EEE 187 Robotics
Velocity Kinematics and Jacobian
(Textbook Chapter 4)

Time to understand the Jacobian matrix.

Columns of the Jacobian matrix are associated with joints of the robot. Each column in the Jacobian matrix represents the effect on end effector velocities due to variation in each joint velocity.

Which means, the first column represents the effect of joint1 velocity ( ) on end-effector velocities ( ), second column is associated with joint2 velocity ( ) and similarly nth column is effect of nth joint velocity ( ) on end-effector velocities .

Hence the number of columns in the Jacobian matrix is equal to the number of joints in the manipulator.

If we closely observe the x matrix, it has two parts.The first three elements of the end-effector velocity matrix are linear velocities [rate of change of position] and the last three elements are the angular velocites [rate of change of orientation] in (x,y,z) direction respectively.

Similarly, rows of the Jacobian matrix can also be split into two part. The first three rows are associated with linear velocities of endeffector and the last three rows are associated with the angular velocities of end-effector due to change in velocities of all the joints combined

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Linear velocity

Angular velocity

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} V \\ \Omega \end{bmatrix} \qquad V = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} \qquad \Omega = \begin{bmatrix} \omega_x = \dot{\phi} \\ \omega_y = \dot{\theta} \\ \omega_z = \dot{\psi} \end{bmatrix} \qquad \dot{q}_{n\times 1} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

$$J_v = \begin{bmatrix} \frac{\partial x}{\partial q_1} & \frac{\partial x}{\partial q_2} & \frac{\partial x}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial x}{\partial q_n} \\ \frac{\partial y}{\partial q_1} & \frac{\partial y}{\partial q_2} & \frac{\partial y}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial y}{\partial q_n} \\ \frac{\partial z}{\partial q_1} & \frac{\partial z}{\partial q_2} & \frac{\partial z}{\partial q_3} & \cdots & \cdots & \cdots & \frac{\partial z}{\partial q_n} \end{bmatrix}_{3\times n}$$

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JACOBIAN OF THE TWO LINK MANIPULATOR

For the two link planar manipulator of figure 1, the coordinates of the end effector are

$$p_x = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$$
$$p_y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

By taking the time derivative, we get

$$\dot{p}_x = -\dot{\theta}_1 a_1 \sin\theta_1 - \dot{\theta}_1 a_2 \sin(\theta_1 + \theta_2) - \dot{\theta}_2 a_2 \sin(\theta_1 + \theta_2)$$
$$\dot{p}_y = \dot{\theta}_1 a_1 \cos\theta_1 + \dot{\theta}_1 a_2 \cos(\theta_1 + \theta_2) + \dot{\theta}_2 a_2 \cos(\theta_1 + \theta_2)$$

which can be re-arranged under matrix form:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

JACOBIAN MATRIX

In general, it is possible to write

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

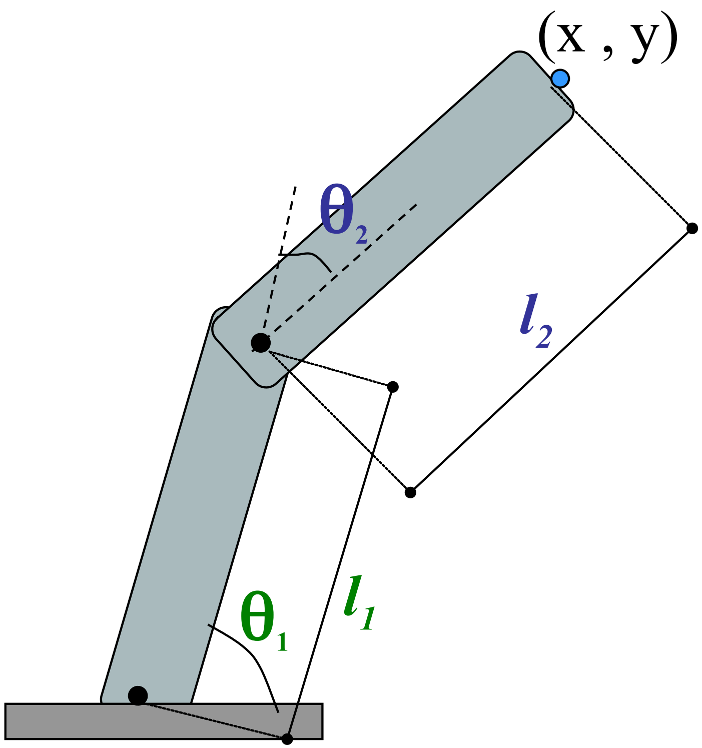
JACOBIAN MATRIX

$$J = \begin{array}{cc} \dfrac{\partial p_x}{\partial \theta_1} & \dfrac{\partial p_x}{\partial \theta_2} \\ \left[\begin{array}{c} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \end{array}\right. & \left.\begin{array}{c} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \end{array}\right] \\ \dfrac{\partial p_y}{\partial \theta_1} & \dfrac{\partial p_y}{\partial \theta_2} \end{array}$$

This is the standard format for the Jacobian. If there were a z component there would be a third row to the matrix

Example

- 2-DOF planar robot arm
  - Given ***l_1, l_2, Find: Jacobian***



$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{bmatrix} = \begin{bmatrix} h_1(\theta_1, \theta_2) \\ h_2(\theta_1, \theta_2) \end{bmatrix}$$

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial \theta_1} & \frac{\partial h_1}{\partial \theta_2} \\ \frac{\partial h_2}{\partial \theta_1} & \frac{\partial h_2}{\partial \theta_2} \end{bmatrix} = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

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- Find the singularity configuration of the 2-DOF planar robot arm

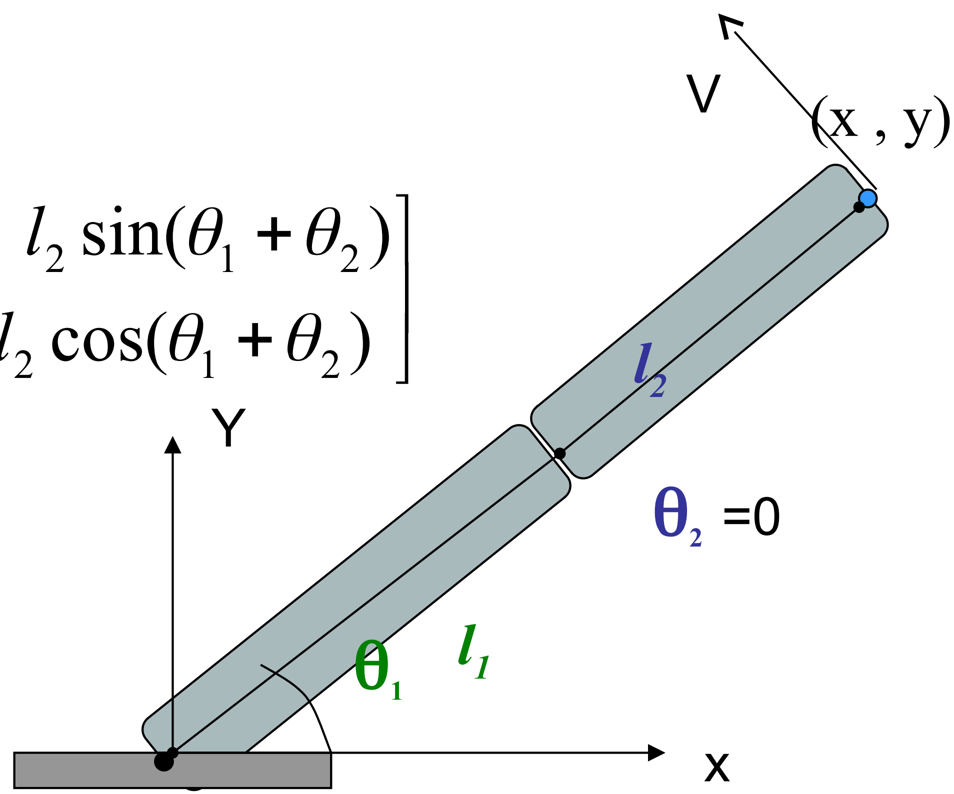
determinant(J)=0 $\Rightarrow$ Not full rank

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$J = \begin{bmatrix} -l_1 \sin\theta_1 - l_2 \sin(\theta_1 + \theta_2) & -l_2 \sin(\theta_1 + \theta_2) \\ l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) & l_2 \cos(\theta_1 + \theta_2) \end{bmatrix}$$

Det(J)=0

$$\theta_2 = 0$$



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Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

$$A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1c_1 \\ s_1 & c_1 & 0 & a_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Where $(\theta_1 + \theta_2)$ denoted by θ_{12} and $\cos(\theta_1 + \theta_2)$ by c_{12}

$$A_2 = \begin{bmatrix} c_2 & -s_2 & 0 & a_2c_2 \\ s_2 & c_2 & 0 & a_2s_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Z_0 = Z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$T_1^0 = A_1.$$

$$T_2^0 = A_1A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1c_1 + a_2c_{12} \\ s_{12} & c_{12} & 0 & a_1s_1 + a_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$O_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, O_1 = \begin{bmatrix} a_1\cos\theta_1 \\ a_1\sin\theta_1 \\ 0 \end{bmatrix}, O_2 = \begin{bmatrix} a_1\cos\theta_1 + a_2\cos(\theta_1+\theta_2) \\ a_1\sin\theta_1 + a_2\sin(\theta_1+\theta_2) \\ 0 \end{bmatrix}$$

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cross product

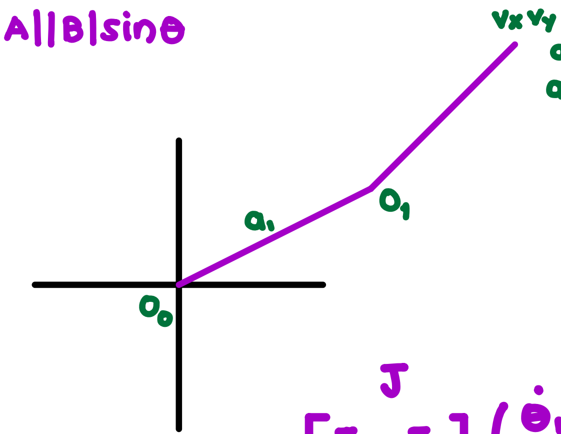
$|A \times B| = |A||B|\sin\theta$

Velocity = $\omega \times r$

$v_i = \omega_i \times r_i$

$r_i = o_{end\text{-}effector} - o_{i-1}$

$\omega_i = \dot{\theta}_i \cdot z_{i-1}$



$a\cos\theta_1 + a\cos(\theta_1+\theta_2) = x$
$a\sin\theta_1 + a\sin(\theta_1+\theta_2) = y$

$$\overset{J}{[J_1 \quad J_2]} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} = \begin{matrix} v_x \\ v_y \end{matrix}$$

$\dot{\theta}_3 \qquad v_z$

The actual velocity of the end effector is the vector sum of all of the vi components

Jacobian Matrix

2-DOF planar robot arm

Given *l1*, *l2* , ***Find: Jacobian***

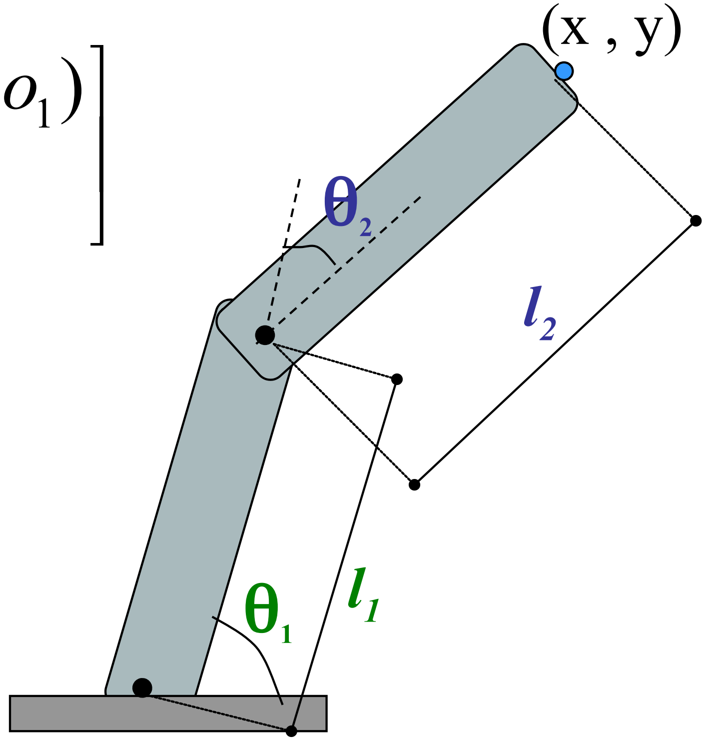
Link	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1^*
2	a_2	0	0	θ_2^*

* variable

- ***Here, n=2***

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix}, J_2 = \begin{bmatrix} z_1 \times (o_2 - o_1) \\ z_1 \end{bmatrix}$$

$$J = \begin{bmatrix} J_1 & J_2 \end{bmatrix}$$



Jacobian Matrix

$$J_1 = \begin{bmatrix} z_0 \times (o_2 - o_0) \\ z_0 \end{bmatrix}$$

$$Z_0 \times (o_2 - o_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

Jacobian Matrix

$$J_2 = \begin{bmatrix} z_1 \times (o_2 - o_1) \\ z_1 \end{bmatrix}$$

$$Z_1 \times (o_2 - o_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} a_2 \cos(\theta_1 + \theta_2) \\ a_2 \sin(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} i & j & k \\ 0 & 0 & 1 \\ a_2 \cos(\theta_1 + \theta_2) & a_2 \sin(\theta_1 + \theta_2) & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \end{bmatrix}$$

Jacobian Matrix

$$J_1 = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad J_2 = \begin{bmatrix} -a_2 \sin(\theta_1 + \theta_2) \\ a_2 \cos(\theta_1 + \theta_2) \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

The required Jacobian matrix **J**

$$J = \begin{bmatrix} J_1 & J_2 \end{bmatrix} = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Jacobian Matrix

$$\dot{X} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} V \\ \Omega \end{bmatrix}$$

Linear velocity

$$V = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix}$$

Angular velocity

$$\Omega = \begin{bmatrix} \omega_x = \dot{\phi} \\ \omega_y = \dot{\theta} \\ \omega_z = \dot{\psi} \end{bmatrix}$$

$$\dot{q}_{n\times 1} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}_{n\times 1}$$

The Jacobian Equation

$$\dot{X} = J_{6\times n}\dot{q}_{n\times 1}$$

Jacobian Matrix

(Three ways to form the Jacobian)

1) Differentiate the link equations for X (or P_x) and Y (or P_y) (Lecture 8)
2) Create a Jacobian by doing partial differentials of X and Y (slides 2,3,4)
3) Use Cross Products to create the Jacobian (slides 7,8,9,10)

Regardless of the method used to create the Jacobian;

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

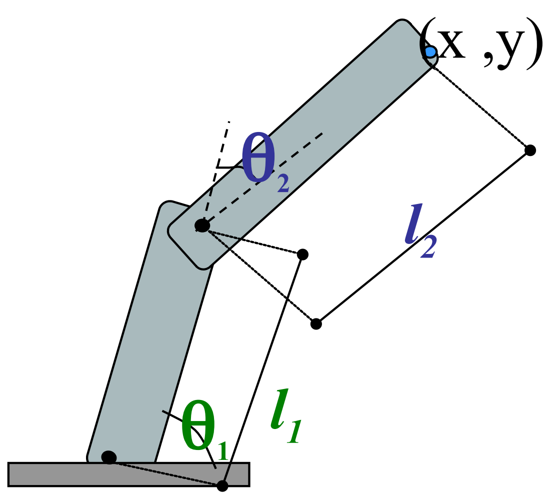
Where:

$\dot{x}$ is the x-component of the velocity of the end effector

$\dot{y}$ is the y-component of the velocity of the end effector

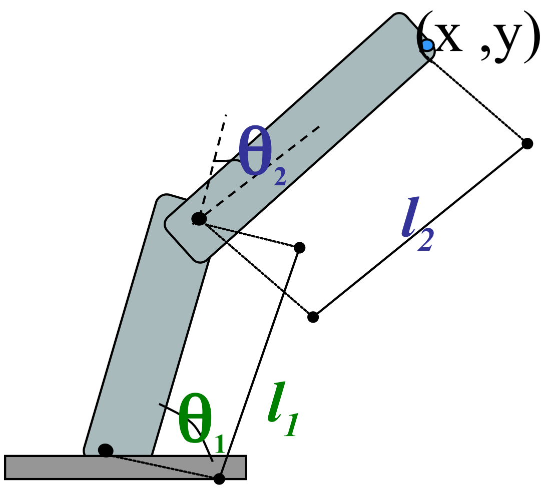
$\dot{\theta}_1$ Is the angular velocity of the first joint

$\dot{\theta}_2$ Is the angular velocity of the second joint



Jacobian Matrix
(Two ways to travel from X_0,Y_0 to X_f,Y_f)

$$\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = J\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$



Method #1

1. Calculate velocity vector from X_0,Y_0 to X_f,Y_f
    1. $V_x=(X_f-X_0)/T$, $V_y=(Y_f-Y_0)/T$ where T= travel time
2. Multiply Inverse Jacobian times velocity vector to calculate angular velocities $\dot{\theta}_1, \dot{\theta}_2$
3. Multiply angular velocities times small percentage of travel time to calculate small angular change to θ_1,θ_2 and new X_1,Y_1
4. Calculate velocity vector from X_1,Y_1 to X_f,Y_f and repeat the 4 steps

Method #2

1. Use Inverse Kinematics to calculate θ_1,θ_2 for both initial position ($\theta_{1initial},\theta_{2initial}$) and the final position ($\theta_{1final},\theta_{2final}$)
2. Calculate $\dot{\theta}_1$ and $\dot{\theta}_2$ so that ($\theta_{1initial},\theta_{2initial}$) rotates into ($\theta_{1final},\theta_{2final}$) in the same time interval

Introduction to ROBOTICS

Velocity Analysis
Jacobian

University of Bridgeport

BOSTON DYNAMICS DANCING ROBOT

https://fb.watch/gDAU6B9JBG/