\left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

$$= \frac{1}{\left(1 + e^{-\frac{sT}{2}}\right)\left(1 - e^{-\frac{sT}{2}}\right)}\left[\frac{A}{s}\left(1 - e^{-\frac{sT}{2}}\right)^2\right]$$

$$\boxed{a^2 - b^2 = (a + b)\,(a - b)}$$

$$= \frac{A}{s}\left(\frac{1 - e^{-\frac{sT}{2}}}{1 + e^{-\frac{sT}{2}}}\right)$$

Example 3.7

Determine the Laplace transform of the following signals using properties of Laplace transform.

a) $x(t) = (t^2 - 2t)\, u(t-1)$ b) Unit ramp signal starting at $t = a$. c)

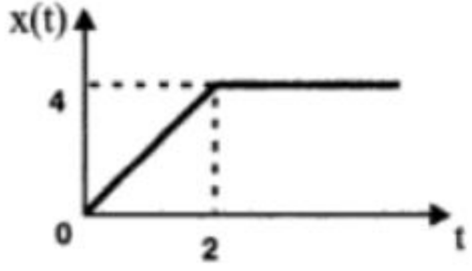


Fig 3.7.1.

Solution

a) Given that, $x(t) = (t^2 - 2t)\, u(t-1) = t^2\, u(t-1) - 2t\, u(t-1)$

From table 3.2 we get,

$$\mathcal{L}\{t^2\, u(t)\} = \frac{2}{s^3} \qquad .....(1)$$

$$\mathcal{L}\{t\, u(t)\} = \frac{1}{s^2} \quad ; \quad \therefore\ \mathcal{L}\{2t\, u(t)\} = \frac{2}{s^2} \qquad .....(2)$$

From time delay property of Laplace transform,

If $\mathcal{L}\{x(t)\, u(t)\} = X(s)$, then $\mathcal{L}\{x(t)\, u(t-a)\} = e^{-as}\, X(s)$

$$\therefore\ X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\left\{\left(t^2 - 2t\right) u(t-1)\right\} = \mathcal{L}\left\{t^2\, u(t-1)\right\} - \mathcal{L}\{2t\, u(t-1)\}$$

$$= e^{-s}\, \mathcal{L}\left\{t^2\, u(t)\right\} - e^{-s}\, \mathcal{L}\{2t\, u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= e^{-s}\, \frac{2}{s^3} - e^{-s}\, \frac{2}{s^2} = 2e^{-s}\left(\frac{1-s}{s^3}\right) = \frac{2e^{-s}(1-s)}{s^3} \qquad \boxed{\text{Using equations (1) and (2)}}$$

b) Given that, Unit ramp signal starting at $t = a$.

The unit ramp starting at $t = a$, is unit ramp delayed by "a" units of time. The unit ramp waveform and the ramp waveform starting at $t = a$ are shown in fig 1 and fig 2 respectively. The equation of unit ramp and delayed ramp are given below.

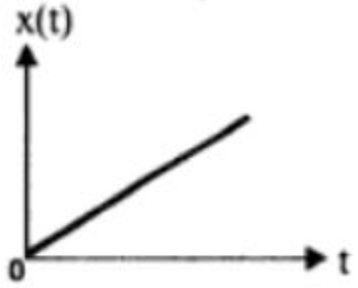


Fig 1 : Ramp.

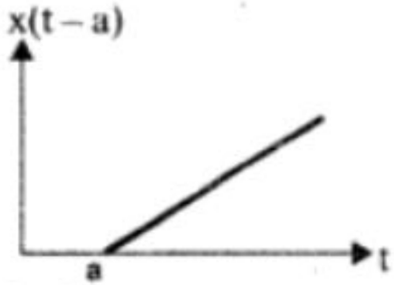


Fig 2 : Ramp starting at $t = a$.

Unit ramp, $x(t) = t\,u(t)$

Delayed unit ramp, $x(t-a) = (t-a)\,u(t-a)$

From table 3.2 we get,

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2} \qquad .....(3)$$

From time delay property of Laplace transform,

If $\mathcal{L}\{x(t)\,u(t)\} = X(s)$, then $\mathcal{L}\{x(t)\,u(t-a)\} = e^{-as}\,X(s)$

Therefore Laplace transform of delayed ramp signal is,

$$\mathcal{L}\{x(t-a)\} = e^{-as}\,\mathcal{L}\{x(t)\}$$

$$= e^{-as}\,\mathcal{L}\{t\,u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= e^{-as}\,\frac{1}{s^2} = \frac{e^{-as}}{s^2} \qquad \boxed{\text{Using equation (3)}}$$

c)

The given signal can be decomposed into two signals as shown in fig 4 and fig 5.

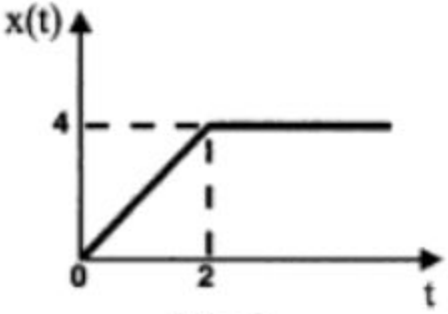


Fig 3.

$\Rightarrow$

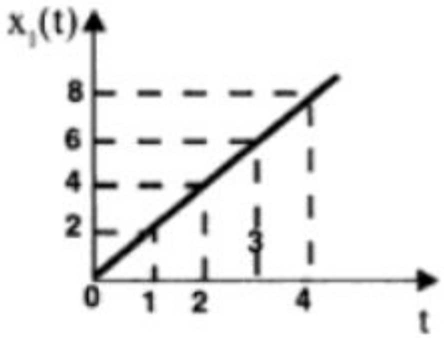


Fig 4.

+

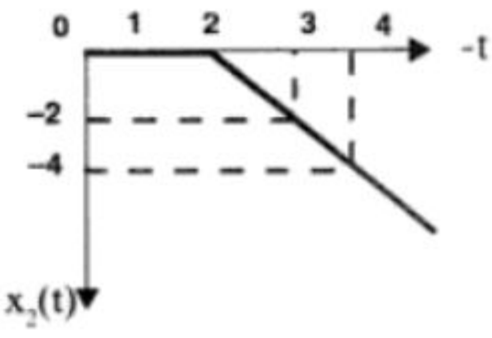


Fig 5.

The mathematical equations of signals, $x_1(t)$ and $x_2(t)$ are given below.

$$x_1(t) = 2t\,u(t)$$

$$x_2(t) = -2(t-2)\,u(t-2)$$

$$\therefore \quad x(t) = x_1(t) + x_2(t) = 2t\,u(t) - 2(t-2)\,u(t-2)$$

From table 3.2 we get,

$$\mathcal{L}\{t\,u(t)\} = \frac{1}{s^2} \quad ; \quad \therefore \mathcal{L}\{2t\,u(t)\} = \frac{2}{s^2} \qquad .....(4)$$

$$\therefore X(s) = \mathcal{L}\{x(t)\} = \mathcal{L}\{x_1(t) + x_2(t)\} = \mathcal{L}\{2t\,u(t) - 2(t-2)\,u(t-2)\}$$

$$= \mathcal{L}\{2t\,u(t)\} - \mathcal{L}\{2(t-2)\,u(t-2)\}$$

$$= \mathcal{L}\{2t\,u(t)\} - e^{-2s}\,\mathcal{L}\{2t\,u(t)\} \qquad \boxed{\text{Using time delay property}}$$

$$= \frac{2}{s^2} - e^{-2s}\frac{2}{s^2} = \frac{2\left(1 - e^{-2s}\right)}{s^2} \qquad \boxed{\text{Using equation (4)}}$$

Example 3.8

Let, $X(s) = \mathcal{L}\{x(t)\}$. Determine the initial value, $x(0)$ and the final value, $x(\infty)$, for the following signals using initial value and final value theorems.

a) $X(s) = \dfrac{1}{s(s-1)}$ b) $X(s) = \dfrac{s+1}{s^2+2s+2}$ c) $X(s) = \dfrac{7s+6}{s(3s+5)}$

d) $X(s) = \dfrac{s^2+1}{s^2+6s+5}$ e) $X(s) = \dfrac{s+5}{s^2(s+9)}$

Solution

a) Given that, $X(s) = \dfrac{1}{s(s-1)}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{\text{Lt}}\ s\,X(s) = \underset{s\to\infty}{\text{Lt}}\ s\,\frac{1}{s(s-1)} = \underset{s\to\infty}{\text{Lt}}\ \frac{1}{(s-1)} = \underset{s\to\infty}{\text{Lt}}\ \frac{1}{s\left(1-\frac{1}{s}\right)}$$

$$= \underset{s\to\infty}{\text{Lt}}\ \frac{1}{s}\,\frac{1}{\left(1-\frac{1}{s}\right)} = \frac{1}{\infty}\,\frac{1}{\left(1-\frac{1}{\infty}\right)} = 0 \times \frac{1}{1-0} = 0$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{\text{Lt}}\ sX(s) = \underset{s\to 0}{\text{Lt}}\ s\,\frac{1}{s(s-1)} = \underset{s\to 0}{\text{Lt}}\ \frac{1}{(s-1)} = \frac{1}{0-1} = -1$$

b) Given that, $X(s) = \dfrac{s+1}{s^2+2s+2}$

$$\text{Initial value, } x(0) = \underset{s\to\infty}{\mathrm{Lt}}\ sX(s) = \underset{s\to\infty}{\mathrm{Lt}}\ s\frac{s+1}{s^2+2s+2} = \underset{s\to\infty}{\mathrm{Lt}}\ s\frac{s\left(1+\frac{1}{s}\right)}{s^2\left(1+\frac{2}{s}+\frac{2}{s^2}\right)}$$

$$= \underset{s\to\infty}{\mathrm{Lt}}\ \frac{1+\frac{1}{s}}{1+\frac{2}{s}+\frac{2}{s^2}} = \frac{1+\frac{1}{\infty}}{1+\frac{2}{\infty}+\frac{2}{\infty}} = \frac{1+0}{1+0+0} = 1$$

$$\text{Final value, } x(\infty) = \underset{s\to 0}{\mathrm{Lt}}\ sX(s) = \underset{s\to 0}{\mathrm{Lt}}\ s\,\frac{s+1}{s^2+2s+2}$$

$$= 0\times\frac{0+1}{0+0+2} = 0$$

Example 3.9

Perform convolution of $x_1(t)$ and $x_2(t)$ using convolution theorem of Laplace transform.

a) $x_1(t) = u(t+5),\ x_2(t) = \delta(t-7)$

b) $x_1(t) = u(t-2),\ x_2(t) = \delta(t+6)$

c) $x_1(t) = u(t+1),\ x_2(t) = r(t-2)$; where $r(t) = t\,u(t)$

Solution

$\mathcal{L}\{u(t)\} = \dfrac{1}{s}$ and $\mathcal{L}\{\delta(t)\} = 1$

if $\mathcal{L}\{x(t)\} = X(s)$ then $\mathcal{L}\{x(t \pm a)\} = e^{\pm as}X(s)$

a) Given that, $x_1(t) = u(t+5),\ x_2(t) = \delta(t-7)$

$$x_1(t) = u(t+5)$$

$$\therefore X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t+5)\} = e^{5s}\,\mathcal{L}\{u(t)\} = e^{5s} \times \frac{1}{s} = \frac{e^{5s}}{s} \quad .....(1)$$

$$x_2(t) = \delta(t-7)$$

$$\therefore X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{\delta(t-7)\} = e^{-7s}\,\mathcal{L}\{\delta(t)\} = e^{-7s} \times 1 = e^{-7s} \quad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{e^{5s}}{s} \times e^{-7s} = \frac{e^{5s-7s}}{s} = \frac{e^{-2s}}{s} \qquad \boxed{\text{Using equations (1) and (2)}}$$

$$\therefore x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Bigg|_{t=t-2} = u(t)\Big|_{t=t-2} = u(t-2)$$

b) Given that, $x_1(t) = u(t-2)$, $x_2(t) = \delta(t+6)$

$$\boxed{\begin{array}{l}\mathcal{L}\{u(t)\} = \dfrac{1}{s} \text{ and } \mathcal{L}\{\delta(t)\} = 1 \\ \text{if } \mathcal{L}\{x(t)\} = X(s) \text{ then } \mathcal{L}\{x(t \pm a)\} = e^{\pm as}X(s)\end{array}}$$

$$x_1(t) = u(t-2)$$

$$\therefore X_1(s) = \mathcal{L}\{x_1(t)\} = \mathcal{L}\{u(t-2)\} = e^{-2s}\,\mathcal{L}\{u(t)\} = e^{-2s} \times \frac{1}{s} = \frac{e^{-2s}}{s} \qquad .....(1)$$

$$x_2(t) = \delta(t+6)$$

$$\therefore X_2(s) = \mathcal{L}\{x_2(t)\} = \mathcal{L}\{\delta(t+6)\} = e^{6s}\,\mathcal{L}\{\delta(t)\} = e^{6s} \times 1 = e^{6s} \qquad .....(2)$$

From convolution theorem of Laplace transform,

$$\mathcal{L}\{x_1(t) * x_2(t)\} = X_1(s)\,X_2(s)$$

$$= \frac{e^{-2s}}{s} \times e^{6s} = \frac{e^{-2s+6s}}{s} = \frac{e^{4s}}{s} \qquad \boxed{\text{Using equations (1) and (2)}}$$

$$\therefore x_1(t) * x_2(t) = \mathcal{L}^{-1}\left\{\frac{e^{4s}}{s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\}\Bigg|_{t=t+4} = u(t)\big|_{t=t+4} = u(t+4)$$

3.4 Poles and Zeros of Rational Function of s

Let X(s) be Laplace transform of x(t). When X(s) is expressed as a ratio of two polynomials in s, then the s-domain signal X(s) is called a ***rational function*** of s.

The zeros and poles are two critical complex frequencies at which a rational function of s takes two extreme values, such as zero and infinity respectively.

Let X(s) is expressed as a ratio of two polynomials in s as shown in equation (3.26).

$$X(s) = \frac{P(s)}{Q(s)}$$

$$= \frac{b_0 s^M + b_1 s^{M-1} + b_2 s^{M-2} + \ldots\ldots + b_{M-1} s + b_M}{a_0 s^N + a_1 s^{N-1} + a_2 s^{N-2} + \ldots\ldots + a_{N-1} s + a_N} \quad \ldots\ldots(3.26)$$

where, P(s) = Numerator polynomial of X(s)

Q(s) = Denominator polynomial of X(s)

In equation (3.26) let us scale the coefficients of numerator polynomial by b_0 and the coefficients of denominator polynomial by a_0, and the equation (3.26) can be expressed in factorized form as shown in equation (3.27).

$$X(S) = \frac{b_0\left(s^M + \frac{b_1}{b_0} s^{M-1} + \frac{b_2}{b_0} s^{M-2} + \ldots\ldots + \frac{b_{M-1}}{b_0} s + \frac{b_M}{b_0}\right)}{a_0\left(s^N + \frac{a_1}{a_0} s^{N-1} + \frac{a_2}{a_0} s^{N-2} + \ldots\ldots + \frac{a_{N-1}}{a_0} s + \frac{a_N}{a_0}\right)}$$

$$= G \frac{(s - z_1)(s - z_2) \ldots\ldots (s - z_M)}{(s - p_1)(s - p_2) \ldots\ldots (s - p_N)} \qquad \ldots\ldots(3.27)$$

where, $G = \frac{b_0}{a_0}$ = Scaling factor

$z_1, z_2, \ldots\ldots, z_M$ = Roots of numerator polynomial, P(s)

$p_1, p_2, \ldots\ldots, p_N$ = Roots of denominator polynomial, Q(s)

In equation (3.27) if the value of s is equal to any one of the root of numerator polynomial then the signal X(s) will become zero.

Therefore the roots of numerator polynomial $z_1, z_2, \ldots\ldots, z_M$ are called ***zeros*** of X(s). Since s is complex frequency, the ***zeros*** can be defined as values of complex frequencies at which the signal X(s) becomes zero.

In equation (3.27) if the value of s is equal to any one of the root of numerator polynomial then the signal X(s) will become zero.

Therefore the roots of numerator polynomial z_1, z_2, ..., z_M are called ***zeros*** of X(s). Since s is complex frequency, the ***zeros*** can be defined as values of complex frequencies at which the signal X(s) becomes zero.

In equation (3.27), if the value of s is equal to any one of the roots of the denominator polynomial then the signal X(s) will become infinite. Therefore the roots of denominator polynomial p_1, p_2, ..., p_N are called ***poles*** of X(s). Since **s** is complex frequency, the ***poles*** can be defined as values of complex frequencies at which the signal X(s) become infinite. *Since the signal X(s) attains infinte value at poles, the ROC of X(s) does not include poles.*

In a realizable system, *the number of zeros will be less than or equal to number of poles.* Also for every zero, we can associate one pole. (When number of finite zeros are less than poles, the missing zeros are assumed to exist at infinity).

Let z_i be the zero associated with the pole p_i. If we evaluate |X(s)| for various values of s, then |X(s)| will be zero for $s = z_i$ and infinite for $s = p_i$. Hence the plot of |X(s)| in a three dimensional plane will look like a pole (or pillar like structure) and so the point $s = p_i$ is called a pole.

3.4.3 Properties of ROC

The various concepts of ROC that has been discussed in section 3.2 and 3.4.2 are summarized as properties of ROC and given below.

Property-1 : The ROC of X(s) consists of strips parallel to the $j\Omega$ - axis in the s-plane.

Property-2 : If x(t) is of finite duration and is absolutely integrable, then the ROC is the entire s- plane.

Property-3 : If x(t) is right sided, and if the line passing through $Re(s) = \sigma_0$ is in ROC, then all values of s for which $Re(s) > \sigma_0$ will also be in ROC.

Property-4 : If x(t) is left sided, and if the line passing through $Re(s) = \sigma_0$ is in ROC, then all values of s for which $Re(s) < \sigma_0$ will also be in ROC.

Property-5 : If x(t) is two sided, and if the line passing through $Re(s) = \sigma_0$ is in ROC, then the ROC will consists of a strip in the s-plane that includes the line passing through $Re(s) = \sigma_0$.

Property-6 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), then its ROC is bounded by poles or extends to infinity.

Property-7 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), then ROC does not include any poles of X(s).

Property-8 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), and if x(t) is right sided, then ROC is the region in s-plane to the right of the rightmost pole.

Property-9 : If X(s) is rational, (where X(s) is Laplace transform of x(t)), and if x(t) is left sided, then ROC is the region in s-plane to the left of the leftmost pole.