CALIFORNIA STATE UNIVERSITY

SACRAMENTO

DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

EEE 117 Network Analysis

Text: Electric Circuits by J. Nilsson and S. Riedel Prentice Hall

Lecture Set 4: Laplace Transform in Circuit Analysis

Instructor: Riaz Ahmad

Laplace Transform in Circuit Analysis

- Preview
- Circuit Elements in the “s” Domain
- Circuit Analysis in the “s” Domain
- The Transfer Function
- The Transfer Function in Partial Fraction Expansions
- The Transfer Function and the Steady-State Sinusoidal Response
- The Impulse Function in Circuit Analysis
- Bode Diagram
- What is Decibel (dB)
- Magnitude in terms of Linear versus dB
- Bode Plot for Real, First-Order Poles and Zeros
- Bode Plot for Complex Poles and Zeros

Preview

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

- The Laplace transform has two unique characteristics:
- First, it transforms a set of linear constant-coefficient differential equations into a set of linear polynomial equations, which are easier to manipulate.
- Second, it automatically introduces into the polynomial equations the initial values of the current and voltage variables. Thus, initial conditions are an inherent part of the transform process.

$$\begin{array}{rcl} \text{Time Domain} & & \text{Freq Domain} \\ \text{Volts} & \longleftrightarrow & \text{Volts sec} \\ \text{Amps} & \longleftrightarrow & \text{Amps sec} \\ \Omega = \dfrac{V}{A} & \longleftrightarrow & \dfrac{V\,\text{sec}}{A\,\text{sec}} = \dfrac{V}{A} = \Omega \end{array}$$

Circuit Elements in the "s" Domain

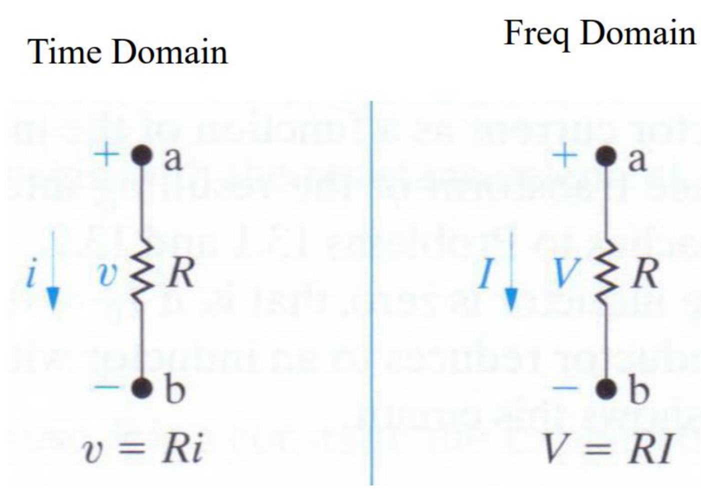
- The procedure for developing an s-domain equivalent for each circuit element is simple and straightforward.
- Find Laplace transform of each element.

Resistor: A resistive circuit in time domain can be represented:

$$v = Ri$$

Take the Laplace of both sides

$$V(s) = R\,I(s) \qquad \textit{Where } V = \mathscr{L}\{v\} \qquad \textit{And } I = \mathscr{L}\{i\}$$



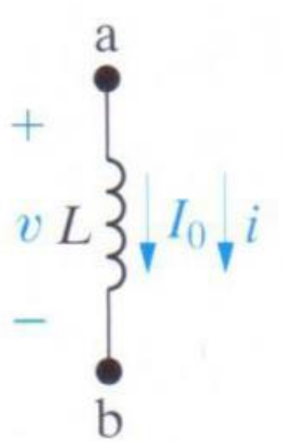
Inductor: An Inductive circuit can be represented as:

$$v = L\frac{di}{dt}$$

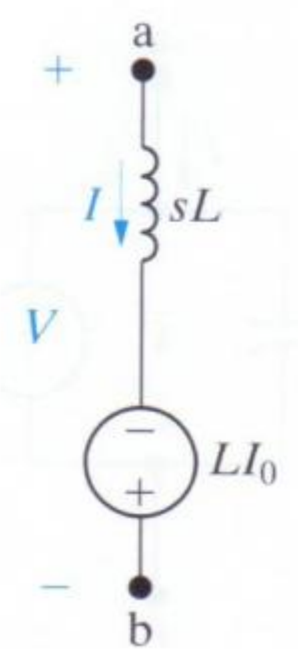
Take the Laplace of both sides

$$V = L\left[sI - i(0^-)\right] \quad = sLI - L\underbrace{i(0^-)}_{I_0}$$

$$= sLI - LI_0$$



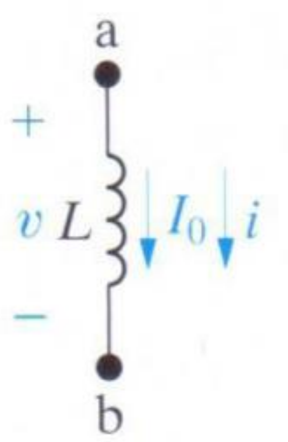
$$V = sLI - LI_0$$



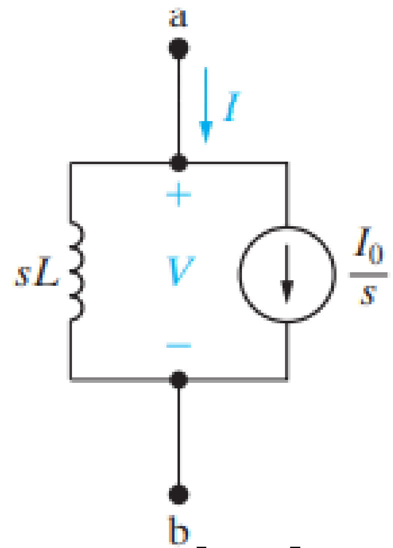
- The same inductor element can be modeled by a parallel connection by solving the previous equation for I. (or by source transformation)

$$V = sLI - LI_0 \quad \Rightarrow I = \frac{V + LI_0}{sL} \quad = \frac{V}{sL} + \frac{I_0}{s}$$

- Thus, the inductor can also be shown as:



$$I = \frac{V}{sL} + \frac{I_0}{s}$$



- Note that the passive sign convention determines the direction of current flow.

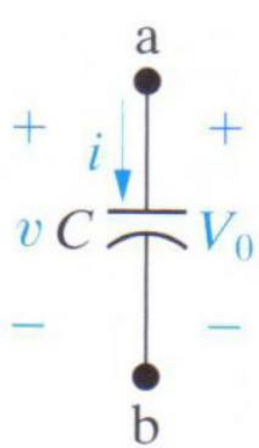
Capacitor: A Capacitive circuit can be represented as:

$$i = C\frac{dv}{dt}$$

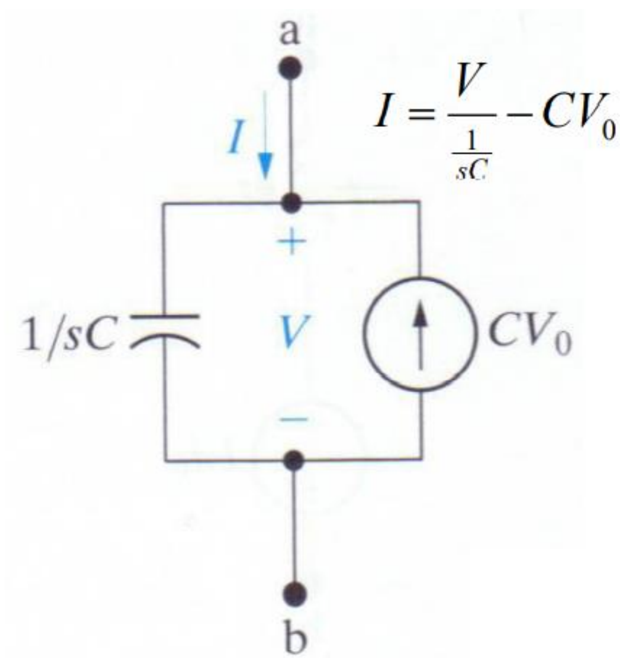
Take the Laplace of both sides

$$I = C\left[sV - v(0^-)\right] \quad = \frac{V}{\frac{1}{sC}} - CV_0$$

Note that $Z_C = \frac{1}{sC}$

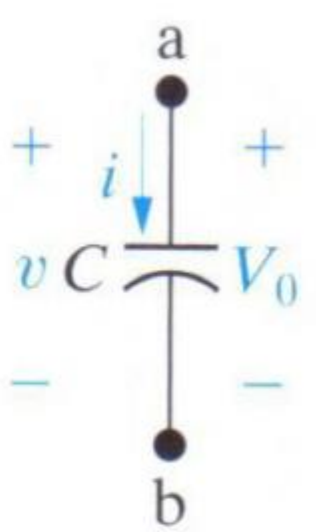


$I = \frac{V}{\frac{1}{sC}} - CV_0$

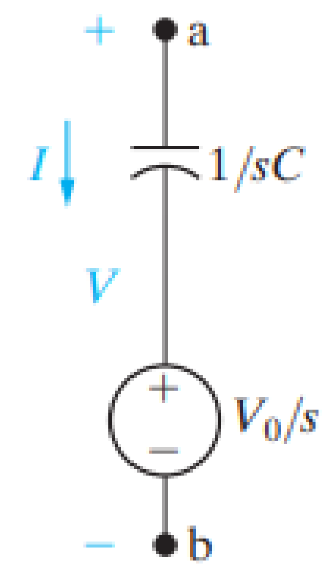


- The same capacitor element can be modeled by a series connection by solving the previous equation for V. (or by source transformation)

$$I = sCV - CV_0 \Rightarrow V = \frac{I + CV_0}{sC} = I\frac{1}{sC} + \frac{V_0}{s}$$

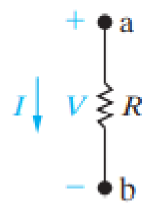
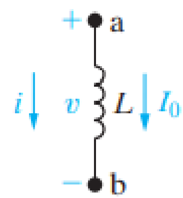
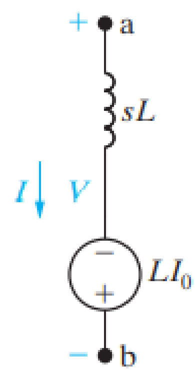
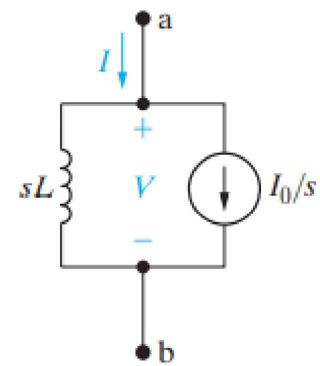
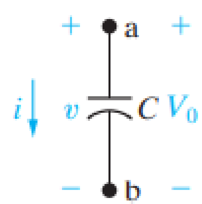
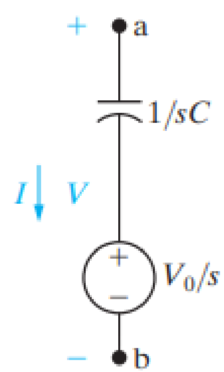
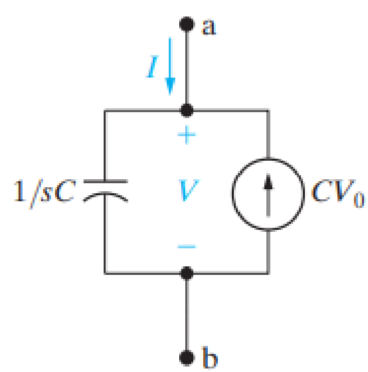


$$V = I\frac{1}{sC} + \frac{V_0}{s}$$



- Note that the passive sign convention again determines the direction of current flow.

TABLE 13.1 **Summary of the *s*-Domain Equivalent Circuits**

TIME DOMAIN	FREQUENCY DOMAIN	
 $v = Ri$	 $V = RI$	
 $v = L\,di/dt,$ $i = \frac{1}{L}\int_{0^-}^{t} v\,dx + I_0$	 $V = sLI - LI_0$	 $I = \frac{V}{sL} + \frac{I_0}{s}$
 $i = C\,dv/dt,$ $v = \frac{1}{C}\int_{0^-}^{t} i\,dx + V_0$	 $V = \frac{I}{sC} + \frac{V_0}{s}$	 $I = sCV - CV_0$

Circuit Analysis in the "s" Domain

- When no initial conditions are present, we can write the following equation in the "s" domain for any of the three circuit elements.

$$V = ZI$$

For the resistor Z = R Ohms.

For the inductor Z = sL Ohms.

For the capacitor Z = 1/(sC) Ohms.

The Transfer Function

- The transfer function is defined as the s-domain ratio of the (Laplace transformed) output to the input with all initial conditions =0

$$H(s) = \frac{Y(s)}{X(s)}$$

- Y(s) is called the response of the system.
- X(s) is the input to the system.
- For linear lumped parameter circuits (which is most of your undergraduate study), H(s) is always a rational function of s.

$$H(s) = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}.$$

  - The poles of are the roots of the denominator.

$$\boxed{\begin{aligned} p_1 &= -3000 - j4000, \\ p_2 &= -3000 + j4000. \end{aligned}}$$

  - The zeros of are the roots of the numerator.

$$\boxed{z_1 = -5000.}$$

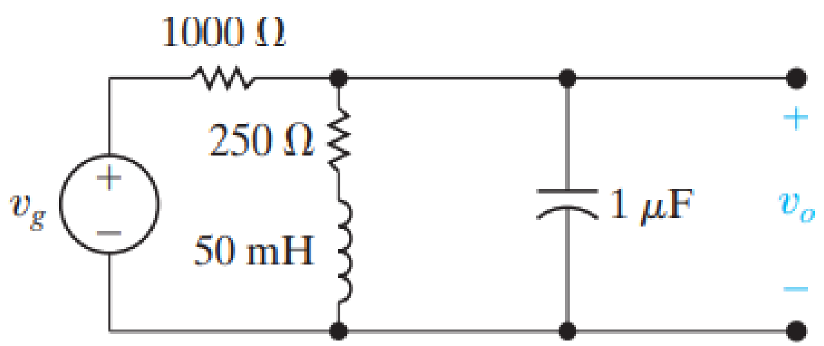
- Complex poles and zeros always appear in conjugate pairs.
- The poles (from the denominator) of H(s) must lie in the left-hand plane (LHP) for a bounded response.
- The zeros (from the numerator) of H(s) may lie anywhere.
- We can write the circuit output Y(s) as the product of the transfer function H(s) and the driving function X(s).

$$Y(s) = H(s)X(s)$$

- For the inputs of common use in circuits, the input X(s) is also a rational function.
- Thus, by partial fraction expansion, we can form the poles of H(s) and X(s) into a summation of terms.
- The terms generated by poles of X(s) give rise to the steady state component of total response.
- The terms generated by poles of H(s) give rise to the transient components of total response.

Ex-1: The circuit shown below is driven by a voltage source whose voltage increases linearly with time, namely $v_g = 50\,t$

a) Use the transfer function to find v_o.
b) Identify the transient component of the response.
c) Identify the steady-state component of the response.
d) Sketch v_o versus t for $0 \le t \le 1.5$ ms.



The Transfer Function and the Steady-State Sinusoidal Response

- Consider the following sinusoidal signal:

$$x(t) = A\cos(\omega t + \phi),$$

$$x(t) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi,$$

- Note, cosφ and sinφ are constant values.
- Take Laplace of both sides.

$$X(s) = \frac{(A\cos\phi)s}{s^2+\omega^2} - \frac{(A\sin\phi)\omega}{s^2+\omega^2} \quad = \frac{A(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2}$$

$$= \frac{A(s\cos\phi - \omega\sin\phi)}{(s-j\omega)(s+j\omega)}$$

- Note that the input transforms to the s-domain is complex conjugate.

- The output can be written as:

$$Y(s) = H(s)X(s) \quad = H(s)\frac{A\left(s\cos\phi - \omega\sin\phi\right)}{\left(s - j\omega\right)\left(s + j\omega\right)}$$

$$Y(s) = \frac{K}{s - j\omega} + \frac{K^*}{s + j\omega} + \sum \text{terms generated by the poles of } H(s)$$

- Now poles of H(s) contributes only to the transient response and not to steady state response.
- Since we are interested in the steady-state solution at the moment, this allows us to simplify the derivation.
- Thus, the first two terms on the right-hand side of equation determine the steady-state response.

$$Y(s) = \frac{K}{s - j\omega} + \frac{K^*}{s + j\omega}$$

$$Y(s) = H(s)\cdot\frac{A(s\cos\phi - \omega\sin\phi)}{(s-j\omega)(s+j\omega)} = \frac{K}{(s-j\omega)} + \frac{K^*}{(s+j\omega)} \quad -(1)$$

$$H(s)\cdot A(s\cos\phi - \omega\sin\phi) = K(s+j\omega) + K^*(s-j\omega)$$

For $s = j\omega$

$$H(j\omega)\cdot A(j\omega\cos\phi - \omega\sin\phi) = K(j\omega + j\omega)$$

$$K = \frac{H(j\omega)\cdot A(j\omega\cos\phi - \omega\sin\phi)}{2j\omega}$$

$$K = \frac{H(j\omega)\cdot A(j\omega\cos\phi + j^2\omega\sin\phi)}{2j\omega}$$

$$K = \frac{H(j\omega)\cdot A(\cos\phi + j\sin\phi)}{2}$$

$$K = H(j\omega)\cdot\frac{A e^{j\phi}}{2}$$

In general 'H' is a Complex quantity. So

$$H(j\omega) = H\angle\theta = H e^{j\theta}$$

$$K = |H| e^{j\theta}\cdot\frac{A e^{j\phi}}{2} = |H| e^{j(\theta+\phi)}\cdot\frac{A}{2}$$

$$K = \frac{A}{2}\cdot|H| e^{j(\theta+\phi)}$$

$$K^* = \frac{A}{2}\cdot|H|\cdot e^{-j(\theta+\phi)}.$$

Put it in eq (1)

$$Y(s) = \frac{\frac{A}{2}|H|e^{j(\theta+\phi)}}{s-j\omega} + \frac{\frac{A}{2}|H|e^{-j(\theta+\phi)}}{s+j\omega}$$

$$Y(s) = \frac{A}{2}|H|\left[e^{j(\theta+\phi)}\cdot\frac{1}{(s-j\omega)} + e^{-j(\theta+\phi)}\cdot\frac{1}{(s+j\omega)}\right]$$

Take inverse laplace of both sides.

$$y(t) = \frac{A}{2}\cdot|H|\left[e^{j(\theta+\phi)}\cdot e^{j\omega t} + e^{-j(\theta+\phi)}\cdot e^{-j\omega t}\right]$$

$$y(t) = \frac{A}{2}\cdot|H|\left[e^{j(\omega t+\theta+\phi)} + e^{-j(\omega t+\theta+\phi)}\right]$$

$$y(t) = A\cdot|H|\cos(\omega t+\theta+\phi)$$

$$\boxed{y_{ss}(t) = A\cdot|H|\cos(\omega t+\theta+\phi)}$$

Here

$|H|$ = Amplitude of transfer function

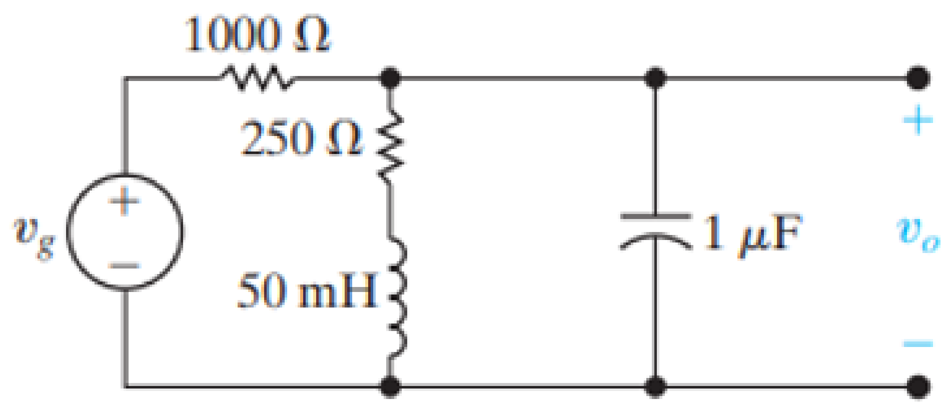
θ = Phase angle of transfer function

ϕ = Phase angle of input

A = Amplitude of input

ω = Frequency of input in rad/sec

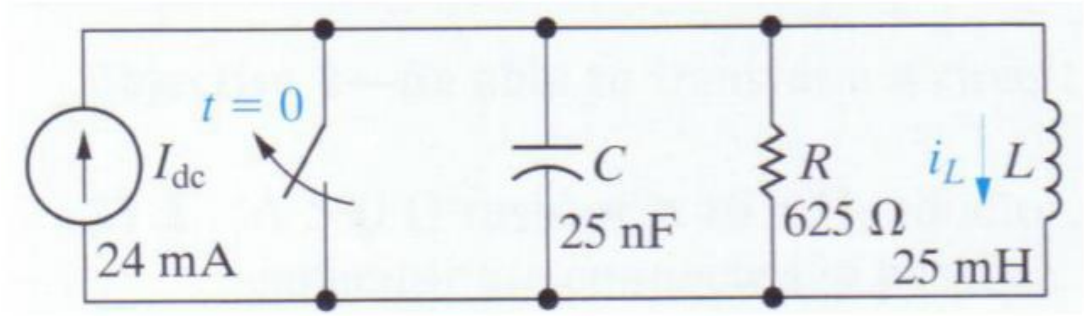
Ex-2: For the circuit shown below, the sinusoidal source voltage is 120 cos(5000t + 30^0) V. Find the steady-state expression for v_o.



The Impulse Function in Circuit Analysis

- Impulse functions occur in circuit analysis either because of a switching operation or because a circuit is excited by an impulsive source.
- If we create the frequency domain equivalent circuit correctly, these impulse functions are handled as a routine matter in the course of solving the frequency domain circuit equations.

Ex-3: Step Response of a Parallel RLC Circuit.

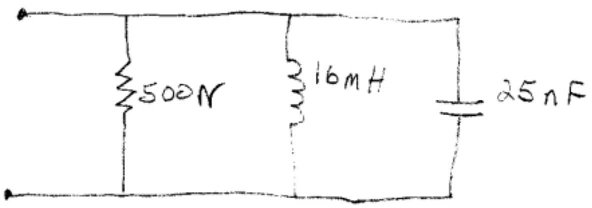


Find the current i_L through the inductor for $t > 0$.

Assume that the switch has been closed for "a long time".

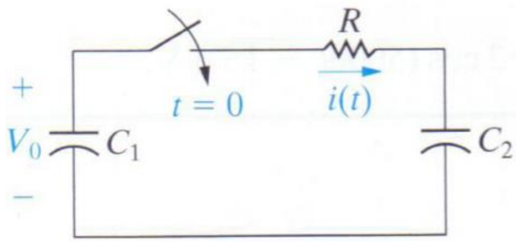
Ex-4: A 500 Ω resistor, a 16 mH inductor, and a 25 nF capacitor are connected in parallel. Find the zeros and poles.

Analyze the circuit and find the zeros and poles.



Ex-5: In the circuit above, the capacitor C_1 is charged to an initial voltage V_0 prior to closing the switch at $t = 0$.

The capacitor C_2 is discharged so that $V_{C2} = 0$ prior to closing the switch at $t = 0$.

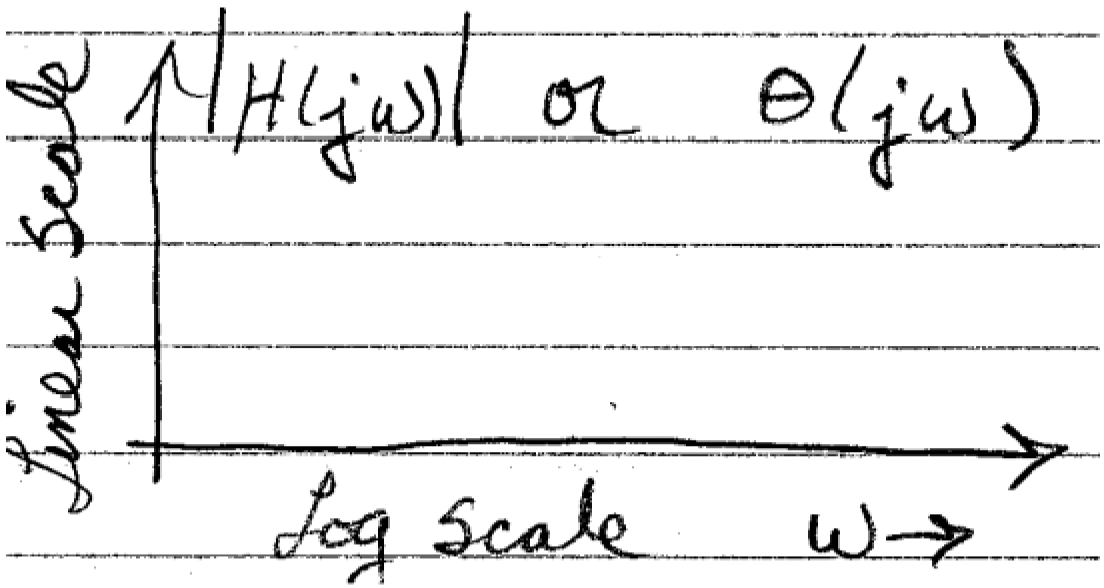


Analyze the circuit and find the current $i(t)$ expression.

Bode Diagram

- A Bode diagram is a graphical technique that gives a feel for the frequency response of the circuit.
- Diagrams are named after Hendrick W. Bode.
- Dr. Bode was an engineer for Bell Telephone Laboratories. He proposed this approach now known by his name.
- We can approximate the complete frequency response by selecting a few appropriate frequencies. Then we draw a straight-line approximation to the complete response.
- There are many software programs that can plot the frequency response very accurately. But as always, the "hand" analysis confirms the validity of the computer program.
- A Bode diagrams consists of two separate plots. One plot shows how the amplitude of $H(j\omega)$ varies with frequency. The other plot shows how the phase $\theta(j\omega)$ varies with frequency.

- The plots are made on semi log paper so that a wide range of frequency values may be plotted.



- The magnitude is given in decibels.

What is Decibel (dB)

- A unit used to measure the intensity of a sound or the power level of an electrical signal.
- The decibel (symbol: dB) is a relative unit of measurement corresponding to one tenth of a bel (B). It is used to express the ratio of one value of a power quantity to another, on a logarithmic scale.

$$1 \text{ dB} = 10 \log \left[\frac{P1}{P2}\right]$$

- dB for a voltage quantity.

$$10\log_{10}\left(\frac{P_1}{P_2}\right) = 10\log_{10}\left(\frac{\frac{V_1^2}{R_1}}{\frac{V_2^2}{R_2}}\right) = 10\log_{10}\left(\frac{V_1^2}{V_2^2}\frac{R_2}{R_1}\right)$$

$$= 10\log_{10}\left(\frac{V_1}{V_2}\right)^2 + 10\log_{10}\left(\frac{R_2}{R_1}\right)$$

$$= 20\log_{10}\left(\frac{V_1}{V_2}\right) + 10\log_{10}\left(\frac{R_2}{R_1}\right)$$

If R1 = R2 then

$$\boxed{dB = 20\log_{10}\left(\frac{V_1}{V_2}\right) + 10\log_{10}(1) \quad = 20\log_{10}\left(\frac{V_1}{V_2}\right)}$$

- dB for a current quantity.

$$10\log_{10}\left(\frac{P_1}{P_2}\right) = 10\log_{10}\left(\frac{I_1^2 R_1}{I_2^2 R_2}\right)$$

$$= 10\log_{10}\left(\frac{I_1}{I_2}\right)^2 + 10\log_{10}\left(\frac{R_1}{R_2}\right)$$

If R1 = R2 then

$$\boxed{dB = 20\log_{10}\left(\frac{I_1}{I_2}\right) + 10\log_{10}(1) \quad = 20\log_{10}\left(\frac{I_1}{I_2}\right)}$$

Magnitude in terms of Linear versus dB

Linear A Voltage or Current	A_{dB}
10^{-6}	-120
10^{-5}	-100
10^{-4}	-80
10^{-3}	-60
10^{-2}	-40
10^{-1}	-20
$\frac{1}{2}$	-6
$\frac{1}{\sqrt{2}} = 0.707$	-3
$10^{0} = 1$	0
$1.41 = \sqrt{2}$	3
2	6
10^{1}	20
10^{2}	40
10^{3}	60
10^{4}	80
10^{5}	100
10^{6}	120

Bode Plot for Real, First-Order Poles and Zeros

- As mentioned before, bode diagrams consists of two separate plots; **Amplitude** plot and the **Phase** plot.

Straight-Line Amplitude Plots:

- The straight-line magnitude plot has the following rules:
- Change the transfer function into standard format which can be one of the following types:

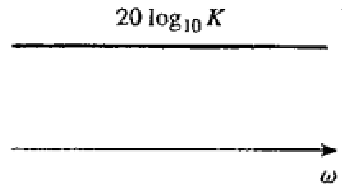
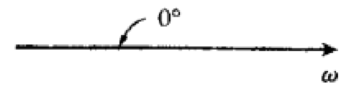
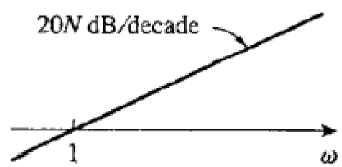
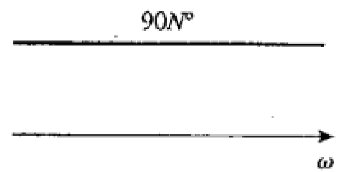
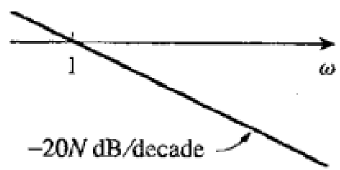
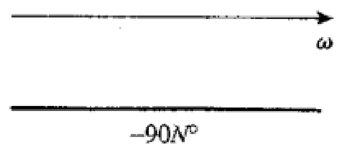
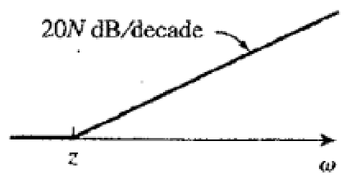
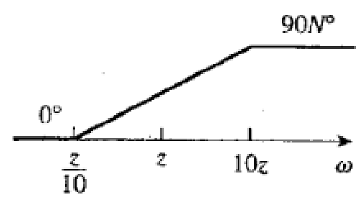
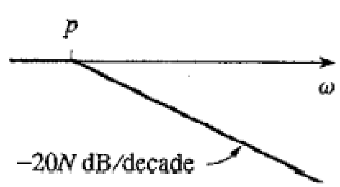
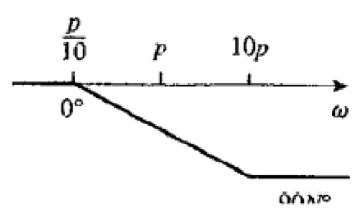
$$H(s)= \frac{K(1+\frac{s}{z1})(1+\frac{s}{z2})(1+\frac{s}{z3})}{(1+\frac{s}{p1})(1+\frac{s}{p2})(1+\frac{s}{p3})} \quad H(s)= \frac{Ks(1+\frac{s}{z1})(1+\frac{s}{z2})(1+\frac{s}{z3})}{(1+\frac{s}{p1})(1+\frac{s}{p2})(1+\frac{s}{p3})} \quad H(s)= \frac{K(1+\frac{s}{z1})(1+\frac{s}{z2})(1+\frac{s}{z3})}{s\,(1+\frac{s}{p1})(1+\frac{s}{p2})(1+\frac{s}{p3})}$$

- First order Zeros will have a slope = 20 dB / decade
- First order Poles will have a slope = -20 dB / decade
- Slopes will start at the corner frequency.
- “K” will contribute a constant value in the plot = 20 log K

Straight-Line Phase Plots:

- The phase angle associated with the constant "K" is zero.
- The phase angle associated with a first-order zero or pole at the origin is a constant, $+90^{o}$ for zeros and -90^{o} for poles.
- For a first-order zero or pole not at the origin, the straight-line approximations are as follows:
  - For frequencies less than one tenth the corner frequency, the phase angle is assumed to be zero.
  - For frequencies greater than 10 times the corner frequency, the phase angle is assumed to be $+90^{o}$ and -90^{o} for zeros and poles respectively.
  - Between one tenth the corner frequency and 10 times the corner frequency, the phase angle plot is a straight line that goes through 0^{o} at one-tenth the corner frequency, $\pm 45^{o}$ at the corner frequency, and $\pm 90^{o}$ at 10 times the corner frequency.

Summary of Bode straight-line magnitude and phase plots.

Factor	Magnitude	Phase
K		
$(j\omega)^N$		
$\dfrac{1}{(j\omega)^N}$		
$\left(1 + \dfrac{j\omega}{z}\right)^N$		
$\dfrac{1}{(1 + j\omega/p)^N}$		

Ex-6: Draw Bode diagram for the following Transfer function.

$$H(s)=\frac{100(s+1)}{(s+10)(s+100)}$$

Ex-7: Draw Bode plot for the following Transfer function.

$$H(s)=\frac{50(s+200)(s+2000)}{(s+20)(s+20000)}$$

Ex-8: Draw Bode plot for the following Transfer function.

$$H(s)=\frac{100s}{(s+10)(s+100)}$$

Ex-9: Draw Bode plot for the following Transfer function.

$$H(s)=\frac{10000(s+1)}{(s+10)(s+100)^2}$$

Bode Plot for Complex Poles and Zeros

- Complex poles/zeros in the expression for H(s) require special attention when you make amplitude and phase angle plots.
- The complex poles and zeros of H(s)always appear in conjugate pairs.
- The first step in making either an amplitude or a phase angle plot of a transfer function that contains complex poles is to combine the conjugate pair into a single quadratic term.
- Express conjugate pair in the form: $s^2 + 2\zeta\omega_n s + \omega_n^2$
- Then change it into standard form: $1 + (s/\omega_n)^2 + 2\zeta(s/\omega_n)$

Straight-Line Amplitude Plots:

- For $\omega < \omega_n$, the straight line lies along the 0 dB axis.
- For $\omega = \omega_n$, the amplitude is 0 dB.
- For $\omega > \omega_n$, the straight line has a slope of -40 dB/decade.

Straight-Line Phase Plots:

- For $\omega = \omega_n$, phase is $\pm 90^o$.
- The straight-line approximations are as follows:

  - For frequencies less than one tenth the corner frequency, the phase angle is assumed to be zero.
  - For frequencies greater than 10 times the corner frequency, the phase angle is assumed to be $+180^o$ and -180^o for zeros and poles respectively.
  - Between one tenth the corner frequency and 10 times the corner frequency, the phase angle plot is a straight line that goes through 0^o at one-tenth the corner frequency, $\pm 90^o$ at the corner frequency, and $\pm 180^o$ at 10 times the corner frequency.

Ex-10: Draw Bode plot for the following Transfer function.

$$H(s) = \frac{4(s+25)}{(s^2+4s+100)}$$