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CSUS College of Engineering and Computer Science
Electrical & Electronic Engineering
ENGR 120 Probability and Random Signals

Final Exam (100 points, 120 min.)

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Question 1 [25 points]

(a) Answer the following questions (True or False).

(i) If A and B are both nonempty events of a sample space S and A and B are mutually exclusive, then A and B are dependent. **T/F** TRUE

(ii) For the random variables X and $Y = 3X+9$, $\mathrm{Var}(Y) = 9\,\mathrm{Var}(X) + 9$. **T/F** TRUE

(iii) The Markov inequity bound only applies to nonnegative random variables. **T/F** TRUE

(iv) The power spectral density of a WSS random process is defined as The Fourier transform of the auto-correlation function of the random process. **T/F** TRUE

(v) The central limit theorem always holds true, regardless of the sample size **T/F** FALSE

(b) Two 6-sided dice are rolled. What is the probability that their sum is at most 3?

$$P(3) = \frac{n(E)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

$E = \{(1,2),(2,1)\}$

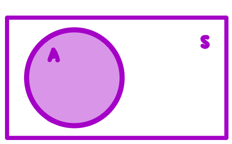
$S = \{(1,1),(1,2),(1,3),\dots,(6.6)\}$

(c) For the following experiment, S is the sample space and A, B, and C are events.
$S = \{s_1, s_2, s_3, s_4, s_5, s_6\}$. $A = \{s_2\}$. $B = \{s_3, s_5, s_5\}$. $C = \{s_2, s_3, s_6\}$. $D = \{s_1, s_2\}$.

Outcome	s_1	s_2	s_3	s_4	s_5	s_6
Probability	1/12	1/6	1/8	1/8	1/6	1/3

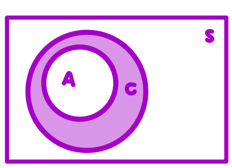
Sketch the Venn diagram of events and find the following probabilities.
(i) Pr(A).
(ii) Pr($A^c \cap C$).
(iii) Which pair of events, A,B,C, and $D,$ (if any) are mutually exclusive?

(i)



$P(A) = P(S_2) = 1/6$

(ii)



$P(A^c \cap C) = P(S_3) + P(S_6) = 1/8 + 1/3 = 11/24$

(iii) $P(A \cap B) = 0$, $P(B \cap D) = 0$

(A,B) and (B,D) are mutual exclusive.

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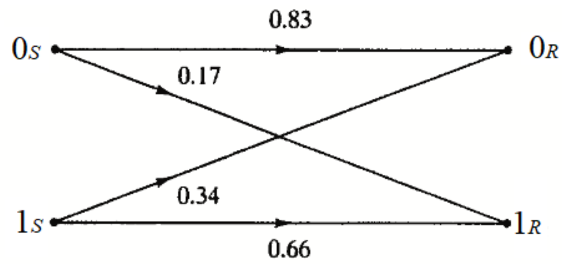
Question 2 [25 points]
Consider the binary communication channel depicted below. Given that the a “1” is observed at the receiver, calculate the probability that a “1” was transmitted, i.e. $\Pr(1_S \mid 1_R)$. Assume $\Pr(0_S) = \Pr(1_S) = 0.5$.



$$\Pr(1_R) = \Pr(0_S)\cdot 0.17 + \Pr(1_S)\cdot 0.66$$
$$= (0.5)(0.17) + (0.5)(0.66)$$
$$= 0.415$$

$$\Pr(1_S \mid 1_R) = \frac{\Pr(1_R \mid 1_S)\cdot \Pr(1_S)}{\Pr(1_R)} = \frac{(0.66)(0.5)}{0.415}$$
$$= 0.795$$

b) Consider the random variable $Y = X + k$, where k is a constant and X is a random variable that is always strictly larger than -10 and has an expected value of $E(X) = -6$. Choose a suitable value for the constant k and determine an upper bound for the probability $\Pr(Y > 8)$. Justify your choice for the constant k.

$$E[Y] = E[X + k + 10]$$
$$= E[X] + E[k] + 10$$
$$= -6 + k + 10$$

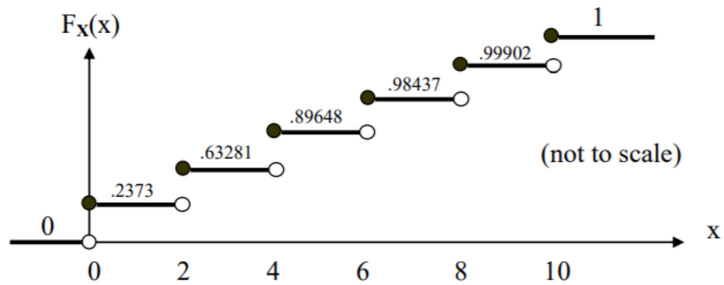
$$\Pr(Y>8) \le \frac{E[Y]}{8} = \frac{4+k}{8}$$

Let $k = -4$
$E[Y] = 0$

$$\Pr(Y>8) \le 0$$

Question 3 [25 points]

a) The distribution (CDF) of a discrete random variable is shown in the figure below.



i) Find the probability Pr(X≤2).

$$P(x \le 2) = P(x=2) + P(x=0)$$
$$= 0.39551 + 0.2373 = 0.63281$$

ii) Calculate the variance of X, ie VAR[X].

$$E(x) = \sum x \cdot P(x) = 0(0.2373) + 2(0.39551) + 4(0.26367)$$
$$+ 6(0.08789) + 8(0.01465) + 10(0.00098)$$
$$= 2.50004$$

iii) Sketch the corresponding probability mass function PMF.

(iii)

b) The PDF of a random variable X is given by:

$$f_X(x) = \begin{cases} 0.4 + kx, & 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

i) Find the value k that makes f_X a valid PDF.
ii) Find P(X>1/2).
iii) Find the CDF, $F_X(x)$.

(i) $\int_0^4 (0.4 + kx)dx = 1 \longrightarrow \left[0.4x + \frac{kx^2}{2}\right]_0^4 = 1$

$1.6 + 8k = 1$

$k = -0.075$

(ii) $Pr[x > 1/2] = \int_{1/2}^{4} (0.4 - 0.075x)dx$

$= \left[0.4x - 0.075x^2/2\right]_{1/2}^{4}$

$= 0.8094$

(iii) for $x<0$, $F_x(x) = 0$

for $0 \le x \le 4$, $F_x(x) = \int_0^x (0.4 - 0.075x)dx$

$= 0.4x - 0.0375x^2$

for $x > 4$, $F_x(x) = 1$

Question 4 [25 points]

a) Let the joint density function $f_{X,Y}(x,y)$ be

$$f_{X,Y}(x,y) = \frac{xy}{9} \qquad ; 0 \le x \le 2,\ 0 \le y \le 3$$

Determine:

i) The marginal PDFs $f_X(x)$ and $f_Y(y)$.

ii) The expected values of X and Y.

iii) Are X and Y statistically independent?

iv) Are X and Y uncorrelated? Justify your answer. *Hint: There is an easier way to solve this question without working-out the double integral*

(i) $f_X(x) = \int_0^3 (xy/9)\,dy = \left[xy^2/18\right]_0^3 = 1/2\, x \qquad 0 \le x \le 2$

$f_Y(y) = \int_0^2 (xy/9)\,dx = \left[x^2y/18\right]_0^2 = 2y/9 \qquad 0 \le y \le 3$

(ii) $E[x] = \int_0^2 x \cdot 1/2\, x \; dx = 1/2 \int_0^2 x^2 dx = 1/2 \left[x^3/3\right]_0^2 = 4/3$

$E[y] = \int_0^3 y \cdot 2y/9 \, dy = 2/9 \int_0^3 y^2 dy = 2/9 \left[y^3/3\right]_0^3 = 2$

(iii) $f(x,y) \overset{?}{=} f(x) \cdot f(y)$

$\frac{xy}{9} \overset{?}{=} \frac{x}{2} \cdot \frac{2y}{9} \longrightarrow \frac{xy}{9} = \frac{xy}{9}$

$\therefore$ x and y are statistically independent

(iv) $COV[x,y] = E[xy] - E[x]E[y]$

$= 0$

$E[xy] = E[x] \cdot E[y]$

$CORR[x,y] = \frac{COV[x,y]}{\sigma_x \sigma_y} = 0$

$\therefore$ x and y are uncorrelated

b) Consider the random process

$$Y(t) = A,$$

where $A \sim N(0,1)$ is a standard Gaussian random variable. Is this process WSS? Justify your answer.

$E[Y(t)] = E[A]$

$= 0$ constant mean

mean = 0

variance = 1

$R_Y(\tau) = E[Y(t_1)\,Y(t_2)]$

$= E[A^2]$

$= \sigma_A^2 \cdot \delta(t_1 - t_2)$

$= \delta(t_1 - t_2)$ time different

$Y(t)$ is WSS