1.1 $\bar{0}$, closed under addition + multiplication.

1.2 $\begin{bmatrix} 1 & 0 & -4 & 0 \\ 1 & 2 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ so $\begin{pmatrix}1\\1\\0\\1\end{pmatrix}\begin{pmatrix}0\\2\\-1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$ works

$\uparrow \ \uparrow \qquad \uparrow$

1.3 yes.

1.4 $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 2 & -2 & 3 \\ 1 & 3 & -2 & t \end{array}\right] \to \begin{matrix} 1 & 1 & 0 & 2 \\ 0 & 2 & -2 & 3 \\ 0 & 2 & -2 & t-2 \end{matrix} \to \begin{matrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & t-5 \end{matrix} \to \begin{matrix} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1 & 3/2 \\ 0 & 0 & 0 & t-5 \end{matrix}$

so $t=5$ (then $\frac{1}{2}\begin{pmatrix}1\\0\\1\end{pmatrix} + \frac{3}{2}\begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\3\\5\end{pmatrix}$ works)

2.1 Spanning + Linearly independent $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}$ is a basis of the 2-dimensional subspace span$\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$.

2.2 The dimension of a vector space = the number of things in its basis (it may have many bases but they will all have the same # of vectors).

ex. the space of deg 2 polynomials, basis = $\{1, x, x^2\}$

2.3 yes. The general alg to extend $\bar{v}_1, \ldots, \bar{v}_m$ to a basis is to first throw in all the standard basis vectors $\bar{e}_i$ (to make sure its spanning) and then row reduce the matrix: $[\bar{v}_1 \cdots \bar{v}_m \, \bar{e}_1 \cdots \bar{e}_n]$ to find a linearly indep. subset that still spans.

put the $\bar{v}_i$ first to make sure they're included!

3.1 $\begin{pmatrix}4 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}10 & 0\\0 & 1\end{pmatrix} = \begin{pmatrix}4\times 10 & 0\\0 & 3\times 1\end{pmatrix} = \begin{pmatrix}10 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}4 & 0\\0 & 3\end{pmatrix}$

$\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}$ vs $\begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$

3.2 yes, $\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} = 0$

4.1 $T(c\bar{v}) = cT(\bar{v})$ and $T(\bar{v}+\bar{w}) = T(\bar{v}) + T(\bar{w})$

$\hookrightarrow$ also implies $T(\bar{0}) = \bar{0}$.

4.2 (a) nope (b) yup $\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ (c) nope (doesn't send $\bar{0}$ to $\bar{0}$)

4.3 so if we had $T(e_1)$ and $T(e_2)$ we'd write the matrix $[T(e_1)\ T(e_2)]$. we want to find $a_1, a_2, b_1, b_2 \in \mathbb{R}$ such that

$* \ a_1\begin{bmatrix}1\\2\end{bmatrix} + a_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ and $b_1\begin{bmatrix}1\\2\end{bmatrix} + b_2\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \bigstar$

solve both these linear systems at once by row reducing this matrix to REF!

$$\left[\begin{array}{cc|cc}1&-1&1&0\\2&1&0&1\end{array}\right] \mapsto \left[\begin{array}{cc|cc}1&0&1/3&1/3\\0&1&-2/3&1/3\end{array}\right]$$

$\ast$ $\bigstar$ a_1 b_1 a_2 b_2

Then $T(e_1) = \frac{1}{3}\begin{bmatrix}3\\4\end{bmatrix} - \frac{2}{3}\begin{bmatrix}4\\0\end{bmatrix}$ and $T(e_2) = \frac{1}{3}\begin{bmatrix}3\\4\end{bmatrix} + \frac{1}{3}\begin{bmatrix}4\\0\end{bmatrix}$

$\hookrightarrow$ matrix is $\begin{bmatrix}-5/3 & 7/3\\ 4/3 & 4/3\end{bmatrix}$

5.1
$$\left[\begin{array}{ccc|cccc}1&5&-3&1&0&0&0\\-1&-4&1&0&1&0&0\\-1&-2&-3&0&0&1&0\\-2&-7&0&0&0&0&1\end{array}\right] \sim \left[\begin{array}{ccc|cccc}1&0&7&-4&-5&0&0\\0&1&-2&1&1&0&0\\0&0&0&-2&-3&1&0\\0&0&0&-1&-3&0&1\end{array}\right]$$

there are many ways to see (1 1 1 1) is not in LNull A. * since for any $\bar{x} \in$ LNulA $A^T\bar{x} = \bar{0}$ we can check that it doesn't become zero when multiplied by either of the columns of A. Also (more work) can see it's not in the span found below for Nul A.

Col A = Span$\left\{\begin{pmatrix}1\\-1\\-1\\-2\end{pmatrix}\begin{pmatrix}5\\-4\\-2\\-7\end{pmatrix}\right\}$ Row A = Span$\{(1\,0\,7)\ (0\,1\,-2)\}$

Nul A: $\begin{matrix}x + 7z = 0\\ y - 2z = 0\end{matrix} \to \begin{pmatrix}x\\y\\z\end{pmatrix} = z\begin{pmatrix}-7\\2\\1\end{pmatrix}$ LNul A = Span$\{(-2,-3,1,0), (-1,-3,0,1)\}$

$$\begin{bmatrix}1&5&-4\\-1&-4&3\\-1&-2&1\\-2&-7&h\end{bmatrix} \sim \begin{bmatrix}1&5&-4\\0&1&-1\\0&3&-3\\0&3&h-8\end{bmatrix} \sim \begin{bmatrix}1&5&-4\\0&1&-1\\0&0&0\\0&0&h-5\end{bmatrix}$$

$\to$ so you have solutions for $h=5$.

6. So rank = the number of pivots, therefore rank = dim Col A
and rank = dim Row A

- we also have Rank A + dim Nul A = n for an $m \times n$ matrix.
- an $n \times n$ matrix is invertible if and only if Rank = n.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 6 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & -6 & 0 & -4 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ \boxed{0} & \boxed{0} & \boxed{0} & 3 & -4 & 1 \end{array}\right] \times \text{ not invertible!}$$

7.

$$\left[\begin{array}{cc|cc} 7 & -3 & 1 & -2 \\ 5 & -1 & -5 & 2 \end{array}\right] \rightarrow \begin{bmatrix} 2 & -2 & 6 & -4 \\ 5 & -1 & -5 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 3 & -2 \\ 0 & 4 & -20 & 12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -5 & 3 \end{bmatrix}$$

note that the order of these columns matters!

So $\underset{B \leftarrow C}{P} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$ then $[x]_B = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$\underset{C \leftarrow B}{P} = \left(\underset{B \leftarrow C}{P}\right)^{-1}$ so $\left[\begin{array}{cc|cc} -2 & 1 & 1 & 0 \\ -5 & 3 & 0 & 1 \end{array}\right] \rightarrow \begin{bmatrix} 1 & -1/2 & -1/2 & 0 \\ 0 & 1/2 & -5/2 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$

so $\underset{C \leftarrow B}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$

sanity check:

$\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix} = \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}$

yay.