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PATH PLANNING USING POLYNOMIAL TRAJECTORIES

It is possible to use polynomial trajectories to control the point-to-point motion of the joint variables. In this case, the joints are controlled independently. The degree of the polynomial depends on the number of constraints or specified values. For example if we have a desired value for the configuration variable and its speed, we will need to use a polynomial of degree 3 (with four terms).

I. TWO END POINTS

Assume that we have constraints on the initial and final values of the configuration variable, as well as its initial and final speed. We want to plan the trajectory of the configuration variable i.e., find a function $q(t)$ that satisfies the following

- At time t_0

$$q(t_0) = q_0 \tag{1}$$
$$\dot{q}(t_0) = v_0 \tag{2}$$

- At time t_f

$$q(t_f) = q_f \tag{3}$$
$$\dot{q}_f(t) = v_f \tag{4}$$

where vector $[q_0, q_f, v_0, v_f]$ is given.

It is also possible to specify constraints on the initial; and final values of the acceleration as follows

$$\ddot{q}(t_0) = \alpha_0$$
$$\ddot{q}_f(t) = \alpha_f$$

where vector $[\alpha_0, \alpha_f]$ is given. The goal is to compute the trajectory using low order polynomial

$q(t) = a_0 + a_1 t + ...$

The order depends on the number of constraints:
- Cubic polynomial allows specifying the initial and final position and velocity
- Quintic polynomial allows specifying the initial and final acceleration in addition to the initial and final position and velocity

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Cubic polynomial

We have four constraints because we specify the initial and final position and speed. Because we have four constraints, we need four parameters; therefore, we use polynomial of degree 3 as follows

$$q(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$
$$\dot{q}(t) = a_1 + 2a_2 t + 3a_3 t^2$$

Now, we have four constraints with four unknowns as follows

$$q_0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3$$
$$v_0 = a_1 + 2a_2 t_0 + 3a_3 t_0^2$$
$$q_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$$
$$\dot{q}(t) = v_f = a_1 + 2a_2 t_f + 3a_3 t_f^2$$

We can write this under matrix form as follows:

$$Q = TA$$

with

$$A = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

and

$$Q = \begin{bmatrix} q_0 \\ v_0 \\ q_f \\ v_f \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & t_0 & t_0^2 & t_0^3 \\ 0 & 1 & 2t_0 & 3t_0^2 \\ 1 & t_f & t_f^2 & t_f^3 \\ 0 & 1 & 2t_f & 3t_f^2 \end{bmatrix}$$

Note, its given $t_0=0$; $t_f=5$

The solution for A is given by

$$A = T^{-1}Q$$

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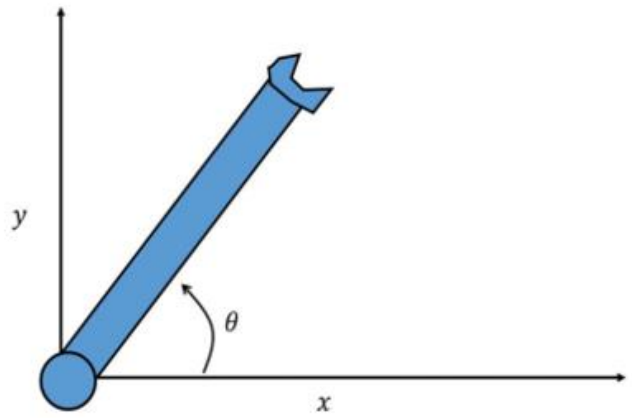
Example

The desired initial and final values for the joint speed are $v_0 = 0; v_f = 0$. Find $q(t)$ when

- Case 1: when $q_0 = 0, q_f = 90^o$
- Case 2: when $q_0 = 0, q_f = -90^o$



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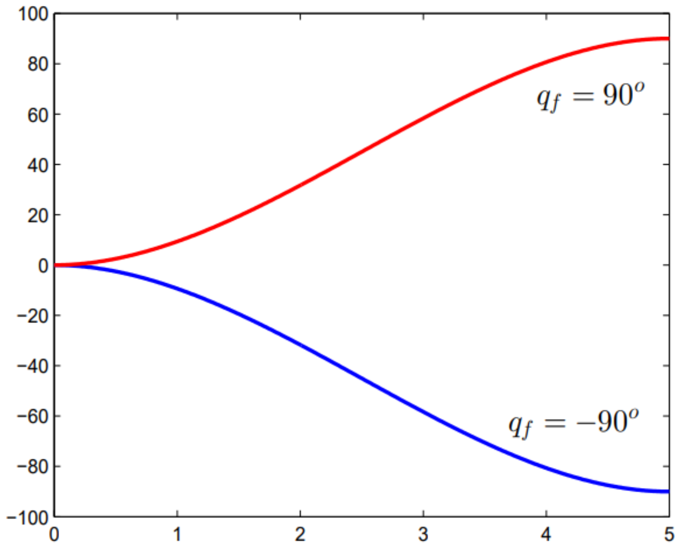
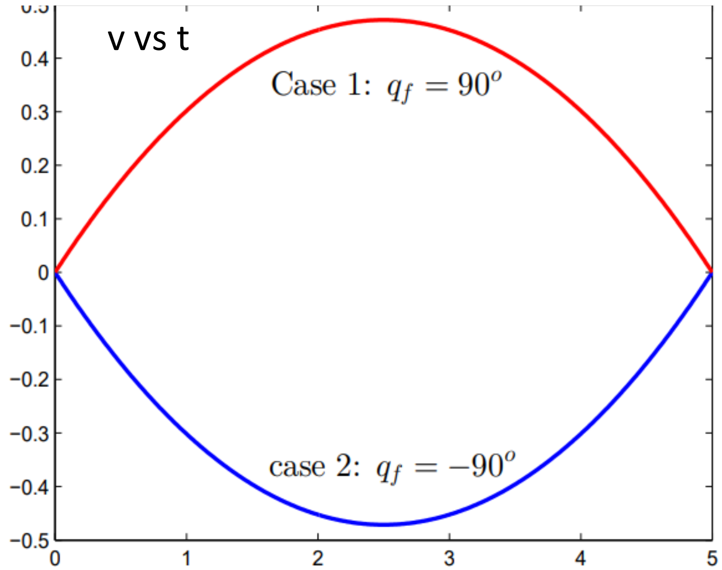
Solution

- Case 1: We have four parameters: $a_0 = 0, a_1 = 0, a_2 = 0.1885, a_3 = -0.0251$

- Case 2: We have four parameters: $a_0 = 0, a_1 = 0, a_2 = -0.1885, a_3 = 0.0251$

The configuration variable and the velocity profiles are shown in figure 1 and 2





$t_0 = 0$

$t_f = 5$

$q_0 = 0 = a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3$

$v_0 = 0 = 0 + a_1 + 2a_2 t_0 + 3a_3 t_0^2$

$q_f = \pi/2 = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3$

$v_f = 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2$

$0 = a_0$

$0 = a_1$

$\pi/2 = 25a_2 + 225a_3$

$0 = 10a_2 + 75a_3$

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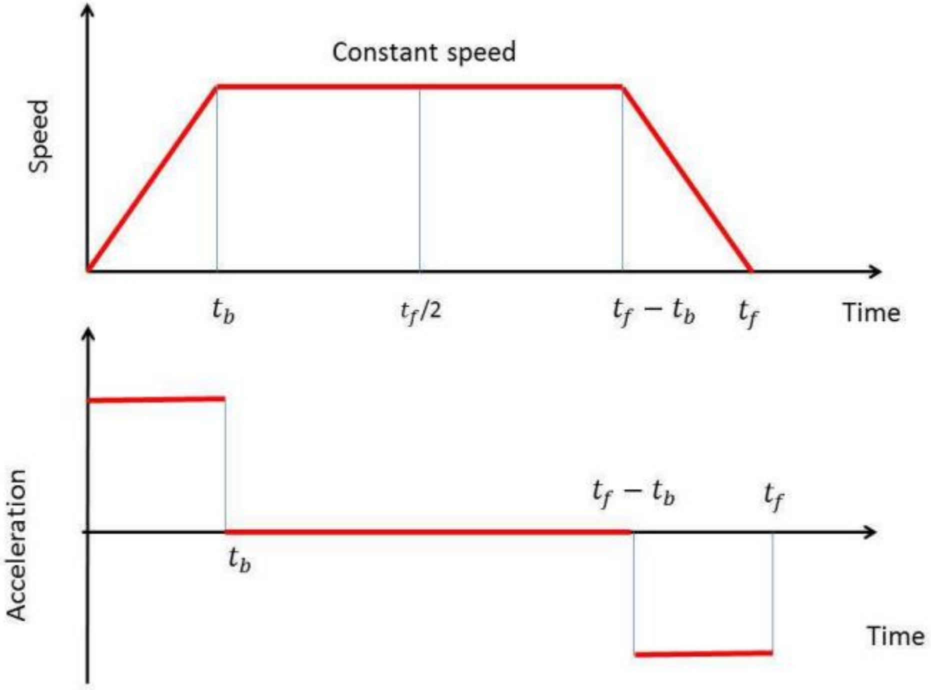
LINEAR SEGMENTS WITH PARABOLIC BLENDS (LSPB)
In this case, we have a specific velocity profile which consists of a trapezoidal velocity profile. The velocity is initially increasing, then constant in the middle interval, then decreasing. The velocity profile is shown n figure 3 where tb is the blend time. The desired trajectory can be divided into three parts;

• Part 1: time interval [t0, tb], the time evolution of the joint variable q(t) consists of a quadratic polynomial

• Part 2: time interval [tb, tf −tb], the time evolution of the joint variable q(t) is linear.

• Part 3: time interval [tf − tb, tf], the joint variable q(t) switches back to the quadratic polynomial.



For simplicity, we assume that

$$t_0 = 0 \tag{20}$$
$$\dot{q}(t_f) = 0 \tag{21}$$
$$\dot{q}(0) = 0 \tag{22}$$

Our goal is determine the trajectory of the joint variable $q(t)$.

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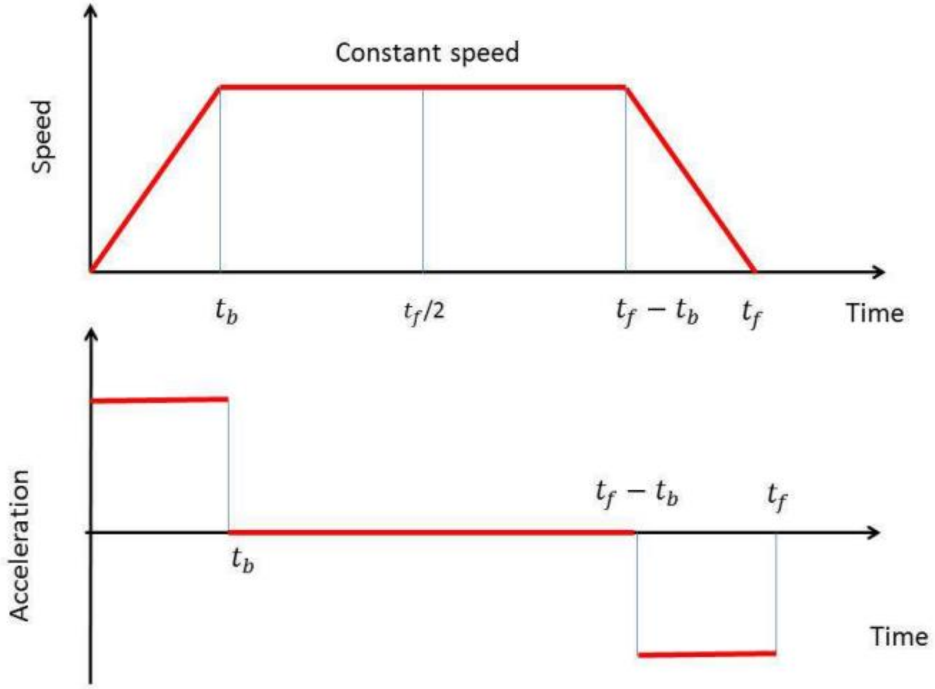

Fig. 3. Trapezoidal velocity profile and corresponding acceleration

- **Part 1: time interval** $[t_0, t_b]$ In this case, the joint variable has a constant positive acceleration, the speed is increasing, and the joint variable is a quadratic function of time, that is

$$q(t) = a_0 + a_1 t + a_2 t^2 \tag{23}$$

$$\dot{q} = a_1 + 2a_2 t \tag{24}$$

Now, by considering the initial condition and the values at time t_b, we have

$$\dot{q}(0) = 0 \Longrightarrow a_1 = 0 \tag{25}$$

$$q(0) = a_0 \tag{26}$$

at time t_b, $\dot{q} = V = constant$

$$\dot{q} = 2a_2 t_b = V \Longrightarrow a_2 = \frac{V}{2t_b} \tag{27}$$

Therefore, in this time interval

$$q(t) = q_0 + \frac{V}{2t_b} t^2 \tag{28}$$

$$\dot{q}(t) = \frac{V}{t_b} t \tag{29}$$

$$\ddot{q}(t) = \frac{V}{t_b} \tag{30}$$

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- **Part 2: time interval** $[t_b, t_f - t_b]$, $q(t)$ is linear with constant speed V. We have

$$\dot{q} = V \tag{31}$$

$$\int_{t_b}^{t} \dot{q} dt = \int_{t_b}^{t} V dt \tag{32}$$

Therefore

$$q(t) - q(t_b) = V(t - t_b) \tag{33}$$

Thus

$$q(t) = q(t_b) + V(t - t_b) \tag{34}$$

Note that t_b is not a free variable, it is constrained by the initial and final configurations, and the speed V. By symmetry we have

$$q(\frac{t_f}{2}) = \frac{q_0 + q_f}{2} \tag{35}$$

Because $q(\frac{t_f}{2})$ happens during the second time interval, equation (34) gives us

$$q(\frac{t_f}{2}) = q(t_b) + V(\frac{t_f}{2} - t_b) \tag{36}$$

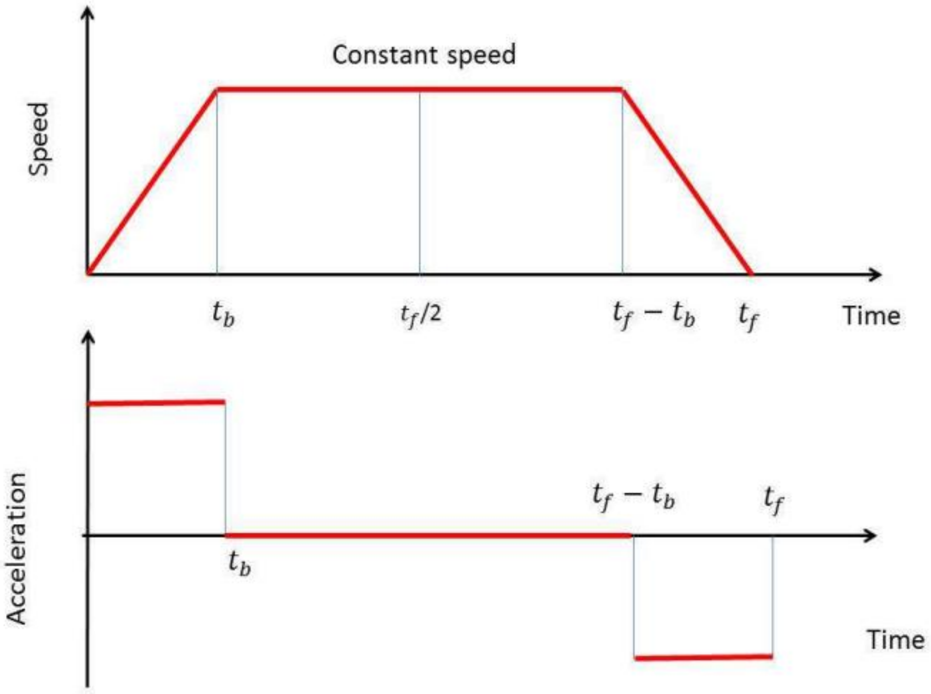

Fig. 3. Trapezoidal velocity profile and corresponding acceleration

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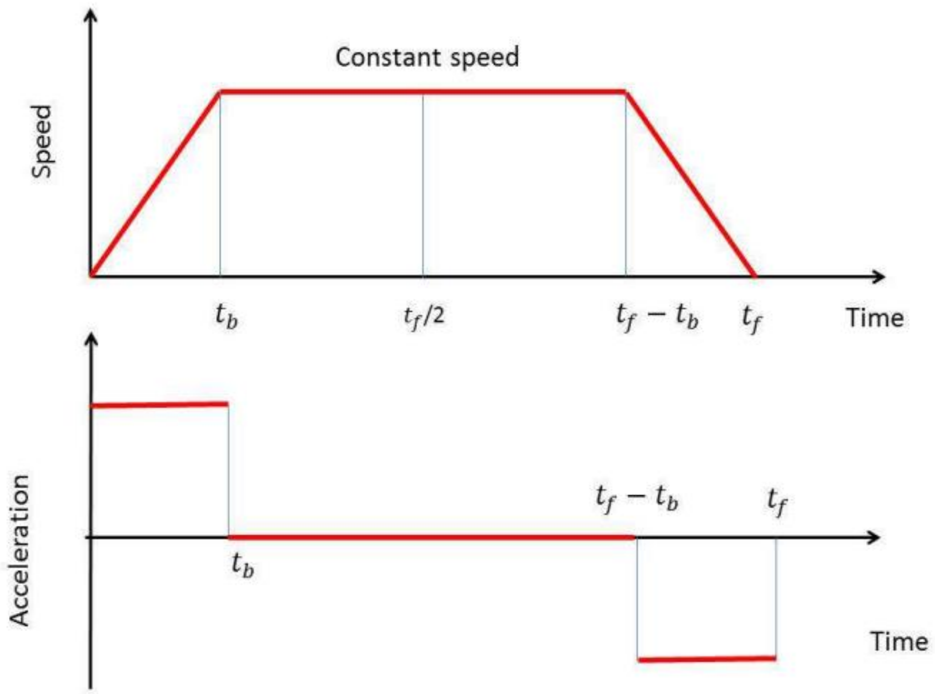

Fig. 3. Trapezoidal velocity profile and corresponding acceleration

Combining equations (36) and (35), we get

$$\frac{q_0 + q_f}{2} = q(t_b) + V(\frac{t_f}{2} - t_b) \tag{37}$$

$$q(t_b) = \frac{q_0 + q_f}{2} - V(\frac{t_f}{2} - t_b) \tag{38}$$

Now $q(t_b)$ can be expressed using the equation of $q(t)$ in the first time interval given by equation (28)

$$q(t_b) = q_0 + \frac{V}{2} t_b \tag{39}$$

and thus by equating the equations for $q(t_b)$

$$q_0 + \frac{V}{2} t_b = \frac{q_0 + q_f}{2} - V(\frac{t_f}{2} - t_b) \tag{40}$$

and finally

$$t_b = \frac{q_0 - q_f + V t_f}{V} \tag{41}$$

so finally for the time interval $[t_b, t_f - t_b]$, we have

$$q(t) = q(t_b) + V(t - t_b) \tag{42}$$

$$q(t) = \frac{q_f + q_0 - V t_f}{2} + V t \tag{43}$$

$$\dot{q}(t) = V \tag{44}$$

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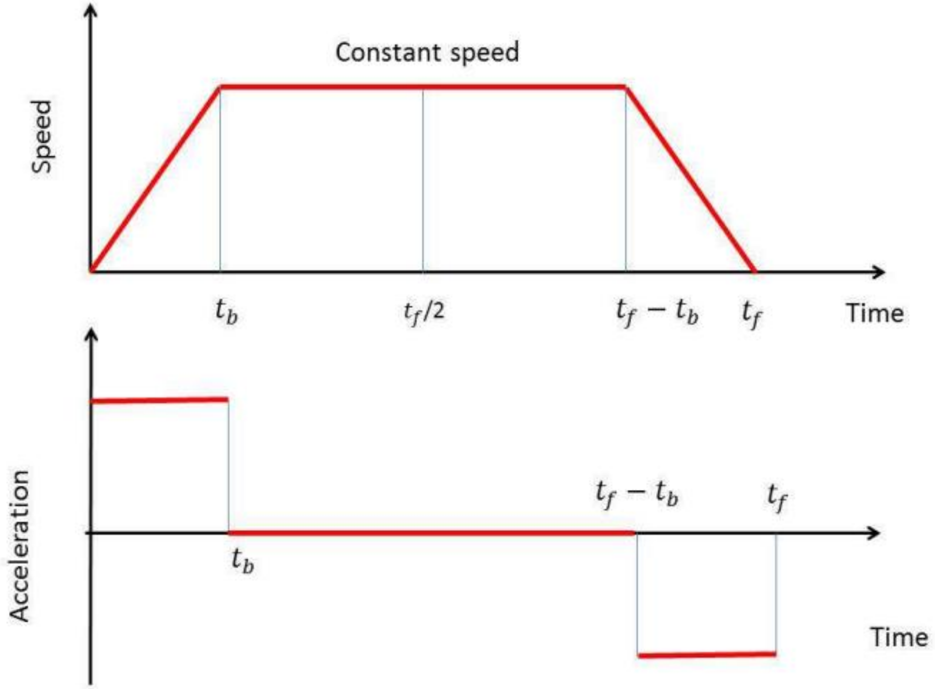


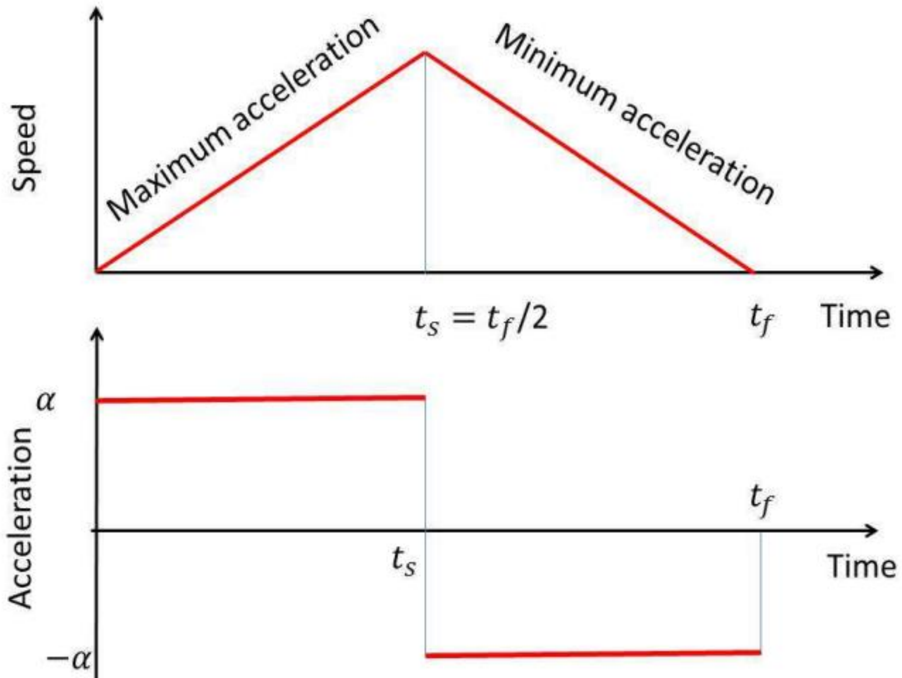
Fig. 3. Trapezoidal velocity profile and corresponding acceleration

- **Part 3: time interval** $[t_f - t_b, t_f]$, $q(t)$ switches back to quadratic, using similar logic, we have

$$q(t) = q_f - \frac{V}{2t_b}t_f^2 + \frac{V}{t_b}t_f t - \frac{V}{2t_b}t^2 \tag{45}$$

It is important to recall that in this case, we assumed that motion begins and ends at rest, we have specified values for the initial and final values of the joint variable and its speed $q_0, q_f, \dot{q}_0, \dot{q}_f$ as well as the final time t_f. We did not specify the blend time t_b or $q(t_b)$.

In the previous case, we specified the final time, it is possible to calculate the shortest time. This case corresponds to a triangular velocity profile as shown in figure 4. The joint variable moves with maximum acceleration (α) in the first time interval and switches at time t_s (switching time) to the minimum acceleration ($-\alpha$). Let V_s be the speed at the switching time, we have



$$V_s = \alpha t_s$$

and

$$t_f = 2t_s$$

The switching time is given by

$$t_s = \sqrt{\frac{q_f - q_0}{\alpha}}$$

Example

This example illustrates motion planning for the planar manipulator using polynomial trajectories. The initial configuration is $q_1(0) = 0, \dot{q}_1(0) = 0, q_2(0) = 0, \dot{q}_2(0) = 0$ and the desired configuration is $q_1(0) = \pi/2, \dot{q}_1(0) = 0, q_2(0) = \pi/4, \dot{q}_2(0) = 0$. Figure 5 shows the manipulator and its path. Figures 6 and 7 show the time evolution for the joint angles and velocities.

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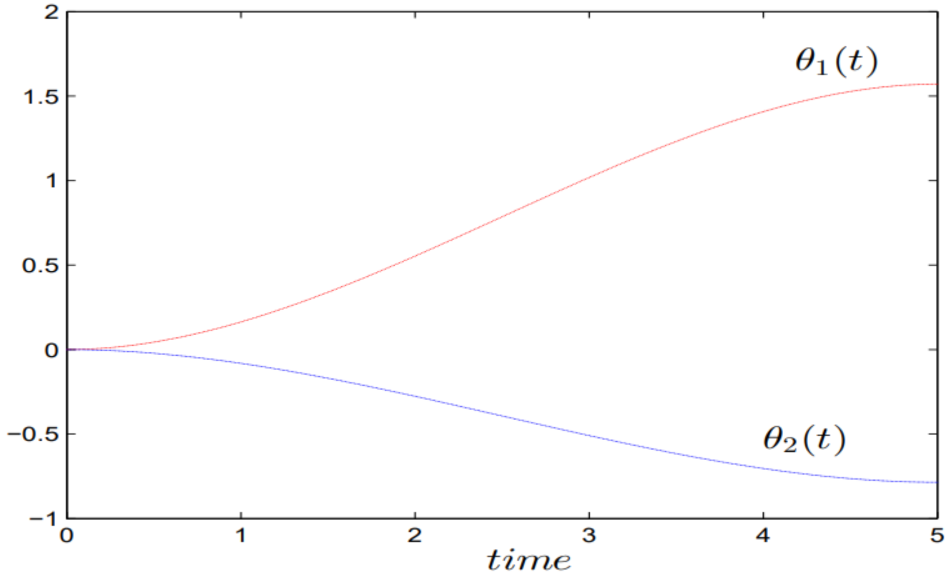

Fig. 6. Time evolution of the joint variables

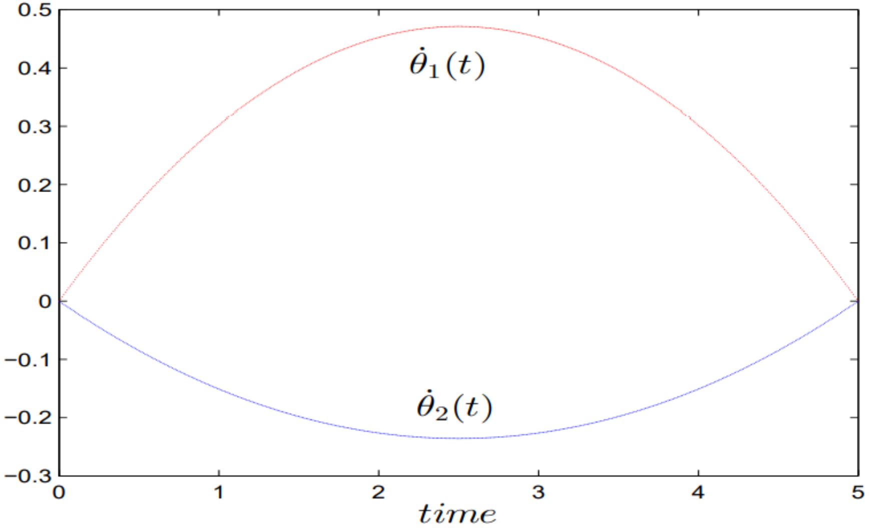

Fig. 7. Time evolution of the velocity of the joint variables

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CODE EXAMPLE An example of the implementation of trajectory planning using polynomials of degree 4 and 6 is shown on the right. The functions are called polynomialtrajectory.m and polynomialtrajectorydegree6.m. These functions are called in the main program by assigning the values of the initial and final values as well as the initial and final time.

```
%Function is saved as polynomialtrajectory.m

function [A]=polynomialtrajectory(q0,dq0,qf,dqf,t0,tf)
q=[q0,dq0,qf, dqf]a'
T=[1 t0, t0^2 t0^3;0 1 2*t0 3*t0^2;1 tf, tf^2 tf^3;0 1 2*tf 3*tf^2 ]
A=[inv(T)]*q
```

```
%Function is saved as polynomialtrajectorydegree6.m

function [A]=polynomialtrajectorydegree6(q0,dq0,q1,dq1,qf, dqf,t0,t1,tf)
 q=[q0,dq0,q1,dq1,qf, dqf]'
T=[1 t0 t0^2 t0^3 t0^4 t0^5
0 1 2*t0 3*t0^2 4*t0^3 5*t0^4
 1 t1 t1^2 t1^3 t1^4 t1^5
  0 1 2*t1 3*t1^2 4*t1^3 5*t1^4
    1 tf tf^2 tf^3 tf^4 tf^5
0 1 2*tf 3*tf^2 4*tf^3 5*tf^4]
A=[inv(T)]*q
```

```
%Calling functions
[A]=polynomialtrajectory(q0,dq0,qf,dqf,t0,tf) %for polynomial of degree 4
[A]=polynomialtrajectorydegree6(q0,dq0,q1,dq1,qf,dqf,t0,t1,tf)%for polynomial of degree
6
```