HW 6

1) Problem 5.7 (a), (b)- (i), (ii)
2) Problem 5.9.
3) Problem 5.12.

4) The random variables *X* and *Y* have the following joint PDF

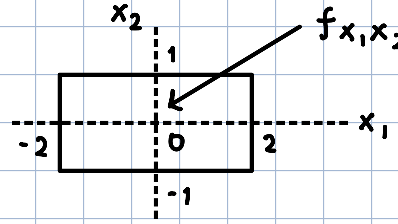
$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   a) Find the variance of *X.*
   b) Find the variance of *Y.*
   c) Find the covariance between *X* and *Y,* i.e. Cov(X,Y).
   d) Find E[X+Y].
   e) Find Var [X+Y]. (Note *X* and *Y* and not independent)

5) For the random variables *X* and *Y* having *E*(*X*) = 1, *E*(*Y*) = 2, Var (*X*) = 6, Var (*Y*) = 9, and ρ_{XY} = -2/3. Find
   a) The covariance of *X* and *Y.*
   b) The correlation of *X* and *Y.*
   c) $E(X^2)$ and $E(Y^2)$.

VIGOMAR KIM ALGADOR
ENGR-120-01 SU
HOMEWORK #06

1. PROBLEM 5.7



$f_{x_1,x_2}(x_1,x_2) = 1/8$

$$f_{x_1,x_2}(x_1,x_2) = \begin{cases} 1/8 & -2 \le x_1 \le 2,\ -1 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) FIND THE MARGINAL PDFs.

$$f_{x_1}(x_1) = \int_{-1}^{1} 1/8 \, dx_2 = 1/4 \qquad -2 \le x_1 \le 2$$

$$f_{x_2}(x_2) = \int_{-2}^{2} 1/8 \, dx_1 = 1/2 \qquad -1 \le x_2 \le 1$$

(b) DETERMINE THE FOLLOWING PROBABILITIES:

(i) $Pr[0 \le x_1 \le 1,\ -0.5 \le x_2 \le 2]$

$$= \int_0^1 \int_{-0.5}^{1} 1/8 \, dx_2 dx_1 = \int_0^1 \left[1/8 \, x_2 \right]_{-0.5}^{1} dx_1$$

$$= \int_0^1 3/16 \, dx_1 = \left[3/16 \, x_1 \right]_0^1 = 3/16$$

(ii) $Pr[x_1 < x_2]$

$$= Pr[x_1 - x_2 < 0]$$

$$= \int_{-2}^{2} \int_{x_1}^{1} 1/8 \, dx_2 dx_1 = \int_{-2}^{2} \left[1/8 \, x_2 \right]_{x_1}^{1} dx_1$$

$$= \int_{-2}^{2} 1/8 (1 - x_1) \, dx_1 = \left[1/8 (x_1 - x_1^2/2) \right]_{-2}^{2}$$

$$= 1/8 (0 + 4) = 1/2$$

2. PROBLEM 5.9

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} c(4-x_1x_2) & 0 \le x_1 \le 4,\ 0 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) FIND C TO MAKE THIS A VALID PDF.

$$c\int_0^4\int_0^1 (4-x_1x_2)\,dx_2\,dx_1 = 1$$

$$c\int_0^4 \left[4x_2 - \frac{x_1x_2^2}{2}\right]_0^1 dx_1 = c\int_0^4 (4 - x_1/2)\,dx_1$$

$$= c\left[4x_1 - \frac{x_1^2}{4}\right]_0^4 = c(16-4) = 12c$$

$$12c = 1 \longrightarrow c = 1/12$$

(b) FIND THE MARGINAL DENSITY FUNCTIONS OF X_1 AND X_2. CLEARLY DEFINE THE RANGES OF VALUES THEY TAKE.

$$f_{x_1}(x_1) = \int_0^1 1/12\,(4-x_1x_2)\,dx_2 = 1/12\left[4x_2 - \frac{x_1x_2^2}{2}\right]_0^1$$

$$= 1/12\,(4 - x_1/2) \quad 0 \le x_1 \le 4$$

$$f_{x_2}(x_2) = \int_0^4 1/12\,(4-x_1x_2)\,dx_1 = 1/12\left[4x_2 - \frac{x_1^2x_2}{2}\right]_0^4$$

$$= 1/12\,(16 - 16x_2/2) = 1/12\,(16 - 8x_2)$$

$$= 2/3\,(2-x_2) \quad 0 \le x_2 \le 1$$

(c) ARE THE RANDOM VARIABLES INDEPENDENT?

$$f_{x_1x_2}(x_1,x_2) \stackrel{?}{=} f_{x_1}(x_1)\,f_{x_2}(x_2)$$

NOT INDEPENDENT

3. PROBLEM 5.12

$$f_{X_1X_2}(x_1,x_2) = \begin{cases} c & 0 \le x_1 \le x_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) FIND C TO MAKE THE JOINT PDF A VALID ONE

$$c\int_0^1\int_0^{x_2} dx_1\,dx_2 = 1$$

$$c\int_0^1\int_0^{x_2} dx_1\,dx_2 = c\int_0^1 [x_1]_0^{x_2}\,dx_2 = c\int_0^1 x_2\,dx_2$$

$$= c\,[x_2^2/2]_0^1 = 1/2\,c$$

$$\frac{1}{2}c = 1 \longrightarrow c = 2$$

(b) DETERMINE THE CONDITIONAL PDF $f_{X_1|X_2}(x_1|x_2)$. BE SURE TO SPECIFY THE RANGES OF VALUES FOR x_1 AND x_2.

$$f_{X_2}(x_2) = 2\int_0^{x_2} dx_1 = 2[x_1]_0^{x_2} = 2x_2 \quad 0 \le x_2 \le 1$$

$$f_{X_1|X_2}(x_1|x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)} = \frac{2}{2x_2} = \frac{1}{x_2} \quad \begin{matrix} 0 \le x_1 \le x_2 \\ 0 \le x_2 \le 1 \end{matrix}$$

4. THE RANDOM VARIABLES X AND Y HAVE THE FOLLOWING JOINT PDF

$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

* MARGINAL PDF FOR $0 \le x \le 1$

$$f_X(x) = \int_0^x 2\,dy = 2x$$

NOTE: $f_X(x) = 0$ FOR $x < 0$ OR $x > 1$

* MARGINAL PDF FOR $0 \le y \le 1$

$$f_Y(y) = \int_y^1 2\,dx = 2(1-y)$$

NOTE: $f_Y(y) = 0$ FOR $y < 0$ OR $y > 1$

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) FIND THE VARIANCE OF X

$$E[X] = \int_0^1 2x^2\,dx = 2/3$$
$$E[X^2] = \int_0^1 2x^3\,dx = 1/2$$
$$VAR[X] = E[X^2] - (E[X])^2 = 1/2 - 4/9 = 1/18$$

(b) FIND THE VARIANCE OF Y

$$E[Y] = \int_0^1 2y(1-y)\,dy = [y^2 - 2y^3/3]_0^1 = 1/3$$
$$E[Y^2] = \int_0^1 2y^2(1-y)\,dy = [2y^3/3 - y^4/2]_0^1 = 1/6$$
$$VAR[Y] = E[Y^2] - (E[Y])^2 = 1/6 - 1/9 = 1/18$$

(C.) FIND THE COVARIANCE BT. X AND Y

$$E[XY] = \int_0^1\int_0^x 2xy\,dy\,dx = \int_0^1 [xy^2]_0^x\,dx = \int_0^1 x^3\,dx = [x^4/4]_0^1$$
$$= 1/4$$
$$COV[X,Y] = E[XY] - E[X]E[Y] = 1/36$$

(d) FIND $E[X+Y]$

$E[X+Y] = E[X] + E[Y] = 2/3 + 1/3 = 1$

(e) FIND $\mathrm{Var}[X+Y]$.

$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{COV}[X,Y] = 1/6$

5. (a) $\rho = \dfrac{C_{XY}}{\sigma_X \sigma_Y} \longrightarrow C_{XY} = \rho \sigma_X \sigma_Y = -\frac{2}{3}\sqrt{6}\sqrt{9} = -2\sqrt{6}$

(b) $C_{XY} = R_{XY} - \bar{X}\bar{Y} \longrightarrow R_{XY} = -2\sqrt{6} + 2 = 2(1-\sqrt{6})$

(c) $\overline{X^2} = \sigma_X^2 + \bar{X}^2 = 6 + 1 = 7$

$\overline{Y^2} = \sigma_Y^2 + \bar{Y}^2 = 9 + 4 = 13$