CSUS College of Engineering and Computer Science
Electrical & Electronic Engineering
ENGR 120 Probability and Random Signals

Practice Exam (100 points, 120 min.)

This exam is meant to give you an idea of the type and difficulty level of questions you might see on the exam.
Answer any four questions

Question 1 [25 points]

(a) Answer the following questions (True or False). No need to justify the answer.

(i) The sets C, D are disjoint if they share common elements T/F
(ii) $P(D) \geq P(C)$ if $C \subset D$, where C and D are sets. T/F
(iii) The CDF of a random variable X is $0 \leq F_X(x) \leq 1$ T/F
(iv) A random process is WSS if its ensemble average is equal to its time average T/F
(v) Let $Z = X+Y$, where $X, Y \sim N(0,1)$ are IID. Then $Z \sim N(0,2)$. T/F

(b) Two 6-sided dice are rolled. What is the probability that
(i) their sum is at least 6
(ii) their sum is at least 6, given that the outcome of both dice is an even number.

(i) $P(S \geq 6) = 1 - \underline{P(S<6)}$

$= 1 - 10/36$ 8/3

(ii) $P(S \geq 6 \mid E) = \dfrac{P(S \geq 6 \cap E)}{P(E) \to 9/36}$

$= \underline{\underline{8/9}}$

11 12 13 14 15 16
21 22 23 24 25 26
31 32 33 34 35 36
41 42 43 44 45 46
51 52 53 54 55 56
61 62 63 64 65 66

Question 2 [25 points]

a) A 2-bit binary sequence is transmitted over a noisy communication channel. The noise corrupts the signal in the sense that a transmitted digit can be flipped with probability 0.1. It has been observed that, across a large number of transmitted signals, the 0s and 1s are transmitted in the ratio 3:4. Given that the sequence 01 is received, calculate the probability that this sequence was transmitted.

$$P(S_1|R_1) = \frac{P(R_1|S_1)\,P(S_1)}{P(R_1)_{*}} = \boxed{0.83}$$

$P(R_1|S_1) = (0.9)(0.9)$, $P(S_1) \to 3/7 \cdot 4/7$

	00	01	10	11
Tx	S_0	S_1	S_2	S_3
Rx	R_0	R_1	R_2	R_3

$P_{flip} = 0.1$

$P(0) = \frac{3}{7}$, $P(1) = \frac{4}{7}$

$P(S_1) = \frac{3}{7} \cdot 4/7$

00, 01, 10, 11 : 01

b) Ten 6-sided dice are rolled. What is the probability that at most one "5" appears?

$n = 6$

$P_5 = 1/6$

$P(X=0) + P(X=1) \Rightarrow P\left(\text{at most 1 "5" appear}\right)$

$\binom{10}{0} p_x^0 (1-p_x)^{10-0}$

$\binom{10}{0} (1/6)^0 (1-1/6)^{10} + \binom{10}{1} (1/6)(1-1/6)^9$

$= \boxed{0.485}$.

$P(R_1) = P(S_0)P(R_1|S_0) + P(S_1)P(R_1|S_1) + P(S_2)P(R_1|S_2) + P(S_3)P(R_1|S_3)$

$(3/7)^2 \cdot (0.1)(0.9) +$

$(3/7)\, 3/7 \cdot 4/7\, (0.9)^2 +$

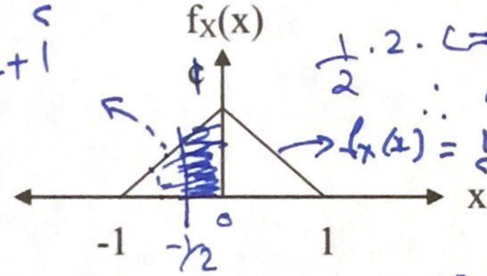
$(3/7 \cdot 4/7)(0.1)^2 +$

$(4/7)^2 (0.1)(0.9) = \underline{*}$

Question 3 [25 points]

a) A random variable X has a PDF defined by

$\frac{1}{\cancel{2}} \cdot \cancel{2} \cdot c \Rightarrow \boxed{1}$



$f_x(x) = x + 1$

$\frac{1}{2} \cdot 2 \cdot c \Rightarrow 1$

$\therefore c = 1$

$f_x(x) = 1 - x$

i) Find a suitable value for c.

ii) Find $P(X>-0.5|-1<X<0)$.

iii) Write an expression for the CDF $F_X(x)$.

$\frac{P(X > -0.5 \cap -1 < X < 0)}{P(-1 < X < 0)} = \frac{3/8}{1/2} = 6/8$

$P(-\frac{1}{2} < X < 0) \to 3/8$

$\int_{-1}^{0} x + c \, dx + \int_{0}^{1} c - x \, dx \Rightarrow 1$

$\int_{-1/2}^{0} x + 1 \, dx = \frac{x^2}{2} + x \Big|_{-1/2}^{0}$

$= \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

$0 - \left[\frac{1/4}{2} - \frac{1}{2}\right]$

$\frac{1}{2} - \frac{1}{8} = 3/8$

b) Let X be an exponential random variable with parameter $\lambda=1$.

i) Find the mean and variance of $|X|$.

ii) Find $P(X \geq 3)$.

iii) Use an appropriate inequality (Chebyshev's one-sided or two-sided, or Markov) to find an approximation for $P(X \geq 3)$. Justify your choice

$f_x(x) = \lambda e^{-x\lambda}, \quad x \geq 0$

(i) $E(X) = 1/\lambda \qquad 1/\lambda^2 = var(X)$

(ii) $P(X \geq 3) = \int_{3}^{\infty} e^{-x} = -e^{-x}\Big|_{3}^{\infty} = e^{-3}$

(iii) $P(X \geq 1 + 2) \leq \frac{\sigma^2}{\sigma^2 + a^2}$

$= \frac{1}{1+4} = \frac{1}{5}$

(ii) $\int_{-1}^{x} \alpha + 1 \, d\alpha = \frac{\alpha^2}{2} + \alpha \Big|_{-1}^{x}$

$= \frac{x^2}{2} + x - \left[\frac{1}{2} - 1\right]$

$= x^2/2 + x + 1/2$

$\Rightarrow \int_{0}^{x} 1 - \alpha \, d\alpha = \alpha - \frac{\alpha^2}{2}\Big|_{0}^{x}$

$= x - \frac{x^2}{2}$

$$\therefore \; F_X(x) = \begin{cases} 0 & x < -1 \\ x^2/2 + x + 1/2 & -1 < x \leq 0 \\ 1/2 + x - x^2/2 & 0 < x \leq 1 \\ 1 & x > 1 \end{cases}$$

$$\boxed{f_{XY}(x,y) = f_X(x) f_Y(y)}$$

$$e^{-x} \neq x e^{-x} \cdot e^{-y}$$

$$M_X(s) = \frac{\lambda}{\lambda - s} \Rightarrow \frac{1}{1-s}$$

$$E(X^2) = \frac{d^2}{ds^2}(1-s)\Big|_{s=0} = \underline{\underline{2}}$$

$$E(X^3) = \frac{d^3}{ds^3}\left(\frac{1}{1-s}\right)\Big|_{s=0} \Rightarrow 6$$

Question 4 [25 points]

Let the joint density function $f_{X,Y}(x,y)$ be

$$f_{X,Y}(x,y) = e^{-x} \quad ; 0<y<x<\infty$$

Find:

a) Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.
b) What is the conditional density $f_{X|Y}(x|y)$
c) Find $Cov(X,Y)$.
d) Find the coefficient of correlation for X and Y. $\rightarrow \rho = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \boxed{\frac{1}{\sqrt{2}}}$

$\sigma_X \rightarrow \sqrt{2}$, $\sigma_Y \rightarrow 1$

e) Are X and Y independent? $\boxed{\text{NO.}}$

a) $f_X(x) = \int f_{X,Y}(x,y)\,dy$

$$= \int_0^x e^{-x}\,dy = y e^{-x}\Big|_0^x$$

$$= \boxed{x e^{-x} \; ; 0 < x < \infty}$$

$$f_Y(y) = \int_y^\infty e^{-x}\,dx = -e^{-x}\Big|_y^\infty = \boxed{e^{-y}} \; ; 0 < y < \infty$$

b) $f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{e^{-x}}{e^{-y}} = \boxed{e^{-x+y}}$

c) $\mathrm{Cov}(X,Y)$

$\hookrightarrow E(XY) - E(X)E(Y)$

$E(XY) \rightarrow 3$, $E(X) \rightarrow 2$, $E(Y) \rightarrow 1$

$\cancel{e^{-x} e^{y}} = e^{-x+y}$

$$f_X(x) = x e^{-x}$$

$$E(X) = \int_0^\infty x^2 e^{-x}\,dx \Rightarrow E(X^2) = 2$$

$\hookrightarrow 2/\lambda^2$

If $x \sim \exp(1)$

$E(X^2) =$

$\int \cancel{\quad} x^2 e^{-x}$

$$E(XY) = \int_0^\infty \int_0^x xy e^{-x}\,dy\,dx \quad \text{OR}$$

$$\cancel{\int_0^\infty \int_y^\infty xy e^{-x}\,dx\,dy}$$

$$\hookrightarrow \int_0^\infty x \int_0^x y e^{-x}\,dy\,dx$$

$$y^2/2\, e^{-x} \Rightarrow \frac{x^2}{2} e^{-x}$$

$$\frac{1}{2}\int_0^\infty x^3 e^{-x}\,dx \Rightarrow \frac{1}{2}E(X^3) \Rightarrow \frac{6}{2} = 3$$

$$\therefore \boxed{E(XY) = 3 - 2 \cdot 1 = \underline{\underline{1}}}$$

$$\mathrm{var}(x) = \underline{E(x^2)} - \left(E(x)\right)^2$$

$$2^2$$

$$E(x^2) = \int_0^\infty x^2 . x e^{-x} \, dx \Rightarrow \underbrace{\int_0^\infty x^3 e^{-x} \, dx}_{E(x^3) = 6}$$

$$\therefore \mathrm{var}(x) = 6 - 4 = 2$$

$$\sigma_x = \sqrt{2}$$

$\Rightarrow E[A^2) + E(A\cancel{X}(t))] + E(A\,X(t+\tau)) + E$

μ_A const. $\quad E(A)\,E(X(t)) \quad + E\,(X(t)\,X(t+\tau))$

$\mu_A \quad \mu_X \quad E(A)\,E(X(t+\tau)) \Rightarrow$ const. $\quad R_X(t, t+\tau)$

Question 5 [25 points]

a) Let $\boxed{A \text{ be a nonnegative random variable}}$ that is independent of any collection of samples $X(t_1), \ldots, X(t_n)$ of a wide sense stationary random process $X(t)$. Is $Y(t) = A + X(t)$ a wide sense stationary process? Justify your answer.

(i) $E(Y(t)) \Rightarrow$ Const.

(ii) $R_Y(t, t+\tau) \Rightarrow$ should not be a function of time "t".

(i) $E(Y(t)) = E[A + X(t)] = \underset{\text{Const.}}{E(A)} + \underset{\text{Const.}}{E(X(t))} \Rightarrow$ Const. μ_Y.

(ii) $E[(A + X(t))(A + X(t+\tau))]$

$= E[A^2 + A\,X(t+\tau) + A\cancel{X}(t) + X(t)X(t+\tau)]$

not a function of t.

$\Rightarrow R_Y(t, t+\tau) = \mu_A + \mu_A \cdot \mu_X + \mu_A \mu_X +$ term (not a function of time)

$\therefore R_Y(t, t+\tau)$ is not a function of t.

$\boxed{\therefore Y(t) \text{ is WSS.}}$

b) The autocorrelation function of a stationary random signal is given by

$$R_X(\tau) = e^{-|\tau|}$$

a) Obtain an expression for the power spectral density $S_X(f)$

b) What is the average power of this signal.

~~c) Show that the average power in the time domain is equivalent to average power in the frequency domain.~~

(a) $S_X(f) = \int_{-\infty}^{\infty} e^{-|\tau|}\, e^{-j2\pi f\tau}\, d\tau$

$= \int_{-\infty}^{0} e^{\tau(1 - j2\pi f)}\, d\tau + \int_{0}^{\infty} e^{-\tau(1+j2\pi f)}\, d\tau$

$= \frac{1}{1-j2\pi f} e^{\tau(1-j2\pi f)}\Big|_{-\infty}^{0} + \frac{-1}{1+j2\pi f} e^{-\tau(1+j2\pi f)}\Big|_{0}^{\infty}$

$= \frac{1}{1-j2\pi f} + \frac{1}{1+j2\pi f} = \boxed{\frac{2}{1+4\pi^2 f^2}}$

(b) $\tau = 0$, $e^{-0} = \boxed{1}$