CALIFORNIA STATE UNIVERSITY
SACRAMENTO

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EEE 117 Network Analysis

Text: Electric Circuits by J. Nilsson and S. Riedel Prentice Hall

Lecture Set 3: Laplace and Inverse Laplace Transformations

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Laplace and Inverse Laplace Transformations

- What is Laplace?
- Why use Laplace?
- Definition of the Laplace Transform
- The Step Function
- The Impulse Function
- Functional Transforms
- Operational Transforms
- Inverse Laplace Transform
- Use of Partial Fraction
- 4-Type of Partial Fraction Expansions
- Poles and Zeros of F(s)
- Initial-Value and Final-Value Theorems

What is Laplace?

- In mathematics, the **Laplace transform**, named after its inventor Pierre-Simon Laplace, is an integral transform that converts a function of a real variable "**t**" (often time) to a function of a complex variable "**s**" (complex frequency).
- The Laplace transform is a tool for analyzing linear, time-invariant, lumped parameter systems.
- This tool transforms a function from the time-domain, in which inputs and outputs are functions of time, into a function in the frequency-domain, where the same inputs and outputs are functions of complex angular frequency.
- Given a simple mathematical or functional description of an input or output to a system, the Laplace transform provides an alternative functional description that often simplifies the process of analyzing the behavior of the system, or in synthesizing a new system based on a set of specifications.

Why use Laplace?

- Analysis of the transient and steady state conditions in multi-node or multi-mesh systems. (Remember: Phasor method is only for steady state)
- Reduce the math complexity of sets of linear differential equations in multi-node or multi-mesh systems.
- Discovery of the transient conditions in the presence of more complicated signal sources.
- Use of Transfer Functions in a system where the frequency of the input varies.
- It is a tool for solving differential equations. In particular, it transforms differential equations into algebraic equations and convolution into multiplication.

Definition of the Laplace Transform

- The Laplace transform of a function is given by the expression:

$$\boxed{\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt}$$

- Symbol $\mathscr{L}$ {f(t)} means "the Laplace transform of "f(t)".

- The Laplace transform is also denoted as F(s).

$$\boxed{F(s) = \mathscr{L}\{f(t)\}}$$

We represent the corresponding s-domain variables with uppercase letters.

$$\boxed{\begin{aligned} \mathscr{L}\{v\} &= V \\ \mathscr{L}\{i\} &= I \end{aligned}}$$

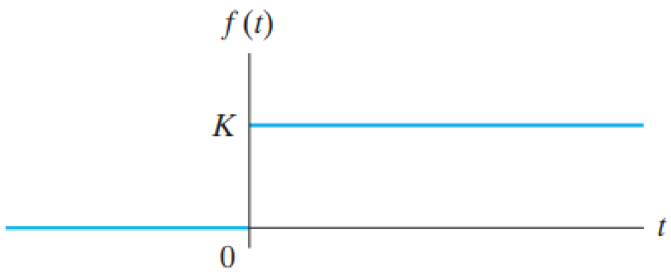
- In circuit analysis, we use the Laplace transform to take the integrodifferential equations of the time-domain into a set of algebraic equations in the frequency domain.

$$\boxed{\begin{array}{c} \textit{Time Domain} \rightarrow \text{t in seconds} \\ \textit{Frequency Domain} \rightarrow s \text{ in} \dfrac{1}{\text{seconds}} \quad \textit{i.e.Hertz} \end{array}}$$

- In this course we will only use the unilateral or one-sided Laplace transform.
- Thus, we integrate from zero to infinity. We will be careful to use sources where the Laplace integral converges.
- The result of circuit behavior prior to t = zero is accounted for by initial conditions.

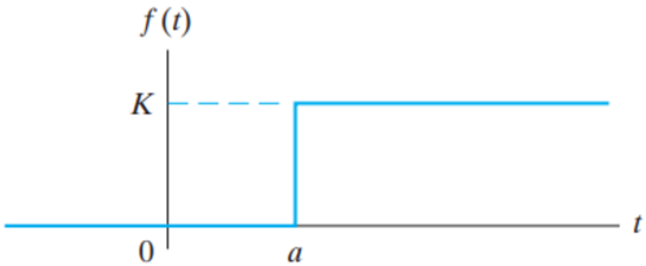
The Step Function

- Step function is an arbitrary signal which has discontinuity.



$$Ku(t) = 0, \quad t < 0,$$
$$Ku(t) = K, \quad t > 0.$$

- The symbol for the step function is Ku(t).
- The step function is not defined at t=0. It is discontinuous.
- A discontinuity may occur at some time other than t=0.



$$Ku(t - a) = 0, \quad t < a,$$
$$Ku(t - a) = K, \quad t > a.$$

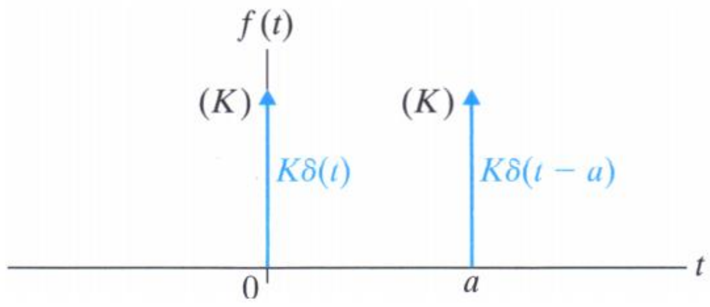
- Hence a step function can be represent as Ku(t ± a), where "a" is the distance of point from origin.

The Impulse Function

- The Impulse function is defined as a signal of infinite amplitude and zero duration.
- The Impulse function is a signal which has some value only at a certain instant and its value is zero for any other instant.
- A unit impulse function is denoted as $\delta(t)$
- The Impulse function is mathematically defined as

$$\int_{-\infty}^{\infty} K\delta(t)\,dt = \begin{cases} K & for\ \ t = 0 \\ 0 & for\ \ t \neq 0 \end{cases}$$

- The impulse can be advanced or retarded in time. Thus, it can have a value at some other time "a" (only at that other time).



- While a “pure” impulse signal does not exist in nature, the mathematical use of the concept is very valuable.
- Since the impulse only exists at a single point in time, it may be used to “sift” the values of another function.

The sifting property is defined as

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,dt = \quad = f(a)$$

We can use the sifting property to find the Laplace transform of δ(t).

$$\mathscr{L}\{\delta(t)\} \equiv \int_{0^-}^{\infty} \delta(t=0)e^{-st}dt = \underbrace{\left(e^{-st}\Big|^{t=0}\right)}_{fct\ only\ exists\ for\ t=0} \int_{0^-}^{\infty} \delta(t)\,dt$$

$$= e^{-0}(1) \qquad = 1$$

$$\boxed{\mathscr{L}\{\delta(t)\} \quad = 1}$$

Functional Transforms

- A functional transform is simply the Laplace transform of a specified function of t.
- Since we will use only the unilateral Laplace transform, we will define all our functions = zero for t < 0-

Laplace transform of the unit function:

$$\mathcal{L}\{u(t)\} = \int_{0^-}^{\infty} f(t)e^{-st}\,dt = \int_{0^+}^{\infty} 1e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_{0^+}^{\infty} = \frac{1}{s}.$$

Laplace transform of the decaying exponential function:

$$\mathcal{L}\{e^{-at}\} = \int_{0^+}^{\infty} e^{-at}e^{-st}\,dt = \int_{0^+}^{\infty} e^{-(a+s)t}\,dt = \frac{1}{s+a}.$$

Laplace transform of a sinusoidal function:

$$\mathcal{L}\{\sin\omega t\} = \int_{0^-}^{\infty} (\sin\omega t)e^{-st}\,dt = \int_{0^-}^{\infty}\left(\frac{e^{j\omega t}-e^{-j\omega t}}{2j}\right)e^{-st}\,dt = \int_{0^-}^{\infty}\frac{e^{-(s-j\omega)t}-e^{-(s+j\omega)t}}{2j}\,dt$$

$$= \frac{1}{2j}\left(\frac{1}{s-j\omega}-\frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}.$$

TABLE 12.1 An Abbreviated List of Laplace Transform Pairs

Type	$f(t)\ (t > 0-)$	$F(s)$
(impulse)	$\delta(t)$	1
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	t	$\frac{1}{s^2}$
(exponential)	e^{-at}	$\frac{1}{s+a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
(damped ramp)	te^{-at}	$\frac{1}{(s+a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

Operational Transforms

- Operational transforms indicate how mathematical operations performed on either f(t) or F(s) are converted to the other domain.

- Differentiation in the time domain corresponds to multiplying F(s) by s then subtracting the initial value of f(t).

$$\mathscr{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

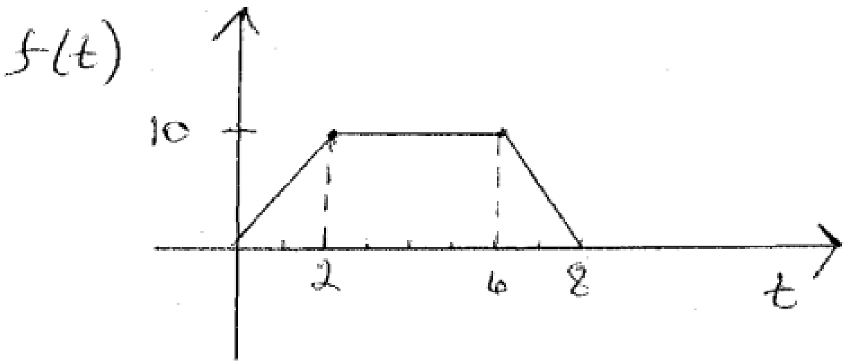
- Integration in the time domain corresponds to dividing F(s) by s.

$$\mathscr{L}\left\{\int_{0^-}^{t} f(x)dx\right\} = \frac{F(s)}{s}$$

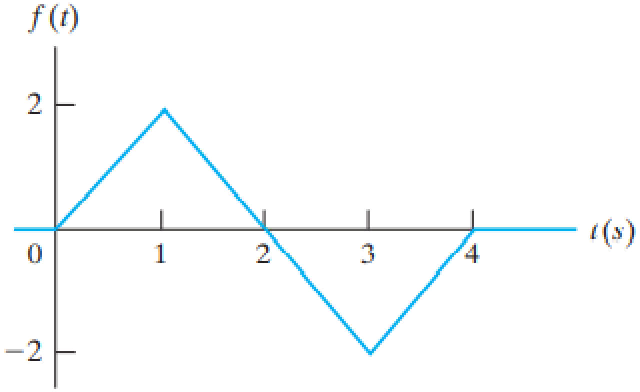
TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \cdots$	$F_1(s) + F_2(s) - F_3(s) + \cdots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
nth derivative (time)	$\frac{d^nf(t)}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{df^2(0^-)}{dt^2} - \cdots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x)\,dx$	$\frac{F(s)}{s}$
Translation in time	$f(t-a)u(t-a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s+a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
nth derivative (s)	$t^nf(t)$	$(-1)^n\frac{d^nF(s)}{ds^n}$
s integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u)\,du$

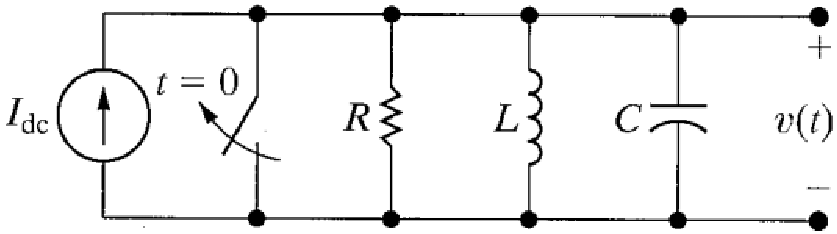
Ex-1: Use step functions to write an expression for the following function.



Ex-2: Use step functions to write an expression for the following function.



Ex-3: Apply the Laplace transform to the following circuit.



Inverse Laplace Transforms

- In general, the Laplace transform resulting from a circuit analysis will yield a rational function in the form:

$$F(s) = \frac{N(s)}{D(s)} \quad = \frac{a_n s^n + a_{n-1} s^{n-1} + \bullet\bullet\bullet + a_1 s + a_0}{b_m s^m + b_{m-1} s^{m-1} + \bullet\bullet\bullet + b_1 s + b_0}$$

- If m > n, this is a proper rational function.
- If m < n, the improper rational function must be divided to create a proper rational function.
- The task now is to simplify the proper rational function into a form which can be identified in the Laplace transform pair.
- We will use partial fraction method to solve the problem. There are four types of partial fraction expansions:
    - ➢The roots of D(s) are real and distinct.
    - ➢The roots of D(s) are real and repeated.
    - ➢The roots of D(s) are complex.
    - ➢The roots of D(s) are complex and repeated.

Ex-4: Find f(t) for $F(s) = \dfrac{s+6}{s(s+3)(s+1)^2}$

Ex-5: Find f(t) for $F(s) = \dfrac{5s^2+29s+32}{(s+2)(s+4)}$

Ex-6: Find f(t) for $F(s) = \dfrac{10(s^2+119)}{(s+5)(s^2+10s+169)}$

Poles and Zeros of F(s)

- The rational function of the Laplace transform can be expressed as the ratio of two factored polynomials.

$$F(s) = \frac{K(s+z_1)(s+z_2)\cdots(s+z_n)}{(s+p_1)(s+p_2)\cdots(s+p_m)}$$

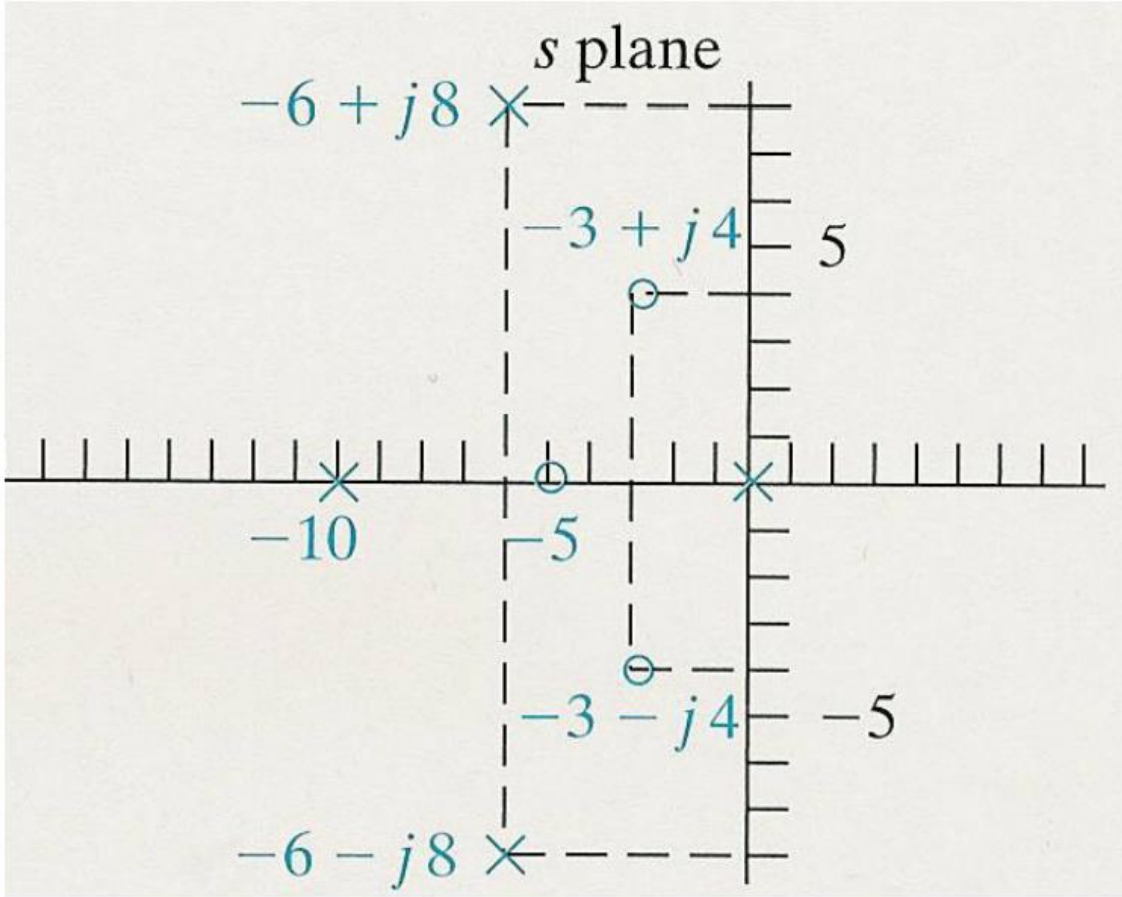
- The roots of the denominator polynomials $-p_1, -p_2, \ldots, -p_m$ are called the poles of F(s).
- The roots of the numerator polynomials $-z_1, -z_2, \ldots, -z_n$ are called the zeros of F(s).
- The zeros of F(s) are the values of s at which F(s) becomes zero.
- The poles and zeros are plotted on a s-plane having real and imaginary axis.

Plot the poles and zeros of the following rational function.

$$F(s)=\frac{10(s+5)(s+3-j4)(s+3+j4)}{s(s+10)(s+6-j8)(s+6+j8)}$$

The zeros of F(s) are
-5, -3 + j4, -3 - j4.

The poles of F(s) are
0, -10, -6 + j8, -6 – j8.



Initial-Value and Final-Value Theorems

- The initial-value and final-value theorems are useful because they enable us to determine from F(s) the behavior of f(t) at t = 0 and t $\rightarrow \infty$.

The initial-value theorem	$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$
The final-value theorem	$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

- The theorems assume (require) that:
  - f(t) contains no impulses
  - All the poles f(t) lies in the left-hand plane.
  - A non-repeated pole (first order) of f(t) may exist at the origin.

Ex-7: Plot the poles and zeros of the following rational function.

$$F(s) = \frac{8s^2 + 120s + 400}{2s^4 + 20s^3 + 70s^2 + 100s + 48}$$

Ex-8: Prove Initial-value and Final-value Theorems for the following: Given: $f(t) = \left[-12e^{-6t} + 20e^{-3t}\cos(4t - 53.13^\circ) \right] u(t)$

$$F(s) = \frac{100(s+3)}{(s+6)(s^2+6s+25)}$$