Chapter 7

Response of First-Order RL and RC Circuits

Text: *Electric Circuits*, 9th Edition, by J. Nilsson and S. Riedel

Prentice Hall

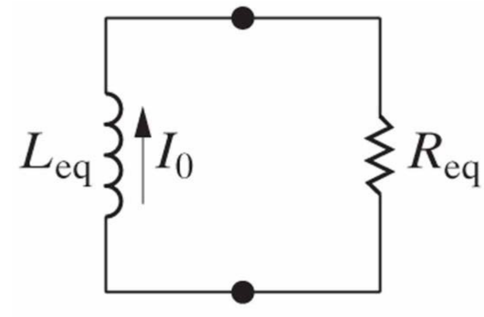
Engr 17 Introductory Circuit Analysis

Instructor: Russ Tatro

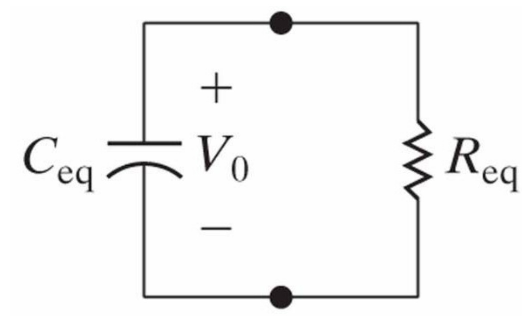
Chapter 7 Overview

This chapter introduces how the energy storage aspect of the inductor and the capacitor leads to useful circuit behavior.

RL – a circuit comprised of resistors and inductors that can be resolved into an equivalent resistor and an equivalent inductor.



RC – a circuit comprised of resistors and capacitors that can be resolved into an equivalent resistor and an equivalent capacitor

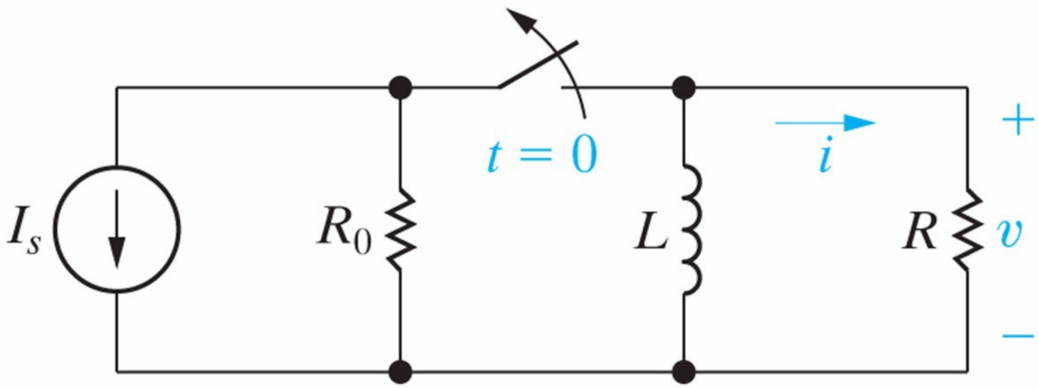


Section 7.1

The Natural Response of an RL Circuit

The Natural Response of an RL Circuit

The model RL circuit is:



Assume that an independent current source generates a constant current.

The switch has been closed for "a long time". You will soon discover that we have a special circuit definition for *a long time*.

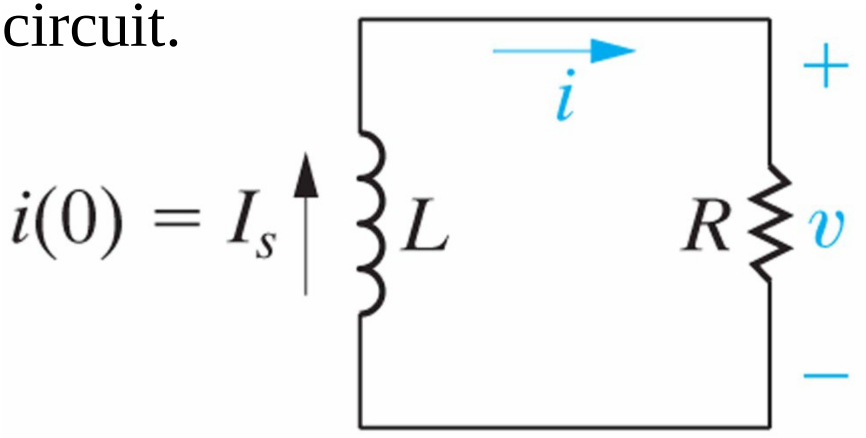
$$v_L = L\frac{di}{dt}$$

The current is given as a constant for a long time.

Thus the voltage across the inductor (and the resistor) is then zero.

The Natural Response of an RL Circuit

Now open the switch and calculate the circuit response of the following equivalent circuit.



Find i(t) by writing KVL for the circuit.

$$L\frac{di}{dt} + Ri = 0$$

The last equation is a first-order ordinary differential equation. Hence the term *first order circuits* for this chapter.

The Natural Response of an RL Circuit

Now resolve the differential equation:

$$L\frac{di}{dt}+Ri=0 \qquad \Rightarrow \frac{di}{dt}dt=-\frac{R}{L}idt$$

$$\frac{di}{i}=-\frac{R}{L}dt$$

Now integrate both sides:

$$\int_{i(t_0)}^{i(t)}\frac{dx}{x}=-\frac{R}{L}\int_{t_0}^{t}dy \qquad \Rightarrow \ln x\Big|_{i(t_0)}^{i(t)}=-\frac{R}{L}\,y\Big|_{t_0}^{t}$$

$$\ln i(t)-\ln i(t_0)=-\frac{R}{L}(t-t_0)$$

The Natural Response of an RL Circuit

$$\ln i(t) - \ln i(t_0) = -\frac{R}{L}(t - t_0)$$

Now simplify and assume $t_0 = 0$.

$$\ln \frac{i(t)}{i(t_0 = 0)} = -\frac{R}{L}\left(t - 0\right)$$

Now take the (natural) exponential of both sides:

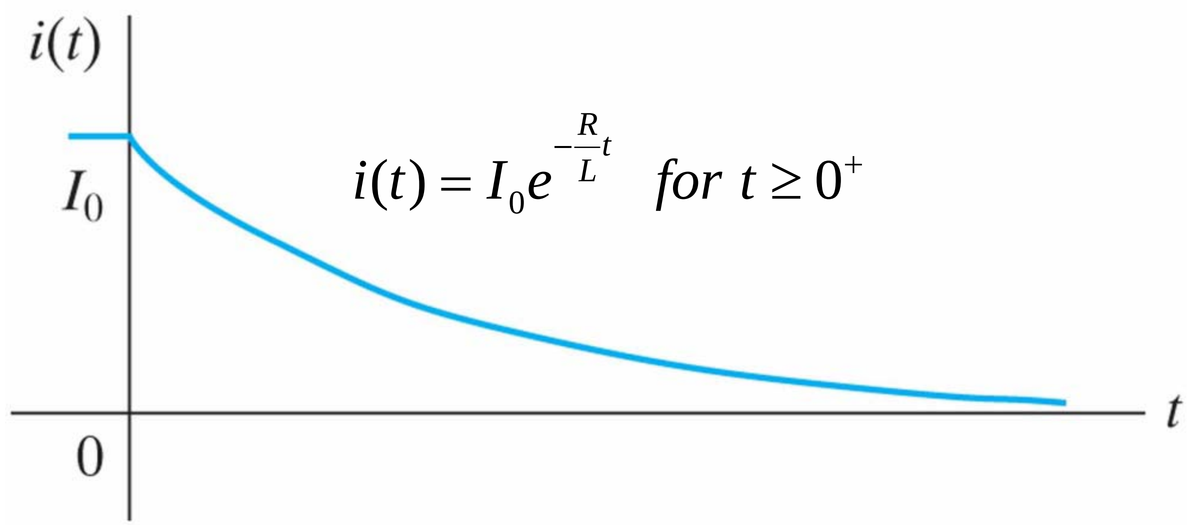
$$\frac{i(t)}{i(0)} = e^{-\frac{R}{L}t} \qquad \Rightarrow i(t) = i(0)e^{-\frac{R}{L}t} \quad for\ t \geq 0^+$$

Let the initial condition for the current at $t = 0^- = I_0$. Thus

$$i(t) = I_0 e^{-\frac{R}{L}t} \quad for\ t \geq 0^+$$

The Natural Response of an RL Circuit

So we see the current in the circuit exponentially decays.



$$i(t=0) = I_0 e^{-\frac{R}{L}(0)} = I_0(1) = I_0$$

Current decreases exponentially $0 \leq t \leq \infty$.

$$i(t \to \infty) = I_0 e^{-\frac{R}{L}(\infty)} = I_0(0) = 0$$

The Natural Response of an RL Circuit

The voltage across the resistor is

$$v(t) = i(t)R = I_0 R e^{-\frac{R}{L}t} \quad for\ t \geq 0^+$$

Note the step change in the resistor voltage at t = 0^+.

$$v_R = \begin{cases} 0 & for\ t \geq 0^- \\ I_0 R e^{-\frac{R}{L}t} & for\ t \geq 0^+ \end{cases}$$

The power dissipated in the resistor is

$$p = vi \quad = \left(I_0 R e^{-\frac{R}{L}t} \right)\left(I_0 e^{-\frac{R}{L}t} \right) \qquad = I_0^2 R e^{-2\frac{R}{L}t} \quad for\ t \geq 0^+$$

The energy delivered to the resistor is (see text for derivation)

$$w = \frac{1}{2} L I_0^2 \left(1 - e^{-2\frac{R}{L}t} \right) \quad for\ t \geq 0^+$$

The energy in the inductor is fully dissipated into the resistor as t $\rightarrow \infty$

The Significance of the Time Constant

Notice the equation for the current has a interesting term as the power of the exponent.

$$i(t) = I_0 e^{-\frac{R}{L}t} = I_0 e^{-\frac{t}{L/R}}$$

Since the exponential must have dimensionless units, recast the equation as

$$i(t) = I_0 e^{-\frac{t}{\tau}} \qquad \frac{t \text{ (in seconds)}}{\tau \text{ (in seconds)}}$$

Where we will call τ the *time constant*.

$$\tau = \frac{L}{R}$$

The Significance of the Time Constant

The time constant τ plays the role of pacing the system from the initial *transient response* to the *steady-state response*.

let $\frac{t}{\tau}$ be an integer i.e. 1, 2, 3, 4, ...

Then the exponential is

$$e^{-1} = 0.36788$$
$$e^{-2} = 0.13534$$
$$e^{-3} = 0.04979$$
$$e^{-4} = 0.01832$$
$$e^{-5} = 0.00673$$

Thus after 5 time constants the transient response has fallen to less than 1% of the initial value.

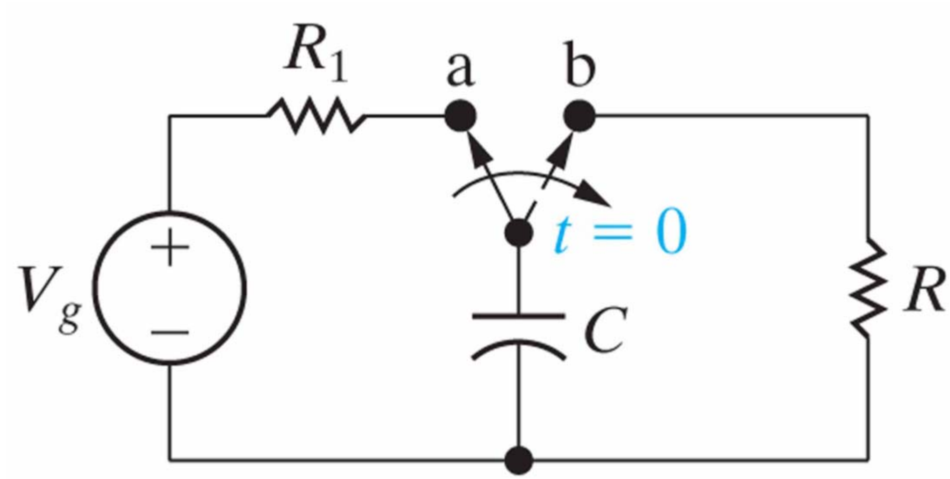
a long time in circuit operation is arbitrarily defined as 5 time constants.

Section 7.2

The Natural Response of an RC Circuit

The Natural Response of an RC Circuit

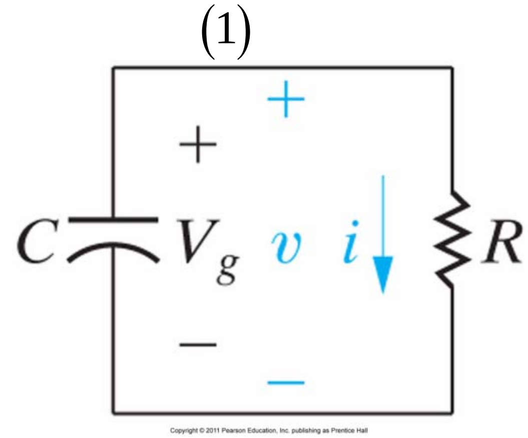
Just like in the RL case, we need a circuit that can give energy to the capacitor and then simplify to just the RC elements.



We can assume *a long time* so that at $t = 0^-$ the capacitor has charged up to the voltage V_g.

At $t = 0^+$ the switch creates the RC circuit on the right (without V_g and R_1 playing any further part in this analysis)

The Natural Response of an RC Circuit



Recall: $i_C = C\dfrac{dv}{dt}$

Write the KCL equation at node (1) for the two currents leaving node (1).

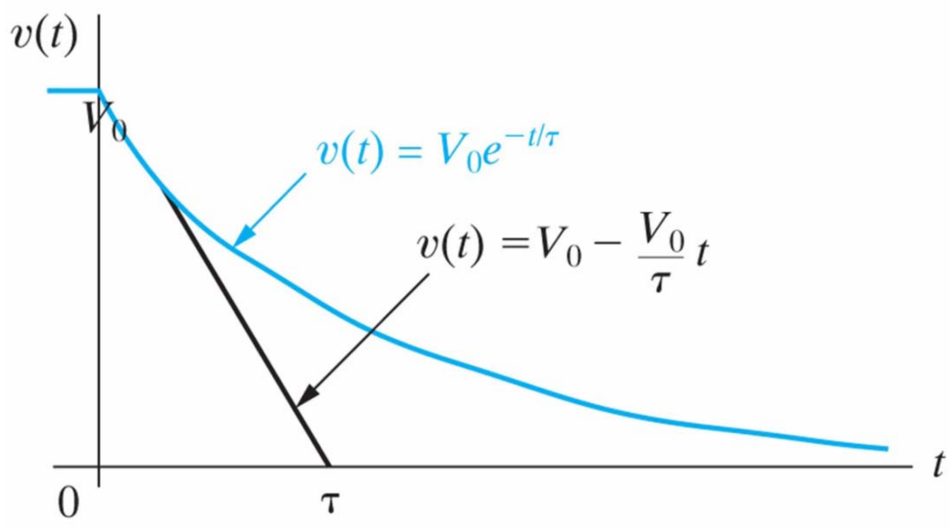
$$C\frac{dv}{dt} + \frac{v}{R} = 0$$

The first order differential equation can be solved just like we did for the inductor but now with the result:

$$v(t) = v(0^-)e^{-\frac{t}{RC}} \quad for\ t \geq 0^+$$

$$v(t) = V_0 e^{-\frac{t}{\tau}} \quad where\ \tau = RC$$

So we see the voltage in the circuit exponentially decays.



$$v(t=0) = V_0 e^{-\frac{0}{RC}} = V_0(1) = V_0$$

Voltage decreases exponentially $0 \leq t \leq \infty$

$$v(t \to \infty) = V_0 e^{-\frac{\infty}{RC}} = V_0(0) = 0$$

The Natural Response of an RC Circuit

The voltage across the resistor is $v(t) = i_C(t)R$

$$i_C(t) = \frac{v(t)}{R} \quad = \frac{V_0 e^{-\frac{t}{RC}}}{R} \quad = \frac{V_0 e^{-\frac{t}{\tau}}}{R} \quad for\ t \geq 0^+$$

Note the step change in the resistor voltage at t = 0^+.

$$v_R = \begin{cases} 0 & for\ t \geq 0^- \\ V_0 e^{-\frac{t}{RC}} & for\ t \geq 0^+ \end{cases}$$

Then the power stored in a capacitor’s electric field is

$$p = vi \quad = \left(V_0 e^{-\frac{t}{\tau}}\right)\left(\frac{V_0}{R} e^{-\frac{t}{\tau}}\right) \quad = \frac{V_0^2}{R} e^{-\frac{2t}{\tau}} \quad for\ t \geq 0^+$$

The energy ω in the RC circuit is (see text for derivation)

$$w = \frac{1}{2} C V_0^2 \left(1 - e^{-\frac{2t}{\tau}}\right) \quad for\ t \geq 0^+$$

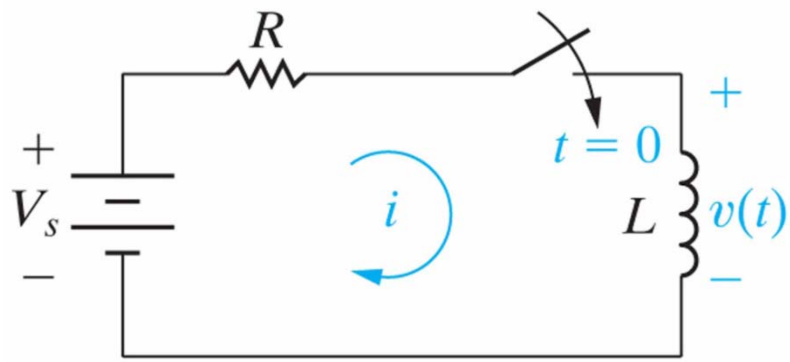
Section 7.3.a

The Step Response of RL Circuits

The Step Response of RL Circuits

This section will look at the response of RL circuits when a dc voltage is suddenly applied.

Let’s start with the RL circuit just after the switch closes.

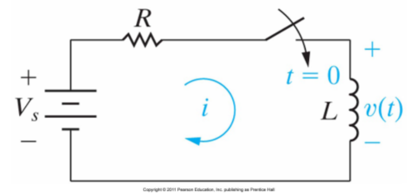


Write the KVL equation for the circuit:

$$-V_S + iR + L\frac{di}{dt} = 0$$

$$iR + L\frac{di}{dt} = V_S$$

The Step Response of RL Circuits



$$iR + L\frac{di}{dt} = V_S$$

Now separate the variables *i* and *t* and then integrate.

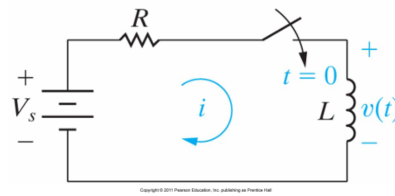
$$\frac{di}{dt} = \frac{V_S - iR}{L} \qquad = \left(-\frac{R}{L}\right)\left(i - \frac{V_S}{R}\right)$$

Multiply both sides by dt

$$\frac{di}{dt}dt = \left(-\frac{R}{L}\right)\left(i - \frac{V_S}{R}\right)dt$$

$$\frac{di}{i - \dfrac{V_S}{R}} = -\frac{R}{L}dt$$

The Step Response of RL Circuits



$$\frac{di}{i-\frac{V_S}{R}} = -\frac{R}{L}dt$$

Integrate the last result:

$$\int_{I_0}^{i(t)} \frac{dx}{x-\frac{V_S}{R}} = -\frac{R}{L}\int_0^t dy$$

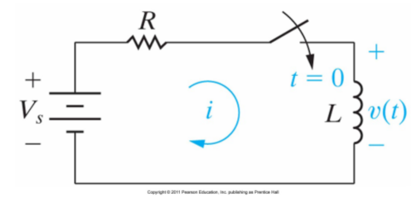
$$use \int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b|$$

Then

$$\ln\left(i(t)-\frac{V_S}{R}\right) - \ln\left(I_0-\frac{V_S}{R}\right) = -\frac{R}{L}(t-0)$$

$$\ln\left(\frac{i(t)-\frac{V_S}{R}}{I_0-\frac{V_S}{R}}\right) = -\frac{R}{L}t$$

The Step Response of RL Circuits



$$\ln\left(\frac{i(t)-\frac{V_S}{R}}{I_0-\frac{V_S}{R}}\right)=-\frac{R}{L}t$$

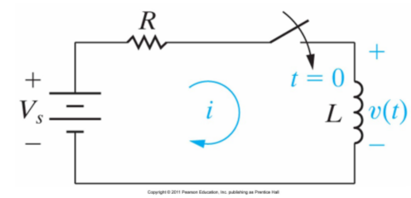
Take the natural exponent of both sides and simplify.

$$\frac{i(t)-\frac{V_S}{R}}{I_0-\frac{V_S}{R}}=e^{-\frac{R}{L}t}$$

$$i(t)=\frac{V_S}{R}+\left(I_0-\frac{V_S}{R}\right)e^{-\frac{R}{L}t}$$

This last equation is the response of the RL circuit to voltage step input.

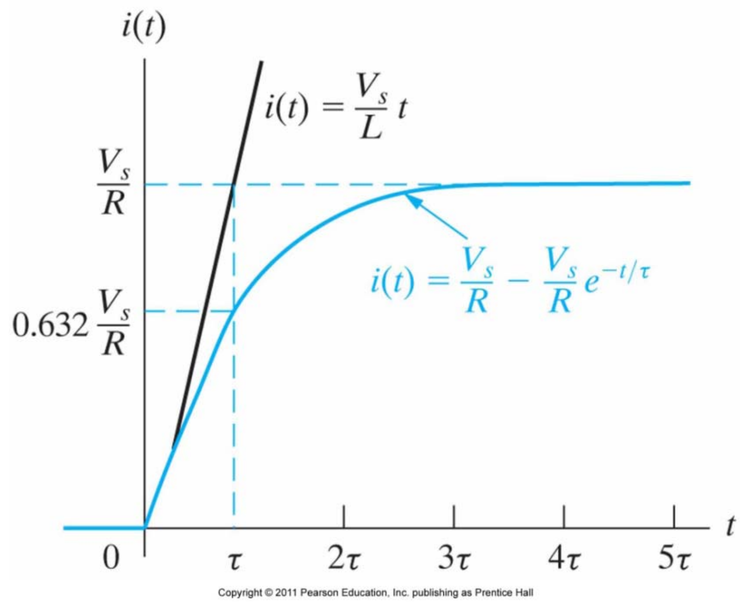
The Step Response of RL Circuits



$$i(t)=\frac{V_S}{R}+\left(I_0-\frac{V_S}{R}\right)e^{-\frac{R}{L}t}$$

For the moment, let $I_0 = 0$, the response is then

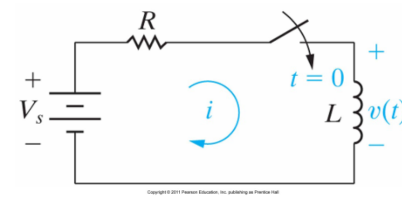
$$i(t=0)=\frac{V_S}{R}-\frac{V_S}{R}e^{-\frac{R}{L}0} \quad =\frac{V_S}{R}-\frac{V_S}{R}(1)=0$$



The current initially increases exponentially $t \geq 0$ and then trends to the steady state value after *a long time*.

$$i(t=\infty)=\frac{V_S}{R}-\frac{V_S}{R}e^{-\frac{R}{L}\infty} \quad =\frac{V_S}{R}$$

The Step Response of RL Circuits



$$i(t) = \frac{V_S}{R} + \left(I_0 - \frac{V_S}{R} \right) e^{-\frac{R}{L}t}$$

The voltage across the resistor is $v = i$R

$$v_R(t=0) = \left[\frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}0} \right] R = 0$$

The resistor voltage increases exponentially (for a short while) for $0 \leq t$

$$v_R(t=\infty) = \left[\frac{V_S}{R} - \frac{V_S}{R} e^{-\frac{R}{L}\infty} \right] R \qquad = \left[\frac{V_S}{R} - \frac{V_S}{R}(0) \right] R \qquad = \left[\frac{V_S}{R} \right] R \qquad = V_S$$

Recall that under a constant current and after *a long time* the inductor voltage goes to zero.

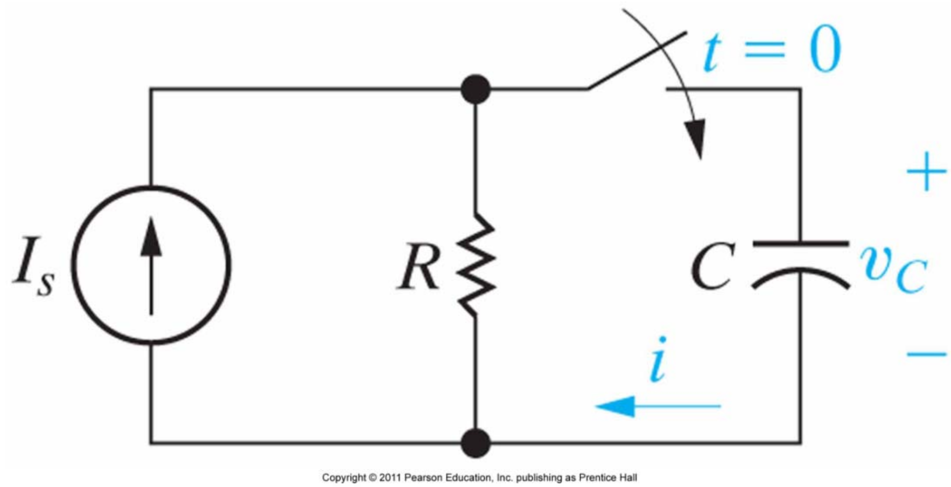
Section 7.3.b

The Step Response of RC Circuits

The Step Response of RC Circuits

This section will look at the response of RC circuits when a dc current is suddenly applied.

Let's start with the RC circuit.



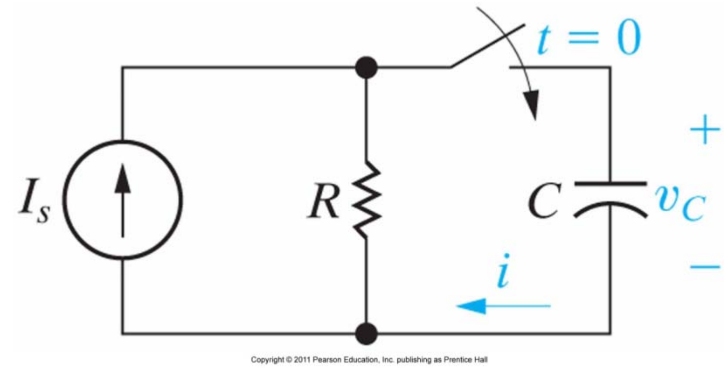
Write the KCL equation for the circuit:

$$-I_S + \frac{v_C}{R} + C\frac{dv}{dt} = 0$$

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_S}{C}$$

The Step Response of RC Circuits

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_S}{C}$$



In the same manner as for the RL circuit (but not shown here), the step response for the RC circuit is

$$v_C(t) = I_S R + \left(V_0 - I_S R\right)e^{-\frac{t}{RC}} \quad for\ t \geq 0^+$$

Where V_0 is the initial voltage across the capacitor.

The final voltage across all circuit elements (since they are in parallel when the switch is closed) is

$$v_C(t \rightarrow \infty) = I_S R$$

Section 7.4

A General Solution for

Step and Natural Responses

General Solution

There are four possible first order circuits.

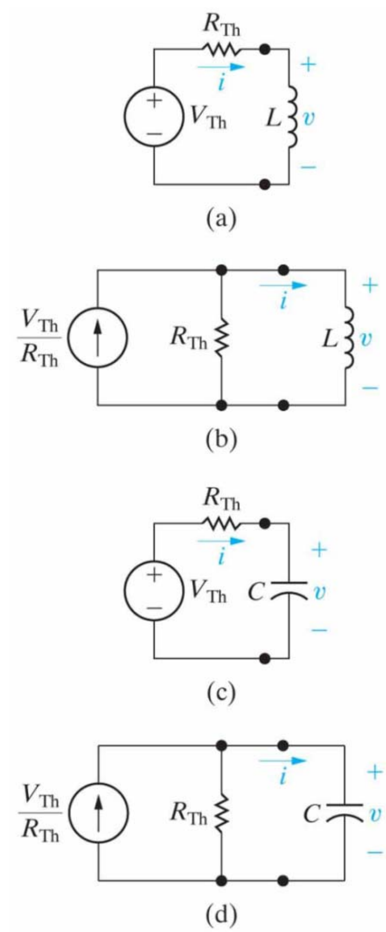
Let x(t) represent the unknown quantity.

The differential equation will then have the form

$$\frac{dx}{dt} + \frac{x}{\tau} = K$$

τ is the constant of the circuit.

K is a constant and might equal zero.



x(t) may be a current or voltage depending on the circuit.

General Solution

Combining the natural response and the transient response yields the general solution of the circuit.

$$x(t) = x_{final} + \left[x_{initial} - x_{final} \right] e^{\frac{-(t-t_0)}{\tau}} \quad for\ t \geq t_0$$

t_0 is the time of switching and might occur at $t_0 = 0$.

Section 7.5

Sequential Switching

Sequential Switching

Sequential Switching will not be covered.

Section 7.6

Unbounded Response

Unbounded Response

It is possible for the response of a circuit to some input stimulus to grow rather than decay under certain circumstances.

An unbounded response is when the response grows exponentially with time which can occur when the circuit contains dependent sources.

You might find the Thévenin equivalent resistance for a circuit is found to be *negative*. This is a sure sign of unbounded response.

Real circuits will eventually either constrain the unbounded response or outright fail.

Section 7.7

The Integrating Amplifier

The Integrating Amplifier

A *reactive* element is able to store energy in some circumstances. Recall that a resistor only gets hot - it cannot store energy for some later time.

Adding *reactive* elements into the operational amplifier circuit creates a considerable amount of math.

We will avoid using reactive elements in an opamp circuit until we have the *phasor* tool of Chapter 9 in our tool kit.

Chapter 7

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