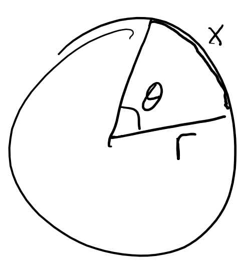
Chapter 10: Rotational Motion

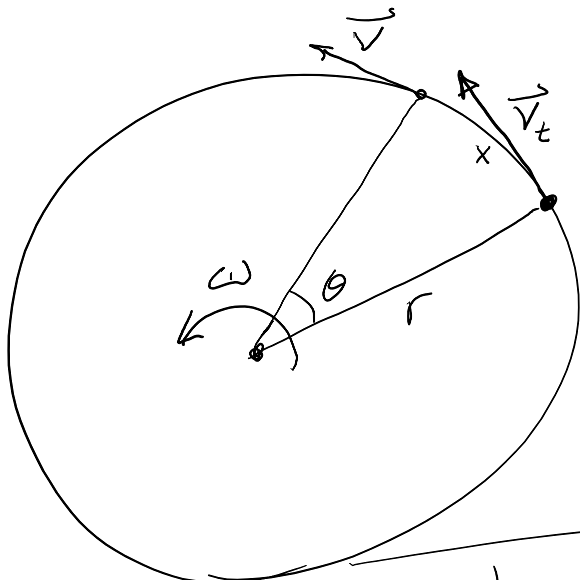
Monday, July 12, 2021 12:46

- HW 6 will be shorter + due THURSDAY at midnight.
- No regrade available for Quizzam 6.

Recall (Ch. 6 – uniform circular motion)

translational		rotational	
displacement	$\vec{x}$	angle	$\theta = \frac{x}{r}$ ← radius
velocity	$\vec{v} = \frac{\Delta \vec{x}}{\Delta t}$	angular velocity	$\omega = \frac{\Delta\theta}{\Delta t} = \frac{v}{r}$





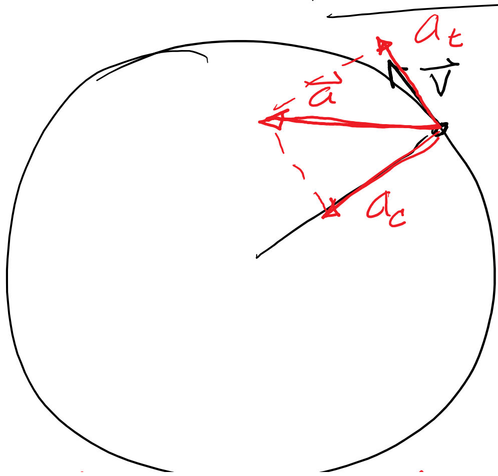
$$\omega = \frac{v_t}{r}$$

$$v_t = r\omega$$

$$\left(\begin{array}{l} +\omega = \text{ccw} \\ -\omega = \text{cw} \end{array}\right)$$

acceleration $\vec{a} = \frac{\Delta \vec{v}}{\Delta t}$

$$\boxed{\text{angular acceleration} \quad \alpha = \frac{\Delta\omega}{\Delta t}}$$



centripetal acceleration - part of the acceleration which is pointed inwards parallel to the radius (a_c)

$$a_c = v^2/r$$

tangential acceleration - part of the acceleration which is perpendicular to the radius (a_t)

centripetal changes DIRECTION of velocity
tangential changes MAGNITUDE of velocity

$$\boxed{\alpha = \frac{a_t}{r}}$$ (unit: radian/s^2)

$\alpha = \dfrac{\Delta\omega}{\Delta t}$ ← if ω is increasing then α is positive
if ω is decreasing then α is negative

10.2 rotational kinematics (compare Ch. 2+3)

translational (ch 2)	rotational
$\Delta x = \bar{v} t$	$\Delta\theta = \bar{\omega} t$
$\bar{v} = \dfrac{v + v_0}{2}$	$\bar{\omega} = \dfrac{\omega + \omega_0}{2}$
$v = v_0 + at$	$\omega = \omega_0 + \alpha t$
$\Delta x = v_0 t + \frac{1}{2} a t^2$	$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t$
$v^2 = v_0^2 + 2a\Delta x$	$\omega^2 = \omega_0^2 + 2\alpha\Delta\theta$

e.g. $v = v_0 + at$

$r\omega = r\omega_0 + r\alpha t$

$$r\omega = r\omega_0 + r\alpha t$$
$$\omega = \omega_0 + \alpha t$$

e.g. a fishing line is unspooling from its reel ($r = 4.5\,cm$)
angular acceleration $\alpha = 110\ rad/s^2$ for $2s$.

a) what is the final angular velocity?

known: $\alpha = 110\ rad/s^2$ unknown: $\omega = ?$
$t = 2s$
$\omega_0 = 0\ rad/s$
$r = 0.045\,m$

$$\omega = \omega_0 + \alpha t = 0 + (110\ rad/s^2)(2s) = \boxed{220\ rad/s}$$

b) what is the final (linear) velocity of the line?

$$v = r\omega = (0.045\,m)(220\ rad/s) = \boxed{9.9\ m/s}$$

c) how many revolutions does the reel make?

$$\theta = \omega_0 t + \tfrac{1}{2}\alpha t^2 = (0)(2s) + \tfrac{1}{2}(110\ rad/s^2)(2s)^2 = 220\ rad$$
$$= 220\ rad \times \frac{1\ rev}{2\pi\ rad} = \boxed{35.0\ \text{revolutions}}$$

d) how much fishing line came off the reel?

$$x = r\theta = (0.045\,m)(220\ rad) = \boxed{9.9\,m}$$

10.3 rotational dynamics (compare Ch. 4)

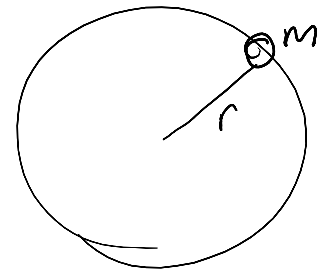
$$F = ma = mr\alpha \longrightarrow rF = mr^2\alpha$$

if $F \perp r$, then $\boxed{\tau = mr^2\alpha}$

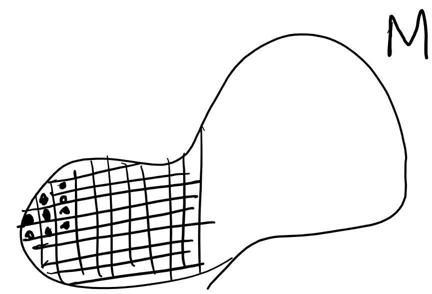
moment of inertia – how easy it is to change an objects angular velocity.

translational		rotational
F	$\longrightarrow$	τ
a	$\longrightarrow$	α
m	$\longrightarrow$	mr^2

- this is for a point mass.



- expand this formula by summing the moments of inertia of all point masses making up the object

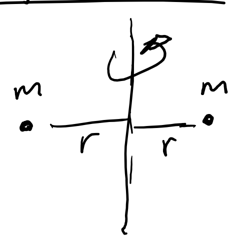


$$I = \sum mr^2$$

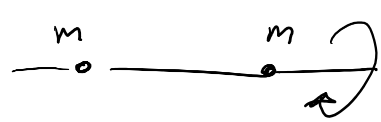
moment of inertia (I)

- the value of I depends on the axis.

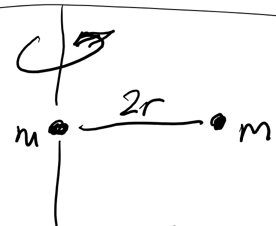
e.g. two point masses



$$I = mr^2 + mr^2 = 2mr^2$$



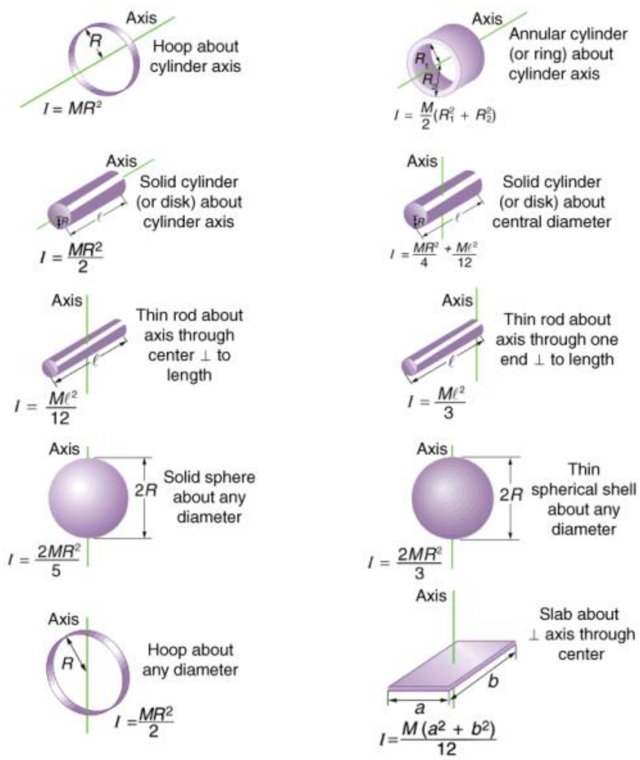
$$I = m(0)^2 + m(0)^2 = 0$$



$$I = m(0)^2 + m(2r)^2 = 4mr^2$$

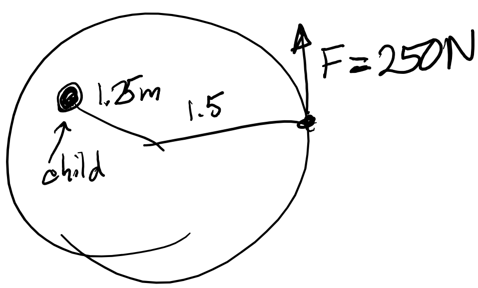
generally $\boxed{\tau = I\alpha}$

Fig. 10.2 in book $\longrightarrow$



unit for I: $kg \cdot m^2$

e.g. a 250N force is exerted at the edge of a 50 kg merry-go-round with radius 1.5m.



a) find the angular acceleration when nothing is on the merry go round.

$$\tau = (1.5\,m)(250\,N)\sin 90° = 375\,N\cdot m$$

disk about axis $\rightarrow$ $I = \frac{MR^2}{2} = \frac{(50\,kg)(1.5\,m)^2}{2} = 56.25\,kg\cdot m^2$

$$\tau = I\alpha \rightarrow \boxed{\alpha = \frac{\tau}{I} = 6.67\ rad/s^2}$$

b) what if an 18kg child is 1.25m from the center.

$$I_{tot} = I_{merry\text{-}go\text{-}round} + \underbrace{I_{child}}_{mr^2} = 56.25\,kg\cdot m^2 + (18\,kg)(1.25\,m)^2$$

$$= 84.375\,kg\cdot m^2$$

$$\alpha = \frac{\tau}{I} = \frac{375\,N\cdot m}{84.375\,kg\cdot m^2} = \boxed{4.44\ rad/s^2}$$

$I = 84.375\ kg \cdot m^2$

10.4 | rotational kinetic energy (comare Ch. 7)

recall work-energy theorem

$$W = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2 = \Delta KE$$

$$W = F\Delta s = (rF)\left(\frac{\Delta s}{r}\right) = \tau\theta = \underbrace{I\alpha\theta}$$

(assuming F is parallel to Δs) (F and r are perpendicular)

kinematics: $\omega^2 = \omega_0^2 + 2\alpha\theta \rightarrow \alpha\theta = \frac{\omega^2 - \omega_0^2}{2}$

$$W = I\left(\frac{\omega^2 - \omega_0^2}{2}\right) = \boxed{\frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_0^2 = W}$$

rotational work-energy theorem

define $\boxed{KE_{rot} = \frac{1}{2}I\omega^2}$ (Compare $KE_{trans} = \frac{1}{2}mv^2$)

for closed systems, mechanical energy is conserved.

i.e. KE + PE = constant

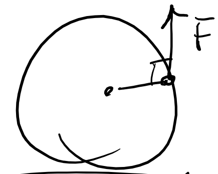
both translational KE and rotational KE

gravitational PE
spring PE

e.g. a 85 kg grindstone with 0.32 m radius.

a) a force of 200N is exerted at the edge over an angular distance of 1 radian.

a) a force of 200N is exerted at the edge over an angular distance of 1 radian.
what is the work done?



$$W = \tau\theta = rF\theta = (0.32m)(200N)(1rad) = \boxed{64J}$$

b) what is the final angular velocity?

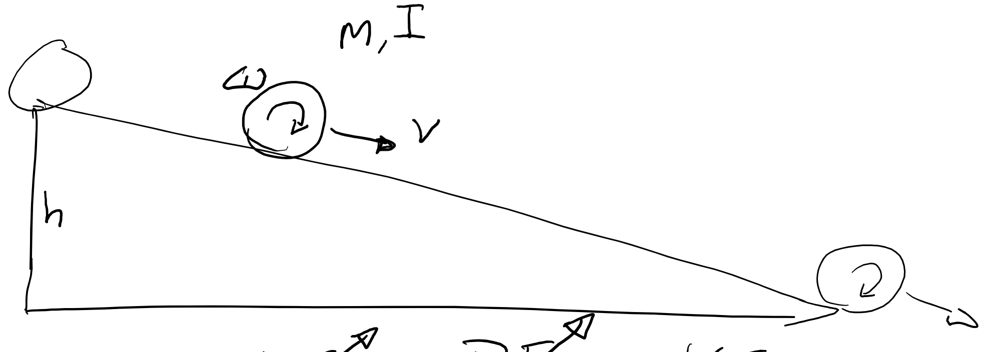
$$\alpha = \frac{\tau}{I} = \frac{rF}{\frac{1}{2}Mr^2} = \frac{(0.32m)(200N)}{\frac{1}{2}(85kg)(0.32m)^2} = 14.7\ rad/s^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \rightarrow \omega = \sqrt{2\alpha\theta} = \boxed{5.42\ rad/s}$$

c) what is the rotational KE?

$$KE = \frac{1}{2}I\omega^2 = \frac{1}{2}\left[\frac{1}{2}(85kg)(0.32m)^2\right](5.42 rad/s)^2$$
$$= \boxed{64J}$$ ✓

application - objects do NOT roll downhill at the same speed.

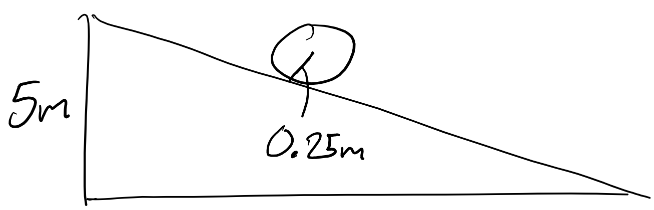


$$PE_i + \cancel{KE_i} = \cancel{PE_f} + KE_f$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

- the gravitational PE must be split between the translational KE and the rotational KE.
- if sliding ($\omega = 0$) $\rightarrow$ all PE goes to translation
  $\rightarrow$ object has largest final velocity
- if rolling it goes slower, some energy is needed to make the object rotate.
  - the larger the moment of inertia, the more rotational KE
    $\rightarrow$ the slower the object rolls.

e.g. a 0.25m radius solid cylinder and mass 1 kg. rolls from a height of 5m



$m = 1\,kg$

$I = \frac{1}{2}mr^2 = 0.031\ kg\cdot m^2$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 \qquad v = r\omega$$

$$= \frac{1}{2}m\,r^2\omega^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(mr^2 + I)\omega^2$$

$$\omega = \sqrt{\frac{2mgh}{mr^2 + I}} = \boxed{32\ rad/s}$$

10.5 angular momentum (compare Ch.8)

translational

$\vec{p} = m\vec{v}$

rotational

$\boxed{L = I\omega}$

$p = \ldots$ $\boxed{L = I\omega}$

angular momentum (unit: $kg \cdot m^2/s$)

net $\vec{F} = \frac{\Delta \vec{p}}{\Delta t}$

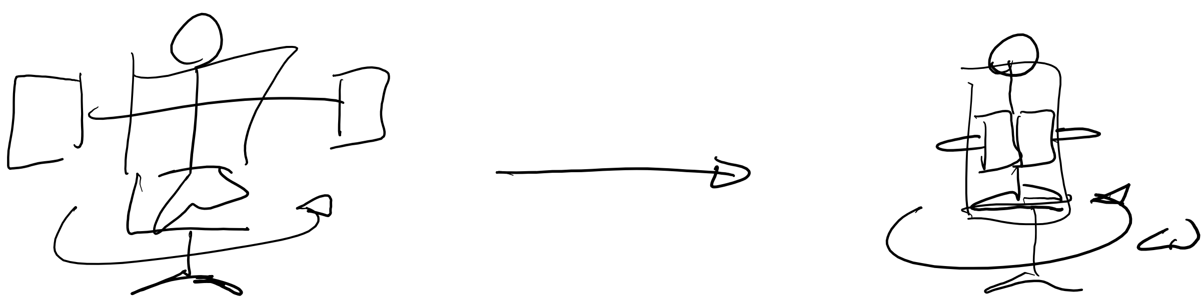
$\boxed{\text{net } \tau = \frac{\Delta L}{\Delta t}}$

(constant $m \rightarrow \vec{F} = m\vec{a}$)

(constant $I \rightarrow \tau = I\alpha$)

angular momentum is conserved if there is no net external torque.

- reduce moment of inertia → increase angular momentum.



spin with two books in outstretched arms

move your arms together

you will spin much faster.

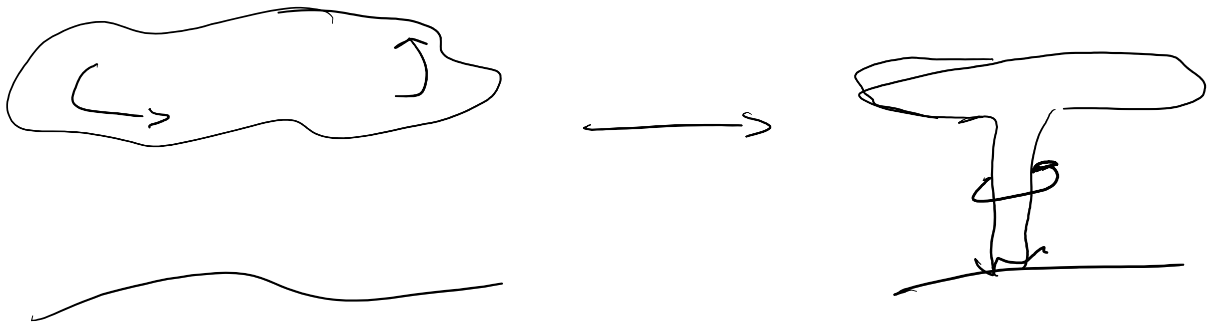
large I
small ω

same $L = I\omega$

small I
large ω

e.g. a spinning ice skater

e.g. tornadoes (before formation, the storm is rotating slowly due to the Coriolis effect)

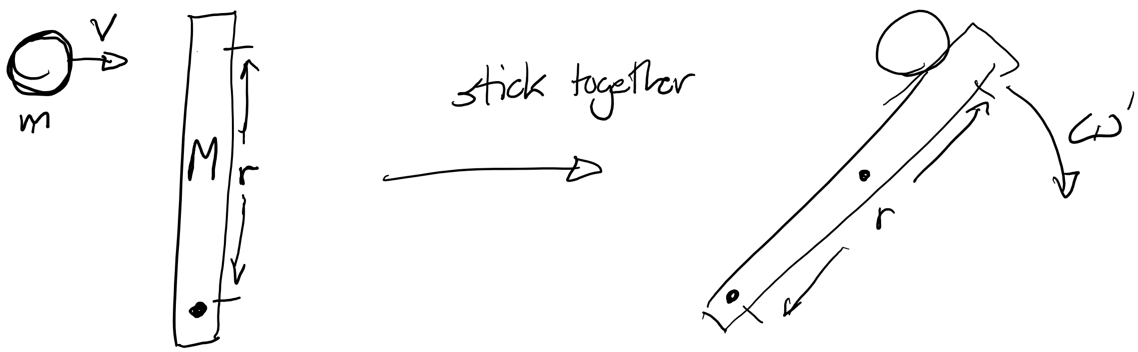


larg I, small ω small I, large ω

10.6) Collisions with extended bodies (compare Ch. 8)

both linear + rotational momentum are conserved.

e.g.



no net $\tau \rightarrow L$ is conserved

before: $L_{stick} = 0$

$$L_{ball} = I\omega = (mr^2)\left(\frac{v}{r}\right) = mrv$$

after: $L'_{stick+ball} = (I_{stick} + I_{ball})\,\omega'$

$$= \left(\frac{Mr^2}{3} + mr^2\right)\omega'$$

ball $\begin{cases} m = 50\,g \\ v = 30\,m/s \end{cases}$

stick $\begin{cases} M = 2\,kg \\ r = 1.2\,m \end{cases}$

$$mrv = \left(\frac{Mr^2}{3} + mr^2\right)\omega'$$

mrv 3mv

$$\omega' = \frac{mrv}{\frac{Mr^2}{3} + mr^2} = \boxed{\frac{3mv}{Mr + 3mr}} = 1.74\ \text{rad/s}$$

$$\Delta KE < 0$$

kinetic energy decreases

before: $KE = \frac{1}{2}mv^2 = 22.5\ \text{J}$

after: $KE = \frac{1}{2}(I_s + I_b)\omega'^2 = 1.57\ \text{J}$

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linear momentum:

<u>before</u>: $P_{stick} = 0$

$P_{ball} = mv$

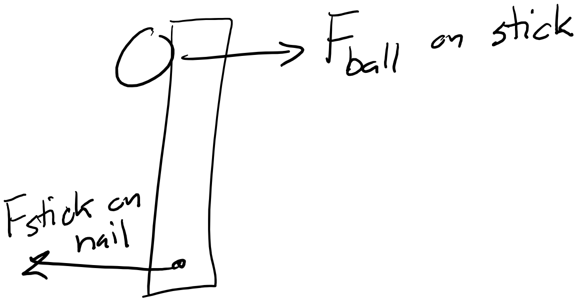
$P_{tot} = 1.5\ \text{kg m/s}$

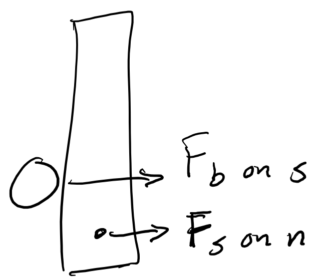
after: $P_{stick} = M\left(\frac{r}{2}\omega'\right)$

$P_{ball} = mr\omega'$

$P_{tot} = 2.2\ \text{kg m/s}$

lin. momentum NOT conserved because there <u>is</u> an external force





there is a "sweet spot" where there is <u>no</u> force on the nail.