# EEE 187 Robotics

## Robotic system components

A robotic system has three major components:

- Actuators: the muscles of the robot
- Sensors: provide information about the environment and also about the internal state of the robot.
- Controller: the brain of the robot.

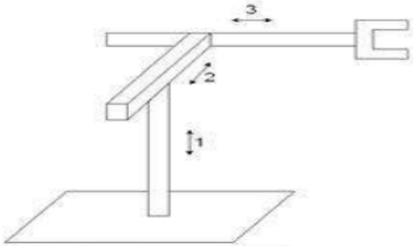
## Robotic manipulators

A robotic manipulator is composed of links connected by joints, together they form a kinematic chain. Manipulators are also called robotic arms. Applications of robotic manipulators include pick and place, welding, painting, etc. Robotic surgery is among the most recent application of robotic manipulators.

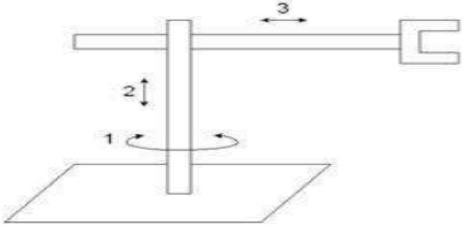
R= Resolute Joint
P-=Prismatic Joint

# EEE 187 Robotics

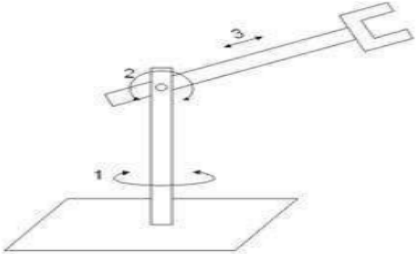
## Manipulators: common arrangements



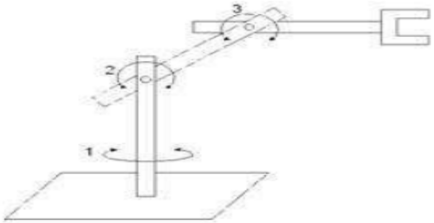

Cartesian: PPP



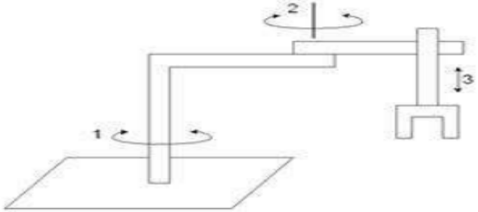

Cylindrical: RPP




Spherical: RRP

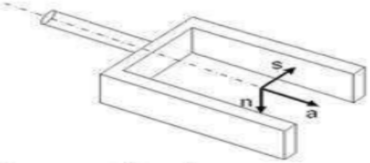



Articulated: RRR




SCARA: RRP

(Selective Compliance Assembly Robot Arm)



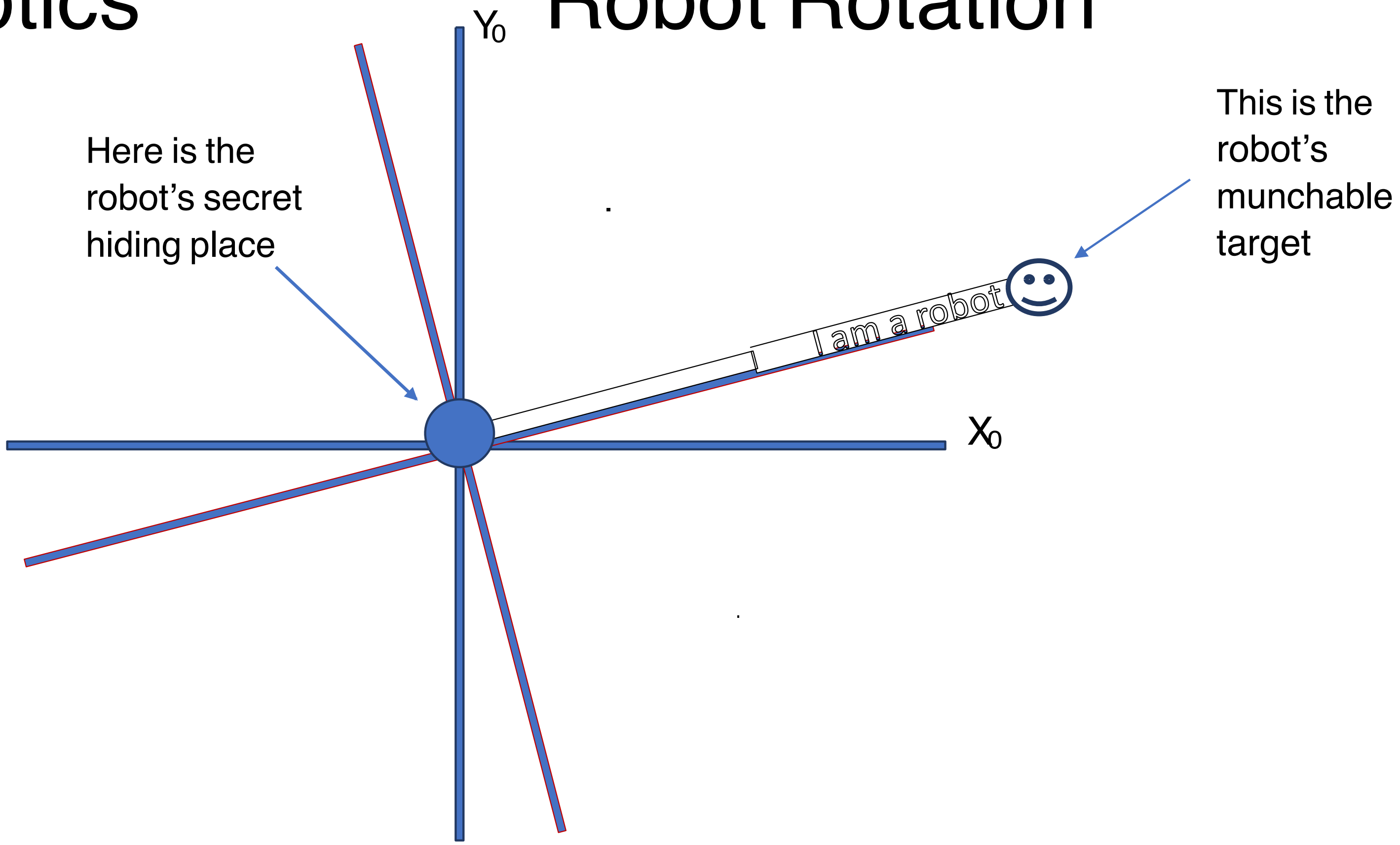

**Hand coordinate:**

**n:** normal vector; **s:** sliding vector;

**a:** approach vector, normal to the tool mounting plate

# EEE 187 Robotics    Robot Rotation

The robot tentacle reaches along an axis that is rotated with respect to the X axis  as it grabs and munches on its target. We need to understand the coordinates of the target in both the original reference frame and the robot's rotated reference frame.



# EEE 187 Robotics Robot Rotation

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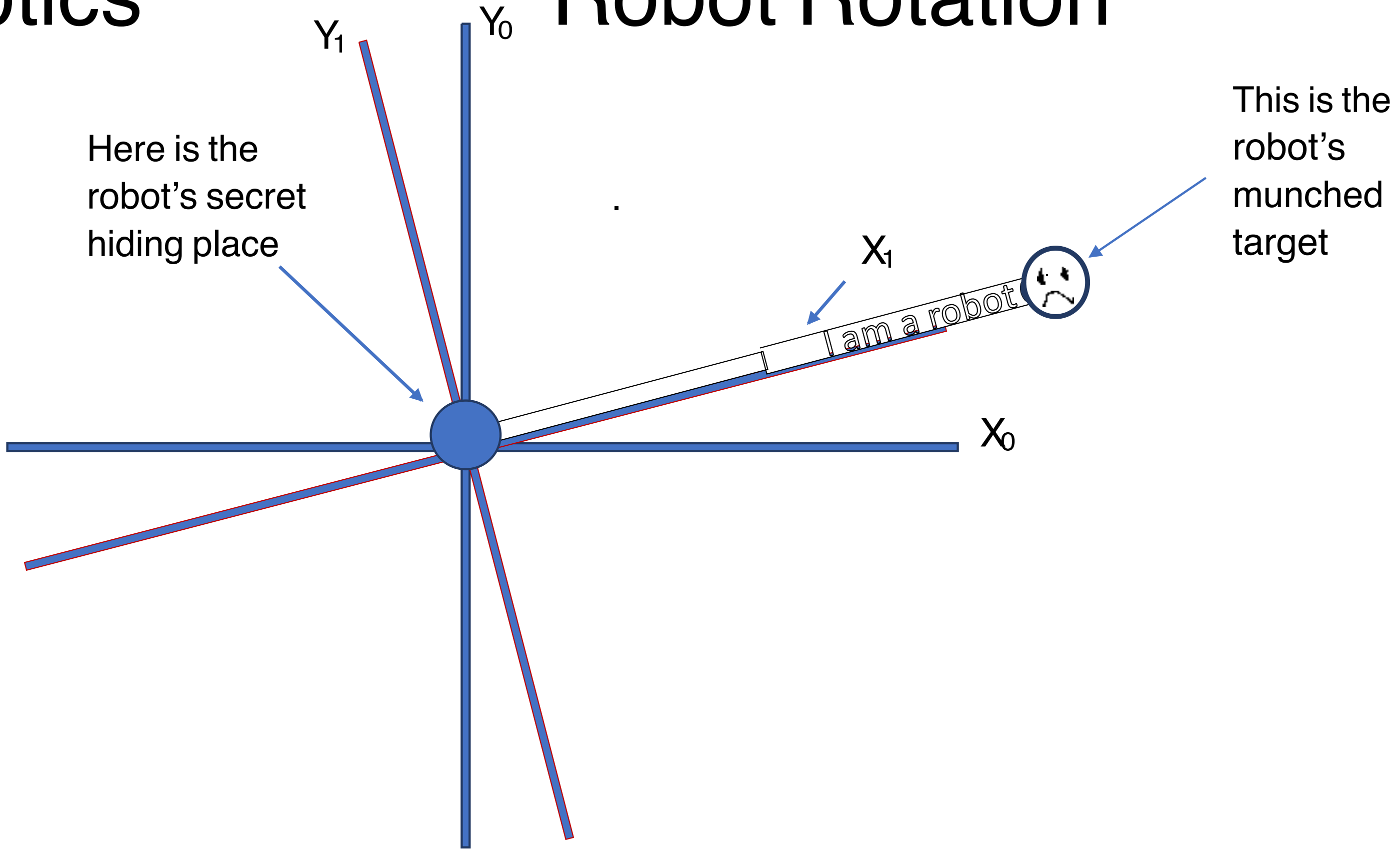


# EEE 187 Robotics
## Reference Frame Rotation

Our goal is to find the relationship between points, e.g. ($X_0Y_0$), described within the original reference frame $O_0$ and points e.g. ($X_1$,$Y_1$) described within the robot's rotated reference frame $O_1$

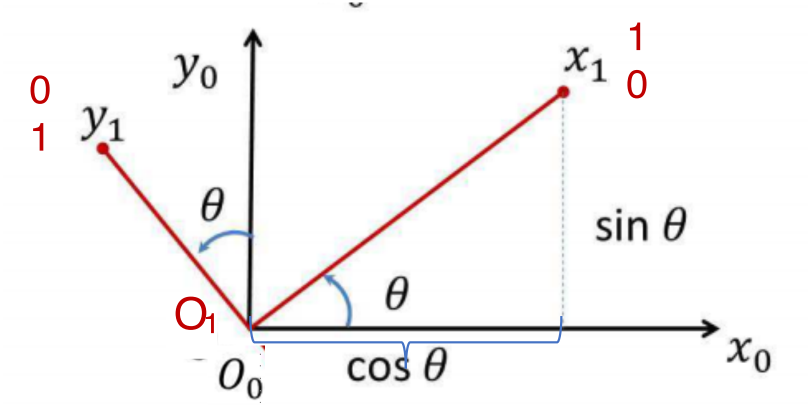
Define:

$X^0_1$ describes the coordinates within the original reference frame $O_0$ of a points whose coordinates within the robot's reference frame is (1, 0).

$$x_1^0 = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$Y^0_1$ describes the coordinates within the original reference frame $O_0$ of a points whose coordinates within the robot's reference frame is (0, 1).

$$y_1^0 = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$




$$R_1^0 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{matrix} 1 \\ 0 \end{matrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{matrix} 0 \\ 1 \end{matrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$

$R_1^0$ corresponds to rotation of reference 1 with respect to 0

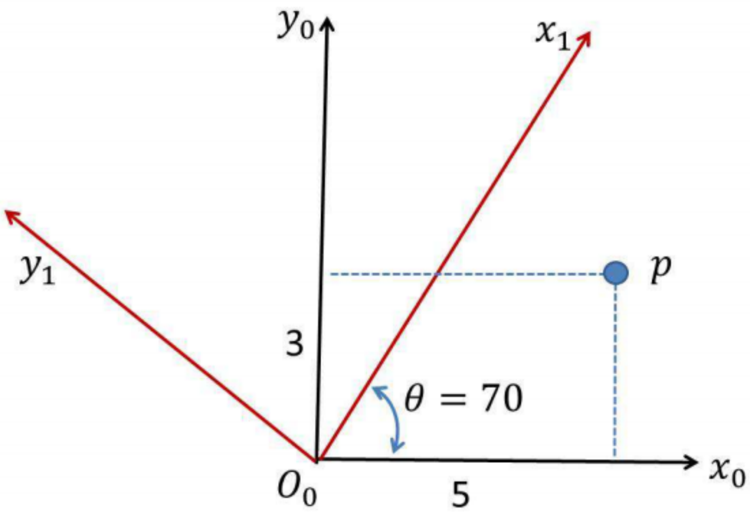
$R_0^1$ corresponds to rotation of reference 0 with respect to 1

$$R^T = R^{-1} \qquad R_0^1 = (R_1^0)^{-1}$$

# EEE 187 Robotics
# **Reference Frame Rotation**

*Example*

Consider figure 5, find the coordinates of point $p$ in the $(x_1, y_1)$ reference frame.




*Solution*

Let $p^0$ be the coordinate of point $p$ in frame $O_0x_0y_0$ and let $p^1$ be the coordinate of point $p$ in frame $O_1x_1y_1$, note that $O_0$ is the same as $O_1$ in this case.

$$p^0 = R_1^0 p^1$$
$$p^1 = R_0^1 p^0$$

and

$$R_1^0 = \begin{bmatrix} \cos(70) & -\sin(70) \\ \sin(70) & \cos(70) \end{bmatrix}$$

Therefore:

$$R_0^1 = \begin{bmatrix} 0.3420 & 0.9397 \\ -0.9397 & 0.3420 \end{bmatrix}$$

and finally

$$p^1 = \begin{bmatrix} 4.5292 \\ -3.6724 \end{bmatrix}$$

Where $$p^0 = \begin{bmatrix} 5.0000 \\ 3.0000 \end{bmatrix}$$

# EEE 187 Robotics

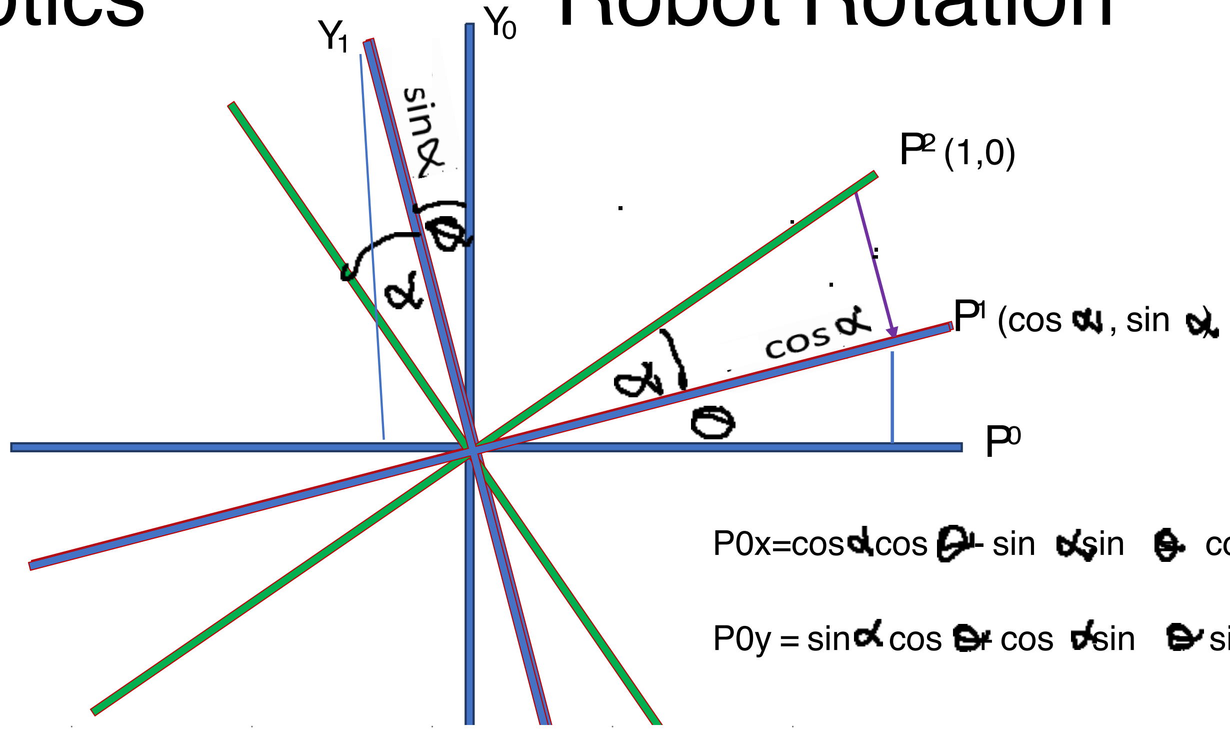
## Reference Frame Rotation

$$\begin{vmatrix} COS(70) & -SIN(70) \\ SIN\ (70) & COS(70) \end{vmatrix} = \begin{vmatrix} COS(30) & -SIN(30) \\ SIN\ (30) & COS(30) \end{vmatrix} \begin{vmatrix} COS(40) & -SIN(40) \\ SIN\ (40) & COS(40) \end{vmatrix}$$

$$= \begin{vmatrix} COS(20) & -SIN(20) \\ SIN\ (20) & COS(20) \end{vmatrix} \begin{vmatrix} COS(25) & -SIN(25) \\ SIN\ (25) & COS(25) \end{vmatrix} \begin{vmatrix} COS(25) & -SIN(25) \\ SIN\ (25) & COS(25) \end{vmatrix}$$

Rotations add!!!!!!!!!!

# EEE 187 Robotics Robot Rotation




$P0x = \cos\alpha\cos\theta - \sin\alpha\sin\theta \quad \cos(\alpha+\theta)$

$P0y = \sin\alpha\cos\theta + \cos\alpha\sin\theta \quad \sin(\alpha+\theta)$

$$R_1^0 R_2^1 = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\alpha+\theta) & -\sin(\alpha+\theta) \\ \sin(\alpha+\theta) & \cos(\alpha+\theta) \end{bmatrix}}_{R_2^0}$$

# ROBOTICS Chapter 2

Consider Figure 2.1, which shows two coordinate frames that differ in orientation by an angle of 45°. Using the synthetic approach, without ever assigning coordinates to points or vectors, one can say that $x_0$ is perpendicular to $y_0$, or that $v_1 \times v_2$ defines a vector that is perpendicular to the plane containing $v_1$ and $v_2$, in this case pointing out of the page.

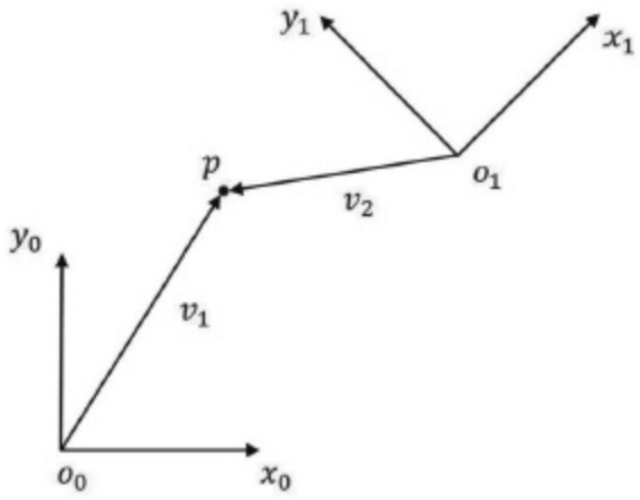



**Figure 2.1** Two coordinate frames, a point $p$, and two vectors $v_1$ and $v_2$.

$\bar{A} \cdot \bar{B} = |A||B|\cos\theta$

# Rotation in the Plane

A slightly less obvious way to specify the orientation is to specify the coordinate vectors for the axes of frame $o_1x_1y_1$ with respect to coordinate frame $o_0x_0y_0$:

$$R_1^0 = \left[x_1^0 \mid y_1^0\right]$$

in which $x_1^0$ and $y_1^0$ are the coordinates in frame $o_0x_0y_0$ of unit vectors $x_1$ and $y_1$, respectively.

A matrix in this form is called a rotation matrix.

# Rotation Matrices Special Properties

## Rotation matrices have a number of special properties

In the two-dimensional case, it is straightforward to compute the entries of this matrix. As illustrated in Figure 2.2,

$$x_1^0 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

which gives

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$R_1^0$ is a matrix whose column vectors are the coordinates of the unit vectors along the axes of frame $o_1x_1y_1$ expressed relative to frame $o_0x_0y_0$.

→ old reference frame

→ new reference frame

# Projecting Axes

An alternative approach is to build the rotation matrix by projecting the axes of frame $o_1x_1y_1$ onto the coordinate axes of frame $o_0x_0y_0$. Recalling that the dot product of two unit vectors gives the projection of one onto the other, we obtain

$$x_1^0 = \begin{bmatrix} x_1 \cdot x_0 \\ x_1 \cdot y_0 \\ \\ \end{bmatrix}, \qquad y_1^0 = \begin{bmatrix} y_1 \cdot x_0 \\ y_1 \cdot y_0 \\ \\ \end{bmatrix}$$

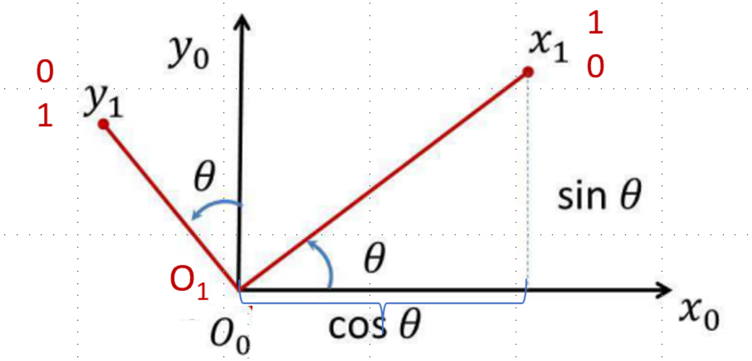
which can be combined to obtain the rotation matrix

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 \\ & \end{bmatrix}$$

# Transpose as Inverse

If we desired instead to describe the orientation of frame $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ (that is, if we desired to use the frame $o_1x_1y_1$ as the reference frame), we would construct a rotation matrix of the form

$$R_0^1 = \begin{bmatrix} x_0 \cdot x_1 & y_0 \cdot x_1 \\ x_0 \cdot y_1 & y_0 \cdot y_1 \end{bmatrix}$$




Since the dot product is commutative, (that is, $x_i \cdot y_j = y_j \cdot x_i$), we see that

$$R_0^1 = (R_1^0)^T$$

In a geometric sense, the orientation of $o_0x_0y_0$ with respect to the frame $o_1x_1y_1$ is the inverse of the orientation of $o_1x_1y_1$ with respect to the frame $o_0x_0y_0$. Algebraically, using the fact that coordinate axes are mutually orthogonal, it can readily be seen that

$$(R_1^0)^T = (R_1^0)^{-1} \quad \text{Which implies that} \quad (R_1^0)^T R_1^0 = I$$

# Rotations in Three Dimensions

- The projection technique described above scales nicely to the three-dimensional case. In three dimensions, each axis of the frame $o_1x_1y_1z_1$ is projected onto coordinate frame $o_0x_0y_0z_0$.

$$R_1^0 = \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix}$$

As was the case for rotation matrices in two dimensions, matrices in this form are orthogonal, with determinant equal to 1

# Example 2.1

Suppose the frame $o_1x_1y_1z_1$ is rotated through an angle θ about the $z_0$-axis, and we wish to find the resulting transformation matrix $R_1^0$.

By convention, the right hand rule (see Appendix B) defines the positive sense for the angle θ to be such that rotation by θ about the z-axis would advance a right-hand threaded screw along the positive z-axis. From Figure 2.3 we see that

$$x_1 \cdot x_0 = \cos\theta, \quad y_1 \cdot x_0 = -\sin\theta,$$
$$x_1 \cdot y_0 = \sin\theta, \quad y_1 \cdot y_0 = \cos\theta$$

$$z_0 \cdot z_1 = 1$$

# Example 2.1 Continued

Since the other dot products are zero the rotation matrix has a particularly simple form in this case, namely

$$R_1^0 = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

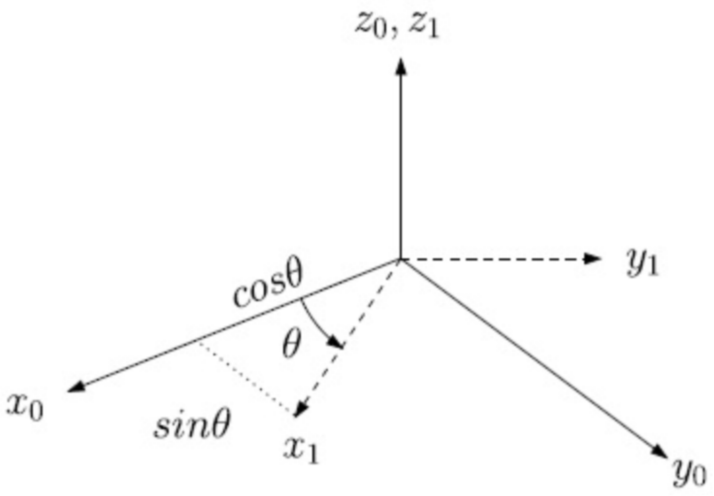



**Figure 2.3** Rotation about $z_0$ by an angle $\theta$.

# Example 2.1 Continued

The rotation matrix given in Equation (2.3) is called a basic rotation matrix (about the z-axis). In this case we find it useful to use the more descriptive notation $R_{z,\theta}$ instead of $R_1^0$ to denote the matrix. It is easy to verify that the basic rotation matrix $R_{z,\theta}$ has the properties (2.4)

$$R_{z,0} = I$$

$$R_{z,\theta} R_{z,\phi} = R_{z,\theta+\phi}$$

Which together imply

$$\left(R_{z,\theta}\right)^{-1} = R_{z,-\theta}$$

$$R_{x,\theta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

which also satisfy properties analogous to Equations (2.4)–(2.6).

$$R_{y,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

# Example 2.2

Consider the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ shown in Figure 2.4. Projecting the unit vectors $x_1$, y1, $z_1$ onto $x_0$, $y_0$, $z_0$ gives the coordinates of $x_1$, $y_1$, $z_1$ in the $o_0x_0y_0z_0$ frame as

$$x_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad y_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{-1}{\sqrt{2}} \end{bmatrix}, \quad z_1^0 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The rotation matrix $R_1^0$ specifying the orientation of $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ has these as its column vectors, that is,

$$R_1^0 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{bmatrix}$$

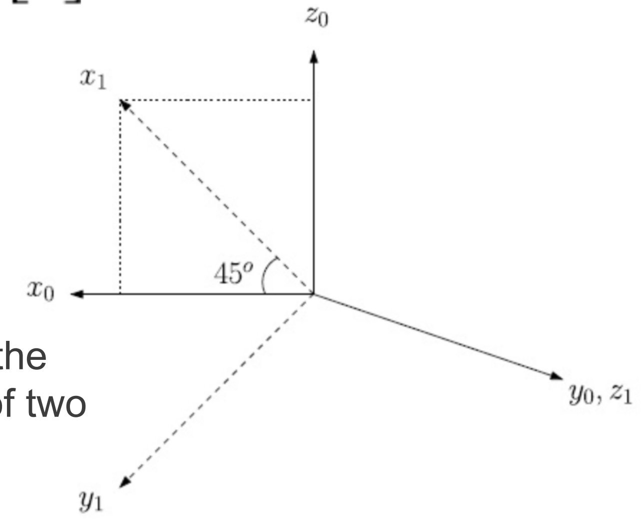



Figure 2.4 Defining the relative orientation of two frames

# Rotational Transformations

Figure 2.5 shows a rigid object S to which a coordinate frame $o_1x_1y_1z_1$ is attached. Given the coordinates $p^1$ of the point p (in other words, given the coordinates of p with respect to the frame $o_1x_1y_1z_1$), we wish to determine the coordinates of p relative to a fixed reference frame $o_0x_0y_0z_0$. The coordinates $p^1 = (u, v, w)$ satisfy the equation

$$p^1 = ux_1 + vy_1 + wz_1$$

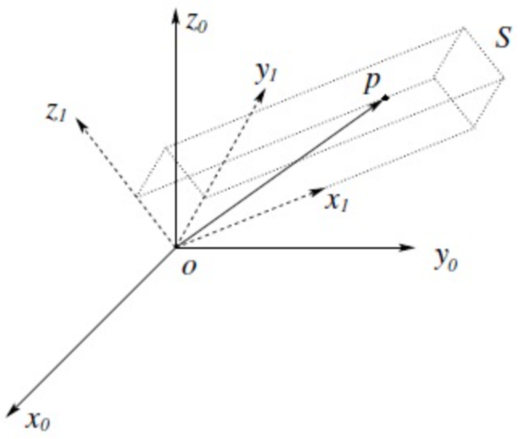



**Figure 2.5** Coordinate frame attached to a rigid body.

In a similar way, we can obtain an expression for the coordinates by projecting the point p onto the coordinate axes of the frame $o_0x_0y_0z_0$, giving

$$p^0 = \begin{bmatrix} p \cdot x_0 \\ p \cdot y_0 \\ p \cdot z_0 \end{bmatrix}$$

# Rotational Transformations (continued)

Combining these two equations we obtain

$$p^0 = \begin{bmatrix} (ux_1 + vy_1 + wz_1) \cdot x_0 \\ (ux_1 + vy_1 + wz_1) \cdot y_0 \\ (ux_1 + vy_1 + wz_1) \cdot z_0 \end{bmatrix}$$

$$= \begin{bmatrix} ux_1 \cdot x_0 + vy_1 \cdot x_0 + wz_1 \cdot x_0 \\ ux_1 \cdot y_0 + vy_1 \cdot y_0 + wz_1 \cdot y_0 \\ ux_1 \cdot z_0 + vy_1 \cdot z_0 + wz_1 \cdot z_0 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \cdot x_0 & y_1 \cdot x_0 & z_1 \cdot x_0 \\ x_1 \cdot y_0 & y_1 \cdot y_0 & z_1 \cdot y_0 \\ x_1 \cdot z_0 & y_1 \cdot z_0 & z_1 \cdot z_0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

But the matrix $R_1^0$ in this final equation is merely the rotation matrix , which leads to $p^0 = R_1^0 p^1$ (2.9)

Thus, the rotation matrix $R_1^0$ can be used not only to represent the orientation of coordinate frame $o_1x_1y_1z_1$ with respect to frame $o_0x_0y_0z_0$, but also to transform the coordinates of a point from one frame to another. If a given point is expressed relative to $o_1x_1y_1z_1$ by coordinates $p^1$, then $R_1^0 p^1$ represents the **same point** expressed relative to the frame $o_0x_0y_0z_0$.

# Rotational Transformations (continued)

We can also use rotation matrices to represent rigid motions that correspond to pure rotation. For example, in Figure 2.6(a) one corner of the block is located at the point $p_a$ in space. Figure 2.6(b) shows the same block after it has been rotated about $z_0$ by the angle π. The same corner of the block is now located at point $p_b$ in space. It is possible to derive the coordinates for $p_b$ given only the coordinates for $p_a$ and the rotation matrix that corresponds to the rotation about $z_0$. To see how this can be accomplished, imagine that a coordinate frame is rigidly attached to the block in Figure 2.6(a), such that it is coincident with the frame $o_0x_0y_0z_0$. After the rotation by π, the block's coordinate frame, which is rigidly attached to the block, is also rotated by π. If we denote this rotated frame by $o_1x_1y_1z_1$, we obtain

$$R_1^0 = R_{z,\pi} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

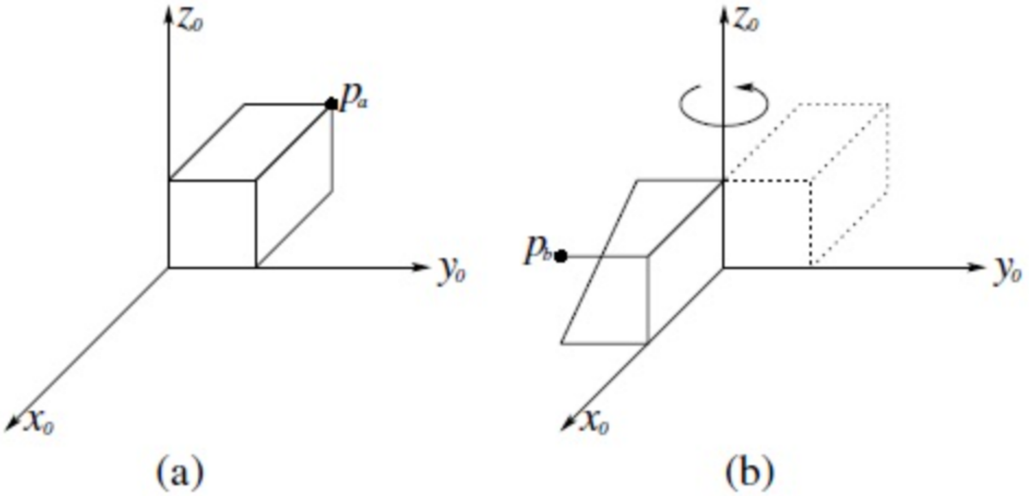



Figure 2.6 The block in (b) is obtained by rotating the block in (a) by π about $z_0$.

In the local coordinate frame $o_1x_1y_1z_1$, the point $p_b$ has coordinate representation $p_b^1$. To obtain its coordinates with respect to frame $o_0x_0y_0z_0$, we merely apply the coordinate transformation Equation (2.9), giving

$$p_b^0 = R_{z,\pi} p_b^1$$

# Rotational Transformations (continued)

It is important to notice that the local coordinates of the corner of the block do not change as the block rotates, since they are defined in terms of the block's own coordinate frame. Therefore, when the block's frame is aligned with the reference frame $o_0x_0y_0z_0$ (that is, before the rotation is performed), the coordinates equals , since before the rotation is performed, the point $p_a$ is coincident with the corner of the block. Therefore, we can substitute into the previous equation to obtain $p_b^0 = R_{z,\pi} p_a^0$

This equation shows how to use a rotation matrix to represent a rotational motion. In particular, if the point $p_b$ is obtained by rotating the point $p_a$ as defined by the rotation matrix , then the coordinates of $p_b$ with respect to the reference frame are given by $p_b^0 = R p_a^0$

This same approach can be used to rotate vectors with respect to a coordinate frame, as the following example illustrates.

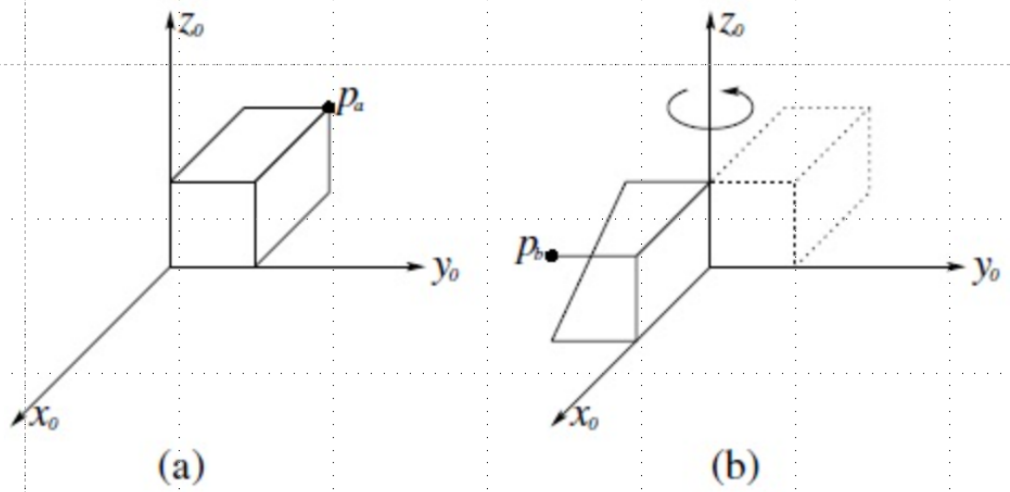



Figure 2.6 The block in (b) is obtained by rotating the block in (a) by π about $z_0$.

# EXAMPLE

The vector v with coordinates $v_0 = (0, 1, 1)$ is rotated about $y_0$ by
as shown in Figure 2.7. The resulting vector $v_1$ is given by

(2.10)

$$v_1^0 = R_{y,\frac{\pi}{2}} v^0$$

$$\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

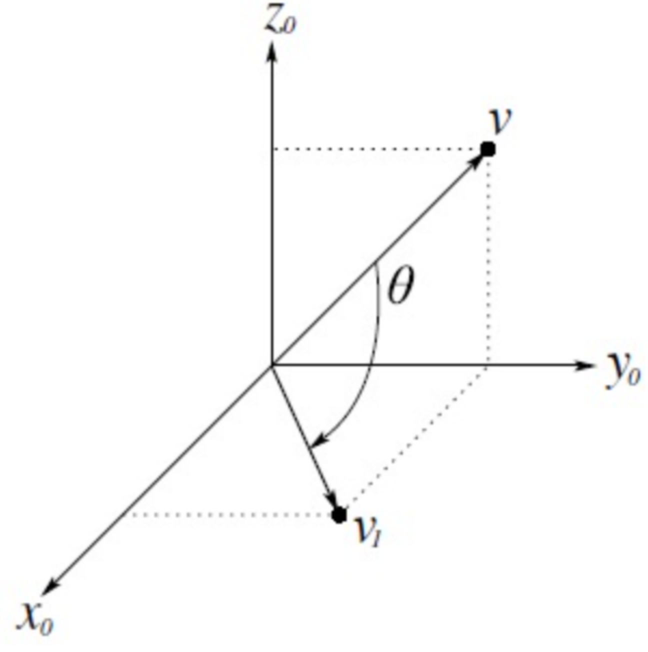



Figure 2.7 Rotating a vector about axis $y_0$

Thus, a third interpretation of a rotation matrix is as an operator acting on vectors in a fixed frame. In other words, instead of relating the coordinates of a fixed vector with respect to two different coordinate frames, Equation (2.10) can represent the coordinates in $o_0x_0y_0z_0$ of a vector $v_1$ that is obtained from a

# ROTATION TRANSFORMATIONS SUMMARIZED

As we have seen, rotation matrices can serve several roles. A rotation matrix can be interpreted in three distinct ways:

1) It represents a coordinate transformation relating the coordinates of a point p in two different frames.

2) It gives the orientation of a transformed coordinate frame with respect to a fixed coordinate frame.

3) It is an operator taking a vector and rotating it to give a new vector in the same coordinate frame.
The particular interpretation of a given rotation matrix should be made clear by the context.

# Similarity Transformations

A coordinate frame is defined by a set of basis vectors, for example, unit vectors along the three coordinate axes. This means that a rotation matrix, as a coordinate transformation, can also be viewed as defining a change of basis from one frame to another. The matrix representation of a general linear transformation is transformed from one frame to another using a so-called similarity transformation.

For example, if A is the matrix representation of a given linear transformation in $o_0x_0y_0z_0$ and B is the representation of the same linear transformation in $o_1x_1y_1z_1$ then A and B are related as

$$B = (R_1^0)^{-1} A R_1^0 \qquad (2.12)$$

Where $R_1^0$ is the coordinate transformation between frames $o_1x_1y_1z_1$ and $o_0x_0y_0z_0$. In particular, if A itself is a rotation, then so is B, and thus the use of similarity transformations allows us to express the same rotation easily with respect to different frames.

# Example 2.4

Henceforth, whenever convenient we use the shorthand notation $c_\theta = \cos\theta$, $s_\theta = \sin\theta$ for trigonometric functions. Suppose frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ are related by the rotation

$$R_1^0 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

If $A = R_{z,\theta}$ relative to the frame $o_0x_0y_0z_0$, then, relative to frame $o_1x_1y_1z_1$ we have

$$B = (R_1^0)^{-1} A R_1^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & s_\theta \\ 0 & -s_\theta & c_\theta \end{bmatrix}$$

In other words, B is a rotation about the $z_0$-axis but expressed relative to the frame $o_1x_1y_1z_1$. This notion will be useful below and in later sections.

# Composition of Rotations

Rotation with Respect to the Current Frame Recall that the matrix in Equation (2.9) represents a rotational transformation between the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$. Suppose we now add a third coordinate frame $o_2x_2y_2z_2$ related to the frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ by rotational transformations. A given point p can then be represented by coordinates specified with respect to any of these three frames: , , and The relationship among these representations of p is

$$p^0 = R_1^0 p^1 \qquad (2.13)$$

$$p^1 = R_2^1 p^2 \qquad (2.14)$$

$$p^0 = R_2^0 p^2 \qquad (2.15)$$

where each $R_j^i$ is a rotation matrix. Substituting Equation (2.14) into Equation (2.13) gives

$$R_2^0 = R_1^0 R_2^1 \qquad (2.16)$$

Note that $R_1^0$ and $R_2^0$ represent rotations relative to the frame $o_0x_0y_0z_0$ while $R_2^1$ represents a rotation relative to the frame $o_1x_1y_1z_1$. Comparing Equations (2.15) and (2.16) we can immediately infer

$$R_2^0 = R_1^0 R_2^1 \qquad (2.17)$$

Equation (2.17) is the composition law for rotational transformations. It states that, in order to transform the coordinates of a point p from its representation $p^2$ in the frame $o_2x_2y_2z_2$ to its representation $p^0$ in the frame $o_0x_0y_0z_0$, we may first transform to its coordinates $p^1$ in the frame $o_1x_1y_1z_1$ using $R_2^1$ and then transform $p^1$ to $p^0$ using $R_1^0$.

# Composition of Rotations (Continued)

We may also interpret $R_2^0 = R_1^0 R_2^1$ as follows.

Suppose that initially all three of the coordinate frames coincide. We first rotate the frame $o_1x_1y_1z_1$ relative to $o_0x_0y_0z_0$ according to the transformation . $R_1^0$.

Then, with the frames $o_1x_1y_1z_1$ and $o_2x_2y_2z_2$ coincident, we rotate $o_2x_2y_2z_2$ relative to $o_1x_1y_1z_1$ according to the transformation $R_2^1$ The resulting frame, $o_2x_2y_2z_2$ has orientation with respect to $o_0x_0y_0z_0$ given by $R_1^0 R_2^1$.

We call the frame relative to which the rotation occurs the current frame.

# Example 2.5.

Suppose a rotation matrix represents a rotation of angle ϕ about the current y-axis followed by a rotation of angle θ about the current z-axis as shown in Figure 2.8. Then the matrix is given by

$$
\begin{aligned}
R &= R_{y,\phi} R_{z,\theta} \\
&= \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\phi c_\theta & -c_\phi s_\theta & s_\phi \\ s_\theta & c_\theta & 0 \\ -s_\phi c_\theta & s_\phi s_\theta & c_\phi \end{bmatrix}
\end{aligned} \tag{2.18}
$$

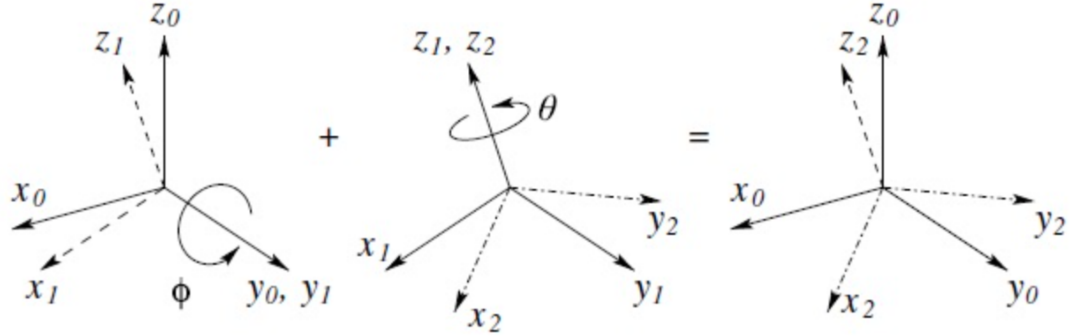



Figure 2.8 Composition of rotations about current axes.

It is important to remember that the order in which a sequence of rotations is performed, and consequently the order in which the rotation matrices are multiplied together, is crucial. The reason is that rotation, unlike position, is not a vector quantity and so rotational transformations do not commute in general.

# Example 2.6

Suppose that the above rotations are performed in the reverse order, that is, first a rotation about the current z-axis followed by a rotation about the current y-axis. Then the resulting rotation matrix is given by

$$
\begin{aligned}
R' &= R_{z,\theta} R_{y,\phi} \\
&= \begin{bmatrix} c_\theta & -s_\theta & 0 \\ s_\theta & c_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\
&= \begin{bmatrix} c_\theta c_\phi & -s_\theta & c_\theta s_\phi \\ s_\theta c_\phi & c_\theta & s_\theta s_\phi \\ -s_\phi & 0 & c_\phi \end{bmatrix}
\end{aligned}
\tag{2.19}
$$

*Comparing Equations (2.18) and (2.19) we see that* $R \neq R'$.

# Rotation with Respect to the Fixed Frame

2.4.2 Many times it is desired to perform a sequence of rotations, each about a given fixed coordinate frame, rather than about successive current frames. For example we may wish to perform a rotation about $x_0$ followed by a rotation about $y_0$ (and not $y_1$!). We will refer to $o_0x_0y_0z_0$ as the fixed frame. In this case the composition law given by Equation (2.17) is not valid. It turns out that the correct composition law in this case is simply to multiply the successive rotation matrices in the reverse order from that given by Equation (2.17). Note that the rotations themselves are not performed in reverse order. Rather they are performed about the fixed frame instead of about the current frame. To see this, suppose we have two frames $o_0x_0y_0z_0$ and $o_1x_1y_1z_1$ related by the rotational transformation $R_1^0$. If R represents a rotation relative to $o_0x_0y_0z_0$, we know from Section 2.3 that the representation for R in the current frame $o_1x_1y_1z_1$ is given by $(R_1^0)^{-1}RR_1^0$.

Therefore, applying the composition law for rotations about the current axis yields

$$R_2^0 = R_1^0\left[(R_1^0)^{-1}RR_1^0\right] = RR_1^0 \qquad (2.20)$$

Thus, when a rotation is performed with respect to the world coordinate frame, the current rotation matrix is premultiplied by to obtain the desired rotation matrix.

# Example 2.7. (Rotations about Fixed Axes)

Referring to Figure 2.9, suppose that a rotation matrix $R$ represents a rotation of angle ϕ about y0 followed by a rotation of angle θ about the fixed $z_0$. The second rotation about the fixed axis is given by $R_{y,-\phi}R_{z,\theta}R_{y,\phi}$

which is the basic rotation about the z-axis expressed relative to the frame $o_1x_1y_1z_1$ using a similarity transformation. Therefore, the composition rule for rotational transformations gives us

$$R = R_{y,\phi}\left[R_{y,-\phi}R_{z,\theta}R_{y,\phi}\right] = R_{z,\theta}R_{y,\phi} \tag{2.21}$$

It is not necessary to remember the above derivation, only to note by comparing Equation (2.21) with Equation (2.18) that we obtain the same basic rotation matrices, but in the reverse order.

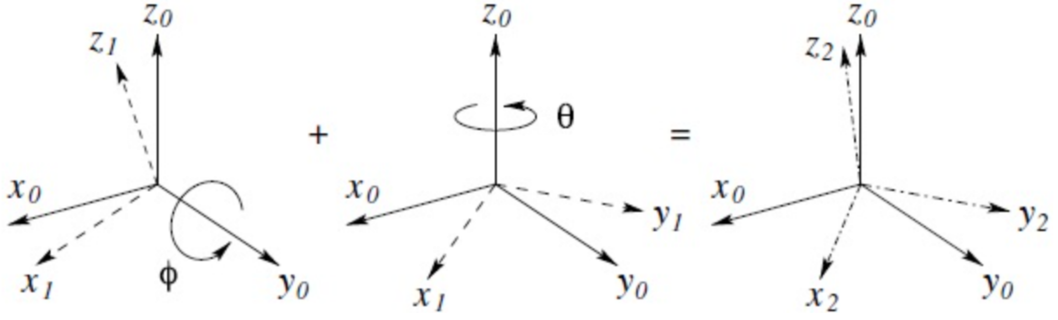



Figure 2.9 Composition of rotations about fixed axes.

# Rules for Composition of Rotations

We can summarize the rule of composition of rotational transformations by the following recipe. Given a fixed frame $o_0x_0y_0z_0$ and a current frame $o_1x_1y_1z_1$, together with rotation matrix $R_1^0$ relating them, if a third frame $o_2x_2y_2z_2$ is obtained by a rotation $R$ performed relative to the current frame then postmultiply $R_1^0$ by $R = R_2^1$ to obtain

$$R_2^0 = R_1^0 R_2^1 \qquad (2.22)$$

If the second rotation is to be performed relative to the fixed frame then it is both confusing and inappropriate to use the notation $R_2^1$ to represent this rotation. Therefore, if we represent the rotation by $R$, we premultiply $R_1^0$ by $R$ to obtain

$$R_2^0 = R R_1^0 \qquad (2.23)$$

In each case $R_2^0$ represents the transformation between the frames $o_0x_0y_0z_0$ and $o_2x_2y_2z_2$. The frame $o_2x_2y_2z_2$ that results from Equation (2.22) will be different from that resulting from Equation (2.23). Using the above rule for composition of rotations, it is an easy matter to determine the result of multiple sequential rotational transformations.

# Example 2.8.

Suppose R is defined by the following sequence of basic rotations in the order specified:

1. A rotation of θ about the current x-axis
2. A rotation of ϕ about the current z-axis
3, A rotation of α about the fixed z-axis
4. A rotation of β about the current y-axis
5. A rotation of δ about the fixed x-axis

In order to determine the cumulative effect of these rotations we simply begin with the first rotation $R_{x,\theta}$ and pre- or postmultiply as the case may be to obtain

$$R = R_{x,\delta} R_{z,\alpha} R_{x,\theta} R_{z,\phi} R_{y,\beta} \qquad (2.24)$$

$R_{x\delta} R_{z\alpha}$ $R_{x\theta}$ $R_{z\phi}$ $R_{y\beta}$

# Find the axis of rotation;

When the magnitude and direction of a vector coincides with its coordinates system's axis of rotation then the magnitude and direction of the vector will not be influenced by the rotation. The vector will point along the axis of rotation before the rotation occurs and will continue to point in that direction after the rotation occurs. Therefore to determine the axis of rotation we need to find the coordinates of a unit vector whose magnitude and direction will remain unchanged by the rotation

Example: What is the axis of rotation that is described by the rotation matrix below;

$$\begin{bmatrix} 0 & 0 & 1.0000 \\ 0.8660 & -0.5000 & 0 \\ 0.5000 & 0.8660 & 0 \end{bmatrix} \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \text{Sqrt }(3/7) \\ \text{Sqrt }(1/7) \\ \text{Sqrt }(3/7) \end{pmatrix}$$

Solving three simultaneous equations for three unknowns

# Rigid Motions

A rigid motion is a pure translation together with a pure rotation.[3] Let $R_1^0$ be the rotation matrix that specifies the orientation of frame $o_1x_1y_1z_1$ with respect to $o_0x_0y_0z_0$, and $d$ the vector from the origin of frame $o_0x_0y_0z_0$ to the origin of frame $o_1x_1y_1z_1$. Suppose the point $p$ rigidly attached to coordinate frame $o_1x_1y_1z_1$, with local coordinates $p^1$. We can express the coordinates of $p$ with respect to frame $o_0x_0y_0z_0$ using

$$p^0 = R_1^0 p^1 + d^0 \qquad (2.58)$$

Now consider three coordinate frames $o_0x_0y_0z_0$, $o_1x_1y_1z_1$, and $o_2x_2y_2z_2$. Let $d_1$ be the vector from the origin of $o_0x_0y_0z_0$ to the origin of $o_1x_1y_1z_1$ and $d_2$ be the vector from the origin of $o_1x_1y_1z_1$ to the origin of $o_2x_2y_2z_2$. If the point p is attached to frame $o_2x_2y_2z_2$ with local coordinates $p^2$, an compute its coordinates relative to frame $o_0x_0y_0z_0$ using and (2.60)

$$p^1 = R_2^1 p^2 + d_2^1 \qquad (2.59)$$

$$p^0 = R_1^0 p^1 + d_1^0 \qquad (2.60)$$

The composition of these two equations defines a third rigid motion, which we can describe by substituting the expression for from Equation (2.59) into Equation (2.60)

$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0 \qquad (2.61)$$

# Rigid Motions

Since the relationship between $p^0$ and $p^2$ is also a rigid motion, we can equally describe it as

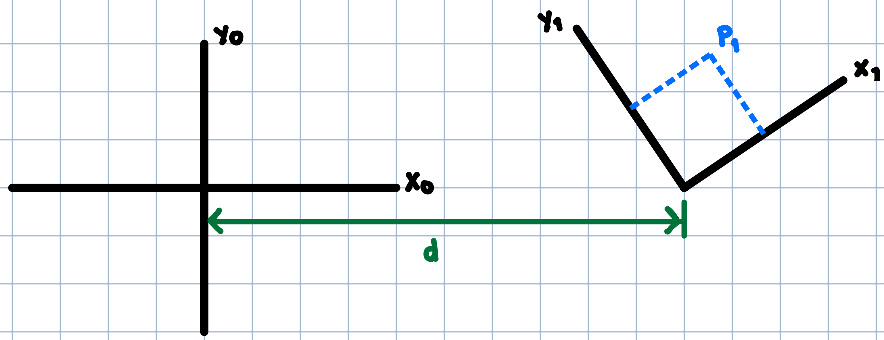
$$p^0 = R_2^0 p^2 + d_2^0 \qquad (2.62)$$

Comparing Equations (2.61) and (2.62) we have the relationships

$$R_2^0 = R_1^0 R_2^1 \qquad (2.63)$$

$$d_2^0 = d_1^0 + R_1^0 d_2^1 \qquad (2.64)$$

Equation (2.63) shows that the orientation transformations can simply be multiplied together and Equation (2.64) shows that the vector from the origin $o_0$ to the origin $o_2$ has coordinates given by the sum of $d_1^0$ (the vector from $o_0$ to $o_1$ expressed with respect to $o_0x_0y_0z_0$) and $R_1^0 d_2^1$ (the vector from $o_1$ to $o_2$, expressed in the orientation of the coordinate frame $o_0x_0y_0z_0$).




$$p^0 = R_1^0 R_2^1 p^2 + R_1^0 d_2^1 + d_1^0$$

$$= R_1^0 \left[ R_2^1 p^2 + d_2^1 \right] + d_1^0$$

$$p^0 = \underbrace{R_1^0 R_2^1 p^2}_{R_2^0 p^2} + \underbrace{R_1^0 d_2^1 + d_1^0}_{d_2^0}$$

# Homogeneous Transformations

One can easily see that the calculation leading to Equation (2.61) would quickly become intractable if a long sequence of rigid motions were considered. In this section we show how rigid motions can be represented in matrix form so that composition of rigid motions can be reduced to matrix multiplication as was the case for composition of rotations. In fact, a comparison of Equations (2.63) and (2.64) with the matrix identity

$$\begin{bmatrix} R_1^0 & d_1^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2^1 & d_2^1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1^0 R_2^1 & R_1^0 d_2^1 + d_1^0 \\ 0 & 1 \end{bmatrix} \tag{2.65}$$

where 0 denotes the row vector (0, 0, 0), shows that the rigid motions can be represented by the set of matrices of the form

$$H = \begin{bmatrix} R & d \\ 0 & 1 \end{bmatrix} \tag{2.66}$$

Transformation matrices of the form given in Equation (2.66) are called homogeneous transformations. A homogeneous transformation is therefore nothing more than a matrix representation of a rigid motion

# Homogeneous Transformations (Continued)

Using the fact that is orthogonal it is an easy exercise to show that the inverse transformation $H^{-1}$ is given by

$$H^{-1} = \begin{bmatrix} R^T & -R^T d \\ 0 & 1 \end{bmatrix} \tag{2.67}$$

In order to represent the transformation given in Equation (2.58) by a matrix multiplication, we must augment the vectors $p^0$ and $p^1$ by the addition of a fourth component of 1 as follows,

$$P^0 = \begin{bmatrix} p^0 \\ 1 \end{bmatrix} \tag{2.68}$$

$$P^1 = \begin{bmatrix} p^1 \\ 1 \end{bmatrix} \tag{2.69}$$

The vectors $p^0$ and $p^1$ are known as homogeneous representations of the vectors $p^0$ and $p^1$, respectively. It can now be seen directly that the transformation given in Equation (2.58) is equivalent to the (homogeneous) matrix equation

$$P^0 = H_1^0 P^1 \tag{2.70}$$

# Homogeneous Transformations (Continued)

A set of basic homogeneous transformations is given by

$$\mathrm{Trans}_{x,a} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{Rot}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha & 0 \\ 0 & s_\alpha & c_\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.71)$$

$$\mathrm{Trans}_{y,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{Rot}_{y,\beta} = \begin{bmatrix} c_\beta & 0 & s_\beta & 0 \\ 0 & 1 & 0 & 0 \\ -s_\beta & 0 & c_\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.72)$$

$$\mathrm{Trans}_{z,c} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}, \mathrm{Rot}_{z,\gamma} = \begin{bmatrix} c_\gamma & -s_\gamma & 0 & 0 \\ s_\gamma & c_\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.73)$$

for translation and rotation about the x, y, z-axes, respectively.

# Example 2.10.

The homogeneous transformation matrix that represents a rotation by angle α about the current x-axis followed by a translation of b units along the current x-axis, followed by a translation of d units along the current z-axis, followed by a rotation by angle θ about the current z-axis, is given by

$$H = \text{Rot}_{x,\alpha}\text{Trans}_{x,b}\text{Trans}_{z,d}\text{Rot}_{z,\theta}$$

$$= \begin{bmatrix} c_\theta & -s_\theta & 0 & b \\ c_\alpha s_\theta & c_\alpha c_\theta & -s_\alpha & -ds_\alpha \\ s_\alpha s_\theta & s_\alpha c_\theta & c_\alpha & dc_\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$