HW 5

1) Problem 3.26
2) Problem 4.12.
3) Problem 4.17.
4) Problem 4.20.
5) Problem 4.39.

6) Let *X* be a random variable whose probability density function is given by

$$f(x) = \begin{cases} e^{-2x} + \frac{e^{-x}}{2} & x > 0 \\ 0 & \text{else} \end{cases}.$$

   a) Write down the moment generating function for *X.*
   b) Compute the first and second moments, i.e $E(X)$ and $E(X^2)$.

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HOMEWORK #5

1.
$$f(x) = \begin{cases} x & 0 \le x \le 1 \\ 2-x & 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the CDF

(i) $x < 0$, $\quad f_x(x) = 0$

(ii) $0 \le x \le 1$, $\quad f_x(x) = \int_0^x z\,dz = x^2/2$

(iii) $1 \le x \le 2$, $\quad f_x(x) = 1/2 + \int_1^x (2-z)\,dz = 2x - 1 + x^2/2$

(iv) $x > 2$, $\quad f_x(x) = 1$

(b) Find the following probabilities

(i) $Pr[1/2 \le x \le 3/2] = F_x(3/2) - F_x(1/2)$
$= 0.875 - 0.125 = 0.75$

(ii) $Pr[1 \le x \le 5] = F_x(5) - F_x(1)$
$= 1 - 0.5 = 0.5$

(iii) $Pr[x < 1/2] = F_x(1/2) = 1/8$

(iv) $Pr[0.75 \le x \le 0.7501] = F_x(0.7501) - F_x(0.75)$
$= 75.005 \times 10^{-6}$

2.
$$f_x(x) = \begin{cases} 1/2\,x & 0 \le x \le 1 \\ 1/6\,(4-x) & 1 < x < 6 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the constant c

$1/2 \int_0^1 x\,dx + 1/6 \int_1^c (4-x)\,dx = 1$

$1/2 [x^2/2]_0^1 + 1/6 [4x - x^2/2]_1^c = 1$

$4c - c^2/2 - 4 + 1/2 = 9/2$

$c^2 - 8c + 16 = 0$

$c = 4$

(b) calculate the expectation: $E[3+2x]$

$E[3+2x] = 3 + 2E[x]$

$E[x] = 1/2 \int_0^1 x^2\,dx + 1/6 \int_1^4 (4x - x^2)\,dx$

$= 1/2 [x^3/3]_0^1 + 1/6 [2x^2 - x^3/3]_1^4$

$= 1/6 + 1/6 (32 - 64/3 - 2 + 1/3)$

$= 5/3$

$$m_K = E\{K\} = \sum_{\forall K} K \cdot f_K[K] = -2(1/6) - 1(1/3) + 1(1/3) + 2(1/6) = 0$$

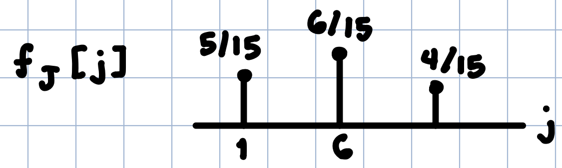
$$E\{K^2\} = \sum_{\forall K} K^2 \cdot f_K[K] = (-2)^2(1/6) + (-1)^2(1/3) + 1(1/3) + (2)^2(1/6) = 2$$

$$\sigma_K^2 = E\{K^2\} - m_K^2 = 2 - 0 = 2$$

3. In a set of independent trials a discrete random variable J takes on the following values : 1, 1, 9, 6, 1, 6, 1, 9, 6, 9, 6, 9, 1, 6, 6

(a) Obtain an estimate and plot the probability mass function of this random variable

value :	1	6	9
Frequency:	5	6	4



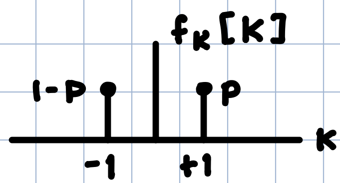
(b) Calculate the mean $E\{J\}$ and the variance $Var[J]$

$$E[J] = 1(5/15) + 6(6/15) + 9(4/15) = 77/15 = 5.1333$$
$$E[J^2] = 1^2(5/15) + 6^2(6/15) + 9^2(4/15) = 545/15 = 36.333$$
$$\sigma_J^2 = 545/15 - (77/15)^2 = 9.9822$$

4. In a certain digital control system, a 0 is represented by a negative 1 volt level and a 1 is represented by a positive 1 volt level. The voltage level is modeled by a discrete random variable K with PMF .

$$f_K[K] = \begin{cases} P & K = +1 \\ 1-P & K = -1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Sketch the PMF



(b) Find $E\{K\}$, $E\{K^2\}$, and $Var[K]$

$$E\{K\} = (-1)(1-P) + (+1)(P) = 2P - 1$$
$$E\{K^2\} = (-1)^2(1-P) + (+1)^2(P) = 1$$
$$Var[K] = 1 - (2P-1)^2 = 4P(P-1)$$

(c) For what value of P is the variance maximized and what is this maximum value?

Var[K] is maximized for $p = 1/2$
Max variance is $4(1/2)(1-1/2) = 1$

5. A random variable Y is defined in terms of another random variable X as $Y = 2X+3$, X is known to be Gaussian random variable with mean $m = 1$ and variance $\sigma^2 = 1$

(a) What is the mean of Y?
$E[Y] = 2E[X] + 3 = 2(1)+3 = 5$

(b) What is the variance of Y?
$\mathrm{Var}[Y] = \mathrm{Var}[2X] = 4\,\mathrm{Var}[X] = 4(1) = 4$

(c) Is Y a Gaussian random variable?
Yes

6. $$f(x) = \begin{cases} e^{-2x} + e^{-x}/2 & x > 0 \\ 0 & \text{else} \end{cases}$$

(a) Write down the moment generating function for X.
(b) Compute the first and second moments

$$M_X(t) = E[e^{tX}] = \int_0^\infty e^{tx}\left(e^{-2x} + e^{-x}/2\right)dx$$
$$= \frac{1}{2-t} + \frac{1}{2(1-t)} \quad \text{for } t < 1$$

$$M'_X(t) = \frac{1}{(2-t)^2} + \frac{1}{2(1-t)^2}$$
$$M''_X(t) = \frac{1}{(2-t)^3} + \frac{1}{2(1-t)^3}$$

so $E[X] = M'_X(0) = 3/4$
$E[X^2] = M''_X(0) = 5/4$