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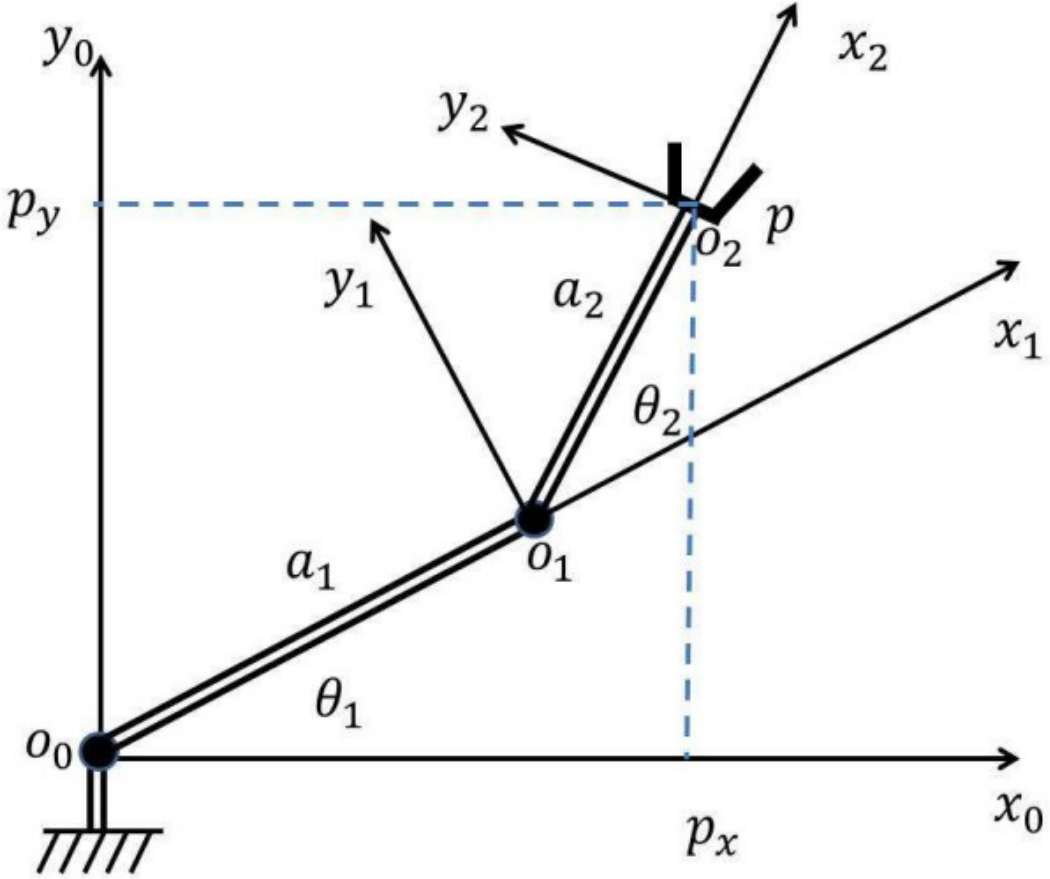
VELOCITY KINEMATICS AND JACOBIAN
The kinematics (and inverse kinematics) equations allowed us to write relationships between the position and orientation of the end effector and the position and orientation of the joint variables of the manipulator. The goal of the velocity kinematics is to establish a relationship between the linear and angular velocities of the end effector and those of the joint variables. Velocity kinematic is important for various reasons, including motion planning, achieving smooth motion and force and torque calculations.



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Velocity Kinematics and Jacobian

JACOBIAN OF THE TWO LINK MANIPULATOR

For the two link planar manipulator of figure 1, the coordinates of the end effector are

$$p_x = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$$
$$p_y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$$

By taking the time derivative, we get

$$\dot{p}_x = -\dot{\theta}_1 a_1 \sin\theta_1 - \dot{\theta}_1 a_2 \sin(\theta_1 + \theta_2) - \dot{\theta}_2 a_2 \sin(\theta_1 + \theta_2)$$
$$\dot{p}_y = \dot{\theta}_1 a_1 \cos\theta_1 + \dot{\theta}_1 a_2 \cos(\theta_1 + \theta_2) + \dot{\theta}_2 a_2 \cos(\theta_1 + \theta_2)$$

which can be re-arranged under matrix form:

$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

JACOBIAN MATRIX

In general, it is possible to write

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

JACOBIAN MATRIX

$$J = \begin{array}{c} \begin{array}{cc} \frac{\partial p_x}{\partial \theta_1} & \frac{\partial p_x}{\partial \theta_2} \end{array} \\ \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \\ \begin{array}{cc} \frac{\partial p_y}{\partial \theta_1} & \frac{\partial p_y}{\partial \theta_2} \end{array} \end{array}$$

This is the standard format for the Jacobian. If there were a z component there would be a third row to the matrix

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For three links the equations would be as follows;

$P_x = a_1COS\ (\theta_1) + a_2COS\ (\theta_1 + \theta_2) + a_3COS\ (\theta_1 + \theta_2 + \theta_3\ )$
$P_y = a_1SIN\ (\theta_1) + a_2SIN\ (\theta_1 + \theta_2) + a_3SIN\ (\theta_1 + \theta_2 + \theta_3\ )$

$$J = \begin{array}{ccc} \dfrac{\partial p_x}{\partial \theta_1} & \dfrac{\partial p_x}{\partial \theta_2} & \dfrac{\partial p_x}{\partial \theta_3} \\ \multicolumn{3}{c}{\begin{bmatrix} -a_1SIN\ (\theta_1) - a_2SIN\ (\theta_1 + \theta_2) - a_3SIN\ (\theta_1 + \theta_2 + \theta_3\ ) & -\ a_2SIN\ (\theta_1 + \theta_2) - a_3SIN\ (\theta_1 + \theta_2 + \theta_3\ ) & -\ a_3SIN\ (\theta_1 + \theta_2 + \theta_3) \\ a_1COS\ (\theta_1) + a_2COS\ (\theta_1 + \theta_2) + a_3COS\ (\theta_1 + \theta_2 + \theta_3\ ) & a_2COS\ (\theta_1 + \theta_2) + a_3COS\ (\theta_1 + \theta_2 + \theta_3) & a_3COS\ (\theta_1 + \theta_2 + \theta_3) \end{bmatrix}} \\ \dfrac{\partial p_y}{\partial \theta_1} & \dfrac{\partial p_y}{\partial \theta_2} & \dfrac{\partial p_y}{\partial \theta_3} \end{array}$$

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

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$$\begin{bmatrix} \dot{p}_x \\ \dot{p}_y \end{bmatrix} = \begin{bmatrix} -a_1 \sin\theta_1 - a_2 \sin(\theta_1 + \theta_2) & -a_2 \sin(\theta_1 + \theta_2) \\ a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2) & a_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

A. Example

At the configuration of the planar manipulator shown in figure 2 ($\theta_1 = \theta_2 = 45^0$), find the speed of the end effector when

$$\left.\begin{aligned} \dot{\theta}_1 &= -32.41 deg/s \\ \dot{\theta}_2 &= 75.38 deg/s \end{aligned}\right\}$$

knowing that $a_1 = 5, a_2 = 4$

θ_1 = 45 degrees

θ_2 = 45 degrees

Needs to be converted to radians/sec

which gives the following speed for the end effector

$$J \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} -7.5 & -4 \\ 3.5 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

OK, how do we calculate the inverse?

Well, for a 2x2 matrix the inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by **ad−bc** .

Note: **ad−bc** is called the determinant.

Let us try an example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4\times6-7\times2}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4\times0.6+7\times-0.2 & 4\times-0.7+7\times0.4 \\ 2\times0.6+6\times-0.2 & 2\times-0.7+6\times0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4-1.4 & -2.8+2.8 \\ 1.2-1.2 & -1.4+2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Our Procedure

We write matrix A on the left and the Identity matrix I on its right separated with a dotted line, as follows. The result is called an **augmented** matrix.

We include row numbers to make it clearer.

$$\begin{array}{c c} \left[\begin{array}{ccc|ccc} 2 & 12 & 7 & 1 & 0 & 0 \\ 4 & 6 & 13 & 0 & 1 & 0 \\ 9 & 10 & 3 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \text{Row[1]} \\ \text{Row[2]} \\ \text{Row[3]} \end{array} \end{array}$$

Next we do several **row operations** on the 2 matrices and our aim is to end up with the identity matrix on the **left**, like this:

$$\begin{array}{c c} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & ? & ? & ? \\ 0 & 1 & 0 & ? & ? & ? \\ 0 & 0 & 1 & ? & ? & ? \end{array}\right] & \begin{array}{l} \text{Row[1]} \\ \text{Row[2]} \\ \text{Row[3]} \end{array} \end{array}$$

(Technically, we are reducing matrix A to **reduced row echelon form**, also called **row canonical form**).

The resulting matrix on the right will be the **inverse matrix** of A.

Our row operations procedure is as follows:

1. We get a "1" in the top left corner by dividing the first row
2. Then we get "0" in the rest of the first column
3. Then we need to get "1" in the second row, second column
4. Then we make all the other entries in the second column "0".

We keep going like this until we are left with the identity matrix on the left.

Let's now go ahead and find the inverse.

Solution

We start with:

$$\left[\begin{array}{ccc:ccc} 2 & 12 & 7 & 1 & 0 & 0 \\ 4 & 6 & 13 & 0 & 1 & 0 \\ 9 & 10 & 3 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \text{Row[1]} \\ \text{Row[2]} \\ \text{Row[3]} \end{array}$$

New Row [1]

Divide Row [1] by 2 (to give us a "1" in the desired position):

This gives us:

$$\left[\begin{array}{ccc:ccc} 1 & 6 & 3.5 & 0.5 & 0 & 0 \\ 4 & 6 & 13 & 0 & 1 & 0 \\ 9 & 10 & 3 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \text{Row[1]} \\ \text{Row[2]} \\ \text{Row[3]} \end{array}$$

New Row [2]

Row[2] − 4 × Row[1] (to give us 0 i

4 − 4 × 1 = 0
6 − 4 × 6 = -18
13 − 4 × 3.5 = -1
0 − 4 × 0.5 = -2
1 − 4 × 0 = 1
0 − 4 × 0 = 0

This gives us our new Row [2]:

$$\begin{array}{cl} \left[\begin{array}{ccc:ccc} 1 & 6 & 3.5 & 0.5 & 0 & 0 \\ 0 & -18 & -1 & -2 & 1 & 0 \\ 9 & 10 & 3 & 0 & 0 & 1 \end{array}\right] & \begin{array}{l} \text{Row[1]} \\ \text{Row[2]} \\ \text{Row[3]} \end{array} \end{array}$$

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Velocity Kinematics and Jacobian

On the previous slide we saw that if we know $\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$ we can use the Jacobian (J) to calculate v_x and v_y

Similarly we can determine v_x and v_y by using J^{-1} to calculate $\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$ for the same link manipulator

Recall $J \cdot J^{-1} = I$

$$\underset{\mathbf{J}}{\begin{bmatrix}-7.5 & -4 \\ 3.5 & 0\end{bmatrix}} \underset{\mathbf{J^{-1}}}{\begin{pmatrix}J^{-1}_{11} & J^{-1}_{12} \\ J^{-1}_{21} & J^{-1}_{22}\end{pmatrix}} = \underset{\mathbf{I}}{\begin{pmatrix}1 & 0 \\ 0 & 1\end{pmatrix}}$$

$$\underset{\mathbf{J^{-1}}}{\begin{bmatrix}0 & 0.28 \\ -.25 & -0.53\end{bmatrix}} \begin{bmatrix}v_x \\ v_y\end{bmatrix} = \begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_2\end{bmatrix}$$

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E. Example

Find the velocity of the joints that allows to move the end effector

- Case 1: Horizontally with speed 1/ms at configuration $(\theta_1, \theta_2) = (45^o, 45^o)$
- Case 2: Vertically with speed 2/ms at configuration $(\theta_1, \theta_2) = (45^o, 90^o)$

F. Solution

- Case 1:

  The inverse Jacobian is calculated first

$$J^{-1} = \begin{bmatrix} 0 & 0.28 \\ -.25 & -0.53 \end{bmatrix} \tag{38}$$

  from which the velocity of the joints is obtained

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0^o \\ -14.3^o \end{bmatrix} \tag{39}$$

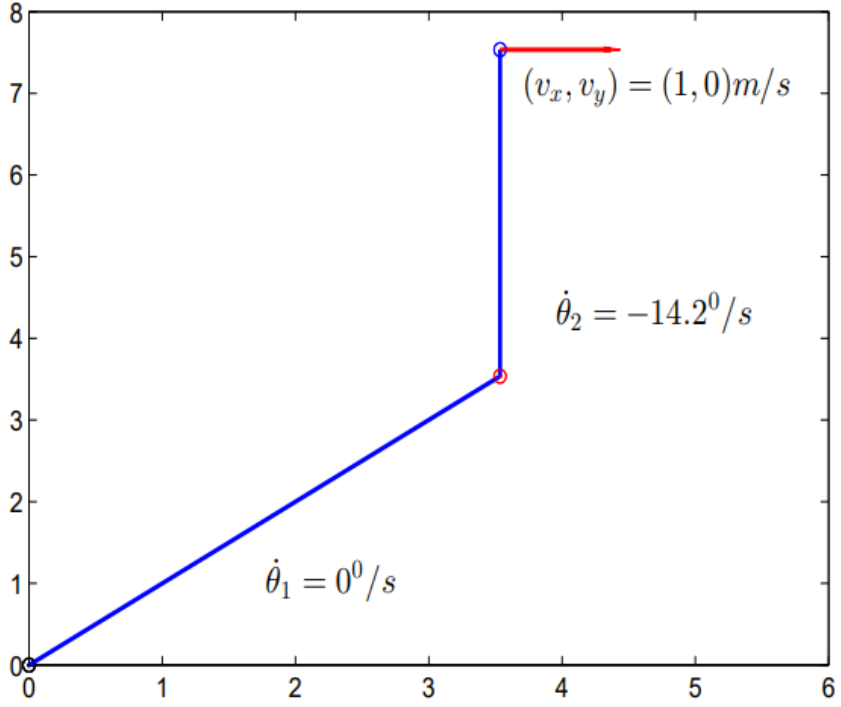

Fig. 3. Planar manipulator

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Velocity Kinematics and Jacobian

- Case 2:
  The inverse Jacobian is

$$J^{-1} = \begin{bmatrix} -0.14 & 0.14 \\ -.03 & -0.32 \end{bmatrix}$$

from which the velocity of the joints is obtained

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 16.2^o \\ -36.4^o \end{bmatrix}$$

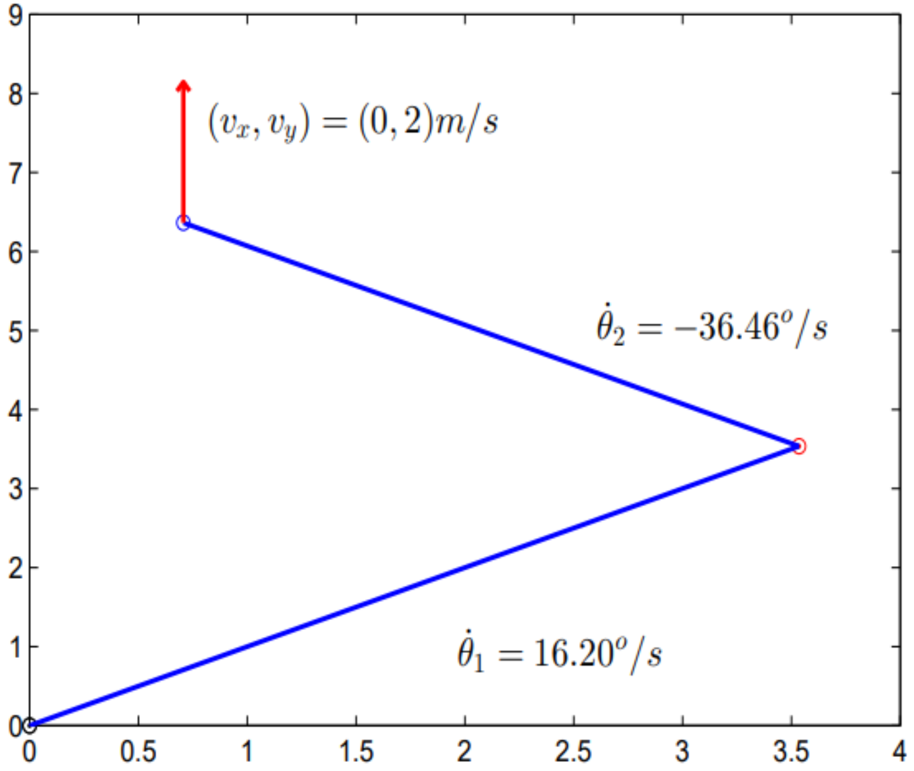

Fig. 4. Planar manipulator

DESCRIBING THE LINEAR TRAVEL OF THE END EFFECTOR

Write the parametric equations (**$X = X_0 + At$, $Y = Y_0 + Bt$**) for the straight line (from X_o,Y_o to X_f,Y_f) indicated in blue

a. **$X = X_0 + At$** where t slowly increases from 1 sec to 10 seconds
b. **$Y = Y_0 + Bt$** where t slowly increases from 1 sec to 10 seconds
c. **$A = (X_f - X_0)/T$** where T = 10 seconds
d. **$B = (Y_f - Y_0)/T$** where T = 10 seconds
e. **a1=5cm, a2=4cm; X_0=6cm, Y_0=6cm; X_f=7cm,Y_f=2cm**

2) Construct the following table using the Jacobian and the Inverse Kinematics equations above

X	Y	θ_1	θ_2	dX/dt	dy/dt	$d\theta_1/dt$	$d\theta_2/dt$
6	6	27.66°	39.19°	0.1	-0.4	-0.050	0.0717
6.5	4						
7	2						

$$x = x_0 + At$$
$$= 6 + \left(\frac{x_F - x_0}{T}\right)t$$
$$= 6 + \left(\frac{7-6}{10}\right)t$$
$$= 6 + (1/10)t$$

$$y = y_0 + Bt$$
$$= 6 + \left(\frac{y_F - y_0}{T}\right)t$$
$$= 6 + \left(\frac{2-6}{10}\right)t$$
$$= 6 + (-4/10)t$$

$$\cos\theta_2 = \frac{P_x^2 + P_y^2 - a_1^2 - a_2^2}{2a_1a_2}$$
$$= \frac{6^2 + 6^2 - 5^2 - 4^2}{2(5)(4)}$$
$$\cos\theta_2 = 0.775$$
$$\theta_2 = \pm 0.684 \approx \pm 39.19^\circ$$

$$\theta_1 = \tan^{-1}\left(\frac{P_y}{P_x}\right) - \tan^{-1}\left(\frac{a_2\sin\theta_2}{a_1 + a_2\cos\theta_2}\right)$$

$$J = \begin{bmatrix} -5\sin(27.66^\circ) - 4\sin(27.66^\circ + 39.19^\circ) & -4\sin(27.66^\circ + 39.19^\circ) \\ 5\cos(27.66^\circ) + 4\cos(27.66^\circ + 39.19^\circ) & -4\cos(27.66^\circ + 39.19^\circ) \end{bmatrix}$$

$$J = \begin{bmatrix} -6 & -3.678 \\ 6 & 1.57 \end{bmatrix}$$

$$\begin{bmatrix} 0.1244 & 0.2910 \\ -0.4747 & -0.4747 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.4 \end{bmatrix} = \begin{bmatrix} -0.050 \\ 0.0717 \end{bmatrix} \text{rad/s}$$

$$J^{-1} = \begin{bmatrix} 0.1244 & 0.2910 \\ -0.4747 & -0.4747 \end{bmatrix}$$