1 Projection Matrices

We continue an example from lecture yesterday. The situation was: we have a matrix

$$B = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 4 & 3 \\ 4 & 7 & 10 & 7 \end{bmatrix}$$

and we want to find another matrix P, that projects vectors onto B's column space.

Question 1: What does it even mean for a matrix to be projecting vectors onto a subspace? If the subspace were a plane, what would that look like geometrically?

The first step is to find a basis for the column space. In the example we actually row reduce $B|I_3$ because knowing the left nullspace will be helpful later. We find:

$$RREF(B|I_3) = \begin{bmatrix} 1 & 0 & -1 & 0 & -3 & 2 & 0 \\ 0 & 1 & 2 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0.5 & -0.5 \end{bmatrix}$$

Question 2: Given $RREF(B|I_3)$ above, what's a basis for $Col(B)$? Let A be the matrix with these vectors as its columns. (write A below)

$$A =$$

So the key thing is, by construction, A has the same column space as B, but its columns are now independent. It turns out we can now plug A into a crazy formula to get the projection matrix P we've been looking for.

$$P = A(A^TA)^{-1}A^T = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}$$

That formula was super weird, so let's sanity-check that P actually does what we want.

Question 3: Show that $Col(P) = Col(B)$. (why do we want this?)
Question 4: Show that P kills vectors in $LNull(B)$. (why do we want this?)
Question 5: Note that P is symmetric. Will $A(A^TA)^{-1}A^T$ always be symmetric?
Question 6: Now use the formula to compute P', the projection matrix onto the left null space.

$$P' =$$

Question 7: Check if $P = I_3 - P'$. Why would this be true? What's going on geometrically?

2 True / False

Write T or F next to each (if true explain why, if false think of a counter example)

1. A change-of-basis/change-of-coordinates matrix is always invertible.
2. If the distance from $\mathbf{u}$ to $\mathbf{v}$ equals the distance to $-\mathbf{v}$ then $\mathbf{u} \perp \mathbf{v}$ (i.e. $\mathbf{u}$ is orthogonal/perpendicular to $\mathbf{v}$).
3. If $||\mathbf{u}+\mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2$ then $\mathbf{u} \perp \mathbf{v}$.
4. Two similar matrices have the same eigenvectors and eigenvalues.
5. The algebraic multiplicity of an eigenvalue (i.e. number of times it appears in the characteristic polynomial) is always greater than the geometric multiplicity (i.e. the dimension of the associated eigenspace).
6. If $x - \lambda$ is a factor of A's characteristic polynomial then there is at least one non-zero $\mathbf{v}$ such that $A\mathbf{v} = \lambda \mathbf{v}$.
7. If $(x - \lambda)^3$ is a factor of A's characteristic polynomial then there are at least 3 linearly independent $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $A\mathbf{v}_i = \lambda \mathbf{v}_i$.

3 Miscellaneous Problems

1. Can you write down a real, diagonalizable 3×3 matrix that has 2 real eigenvalues, each with algebraic multiplicity 1?
2. A basis is orthonormal if it is both orthogonal (i.e. the inner product of each pair of vectors is zero) and normalized (i.e. the length of each vector is 1). Check if $[-1/\sqrt{5}, 2/\sqrt{5}]^T$, $[2/\sqrt{5}, 1/\sqrt{5}]^T$ is an orthonormal basis for $\mathbb{R}^2$. Let U be the 2×2 matrix whose columns are these vectors. What is $U^T U$?

   We say a matrix is orthogonal if its columns are orthonormal (why don't we call these matrices ortho*normal* instead? good question). It turns out this condition is equivalent to $A^T A = I$. You may want to use this fact in the following questions.
3. Suppose A is a square matrix with orthonormal columns. Is A invertible?
4. Suppose U, V are orthogonal matrices. Is UV an orthogonal matrix as well?
5. Let U be an orthogonal matrix and construct V by interchanging some of the rows of U. Is V an orthogonal matrix?