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1. F (unless A is Δ)
2. F (unless you're just adding one row to another)
3. T
4. T
5. F, T if they meant |det A|.
6. F det A^T = det A
7. T
8. F $\begin{pmatrix}1&0\\0&0\end{pmatrix}$ has $\lambda_1=1, \lambda_2=0$, $\begin{pmatrix}0&0\\1&0\end{pmatrix}$ has $\lambda=0$ only.
9. T
10. F (T if A is diagonalizable!)
11. F (T if D is diagonal)
12. Edit: True if they mean geometric multiplicity (e.g. the dimension of the eigenspace), False if they mean algebraic multiplicity.
13. F ($\begin{smallmatrix}0&0\\0&0\end{smallmatrix}$ is diagonalizable)
14. F (T if they are linearly indep)
15. F (eg $\begin{smallmatrix}2&0\\0&2\end{smallmatrix}$)
16. F (P could be the zero matrix)
17. F ($\begin{smallmatrix}2&1\\0&2\end{smallmatrix}$ is invertible b/c det $\neq 0$ but not diagonalizable)
18. F (P is made up of a basis of eigenvectors, and any other basis of eigenvectors would work too eg scale all the vectors by 2)

Diagonalization

1. A can have 0 to 3 eigenspaces, and their dimensions can be any 3 (or fewer) numbers whose sum is less than or equal to 3.

2. Det(A) = 1·2·3 = 6, Tr(A) = 1+2+3 = 6

3. $Ax = \lambda x$ for some x, conjugating: $\bar{A}\bar{x} = \bar{\lambda}\bar{x}$. Assuming A is real, $\bar{A} = A$ so $A\bar{x} = \bar{\lambda}\bar{x}$.
$\rightarrow$ [1-2i, -1+i, 0].

4. (a) $\lambda_1 = 4+i, \quad \lambda_2 = 4-i$

$$\lambda_1 \downarrow \quad v_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} \qquad \lambda_2 \downarrow \quad v_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix}$$

(b) $$A = \begin{bmatrix} 1+i & 1-i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4+i & 0 \\ 0 & 4-i \end{bmatrix} \begin{bmatrix} -i/2 & (1+i)/2 \\ i/2 & (1-i)/2 \end{bmatrix}$$

$\lambda = \lambda_1$ so $a = 4, \ b = -1$

(c) $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$

Check: $\begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ -1 & 4 \end{bmatrix}$ ✓

Inner Products and Orthogonality

1. (a) T (b) F (they could be orthogonal!)
(c) T (b/c it'll all be squares)

2. $BA = \begin{matrix} b_1 \cdot a_1 & b_1 \cdot a_2 & b_1 \cdot a_3 \\ b_2 \cdot a_1 & b_2 \cdot a_2 & b_2 \cdot a_3 \\ b_3 \cdot a_1 & b_3 \cdot a_2 & b_3 \cdot a_3 \end{matrix}$

3. Consider a vector $\bar{v} = \begin{pmatrix} a \\ b \end{pmatrix}$. Pictorally:

by pythagoras $|v|^2 = a^2 + b^2$ so indeed $|v| = \sqrt{a^2+b^2}$

4. $\bar{u} - \bar{v} = (3, 4, 5)$ so $\|\bar{u} - \bar{v}\| = \sqrt{9 + 16 + 25}$

5. $\|(-30, 40)\| = 50 \rightarrow \left[\frac{-30}{50}, \frac{40}{50} \right]$

6. $\bar{u}+\bar{v} = (2,1,1)$ so $\|\bar{u}+\bar{v}\| = \sqrt{4+1+1} = \sqrt{6}$
$\|\bar{u}\| = \sqrt{1+4} = \sqrt{5}$, $\|\bar{v}\| = \sqrt{3}$
→ not equal, but they would be if $\bar{v}\cdot\bar{u}=0$.

$\bar{u}\cdot\bar{v} = 1-2 = -1$ so $|\bar{u}\cdot\bar{v}| = 1$. $\|u\|\|v\| = \sqrt{15}$.
→ not equal, but would be if $\bar{v},\bar{u}$ were parallel.

7. $8\cdot(-2)+(-5)(-3) = -1$ so no.

8. Any vector in the span is of the form $c_1\bar{u}+c_2\bar{v}$ so
$\bar{w}\cdot(c_1\bar{u}+c_2\bar{v}) = c_1\underbrace{\bar{w}\cdot\bar{u}}_{=0} + c_2\underbrace{\bar{w}\cdot\bar{v}}_{=0} = 0$
b/c $\bar{w}$ orthogonal to $\bar{u},\bar{v}$.

9. Say $\bar{x}\in W$ and $W^\perp$.
b/c $\bar{x}\in W^\perp$, for all $\bar{y}\in W$, $\bar{x}\bar{y}=0$. In particular, $\bar{x}\bar{x}=0$. But this can only happen when $\bar{x}=\bar{0}$ (b/c $\bar{x}\bar{x} = x_1^2+x_2^2+\dots x_n^2$.)

10. want (x,y,z) such that $x+y+z=0$ and $2x-z=0$ eg something in the null space of

$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}$. Reduce: $A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \end{pmatrix}$

→ $x = 1/2 z$, $y = -3/2 z$, $z = z$ → $\begin{pmatrix} 1/2 \\ -3/2 \\ 1 \end{pmatrix}$ is a basis for $W^\perp$.

$x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + z\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ ← scaled by 2 for neatness

row reduce

$\underbrace{-\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{3}{2}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}}_{W} \underbrace{-\frac{1}{2}\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}}_{W^\perp} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$