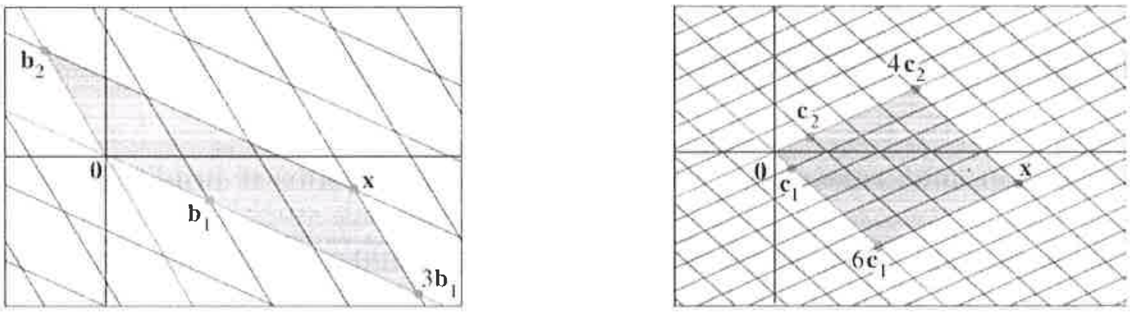
Recall last week we introduced coordinate mapping, and we found a matrix that transformed vectors from the standard basis into terms of a basis of our choice. If you don't remember, that's okay too, because we're basically going to do a more general version of it this week. First, let's try and get a better idea what we're actually doing by stealing an example from the textbook (page 241). They suppose $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ is a basis for $\mathbb{R}^2$ and that $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ is another basis. Let $\mathbf{x}$ be a vector in $\mathbb{R}^2$ such that $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$. Then we said that $\mathbf{x}$ in terms of $\mathcal{B}$ is $[\mathbf{x}]_\mathcal{B} = \begin{bmatrix}3\\1\end{bmatrix}$. If we also have $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$ then $\mathbf{x}$ in terms of $\mathcal{C}$ is $[\mathbf{x}]_\mathcal{C} = \begin{bmatrix}6\\4\end{bmatrix}$.

The vector $\mathbf{x}$ appears in both figures below, note that indeed in figure (a) $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$ while in figure (b) we see $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$.



(a) (b)

FIGURE 1 Two coordinate systems for the same vector space.

So in the above example $\mathbf{b}_1$, $\mathbf{b}_2$ are one basis for $\mathbb{R}^2$ and $\mathbf{c}_1$, $\mathbf{c}_2$ are another.

1. What do the standard basis $\mathbf{e}_1$, $\mathbf{e}_2$ look like in these pictures? roughly:
2. If I told you $\mathbf{b}_1 = \begin{bmatrix}1\\-1/2\end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix}-1/2\\1\end{bmatrix}$ (in the standard basis), then can you write down $\mathbf{x}$ in the standard basis? then $\bar{x} = 3\begin{bmatrix}1\\-1/2\end{bmatrix} + \begin{bmatrix}-1/2\\1\end{bmatrix} = \begin{bmatrix}5/2\\-1/2\end{bmatrix}$

Note that the notation we've been using all along for $\mathbf{x}$ as a column vector was secretly $\mathbf{x}$ written in terms of the standard basis.

Problems

1. We denote by $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ the change of basis matrix (or "change of coordinates" matrix) transforming vectors in terms of basis $\mathcal{B}$ to basis $\mathcal{C}$. So with the

notation $[\mathbf{x}]_{\mathcal{B}}$, $[\mathbf{x}]_{\mathcal{C}}$ as above we have $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$. Find the following:

(a) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$. Let $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$. What is $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$? → it's just $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

(b) Use $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}$ to find $\mathbf{x}$ given that $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 & -2 & 1 \end{bmatrix}^t$. → calculate $\underset{\mathcal{E}\leftarrow\mathcal{B}}{P}[\bar{x}]_{\mathcal{B}}$ = I got $\begin{bmatrix} 0 \\ 7 \\ 30 \end{bmatrix}$

(c) Recall that for any 2 bases $\mathcal{B}$ and $\mathcal{C}$, $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{C}}{P}$ (or see the textbook page 242). What is $\underset{\mathcal{B}\leftarrow\mathcal{E}}{P}$? we invert the above matrix to get: $\begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}$

(d) Using the matrix you found in the previous part, find $[\mathbf{y}]_{\mathcal{B}}$ for $\mathbf{y} = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}^t$. I found $[y]_{\mathcal{B}} = \begin{pmatrix} -9 \\ -5 \\ 3 \end{pmatrix}$

2. Let $\mathcal{D} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V and suppose $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$ and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{b}_3$. Find the change of coordinates matrix from $\mathcal{F}$ to $\mathcal{D}$, $\underset{\mathcal{D}\leftarrow\mathcal{F}}{P}$. Also find $[\mathbf{x}]_{\mathcal{D}}$ for $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$.

See chapter 4.7 for a better explanation of this, but from the equations of f in terms of d you know

$\underset{D\leftarrow F}{P}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_F = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}_D$

$\underset{D\leftarrow F}{P}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_F = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}_D$

$\underset{D\leftarrow F}{P}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_F = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}_D$

and these 3 equations force the 3 columns of $\underset{D\leftarrow F}{P}$ to be $\begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

3. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for $\mathbb{R}^2$. Write an equation that relates $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, $\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}$ and $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}$. Why does it hold? → $\underset{D\leftarrow B}{P} = \underset{D\leftarrow C}{P}\,\underset{C\leftarrow B}{P}$ (think about what the matrices are doing)

4. Okay lets switch gears. Let A be an $m \times n$ matrix. What are the definitions of the Row space of A and the Column space of A? Which is a subspace of $\mathbb{R}^n$ or $\mathbb{R}^m$? → Row A = span of the rows and is inside $\mathbb{R}^n$
Col A = span of the columns and is inside $\mathbb{R}^m$

5. Let A be an $m \times n$ matrix and let U be an $m \times n$ matrix in row echelon form which is obtained from A by row operations. Answer the following true or false. Explain your reasoning, or give a counterexample. (a) Row A = Row U (true see posted reading) (b) Col A = Col U (false) (c) dim Row A = dim Row U (true by (a)) (d) dim Col A = dim Col U (true) (e) dim Row A = dim Col A. (false, think of a non-square matrix.)
→ for (b) and (d) remember our algorithm to find a basis of the column space.

6. Suppose A is an invertible $n \times n$ matrix. What is the rank of A? What is the dimension of the Null space? → n → 0!

7. Find the rank of these matrices: $\begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$, $\begin{bmatrix} t & -1 & 2 \\ t & t & 1 \\ t & t & 1 \end{bmatrix}$ (depending on t)

this became EF $\begin{pmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 0 & (2-t)(1+t) \end{pmatrix}$ → rank = 3, unless t = 1 or 2.

the EF is $\begin{pmatrix} t & -1 & 2 \\ 0 & t+1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ → rank is always 2.

8. Bonus: What does it mean for a vector to be in the left nullspace of a matrix A? Can you rewrite this condition to involve only A? (and not A transpose).

→ see the posted reading for F20 Sept. The alternative condition is $\bar{x}$ in LNul A if $\bar{x}A = \bar{0}$ (ie multiplication on the left!)