This worksheet assumes you've seen the algorithm for computing the determinant. If you haven't seen this yet (or saw it but didn't instantly memorize it), it can be found on pages 167 and 168 of your text (or on google) or you can just ask your friendly GSI who'd be happy to talk about it.

Multiplies the det by −1.

1 Questions

~~det~~ scaling a row by c also scales the det by c.

no change to det!

1. How do row operations (scaling rows, interchanging rows, and adding one row to another) affect the determinant of a matrix?
2. Let A be a square matrix. List at least four different statements which are equivalent to the statement $\det A \neq 0$. → A is invertible, A has n pivots, the column space spans $\mathbb{R}^n$, ~~rank~~ Dim Nul A=0 etc.
3. Let A and B be $n \times n$ matrices. Answer the following True or False. If False give a counterexample.

   $A=\begin{pmatrix}1&0\\0&1\end{pmatrix}$ $B=\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$ $A+B=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

   det A + det B = 2 det(A+B) = 0

   (a) $\det(AB) = \det(BA)$ T
   (b) $\det(AB) = \det A \det B$ T
   (c) $\det(A + B) = \det A + \det B$ F
   (d) If A and B are both invertible, then AB is invertible. T (think about determinants of A and B being non-zero.)
   (e) $\det A = \det A^T$ T
   (f) If A is invertible, then $\det(A^{-1}) = (\det A)^{-1}$. T
   (g) If a is a scalar then $\det(aA) = a^n \det(A)$. T
4. Suppose A is a 3×3 matrix with determinant 5. What is $\det(3A)$? $3^3 \cdot 5$ $\det(A^{-1})$? $1/5$ $\det(2A^{-1})$? $2^3 \frac{1}{5}$ $\det((2A)^{-1})$? $\frac{1}{\det(2A)} = \frac{1}{2^3 \cdot 5}$
5. Here are the true/false questions from chapter 3.1, try them out if you haven't already:

   (a) An $n \times n$ determinant is defined by determinants of $(n-1) \times (n-1)$ submatrices. T
   (b) The (i, j)-cofactor of a matrix is the matrix $A_{i,j}$ obtained by deleting from A its i^{th} row and j^{th} column. F (not quite the definition)
   (c) The cofactor expansion of $\det A$ down a column is equal to the cofactor expansion along a row. T
   (d) The determinant of a triangular matrix is the sum of the entries on the main diagonal. T

det $A = 0$ if and only if the rows are linearly dependent.
and 2 vectors are linearly dependent if and only if one is a multiple of the other.

2 Problems

1. Prove that the determinant of a 2×2 matrix is 0 if and only if one row is a multiple of the other.

2. Compute the following:

   (a) $\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} = -1$ (b) $\begin{vmatrix} 1 & -1 & -2 \\ 3 & 0 & 1 \\ -1 & 1 & 1 \end{vmatrix} = -3$ (c) $\begin{vmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$

3. By inspection, evaluate the following determinants. Are these matrices invertible?

   (a) $\begin{vmatrix} 3 & 0 & 0 \\ 1 & 4 & 0 \\ 1 & 5 & 9 \end{vmatrix} = 3 \times 4 \times 9$ (b) $\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 4 & 9 \\ 0 & 16 & 25 & 36 \\ 49 & 64 & 81 & 100 \end{vmatrix} = 4 \times 16 \times 49$ (c) $\begin{vmatrix} 0 & \pi & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 22 \end{vmatrix} = \pi(-\sqrt{2})22$

4. Compute the determinant of the following matrices: (hint you may want to use row operations..)

   (a) $\begin{bmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & 4 & 1 \\ 0 & 1 & -2 & 2 \\ 3 & 3 & 1 & 4 \\ 0 & 1 & -3 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix}$

5. Show that if A is a square matrix with a row of zeros, then $\det A = 0$. What if A has a column of zeros? → same. → this follows right from the "cofactor expansion" eg $\det A = 0C_{11} + 0C_{21} + 0C_{31} = 0$.

6. (a) Let A be a square matrix. If $\det A \neq 0$, how many solutions does the equation $A\mathbf{x} = \mathbf{0}$ have? → 1!

   (b) Let B be a 3×3 matrix such that the entries in each row of B add up to 0. Use part (a) to show $\det B = 0$. → note that b/c of this $B\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$ so can't have $\det B \neq 0$.

7. Prove that the rank of an $n \times n$ matrix A is the largest integer k for which there is a $k \times k$ sub-matrix of A that has a nonzero determinant.

   if a matrix has a $k \times k$ submatrix with non-zero determinant then you know the vectors in that submatrix are linearly independent → so rank $\geq k$.
   Conversely if rank $= k$ then you can find a submatrix formed by k linearly independent columns. This submatrix also has rank k so you can find k linearly indep. rows. This is your submatrix.

8. Show that $\{v_1, ..., v_n\}$ forms a basis for $\mathbb{R}^n$ if and only if the determinant of the coordinate matrix is not zero.

   → If $\det A \neq 0$ then we know the vectors are linearly independent. Since there are n of them we know they also span $\mathbb{R}^n$.

You won't be tested on this :)