Chapter 10

Sinusoidal Steady-State Power Calculations

Text: *Electric Circuits* by J. Nilsson and S. Riedel
Prentice Hall

Engr 17 Introductory Circuit Analysis
Instructor: Russ Tatro

Chapter 10 Overview

Nearly all electric energy is currently supplied in the form a sinusoidal voltages and currents.

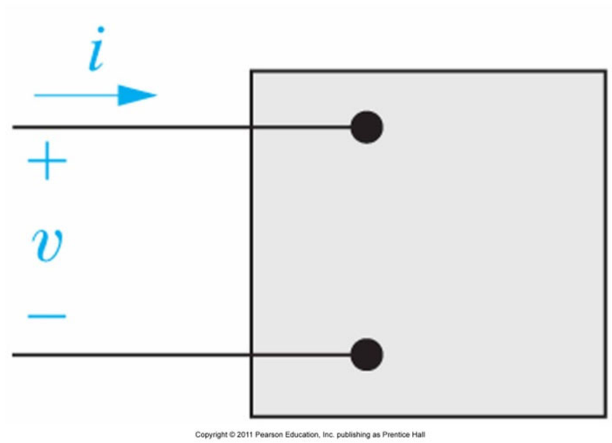
This chapter examines:

- Average Power (real) delivered to or supplied from a circuit
- Reactive Power
- Complex Power
- Apparent Power

Section 10.1
Instantaneous Power

Instantaneous Power

Always calculate power I.A.W. the passive sign convention!.



Current in the direction of voltage drop $\rightarrow$ +

Instantaneous power is the power at any one moment. In terms of the sinusoids:

$$p = vi = V_m \cos(\omega t + \theta_v)\; I_m \cos(\omega t + \theta_i)$$

The phase angle θ_v is the offset to zero time for the maximum voltage.

The phase angle θ_i is the offset to zero time for the maximum current.

Instantaneous Power – Derivation of p(t)

$$p = vi = V_m \cos(\omega t + \theta_v)\ I_m \cos(\omega t + \theta_i)$$

Power engineers prefer to write the current as the zero angle (positive maximum power). So we can slightly simplify the last equation as

$$p = V_m \cos(\omega t + \theta_v - \theta_i)\ I_m \cos(\omega t + \theta_i - \theta_i)$$

$$= V_m \cos(\omega t + \theta_v - \theta_i)\ I_m \cos(\omega t)$$

$$= V_m I_m \cos(\omega t + \theta_v - \theta_i)\ \cos(\omega t)$$

Instantaneous Power – Derivation of p(t)

$$p = vi = V_m I_m \cos(\omega t + \theta_v - \theta_i)\cos(\omega t)$$

Now use a trig identity

$$\cos\alpha\cos\beta = \tfrac{1}{2}\cos(\alpha - \beta) + \tfrac{1}{2}\cos(\alpha + \beta)$$

Let $\alpha = \omega t + \theta_v - \theta_i$

$\beta = \omega t$

And we get

$$p = V_m I_m \left[\tfrac{1}{2}\cos\left(\underbrace{\omega t + \theta_v - \theta_i}_{\alpha} - \underbrace{\omega t}_{\beta} \right) + \tfrac{1}{2}\cos(\underbrace{\omega t + \theta_v - \theta_i}_{\alpha} + \underbrace{\omega t}_{\beta}) \right]$$

$$= \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

Instantaneous Power

$$p = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + \frac{V_m I_m}{2}\cos(2\omega t + \theta_v - \theta_i)$$

We will again use another trig identity to simplify the 2nd term.

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

Where we let $\alpha = \theta_v - \theta_i$, $\beta = 2\omega t$

And now rewriting the last result.

$$p = \underbrace{\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)}_{P_{avg} = p \text{ at } DC} + \underbrace{\frac{V_m I_m}{2}\cos(\underbrace{\theta_v - \theta_i}_{\alpha})\cos(\underbrace{2\omega t}_{\beta})}_{p \text{ at } 2\omega}$$

$$- \underbrace{\frac{V_m I_m}{2}\sin(\underbrace{\theta_v - \theta_i}_{\alpha})\sin(\underbrace{2\omega t}_{\beta})}_{\text{Reactive power from C or L}}$$

Section 10.2
Average and Reactive Power

Average Power

$$p = \underbrace{\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)}_{P_{avg}=p \ at\ DC} + \underbrace{\frac{V_m I_m}{2}\cos(\theta_v - \theta_i)\cos(2\omega t)}_{p\ at\ 2\omega} - \underbrace{\frac{V_m I_m}{2}\sin(\theta_v - \theta_i)\sin(2\omega t)}_{\text{Reactive power from C or L}}$$

From our last major results, we see

$$p = P_{avg} + P_{avg}\cos(2\omega t) - Q\sin(2\omega t)$$

P_{avg} is defined as the average power delivered to a circuit.

Q is defined as the reactive power that is (temporarily) stored by the circuit.

Power for Purely Resistive Circuit

$$p = P_{avg} + P_{avg}\cos(2\omega t) - Q\sin(2\omega t)$$

For a resistor, the voltage and current are in phase – i.e. the maximum current occurs at same time as the maximum voltage across the resistor.

$$\theta_v = \theta_i$$

$$\cos(\theta_v - \theta_i) = \cos(0) = 1$$

$$\sin(\theta_v - \theta_i) = \sin(0) = 0$$

So for a resistor (which can only absorb power "+"), we have the instantaneous power as

$$p = P_{avg} + P_{avg}\cos(2\omega t)$$

Power for Purely Inductive Circuit

$$p = P_{avg} + P_{avg}\cos(2\omega t) - Q\sin(2\omega t)$$

For an inductor, the current lags the voltage by exactly 90°.

$$\theta_i = \theta_v - 90^\circ \qquad \Rightarrow \theta_v - \theta_i = +90^\circ$$

Then $\cos(\theta_v - \theta_i) = \cos(90^\circ) = 0 \qquad \sin(\theta_v - \theta_i) = \sin(90^\circ) = 1$

Average power = zero!

So an ideal inductor does not dissipate real power but only stores or delivers reactive power as required by the circuit.

$$p = -\underbrace{\frac{V_m I_m}{2}\underbrace{\sin(\theta_v - \theta_i)}_{=1}}_{=Q}\sin(2\omega t) = -Q\sin(2\omega t)$$

This reactive power as called Volt-Amp Reactive – “VARs”

Inductors *demand* (absorb) magnetizing VARs.

Power for Purely Capacitive Circuit

For a capacitor, the current leads the voltage by exactly 90°.

$$\theta_i = \theta_v + 90^\circ \qquad \Rightarrow \theta_v - \theta_i = -90^\circ$$

Then $\cos(\theta_v - \theta_i) = \cos(-90^\circ) = 0 \qquad \sin(\theta_v - \theta_i) = \sin(-90^\circ) = -1$

Average power = zero!

An ideal capacitor does not dissipate real power but only stores or delivers reactive power as required by the circuit.

$$p = \underbrace{-\frac{V_m I_m}{2} \underbrace{\sin(\theta_v - \theta_i)}_{=-1}}_{=-Q} \sin(2\omega t) = Q \sin(2\omega t)$$

Capacitors *furnish* (deliver) magnetizing VARs.

Power Factor

The phase angle between the voltage and current is called the *power factor angle.*

The sinusoidal *power factor (pf)* is defined as

$$pf = \cos(\theta_v - \theta_i)$$

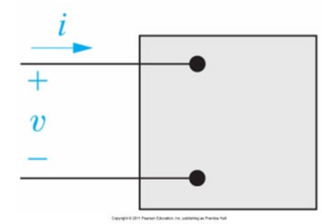
Since cosine(+ θ) = cosine(- θ), we need to add the term *lagging* to the power factor for an inductive load where current lags the voltage.

We add the term *leading* to the power factor for a capacitive load where current leads the voltage.

The sinusoidal *reactive factor* is defined as

$$rf = \sin(\theta_v - \theta_i)$$

Example



Given: $v = 100\cos(\omega t + 15^\circ)V$ $i = 4\sin(\omega t - 15^\circ)\,\text{A}$

Find the power factor at the terminals of this circuit network.

We must first put the given voltage and current into a common form or the phasor's phase angles will be meaningless.

$$i = 4\sin(\omega t - 15^\circ) = 4\cos(\omega t - 15^\circ - 90^\circ) = 4\cos(\omega t - 105^\circ)\,\text{A}$$

Thus the power factor is

$$pf = \cos(\theta_v - \theta_i) \quad = \cos\left[15^\circ - (-105^\circ)\right] \quad = \cos\left(120^\circ\right) \quad = -0.5 \; Lagging$$

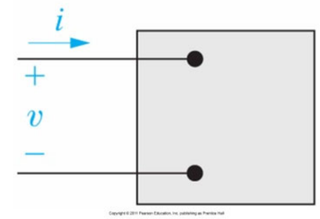
Calculate the average power at the terminals of this circuit network.

$$P_{avg} = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) \quad = \frac{100(4)}{2}\cos(120^\circ) \quad = 200(-0.5) = -100W$$

So we see that the circuit is delivering power to some (other) load not shown in the figure.

Example

Given: $v = 100\cos(\omega t + 15^{\circ})V \qquad i = 4\cos(\omega t - 105^{\circ})\,\mathrm{A}$



Calculate the reactive power.

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i) \quad = \frac{100(4)}{2}\sin[15^{\circ} - (-105^{\circ})] \quad = 200\sin(120^{\circ})$$

$$= 200(0.866) = (+)173.2\,VAR$$

So we see that the network is absorbing VARs which is an inductive network as indicated by the lagging power factor.

Section 10.3

The rms Value and Power Calculations

Average and rms

Recall that the average of function f(x) is defined by

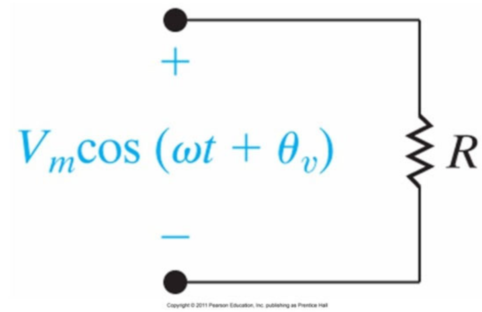
$$Average = \frac{1}{T}\int_{t_0}^{t_0+T} f(x)\,dx$$

And the *root mean squared* of a function f(x) is defined as

$$rms = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f^2(x)\,dx}$$

Average and rms

Assume that a sinusoidal voltage (or current) is applied to the terminals of a resistor.



The average power is

$$P_{avg} = \frac{1}{T}\int_{t_0}^{t_0+T} \frac{v^2(t)}{R}\,dx \quad = \frac{1}{R}\left[\underbrace{\frac{1}{T}\int_{t_0}^{t_0+T} \left[V_m\cos(\omega t + \theta_v\right]^2 dx}_{=V_{rms}^2}\right]$$

Thus we see that $P_{avg} = \dfrac{V_{rms}^2}{R}$

And by similar logic $P_{avg} = I_{rms}^2 R$

Sinusoid rms

Find the rms value of a sinusoidal input (voltage or current).

$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T}\left[V_m \sin(\omega t+\theta_v)\right]^2 d\omega t}$$

Since we are integrating over a full period, let $\theta_v = 0°$

$$= V_m\sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left[\sin(\omega t)\right]^2 d\omega t}$$

From the integral tables:

$$\int\left[\sin(\mathrm{x})\right]^2 dx = \frac{x}{2}-\frac{1}{4}\sin(2x)$$

$$= V_m\sqrt{\frac{1}{2\pi}\left\{\frac{2\pi}{2}-\frac{1}{4}\underbrace{\sin\left[2(2\pi)\right]}_{=0}\right\}-\left\{\frac{0}{2}-\frac{1}{4}\underbrace{\sin\left[2(0)\right]}_{=0}\right\}}$$

$$= V_m\sqrt{\frac{2\pi}{2\pi(2)}} = \frac{V_m}{\sqrt{2}}$$

The rms value is often called the *effective value.*

The average power in terms of the effective value is

$$P_{avg} = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i)$$

We now recognize $\frac{V_m}{\sqrt{2}} = V_{rms}$ and $I_{rms} = \frac{I_m}{\sqrt{2}}$

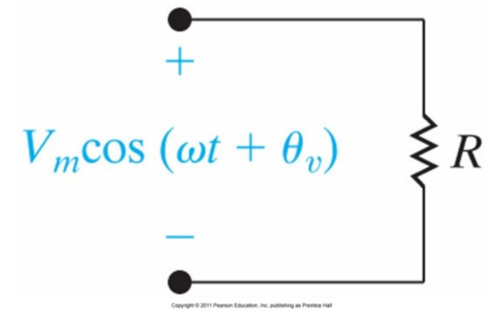
$$P_{avg} = V_{eff} I_{eff} \cos(\theta_v - \theta_i)$$

The reactive power can also be stated in terms of effective values.

$$Q = V_{eff} I_{eff} \sin(\theta_v - \theta_i)$$

Example

Given: $v = 625\cos(\omega t)\,V$ $R = 50\Omega$



Find the average power delivered to the resistor.

The voltage in terms of rms is

$$v_{rms} = \frac{625}{\sqrt{2}}$$

The power absorbed by the 50Ω resistor is then

$$P = \frac{v_{rms}^2}{50\Omega} = \frac{\left(\frac{625}{\sqrt{2}}\right)^2}{50} = \frac{390{,}625}{50(2)} = 3{,}906.25W$$

Section 10.4
Complex Power

Complex Power

Complex power is the sum of real and reactive power.

$$S\angle\theta = \underbrace{S\cos(\theta)}_{=P} + j\,\underbrace{S\sin(\theta)}_{=Q} \qquad \textit{Where}\ \ \theta = \theta_v - \theta_i$$

$$S = P + jQ$$

$$P = \text{real part}\{S\} \qquad Q = \text{Imaginary part}\{S\}$$

Apparent Power is defined as the magnitude of the complex power S.

$$|S| = \sqrt{P^2 + Q^2}$$

The last definition leads to two very useful relationships.

$$P = |S|\cos\theta$$

$$Q = |S|\sin\theta$$

Example

An electrical load operates at 240 V_{rms}.
The load absorbs an average power of 8 kW at a lagging power factor of 0.8

Calculate the complex power of the load.

The “lagging power factor” means the current lags the voltage = inductive load.

So $\theta_v - \theta_i$ is a positive angle.

$$P = |S|\cos(\theta)$$

$$|S| = \frac{P}{\cos(\theta)} \qquad = \frac{8,000W}{0.8} \qquad = 10kVA$$

The power factor angle and sin(θ) are

$$\theta = \cos^{-1}(0.8) = +36.87^\circ \qquad\qquad \sin(\theta) = \sin(36.87^\circ) = 0.6$$

The reactive power is

$$Q = |S|\sin(\theta) = 10\,\text{kVA}(0.6) = 6\,\text{kVAR}$$

The complex power is then

$$S = P + jQ = (8 + j6)\,kVA$$

Section 10.5
Power Calculations

Power Calculations

Given:

$$P = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i)$$

$$Q = \frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

And that $S = P + jQ$

Then

$$S = \frac{V_m I_m}{2}\cos(\theta_v - \theta_i) + j\frac{V_m I_m}{2}\sin(\theta_v - \theta_i)$$

$$= \frac{V_m I_m}{2}\left[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right]$$

Power Calculations

$$S = \frac{V_m I_m}{2}\left[\cos(\theta_v - \theta_i) + j\sin(\theta_v - \theta_i)\right]$$

Express the last result by using Euler's Identity.

$$S = \frac{V_m I_m}{2} e^{j(\theta_v - \theta_i)} \qquad = V_{eff} e^{j\theta_v} I_{eff} e^{-j\theta_i}$$

The last result can be written in phasor form as

$$S = V_{eff} I_{eff}^{*}$$

Where * indicates the complex conjugate of the angle.

In terms of the maximum voltage (V_m) and current (I_m), we have

$$S = \tfrac{1}{2} V I^{*}$$

Alternate Forms

As long as we carefully write the complex form of the voltage and current, we can write

$$V_{eff} = ZI_{eff} \qquad Where \;\; Z = R + jX$$

The powers can then be expressed as

$$P = \left|I_{eff}\right|^2 R \qquad S = \left|I_{eff}\right|^2 Z$$

$$Q = \left|I_{eff}\right|^2 X \qquad S = \frac{\left|V_{eff}\right|^2}{Z^*}$$

Example

Given: $v = 100\cos(\omega t + 15^\circ)V$ $\quad i = 4\cos(\omega t - 105^\circ)\,\mathrm{A}$

The phasor form of the inputs are

$$V = 100\angle 15^\circ\,\mathrm{V} \qquad I = 4\angle -105^\circ\,\mathrm{A}$$

$$I^* = 4\angle +105^\circ\,\mathrm{A}$$

Calculate the complex power.

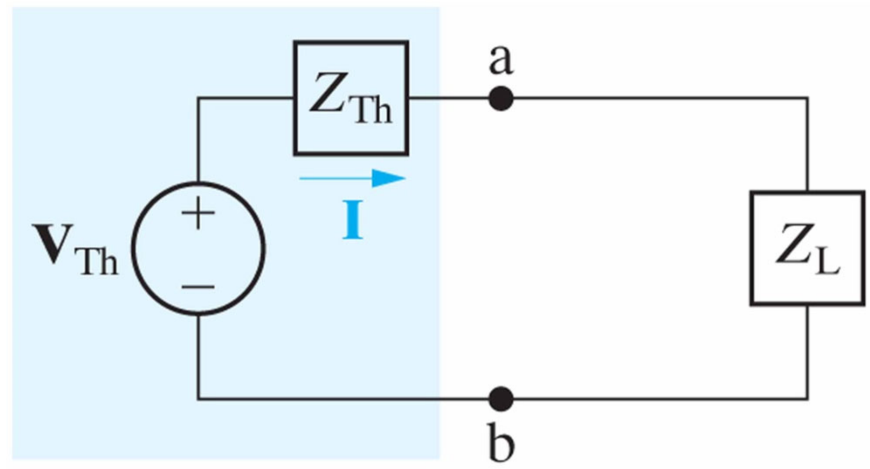
$$S = \frac{1}{2}VI^* = \frac{1}{2}(100\angle 15^\circ\,\mathrm{V})(4\angle 105^\circ\,\mathrm{A}) \quad = 200\angle 120^\circ VA \quad = -100 + j173.2VA$$

Section 10.6
Maximum Power Transfer

Maximum Power Transfer

Recall from earlier work that the maximum power is transferred to a load when the load resistance equals the Thévenin resistance of the source.

$$R_{load} \equiv R_{Th}$$



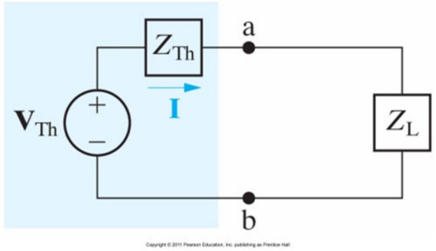
In the complex case, we must "cancel out" the reactive components.

$$Z_{load} \equiv Z_{Th}^{*}$$

We will start the derivation of this result by defining the two impedances:

$$Z_{load} = R_L + jX_L \qquad\qquad Z_{Th} = R_{Th} + jX_{Th}$$

Maximum Power Transfer



The phasor current in the circuit can be immediately written from

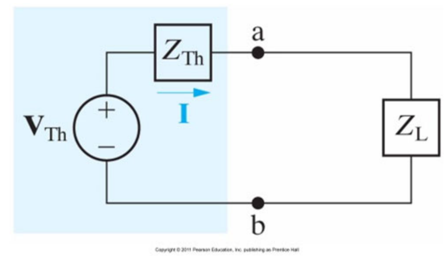
$$I = \frac{V_{Th}}{Z_{load} + Z_{Th}} = \frac{V_{Th}}{\left(R_L + R_{Th}\right) + j\left(X_L + X_{Th}\right)}$$

The average power delivered to the load is

$$P = \left|I\right|^2 R_L = \left|\frac{V_{Th}}{\left(R_L + R_{Th}\right) + j\left(X_L + X_{Th}\right)}\right|^2 R_L$$

$$= \frac{\left|V_{Th}\right|^2 R_L}{\left(R_L + R_{Th}\right)^2 + \left(X_L + X_{Th}\right)^2}$$

Maximum Power Transfer



We now take the partial derivative of the last result with respect to X_L.

$$\frac{\partial P}{\partial X_L} = \frac{\partial}{\partial X_L}\left[\frac{|V_{Th}|^2 R_L}{(R_L + R_{Th})^2 + (X_L + X_{Th})^2}\right]$$

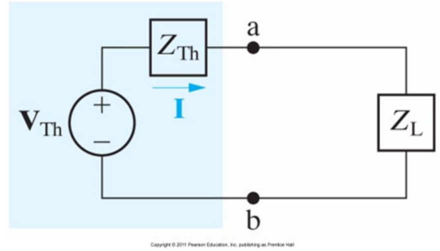
$$= \frac{|V_{Th}|^2 R_L\left[(-2)(X_L + X_{Th})\right]}{\left[(R_L + R_{Th})^2 + (X_L + X_{Th})^2\right]^2}$$

From the last result, we see that the partial derivative is zero when

$X_L = -X_{Th}$

We ignored the trivial result where V_{Th} = zero or R_L = zero.

Maximum Power Transfer



Note on the partial derivative.

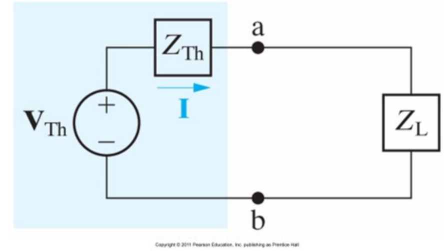
We have a function of multiple variables in the form:

$$f = \frac{u(x, y)}{v(x, y)}$$

The partial derivative is then found by using the quotient rule.

$$\frac{\partial f}{\partial x} = \frac{v\dfrac{\partial u}{\partial x} - u\dfrac{\partial v}{\partial x}}{v^2}$$

Maximum Power Transfer

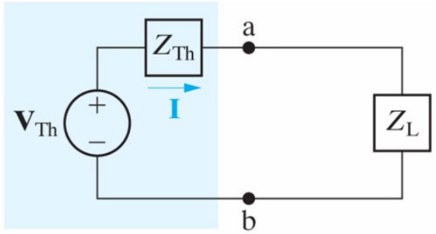


We now take the partial derivative with respect to R_L.

$$\frac{\partial P}{\partial R_L} = \frac{\partial}{\partial R_L}\left[\frac{|V_{Th}|^2 R_L}{(R_L + R_{Th})^2 + (X_L + X_{Th})^2}\right]$$

$$= \frac{|V_{Th}|^2\left[(R_L + R_{Th})^2 + (X_L + X_{Th})^2 - 2R_L(R_L + R_{Th})\right]}{\left[(R_L + R_{Th})^2 + (X_L + X_{Th})^2\right]^2}$$

Maximum Power Transfer



From the last result, we see that the partial derivative is zero when

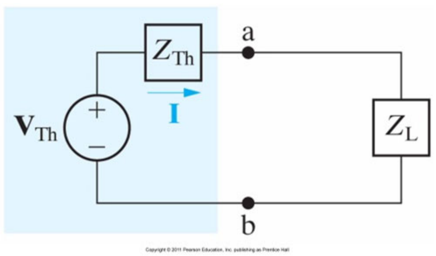
$$\left(R_L + R_{Th}\right)^2 + \left(X_L + X_{Th}\right)^2 - 2R_L\left(R_L + R_{Th}\right) = 0$$

$$R_L^2 + 2R_L R_{Th} + R_{Th}^2 + \left(X_L + X_{Th}\right)^2 - 2R_L^2 - 2R_L R_{Th} = 0$$

$$R_L^2 = R_{Th}^2 + \left(X_L + X_{Th}\right)^2$$

$$R_L = \sqrt{R_{Th}^2 + \left(X_L + X_{Th}\right)^2}$$

Maximum Power Transfer



The two partial derivatives when set to zero yielded these two equations:

$$X_L = -X_{Th}$$

$$R_L = \sqrt{R_{Th}^2 + \left(X_L + X_{Th}\right)^2}$$

Combining the two results yields

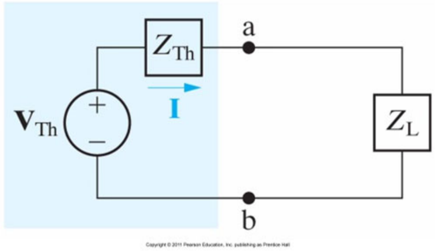
$$R_L = \sqrt{R_{Th}^2 + \left(-X_{Th} + X_{Th}\right)^2} \Rightarrow R_L = \sqrt{R_{Th}^2} \Rightarrow R_L = R_{Th}$$

Thus maximum power transfer occurs when

$$R_{Th} - X_{Th} = R_L + X_L$$

$$Z_{Th}^* = Z_L$$

Maximum Average Power Absorbed



The condition for maximum average power delivered to load Z_L is

$$Z_L = Z_{Th}^*$$

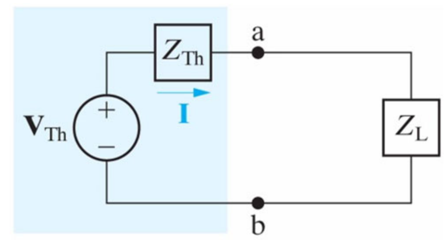
Thus (from chapter 4)

$$P_{\max} = \frac{|V_{Th}|^2 R_L}{4R_L^2} = \frac{|V_{Th}|^2}{4R_L}$$

If V_{Th} is expressed in terms of the maximum voltage V_m rather than rms values then

$$P_{\max} = \frac{V_m^2}{8R_L}$$

Maximum Power Transfer when Z is Restricted



What about the case when the load cannot be fully manipulated by varying both R_L and X_L but the magnitude can be adjusted?

Then the closest approach to maximum power is when the magnitudes of the load and the Thévenin equivalent are made equal.

$$|Z_L| = |Z_{Th}|$$

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