* Residues :

If z_0 be the singular point of $f(z)$ then the coefficient of $\frac{1}{z-z_0}$ in the Laurent series expansion of $f(z)$ is called Residue of $f(z)$ at z_0

that is

$$\boxed{\text{Residue of } f(z) \left(\text{at } z=z_0\right) = b_1 = \text{coefficient of } \frac{1}{z-z_0}}$$

Note that :

① If z_0 is simple pole of $f(z)$ then

$$\boxed{\text{Residue of } f(z) \text{ at } z_0 = \lim_{z \to z_0} (z-z_0) \cdot f(z)}$$

② If z_0 is pole of order 'm' then

$$\boxed{\text{Residue of } f(z) \text{ at } z_0 = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z)\right]}$$

Examples :

Find the Residues at each pole of the following function

① $\dfrac{e^z}{(1-z)^3}$ ② $\dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$

Solution:

(1) Given: $f(z) = \dfrac{e^{z}}{(z-1)^3}$

clearly, the point $z=1$ is pole of $f(z)$ of order 3

$\therefore$ Residue of $f(z)$ at $z = z_0 = 1$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z) \right] \quad \left(\begin{matrix}\text{for pole}\\ \text{of order } m\end{matrix}\right)$$

$$= \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} \left[(z-1)^3 \cdot \frac{e^{z}}{(z-1)^3} \right] \quad \left(\begin{matrix}\text{if } m=3\\ \& \ z_0 = 1\end{matrix}\right)$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} (e^{z})$$

$$= \frac{1}{2} \lim_{z \to 1} e^{z}$$

$$= \frac{1}{2} \cdot e^{1}$$

$\therefore$ Residue of $f(z)$ at $z=1$ is $\frac{1}{2} e$

(2) Given: $f(z) = \dfrac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2}$

clearly, $z=1$ is simple pole of $f(z)$ and $z=2$ is pole of $f(z)$ of order 2

* Residue of $f(z)$ at $z = z_0 = 1$

$$= \lim_{z \to z_0} (z - z_0) f(z)$$

$$= \lim_{z \to 1} (z-1) \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2} \quad (\because z_0 = 1)$$

$$= \lim_{z \to 1} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-2)^2}$$

$$= \frac{\sin \pi + \cos \pi}{(1-2)^2}$$

$$= -1$$

* Residue at f(z) at z = 2

$$= \frac{1}{(-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$$

$$= \frac{1}{1!} \lim_{z \to 2} \frac{d}{dz} \left[(z-2)^2 \cdot \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)^2} \right]$$

$$= \lim_{z \to 2} \frac{d}{dz} \left[\frac{\sin \pi z^2 + \cos \pi z^2}{z-1} \right]$$

$$= \lim_{z \to 2} \left[\frac{(z-1)\left[(\cos \pi z^2) \cdot (2\pi z) - (\sin \pi z^2)(2\pi z) \right] - (\sin \pi z^2 + \cos \pi z^2)(}{(z-1)^2} \right]$$

$$\left(\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{d}{dx} v}{v^2} \right)$$

$$= \frac{(2-1)\left[\cos 4\pi \cdot (4\pi) - \sin 4\pi \, (4\pi) \right] - (\sin 4\pi + \cos 4\pi)}{(2-1)^2}$$

$$= \frac{((1)\, 4\pi - 0) - (0 + 1)}{1}$$

$$= 4\pi - 1$$

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Ex ③ Find the residue of $f(z) = \frac{1}{z - \sin z}$ at its singularity using laurent's series expansion

Solution: Given: $f(z) = \frac{1}{z - \sin z}$

clearly, $z = 0$ is pole of $f(z)$

Now,

$$f(z) = \frac{1}{z - \sin z}$$

$$= \frac{1}{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right]}$$

$$= \frac{1}{\left[\frac{z^3}{3!} - \frac{z^5}{5!} + \frac{z^7}{7!} - \cdots\right]}$$

$$= \frac{1}{\frac{z^3}{3!}\left[1 - \frac{z^2}{4\times5} + \frac{z^4}{4\times5\times6\times7} - \cdots\right]}$$

$$= \frac{1}{\frac{z^3}{3!}} \cdot \left[1 - \frac{z^2}{20} + \frac{z^4}{840} - \cdots\right]^{-1}$$

$$= \frac{3!}{z^3}\left[1 - \left(\frac{z^2}{20} + \frac{z^4}{840} - \cdots\right)\right]^{-1}$$

$$= \frac{6}{z^3}\left[1 + \frac{z^2}{20} + \frac{z^4}{840} \cdots\right] \quad \left(\because [1-z]^{-1} = 1 + z + z^2 + \cdots\right)$$

$$f(z) = \frac{6}{z^3} + \frac{6}{20}\frac{1}{z} + \frac{6z}{840} + \cdots$$

$\therefore$ Residue of $f(z)$ (at $z=0$) $= b_1 =$ coefficient of $\frac{1}{z}$

$$= \frac{6}{20}$$

$$= \frac{3}{10}$$

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Ex. 4 find the sum of all the residues at singular point of $f(z) = \dfrac{z}{(z-1)^2 (z^2-1)}$

solution: Given: $f(z) = \dfrac{z}{(z-1)^2(z^2-1)} = \dfrac{z}{(z-1)^3 \cdot (z+1)}$

clearly, $z=1$ is pole of $f(z)$ of order 3 and $z=-1$ is a simple pole of $f(z)$

Now, Residue of $f(z)$ at $z = z_0 = 1$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m \cdot f(z) \right] \quad \left(\because \text{for pole of order } m \right)$$

$$= \frac{1}{2!} \lim_{z \to 1} \frac{d^2}{dz^2} \left[(z-1)^3 \frac{z}{(z-1)^3 (z+1)} \right]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d^2}{dz^2} \left(\frac{z}{z+1} \right)$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz} \left[\frac{(z+1)(1) - z(1)}{(z+1)^2} \right]$$

$$= \frac{1}{2} \lim_{z \to 1} \frac{d}{dz} \left(\frac{1}{(z+1)^2} \right)$$

$$= \frac{1}{2} \lim_{z \to 1} \left[-\frac{2}{(z+1)^3} \right]$$

$$= \frac{1}{2} \left[-\frac{2}{(1+1)^3} \right]$$

$$= -\frac{1}{8}$$

Now, Residue of $f(z)$ at $z = z_0 = -1$

$$= \lim_{z \to z_0} (z-z_0) f(z)$$

$$= \lim_{z \to -1} (z+1) \frac{z}{(z-1)^3 (z+1)} = \frac{-1}{(-1-1)^3} = \frac{1}{8}$$

Therefore, Sum of the Residues $= -\frac{1}{8} + \frac{1}{8} = \underline{\underline{0}}$

Homework:

* Find the residues of following function at its singularities.

① $\dfrac{1-e^{2z}}{z^3}$ ② $\dfrac{1-z}{1-\cos z}$ ③ $\dfrac{\sin \pi z}{(z-1)^2 (z-2)}$

④ find the sum of the Residues at singular point of the function $\dfrac{z}{z^3+1}$