3.3.5 Stanford Manipulator

Consider now the Stanford Manipulator shown in Figure 3.10. This manipulator is an example of a spherical (RRP) manipulator with a spherical wrist. This manipulator has an offset in the shoulder joint that slightly complicates both the forward and inverse kinematics problems.

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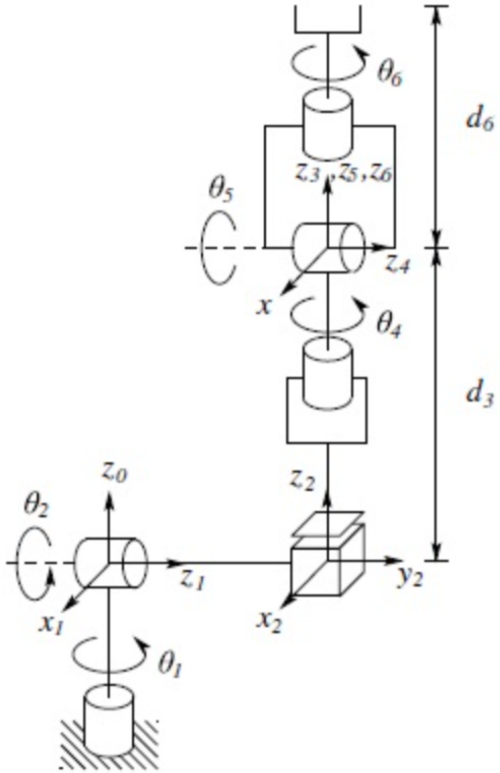

Figure 3.10 DH coordinate frame assignment for the Stanford manipulator.

From Lecture 6…….

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A commonly used convention for selecting frames of reference in robotic applications is the **Denavit–Hartenberg**, or **DH convention**. In this convention, each homogeneous transformation A_i is represented as a product of four basic transformations

$$
\begin{aligned}
A_i &= \mathrm{Rot}_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}\mathrm{Rot}_{x,\alpha_i} \qquad (3.10)\\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&\quad \times \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

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Link	d_i	a_i	α_i	θ_i
1	0	0	− 90	θ_1
2	d_2	0	+ 90	θ_2
3	d_3	0	0	0
4	0	0	− 90	θ_4
5	0	0	+ 90	θ_5
6	d_6	0	0	θ_6

It is straightforward to compute the matrices A_i as

$$A_1 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_4 = \begin{bmatrix} c_4 & 0 & -s_4 & 0 \\ s_4 & 0 & c_4 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} c_5 & 0 & s_5 & 0 \\ s_5 & 0 & -c_5 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_6 = \begin{bmatrix} c_6 & -s_6 & 0 & 0 \\ s_6 & c_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_6^0 is then given as

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$$T_6^0 = A_1 \cdots A_6 = \begin{bmatrix} r_{11} & r_{12} & r_{13} & d_x \\ r_{21} & r_{22} & r_{23} & d_y \\ r_{31} & r_{32} & r_{33} & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$r_{11} = c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6)$$
$$r_{21} = s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6)$$
$$r_{31} = -s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6$$
$$r_{12} = c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6)$$
$$r_{22} = -s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6)$$
$$r_{32} = s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6$$
$$r_{13} = c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5$$
$$r_{23} = s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5$$
$$r_{33} = -s_2c_4s_5 + c_2c_5$$
$$d_x = c_1s_2d_3 - s_1d_2 + +d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5)$$
$$d_y = s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2)$$
$$d_z = c_2d_3 + d_6(c_2c_5 - c_4s_2s_5)$$

Example 5.1. (The Stanford Manipulator).

Recall the Stanford Manipulator of Section 3.3.5. Suppose that the desired position and orientation of the final frame are given by

$$H = \begin{bmatrix} 0 & 1 & 0 & -0.154 \\ 0 & 0 & 1 & 0.763 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find the corresponding joint variables θ_1, θ_2, d_3, θ_4, θ_5, and θ_6 we must solve the following simultaneous set of nonlinear trigonometric equations:

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$$\begin{aligned}
c_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] - s_1(s_4c_5c_6 + c_4s_6) &= 0 \\
s_1[c_2(c_4c_5c_6 - s_4s_6) - s_2s_5c_6] + c_1(s_4c_5c_6 + c_4s_6) &= 0 \\
-s_2(c_4c_5c_6 - s_4s_6) - c_2s_5c_6 &= 1 \\
c_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] - s_1(-s_4c_5s_6 + c_4c_6) &= 1 \\
s_1[-c_2(c_4c_5s_6 + s_4c_6) + s_2s_5s_6] + c_1(-s_4c_5s_6 + c_4c_6) &= 0 \\
s_2(c_4c_5s_6 + s_4c_6) + c_2s_5s_6 &= 0 \\
c_1(c_2c_4s_5 + s_2c_5) - s_1s_4s_5 &= 0 \\
s_1(c_2c_4s_5 + s_2c_5) + c_1s_4s_5 &= 1 \\
-s_2c_4s_5 + c_2c_5 &= 0 \\
c_1s_2d_3 - s_1d_2 + d_6(c_1c_2c_4s_5 + c_1c_5s_2 - s_1s_4s_5) &= -0.154 \\
s_1s_2d_3 + c_1d_2 + d_6(c_1s_4s_5 + c_2c_4s_1s_5 + c_5s_1s_2) &= 0.763 \\
c_2d_3 + d_6(c_2c_5 - c_4s_2s_5) &= 0
\end{aligned}$$

If the values of the nonzero DH parameters are $d_2 = 0.154$ and $d_6 = 0.263$, one solution to this set of equations is given by:

$$\theta_1 = \pi/2,\ \theta_2 = \pi/2,\ d_3 = 0.5,\ \theta_4 = \pi/2,\ \theta_5 = 0,\ \theta_6 = \pi/2.$$

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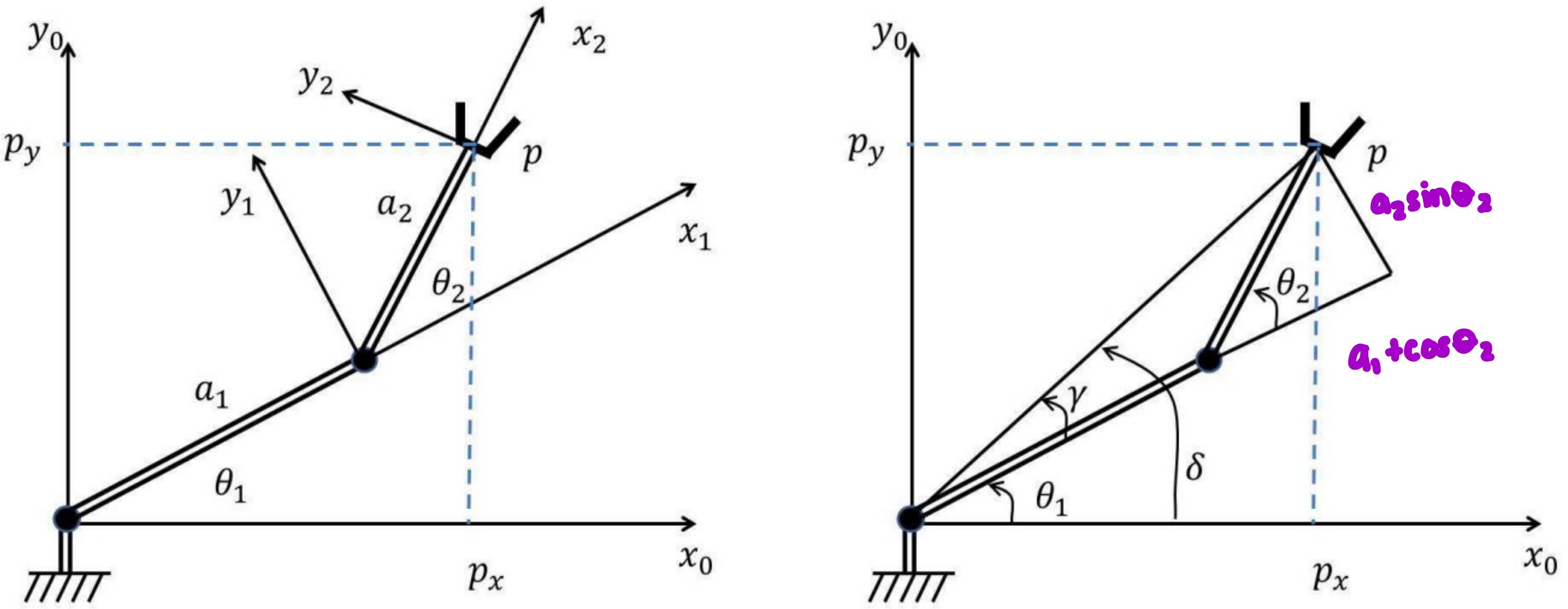
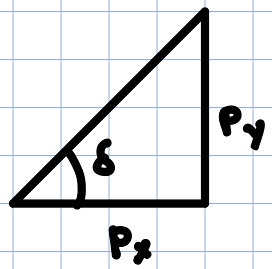
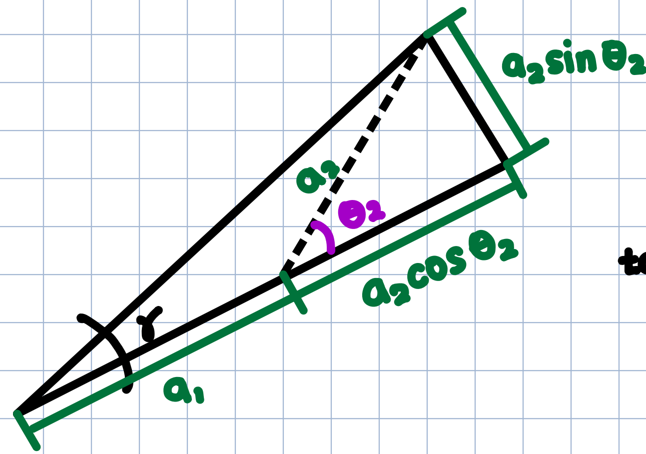
Consider the example of a two link manipulator. How will the values that we choose for a_1 and a_2 determine the range of values for p_x and p_y that are reachable by our manipulator?





$$\tan\delta = P_y / P_x$$



$$\tan\gamma = \frac{a_2 \sin\theta_2}{a_1 + a_2\cos\theta_2}$$

$$\theta_1 = \delta - \gamma = \tan^{-1}(P_y/P_x) - \tan^{-1}\left(\frac{a_2\sin\theta_2}{a_1 + a_2\cos\theta_2}\right)$$

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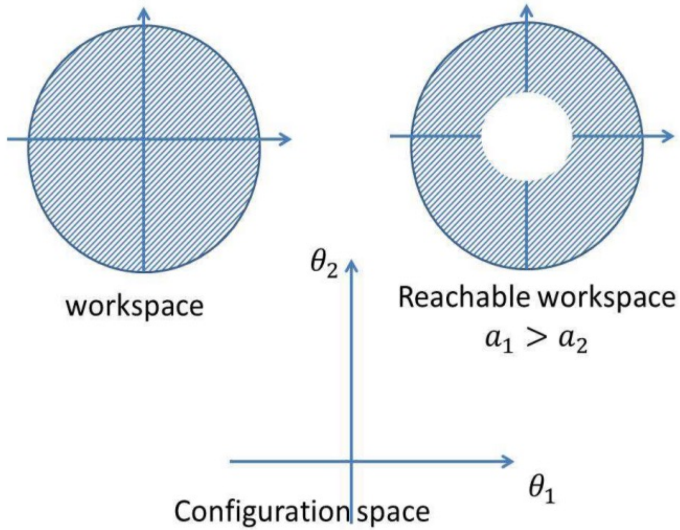
For the two link manipulator the workspace, the reachable workspace and the configuration space are shown. It is possible to write;

$p_x = a_1 \cos(\theta_1) + a_2 \cos(\theta_1 + \theta_2)$
$p_y = a1 \sin(\theta_1) + a_2 \sin(\theta_1 + \theta_2)$

The forward kinematics problem can be formulated as follows: Given θ_1 and θ_2, solve for p_x and p_y. The inverse kinematics problem can be formulated as follows: Given p_x and p_y solve for θ_1 and θ_2.



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It is possible to obtain the homogeneous transformation between reference frames 0 and 2 as follows

$$P^0 = H_1^0 H_2^1 P^2$$

$$\begin{bmatrix} \mathrm{COS}\ (\theta_1 + \theta_2) & -\mathrm{SIN}\ (\theta_1 + \theta_2) & 0 & a_1\mathrm{COS}\ (\theta_1) + a_2\mathrm{COS}\ (\theta_1 + \theta_2) \\ \mathrm{SIN}\ (\theta_1 + \theta_2) & \mathrm{COS}\ (\theta_1 + \theta_2) & 0 & a_1\mathrm{SIN}\ (\theta_1) + a_2\mathrm{SIN}\ (\theta_1 + \theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Using homogeneous transformation

$$H_1^0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & a_1\cos\theta_1 \\ \sin\theta_1 & \cos\theta_1 & 0 & a_1\sin\theta_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$H_2^1 = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & a_2\cos\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & a_2\sin\theta_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can say;

$P_x = a_1\mathrm{COS}\ (\theta_1) + a_2\mathrm{COS}\ (\theta_1 + \theta_2)$

$P_y = a_1\mathrm{SIN}\ (\theta_1) + a_2\mathrm{SIN}\ (\theta_1 + \theta_2)$

and solve for the angles or lengths.

We will then find that there are two possible solutions for each angle

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If as an example, we specify that ;

$$P^0 = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 1 \end{bmatrix}$$

We can then solve for the angles if we assume the a1, a2 lengths or solve.

We can say;

$P_x = a_1 COS(\theta_1) + a_2 COS(\theta_1 + \theta_2) = 5.0$

$P_y = a_1 SIN(\theta_1) + a_2 SIN(\theta_1 + \theta_2) = 6.0$

and solve for the angles or lengths.

The inverse kinematics problem for our popular two-link planar manipulator has two solutions in general. We want to solve the inverse kinematics problem by finding **both solutions**. You can use the following equations:

$$\cos\theta_2 = \frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1a_2} \qquad (1)$$

and

$$\sin\theta_2 = \pm\sqrt{1 - cos^2\theta_2} \qquad (2)$$

Now we can write the solution for θ_2:

$$\theta_2 = atan2(\sin\theta_2, \cos\theta_2) \qquad (3)$$

The solution for θ_1 is

$$\theta_1 = atan2(p_y, p_x) - atan2(a_2\sin\theta_2, a_1 + a_2\cos\theta_2) \qquad (4)$$

$$\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

$$P_x = a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2)$$
$$P_y = a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2)$$

$$P_x^2 + P_y^2 = a_1^2\cos^2\theta_1 + 2a_1a_2\cos\theta_1\cos(\theta_1+\theta_2) + a_2^2\cos^2(\theta_1+\theta_2)$$
$$+ a_1^2\sin^2\theta_1 + 2a_1a_2\sin\theta_1\sin(\theta_1+\theta_2) + a_2^2\sin^2(\theta_1+\theta_2)$$
$$= a_1^2 + 2a_1a_2[\underbrace{\cos\theta_1\cos(\theta_1+\theta_2) + \sin\theta_1\sin(\theta_1+\theta_2)}_{\cos[\theta_1-(\theta_1+\theta_2)] = \cos\theta_2}] + a_2^2$$

$$= a_1^2 + 2a_1a_2\cos\theta_2 + a_2^2$$

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Matlab code:

Create an m-file with:

```
function f=fsolvenle(x)
a1=5
a2=4
px=5
py=6
f(1)=a1*cos(x(1))+a2*cos(x(1)+x(2))-px
f(2)=a1*sin(x(1))+a2*sin(x(1)+x(2))-py
```

then, call the function in the Matlab command window as follows:

NEXT SLIDE

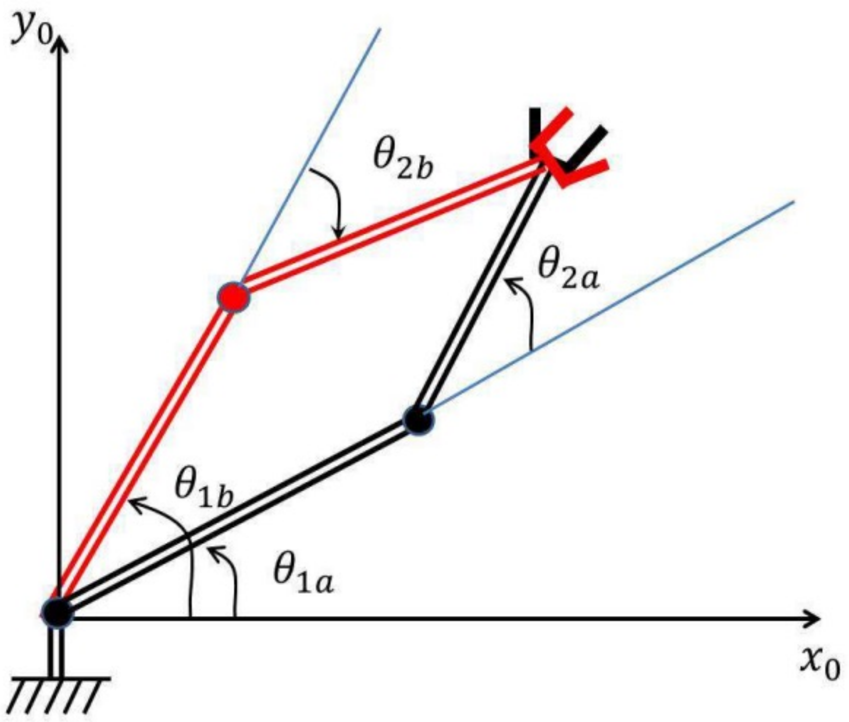

Fig. 5. Two possible solutions for the inverse kinematics of the planar manipulator

1. $\cos\theta_2 = \dfrac{(5)^2+(6)^2-(5)^2-(4)^2}{2(5)(4)}$

$\cos\theta_2 = 0.5$

$\theta_2 = \pm 60°$

$\sin\theta_2 = \pm\sqrt{1-\cos^2\theta_2}$

$= \pm\sqrt{1-}$

$\sin\theta_2 = \pm 0.893$

$\theta_2 = \text{atan}^2(\sin\theta_2/\cos\theta_2)$

$=$

$\theta_1 = \tan^{-1}(P_y/P_x) - \tan^{-1}\left(\dfrac{a_2\sin\theta_2}{a_1+a_2\cos\theta_2}\right)$

$= \tan^{-1}(6/5) - \tan^{-1}\left(\dfrac{4\sin 60°}{5+4\cos 60°}\right) = 23.86°$

50.19 26.32

0.876 0.460 = 0.416

$\theta_2 = \tan^{-1}(6/5) - \tan^{-1}\left[\dfrac{4\sin(-60)}{5+4\cos(-60)}\right] = 1.336 \approx 76.5°$

0.876 − −0.460

$P_x = 5\cos(23.8) + 4\cos(23.8+60) = 5$

$P_y = 5\sin(23.8) + 4\sin(23.8+60) = 6$

$P_x = 5\cos(76.5) + 4\cos(76.5-60) = 5$

$P_y = 5\sin(76.5) + 4\sin(76.5-60) = 6$

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From;

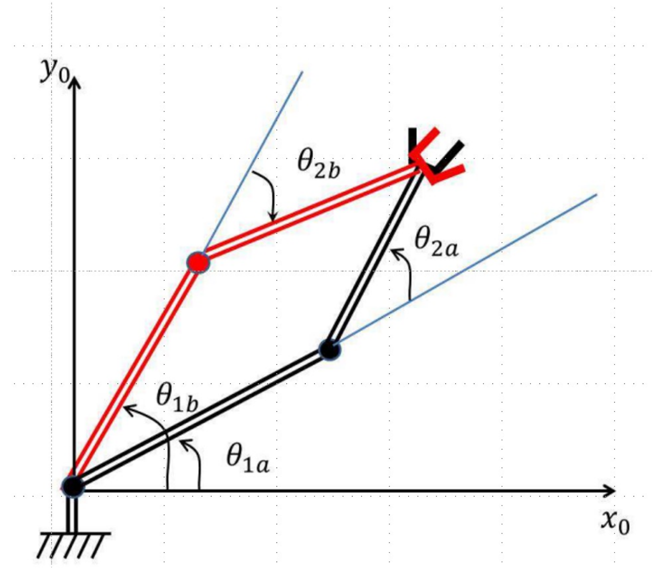
$P_x = a_1COS(\theta_1) + a_2COS(\theta_1 + \theta_2)$
$P_y = a_1SIN(\theta_1) + a_2SIN(\theta_1 + \theta_2)$

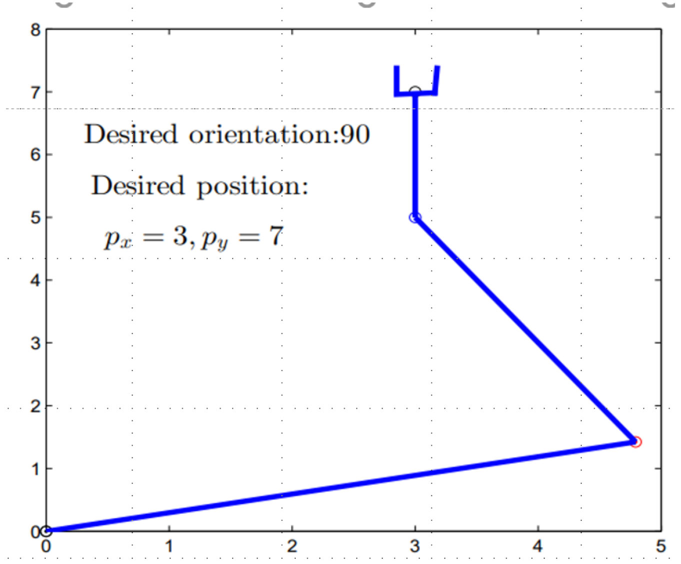
$COS(\theta_2) = (Px^2 + Py^2 - a_1^2 - a_2^2)/ 2a_1a_2$

Valid solutions range from $(a_1 - a_2)$ to $(a_1 + a_2)$

For three links the equations would be as follows;

$P_x = a_1COS(\theta_1) + a_2COS(\theta_1 + \theta_2) + a_3COS(\theta_1 + \theta_2 + \theta_3)$
$P_y = a_1SIN(\theta_1) + a_2SIN(\theta_1 + \theta_2) + a_3SIN(\theta_1 + \theta_2 + \theta_3)$





$$
\begin{aligned}
A_i &= R_{z,\theta_i}\mathrm{Trans}_{z,d_i}\mathrm{Trans}_{x,a_i}R_{x,\alpha_i} \qquad (3.10)\\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i} & 0 & 0 \\ s_{\theta_i} & c_{\theta_i} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{\alpha_i} & -s_{\alpha_i} & 0 \\ 0 & s_{\alpha_i} & c_{\alpha_i} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

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x = fsolve('fsolvenle', [1.2 ;1])*180/pi

```
a1=5; a2=4;
px=5; py=6;
cL=(px^2+py^2-a1^2-a2^2)/(2*a1*a2);
sLa=sqrt(1-cL^2);
sLb=-sqrt(1-cL^2);

x2a=atan2(sLa,cL)
x1a=atan2(py,px)-atan2(a2*(sLa),a1+a2*(cL) )

x2b=atan2(sLb,cL)
x1b=atan2(py,px)-atan2(a2*(sLb),a1+a2*(cL) )

xa=[x1a,x2a]
xb=[x1b,x2b]

x1=a1*cos(xa(1))
y1=a1*sin(xa(1))

x2=x1+a2*cos(xa(1)+xa(2))
y2=y1+a2*sin(xa(1)+xa(2))

x3=a1*cos(xb(1))
y3=a1*sin(xb(1))

x4=x3+a2*cos(xb(1)+xb(2))
y4=y3+a2*sin(xb(1)+xb(2))
```

```
figure(1)

plot(0,0,'ko'),
  hold on;
plot(0,0,'k.'),
  hold on;
plot(x1,y1,'ro'),
  hold on;
  plot(x2,y2,'bo'),
  hold on;
 plot(x3,y3,'ko'),
  hold on;
plot(x4,y4,'k+'),
  hold on;
plot(x1,y1,'r.'),
  hold on;


line([0,x1],[0,y1])
line([x1,x2],[y1,y2])

line([0,x3],[0,y3])
line([x3,x4],[y3,y4])
```