CALIFORNIA STATE UNIVERSITY

SACRAMENTO

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EEE 117 Network Analysis

Text: Electric Circuits by J. Nilsson and S. Riedel Prentice Hall

Lecture Set 5: Frequency Selective Circuits

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Frequency Selective Circuits

- Preview
- Some Preliminaries
- Types of Filters
- Passive Low-pass Filter
- Passive High-pass Filter
- Passive Band-pass Filter
- Passive Band-reject Filter

Preview

- Up to this point, our circuit analysis was at a fixed frequency. We now begin the analysis of circuits when the input frequency is varied.
- Frequency selective circuits are called filters.
- Essentially a filter is a circuit, designed to pass a specific range of frequencies and stopped or block the remaining range of frequencies.
- Passband – the band of frequencies that appear in the output.
- Stopband – Frequencies that do not appear in the output.
- Frequency selective circuits are usually characterized by the location of their passband.
- The frequency response plot shows how a circuit's transfer function (both amplitude and phase) changes as the source frequency changes.

Some Preliminaries

- The transfer function can be used to find the steady state response of a circuit to sinusoidal inputs.
- We will now let the input be a variable frequency but with constant amplitude and phase unless we specifically wish to change the amplitude or phase.
- Thus, for linear, lumped parameter, time-invariant systems, if the input is a sinusoid of frequency ω, then the response will also be a sinusoid at the same frequency ω.
- Note that there may a decaying exponential involved (transient response), but the resulting steady state sinusoid still has the frequency ω.
- We will look at the transfer functions and define them as the ratio of some input to some output.

- In the case of input/output voltages this can written as:

$$H(s) = \frac{V_{output}(s)}{V_{input}(s)}$$

- Or in terms of current as:

$$H(s) = \frac{I_{output}(s)}{I_{input}(s)}$$

- There are many other possible forms for the ratio of an arbitrary input to an arbitrary output.

Type of Filters

FILTERS

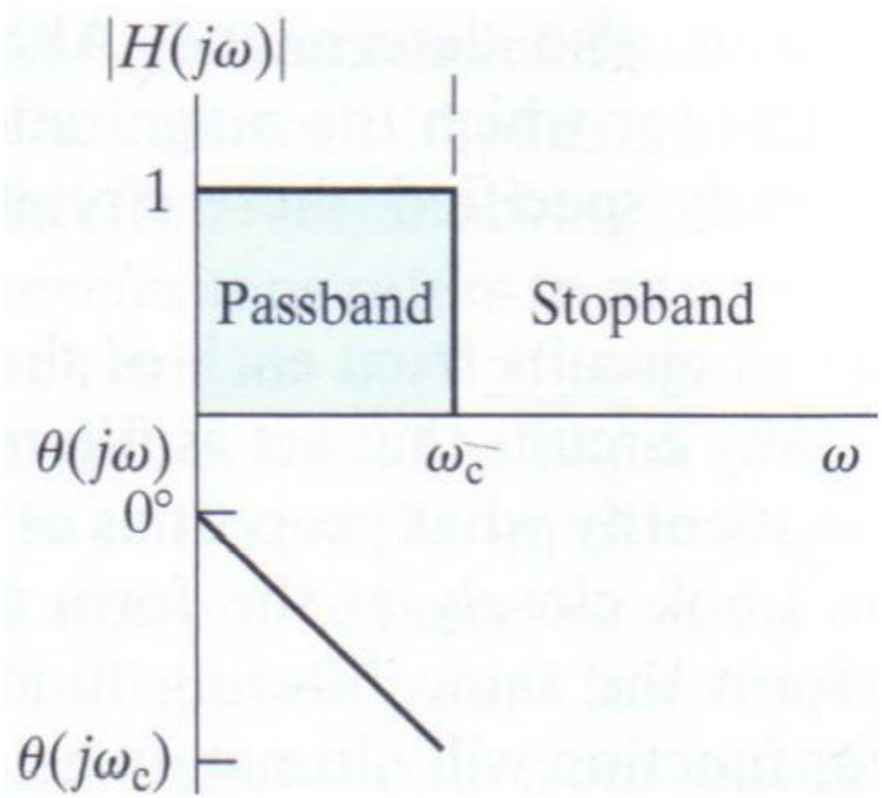
- **PASSIVE FILTERS**
  - Low Pass Filter
  - High Pass Filters
  - Band Pass Filters
  - Band Reject Filters
- **ACTIVE FILTERS**
  - Low Pass Filter
  - High Pass Filters
  - XXXXXX
  - XXXXXXXX

- ***In lecture set 5, we will examine passive filters only. The gain will thus be equal to or less than 1.***

Passive Low Pass Filters

- A low-pass filter selects the low frequencies $0 \leq \omega \leq \omega_c$ for the passband. where ω_c is called the cutoff frequency.

- For an Ideal Low Pass Filter, frequency plot is as shown below;



Corner or Cutoff Frequency

- Cutoff frequency is the frequency at which we observe the following:
  - The half-power point.

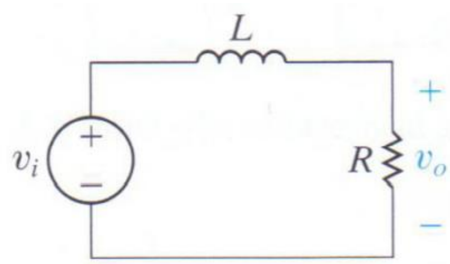
    $$P = 1/2\ P_{max}$$

  - The transfer function relationship.

    $$H(j\omega_c) = \frac{1}{\sqrt{2}} H_{max}$$

  - The inductive reactance $X_L = R$
  - Phase angle of response is -45

Low-pass filter with a RL circuit:



- For $\omega = 0$, $Z_L = 0$, inductor act as a short and we have the equivalent circuit as shown in Fig.(a)
- As the frequency rises, the behavior of the inductor will change. However, the behavior of the resistor is the same for all frequencies where linear, lumped parameter applies.
- For $\omega = \infty$, $Z_L = \infty$, inductor acts as an open thus no current flows through the resistor. $V_R = 0$ and we have the equivalent circuit as shown in Fig. (b)

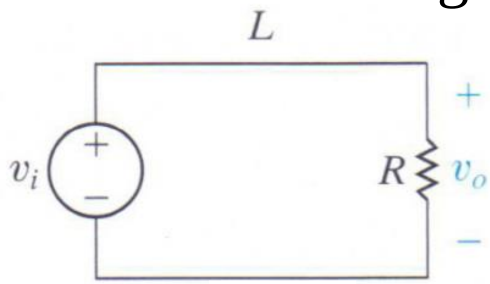


Fig (a)

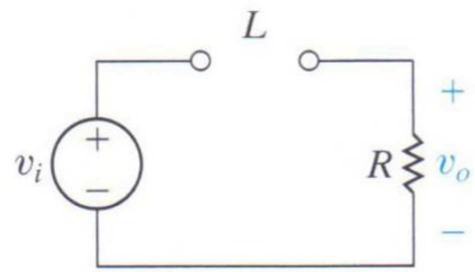
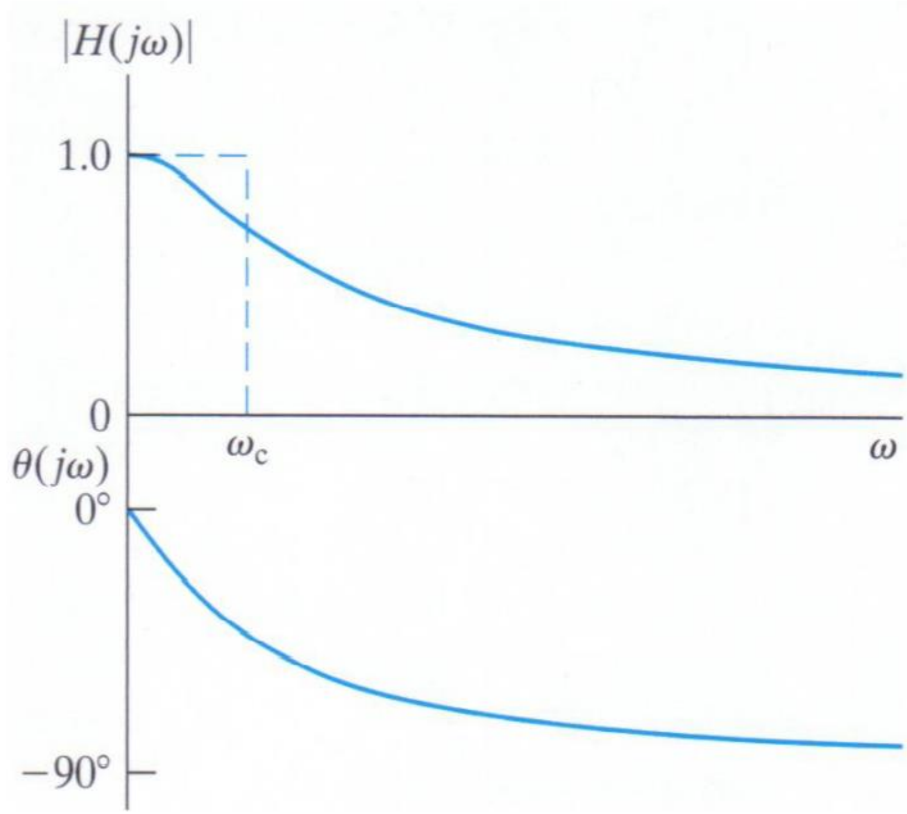


Fig (b)

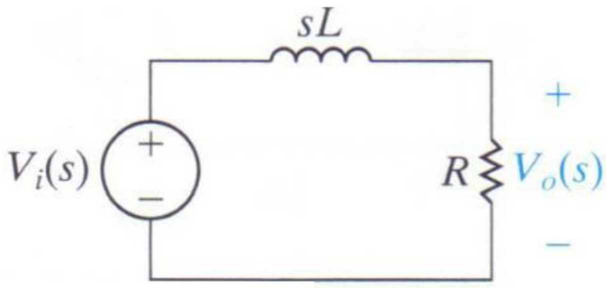
- The frequency response plot is as follows:



- Note here that the transition from passband to stopband is not abrupt but occupies a certain range of frequencies.
- Thus, we need to define a point known as cutoff frequency that identifies where the circuit behavior changes.

Expression of Transfer Function of LPF:

- In the s-domain, the series RL circuit will be:



$$I(s)R = V_o(s)$$

$$I(s) = \frac{V_o(s)}{R}$$

By KVL, we have

$$-V_i(s) + I(s)sL + I(s)R = 0$$

$$V_i(s) = I(s)sL + I(s)R \quad = I(s)\left(sL + R\right) \quad = \frac{V_o(s)}{R}\left(sL + R\right)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} \quad = \frac{R}{sL + R} \quad = \frac{\frac{R}{L}}{s + \frac{R}{L}}$$

We can also find H(s) by the use of a voltage divider.

$$V_o(s) = \frac{R}{sL+R} V_i(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{sL+R} = \frac{\frac{R}{L}}{s+\frac{R}{L}}$$

$$\boxed{H(j\omega) = \frac{\frac{R}{L}}{j\omega + \frac{R}{L}} = \frac{\frac{R}{L}}{\sqrt{\omega^2 + \left(\frac{R}{L}\right)^2} \angle \tan^{-1}\left(\frac{\omega}{\frac{R}{L}}\right)}}$$

EQ_1

Expression of Cutoff Frequency:

At cut off frequency, $$\left|H(j\omega_c)\right| = \frac{1}{\sqrt{2}}\left|H_{\max}\right|$$

and $H_{max} = 1$ when $\omega=0$

Thus at $\omega = 0$

$$H(j\omega = 0) = \frac{\frac{R}{L}}{\sqrt{0^2 + \left(\frac{R}{L}\right)^2}\angle \tan^{-1}\left(\frac{0}{\frac{R}{L}}\right)} = \frac{\frac{R}{L}}{\sqrt{\left(\frac{R}{L}\right)^2}\angle 0^\circ} = 1\angle 0^\circ = H_{\max}$$

Hence at cutoff frequency ω_c:

$$\left|H(j\omega_c\right| \equiv \frac{1}{\sqrt{2}} H_{Max} \quad = \frac{1}{\sqrt{2}}(1)$$

Or rewrite it as:

$$\frac{1}{\sqrt{2}} = \frac{\frac{R}{L}}{\sqrt{\omega_c^2 + \left(\frac{R}{L}\right)^2}} \Rightarrow \frac{1}{2} = \frac{\left(\frac{R}{L}\right)^2}{\omega_c^2 + \left(\frac{R}{L}\right)^2} \Rightarrow \omega_c^2 + \left(\frac{R}{L}\right)^2 = 2\left(\frac{R}{L}\right)^2$$

$$\omega_c^2 = 2\left(\frac{R}{L}\right)^2 - \left(\frac{R}{L}\right)^2 \quad = \left(\frac{R}{L}\right)^2$$

So we have $\boxed{\omega_c = \frac{R}{L}}$ For the series RL circuit.

- Deduced from bullet R= X_L

X_L=R

ωL = R

ω = R/L

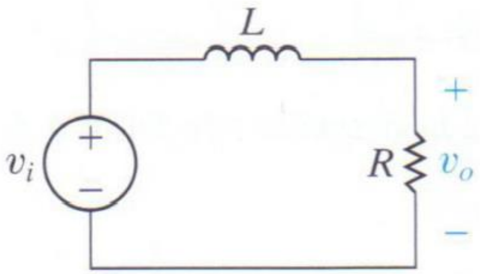
$$\boxed{\omega_c = \frac{R}{L}}$$

Put the value of ω_c in EQ_1,

phase Θ = 0 - 45 = -45 degree

Example-1 Design a low pass filter with a corner frequency of 10 Hz.

a. Find the response at 1 Hz, 10 Hz and 60 Hz.
b. Draw bode diagram.



The corner frequency was specified in Hertz. So

$$\omega_c = 2\pi f_c \quad = 2\pi(10) \quad = 20\pi \ \tfrac{rads}{\text{sec}}$$

$$\omega_c = \frac{R}{L}$$ So we have one equation and two unknowns, R and L.

Often in engineering, usually in such situations, we start with one assumed value and simulate the circuit to determine suitability of the other unknown.

So let L = 100 mH. $\omega_c = \dfrac{R}{L} \quad \Rightarrow \omega_c L = R$

$$R = 20\pi \tfrac{rad}{\text{sec}}\left(100\times10^{-3} H\right) \quad \approx 6.283\ \Omega$$

$$\left|H(j\omega)\right| = \frac{\left|V_o(j\omega)\right|}{\left|V_i(j\omega)\right|} \qquad \Rightarrow \left|V_o(\omega)\right| = \left|H(\omega)\right| \cdot \left|V_i(\omega)\right|$$

$$\left|V_o(\omega)\right| = \frac{\frac{R}{L}}{\sqrt{\omega^2 + \left(\frac{R}{L}\right)^2}}\left|V_i(\omega)\right| \qquad = \frac{\frac{R}{L}}{\sqrt{\omega^2 + \omega_c^2}}\left|V_i(\omega)\right|$$

$$= \frac{20\pi}{\sqrt{\omega^2 + (20\pi)^2}}\left|V_i(\omega)\right| \qquad = \frac{20\pi}{\sqrt{\omega^2 + 400\pi^2}}\left|V_i(\omega)\right|$$

So now we can find the response at 1 Hz, 10 Hz and 60 Hz.

f = 1Hz	ω = 2π rad/sec
f = 10 Hz	ω = 20π rad/sec
f = 60 Hz	ω = 120π rad/sec

$$\left|V_o(\omega)\right| = \frac{20\pi}{\sqrt{\omega^2 + 400\pi^2}}\left|V_i(\omega)\right|$$

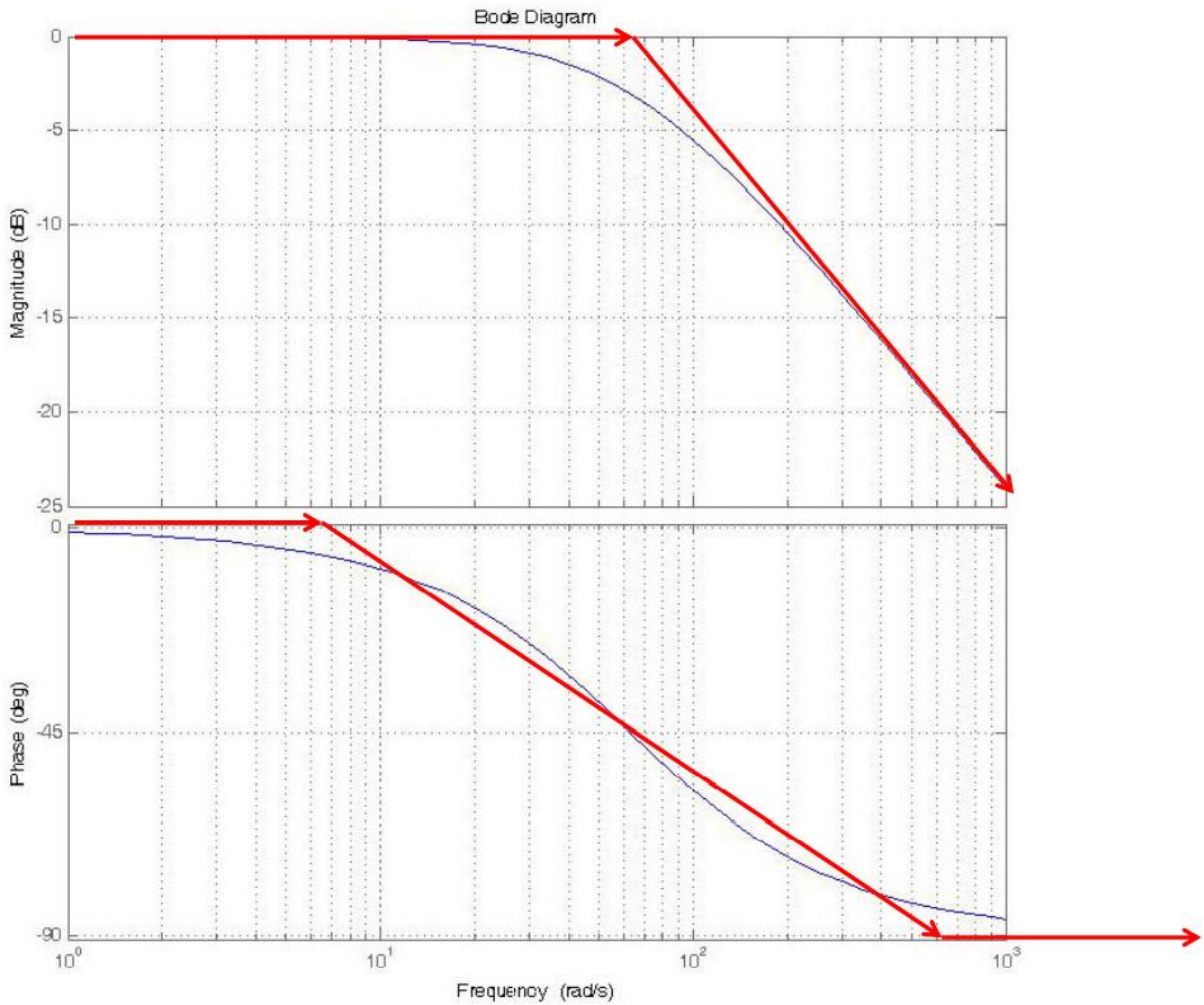
$$\left|V_o(\omega=2\pi)\right| = \frac{20\pi}{\sqrt{(2\pi)^2+400\pi^2}}\left|V_i(\omega)\right| \quad = 0.99504\left|V_i(\omega)\right|$$

$$\left|V_o(\omega=20\pi)\right| = \frac{20\pi}{\sqrt{(20\pi)^2+400\pi^2}}\left|V_i(\omega)\right| \quad = \frac{1}{\sqrt{2}}\left|V_i(\omega)\right| \quad = 0.70711\left|V_i(\omega)\right|$$

$$\left|V_o(\omega=120\pi)\right| = \frac{20\pi}{\sqrt{(120\pi)^2+400\pi^2}}\left|V_i(\omega)\right| \quad = 0.16440\left|V_i(j\omega)\right|$$

Let’s create the straight-line Bode Diagram for this low-pass filter.

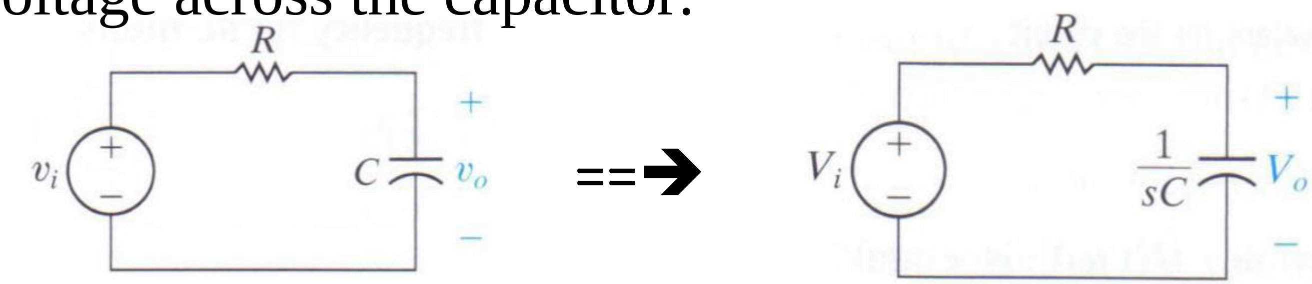
$$\frac{R}{L} = \frac{6.283\Omega}{100mH} = 62.83\,\text{sec} \qquad H(s) = \frac{V_o(s)}{V_i(s)} \quad = \frac{\frac{R}{L}}{s+\frac{R}{L}} \quad = \frac{62.83}{s+62.83}$$



Bode Diagram
Magnitude (dB)
0
-5
-10
-15
-20
-25
Phase (deg)
0
-45
-90
10^0
10^1
10^2
10^3
Frequency (rad/s)

Low-pass filter with a RC circuit:

- The series RC circuit also acts as a low-pass filter when in the following configuration and assuming the output is defined as the voltage across the capacitor.



- When $\omega = 0$, the capacitor acts as an open circuit. $V_o = V_i$.
- When $\omega = \infty$, the capacitor acts as a short circuit. $V_o = 0$.
- We can find the output voltage V_o by using the voltage divider.

$$\boxed{V_o(s) = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} V_i(s) \quad = \frac{\frac{1}{SRC}}{1 + \frac{1}{sRC}} V_i(s) \quad = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} V_i(s)}$$

Expression of Transfer Function:

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \quad \Longrightarrow \quad H(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

Magnitude:

$$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}.$$

Phase:

$\Theta = 0 - \tan^{-1}(\omega RC)$

Expression of Cutoff Frequency:

At cut off frequency, $|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{\max}|$

and $H_{max} = 1$ when ω=0

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}(1) = \frac{\frac{1}{RC}}{\sqrt{\omega_c^2 + \left(\frac{1}{RC}\right)^2}}$$

$$\boxed{\omega_c = \frac{1}{RC}.}$$

From the previous discussion we know that at corner frequency

$X_C = R$ ====➔ $\frac{1}{\omega * C} = R$ ==➔ $\boxed{\omega_C = \frac{1}{RC}}$

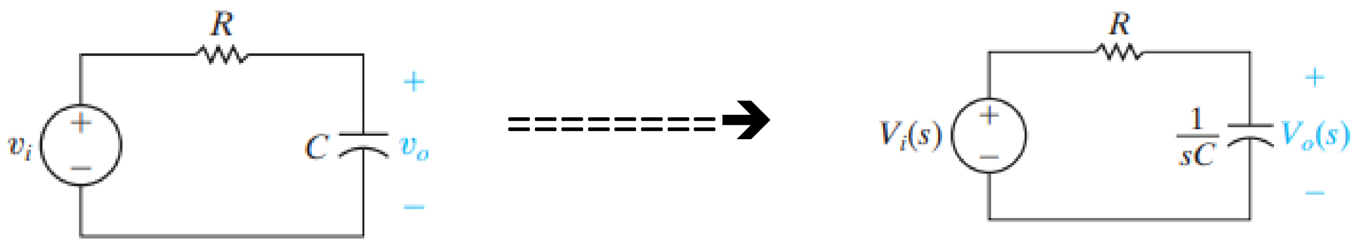
Phase at cutoff frequency

$\Theta = 0 - \tan^{-1}(\omega RC) = 0 - 45 = \boxed{-45^o}$

Note: It should be kept in mind that a passive low pass filter will have the phase angle = -45° at cutoff frequency regardless of whether the circuit is constructed with inductor or capacitor.

Example-2: Design a Series RC Low-Pass Filter.

a) Find the transfer function between the source and the output voltages.
b) Determine an equation for the cutoff frequency in the series RC circuit.
c) Choose values for R and C that will yield a lowpass filter with a cutoff frequency of 3 kHz.



a) Construct the s-domain equivalent circuit. Using s-domain voltage division on the equivalent circuit, find transfer function, substitute s=jω and then find the magnitude and phase angle expressions:

$$H(s) = \frac{\frac{1}{RC}}{s + \frac{1}{RC}} \quad \Longrightarrow \quad H(j\omega) = \frac{\frac{1}{RC}}{j\omega + \frac{1}{RC}}$$

Magnitude: $$|H(j\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \left(\frac{1}{RC}\right)^2}}.$$

Phase: $\Theta = 0 - \tan^{-1}(\omega RC)$

b) At the cutoff frequency ω_c, $|H(j\omega)|$ is equal to $(1/\sqrt{2})H_{\max}$.

$$|H(j\omega_c)| = \frac{1}{\sqrt{2}}(1) = \frac{\frac{1}{RC}}{\sqrt{\omega_c^2 + \left(\frac{1}{RC}\right)^2}}.$$

$$\boxed{\omega_c = \frac{1}{RC}.}$$

c) Choose a capacitor value first.

Let C= 1 μF

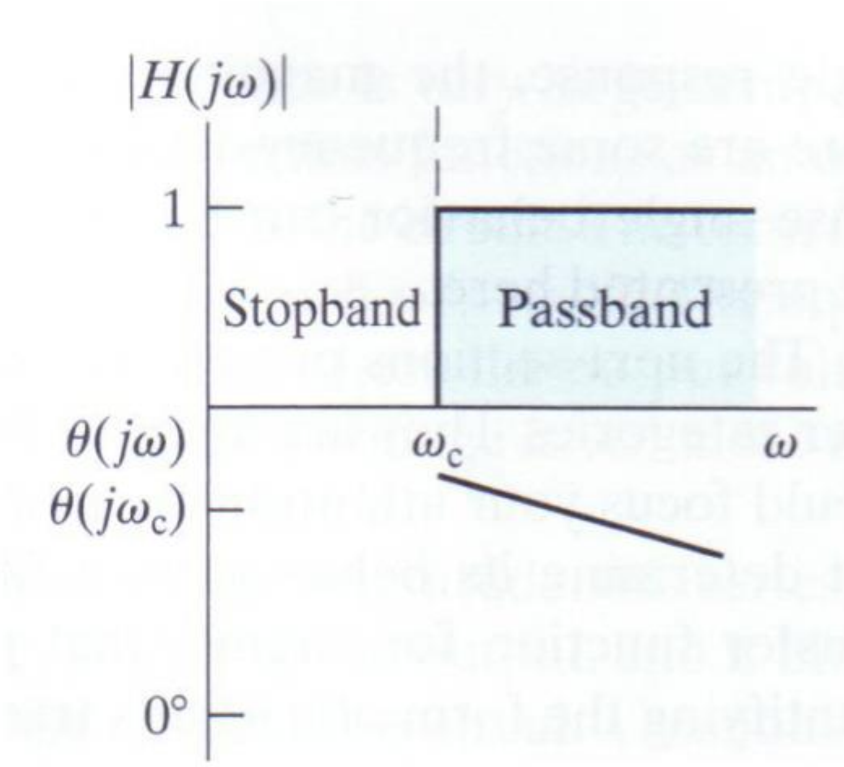
For fc = 3 kHz ==➔ ω_C = 3000 x 2π rad/s

$$R = \frac{1}{\omega_c C} = \frac{1}{(2\pi)(3 \times 10^3)(1 \times 10^{-6})} = 53.05\ \Omega.$$

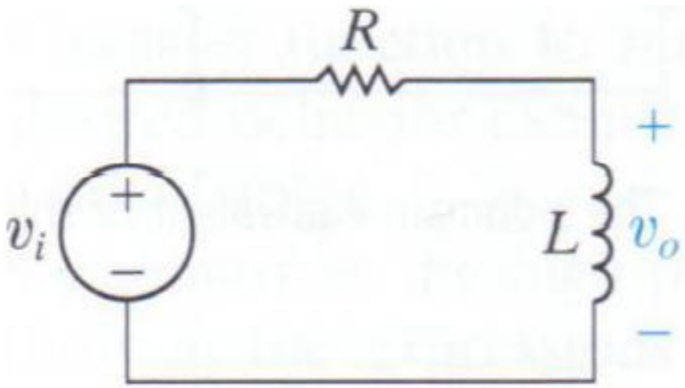
Note: Many other combinations of C and R can be found which will give you the corner frequency 3 kHz.

Passive High Pass Filters

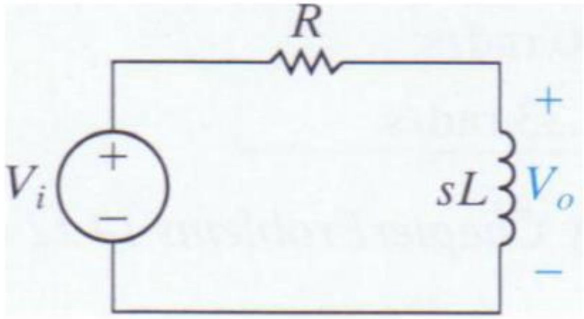
- A high-pass filter selects the high frequencies $\omega_c \leq \omega \leq \infty$ for the passband. where ω_c is the cutoff frequency.
- For an Ideal high Pass Filter, frequency plot is as shown below;



High-pass filter with a RL circuit:



- When ω = 0, the inductor acts as a short circuit. $V_o = 0$.
- When ω = ∞, the inductor acts as an open circuit. $V_o = V_i$.
- We can find the output voltage V_o by using the voltage divider.



$$V_o(s) = \frac{sL}{R + sL} V_i(s)$$

Expression of Transfer Function:

Once again, we can find the transfer function by the voltage divider.

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{sL}{R + sL} = \frac{s}{s + \frac{R}{L}}$$

$$H(j\omega) = \frac{j\omega}{j\omega + \frac{R}{L}} = \frac{\omega}{\sqrt{\omega^2 + (\frac{R}{L})^2}} \frac{\angle\ 90^\circ}{\angle\ \tan^{-1}(\frac{\omega L}{R})}$$

Magnitude of response

$$|H(j\omega)| = \frac{|j\omega|}{|j\omega + \frac{R}{L}|} = \frac{\omega}{\sqrt{\omega^2 + (\frac{R}{L})^2}}$$

Phase of response

$$\theta(j\omega) = 90^\circ - \tan^{-1}(\frac{\omega L}{R})$$

Expression of Cutoff Frequency:

At cut off frequency, $\left|H(j\omega_c)\right| = \frac{1}{\sqrt{2}}\left|H_{\max}\right|$

and $H_{max} = 1$ when $\omega=\infty$

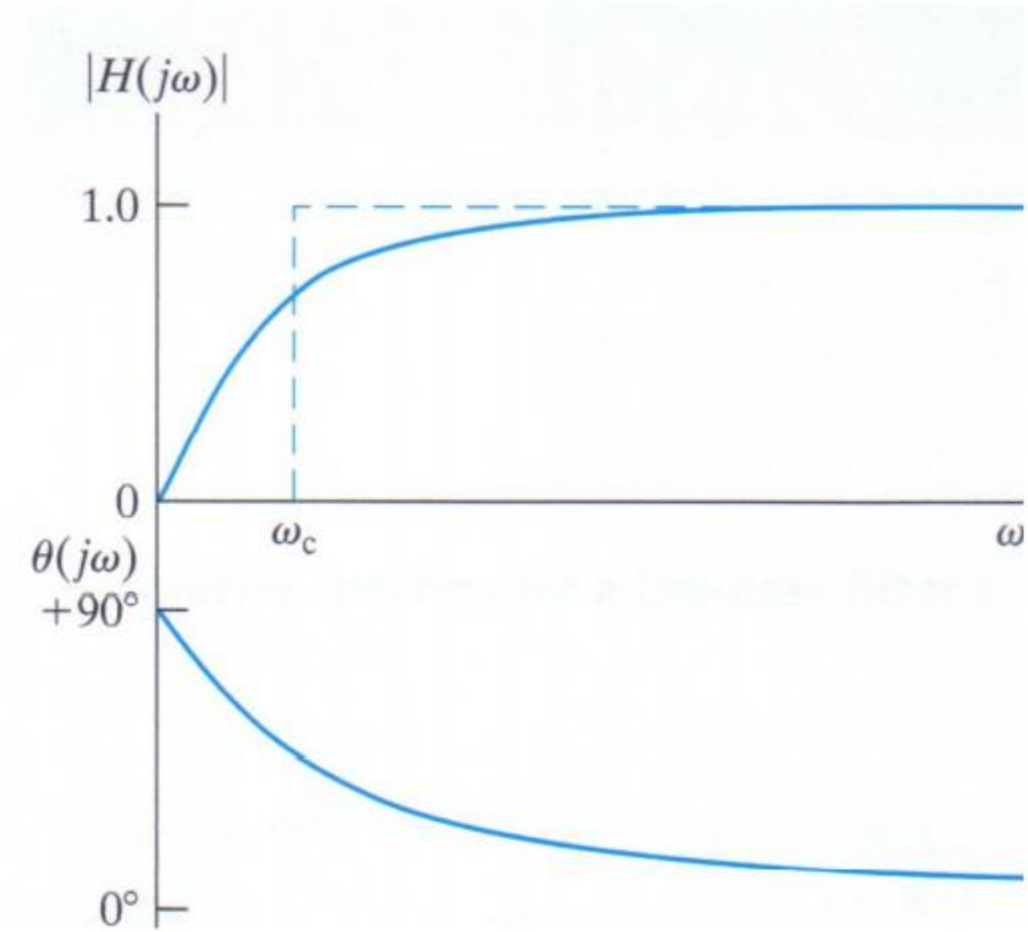
So

$$\frac{1}{\sqrt{2}} = \frac{\omega_c}{\sqrt{\omega_c^2 + (R/L)^2}}, \quad \boxed{\omega_c = \frac{R}{L}}$$

Phase at cutoff frequency

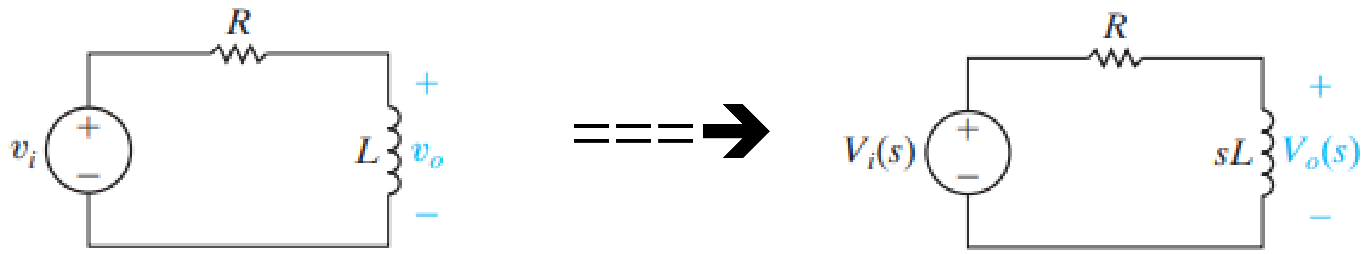
$\Theta = 90 - \tan^{-1}(\omega L/R) = 90 - 45 \boxed{= 45^{\circ}}$

The frequency response plot is



Example-3: Design a Series RL High-Pass Filter.

a) Derive an expression for the circuit's transfer function.
b) Use the result from (a) to determine an equation for the cutoff frequency.
c) Choose values for R and L that will yield a high pass filter with a cutoff frequency of 15 kHz



Construct the s-domain equivalent circuit. Using s-domain voltage division on the equivalent circuit, find transfer function, substitute s=jω and then find the magnitude and phase angle expressions:

$$H(s) = \frac{s}{s + R/L} \implies H(j\omega) = \frac{j\omega}{j\omega + R/L}$$

Magnitude: $|H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}}$ **_Phase:_** $\Theta = 90^o - \tan^{-1}(\omega L/R)$

b) To find an equation for the cutoff frequency, first compute the magnitude of $H(j\omega)$:

$$|H(j\omega)| = \frac{\omega}{\sqrt{\omega^2 + (R/L)^2}}.$$

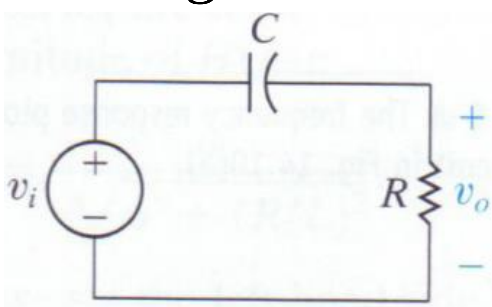
$$\frac{1}{\sqrt{2}} = \frac{\omega_c}{\sqrt{\omega_c^2 + (R/L)^2}}, \quad \boxed{\omega_c = \frac{R}{L}.}$$

c) Using the equation for ω_c computed in (b), we recognize that it is not possible to specify values for R and L independently. Therefore, let's arbitrarily select a value of 500 Ω for R. Remember to convert the cutoff frequency to radians per second:

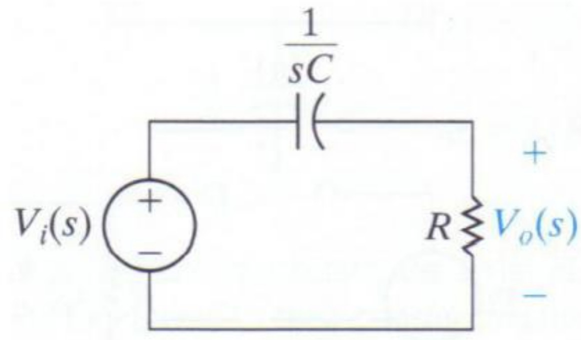
$$L = \frac{R}{\omega_c} = \frac{500}{(2\pi)(15{,}000)} = 5.31 \text{ mH}.$$

High-pass filter with a RC circuit:

- The series RC circuit also acts as a high-pass filter when in the following configuration and assuming the output is defined as the voltage across the resistor.



====➔



- When $\omega = 0$, the capacitor acts as an open circuit. $V_o = 0$.
- When $\omega = \infty$, the capacitor acts as a short circuit. $V_o = V_i$.
- We can find the output voltage V_o by using the voltage divider.

$$\boxed{V_o(s) = \frac{R}{\frac{1}{sC} + R} V_i(s)}$$

Expression of Transfer Function:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{\frac{1}{sC} + R} = \frac{sR}{\frac{1}{C} + sR} = \frac{s}{s + \frac{1}{RC}}$$

$$H(j\omega) = \frac{j\omega}{j\omega + \frac{1}{RC}}$$

Magnitude:

$$|H(j\omega)| = \frac{|j\omega|}{\left|j\omega + \frac{1}{RC}\right|} = \frac{\omega}{\sqrt{\omega^2 + (\frac{1}{RC})^2}}$$

Phase:

$$\theta(j\omega) = \frac{\angle \tan^{-1}\left(\frac{\omega}{0}\right)}{\angle \tan^{-1}\left(\frac{\omega}{\frac{1}{RC}}\right)} = \frac{\angle \tan^{-1}(\infty)}{\angle \tan^{-1}(\omega RC)} = 90^\circ - \tan^{-1}(\omega RC)$$

Expression of Cutoff Frequency:

At cut off frequency, $$|H(j\omega_c)| = \frac{1}{\sqrt{2}}|H_{\max}|$$

and $H_{max} = 1$ when $\omega = \infty$

So
$$\omega_c = \frac{1}{RC}$$

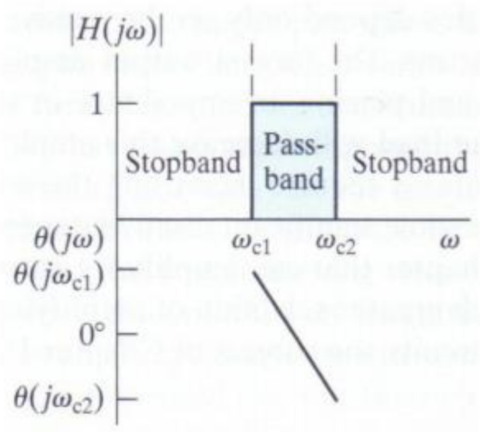
Phase at cutoff frequency

$\Theta = 90 - \tan^{-1}(\omega RC) = 90 - 45 = \boxed{45^o}$

Note: It should be kept in mind that a passive high pass filter will have the phase angle = +45° at cutoff frequency regardless of whether the circuit is constructed with inductor or capacitor.

Passive Band Pass Filters

- The ideal band-pass filter allows a limited region of frequencies to pass through the filter.



- Five main parameters associated with band pass filter, which need to be explored.

Corner Frequency: There are two stop-bands. One stopband in the low frequency area, and one stopband in the high frequency region. These frequencies are corner frequencies.

Center Frequency: Center frequency "ω_0" is also called the resonant frequency. ω_0 is the geometric center of the pass-band and defined as:

$$\omega_0 = \sqrt{\omega_{c1}\ \omega_{c2}}$$

Transfer function: The max. of the filter's transfer function H_{max} occurs at ω_0 and is given as:

$$H_{\max} = \left| H(j\omega_0) \right|$$

Bandwidth: The bandwidth B of the filter is expressed as the width of the passband (separation of the two cutoff frequencies).

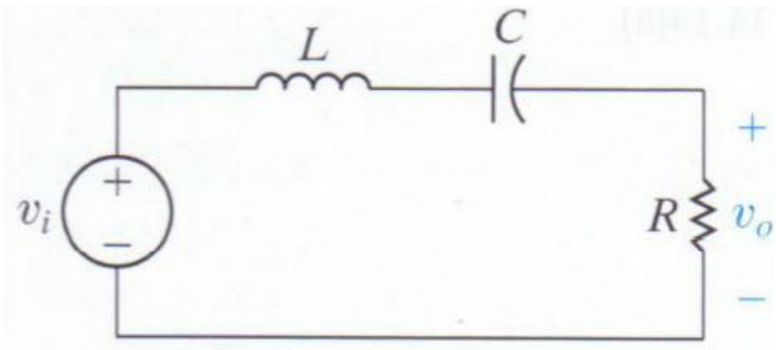
$$B = \omega_{c2} - \omega_{c1} \ \ in \ \tfrac{rad}{\sec} \qquad \Rightarrow B = f_{c2} - f_{c1} \ \ in \ \ Hz$$

Quality factor: The quality factor Q is a measure of how strongly the filter resonates at the center frequency. You can think of Q as an indicator of the power of the input signal versus width of the passband.

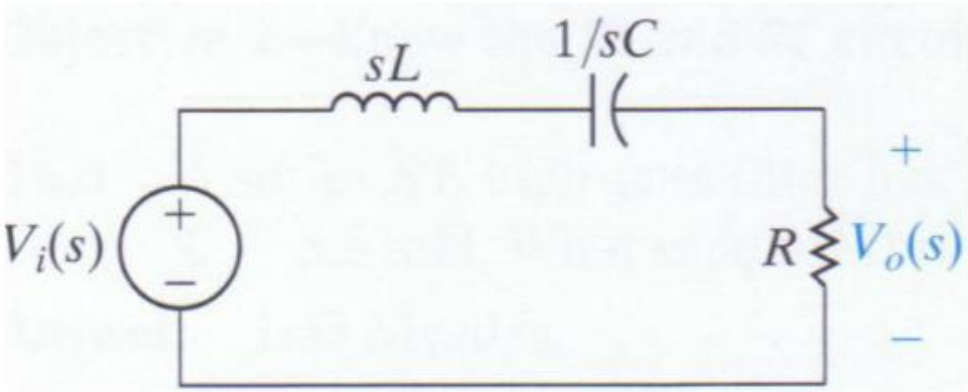
$$Q = \frac{\omega_0}{\beta \, in \, \tfrac{rad}{\sec}} \quad = \frac{f_0}{\beta \, in \ \ Hz}$$

Band-pass filter circuit

Series form of the RLC band-pass filter: A series RLC circuit acts as a bandpass filter when output is defined as the voltage across the resistor R.



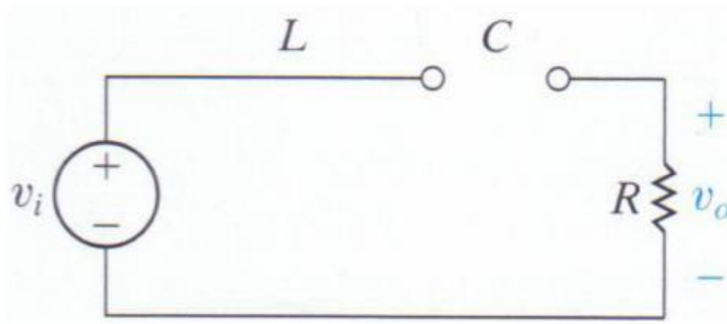
- Draw the s-domain circuit.



$$H(s) = \frac{V_o(s)}{V_i(s)}$$

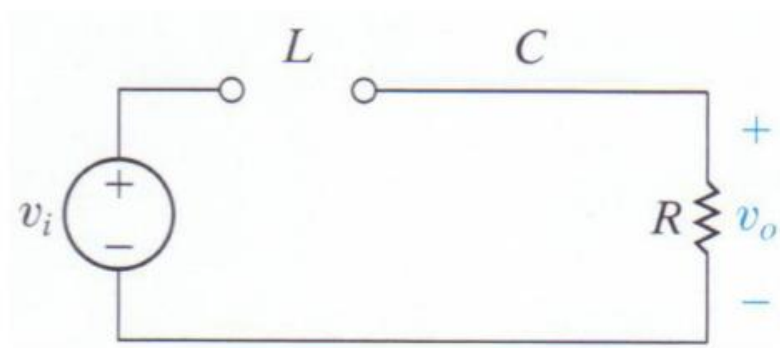
Some qualitative circuit behavior

- At DC (ω = 0), the capacitor acts as an open circuit and the inductor acts like a short circuit. No current flows through the resistor R.



$$v_0 = 0$$

- At high frequency ($\omega = \infty$), the capacitor acts as a short circuit and the inductor acts like an open circuit. No current flows through the resistor R.



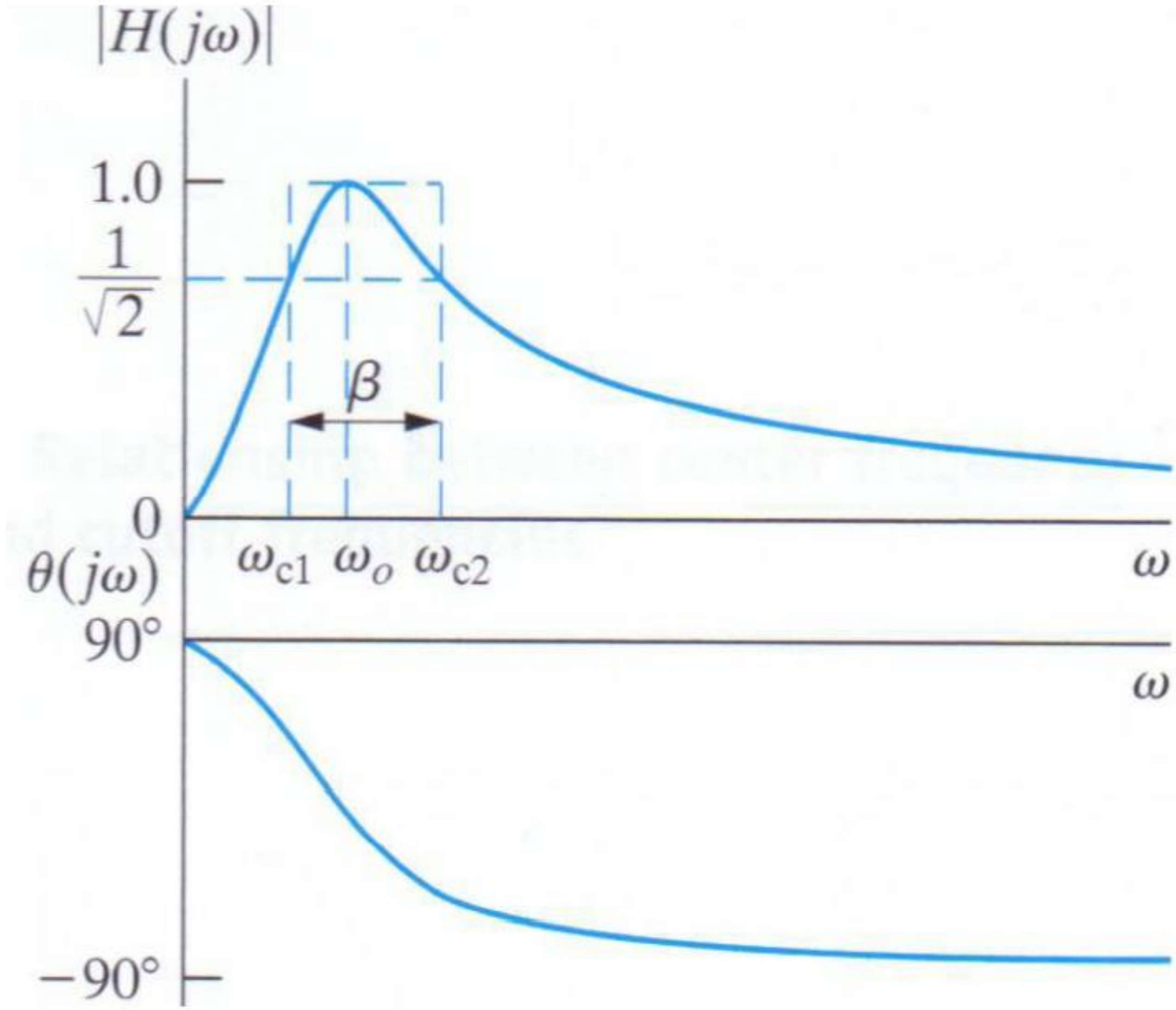
$$v_0 = 0$$

- Thus, qualitatively we see that there are two stop-bands. One stopband in the low frequency area, and one stopband in the high frequency region.

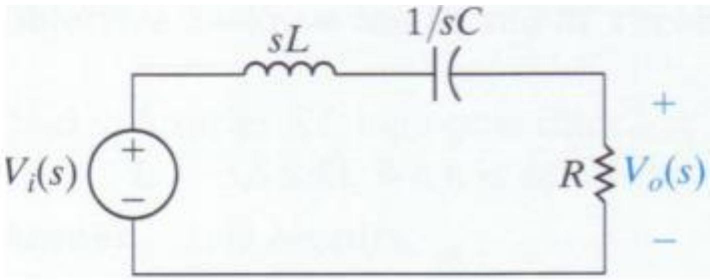
- Note at some frequency, the impedances Z_L and Z_C will be of the same magnitude and opposite sign. Thus, the output voltage will be only real at this frequency and it will be the maximum output voltage. This frequency is known as center frequency.

- At center frequency (ω_0), the phase is zero due to the net reactance = 0 at that frequency and circuit act as pure resistive circuit.

Thus the magnitude and phase frequency response plot is



Expression of Transfer Function:



$$V_o(s) = \frac{R}{sL + \frac{1}{sC} + R} V_i(s)$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{sL + \frac{1}{sC} + R} = \frac{Rs}{s^2 L + \frac{1}{C} + Rs} = \frac{\frac{R}{L} s}{s^2 + \frac{1}{LC} + s\frac{R}{L}} = \frac{\frac{R}{L} s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

- The transfer function H(s) is a second order system as expected.
- Depending on the placement of the poles, we will expect exponential and sinusoidal circuit behavior.
- It is the specific component values that place the poles. The transfer function then shows the circuit response to a driving input frequency.

The transfer function can be written in terms of jω.

$$H(s) = \frac{\frac{R}{L}s}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$H(s = j\omega) = \frac{\frac{R}{L}j\omega}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{\frac{R}{L}j\omega}{-\omega^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{\frac{R}{L}j\omega}{(\frac{1}{LC} - \omega^2) + j\omega\frac{R}{L}}$$

The magnitude and phase of H(jω) are

$$\left|H(j\omega)\right| = \frac{\frac{R}{L}\omega}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + (\omega\frac{R}{L})^2}}$$

$$\theta(j\omega) = \tan^{-1}\left(\frac{\frac{R}{L}\omega}{0}\right) - \tan^{-1}\left(\frac{\frac{R}{L}\omega}{\frac{1}{LC} - \omega^2}\right) = 90^\circ - \tan^{-1}\left(\frac{\frac{R}{L}\omega}{\frac{1}{LC} - \omega^2}\right)$$

Parameters that characterize this RLC band-pass filter

- **Center frequency ω_0:**

At center frequency ω_0, $Z_L + Z_C = \text{zero}$.

$$j\omega_0 L + \frac{1}{j\omega_0 C} = 0 \qquad \Rightarrow (j\omega_0 C)\, j\omega_0 L + j\omega_0 C \frac{1}{j\omega_0 C} = 0$$

$$-\omega_0^2 LC + 1 = 0 \qquad \Rightarrow \omega_0^2 LC = 1 \qquad \Rightarrow \omega_0^2 = \frac{1}{LC}$$

$$\boxed{\omega_0 = \sqrt{\frac{1}{LC}}}$$

- **Cutoff frequencies:**

$$\omega_{C1} = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad \omega_{C2} = \frac{+R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

- **Band width (B):**

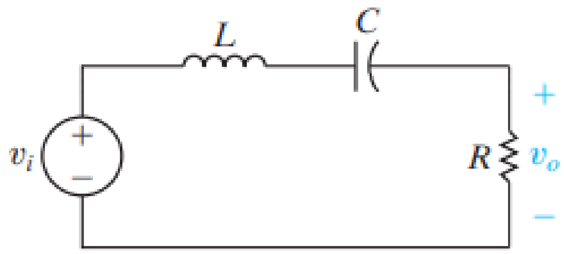
$$\beta = \omega_{C2} - \omega_{C1} \quad = \frac{R}{2L} - \frac{-R}{2L} \quad = \frac{R}{L}$$

- **Quality Factor (Q):**

$$Q = \frac{\omega_0}{\beta} \quad = \frac{\sqrt{\frac{1}{LC}}}{\frac{R}{L}} \quad = \sqrt{\frac{L^2}{LCR^2}} \quad = \sqrt{\frac{L}{CR^2}}$$

Example-4: Design a Bandpass Filter. Choose values for R, L, and C that yield a bandpass circuit able to select inputs within the 1–10 kHz frequency band.

Solution



Note any approach we choose will provide only two equations insufficient to solve for the three unknowns. We need to select a value for either R, L, or C and use the two equations we've chosen to calculate the remaining component values.

Let's choose 1 uF as the capacitor value.

Find center frequency:

$$f_o = \sqrt{f_{c1} f_{c2}} = \sqrt{(1000)(10{,}000)} = 3162.28 \text{ Hz}.$$

Compute the value of L:

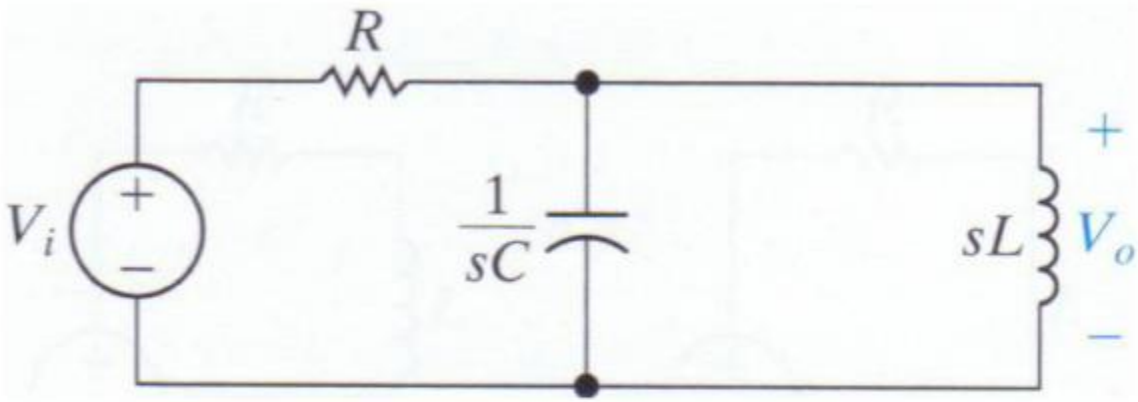
$$L = \frac{1}{\omega_o^2 C} = \frac{1}{[2\pi(3162.28)]^2(10^{-6})} = 2.533 \text{ mH}.$$

Find quality factor, Q: $$Q = \frac{f_o}{f_{c2} - f_{c1}} = \frac{3162.28}{10{,}000 - 1000} = 0.3514.$$

Calculate R: $$R = \sqrt{\frac{L}{CQ^2}} = \sqrt{\frac{0.0025}{(10^{-6})(0.3514)^2}} = 143.24\ \Omega.$$

This example reminds us that only two of the five bandpass filter parameters can be specified independently. The other three parameters can always be computed from the two that are specified. In turn, these five parameter values depend on the three component values, R, L, and C, of which only two can be specified independently.

Parallel form of the RLC band-pass filter:



- The transfer function H(s) is identical to the series form. (Students need to verified)
- The parameters are only slightly different.

- **Center frequency ω_0:**

$$\boxed{\omega_0 = \sqrt{\frac{1}{LC}}}$$

- **Cutoff frequencies:**

$$\omega_{C1} = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad \omega_{C2} = \frac{+1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

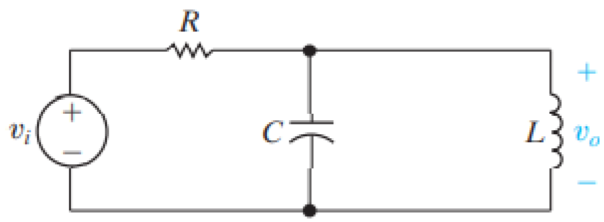
- **Band width (B):**

$$\boxed{\beta = \omega_{C2} - \omega_{C1} = \frac{1}{RC}}$$

- **Quality Factor (Q):**

$$\boxed{Q = \frac{\omega_0}{\beta} = \sqrt{\frac{R^2 C}{L}}}$$

Example-5: Designing a Bandpass Filter. Compute values for R and L to yield a bandpass filter with a center frequency of 5 kHz and a bandwidth of 200 Hz, using a 5 uF capacitor.



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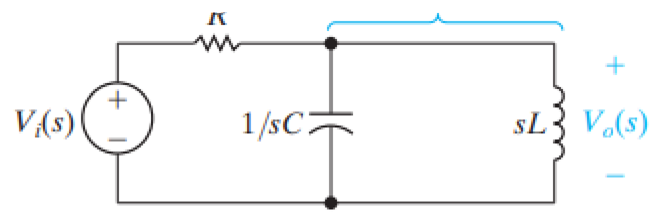


Solution:

Find R:

$$R = \frac{1}{\beta C} = \frac{1}{(2\pi)(200)(5 \times 10^{-6})} = 159.15\ \Omega.$$

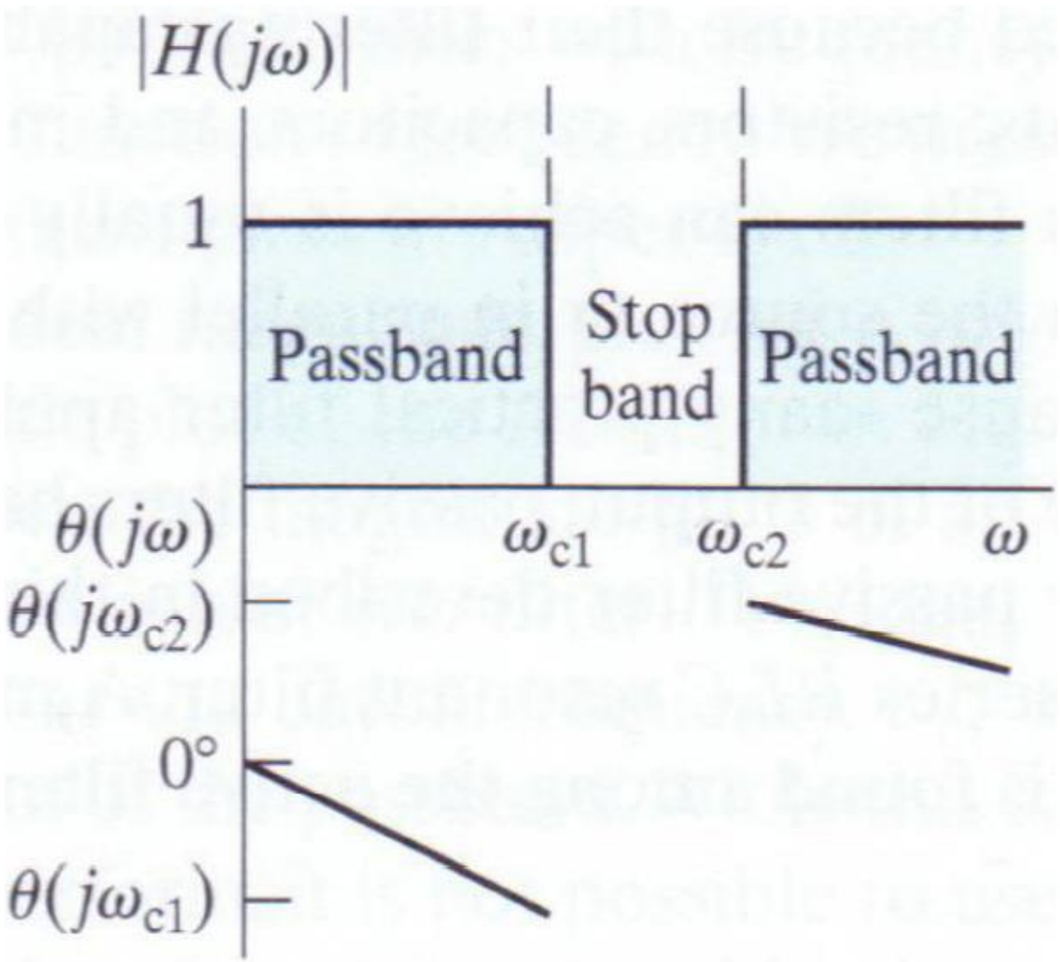
Find L:

$$L = \frac{1}{\omega_o^2 C} = \frac{1}{[2\pi(5000)]^2(5 \times 10^{-6})}$$

$$= 202.64\ \mu\text{H}.$$

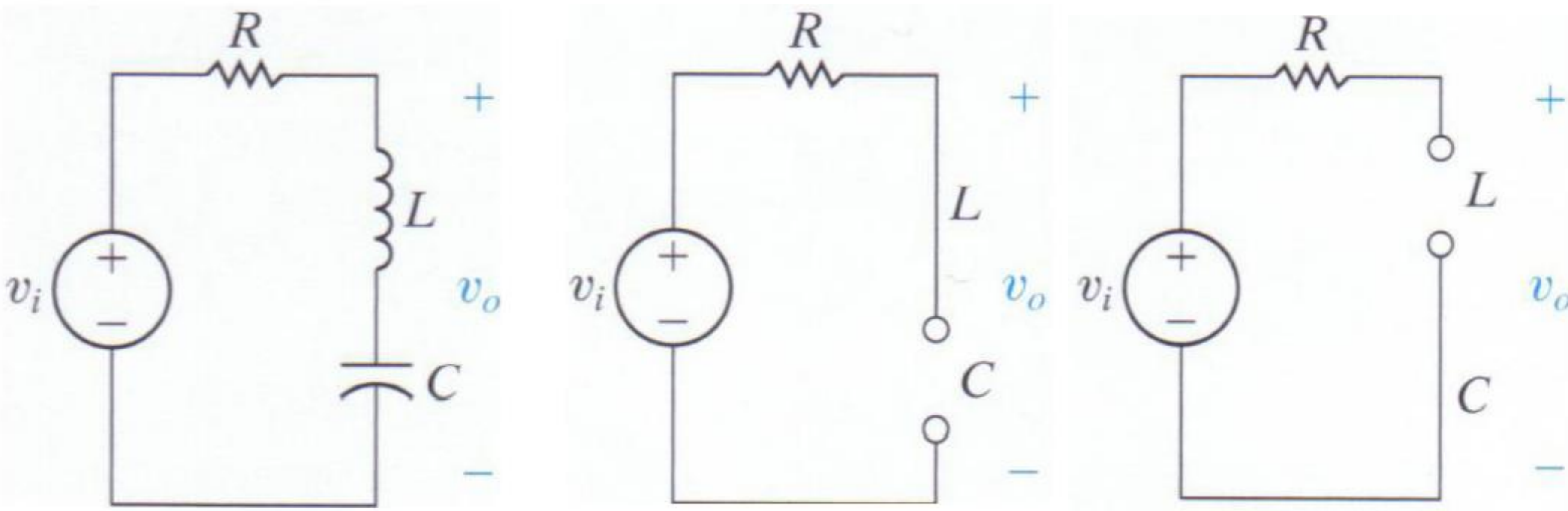
Passive Band Reject Filters

- The band reject filter passes source voltages outside the band between two cutoff frequencies.
- Ideal Band Reject Filter frequency plot.

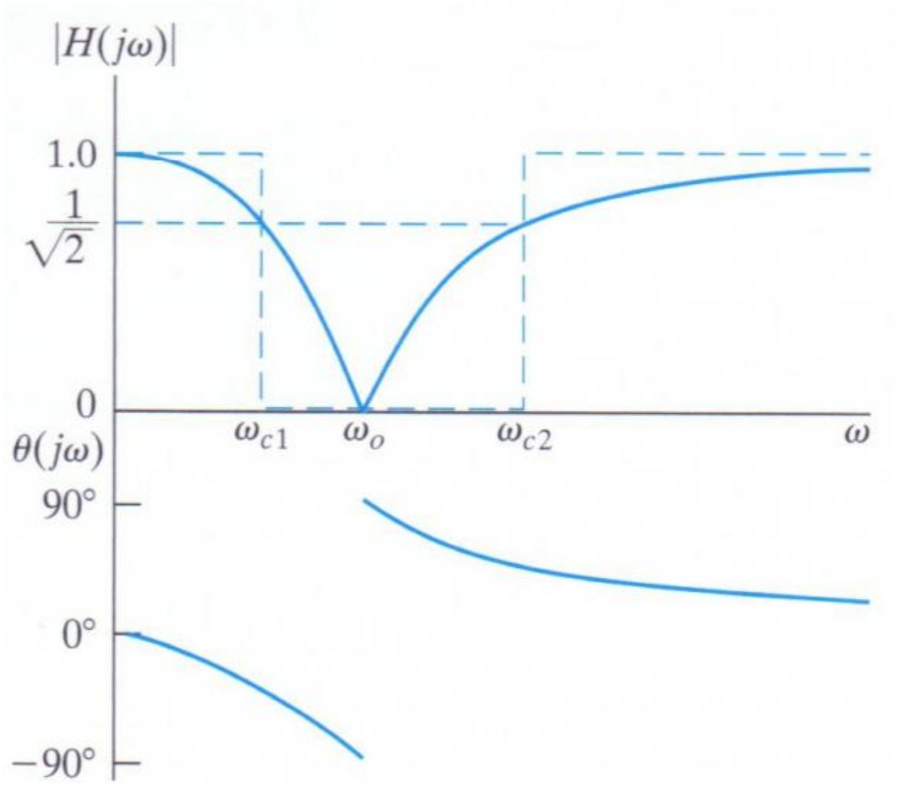


Band-reject filter circuit

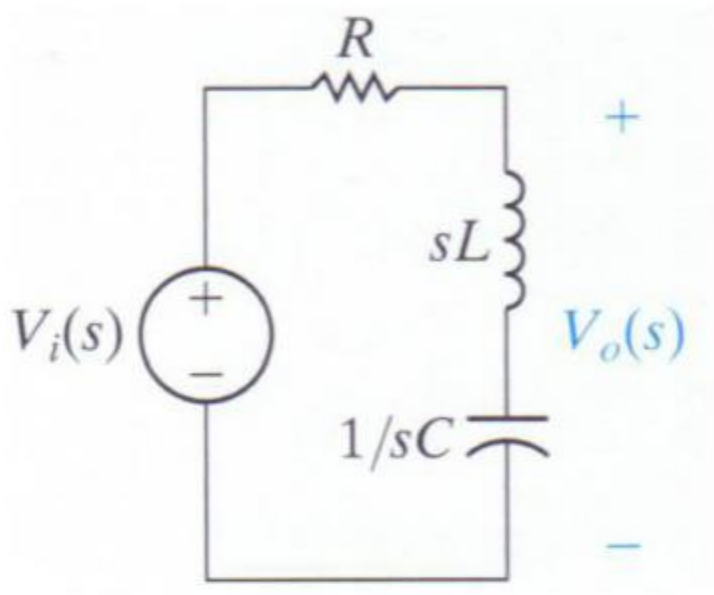
Series form of the RLC band-reject filter: A series RLC circuit can also be a band reject filter when the output voltage is defined as that across the inductor and capacitor pair.



- For low frequencies the capacitor is an open circuit and there is a voltage across the capacitor.
- For high frequencies the inductor acts like an open circuit and there is a voltage across the inductor.
- At center frequency "ω_0", the impedances Z_L and Z_C cancel each other and there is zero voltage.
- The frequency response plot is



- We can find the output voltage V_o by using the voltage divider.



$$V_o(s) = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} V_i(s)$$

$$H(s) = \frac{sL + \frac{1}{sC}}{R + sL + \frac{1}{sC}} = \frac{s^2L + \frac{1}{C}}{s^2L + sR + \frac{1}{C}} = \frac{s^2 + \frac{1}{LC}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

- For the steady-state response, s = jω and the transfer function can be written as:

$$H(j\omega)=\frac{(j\omega)^2+\frac{1}{LC}}{(j\omega)^2+j\omega\frac{R}{L}+\frac{1}{LC}}=\frac{-\omega^2+\frac{1}{LC}}{-\omega^2+j\omega\frac{R}{L}+\frac{1}{LC}}=\frac{\frac{1}{LC}-\omega^2}{\frac{1}{LC}-\omega^2+j\omega\frac{R}{L}}$$

The magnitude and phase are

$$\left|H(j\omega)\right|=\frac{\left|\frac{1}{LC}-\omega^2\right|}{\sqrt{(\frac{1}{LC}-\omega^2)^2+(\omega\frac{R}{L})^2}}$$

$$\theta(j\omega)=\tan^{-1}\left(\frac{0}{\frac{1}{LC}-\omega^2}\right)-\tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC}-\omega^2}\right)=-\tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC}-\omega^2}\right)$$

Parameters that characterize this RLC band-reject filter

- **Center frequency ω_0:**

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

- **Cutoff frequencies:**

$$\omega_{C1} = \frac{-R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}} \qquad \omega_{C2} = \frac{+R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

- **Band width (B):**

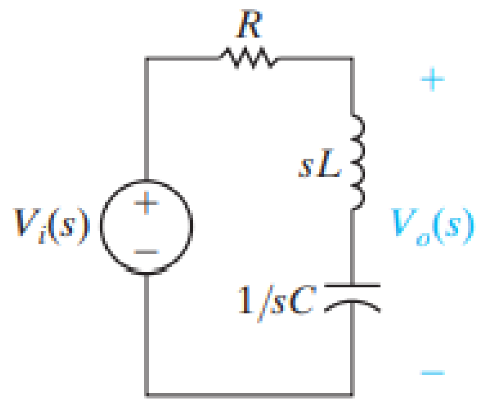
$$\beta = \omega_{C2} - \omega_{C1} = \frac{R}{2L} - \frac{-R}{2L} = \frac{R}{L}$$

- **Quality Factor (Q):**

$$Q = \frac{\omega_0}{\beta} = \frac{\sqrt{\frac{1}{LC}}}{\frac{R}{L}} = \sqrt{\frac{L^2}{LCR^2}} = \sqrt{\frac{L}{CR^2}}$$

Example-6: Design a Series RLC Bandreject Filter. Use the series RLC circuit. compute the component values that yield a band reject filter with a bandwidth of 250 Hz and a center frequency of 750 Hz. Use a 100 nF capacitor. Compute values for R, L, and Q.

Solution:



Find L:

$$L = \frac{1}{\omega_o^2 C}$$

$$= \frac{1}{[2\pi(750)]^2(100 \times 10^{-9})}$$

$$= 450 \text{ mH}.$$

Find R:

$$R = \beta L$$

$$= 2\pi(250)(450 \times 10^{-3})$$

$$= 707\ \Omega.$$

Find Corner Frequencies:

$$\omega_{c1} = -\frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}$$

$$= 3992.0 \text{ rad/s},$$

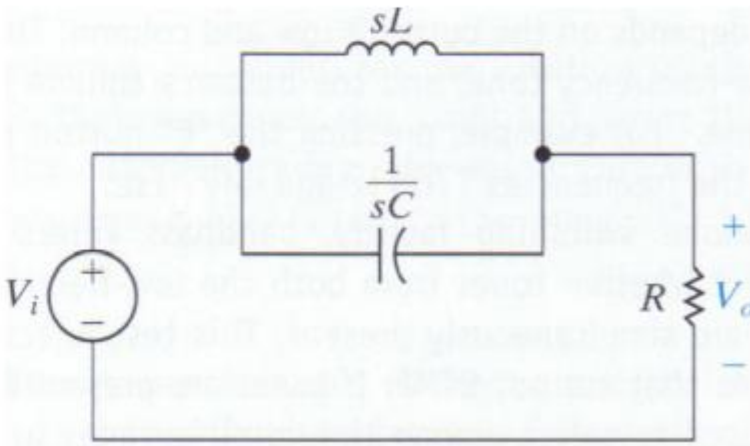
$$\omega_{c2} = \frac{\beta}{2} + \sqrt{\left(\frac{\beta}{2}\right)^2 + \omega_o^2}$$

$$= 5562.8 \text{ rad/s}.$$

Find Quality Factor:

$$Q = \omega_o/\beta = 3.$$

Parallel form of the RLC band-reject filter:



$$Z_{\|} = \frac{sL(\frac{1}{sC})}{sL + \frac{1}{sC}} = \frac{sL}{s^2LC + 1}$$

$$V_0(s) = V_i(s)\frac{R}{Z_{\|} + R}$$

$$H(s) = \frac{V_0(s)}{V_i(s)} = \frac{R}{\frac{sL}{s^2LC+1} + R} = \frac{R(s^2LC+1)}{sL + R(s^2LC+1)} = \frac{RLC(s^2 + \frac{1}{LC})}{RLC\left[s\frac{1}{RC} + s^2 + \frac{1}{LC}\right]}$$

$$H(s) = \frac{s^2 + \frac{1}{LC}}{s^2 + s\frac{1}{RC} + \frac{1}{LC}}$$

- For the steady-state response, s = jω and the transfer function can be written as:

$$H(j\omega) = \frac{(j\omega)^2 + \frac{1}{LC}}{(j\omega)^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{-\omega^2 + \frac{1}{LC}}{-\omega^2 + j\omega\frac{R}{L} + \frac{1}{LC}} = \frac{\frac{1}{LC} - \omega^2}{\frac{1}{LC} - \omega^2 + j\omega\frac{R}{L}}$$

The magnitude and phase are

$$|H(j\omega)| = \frac{\left|\frac{1}{LC} - \omega^2\right|}{\sqrt{(\frac{1}{LC} - \omega^2)^2 + (\omega\frac{R}{L})^2}}$$

$$\theta(j\omega) = \tan^{-1}\left(\frac{0}{\frac{1}{LC} - \omega^2}\right) - \tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC} - \omega^2}\right) = -\tan^{-1}\left(\frac{\omega\frac{R}{L}}{\frac{1}{LC} - \omega^2}\right)$$

Parameters that characterize this RLC band-reject filter

- **Center frequency ω_0:**

$$\omega_0 = \sqrt{\frac{1}{LC}}$$

- **Cutoff frequencies:**

$$\omega_{C1} = \frac{-1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}} \qquad \omega_{C2} = \frac{+1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 + \frac{1}{LC}}$$

- **Band width (B):**

$$\beta = \omega_{C2} - \omega_{C1} = \frac{1}{RC}$$

- **Quality Factor (Q):**

$$Q = \frac{\omega_0}{\beta} = \sqrt{\frac{R^2 C}{L}}$$