Chapter 9

Sinusoidal Steady-State Analysis

Text: *Electric Circuits* by J. Nilsson and S. Riedel

Prentice Hall

Engr 17 Introductory Circuit Analysis

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Chapter 9 Overview

This chapter introduces time-varying voltages and currents.

One particularly useful time-varying signal is the sinusoidal source.

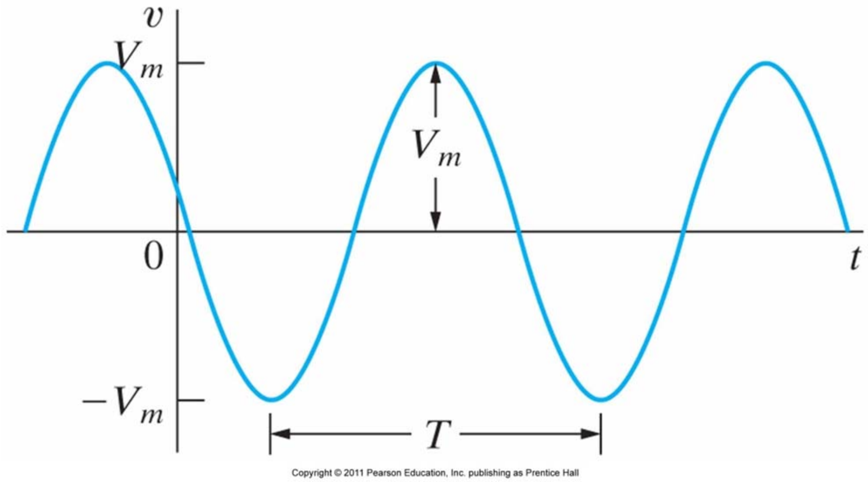
To accomplish the time-varying view, we will now use complex numbers in the form of the steady-state phasor method.

Section 9.1

The Sinusoidal Source

Sinusoidal Source

A sinusoidal voltage source produces a voltage that varies sinusoidally with time.



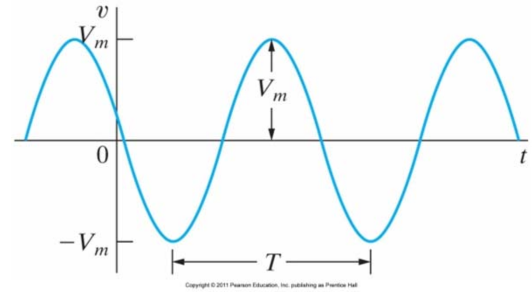
$$v = V_m \cos(\omega t + \phi)$$

The period of the sinusoid function is T in seconds.

The frequency is f (in Hz). $f = \dfrac{1}{T}$

The angular frequency is $\omega = 2\pi f = \dfrac{2\pi}{T}\, radians/\sec$

Sinusoidal Source



$$v = V_m \cos(\omega t + \phi)$$

The phase angle ϕ determines the value of the sinusoid at t = 0.

For example, let ϕ = 90° and V_m = 8 Volts.

$$v(t=0) = 8\cos(\omega * 0 + 90^\circ) = 0\,Volts$$

We usually report the phase angle ϕ in degrees for readability. But the angle is properly written as radians.

$$y(in\ radians) = \frac{\pi}{180^\circ} x(\text{in degrees})$$

For example, find the radians version of 45°

$$\frac{\pi}{180^\circ}(45^\circ) = \frac{\pi}{4} \approx 0.7854\,\text{radians}$$

Sinusoidal Source Example

A sinusoidal current has a maximum amplitude of 20 A.
The period of the current sinusoidal is 1 ms.

The magnitude of the current at t = 0 is 10 A.

What is the frequency of the current?

$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{1ms} = 2,000\pi \frac{rad}{s}$$

$$f = \frac{1}{T} = \frac{1}{1ms} = 1,000\ Hz$$

What is the expression for i(t) ?

$$i(t) = I_m \cos(\omega t + \phi) = 20\cos(2,000\pi t + \phi)\ Amp$$

Sinusoidal Source Example

What is ϕ ? Use the given value at t = 0 and solve for ϕ.

$$i(t=0) = 20\cos(2,000\pi[0] + \phi) = 10\,Amp$$

$$20\cos(\phi) = 10\,Amp$$

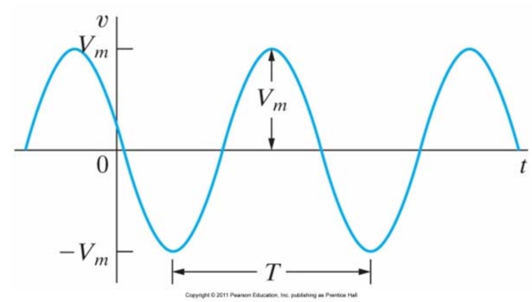
$$\cos\phi = \frac{1}{2}$$

$$\phi = \cos^{-1}\frac{1}{2} \quad = 60^{\circ}$$

So i(t) is

$$i(t) = 20\cos(2,000\pi t + 60^{\circ})\,Amp$$

Root Mean Square



A sinusoid has zero average value.

$$\bar{v}(t) = \frac{1}{T}\int_0^T \cos(x)\,dx \qquad = \frac{1}{2\pi}\int_0^{2\pi} \cos(x)\,dx$$

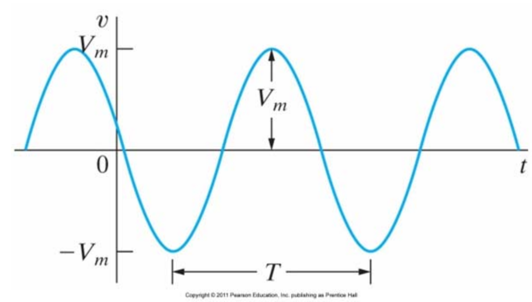
$$= \frac{1}{2\pi}(-)\sin(x)\Big|_0^{2\pi} = \frac{-1}{2\pi}\left(\sin 2\pi - \sin 0\right) = 0$$

But a sinusoid will deliver power to a real load. Remember "a resistor only gets hot!" It does not matter which direction the current flows.

So a very useful metric is the rms value = root mean square.

$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)\,dt}$$

Root Mean Square



$$V_{rms} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} V_m^2 \cos^2(\omega t + \phi)\, dt}$$

Note that if we integrate over one complete period, the value of the phase ϕ does not matter and can be assumed = zero for the integral.

$$= V_m \sqrt{\frac{1}{2\pi}\int_0^{2\pi} \cos^2(\omega t)\, dt} \qquad = V_m \sqrt{\frac{1}{2\pi}\left[\frac{t}{2} + \frac{1}{4\omega}\sin(2\omega t)\right]_0^{2\pi}}$$

$$= V_m \sqrt{\frac{1}{2\pi}\left[\frac{2\pi}{2} - \frac{0}{2} + 0\right]} \quad = V_m \sqrt{\frac{1}{2}} \quad = \frac{V_m}{\sqrt{2}}$$

The $\frac{1}{\sqrt{2}}$ result is only valid for a sinusoid. You must repeat the calculation for a sinusoid with a dc offset or any other signal type.

Root Mean Square Example

Find the rms of a sinusoid with a DC offset.

Let $V_{offset} = V_{DC} = 5$ Volts

Let the sinusoidal component be 2 cos(ωt) Volts

$$v(t) = 5 + 2\cos(\omega t)$$

Now find the rms value of v(t)

$$V_{rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left[5 + 2\cos(\omega t)\right]^2 dt}$$

$$= \sqrt{\frac{1}{2\pi}\int_0^{2\pi}\left[25 + \underbrace{20\cos(\omega t)}_{\text{integral} = 0} + 4\cos^2(\omega t)\right]dt}$$

Root Mean Square Example

$$V_{rms} = \sqrt{\frac{1}{2\pi}\int_0^{2\pi} 25dt + \frac{1}{2\pi}\int_0^{2\pi} 4\cos^2\left(\omega t\right)dt}$$

$$= \sqrt{\frac{25(2\pi)}{2\pi} + \frac{4}{2\pi}\left(\frac{2\pi}{2}\right)} \quad = \sqrt{25 + \frac{4}{2}}$$

Note the form of the answer before we evaluate the rms value.

$$V_{rms} = \sqrt{V_{DC}^2 + \frac{V_m^2}{2}} \quad = \sqrt{25+2} \quad = \sqrt{27} \approx 5.1962 V_{rms}$$

Section 9.2

The Sinusoidal Response

Sinusoidal Response

When we input any signal into a circuit, we expect to see an initial response and then, after *a long time*, a response that persists forever (until the input signal goes away).

The initial immediate response to the signal input is called the *transient response*.

The long lasting response to the signal input is called the *steady-state response*.

The total response to the signal input is a combination of the transient and the steady-state responses.

Sinusoidal Response

For a sinusoidal input to a linear circuit:

1. The steady-state response of a circuit is also a sinusoid function.

2. The frequency of the circuit's response is equal to the frequency of the input signal.

3. The amplitude of the response depends on the circuit and most likely will NOT be the amplitude of the sinusoidal input.

4. The phase of the circuit's response will most likely NOT be the phase of the sinusoidal input. (Be careful – phase is defined in relation to something).

Section 9.3

The Phasor

The Phasor

The phasor and the Laplace (s domain – not covered in this course) representation of a circuit element is the most common way to mathematically describe a circuit.

So we will first define what is the phasor.

Phasor – a complex number that carries the amplitude and phase angle information of a sinusoidal function.

Start with Euler's Identity

$$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

$$Ae^{\pm j\theta} = A\cos\theta \pm jA\sin\theta$$

The Phasor

The authors show that we can let the magnitude and phase angle be represented in polar form by

$$V_m e^{j\phi} \text{ as the phasor transform of } V_m \cos(\omega t+\phi)$$

The frequency component is represented by $e^{j\omega t}$

The phasor’s trigonometric form of magnitude and phase angle is given as

$$\mathrm{V} = V_m\left[\cos\phi + j\sin\phi\right] \quad = V_m e^{j\phi} \quad = V_m \angle\phi$$

We will largely ask that you accept this representation and move quickly to the way we can now express passive circuit elements.

$$V_m \cos(\omega t + \phi) \Rightarrow V_m \angle\phi$$

Section 9.4
Passive Circuit Elements
in the Frequency Domain

Passive Circuit Elements in the Frequency Domain

Since we have signals that change with time and where that change is described in terms of frequency, we now need to represent circuit elements in the frequency domain.

Start with the V-I relationship for a resistor:

$$v = R\left[I_m \cos(\omega t + \theta_i)\right]$$

$$= RI_m\left[\cos(\omega t + \theta_i)\right]$$

The phasor form is (**V** and **I** are phasors)

$$\mathrm{V} = RI_m e^{j\theta_i} = R\underbrace{I_m\angle\theta_i}_{\substack{polar\ \ form \\ of\ \ current}} = R\mathrm{I}$$

There is no phase shift between the voltage and the current in a resistor.

V-I Relationship for an Inductor

Assume $i_L = I_m \cos(\omega t+\theta_i)$, the inductor voltage is found by

$$v = L\frac{di}{dt} = L\frac{d}{dt}\left[I_m \cos(\omega t + \theta_i)\right] \;\; = LI_m(-\omega)\sin(\omega t + \theta_i)$$

So that the phase angles have a common reference form, we will rewrite the last result in terms of cosine.

$$v = -\omega L I_m \cos(\omega t + \theta_i - 90^\circ)$$

The phasor representation is

$$\mathrm{V} = -\omega L I_m e^{j(\theta_i - 90^\circ)} = -\omega L I_m e^{j\theta_i} e^{-j90^\circ}$$

Recall that e^{-j90° on the real versus imaginary plot is at –j1

$$\mathrm{V} = -\omega L I_m e^{j\theta_i)}\left(-j1\right) = j\omega L I_m e^{j\theta_i}$$

V-I Relationship for an Inductor

Another way of looking at $e^{-j90°}$ is to use Euler's Identity.

$$e^{-j90^\circ} = \cos 90^\circ - j\sin 90^\circ = 0 - j1 = -j$$

Now back to the phasor voltage of an inductor:

$$\mathrm{V} = j\omega L I_m e^{j\theta_i} = \left(\omega L \angle 90^\circ\right) I_m \angle \theta_i$$

The last result shows that the voltage across on inductor *leads* the current through the inductor by 90° (π/2 of the period of the waveform).

Power engineers usually state that the current *lags* the voltage by 90°. Same result with just a slightly different viewpoint.

$$\mathrm{V} = j\omega L\,\mathrm{I}$$

V-I Relationship for a Capacitor

Assume $v_C = V_m \cos(\omega t + \theta_v)$, the capacitor current is found by

$$i = C\frac{dv}{dt} = C\frac{d}{dt}\left[V_m\cos(\omega t+\theta_v)\right] \quad = CV_m(-\omega)\sin(\omega t+\theta_v)$$

So that the phase angles have a common reference form, we will rewrite the last result in terms of cosine.

$$i = -\omega C V_m \cos(\omega t + \theta_v - 90^\circ)$$

The phase representation is

$$\mathrm{I} = -\omega C V_m e^{j\theta_v} e^{-j90^\circ} = -\omega C V_m e^{j\theta_v}(-j1) = j\omega C \underbrace{V_m e^{j\theta_v}}_{\mathrm{V}}$$

$$\mathrm{I} = j\omega C V$$

Rewrite the last result to resemble Ohm's Law then

$$\mathrm{V} = \frac{1}{j\omega C}\,\mathrm{I}$$

Impedance and Reactive

We are now in a position to look at a VERY important concept.

The resistor, capacitor and inductor now all have a very similar looking form:

$$V = I\,Z$$

Z represents the *impedance* of the circuit element at the frequency of the driving source. The imaginary part is called the *reactance*.

In other words, the capacitor and inductor have a specific response (impedance) once we know the frequency of the input signal.

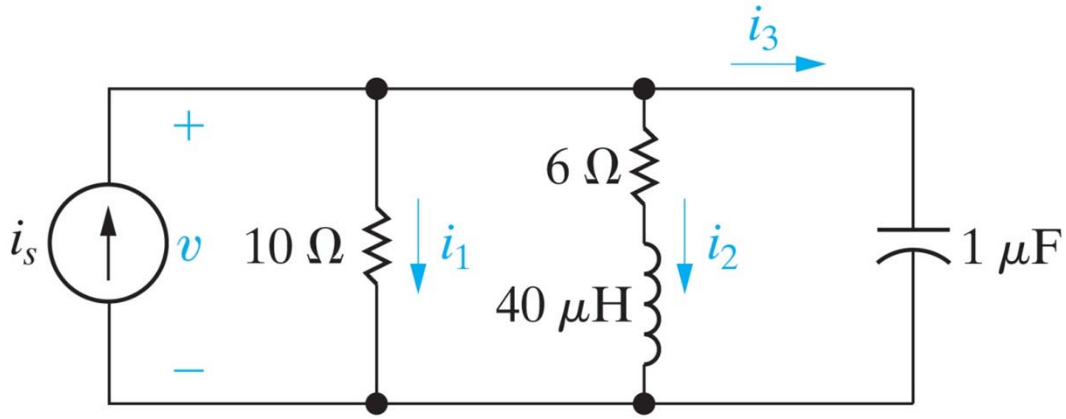
$$V = IR$$

$$V = I(j\omega L)$$

$$V = I\left(\frac{1}{j\omega C}\right)$$

Sinusoidal Response Example

Construct the frequency-domain equivalent of the following circuit.



$$i_s = 8\cos(2\times10^5 t)\,A \qquad\qquad I_s = 8\angle 0^\circ\,A$$

For the inductor

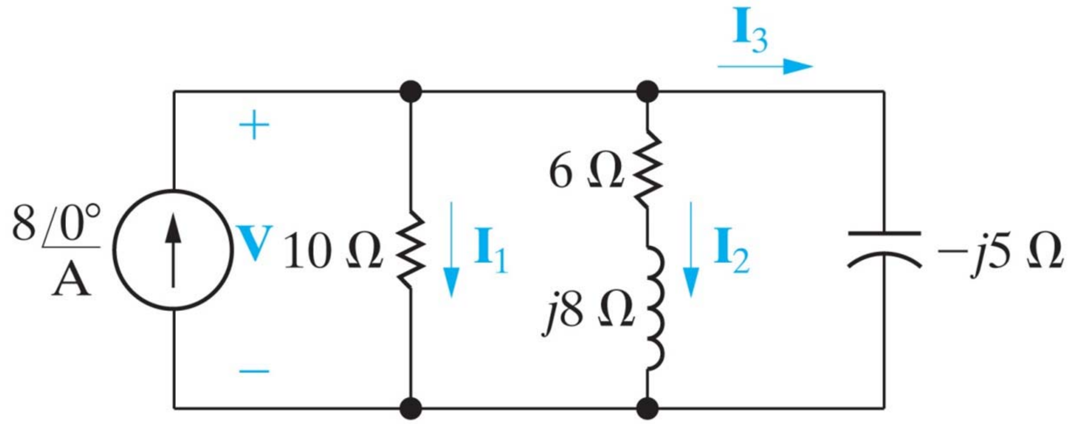
$$Z_L = j\omega L \quad = j(2\times10^5 rad)(40\times10^{-6} H) \quad = j8\Omega$$

For the capacitor

$$Z_C = \frac{1}{j\omega C} = \frac{-j}{\omega C} \quad = \frac{-j}{(2\times10^5 rad)(1\times10^{-6} F)} \quad = -j5\Omega$$

Sinusoidal Response Example

The frequency-domain equivalent circuit.



Find the steady-state expressions for v, i_1, i_2, i_3.

Since admittances add in parallel, first find the admittance of each branch.

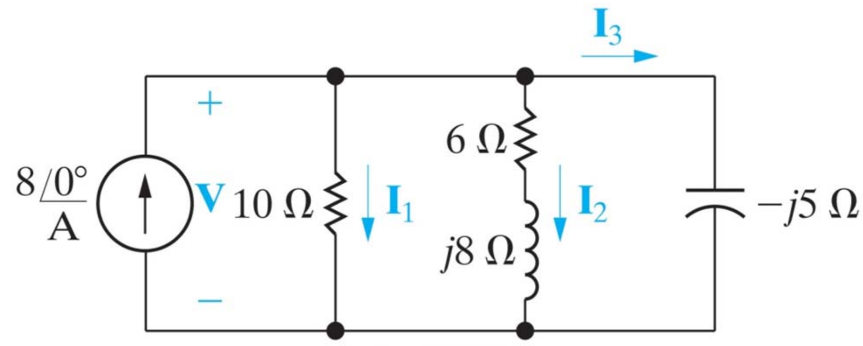
$$Y_R = \frac{1}{R} \qquad = 0.1\,\mho$$

$$Y_{R+L} = \frac{1}{R+Z_L} = \frac{1}{6+j8\,\Omega} \quad = \frac{1}{6+j8\,\Omega}\left(\frac{6-j8\,\Omega}{6-j8\,\Omega}\right) \quad = \frac{6-j8}{36+64}\,\mho$$

$$= \frac{6-j8}{100}\,\mho \qquad = 0.06 - j0.08\,\mho$$

$$Y_C = \frac{1}{Z_C} = \frac{1}{\frac{1}{j\omega C}} \quad = \frac{1}{-j5\Omega} \quad = j0.2\,\mho$$

Sinusoidal Response Example



The equivalent admittance is

$$Y_{equiv} = Y_R + Y_{R+L} + Y_C \qquad = \left(0.1 + 0.06 - j0.08 + j0.2\right)\mho$$

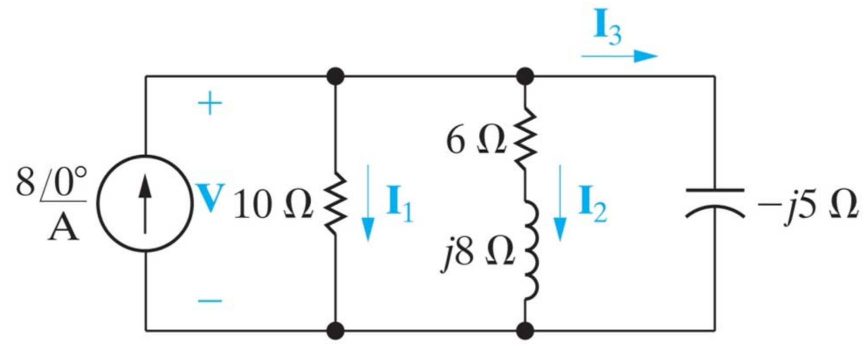
$$= 0.16 + j0.12\,\mho \qquad = \sqrt{0.16^2 + 0.12^2}\angle \tan^{-1}\tfrac{0.12}{0.16}$$

$$= 0.2\angle 36.87^\circ\,\mho$$

The equivalent impedance is

$$Z_{equiv} = \frac{1}{Y_{equiv}} \quad = \frac{1}{0.2\angle 36.87^\circ\,\mho} \quad = 5\angle -36.87^\circ\,\Omega$$

Sinusoidal Response Example



Now we can write the expression for v.

$$\mathbf{V} = Z_{equiv}\mathbf{I} \quad = \left(5\angle -36.87^{\circ}\,\Omega\right)\left(8\angle 0^{\circ}\,A\right)$$

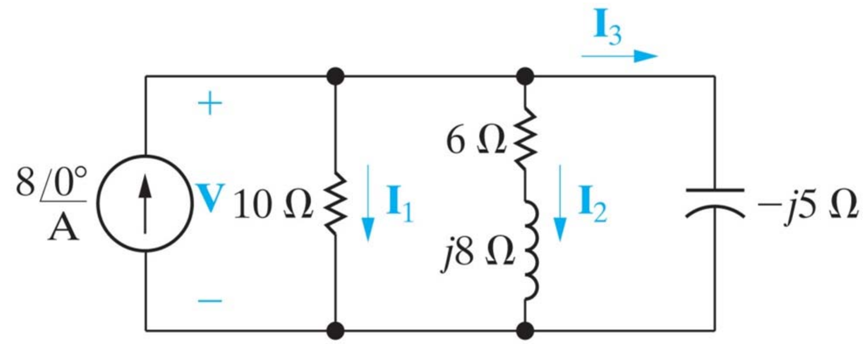
$$= 40\angle -36.87^{\circ}\,Volts$$

The time-domain sinusoid is

$$v(t) = 40\cos(2\times 10^{5}t - 36.87^{\circ})\;Volts$$

Sinusoidal Response Example

$$\mathbf{V} = 40\angle -36.87^{\circ}\ Volts$$



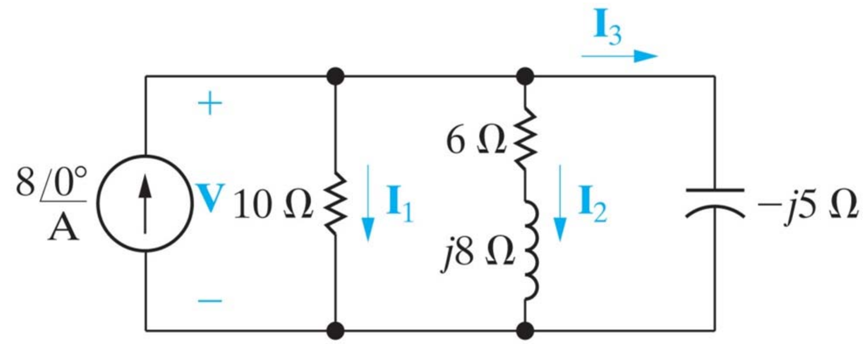
Now that we know the voltage, we can find the branch currents.

$$\mathbf{I}_1 = \frac{\mathbf{V}}{R} = \frac{40\angle -36.87^{\circ} Volts}{10\,\Omega} = 4\angle -36.87^{\circ}\ A$$

The time-domain sinusoid is

$$i_1(t) = 4\cos(2\times 10^5 t - 36.87^{\circ})\ \ Amp$$

Sinusoidal Response Example



$$\mathbf{V} = 40\angle -36.87^\circ\, Volts$$

$$\mathbf{I}_2 = \frac{\mathbf{V}}{Z_{R+L}} = \frac{40\angle -36.87^\circ Volts}{6 + j8\,\Omega} = \frac{40\angle -36.87^\circ Volts}{10\angle 53.13^\circ\ \Omega}$$

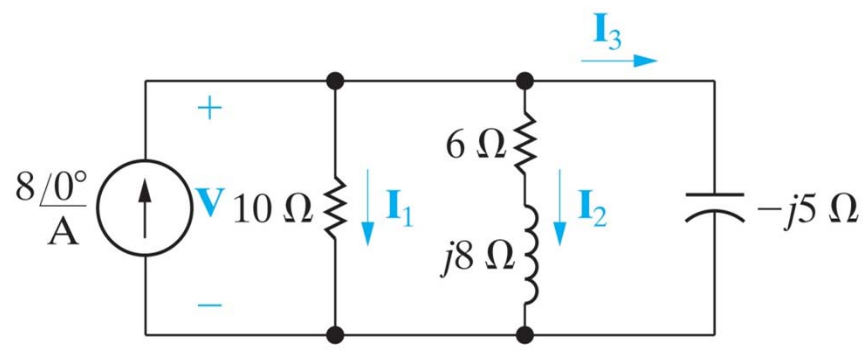
$$= 4\angle -90^\circ\ Amp$$

The time-domain sinusoid is

$$i_2(t) = 4\cos(2\times 10^5 t - 90^\circ)\ Amp$$

Sinusoidal Response Example

$$i_2(t) = 4\cos(2\times10^5 t - 90^\circ)\ Amp$$



A trig identity may help you move from cosine to sine when desired.

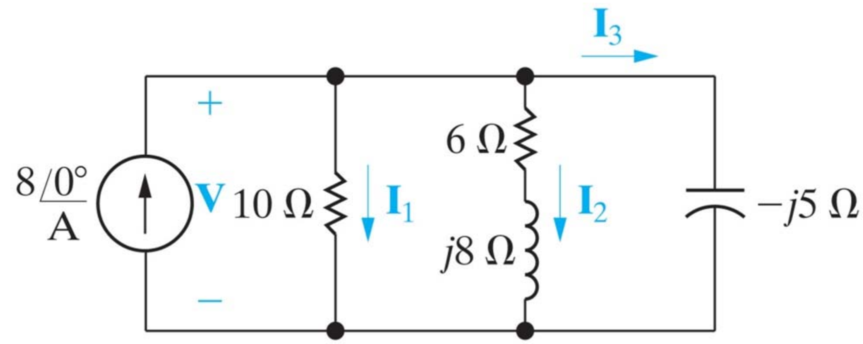
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\cos(2\times10^5 t - 90^\circ) = \cos\left(2\times10^5 t\right)\underbrace{\cos\left(90^\circ\right)}_{=0} + \sin\left(2\times10^5 t\right)\underbrace{\sin\left(90^\circ\right)}_{=1}$$

Thus

$$i_2(t) = 4\sin(2\times10^5 t)\ Amp$$

Sinusoidal Response Example



$$\mathbf{V} = 40\angle -36.87^{\circ}\ Volts$$

$$\mathbf{I}_3 = \frac{\mathbf{V}}{Z_C} = \frac{40\angle -36.87^{\circ} Volts}{-j5\,\Omega} = \frac{40\angle -36.87^{\circ} Volts}{5\angle -90^{\circ}\ \Omega}$$

$$= 8\angle 53.13^{\circ}\ Amp$$

The time-domain sinusoid is

$$i_3(t) = 8\cos(2\times 10^5 t + 53.13^{\circ})\ Amp$$

Section 9.5
Kirchhoff's Laws
in the Frequency Domain

Kirchhoff's Laws in the Frequency Domain

Kirchhoff’s Laws all apply in the frequency domain.

But with the tools we have at this time, we must solve for the response for each frequency in/of a source individually.

Supposition is assumed to apply.

In most problems at this stage, we will have only one input and will solve the circuit at that one frequency.

Section 9.6

Series and Parallel Simplifications

Series and Parallel Impedances

The good news is that all the rules for resistors apply to impedances.

The bad news is that you now have to solve the series/parallel equations with complex numbers.

You will become very adept at both the rectangular and polar form of complex numbers.

$$\text{rectangular form} \rightarrow \mathrm{x + j\ y} \qquad A\angle\theta = \sqrt{x^2 + y^2}\angle\tan^{-1}\frac{y}{x}$$

$$\text{polar form} \rightarrow A\angle\pm\theta = A\left[\cos\theta \pm j\sin\theta\right]$$

You must absolutely know in which quadrant the angle θ belongs!

Complex Numbers Example

Given: -3 – j4.

What is the polar form?

$$A\angle\theta = \sqrt{3^2 + 4^2}\angle\tan^{-1}\frac{-4}{-3}$$

$$\text{A}=\sqrt{9+16} = \sqrt{25} = 5$$

The angle is in the 3rd quadrant (-x and –y).

$$\tan^{-1}\frac{-4}{-3} = 53.13^{\circ}??$$

My calculator gave me an answer in the first quadrant – i.e. $\tan^{-1}(4/3)$

Correct angle is 53.13° + 180° = 233.13°
or equally 53.13° - 180° = -126.87°

Complex Numbers Example

What about going from polar to rectangular?

Given: $5\angle -126.87^{\circ}$

$$5\angle -127^{\circ} = 5\cos(126.87^{\circ}) - j5\sin(126.87^{\circ})$$

$$= 5(-0.6) - j5(0.8) = -3 - j4$$

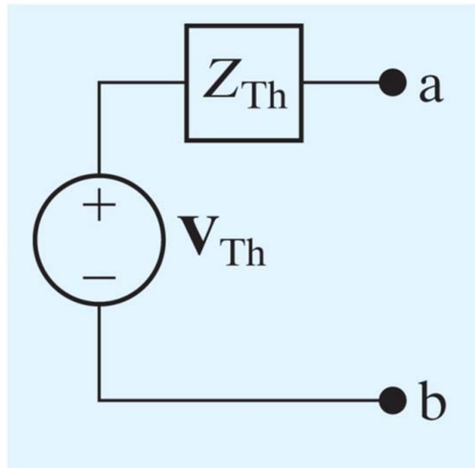
Given: $5\angle 233.13^{\circ}$

$$5\angle 233.13^{\circ} = 5\cos(233.13^{\circ}) + j5\sin(233.13^{\circ})$$

$$= 5(-0.6) + j5(-0.8) = -3 - j4$$

Section 9.7
Source Transformations and
Thévenin-Norton Equivalent Circuits

The Thévenin equivalent circuit is in the form

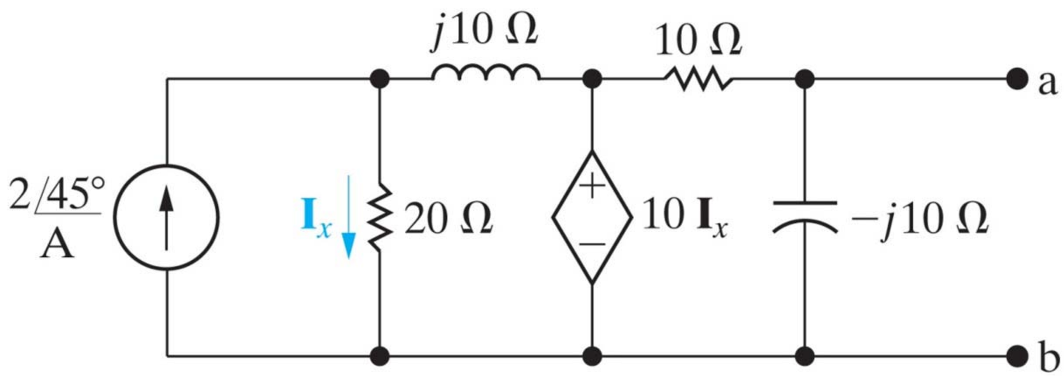


The Thévenin voltage is found under the open-circuit condition at terminals ab.

The Thévenin equivalent impedance is found look into terminals ab.

Thévenin Equivalent Example – Phasor Domain

Find the frequency-domain Thévenin equivalent of the following circuit.



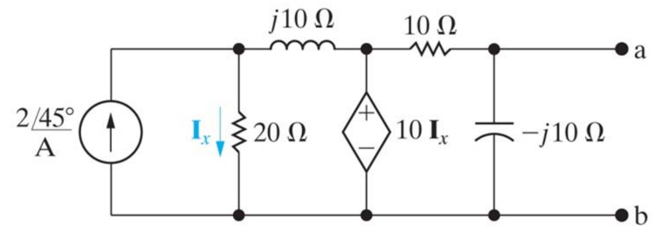
Since we have an independent current source, first write the node equation and let $V_{20\Omega} = V_1$.

$$\frac{V_1}{20} - 2\angle 45^\circ + \frac{V_1 - 10I_x}{j10} = 0$$

The constraint equation for the dependent voltage source is

$$10I_x(Volts) = 10\frac{V_1}{20} \quad = \frac{V_1}{2}$$

Thévenin Equivalent Example – Phasor Domain



$$10I_x = \frac{V_1}{2}Volts$$

Substitute in for the constraint equation.

$$\frac{V_1}{20} - 2\angle 45^\circ + \frac{V_1 - 10I_x}{j10} = 0$$

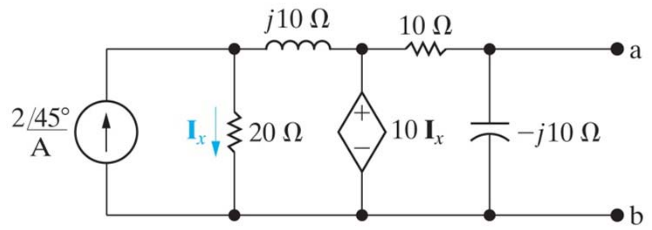
$$\frac{V_1}{20} - 2\angle 45^\circ + \frac{V_1 - \frac{V_1}{2}}{j10} = 0$$

Now solve for = V_1.

$$V_1\left(\frac{1}{20} + \frac{1}{j10} - \frac{1}{j20}\right) = 2\angle 45^\circ$$

$$V_1\left(\frac{1}{20} + \frac{1}{j20}\right) = 2\angle 45^\circ \qquad \Rightarrow V_1\left(0.05 - j0.05\right) = 2\angle 45^\circ$$

Thévenin Equivalent Example – Phasor Domain



$$V_1\left(0.05 - j0.05\right) = 2\angle 45^\circ$$

Continue to simplify.

$$V_1 = \frac{2\angle 45^\circ}{0.05 - j0.05} = \frac{2\angle 45^\circ}{\frac{1}{10\sqrt{2}}\angle -45^\circ} = 20\sqrt{2}\angle 90^\circ = j20\sqrt{2}$$

Now solve for V_{Th}.

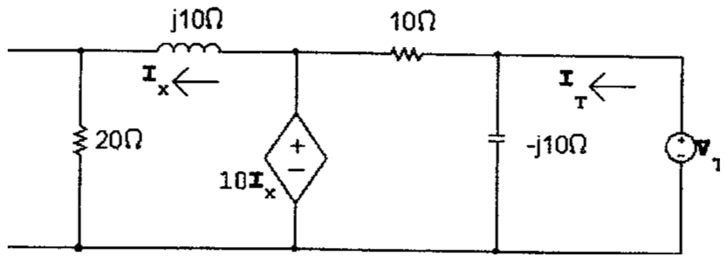
$$V_{Th} = \frac{-j10}{10 - j10}10I_x = \frac{-j10}{10 - j10}\left(\frac{V_1}{2}\right) = \frac{-j10}{10 - j10}\left(\frac{20\sqrt{2}\angle 90^\circ}{2}\right)$$

$$= \frac{10\angle -90^\circ}{10\sqrt{2}\angle -45^\circ}\left(10\sqrt{2}\angle 90^\circ\right) = \left(\frac{1}{\sqrt{2}}\angle -45^\circ\right)\left(10\sqrt{2}\angle 90^\circ\right)$$

$$= 10\angle 45^\circ \, Volts$$

Thévenin Equivalent Example – Phasor Domain

To find the Thévenin impedance, deactivate all independent sources and use the following circuit and apply a test voltage at terminals ab.



First find the constraint equation for the dependent voltage source.

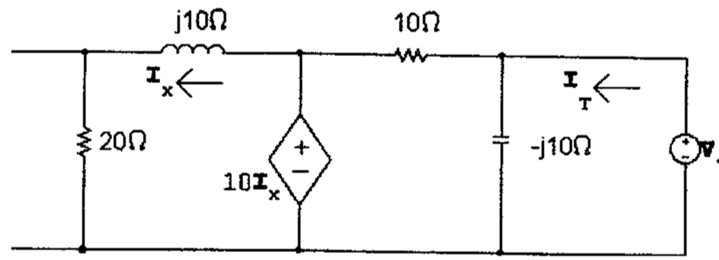
$$10I_x(voltage) = (20 + j10\Omega)I_{x\_current}$$

Only when $I_x = 0$ is the above equation is true.

If $I_x = 0$, then $10I_x = 0$. The entire left-side of the circuit is "grounded". By node analysis we have

$$-I_T + \frac{V_T}{-j10} + \frac{V_T}{10} = 0 \qquad \Rightarrow V_T\left(\frac{1}{-j10} + \frac{1}{10}\right) = I_T$$

Thévenin Equivalent Example – Phasor Domain



$$V_T\left(\frac{1}{10}-\frac{1}{j10}\right)=I_T$$

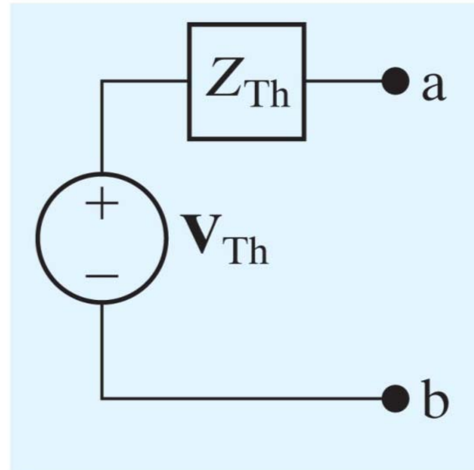
The Thévenin impedance is the ratio of V_T to I_T.

$$Z_{Th}=\frac{V_T}{I_T}=\left(\frac{1}{10}-\frac{1}{j10}\right)^{-1}=\left(0.1+j0.1\right)^{-1}=\left(0.1\sqrt{2}\angle 45^\circ\right)^{-1}$$

$$=\frac{10}{\sqrt{2}}\angle -45^\circ=\frac{10}{\sqrt{2}}\left[\cos 45^\circ - j\sin 45^\circ\right]=\frac{10}{\sqrt{2}}\left[\tfrac{1}{\sqrt{2}}-j\tfrac{1}{\sqrt{2}}\right]$$

$$=5-j5\,\Omega$$

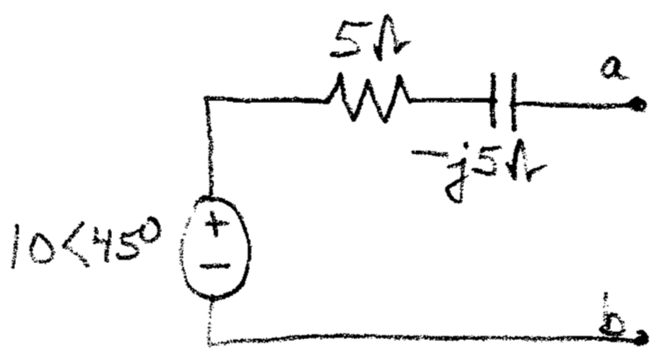
Thévenin Equivalent Example – Phasor Domain



$$V_{Th} = 10\angle 45^{\circ}\,Volts$$

$$Z_{Th} = 5 - j5\,\Omega$$

The Thévenin equivalent circuit is



Section 9.8
The Node-Voltage Method

Section 9.9
The Mesh-Current Method

The phasor form of the impedance is very useful as long as we want only the steady-state response.

All the rules and conventions acquired so far also apply to the phasor form – but remember – phasors are only for the steady state response!

Later work (in other courses) will include Laplace and Fourier forms which will allow transient and steady-state response analysis of time-varying systems.

Section 6.4

Mutual Inductance

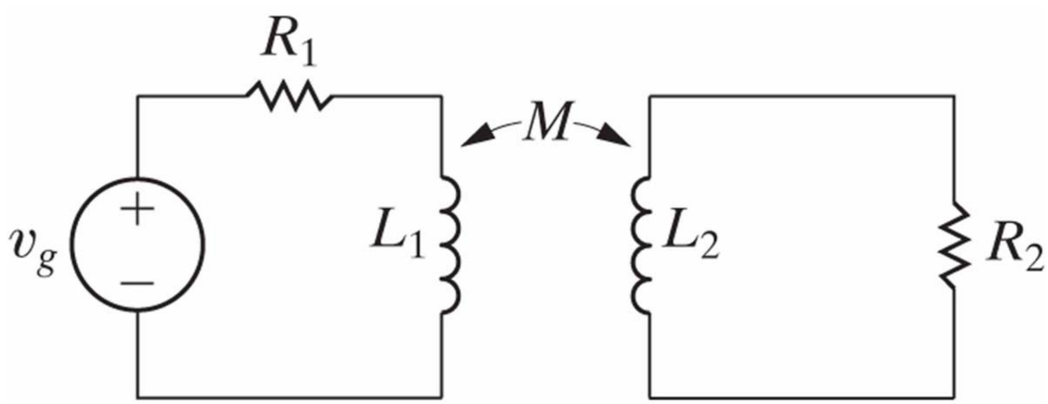
Mutual Inductance

We previously looked at the effect of a moving charge (current) creating a magnetic field which is called inductance.

Since the effect was the current in a single circuit, it should be properly called *self-inductance.*

This section looks at the situation where the magnetic field of two or more circuits are linked.

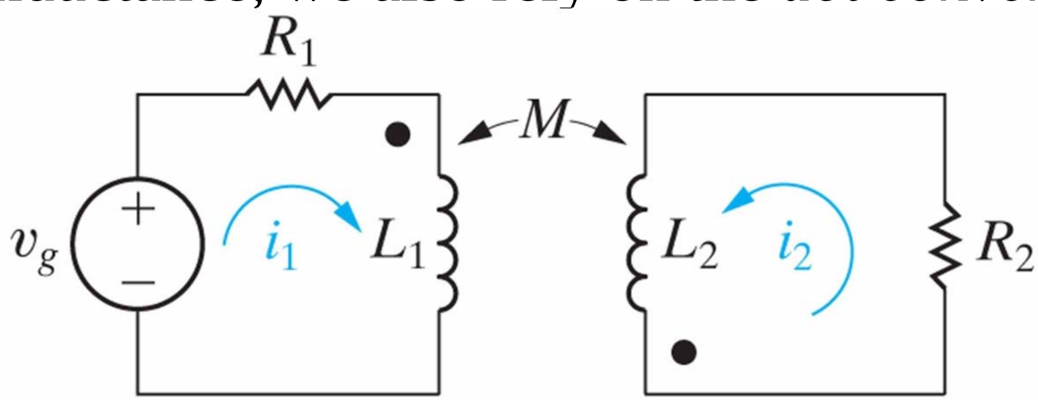
When the magnetic field is linked among circuits by a time varying current then there is *mutual inductance.*



Mutual Inductance

We rely on the passive sign convention to determine voltage versus current direction.

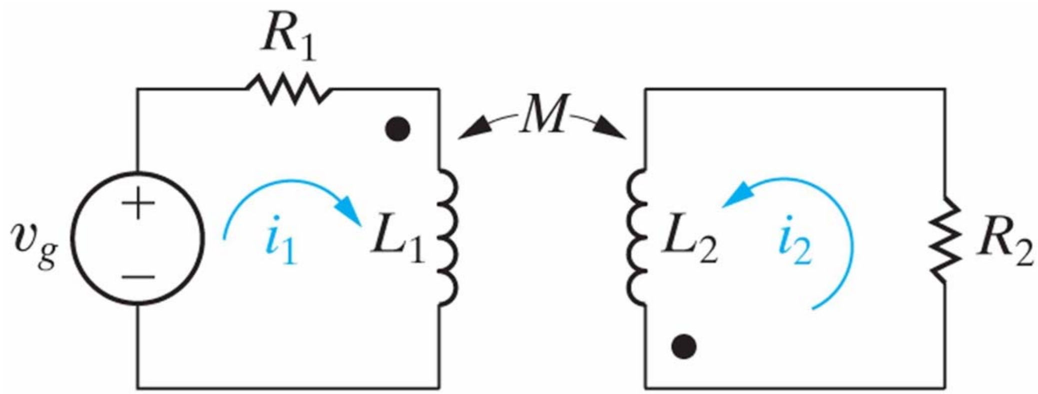
For mutual inductance, we also rely on the *dot convention*..



The polarity of mutually induced voltage depends on the way the coils are wound in relation to the reference direction of coil currents.

So *dots* are placed on the terminals to carry the polarity information schematically (rather than knowing righty or lefty directions).

Dot Convention



Dot Convention

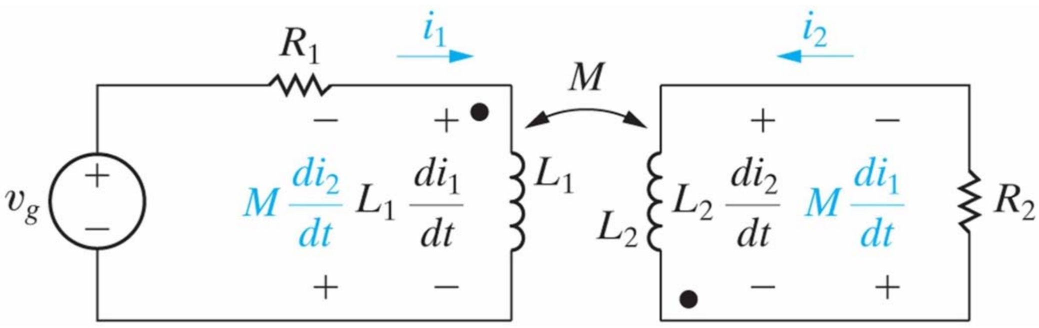
When the reference direction for a current enters the dotted terminal of a coil,

the reference polarity of the voltage that it induces in the other coil is positive at its (the other coil) dotted terminal.

In this course, we will use the dots. Figuring out how to correctly place the dots is left to other courses (such as EEE 130).

Mutual Inductance

In a mutual inductance circuit, we have new voltages to now track.



KVL on the left hand side gives us

$$-v_g + i_1 R_1 + L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = 0$$

KVL on the right hand side gives us

$$-L_2\frac{di_2}{dt} - i_2 R_2 + M\frac{di_1}{dt} = 0$$

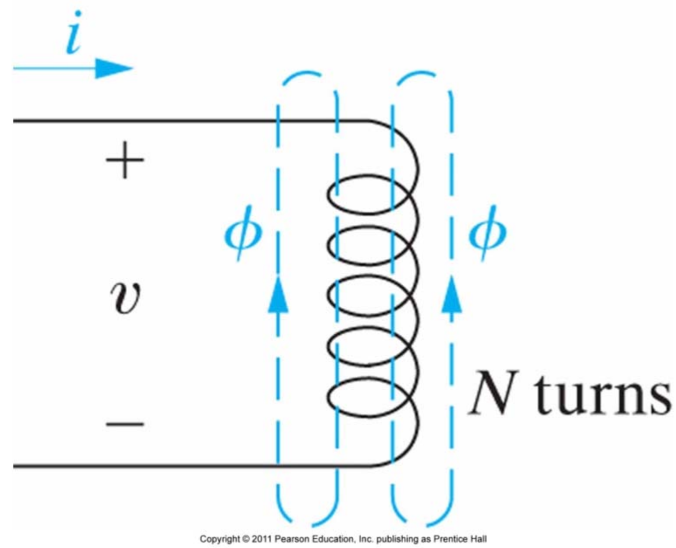
Or equally (multiply last result by -1)

$$L_2\frac{di_2}{dt} + i_2 R_2 - M\frac{di_1}{dt} = 0$$

Closer Look at Mutual Inductance (briefly)

The voltage induced by the magnetic field surrounding a current carrying conductor is described by Faraday's Law:

$$v = \frac{d\lambda}{dt}$$



Where λ is called the magnetic flux linkage – measured in weber-turns.

The flux linkage λ is the product of the magnetic field (ϕ) and the number of turns linked (N)

$$\lambda = N\phi$$

Closer Look at Mutual Inductance (briefly)

The magnetic field strength per unit volume depends on the material in that space - described by the *permeance* of the material.

The text has more details, but the important message is the magnetic coupling M may not be perfect.

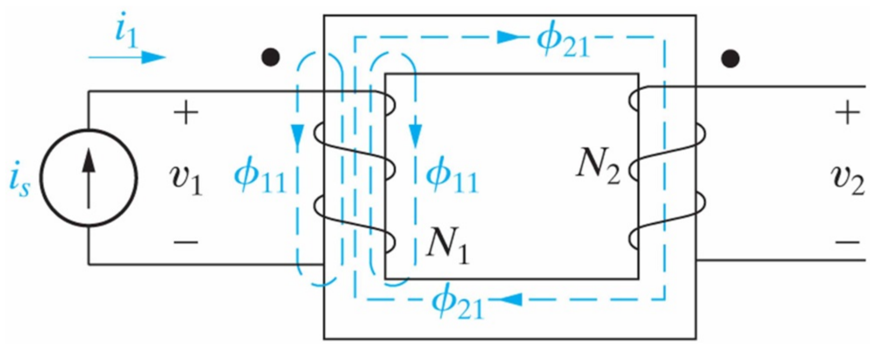
Thus we introduce a coefficient of coupling *k* which can be used to better model a specific physical circumstance.

$$M = k\sqrt{L_1 L_2}$$

Where $0 \leq k \leq 1$

In general, *k* is found by experimental measurement.

We thus have the following view of a mutually coupled circuit:

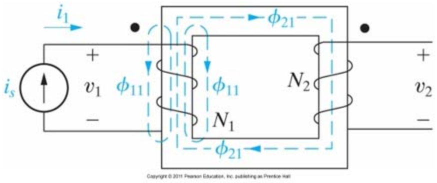


$$M = k\sqrt{L_1 L_2}$$

The energy stored in these two coils (self-inductance) and the energy stored in the coupled magnetic fields (mutual inductance) has the form:

$$w(t) = \frac{1}{2}L_1 i_1^2 + \frac{1}{2}L_2 i_2^2 \pm M i_1 i_2$$

Example – Problem 6.46 (9th edition)



Two magnetically coupled coils have:

$$L_1 = 60mH \qquad L_2 = 9.6mH \qquad M = 22.8mH$$

What is the coefficient of coupling k?

$$M = k\sqrt{L_1L_2} \qquad \Rightarrow k = \frac{M}{\sqrt{L_1L_2}} = \frac{22.8mH}{\sqrt{(60mH)(9.6mH)}} = 0.95$$

What is the largest value of M?

The largest value of M occurs when k = 1 (perfect coupling).

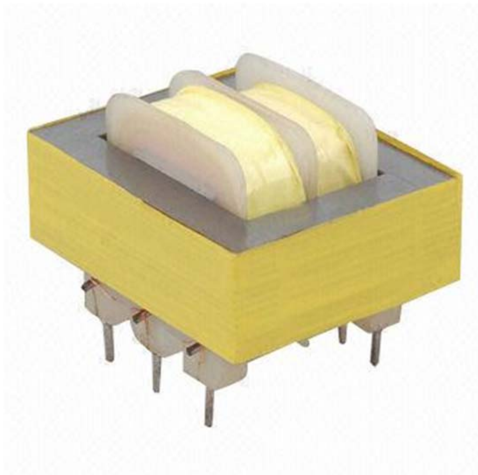
$$M = k\sqrt{L_1L_2} \quad = (\mathrm{k}=1)\sqrt{L_1L_2} \quad = \sqrt{(60mH)(9.6mH)} = 24.0mH$$

Section 9.10
The Transformer

The Transformer

A transformer is a device that is based on magnetic coupling.

In communications, a transformer can be used to match impedance between two circuits and to eliminate dc signals.



http://hzsmart.manufacturer.globalsources.com/si/6008846485368/pdtl/LAN-coil/1060429948/Linear-Transformer.htm

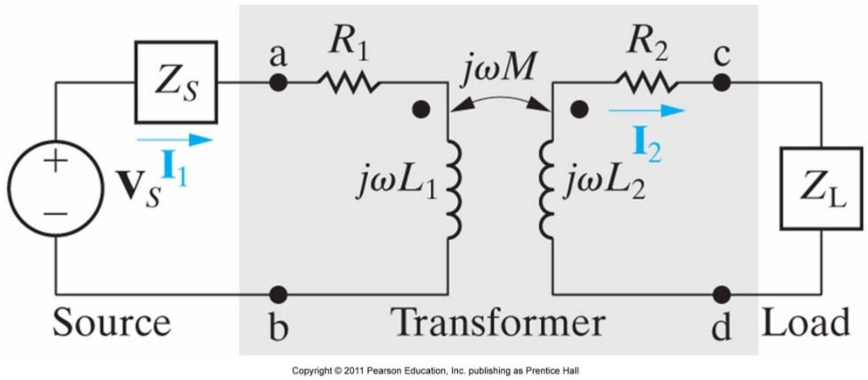
A very common use of transformers is in power systems to raise or lower voltage levels.

http://en.wikipedia.org/wiki/Transformer

The Transformer

Here is a typical transformer circuit.

Primary Side



Secondary Side

R_1 = the resistance in the primary winding

L_1 = the self-inductance of the primary winding

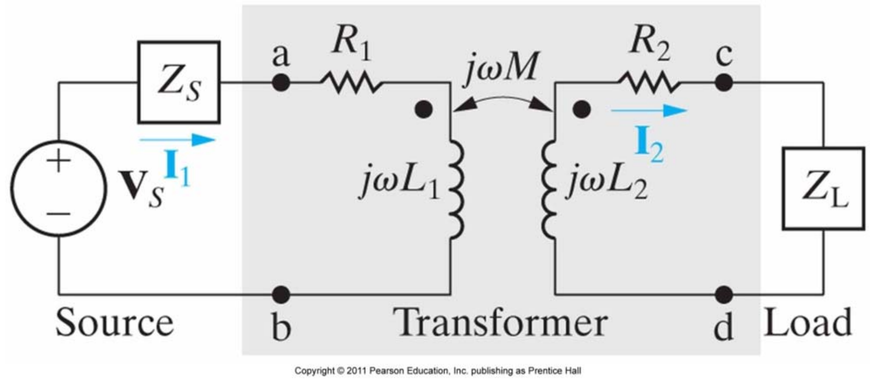
$j\omega M$ = the frequency dependent mutual inductance

R_2 = the resistance in the secondary winding

L_2 = the self-inductance of the secondary winding

The Transformer

Primary Side



Secondary Side

The internal voltage of the sinusoidal source is $\mathbf{V}_S$.

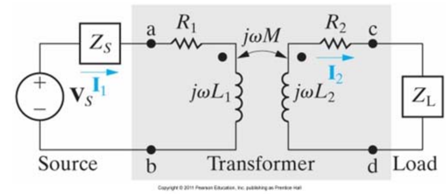
The internal impedance of the source is Z_S.

The impedance Z_L is the load connected to the secondary side.

The phasor current $\mathbf{I}_1$ is on the primary side.

The phasor current $\mathbf{I}_2$ is on the secondary side.

The Transformer



Mesh analysis of the transformer finds the two mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$.

On the primary side:

$$-V_S + I_1 Z_S + I_1 R_1 + I_1 j\omega L_1 - j\omega M I_2 = 0$$

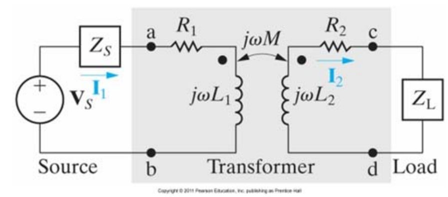
$$I_1\left(\underbrace{Z_S + R_1 + j\omega L_1}_{Z_{11}}\right) - j\omega M I_2 = V_S$$

On the secondary side:

$$I_2 j\omega L_2 - j\omega M I_1 + I_2 R_2 + I_2 Z_L = 0$$

$$I_2\left(\underbrace{j\omega L_2 + R_2 + Z_L}_{Z_{22}}\right) - j\omega M I_1 = 0$$

The Transformer



Now solve for the two mesh currents $\mathbf{I}_1$ and $\mathbf{I}_2$.

From the secondary side:

$$I_2\left(\underbrace{j\omega L_2 + R_2 + Z_L}_{Z_{22}}\right) - j\omega M I_1 = 0 \qquad \Rightarrow I_2 = \frac{j\omega M I_1}{Z_{22}}$$

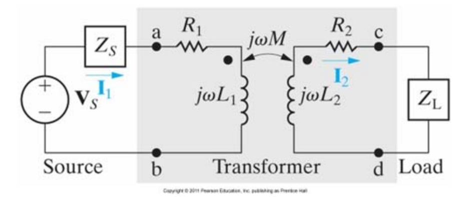
Substitute the last result back into the equation for $\mathbf{I}_1$.

$$I_1 Z_{11} - j\omega M I_2 = V_S \qquad \Rightarrow I_1 Z_{11} - j\omega M\left(\frac{j\omega M I_1}{Z_{22}}\right) = V_S$$

$$I_1\left(Z_{11} + \frac{\omega^2 M^2}{Z_{22}}\right) = V_S \qquad \Rightarrow I_1\left(\frac{Z_{11}Z_{22} + \omega^2 M^2}{Z_{22}}\right) = V_S$$

The Transformer

Simplify the last result and write the primary current



$$I_1\left(\frac{Z_{11}Z_{22}+\omega^2M^2}{Z_{22}}\right)=V_S \qquad \Rightarrow I_1=\left(\frac{Z_{22}}{Z_{11}Z_{22}+\omega^2M^2}\right)V_S$$

For the secondary side:

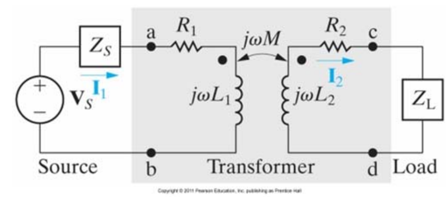
$$I_2Z_{22}=j\omega MI_1 \qquad \Rightarrow I_2=\frac{j\omega MI_1}{Z_{22}}$$

Or equivalently

$$I_2=\left(\frac{Z_{22}}{Z_{11}Z_{22}+\omega^2M^2}\right)V_S\left(\frac{j\omega M}{Z_{22}}\right) \qquad =\left(\frac{j\omega M}{Z_{11}Z_{22}+\omega^2M^2}\right)V_S$$

The Transformer

Now we turn our attention to terminal behavior.



To the internal source voltage $\mathbf{V}_S$, the impedance appears as

$$\frac{V_S}{I_1} = \frac{Z_{11}Z_{22} + \omega^2 M^2}{Z_{22}} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}}$$

At the external terminals ab, the impedance is

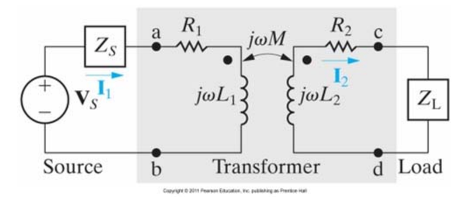
$$Z_{ab} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}} - Z_S \quad = \left(\underbrace{Z_S + R_1 + j\omega L_1}_{Z_{11}} \right) + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L} - Z_S$$

$$= R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}$$

So we see that the impedance seen by the source at terminals ab depends on both the primary and secondary side impedances.

Reflected Impedance

Look at the right hand term of the input impedance

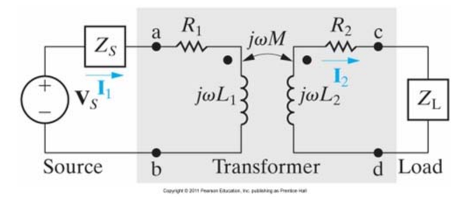


$$Z_{ab} = R_1 + j\omega L_1 + \underbrace{\frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}}_{reflected\ \ impedance = Z_r}$$

That right hand term is the equivalent impendence Z_r of the secondary coil and the load impedance reflected back to the primary side by the coupling of the magnetic fields.

Note that the equation is, in effect, balancing the source's energy input into both the primary and secondary

Reflected Impedance



$$Z_r = \frac{\omega^2 M^2}{R_2 + j\omega L_2 + Z_L}$$

By power engineering custom, we call the impedance of the secondary side Z_{22}.

$$Z_{22} = R_2 + j\omega L_2 + Z_L \qquad \text{thus } Z_r = \frac{\omega^2 M^2}{Z_{22}}$$

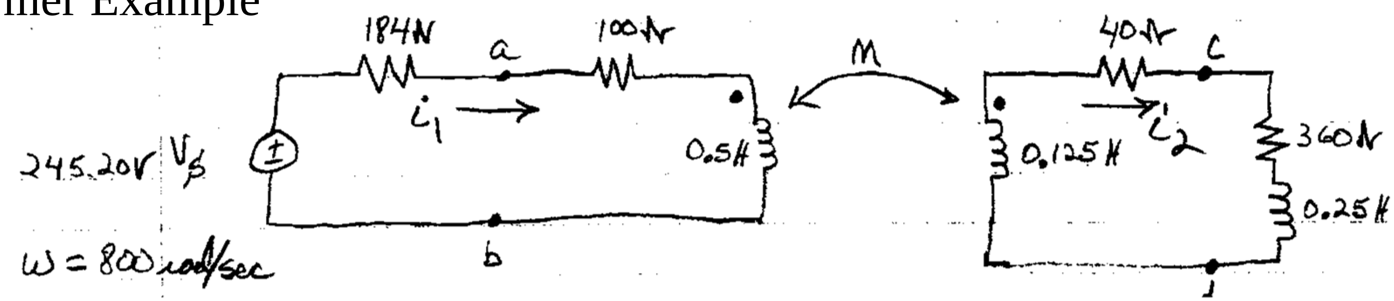
We can write Z_L in rectangular form $Z_L = R_L + jX_L$ so we can rewrite Z_r into a slightly more convenient form.

$$Z_{22} = \left(R_2 + R_L\right) + j\left(\omega L_2 + X_L\right)$$

$$Z_r = \frac{\omega^2 M^2}{\left(R_2 + R_L\right) + j\left(\omega L_2 + X_L\right)} \frac{\left(R_2 + R_L\right) - j\left(\omega L_2 + X_L\right)}{\left(R_2 + R_L\right) - j\left(\omega L_2 + X_L\right)}$$

$$Z_r = \frac{\omega^2 M^2}{\left|Z_{22}\right|^2}\left[\left(R_2 + R_L\right) - j\left(\omega L_2 + X_L\right)\right] \quad = \frac{\omega^2 M^2}{\left|Z_{22}\right|^2} Z_{22}^*$$

Transformer Example



Given k = 0.4

$$M = k\sqrt{L_1 L_2} \quad = 0.4\sqrt{(0.5H)(0.125H)} \quad = 0.1H$$

a) Find the reflected impedance Z_r from the secondary side to the primary side.

$$\omega M = 800\,\tfrac{rad}{\sec}(0.1H) \quad = 80\Omega$$

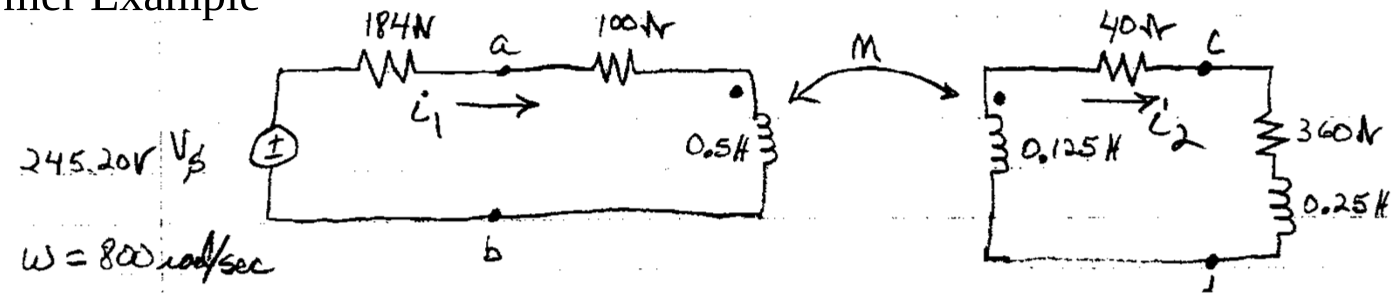
$$Z_{22} = j\omega L_2 + R_2 + Z_L \quad = j(800)(0.125)\Omega + 40\Omega + 360\Omega + j(800)(0.25)\Omega$$

$$= 400 + j300\Omega$$

$$Z_r = \frac{(\omega M)^2}{Z_{22}} \quad = \frac{(80\Omega)^2}{400 + j300\Omega} = \frac{6400\Omega^2}{500\angle 36.87^\circ\Omega} \quad = \underline{\underline{12.8\angle -36.87^\circ\Omega}}$$

$$= \underline{\underline{10.24 - j7.68\Omega}}$$

Transformer Example



b) Find the primary current I_1.

$$\frac{V_S}{I_1} = \frac{Z_{11}Z_{22} + \omega^2 M^2}{Z_{22}} = Z_{11} + \frac{\omega^2 M^2}{Z_{22}}$$

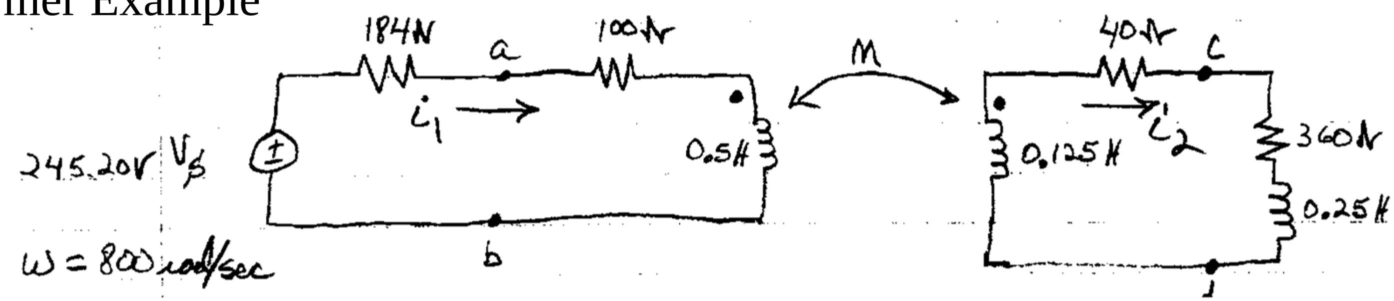
$$I_1 = \frac{V_s}{Z_{11} + \frac{(\omega M)^2}{Z_{22}}}$$ where $Z_{11} = (184 + 100 + j\omega L_1)\Omega$

$$= \frac{245.20\angle 0^\circ V}{(184 + 100 + j(800)(0.5H) + 10.24 - j7.68)\Omega}$$

$$= \frac{245.20\angle 0^\circ}{294.24 + j(400 - 7.68)} = \frac{245.20\angle 0^\circ}{294.24 + j392.32} = \frac{245.20\angle 0^\circ}{490.4\angle 53.13^\circ}$$

$$= \underline{\underline{0.5\angle -53.13^\circ}}\, Amps$$

Transformer Example

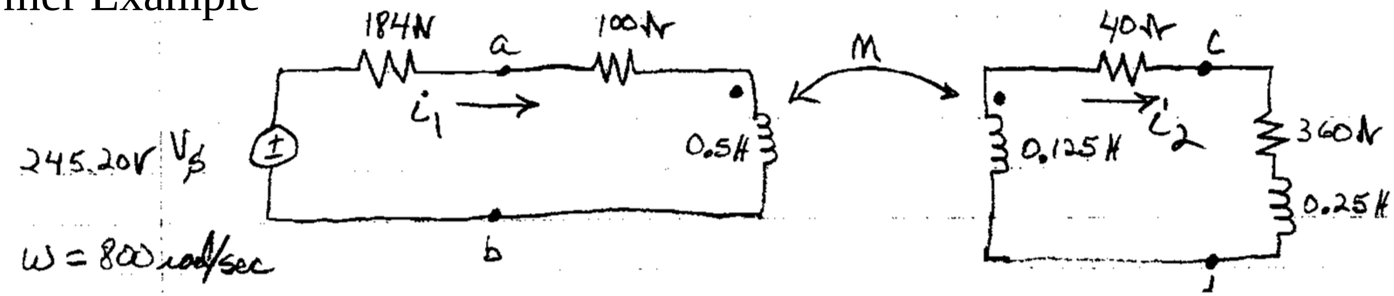


$$I_1 = \underline{\underline{0.5\angle -53.13^\circ}}\, Amps$$

What is the time domain expression for $i_1(t)$?

$$i_1(t) = 0.5\cos(800t - 53.13^\circ)\ Amps$$

Transformer Example



b) Find the secondary current I_2.

$$Z_{22} = \left(40 + j(800\tfrac{rad}{\text{sec}})(0.125H)\right) + \left(360 + j(800\tfrac{rad}{\text{sec}})(0.25H)\right)\Omega \;\; = 400 + j300\,\Omega$$

$$I_2 = \frac{j\omega M}{Z_{22}} I_1 \quad = \frac{j80\Omega}{400 + j300\Omega} 0.5\angle -53.13^\circ$$

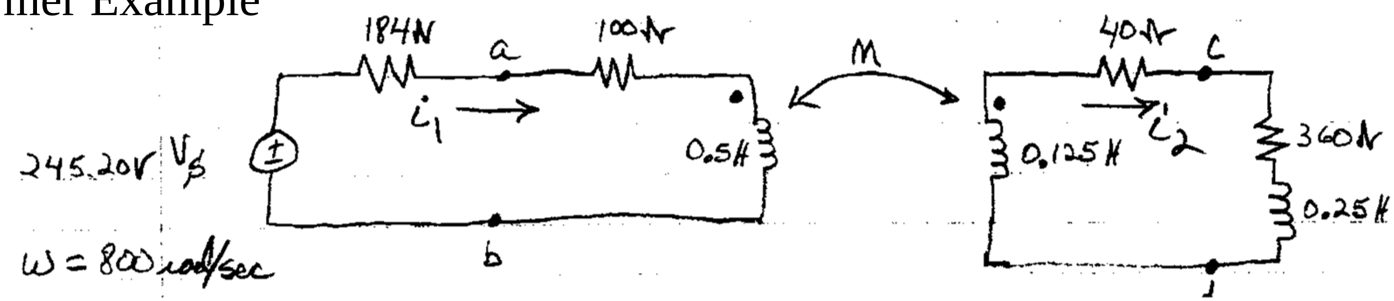
$$= \frac{80\angle 90^\circ}{500\angle 36.87^\circ} 0.5\angle -53.13^\circ \quad = (0.16\angle 53.13^\circ)(0.5\angle -53.13^\circ)$$

$$= 0.08\angle 0^\circ \; Amps \quad = \underline{\underline{80\angle 0^\circ \; mA}}$$

What is the time domain expression for $i_2(t)$?

$$i_2(t) = \underline{\underline{80\cos(800t)\, mA}}$$

Transformer Example



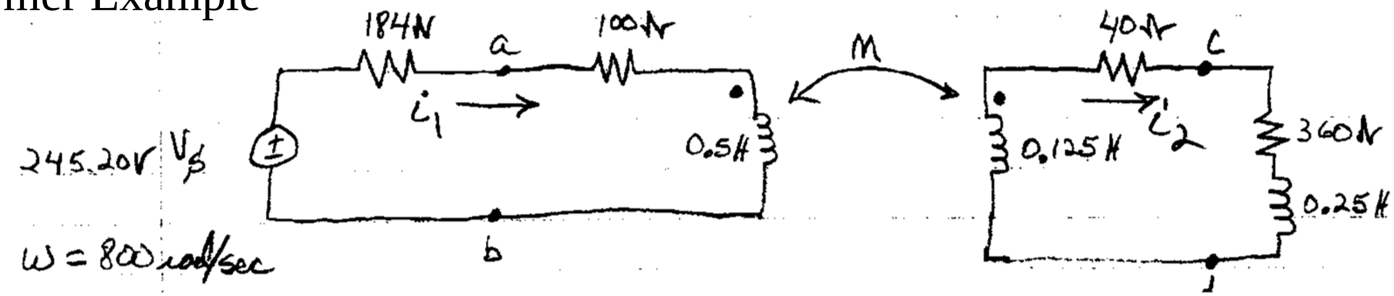
c) What is the impedance seen at terminals ab?

$$Z_{ab} = R_1 + j\omega L_1 + Z_r \qquad = 100 + j(800)(0.5) + (10.24 - j7.68)$$

$$= 100 + j400 + (10.24 - j7.68) \quad = 110.24 + j392.32\Omega$$

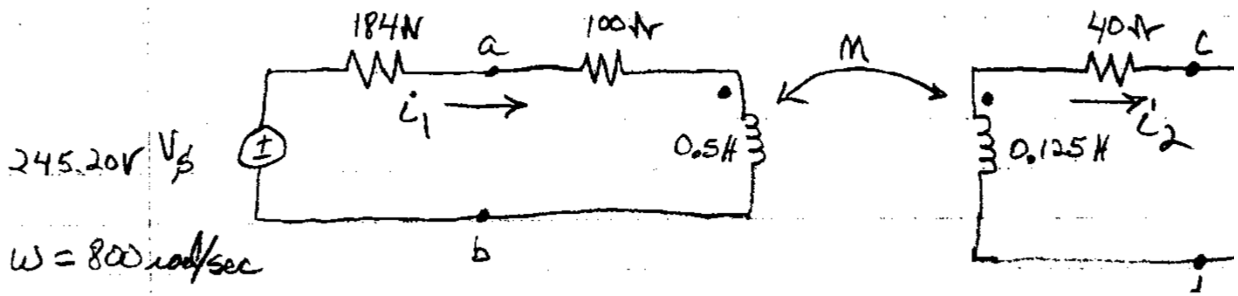
$$= \underline{\underline{407.51\angle 74.3^\circ \,\Omega}}$$

Transformer Example



d) What is the Thévenin equivalent at the terminals cd?

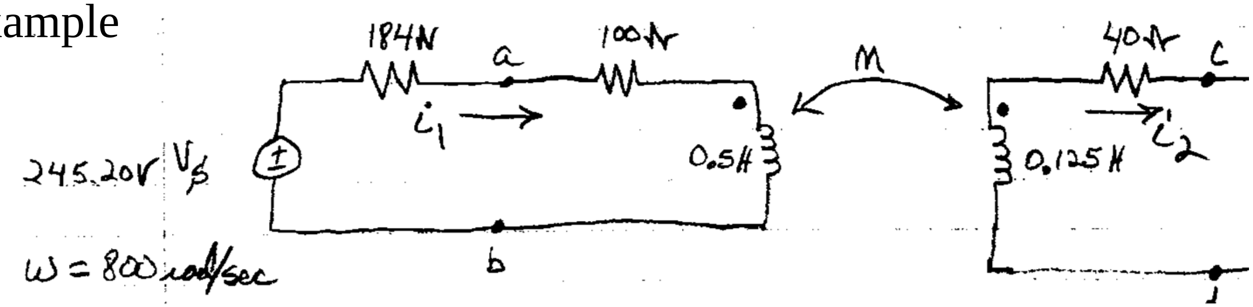
The terminals cd must be open circuited to find the Thévenin voltage.



Thus current I_2 = zero so there is no self-induced voltage across the coil 2.

$$V_{L,2} = j\omega M I_1 \qquad \text{Thus } V_{TH} = (j\omega M)I_{1(open \quad circuit)}$$

Transformer Example



The current I_1 in the primary side with the secondary side open circuited is

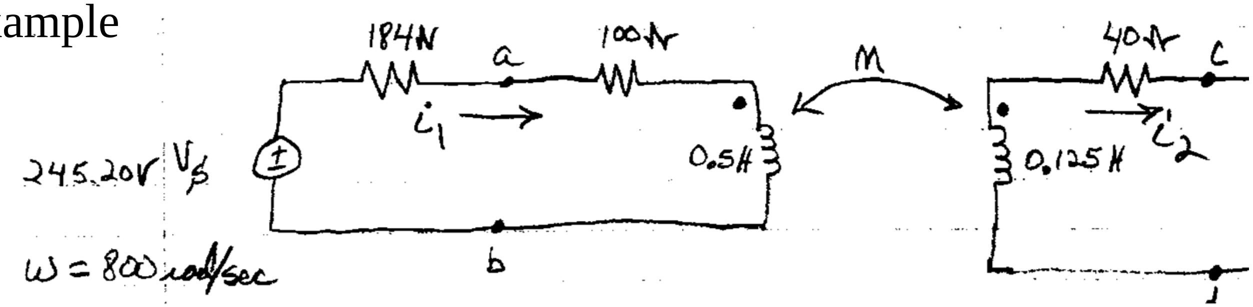
$$I_{1,oc} = \frac{V_s}{Z_{11}} = \frac{245.2\angle 0^\circ}{184+100+j(800)(0.5)} = \frac{245.2\angle 0^\circ}{284+j400} = \frac{245.2\angle 0^\circ}{490.57\angle 54.63^\circ}$$

$$= 0.5\angle -54.63^\circ\ Amps$$

So the Thévenin voltage is

$$V_{TH} = j\omega M I_{1,oc} = (80\angle 90^\circ\,\Omega)(0.5\angle -54.63^\circ\,A) = \underline{\underline{40\angle 35.37^\circ\ Volts}}$$

Transformer Example



Lastly we need the Thévenin equivalent impedance.

Z_{TH} is equal to the impedance of the secondary side PLUS the reflected impedance from the primary side.

Remember that the primary side creates a voltage on the secondary side. This voltage is due to the current I_1 flow thru the primary side impedance.

Thus we have a reflected impedance from the primary side!

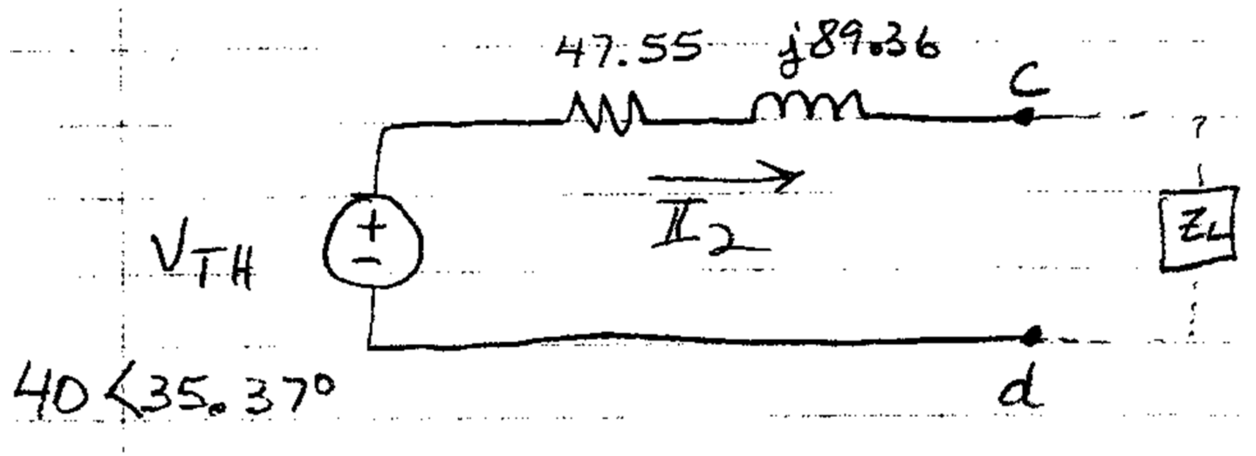
$$Z_{TH} = j\omega L_2 + R_2 + \frac{(\omega M)^2}{Z_{11}} = j(800)(0.125) + 40 + \frac{(80)^2}{284 + j400}$$

$$= j100 + 40 + \frac{6,400}{490.57\angle 54.63^\circ} = 40 + j100 + 13.05\angle -54.63^\circ$$

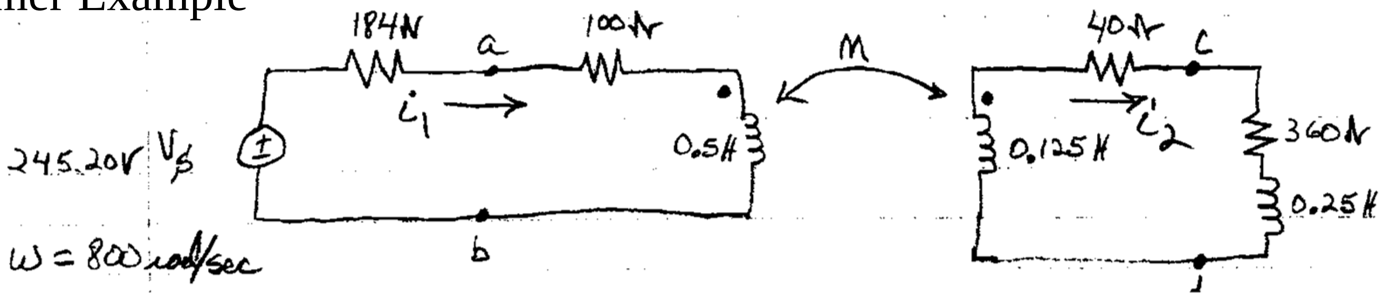
$$= 40 + j100 + 7.55 - j10.64 = \underline{\underline{47.55 + j89.36\,\Omega}}$$

Transformer Example

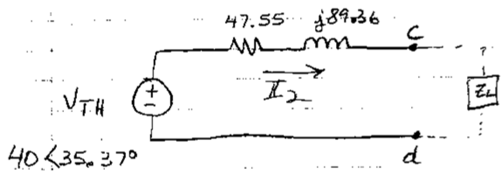
So the Thévenin equivalent circuit is



Transformer Example



As a check on the work, compare the I_2 found previously with the Thévenin equivalent values.



From the Thévenin circuit we have

$$I_2 = \frac{V_{TH}}{Z_{TH} + Z_L} = \frac{40\angle 35.37^\circ}{47.55 + j89.36 + 360 + j(800)(0.25)}$$

$$= \frac{40\angle 35.37^\circ}{407.55 + j89.36 + j200} = \frac{40\angle 35.37^\circ}{407.55 + j289.36}$$

$$= \frac{40\angle 35.37^\circ}{499.83\angle 35.37^\circ} = 0.08\angle 0^\circ \; Amps \quad = \underline{\underline{80\angle 0^\circ \; mA}} \;\; as \;\; found\ before!$$

Section 9.11

The Ideal Transformer

Ideal Transformer

An *ideal transformer* consists of two magnetically coupled coils having:

N_1 *turns* around the coil (all of which couple to the magnetic field) on the primary side.

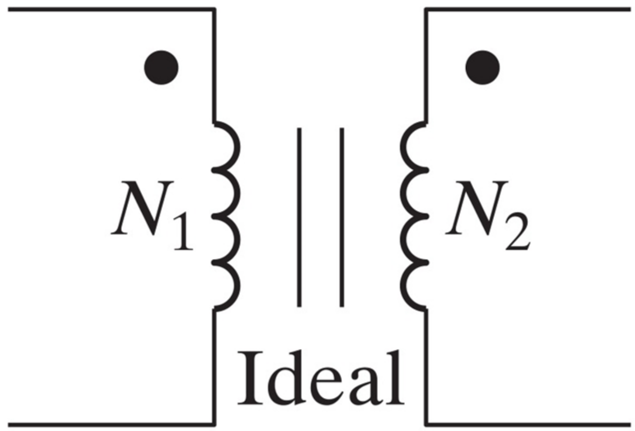
N_2 *turns* around the coil (all of which couple to the magnetic field) on the secondary side.

Coefficient of coupling $k = 1$ which is perfect coupling.

The self-inductance of each coil is infinite $\rightarrow L_1 = L_2 = \infty$

The resistance of each coil is negligible. Thus the *Parasitic* resistance $\rightarrow$ zero.

The text presents a fairly detailed derivation of the ideal transformer's *turns ratio*. Please review that derivation on your own.



The voltage relationship for an ideal transformer is $\dfrac{V_1}{N_1} = (\pm)\dfrac{V_2}{N_2}$

The current relationship is $I_1 N_1 = (\pm) I_2 N_2$

The voltage and current relationships obey the *conservation of energy* requirement.

Section 9.11 The Ideal Transformer

$$\frac{V_1}{N_1} = (\pm)\frac{V_2}{N_2} \qquad\qquad I_1 N_1 = (\pm) I_2 N_2$$

To determine the correct polarity of the voltage and current ratios use the *dot convention* for ideal transformers.

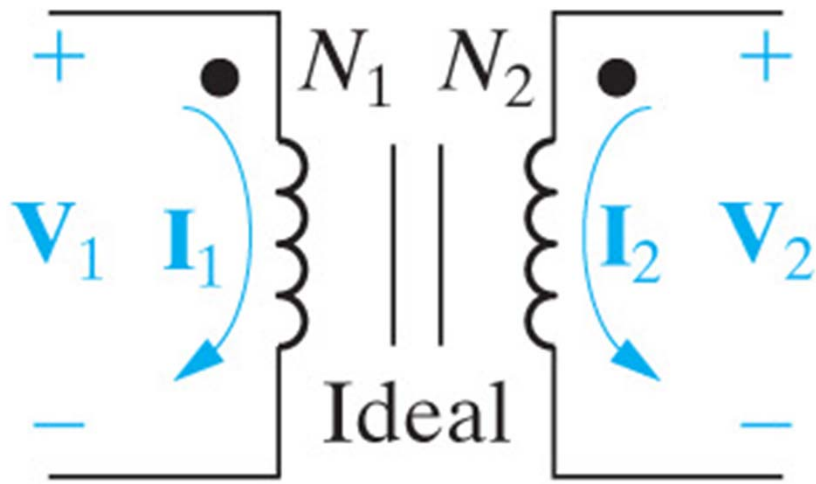
If the coil voltages V_1 and V_2 are both positive or both negative at the dot-marked terminal, then use a plus sign.
Otherwise, use a minus sign.

If the coil currents I_1 and I_2 are both directed into or both directed out of the dot-marked terminal, then use a minus sign.
Otherwise, use a plus sign.

Section 9.11 The Ideal Transformer

$$\frac{V_1}{N_1} = (\pm)\frac{V_2}{N_2} \qquad I_1 N_1 = (\pm) I_2 N_2$$

Case 1. Both voltages are positive at the dot-marked terminal.
Both currents are into the dot-marked terminal.



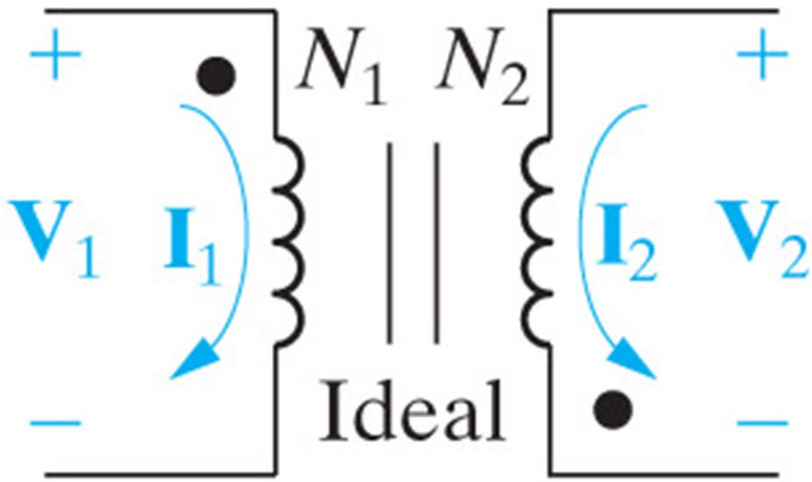
$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2},$$

$$N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$$

Section 9.11 The Ideal Transformer

$$\frac{V_1}{N_1} = (\pm)\frac{V_2}{N_2} \qquad I_1 N_1 = (\pm) I_2 N_2$$

Case 2. Only one voltage is positive at the dot-marked terminal.
Only one current is into the dot-marked terminal.

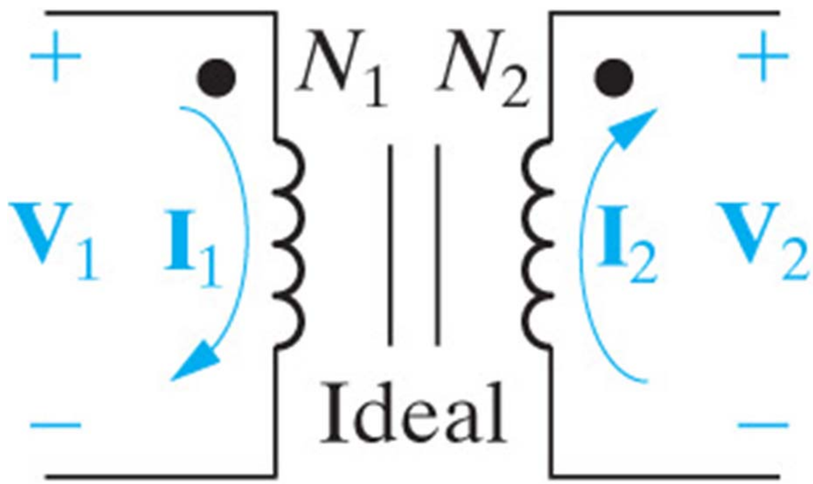


$$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2},$$

$$N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

$$\frac{V_1}{N_1} = (\pm)\frac{V_2}{N_2} \qquad I_1N_1 = (\pm)I_2N_2$$

Case 3. Both voltages are positive at the dot-marked terminal. Only one current is into the dot-marked terminal.

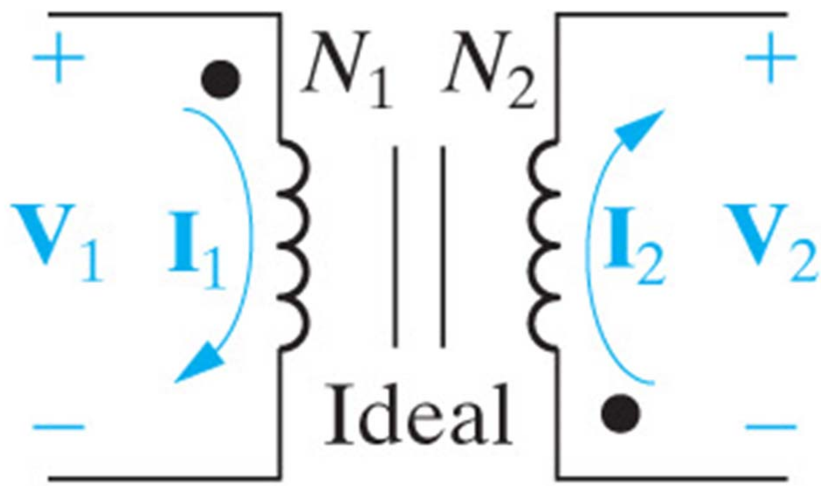


$$\frac{\mathbf{V}_1}{N_1} = \frac{\mathbf{V}_2}{N_2},$$

$$N_1\mathbf{I}_1 = N_2\mathbf{I}_2$$

Section 9.11 The Ideal Transformer $\quad \dfrac{V_1}{N_1} = (\pm)\dfrac{V_2}{N_2} \qquad I_1 N_1 = (\pm) I_2 N_2$

Case 4. Only one voltage is positive at the dot-marked terminal.
Both currents are into the dot-marked terminal.



$$\frac{\mathbf{V}_1}{N_1} = -\frac{\mathbf{V}_2}{N_2},$$

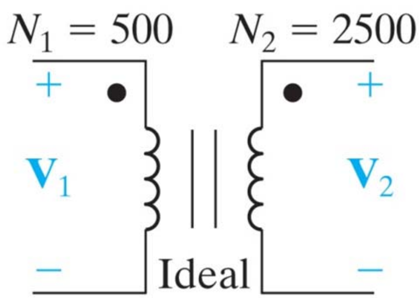
$$N_1\mathbf{I}_1 = -N_2\mathbf{I}_2$$

The ratio of turns on the two windings may be defined as either
N_1/N_2 or as N_2/N_1.

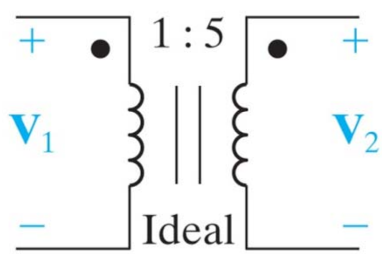
In this text, the authors use

$$a = \frac{N_2}{N_1}$$

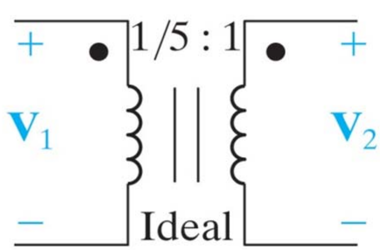
Equivalent ways to show the turns ratio.



$$a = \frac{N_2}{N_1} = \frac{2,500}{500} = 5$$



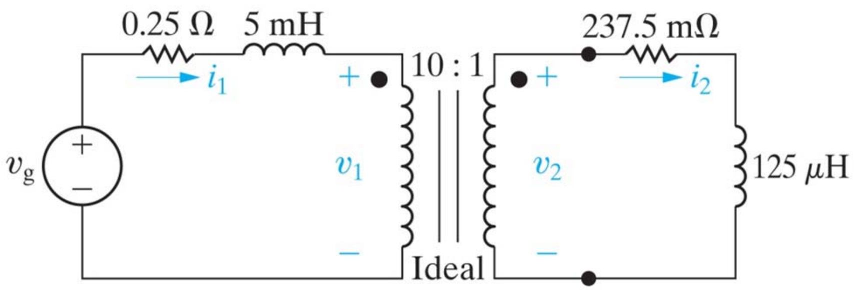
$$1 : a$$



$$\frac{1}{a} : 1$$

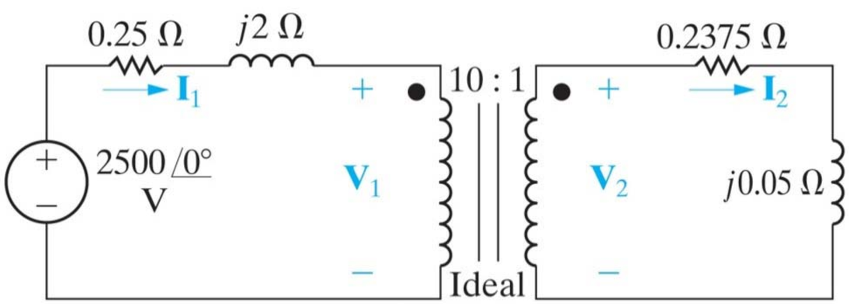
Ideal Transformer Example

Given that v_g = 2,500 cos 400t. Draw the Phasor domain circuit.

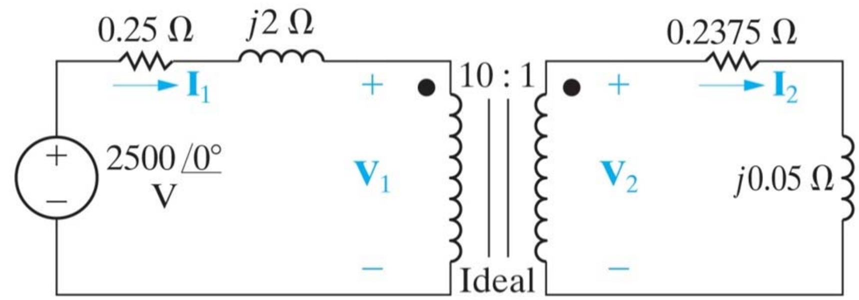


$$j\omega(5mH) = j(400\tfrac{rad}{\sec})(5mH) = j2\Omega$$

$$j\omega(125\mu H) = j(400\tfrac{rad}{\sec})(125\mu H) = j0.05\Omega$$



Ideal Transformer Example



a) Find the steady-state current $i_1(t)$.

By KVL of the primary side

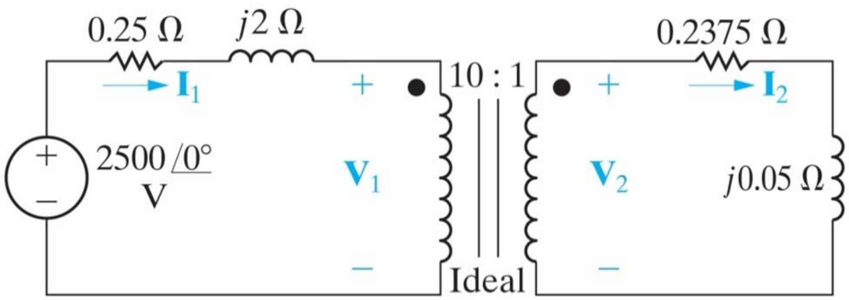
$$-2,500\angle 0^\circ + I_1(0.25 + j2) + V_1 = 0$$

To solve the KVL equation, we need to find V_1.

$$\frac{V_1}{N_1} = (?)\frac{V_2}{N_2}$$ Both voltages are positive at the dot-marked terminal. So use +

$$\frac{V_1}{N_1} = (+)\frac{V_2}{N_2} \quad \Rightarrow \frac{V_1}{10} = \frac{V_2}{1} \quad \Rightarrow V_1 = 10V_2$$

Ideal Transformer Example



Now we shift to finding currents for a moment.

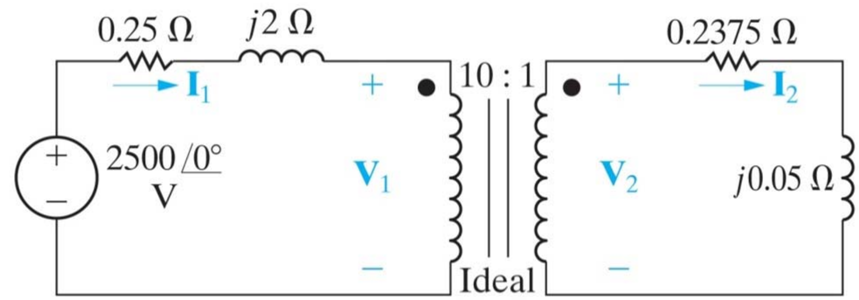
$$V_2 = I_2(0.2375 + j0.05\Omega)$$

And the currents I_1 and I_2 are related by the ideal transformer equation

$$I_1 N_1 = (?) I_2 N_2$$ Only one current is into the dot-marked terminal. So use +

$$I_1 N_1 = (+) I_2 N_2 \quad \Rightarrow I_1 10 = I_2 1 \quad \Rightarrow I_2 = 10 I_1$$

Ideal Transformer Example



$$V_1 = 10V_2 \qquad V_2 = I_2(0.2375 + j0.05\Omega)$$

$$I_2 = 10I_1$$

Now we gather up these results are rewrite the original KVL equation.

$$-2,500\angle 0^\circ + I_1(0.25 + j2) + V_1 = 0 \quad \text{original KVL equation.}$$

$$I_1(0.25 + j2) + V_1 = 2,500\angle 0^\circ$$

$$I_1(0.25 + j2) + 10V_2 = 2,500\angle 0^\circ \qquad \text{Substitute in for } V_1$$

$$I_1(0.25 + j2\Omega) + 10[I_2(0.2375 + j0.05\Omega)] = 2,500\angle 0^\circ \qquad \text{Substitute in for } V_2$$

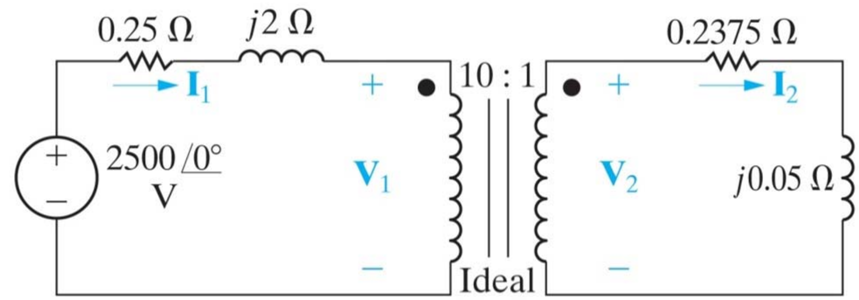
$$I_1(0.25 + j2\Omega) + 10[(10I_1)(0.2375 + j0.05\Omega)] = 2,500\angle 0^\circ \quad \text{Substitute in for } I_2$$

Solve for I_1

$$I_1[(0.25 + j2\Omega) + 100(0.2375 + j0.05\Omega)] = 2,500\angle 0^\circ$$

$$I_1(24 + j7\Omega) = 2,500\angle 0^\circ$$

Ideal Transformer Example



$$I_1(24 + j7\Omega) = 2{,}500\angle 0^\circ$$

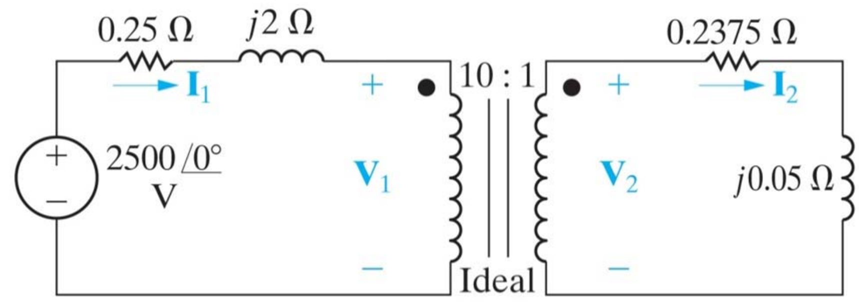
Now we can finally solve for I_1.

$$I_1 = \frac{2{,}500\angle 0^\circ}{24 + j7\Omega} = \frac{2{,}500\angle 0^\circ}{25\angle 16.26^\circ\,\Omega} = 100\angle -16.26^\circ\ Amps$$

We can immediately write the time-domain current from the phasor above.

$$i_1(t) = \underline{\underline{100\cos(400t - 16.26^\circ)\ Amps}}$$

Ideal Transformer Example



$$I_1 = 100\angle -16.26^\circ\ Amps$$

b) Find the steady-state voltage $v_1(t)$.

Start with the original KVL equation we have

$$I_1(0.25 + j2) + V_1 = 2{,}500\angle 0^\circ \qquad \Rightarrow V_1 = 2{,}500\angle 0^\circ - I_1(0.25 + j2)$$

$$V_1 = 2{,}500\angle 0^\circ - (100\angle -16.26^\circ)(0.25 + j2\Omega)$$ Use result of I_1 and solve for V_1

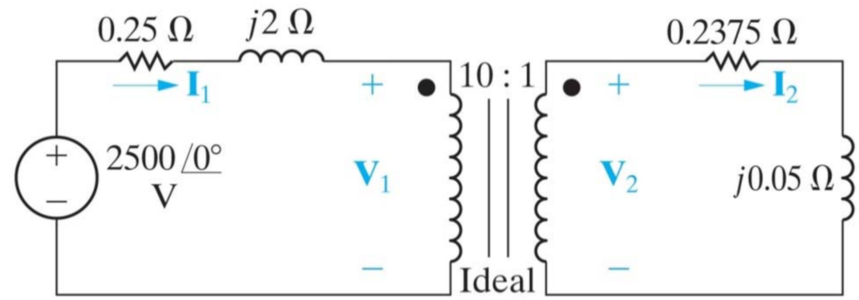
$$= 2{,}500\angle 0^\circ - (100\angle -16.26^\circ A)(2.02\angle 82.87^\circ \Omega)$$

$$= 2{,}500\angle 0^\circ - 202\angle 66.61^\circ V \qquad = 2{,}500 + j0 - 80.2 - j185.4$$

$$= 2{,}419.8 - j185.4 \qquad = 2{,}426.9\angle -4.38^\circ\ Volts$$

$$v_1(t) = \underline{\underline{2{,}426.9\cos(400t - 4.38^\circ)Volts}}$$

Ideal Transformer Example



$$I_1 = 100\angle -16.26^\circ \; Amps$$

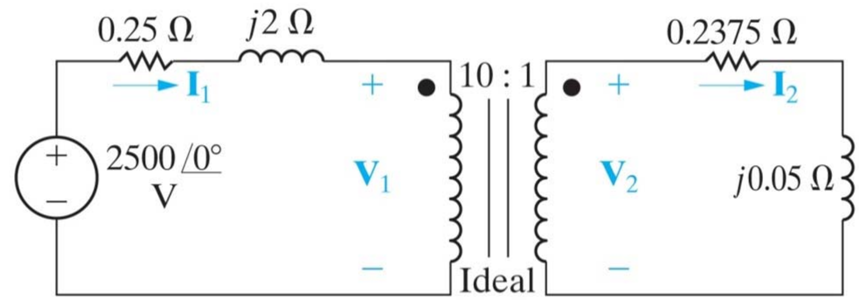
c) Find the steady-state current $i_2(t)$.

From the turns ratio, we have

$$I_2 = 10 I_1 \quad = 10(100\angle -16.26^\circ) \quad = 1,000\angle -16.26^\circ \; Amps$$

$$i_2(t) = \underline{\underline{1,000\cos(400t - 16.26^\circ)\; Amps}}$$

Ideal Transformer Example



d) Find the steady-state voltage $v_2(t)$.

Using the given turns ratio, we have

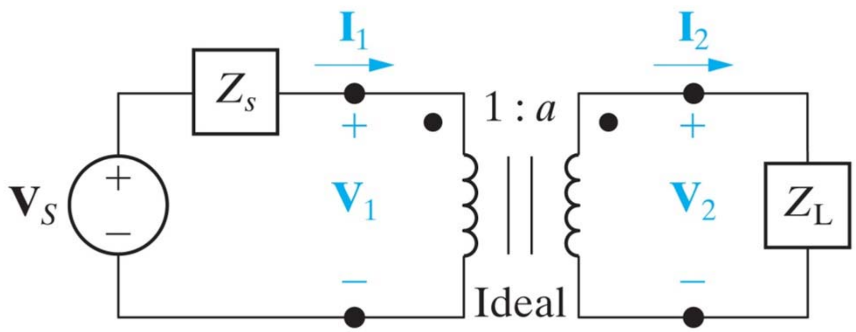
$$\frac{V_1}{10} = \frac{V_2}{1}$$

$$V_2 = \frac{V_1}{10} \qquad = \frac{2,426.9\angle -4.38^\circ}{10} \qquad = 242.7\angle -4.38^\circ$$

$$v_2(t) = \underline{\underline{242.7\cos(400t - 4.38^\circ)Volts}}$$

Ideal transformers can be used to raise or lower the impedance of load as seen by the source.

$$\frac{V_1}{1} = (+)\frac{V_2}{a}$$



$$I_1(1) = (+)I_2(a)$$

Thus the input impedance is (leave out the source impedance Z_s)

$$Z_{in} \equiv \frac{V_1}{I_1} \qquad Z_L \equiv \frac{V_2}{I_2}$$

$$Z_{in} = \frac{V_1}{I_1} = \frac{\frac{V_2}{a}}{aI_2} = \frac{1}{a^2}\frac{V_2}{I_2} = \frac{1}{a^2}Z_L$$

Impedance Matching

$$Z_{in} = \frac{1}{a^2} Z_L$$

Thus the ideal transformer's secondary coil reflects the load impedance back to the primary coil with a scaling factor.

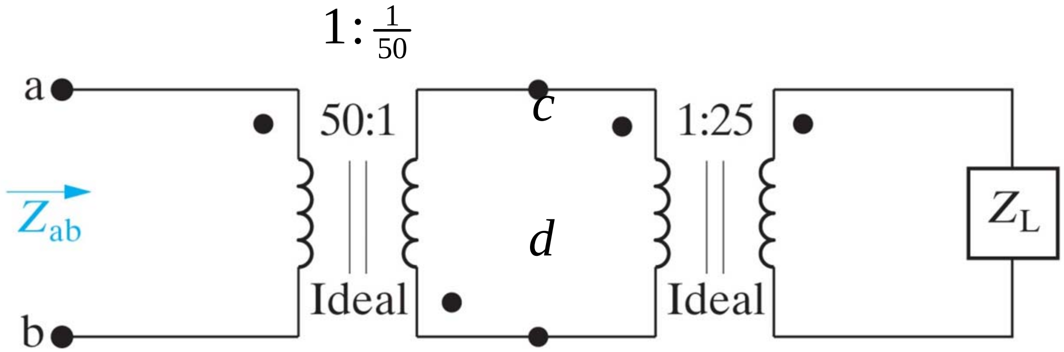
The magnitude is scaled, but the phase induced by the load is not altered!

In section 10.6 you will find the maximum power transfer. If the phase of Z_L cannot be changed, the greatest power is delivered to the load when the magnitudes of the Thévenin and load impedances are equal.

$$|Z_{TH}| = |Z_L|$$

Impedance matching with ideal transformers allow us to create the above condition.

Impedance Matching Example



$$Z_L = 200 + j150\,\Omega$$

Find the impedance Z_{ab}.

Use the impedance matching equations to quickly find the solution.

$$Z_{cd} = \frac{1}{25^2}(200 + j150\,\Omega)$$

$$Z_{ab} = \frac{1}{\frac{1}{50^2}} Z_{cd} \quad = 50^2 Z_{cd} \quad = 50^2\left[\frac{1}{25^2}(200 + j150\,\Omega)\right]$$

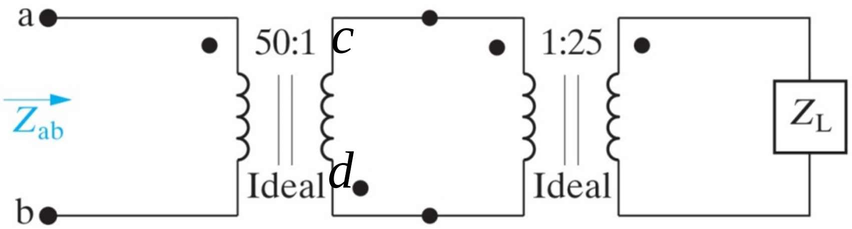
$$= \left(\frac{50}{25}\right)^2 (200 + j150\,\Omega) \; = (2)^2 (200 + j150\,\Omega) \quad = 4(200 + j150\,\Omega)$$

$$= \underline{\underline{800 + j600\,\Omega}}$$

Impedance Matching Example

$1:\frac{1}{50}$

$Z_{ab} = \underline{\underline{800 + j600\,\Omega}}$



$Z_L = 200 + j150\,\Omega$

To deliver maximum power to the load Z_L, then

$$|Z_{Th}| = |Z_{ab}| = \sqrt{800^2 + 600^2} = \sqrt{10 \times 10^6} = 1{,}000\,\Omega$$

Usually we have a specific load that must be matched to an existing source.

We would then find a linear transformer with the appropriate turns ratio that achieves the desired impedance matching.

$$\left|Z_{Th}\right| = \frac{1}{a^2}\left|Z_L\right| \quad \Rightarrow a^2 = \frac{\left|Z_L\right|}{\left|Z_{Th}\right|} \quad \Rightarrow a = \sqrt{\frac{\left|Z_L\right|}{\left|Z_{Th}\right|}}$$

Chapter 9

Sinusoidal Steady-State Analysis

Text: *Electric Circuits* by J. Nilsson and S. Riedel

Prentice Hall

Engr 17 Introductory Circuit Analysis

Instructor: Russ Tatro