Instructions: Read one of the sections on the other sheet. Then flip that sheet over and with a partner / group answer as many of the associated questions on this sheet as you can, without looking at the notes (think of it as a mini zero-stakes, team quiz :).

1 Cramer's Rule Questions

all the $A_i(b)$'s plus $\det A$ so $n+1$

1. How many determinants do you have to compute to solve $A\mathbf{x} = \mathbf{b}$ for an $n \times n$ matrix A with Cramer's rule?

   no, b/c then A isn't invertible (b/c invertible matrices are one-to-one)

2. What if $A\mathbf{x} = \mathbf{b}$ has two solutions? Can you apply Cramer's rule?

3. Use Cramer's rule to find $\mathbf{x}$ in $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

   $\bar{x} = \begin{pmatrix} 8/5 \\ 1/5 \end{pmatrix}$

2 Determinants as Area Questions

1. Can the determinant formula for an area or volume ever give you a negative area? Why or why not? → no b/c we take the absolute value.

2. Does the formula still make sense if the matrix is not invertible? What does it say? What is happening geometrically? → yes, then you get area zero. In $\mathbb{R}^2$ this is the case where your parallelogram is actually a line.

area $= \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3$

3. Draw the vectors $[1, 2]^T$ and $[0, 3]^T$ in $\mathbb{R}^2$, shade in the parallelogram they form and compute its area.

4. Suppose S is the portion of $\mathbb{R}^2$ inside a circle, and you know it has area 9π. Let T be the linear transformation defined by matrix $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$. What is $Area(T(S))$?

   $4 \times (9\pi) = 36\pi$

5. Suppose S is a subset of $\mathbb{R}^{100}$ with volume 14. Suppose A is a 100×100 diagonal matrix with -2 down the diagonal. What is the volume of $A(S)$?

   $|(-2)^{100}| = 2^{100}$ so $2^{100} \times 14$.

3 Eigenvector Questions

1. Is zero always an eigenvector? → no, in fact we don't allow it to ever be an eigenvector (the definition is a non-zero $\bar{v}$ s.t. $A\bar{v} = \lambda\bar{v}$)

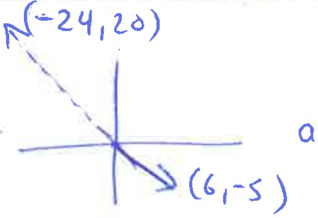
2. Is $\mathbf{v}$ an eigenvector of A if $\mathbf{v}$ is in the null space of A?

   yes! (as long as $\bar{v} \neq \bar{0}$) being in the nullspace means $A\bar{v} = \bar{0} = 0 \cdot \bar{v}$ so it has eigenvalue 0.

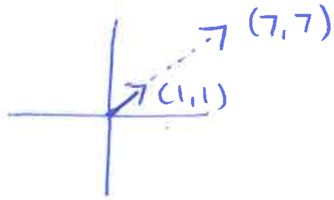
$A\begin{bmatrix}6\\-5\end{bmatrix}=\begin{bmatrix}-24\\20\end{bmatrix}$

$A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}7\\7\end{bmatrix}$

So:



and



3. Let $A = \begin{bmatrix}1 & 6\\5 & 2\end{bmatrix}$. I claim it has eigenvectors $[6,-5]^T$ and $[1,1]^T$. Check this by drawing these vectors in the plane, as well as $A[6,-5]^T$ and $A[1,1]^T$.

4. Suppose I tell you that the matrix $A = \begin{bmatrix}10 & -9\\4 & -2\end{bmatrix}$ has eigenvalue $\lambda = 4$. Can you find the eigenvectors associated to that eigenvalue? Yes, I found $\bar{x} = \begin{pmatrix}3/2\\1\end{pmatrix}$ and in general the set of all eigenvectors associated to that λ (ie the "eigenspace") is $\left\{c\begin{pmatrix}3/2\\1\end{pmatrix} : c \in \mathbb{R}\right\}$

4 Eigenvalue Questions

this question was harder than intended :)

note section 2.4, I changed this matrix to make the cubic friendlier...

1. Find the eigenvalues for $A = \begin{bmatrix}2 & -1 & -1\\0 & 3 & -3\\-1 & 0 & 1\end{bmatrix}$. Then, for each eigenvalue, find the associated eigenvectors.

I found characteristic poly $\lambda^3 - 6\lambda^2 + 10\lambda$ (or the negation of this) $= \lambda(\lambda-(3+i))(\lambda+(3_i)$

so eigenvalues are $0, 3+i, 3-i$

for $\lambda = 0$: get $\left\{x\begin{pmatrix}1\\1\\1\end{pmatrix} : x \in \mathbb{R}\right\}$

for $\lambda = 3-i$ get $\left\{x\begin{pmatrix}-2+i\\-3i\\1\end{pmatrix}\right\}$

for $\lambda = 3+i$ get $\left\{x\begin{pmatrix}-2-i\\3i\\1\end{pmatrix}\right\}$

2. Let A be an $n \times n$ matrix. Explain why A is invertible if and only if 0 is not an eigenvalue of A. b/c 0 is an eigenvalue if and only if $A\bar{x} = \bar{0}$ has a non-trivial soln.

3. For $A = \begin{bmatrix}1 & 2 & 3\\1 & 2 & 3\\1 & 2 & 3\end{bmatrix}$ find one eigenvector and eigenvalue (hint: no computation needed). $\bar{v} = \begin{pmatrix}1\\1\\1\end{pmatrix}$ works!

4. (a) Show that the eigenvalues of an upper triangular $n \times n$ matrix are the entries on the main diagonal. this follows from the formula for the determinant of an upper triangular matrix.

   (b) Show that if λ is an eigenvalue of an $n \times n$ matrix A then λ^2 is an eigenvalue of A^2. More generally, show that λ^k is an eigenvalue of A^k if k is a positive integer. if $A\bar{x} = \lambda\bar{x}$ then $A^k\bar{x} = A^{k-1}(A\bar{x}) = A^{k-1}(\lambda\bar{x})$

   $\vdots$

   $= \lambda^k\bar{x}$.

   eg just apply them one at a time.

   (c) Use (a) and (b) to find the eigenvalues of A^9, where

$$A = \begin{bmatrix}1 & 3 & 7 & 11\\0 & -1 & 3 & 8\\0 & 0 & -2 & 4\\0 & 0 & 0 & 2\end{bmatrix}.$$

so $\det(A - \lambda I)$
$= (1-\lambda)(-1-\lambda)(-2-\lambda)(2-\lambda)$

so indeed A's eigenvalues are $1, -1, 2, -2$, then by (b)

A^9 has eigenvalues $1^9 = 1$, $-1^9 = -1$, $2^9 = 512$, $-2^9 = -512$.