Electronics and Telecommunication Engineering

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Engineering Mathematics IV

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Syllabus Content

Module: Linear Algebra: Quadratic Forms

5.1 Quadratic forms over real field, Linear Transformation of Quadratic form, Reduction of Quadratic form to diagonal form using congruent transformation.

5.2 Rank, index, signature of quadratic form, Sylvester law of inertia, Value class of a quadratic form-Definite, Semidefinite and Indefinite.

5.3 Reduction of Quadratic form to a canonical form using congruent transformations.

5.4 Singular Value Decomposition.

Self-learning Topics: Orthogonal Transformations, Applications of Quadratic forms and SVD in Engineering.

Course Outcome

At the end of the topic, student should be able to Reduce the Quadratic form to a canonical form using congruent and orthogonal transformations

Homogeneous polynomial of second degree in N variables is called a quadratic form

1. $ax^2 + by^2 + 2cxy$ $\quad 1+1=2$

$$\frac{2c}{2} = c$$

2. Every quadratic form can be expressed in matrix notation as XAX', where X is a column matrix X' is its transpose

$$\underbrace{\begin{bmatrix} x & y \end{bmatrix}}_{X_{1\times 2}} \underbrace{\begin{bmatrix} a & c \\ c & b \end{bmatrix}}_{A_{2\times 2}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X'_{2\times 1}}$$

Homogeneous polynomial of second degree in N variables is called a quadratic form

1. Express $2x^2 + 3y^2 - 5z^2 - 2xy + 4xz - 6yz$ in matrix form

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{array}{c c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x \\ y \\ z \end{array} & \begin{bmatrix} 2 & -1 & 2 \\ -1 & 3 & -3 \\ 2 & -3 & -5 \end{bmatrix} \end{array} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$1\times\cancel{3}$ $\cancel{3}\times 3$ $\cancel{3}\times 1$

$1\times\cancel{3}$ $\boxed{1\times 1}$

Homogeneous polynomial of second degree in N variables is called a quadratic form

1. Express $2x_1^2 - 3x_2^2 + 4x_3^2 + x_4^2 - 2x_1x_2 + 3x_1x_3 - 4x_1x_4 - 5x_2x_3 + 6x_2x_4 + x_3x_4$ *in matrix form*

$$\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix} \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 2 & \boxed{-1} & \boxed{3/2} & -2 \\ \boxed{-1} & -3 & -5/2 & 3 \\ \boxed{3/2} & -5/2 & 4 & 1/2 \\ -2 & 3 & 1/2 & 1 \end{bmatrix} \end{array} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Symmetric

$X' \quad A \quad X$

Linear Transformation of a ~~Quadratic Form~~

Example: Express each of the following transformations

$$x_1 = 2y_1 - 3y_2, x_2 = 4y_1 + y_2 \; and \; y_1 = z_1 - 2z_2, y_2 = 2z_1 + 3z_2$$

In the form and find the composite transformation which expresses $x_1 \; and \; x_2 in \; terms \; of \; z_1 and \; z_2$

$x_1 = 2y_1 - 3y_2$

$x_2 = 4y_1 + y_2$

$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$y_1 = z_1 - 2z_2$

$y_2 = 2z_1 + 3z_2$

$\Rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$= \begin{bmatrix} 2-6 & -4-9 \\ 4+2 & -8+3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$

$\therefore \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4 & -13 \\ 6 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \Rightarrow$ $x_1 = -4z_1 - 13z_2$ ✓

$x_2 = 6z_1 - 5z_2$

$x_1 = 2y_1 - 3y_2 = 2(z_1 - 2z_2) - 3(2z_1 + 3z_2)$

$= 2z_1 - 4z_2 - 6z_1 - 9z_2$

$= -4z_1 - 13z_2$

Linear Transformation of a Quadratic Form

Consider the quadratic form $X'AX$, and the non singular transformation $X = PY$. [A is symmetric]

$(AB)' = B'A'$

$$X'AX = (PY)'APY = (Y'P')APY = Y'(P'AP)Y = Y'BY, where\ \boldsymbol{B} = \boldsymbol{P'AP}$$

$$\boxed{\boldsymbol{Y'BY}}\ \boldsymbol{is\ called\ linear\ transform\ of\ quadratic\ form\ X'AX\ under\ X = PY}$$

Congruence of a Square Matrix: A square matrix B of order n is said to be congruent to another square matrix A of the same order, if there exists a non-singular matrix P such that $B = P'AP$

Example 01: Reduce the following quadratic form to a diagonal form through congruent transformation ; $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 + 4x_1x_3 - 2x_2x_3$ [Find diagonal matrix=P'AP]

$$X'AX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = I_3 A I_3$$

$$D = P' A P$$

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ -2 & -1 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R} A \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C}$$

$R_2 + \frac{1}{3}R_1$

$R_3 - \frac{1}{3}R_1$

$C_2 + \frac{1}{3}C_1$

$C_3 - \frac{1}{3}C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-\frac{1}{3} + \frac{1}{7}\cdot\frac{1}{3} = \frac{1}{3}\left(-1 + \frac{1}{7}\right) = -\frac{6}{21} = -\frac{2}{7}$$

$$0 + \frac{1}{7}\,1 = 1/7$$

$$1 + \frac{1}{7}0 = 1$$

$R_3 + \frac{1}{7}R_2$

$C_3 + \frac{1}{7}C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\text{(diagonal)}}{B} = P' A P$$

$\therefore$ A quadratic form $X'AX$ is transformed to

$$Y'BY = 6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2$$

under the congruent transformation $X = PY$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$\therefore$

$$\left.\begin{aligned} x_1 &= y_1 + \tfrac{1}{3}y_2 - \tfrac{2}{7}y_3 \\ x_2 &= y_2 + \tfrac{1}{7}y_3 \\ x_3 &= y_3 \end{aligned}\right\}$$

Example 02: Reduce the following quadratic form to a diagonal form through congruent transformation ;
$6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 18x_1x_3 + 4x_2x_3$ [Find diagonal matrix=P'AP]

$$\rightarrow X'AX = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$A = I_3 A I_3$$

$$\begin{bmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 - \frac{1}{3}R_1$

$R_3 - \frac{3}{2}R_1$

$C_2 - \frac{1}{3}C_1$

$C_3 - \frac{3}{2}C_1$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1 \\ 0 & -1 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{3}{2} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -3/2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 + \frac{3}{7}R_2$

$R_3 + \frac{3}{7}C_2$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 1/14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/3 & 1 & 0 \\ -\frac{23}{14} & \frac{3}{7} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1/3 & -23/14 \\ 0 & 1 & 3/7 \\ 0 & 0 & 1 \end{bmatrix}$$

$B = P' \; A \; P$, a diagonal form

$\therefore$ $X'AX$ is transformed to $Y'BY = 6y_1^2 + \frac{7}{3}y_2^2 + \frac{1}{14}y_3^2$

under the transf^n: $X = PY$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & -1/3 & -23/14 \\ 0 & 1 & 3/7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\therefore \quad x_1 = y_1 - \frac{1}{3}y_2 - \frac{23}{14}y_3$$

$$x_2 = y_2 + \frac{3}{7}y_3$$

$$x_3 = y_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Normal $r=2$

Canonical Echelon $\boxed{r=3}$

$$\underline{y_1^2+5y_2^2-2y_3^2}$$

positive definite: $y_1^2+y_2^2+y_3^2$ ✓

Neg. definite : $-y_1^2-y_2^2-y_3^2$ ✓

positive semi definite: $y_1^2+y_2^2+0y_3^2$ ✓

Neg. semi definite: $-y_1^2-y_3^2+0y_2^2$ ✓

Indefinite : Otherwise

$y_1^2-y_2^2-y_3^2$

s = Index = 3 : no. of +ve squares

$r-s = 0$: no. of -ve squares

$\therefore$ Signature $= s-(r-s)$

$= \underline{\underline{2s-r}}$

Rank, Index, Signature and Value class

- Rank (r): Number of non zero rows in B matrix is the rank of given quadratic form matrix
- Index (s): Number of positive square in $Y'BY$ is called Index
- Signature: Let s be the number of positive squares then $r - s$ will be the number of negative squares. Then difference between positive and negative squares is called signature of the quadratic form: $Signarure = s - (r - s) = 2s - r$
- Value Class: 1) Positive definite: If all squared term of $Y'BY$ are positive
  2) Negative definite: If all squared term of $Y'BY$ are negative
  3) Positive semi definite: If all squared term of $Y'BY$ are either positive or zero
  4) Negative semi definite: If all squared term of $Y'BY$ are either Negative or zero
  5) Indefinite: Otherwise

$X'AX$. $Y'BY = 1y_1^2 - 2y_2^2 + \frac{3}{2}y_3^2$

Sylvester's Law of Inertia: If s is a number of positive squares and r-s is number of negative squares then law states that, ***"The signature of a real quadratic form is invariant "***

Ex.01: Reduce the following quadratic form $2x_1^2 + x_2^2 - 3x_3^2 - 8x_2x_3 - 4x_3x_1 + 12x_1x_2$ to $\boxed{\text{normal form}}$ through congruent transformations. Also find its rank, signature and value class

$$\rightarrow X'AX = [x_1 \ x_2 \ x_3]\begin{bmatrix}2 & 6 & -2\\ 6 & 1 & -4\\ -2 & -4 & -3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$A = I_3 A I_3$$

$$\begin{bmatrix}2 & 6 & -2\\ 6 & 1 & -4\\ -2 & -4 & -3\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\vdots$$

$$\begin{bmatrix}2 & 0 & 0\\ 0 & -17 & 0\\ 0 & 0 & \frac{81}{17}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ -3 & 1 & 0\\ \frac{11}{17} & \frac{2}{17} & 1\end{bmatrix} A \begin{bmatrix}1 & -3 & 11/17\\ 0 & 1 & 2/17\\ 0 & 0 & 1\end{bmatrix}$$

$\frac{1}{\sqrt{2}}R_1, \ \frac{1}{\sqrt{17}}R_2, \ \frac{\sqrt{17}}{9}R_3$

$\frac{1}{\sqrt{2}}C_1, \ \frac{1}{\sqrt{17}}C_2, \ \frac{\sqrt{17}}{9}\cdot C_3$

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{B} = \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & 0\\ -\frac{3}{\sqrt{17}} & \frac{1}{\sqrt{17}} & 0\\ \frac{\sqrt{17}}{9}\cdot\frac{11}{17} & \frac{\sqrt{17}}{9}\cdot\frac{2}{17} & \frac{\sqrt{17}}{9}\end{bmatrix}}_{P'} A \underbrace{\begin{bmatrix}\frac{1}{\sqrt{2}} & -3/\sqrt{17} & \frac{\sqrt{17}}{9}\cdot\frac{11}{17}\\ 0 & 1/\sqrt{17} & \frac{\sqrt{17}}{9}\cdot\frac{2}{17}\\ 0 & 0 & \sqrt{17}/9\end{bmatrix}}_{P}$$

$\therefore$ This is in normal form.

$\therefore X'AX$ is transformed to $y'By = y_1^2 - y_2^2 + y_3^2$ under $X = PY$

$$\left.\begin{aligned} x_1 &= \frac{1}{\sqrt{2}}y_1 - \frac{3}{\sqrt{17}}y_2 + \frac{11}{9\sqrt{17}}y_3\\ x_2 &= \frac{1}{\sqrt{17}}y_2 + \frac{2}{9\sqrt{17}}y_3\\ x_3 &= \frac{\sqrt{17}}{9}y_3 \end{aligned}\right\}$$

Rank: $r = 3$

Index = no. of +ve squares

$s = 2$

Signature $= 2s - r$
$= 4 - 3 = 1$

Value class:
indefinite class.

Ex.02: Reduce the following quadratic form $21x_1^2 + 11x_2^2 + 2x_3^2 - 8x_2x_3 + 12x_3x_1 - 30x_1x_2$ to normal form / canonical form through congruent transformations. Also find its rank, signature, Also show that it is positive semi definite. Find the non zero set of values of x_1, x_2 and x_3 which will make quadratic form zero.

Ex.02: Reduce the following quadratic form $21x_1^2 + 11x_2^2 + 2x_3^2 - 8x_2x_3 + 12x_3x_1 - 30x_1x_2$ to normal form / canonical form through congruent transformations. Also find its rank, signature, Also show that it is positive semi definite. Find the non zero set of values of x_1, x_2 *and* x_3 which will make quadratic form zero.

Ex.03: Reduce the following quadratic form $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2$ to indefinite form. Also find its rank, signature, Ex.03: Reduce the following quadratic form $x_1^2 + 2x_2^2 + 3x_3^2 + 2x_2x_3 - 2x_3x_1 + 2x_1x_2$ to indefinite form. Also find its rank, signature.

Ex.04: Reduce the following quadratic form $x^2 - 2y^2 + 10z^2 - 10xy + 4xz - 2yz$, to canonical form. Also find its rank, signature, index and value class.

$\rightarrow X'AX = [x \;\; y \;\; z]\begin{bmatrix}1 & -5 & 2\\ -5 & -2 & -1\\ 2 & -1 & 10\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}$

$A = I_3 A I_3$

$$\begin{bmatrix}1 & -5 & 2\\ -5 & -2 & -1\\ 2 & -1 & 10\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$R_2 + 5R_1$, $R_3 - 2R_1$, $C_2 - 5C_1$, $C_3 - 2C_1$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -27 & 9\\ 0 & 9 & 6\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 5 & 1 & 0\\ -2 & 0 & 1\end{bmatrix} A \begin{bmatrix}1 & 5 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$R_3 + \frac{1}{3}R_2$, $C_3 + \frac{1}{3}C_2$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & -27 & 0\\ 0 & 0 & 9\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 5 & 1 & 0\\ -1/3 & 1/3 & 1\end{bmatrix} A \begin{bmatrix}1 & 5 & -1/3\\ 0 & 1 & 1/3\\ 0 & 0 & 1\end{bmatrix}$$

$R_2 \times \frac{1}{\sqrt{27}}$, $R_3 \times \frac{1}{3}$, $C_2 \times \frac{1}{\sqrt{27}}$, $C_3 \times \frac{1}{3}$

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}}_{B} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 5/\sqrt{27} & 1/\sqrt{27} & 0\\ -1/9 & 1/9 & 1/3\end{bmatrix}}_{P'} \underbrace{A}_{A} \underbrace{\begin{bmatrix}1 & 5/\sqrt{27} & -1/9\\ 0 & 1/\sqrt{27} & 1/9\\ 0 & 0 & 1/3\end{bmatrix}}_{P}$$

$\therefore X'AX$ is transformed to $Y'BY = y_1^2 - y_2^2 + y_3^2$

under transf^n $X = PY$

$$\left.\begin{aligned} x &= y_1 + \frac{5}{\sqrt{27}}y_2 - \frac{1}{9}y_3\\ y &= 0y_1 + \frac{1}{\sqrt{27}}y_2 + \frac{1}{9}y_3\\ z &= 0y_1 + 0y_2 + \frac{1}{3}y_3 \end{aligned}\right\}$$

Rank $= r = 3$

Index = no. of +ve squares in $Y'BY = s = 2$

Signature $= 2s - r = 4 - 3 = 1$

Value class: indefinite. class.

Ex.04: Reduce the following quadratic form $x^2 - 2y^2 + 10z^2 - 10xy + 4xz - 2yz,$ to canonical form. Also find its rank, signature, index and value class.

Singular Value Decomposition [SVD]

1. $Find\ A'A$ $= B$
2. Find Eigen values of $A'A$
3. Arrange them in descending order, find their square roots and denote them by σ_1 $and\ \sigma_2$ and find diagonal matrix D with them $\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \end{bmatrix} = D$
4. Find Eigen vectors of $A'A$, call them $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$
   v_1 $and\ v_2$. $Note\ that\ they\ are\ orthogonal.$ v_1 v_2 , $\langle v_1, v_2 \rangle = v_1 \cdot v_2 = 0$
5. Normalise $v_1\ and\ v_2$ by dividing by their norms. And find the matrix $V = [v_1,\ v_2]$ $\frac{v_1}{\|v_1\|}$, $\frac{v_2}{\|v_2\|}$
6. To find $U = [u_1,\ u_2], find\ \ u_1 = \frac{1}{\sigma_1} A v_1, u_2 = \frac{1}{\sigma_2} A v_2$
7. Then A can be expressed as $A = UDV'$

Ex.01: Find Singular value decomposition of $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$\rightarrow A'A = \begin{bmatrix} 4 & -3 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 16+9 & 16-9 \\ 16-9 & 16+9 \end{bmatrix} = \begin{bmatrix} 25 & 7 \\ 7 & 25 \end{bmatrix} = B$

Eigen values of $A'A = B$; $|A'A - \lambda I| = 0$

$\begin{vmatrix} 25-\lambda & 7 \\ 7 & 25-\lambda \end{vmatrix} = 0 = (25-\lambda)^2 - 49 = 0$

$\lambda^2 - 50\lambda + 625 - 49 = 0$

$\lambda^2 - 50\lambda + 576 = 0$

$\lambda_1 = 32, \; \lambda_2 = 18$

$\sigma_1 = \sqrt{\lambda_1} = 4\sqrt{2}, \; \sigma_2 = \sqrt{\lambda_2} = 3\sqrt{2}$

$\therefore D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix}$

Eigen vectors of $A'A$: $[A'A - \lambda]X = 0$

i) $\lambda_1 = 32$: $\begin{bmatrix} 25-32 & 7 \\ 7 & 25-32 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 + R_1 \begin{bmatrix} -7 & 7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\therefore -7x + 7y = 0$

$-x + y = 0$

$x = y$

If $y = 1, \; x = 1$

$\therefore$ vector is $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

ii) $\lambda = 18$; $[A'A - \lambda]X = 0$

$\begin{bmatrix} 25-18 & 7 \\ 7 & 25-18 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 - R_1 \begin{bmatrix} 7 & 7 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$7x + 7y = 0$

or $x + y = 0$

or $x = -y$

so if, $y = 1, \; x = -1$

$\therefore$ vector is $= \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

* Normalise vectors:

$\|v_1\| = \sqrt{1^2 + 1^2} = \sqrt{2}$; $\|v_2\| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\therefore$ normalised vector are $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = v_2$

* Orthogonality: $v_1 \cdot v_2 = (1, 1) \cdot (-1, 1)$

$= -1 + 1 = 0$

$\Rightarrow$ they are orthogonal.

* $V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

* To find U :-

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4\sqrt{2}} \cdot \underset{2\times2}{\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}$

$u_1 = \frac{1}{4\sqrt{2}} \begin{bmatrix} \frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} \\ \frac{-3}{\sqrt{2}} \; \frac{3}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ex.01: Find Singular value decomposition of $A = \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix}$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} -\frac{4}{\sqrt{2}} + \frac{4}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} + \frac{3}{\sqrt{2}} \end{bmatrix} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 0 \\ \frac{6}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \quad U = [u_1, u_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ SVD is $A = UDV'$ ✓

$$\begin{bmatrix} 4 & 4 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$=$$

Ex.02: Find Singular value decomposition of $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$

$\rightarrow A'A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix}$

Eigen values of $A'A$, $|A'A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = 0 = (4-\lambda)(13-\lambda) - 36 = 0$

$52 - 17\lambda + \lambda^2 - 36 = 0$

$\lambda^2 - 17\lambda + 16 = 0$

$(\lambda - 16)(\lambda - 1) = 0$

$\lambda = 16, 1$

$\therefore \sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4$; $\sigma_2 = \sqrt{\lambda_2} = 1$

$D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

Eigen vectors; $[A'A - \lambda I]X = 0$

i) $\lambda = 16$; $\begin{bmatrix} 4-16 & 6 \\ 6 & 13-16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 + \frac{1}{2}R_1 \begin{bmatrix} -12 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$-12x + 6y = 0$

$2x - y = 0$

put $y = 1$

$2x = 1 \Rightarrow x = 1/2$

$\therefore \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

ii) $\lambda = 1$; $\begin{bmatrix} 4-1 & 6 \\ 6 & 13-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 - 2R_1 \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$3x + 6y = 0$

$x + 2y = 0$

put $y = 1$, $x = -2$

vector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$\therefore$ These two vectors are orthogonal

$(1,2)\cdot(-2,1) = -2+2 = 0$ ✓

Normalise the vectors

$v_1 = (1,2)$; $\|v_1\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\therefore$ vector is $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$v_2 = (-2,1)$, $\|v_2\| = \sqrt{4+1} = \sqrt{5}$

$\therefore$ vectors is $\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

Now $U = [u_1, u_2]$

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{4}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8/\sqrt{5} \\ 4/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} A v_2 = 1\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$\therefore U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$\therefore \boxed{A = UDV'}$

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$A'A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 13 \end{bmatrix} = B$

Eigen values of B: $|B-\lambda I| = 0$, ch. eqn

$\begin{vmatrix} 4-\lambda & 6 \\ 6 & 13-\lambda \end{vmatrix} = 0 = (4-\lambda)(13-\lambda) - 36 = 0$

$\lambda^2 - 17\lambda + 52 - 36 = 0$

$\lambda^2 - 17\lambda + 16 = 0$

$\lambda = 16, 1$ {descending}

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{16} = 4$; $\sigma_2 = \sqrt{\lambda_2} = \sqrt{1} = 1$

$\therefore D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$

Eigen vectors: $[B - \lambda I]x = 0$

i) $\lambda = 16$: $\begin{bmatrix} 4-16 & 6 \\ 6 & 13-16 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} -12 & 6 \\ 6 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 + \frac{1}{2}R_1 \begin{bmatrix} -12 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$-12x + 6y = 0$

$-2x + y = 0$

If $y = 1$; $x = 1/2$

vector is $V_1 = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

ii) $\lambda = 1$: $[B - \lambda I]x = 0$

$\begin{bmatrix} 4-1 & 6 \\ 6 & 13-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 - 2R_1 \begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\therefore 3x + 6y = 0$

$x + 2y = 0$

$x = -2y$

vector is $V_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Orthogonality: $V_1 \cdot V_2 = (1,2)\cdot(-2,1) = -2 + 2 = 0$ ✓

Normalise: $\|V_1\| = \sqrt{1+4} = \sqrt{5}$

$\|V_2\| = \sqrt{4+1} = \sqrt{5}$

$\therefore$ Normalised vectors are $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$\therefore V = [v_1, v_2] = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

Now To find $U = [u_1, u_2]$

$u_1 = \frac{1}{\sigma_1} A V_1 = \frac{1}{4}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8/\sqrt{5} \\ 4/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} A V_2 = \frac{1}{1}\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$\therefore U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$

$\therefore A = UDV'$

$\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

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Ex.03: Find Singular value decomposition of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\rightarrow A'A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = B$

2×3 3×2

Eigen values: $|A'A - \lambda I| = 0$

$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 = (3-\lambda)^2 - 1$

$9 - 6\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 6\lambda + 8 = 0$

$(\lambda-4)(\lambda-2) = 0$

$\lambda = 4, 2$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{4} = 2$; $\sigma_2 = \sqrt{\lambda_2} = \sqrt{2}$

$\therefore D = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix}$

Eigen vectors: $[A'A - \lambda I]X = 0$

i) $\lambda = 4$; $\begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 + R_1 \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$-x + y = 0 \Rightarrow x = y$

$\therefore$ vector is $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now ii) $\lambda = 2$:

$\begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$R_2 - R_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$x + y = 0$

$x = -y$

$\therefore$ vector is $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Orthogonal: $V_1 \cdot V_2 = (1,1)\cdot(1,-1) = 1 - 1 = 0$

Normalise: $\|V_1\| = \sqrt{1^2+1^2} = \sqrt{2}$; $\|V_2\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\therefore$ Vectors are $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$u_1 = \frac{1}{\sigma_1} A V_1 = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$

$u_2 = \frac{1}{\sigma_2} \cdot A V_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\therefore U = [u_1, u_2] = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A = UDV' = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

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Ex.03: Find Singular value decomposition of $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$