Chapter 8

Natural and Step Responses of RLC Circuits

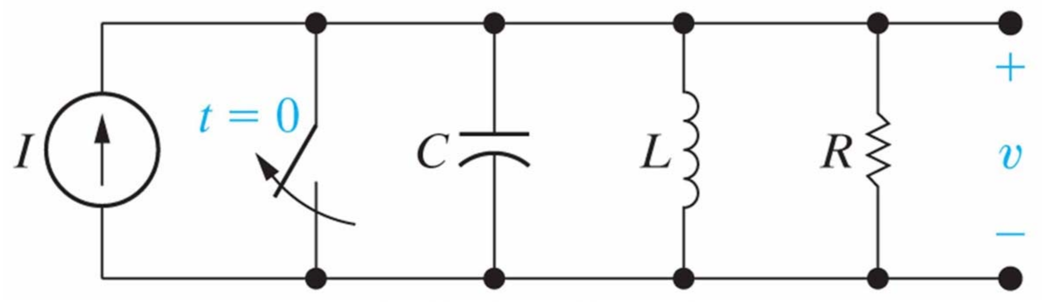
Text: *Electric Circuits*, 9th Edition, by J. Nilsson and S. Riedel
Prentice Hall

Engr 17 Introductory Circuit Analysis
Instructor: Russ Tatro

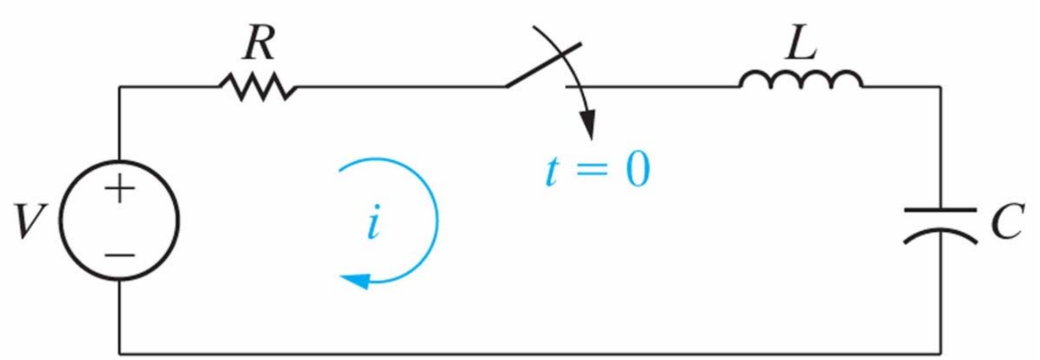
Chapter 8 Overview

This chapter introduces how two circuits with both inductors and capacitors respond.

Parallel RLC



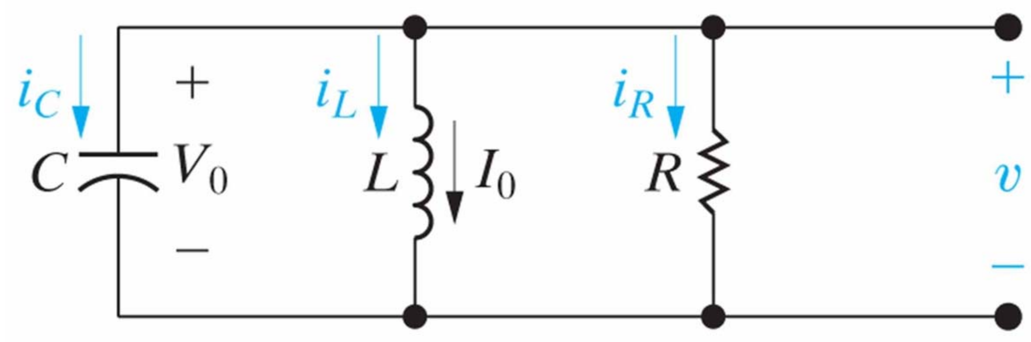
Series RLC



Section 8.1

Natural Response of a Parallel RLC Circuit

Natural Response of a Parallel RLC Circuit



We ultimately want to find the response of this parallel element circuit to the presence of initial conditions in the capacitor and the inductor.

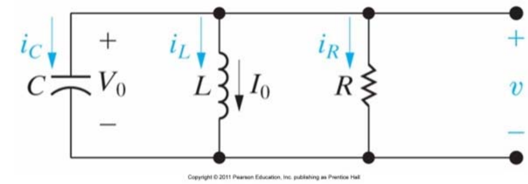
But first we will derive the behavior of this *second-order system* to show how the circuit responds in general.

By KCL:

$$i_C + i_L + i_R = 0$$

$$C\frac{dv}{dt} + \frac{1}{L}\int_0^t v\,dx + \frac{v}{R} = 0$$

Natural Response of a Parallel RLC Circuit



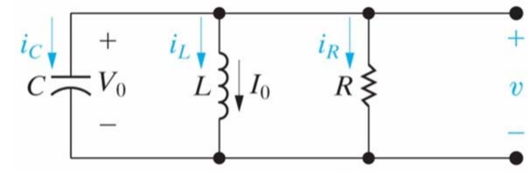
$$C\frac{dv}{dt}+\frac{1}{L}\int_0^t v\,dx+\frac{v}{R}=0$$

Now differentiate the last result in order to express the form as a differential (no integrals in the expression).

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0$$

This is the second-order differential equation for the behavior of the parallel RLC circuit.

Natural Response of a Parallel RLC Circuit



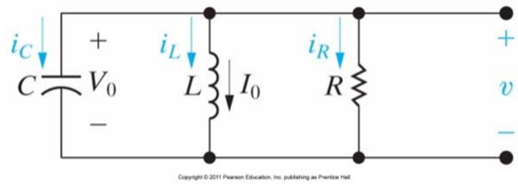
$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

The text's authors follow the usual approach for second order equations – that is – assume the solution (the desired response) has an exponential form:

$$v = Ae^{st}$$

We will have to determine A and s from the boundary conditions and known circuit element values.

Natural Response of a Parallel RLC Circuit



$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{v}{LC}=0 \qquad v = Ae^{st}$$

We will now insert this *trial solution* into the differential equation.

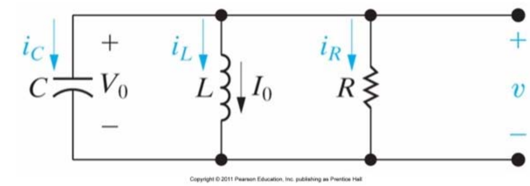
$$\frac{d^2\left(Ae^{st}\right)}{dt^2}+\frac{1}{RC}\frac{d\left(Ae^{st}\right)}{dt}+\frac{Ae^{st}}{LC}=0$$

$$As^2e^{st}+\frac{1}{RC}Ase^{st}+\frac{Ae^{st}}{LC}=0$$

$$Ae^{st}\left(s^2+\frac{s}{RC}+\frac{1}{LC}\right)=0$$

In general, neither A nor e^{st} will equal zero.

Natural Response of a Parallel RLC Circuit



$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

This last result is called the *characteristic equation* whose roots determine the behavior of v(t).

This result is in the form of a quadratic equation

$$ax^2 + bx + c = 0$$

The solution to the quadratic equation has two roots:

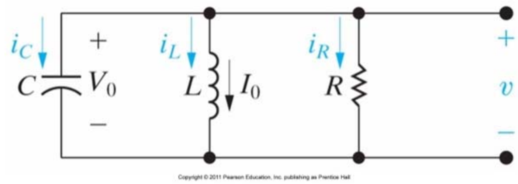
$$x = -\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a = 1$$

$$b = \frac{1}{RC}$$

$$c = \frac{1}{LC}$$

Natural Response of a Parallel RLC Circuit



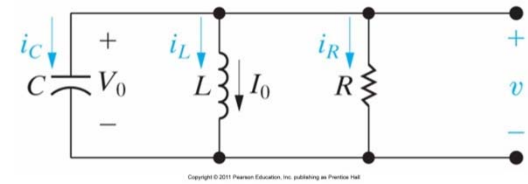
$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

The two roots of this quadratic equation are

$$s_1 = \frac{-\frac{1}{RC} + \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2} = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = \frac{-\frac{1}{RC} - \sqrt{\left(\frac{1}{RC}\right)^2 - \frac{4}{LC}}}{2} = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Natural Response of a Parallel RLC Circuit



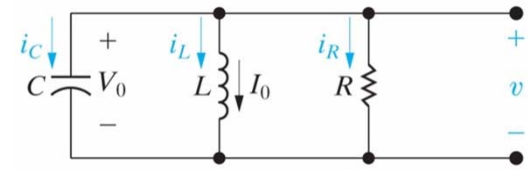
$$s^2 + \frac{s}{RC} + \frac{1}{LC} = 0$$

We can finally write the general form of the parallel RLC response:

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Where A_1 and A_2 are found from the initial conditions.

Natural Response of a Parallel RLC Circuit



The roots of the characteristic equation are often written in the following form:

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

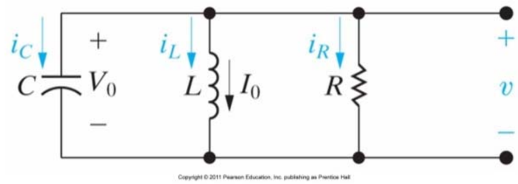
Where

$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

The last term ω_0 is known as the *resonant frequency* with units of radians/second.

Natural Response of a Parallel RLC Circuit



There are three possible types for the RLC response:

The response will be *overdamped* when the roots are *real and distinct*.

$$\omega_0^2 < \alpha^2$$

The response will be *underdamped* when the roots are *complex conjugates* of each other.

$$\omega_0^2 > \alpha^2$$

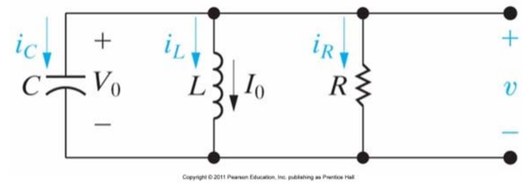
The response will be *critically damped* when the roots are *real and repeated* (equal to each other).

$$\omega_0^2 = \alpha^2$$

Section 8.2
Forms of the Natural Response of a Parallel RLC Circuit

Overdamped

Natural Response of a Parallel RLC Circuit



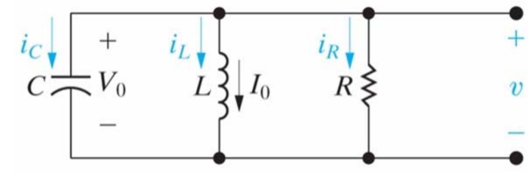
The behavior of the second-order RLC circuit depends on the roots of the characteristic equations which are the roots of s_1 and s_2.

These two roots depend on the circuit element values – what are the values of the R, L and C.

The task remaining is to determine the magnitude of the response which is described by the coefficients A_1 and A_2. which arise from initial conditions.

$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Overdamped Voltage Response



$$v = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Since the three elements are in parallel for this circuit, it makes sense to discover the voltage across all three elements.

The initial conditions are then $v(0^+)\ and\ \dfrac{dv(0^+)}{dt}$

At t = 0, the characteristic equation is

$$v = A_1 e^{s_1 0} + A_2 e^{s_2 0} = A_1 + A_2$$

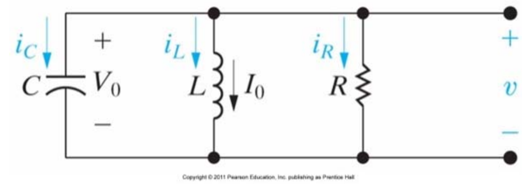
Then take the differential (first) and set t = 0 (second)

$$\frac{dv(0^+)}{dt} = \frac{d}{dt}\left[A_1 e^{s_1 t} + A_2 e^{s_2 t}\right]_{t=0} = A_1 s_1 e^{s_1 t} + A_2 s_2 e^{s_2 t}\Big|_{t=0}$$

$$= A_1 s_1 e^{s_1 0} + A_2 s_2 e^{s_2 0} = A_1 s_1 + A_2 s_2$$

Overdamped Voltage Response

$$v = A_1 + A_2 \qquad \frac{dv(0^+)}{dt} = A_1 s_1 + A_2 s_2$$



Recall that for the overdamped case, the roots s_1 and s_2 are real and distinct.

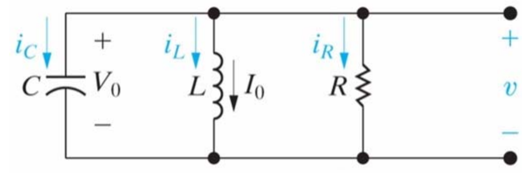
Thus we can now write a procedure to find the overdamped case:

1. Find the roots of the characteristic equation.

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}} \qquad s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

2. Find $v(0^+)$ and $\dfrac{dv(0^+)}{dt}$

Overdamped Voltage Response



3. Find the values of A_1 and A_2 from the two independent equations.

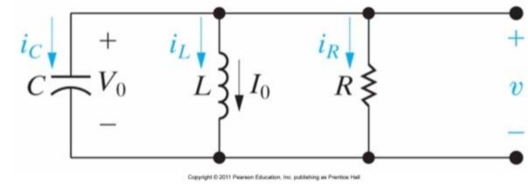
$$v = A_1 + A_2 \qquad\qquad \frac{dv(0^+)}{dt} = \frac{i_C(0^+)}{C} = A_1 s_1 + A_2 s_2$$

4. Substitute all these variables into the characteristic equation.

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad for \;\; t \geq 0$$

Underdamped

Underdamped Voltage Response



Underdamped response is when $\omega_0^2 > \alpha^2$

$$\alpha = \frac{1}{2RC}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

The roots of the characteristic equation are

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha + \sqrt{-\left(\omega_0^2 - \alpha^2\right)}$$

$$= -\alpha + j\sqrt{\omega_0^2 - \alpha^2}$$

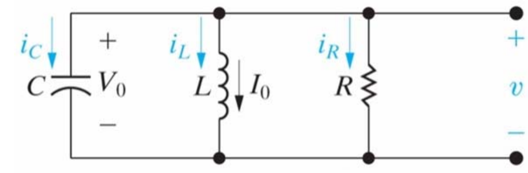
The authors now define the damped radian frequency ω_d.

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

Thus the roots can now be written as

$$s_1 = -\alpha + j\omega_d \qquad s_2 = -\alpha - j\omega_d$$

Underdamped Voltage Response



The roots of the characteristic equation are complex conjugates for the underdamped case.

Recall that complex conjugates always come in pairs:

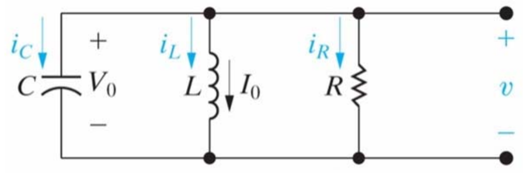
$$s_1 = -\alpha + j\omega_d \qquad\qquad s_2 = -\alpha - j\omega_d$$

So it now becomes useful to invoke Euler's Identity.

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

The next step is to rewrite the characteristic equation by translating the exponentials into sinusoids.

Underdamped Voltage Response



$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad for \;\; t \ge 0$$

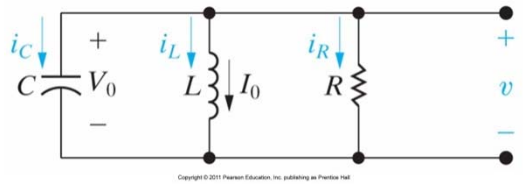
Substitute in the roots s_1 and s_2.

$$\begin{aligned} v(t) &= A_1 e^{(-\alpha + j\omega_d)t} + A_2 e^{(-\alpha - j\omega_d)t} \\ &= A_1 e^{-\alpha t} e^{j\omega_d t} + A_2 e^{-\alpha t} e^{-j\omega_d t} \\ &= A_1 e^{-\alpha t} \left(\cos \omega_d t + j \sin \omega_d t \right) \\ &\quad + A_2 e^{-\alpha t} \left(\cos \omega_d t - j \sin \omega_d t \right) \end{aligned}$$

We can now rewrite the characteristic equation in terms of sinusoids with the decaying exponential.

$$v(t) = \left(A_1 + A_2 \right) e^{-\alpha t} \cos \omega_d t + j \left(A_1 - A_2 \right) e^{-\alpha t} \sin \omega_d t$$

Underdamped Voltage Response



$$v(t) = (A_1 + A_2)e^{-\alpha t}\cos\omega_d t + j(A_1 - A_2)e^{-\alpha t}\sin\omega_d t$$

Now define new constants B1 and B2 which are real (and not complex)

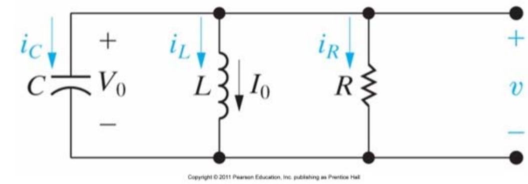
$$B_1 = A_1 + A_2 \qquad\qquad B_2 = j(A_1 - A_2)$$

So that the underdamped equation is now

$$v(t) = B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t$$

Recall that α and ω_d are found from the circuit element values. Thus the remaining task is to find B_1 and B_2.

Underdamped Voltage Response



We need to find $v(0^+)$ *and* $\dfrac{dv(0^+)}{dt}$

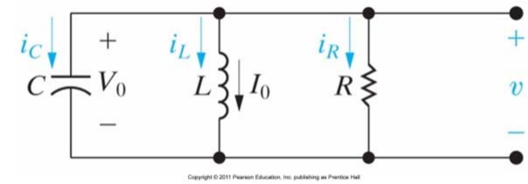
$$v(t=0^+) = B_1 e^{-\alpha 0}\cos 0 + B_2 e^{-\alpha 0}\sin 0 = B_1$$

Then take the differential (first) and set t = 0 (second)

$$\frac{dv(0^+)}{dt} = \frac{d}{dt}\left[B_1 e^{-\alpha t}\cos\omega_d t + B_2 e^{-\alpha t}\sin\omega_d t\right]_{t=0}$$

$$= B_1(-\alpha)e^{-\alpha t}\cos\omega_d t + B_2(-\alpha)e^{-\alpha t}\sin\omega_d t$$
$$+B_1 e^{-\alpha t}(-\omega_d)\sin\omega_d t + B_2(\omega_d)e^{-\alpha t}\cos\omega_d t\Big|_{t=0}$$

Underdamped Voltage Response



Now let t = 0

$$\frac{dv(0^+)}{dt} = B_1(-\alpha)e^{-\alpha 0}\cos 0 + B_2(-\alpha)e^{-\alpha 0}\sin 0$$

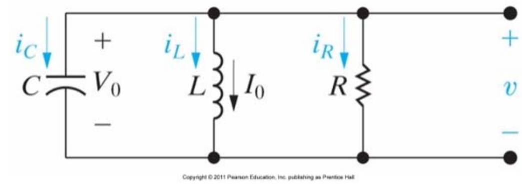
$$+B_1 e^{-\alpha 0}(-\omega_d)\sin 0 + B_2(\omega_d)e^{-\alpha 0}\cos 0$$

$$= B_1(-\alpha) + B_2(\omega_d) = -\alpha B_1 + \omega_d B_2$$

Remember

$$\frac{dv(0^+)}{dt} = \frac{i_C(t=0^+)}{C}$$

Underdamped Voltage Response



General observations

$$v(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

Since the circuit contains a resistor, the response will exponentially *die out.*

$$e^{-\alpha t} \qquad \alpha = \frac{1}{2RC}$$

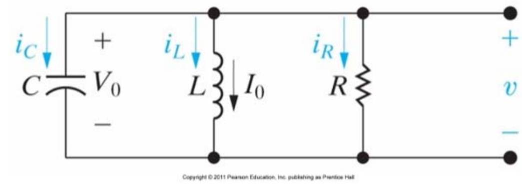
The system will *oscillate* at the frequency ω_d.

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} \qquad \omega_0 = \sqrt{\frac{1}{LC}}$$

This oscillation is called *ringing*

Critically Damped

Critically Damped Voltage Response



Critically damped response is when $\omega_0^2 = \alpha^2$

The roots of the characteristic equation are

$$\alpha = \frac{1}{2RC}$$

$$s_1 = s_2 = -\alpha$$

The characteristic equation thus needs a tweak from differential calculus which yields

$$v(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

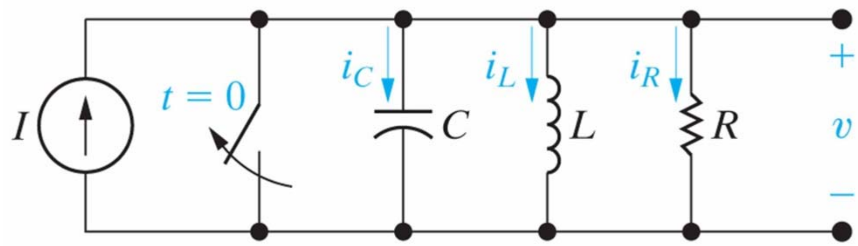
We find D_1 and D_2 from the initial conditions

$$v(t = 0^+) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t} = D_1(0)e^{-\alpha 0} + D_2 e^{-\alpha 0} = D_2$$

$$\left.\frac{dv(t = 0^+)}{dt}\right|_{t=0} = \frac{i_C(0^+)}{C} = D_1 - \alpha D_2$$

Section 8.3
Step Response of a Parallel RLC Circuit

Step Response of Parallel RLC Circuit



This circuit motivates the finding of a dc current step response to the parallel RLC circuit.

The current I is called the *forcing function.*

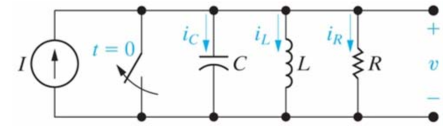
We will focus on finding the inductor current i_L.

By KCL after the switch *opens*

$$-I + i_L + i_C + i_R = 0$$

$$i_L + C\frac{dv}{dt} + \frac{v}{R} = I$$

Step Response of Parallel RLC Circuit



$$i_L + C\frac{dv}{dt} + \frac{v}{R} = I$$

We can relate v to the current i_L with

$$v = L\frac{di_L}{dt}$$

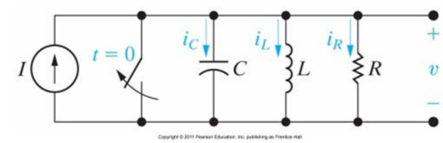
And

$$\frac{dv}{dt} = L\frac{d^2 i_L}{dt^2}$$

Which yields a useful form of the KCL equation:

$$i_L + \underbrace{\frac{L}{R}\frac{di_L}{dt}}_{i_R} + \underbrace{LC\frac{d^2 i_L}{dt^2}}_{i_C} = I$$

Step Response of Parallel RLC Circuit



$$i_L + \underbrace{\frac{L}{R}\frac{di_L}{dt}}_{i_R} + \underbrace{LC\frac{d^2 i_L}{dt^2}}_{i_C} = I$$

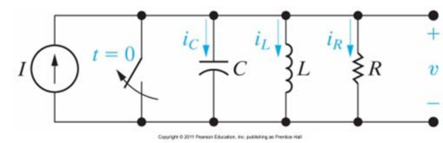
To ease the calculation, rearrange the terms

$$\frac{d^2 i_L}{dt^2} + \frac{1}{RC}\frac{di_L}{dt} + \frac{i_L}{LC} = \frac{I}{LC}$$

We now need to find the solution to a second-order differential equation with a constant forcing function.

Please see the text for details.

Step Response of Parallel RLC Circuit



The authors show (by their indirect method) that the solution to the step response is

$$i(t) = I_{final} + \{current\ natural\ response\}$$

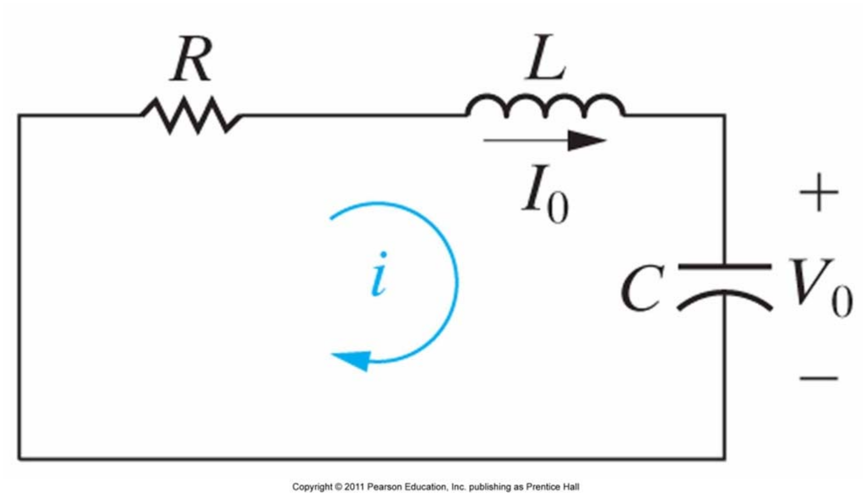
$$v(t) = V_{final} + \{voltage\ natural\ response\}$$

See the worked examples for full details on how to follow the process on finding both the natural response and the step response.

Section 8.4

Natural and Step Response of a Series RLC Circuit

Series RLC Circuit



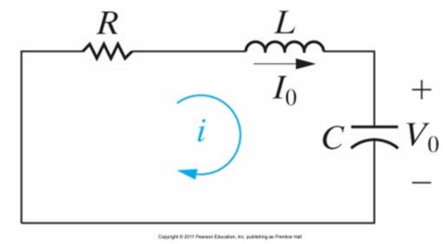
The series parallel RLC circuit can be found by the same process as was followed for the parallel RLC circuit.

By KVL, we write

$$v_R + v_L + v_C = 0$$

$$iR + L\frac{di}{dt} + \frac{1}{C}\int_0^t i_C \, dx + v_0 = 0$$

Series RLC Circuit



$$iR + L\frac{di}{dt} + \frac{1}{C}\int_0^t i_C \, dx + v_0 = 0$$

Differentiate the last equation

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{i}{C} = 0$$

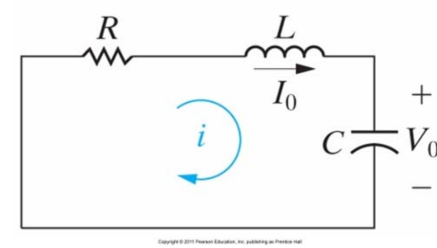
Rearrange the terms to write

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

Compare this result to the equation for the parallel RLC circuit:

$$\frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

Series RLC Circuit



$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0 \qquad \frac{d^2 v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{v}{LC} = 0$$

The form of the response is the same but with different coefficients (and that one is for current and the other for voltage).

The solution of the differential equation must be likewise similar.

So the authors immediately leap to writing the characteristic equation.

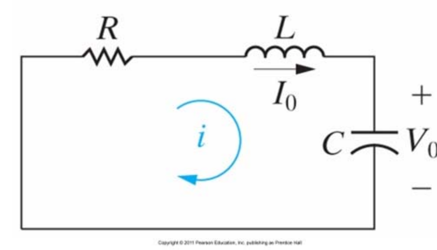
$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \alpha = \frac{R}{2L}\, rad/\sec$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \qquad \omega_0 = \sqrt{\frac{1}{LC}}\, rad/\sec$$

Series RLC Circuit

The series RLC circuits has the same three types of natural response (just like the parallel circuit):

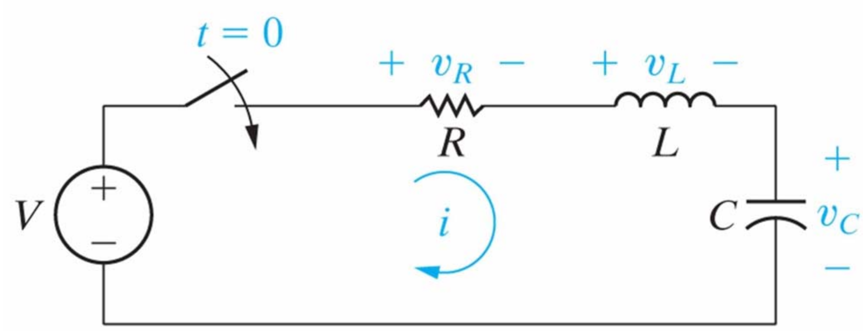


Overdamped $$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Underdamped $$i(t) = B_1 e^{-\alpha t} \cos \omega_d t + B_2 e^{-\alpha t} \sin \omega_d t$$

Critically Damped $$i(t) = D_1 t e^{-\alpha t} + D_2 e^{-\alpha t}$$

Step Response of Series RLC Circuit



The step response of the Series RLC circuit must come from a voltage source (why? – hint: can the current in the circuit instantaneously change?)

By KVL

$$-V + v_R + v_L + v_C = 0$$

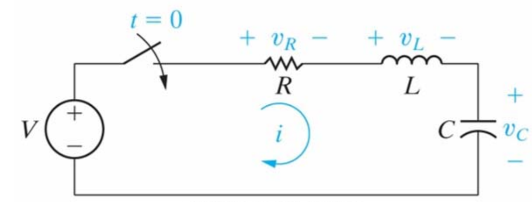
$$Ri + L\frac{di}{dt} + v_C = V$$

$$\frac{d^2 v_c}{dt^2} + \frac{R}{L}\frac{dv_c}{dt} + \frac{v_C}{LC} = \frac{V}{LC}$$

$$i = i_C = C\frac{dv_C}{dt}$$

$$\frac{di}{dt} = C\frac{d^2 v_C}{dt^2}$$

Step Response of Series RLC Circuit



$$\frac{d^2v_c}{dt^2}+\frac{R}{L}\frac{dv_c}{dt}+\frac{v_C}{LC}=\frac{V}{LC}$$

The step response of the series RLC circuit has the same second-order differential equation (with different constants), so the response must be similar:

Overdamped $\quad v_c(t)=V_{c,final}+A_1^{'}e^{s_1t}+A_2^{'}e^{s_2t}$

Underdamped $\quad v_c(t)=V_{c,final}+B_1^{'}e^{-\alpha t}\cos\omega_d t+B_2^{'}e^{-\alpha t}\sin\omega_d t$

Critically Damped $\quad v_c(t)=V_{c,final}+D_1^{'}te^{-\alpha t}+D_2^{'}e^{-\alpha t}$

Section 8.5

A Circuit with

Two Integrating Amplifiers

Section 8.5 Integrating Amplifiers will not be covered.

After phasors are introduced, we will briefly examine that approach to circuits with reactive elements.

Chapter 8

Natural and Step Responses of RLC Circuits

Text: *Electric Circuits*, 9th Edition, by J. Nilsson and S. Riedel
Prentice Hall

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