VIGOMAR KIM ALGADOR HOMEWORK 02

CSC28 — 01 22SU 19 JUNE 2022

1. $S(n) = 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1 \qquad n >= 1$

INDUCTION:

① $p(1): \quad 1 \cdot 1! = (1+1)! - 1$

$1 = 2! - 1 \longrightarrow 1 = 1$ ✓

② $p(k): \quad 1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! = (k+1)! - 1$

③ $p(k+1): \quad 1 \cdot 1! + 2 \cdot 2! + \ldots + k \cdot k! + (k+1) \cdot (k+1)! = (k+2)! - 1$

$(k+1)! - 1 \quad + (k+1) \cdot (k+1)! = (k+2)! - 1$

LHS: $(k+1)! - 1 + (k+1) \cdot (k+1)!$

$= (k+1+1)(k+1)! - 1$

$= (k+2)(k+1)! - 1$

$= (k+2)! - 1$ ✓

2. $n! >= 2^{n-1}$ for $n >= 3$

INDUCTION:

① $p(1): \quad 1 \geq 2^{1-1} \longrightarrow 1 \geq 1$ ✓

② $p(k): \quad k! \geq 2^{k-1}$

③ $p(k+1): \quad (k+1)\,k! \geq 2^{k}$

$k! \geq 2^{k-1} \xrightarrow{\cdot (k+1)} (k+1)\,k! \geq 2^{k-1}(k+1)$ ①

$k+1 \geq 2 \xrightarrow{\cdot 2^{k-1}} (k+1)\,2^{k-1} \geq 2 \cdot 2^{k-1}$

$(k+1)\,2^{k-1} \geq 2^{k}$ ②

① $(k+1)\,k! \geq 2^{k-1}(k+1)$

② $(k+1)\,2^{k-1} \geq 2^{k}$

$(k+1)\,k! \geq 2^{k}$ ✓

3. $7^n - 1$ is divisible by 6

INDUCTION:

① $p(1): \quad 7^1 - 1 = 6 \cdot 1$

② $p(k): \quad 7^k - 1 = 6n$

③ $p(k+1): \quad 7^{k+1} - 1 = 6m$

$$\text{LHS}: \; 7^{k+1} - 1 = 7^k \cdot 7 - 1 = 7^k(6+1) - 1$$
$$= 7^k \cdot 6 + 7^k - 1 = 7^k \cdot 6 + 6n$$
$$= 6\underbrace{(7^k + n)}_{m} = 6m$$

4. $(1+x)^n >= 1 + n \cdot x$ for $x >= -1$ and $n >= 1$

INDUCTION:

① $p(1): (1+x)^1 \geq 1 + x \longrightarrow 1 + x \geq 1 + x$ ✓

② $p(k): \; (1+x)^k \geq 1 + kx$

③ $p(k+1): \; (1+x)^{k+1} \geq 1 + (k+1)x$

$$(1+x)^k \geq 1 + kx \xrightarrow{\cdot (1+x)} (1+x)^k(1+x) \geq (1+kx)(1+x)$$
$$(1+x)^{k+1} \geq 1 + x + kx + kx^2 \qquad kx^2 > 0$$
$$(1+x)^{k+1} \geq 1 + x + kx$$
$$(1+x)^{k+1} \geq 1 + (k+1)x$$

5. $2n + 1 \leq 2^n \quad n \geq 3$

INDUCTION:

① $p(3): \; 2(3) + 1 \leq 2^3 \longrightarrow 7 \leq 8$ ✓

② $p(k): \; 2k + 1 \leq 2^k$

③ $p(k+1): \; 2(k+1) + 1 \leq 2^{k+1}$

$$2(k+1) + 1 \leq 2^k \cdot 2$$

$$2k + 1 \leq 2^k \xrightarrow{+2} 2k + 1 + 2 \leq 2^k + 2$$

$a < b, \; b < c$ $\qquad$ $2(k+1) + 1 \leq 2^k + 2$

$a < c$ $\qquad$ ② $2^k + 2 \leq 2^k \cdot 2$

$$2(k+1) + 1 \leq 2^k \cdot 2$$
$$2(k+1) + 1 \leq 2^{k+1} \; ✓$$

6. $p(n) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

INDUCTION:

① $p(1): \frac{1}{1\cdot 3} = \frac{1}{2(1)+1} \longrightarrow \frac{1}{3} = \frac{1}{3}$

② $p(k): \frac{1}{1\cdot 3} + \dots + \frac{1}{(2k-1)(2k+1)} = \frac{k}{2k+1}$

③ $p(k+1): \frac{1}{1\cdot 3} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]} = \frac{k+1}{2(k+1)+1}$

$$\text{LHS}: \frac{1}{1\cdot 3} + \dots + \frac{1}{(2k-1)(2k+1)} + \frac{1}{[2(k+1)-1][2(k+1)+1]}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)}$$

$$= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$= \frac{2k^2+3k+1}{(2k+1)(2k+3)} = \frac{\cancel{(2k+1)}(k+1)}{\cancel{(2k+1)}(2k+3)} = \frac{k+1}{2k+3} = \frac{k+1}{2k+2+1}$$

$$= \frac{k+1}{2(k+1)+1} \checkmark$$

7. For any integer value n, if n is odd, then n^2 is odd

DIRECT PROOF:

let $n = 2k+1$

$$n^2 = (2k+1)^2 = 4k^2+4k+1$$

$$= 2(2k^2+2k)+1$$

$$= 2z+1 \quad \text{(odd number)}$$

8. For all x in R, if x^3 is irrational then x is irrational too

CONTRAPOSITIVE:

let p: x^3 is irrational

q: x is irrational

$\sim q \longrightarrow \sim p$

x is rational $\longrightarrow$ x^3 is rational

$x = \frac{a}{b} \qquad x^3 = \left[\frac{a}{b}\right]^3 = \frac{a^3}{b^3} = \frac{c}{d}$

$x^3 = \frac{c}{d}$

Since we proved that $\sim q \rightarrow \sim p$ is true therefore $p \rightarrow q$ is also true.

9. For all real numbers x and y, if $x \cdot y \le 2$ then either $x < \sqrt{2}$ or $y \le \sqrt{2}$

CONTRADICTION: $\sim(p \to q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

$p : x \cdot y \le 2$

$q : x < \sqrt{2} \vee y \le \sqrt{2}$

$\sim q : x \ge \sqrt{2} \wedge y > \sqrt{2}$

$\left.\begin{array}{l} x \ge \sqrt{2} \\ y > \sqrt{2} \end{array}\right] x \cdot y > \sqrt{2} \cdot \sqrt{2} = 2$

$x \cdot y > 2$

10. If 100 balls are placed in nine boxes, some boxes contains 12 or more balls

CONTRADICTION:

p: 100 balls place in 9 boxes

q: Some boxes contain 12 or more

~q: All boxes contain less than 12

11 balls · 9 boxes

= 99 balls

11. $\max(x,y) = \dfrac{x+y+|x-y|}{2}$ for all x and y in R.

PROOF BY CASES:

① $x > y$: $|x-y| = x-y$ $\qquad x+y+x-y/2 = 2x/2 = x$

② $x < y$: $|x-y| = -x+y$ $\qquad x+y-x+y/2 = 2y/2 = y$

③ $x = y$: 0 $\qquad x+y+0/2 \to x+x/2 = 2x/2 = x$

$\qquad\qquad\qquad\qquad\quad \to y+y/2 = 2y/2 = y$

12. If four teams play seven games, some pair of the teams play at least two times.

CONTRADICTION:

p: 4 teams play 7 games

q: some pair of the teams play at least 2 times

~q: No pair of the teams play at least 2 times

let teams be: A, B, C, D

Games: AB, AC, AD, BC, BD, CD

There are only 6 games