Math 54 Worksheet **#4**

Notation: Throughout this note matrices will be denoted by capital letters such as A or B, and vectors by lowercase boldface letters like $\mathbf{x}$ and $\mathbf{b}$.

Questions

1. (a) Given a vector $\mathbf{v}$ in $\mathbb{R}^2$, what is meant by $Span\{\mathbf{v}\}$?
   (b) Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, what is meant by $Span\{\mathbf{u}, \mathbf{v}\}$?
   (c) If $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^2$, under what conditions is $Span\{\mathbf{u}, \mathbf{v}\}$ all of $\mathbb{R}^2$?
2. If $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ are three vectors in $\mathbb{R}^3$, describe in geometric terms all of the possibilities for $Span\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.
3. Circle the sets of vectors below that form linearly dependent sets (these can be found just by inspection i.e. staring at them).
   (a) $\begin{bmatrix}1\\7\\6\end{bmatrix}, \begin{bmatrix}2\\0\\9\end{bmatrix}, \begin{bmatrix}3\\1\\5\end{bmatrix}, \begin{bmatrix}4\\1\\8\end{bmatrix}$ (b) $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$ (c) $\begin{bmatrix}-2\\4\\6\\10\end{bmatrix}, \begin{bmatrix}3\\-6\\-9\\15\end{bmatrix}$ (d) $\begin{bmatrix}2\\3\\5\end{bmatrix}, \begin{bmatrix}3\\4\\13\end{bmatrix}, \begin{bmatrix}1\\1\\8\end{bmatrix}$
4. Which of the following is a subspace of $\mathbb{R}^2$?
   (a) Span$\{\ \mathbf{u}, \mathbf{v}\ \}$ where $\mathbf{u} = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix}0\\0\end{bmatrix}$
   (b) The set consisting of just the vector $\begin{bmatrix}0\\0\end{bmatrix}$
   (c) Nul A where A is a 2×2 matrix.

Problems

1. Use row reduction to test if the following sets of vectors are linearly independent.
   (a) $\begin{bmatrix}0\\1\\5\end{bmatrix}, \begin{bmatrix}1\\2\\8\end{bmatrix}, \begin{bmatrix}4\\-1\\0\end{bmatrix}$ (b) $\begin{bmatrix}5\\0\\0\end{bmatrix}, \begin{bmatrix}7\\2\\-6\end{bmatrix}, \begin{bmatrix}9\\4\\-8\end{bmatrix}$ (c) $\begin{bmatrix}0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\5\\-8\end{bmatrix}, \begin{bmatrix}-3\\4\\1\end{bmatrix}$
   (Hint: let A be the matrix whose columns are the three vectors. Can you explain why finding a linear dependence between those three vectors would

be the same as finding a non-trivial solution $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{0}$? Why is this the same as A having a free variable?)

2. You have the following set of vectors: $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-3\\2\\0\\0\end{bmatrix}, \begin{bmatrix}5\\-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ Their span is a **subspace** of $\mathbb{R}^4$ (why?). We want to find a **basis** for that subspace (why aren't the 5 vectors above already a basis?).

   (a) Can you find a basis by inspection? Why is it a basis?

   (b) Ok great. Notice that this is also a basis of the column space of the following matrix: $B = \begin{bmatrix}1 & 0 & -3 & 5 & 0\\0 & 1 & 2 & -1 & 0\\0 & 0 & 0 & 0 & 1\\0 & 0 & 0 & 0 & 0\end{bmatrix}$ What was special about the columns that ended up in the basis? Will the same trick work on any matrix (i.e. will the same trick always work to find a basis for its column space)?

   (c) Now say we have the matrix: $A = \begin{bmatrix}1 & 3 & 3 & 2 & -9\\-2 & -2 & 2 & -8 & 2\\2 & 3 & 0 & 7 & 1\\3 & 4 & -1 & 11 & 8\end{bmatrix}$, which it turns out is actually row equivalent to matrix B. Does it have the same column space as B?

   (d) Can you give a basis for the column space of A? (Hint: Recall that the solution sets of $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ are identical i.e. if $x_1 = 4, x_2 = 5, x_3 = 1$ is a solution to A then it must be one for B as well. In other words if $4\mathbf{b}_1 + 5\mathbf{b}_2 - \mathbf{b}_3 = \mathbf{0}$ where $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are columns of B, then $4\mathbf{a}_1 + 5\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}$ too, where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are columns of A. Thus if $\mathbf{b}_3$ is a linear combination of the rows $\mathbf{b}_1, \mathbf{b}_2$, then $\mathbf{a}_3$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$. How can this help us find a basis for the column space of A?)

   (e) Based on the above, what are the steps to find a basis for a subspace given as the span of a set of vectors? (These steps will probably come in handy on midterms and/or exams).

3. Find a basis for the following sets with the technique from question 2:

(a) $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -13 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ -3 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 4 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ -1 \\ 7 \end{bmatrix}, \begin{bmatrix} -4 \\ -13 \\ 11 \\ -11 \end{bmatrix},$

4. As in question 2, we will now find a basis for a subspace. However, in this question, the subspace will be given by a system of homogeneous equations rather than the span of several vectors. E.g. consider the set of x_1, x_2, x_3, x_4, x_5 such that

$$\begin{aligned} -3x_1 + 6x_2 - x_3 + x_4 - 7x_5 &= 0 \\ x_1 - 2x_2 + 2x_3 + 3x_4 - x_5 &= 0 \\ 2x_1 - 4x_2 + 5x_3 + 8x_4 - 4x_5 &=0 \end{aligned}$$

(a) Why is the solution set a valid subspace anyways? Would it still be one if the right hand size of the equations included non-zero values? (Bonus: What if the equations were non-linear?)

(b) Write down the augmented matrix of this system. (Call the coefficient matrix A so the augmented matrix is $(A|\mathbf{0})$.

(c) Put the augmented matrix into reduced echelon form (why doesn't this change the basis we will eventually find?)

(d) Now that we've reduced, we can write out the solution set (e.g. that subspace we were looking for) in parametric form! Do so.

(e) Staring at the parametric form of the solution set, can you see a basis for the solution set?

(f) Based on the above, what are the steps to find a basis for a subspace given by equations? (Again, these steps are likely to be important on tests).

5. Use your technique from question 4 to find a basis for the solution sets to the following system.

$$\begin{aligned} 2x_1 + 5x_2 - 3x_3 + -4x_4 + 8x_5 &= 0 \\ 4x_1 + 7x_2 - 4x_3 - 3x_4 + 9x_5 &= 0 \\ 6x_1 + 9x_2 - 5x_3 + 2x_4 + 4x_5 &= 0 \\ -9x_2 + 6x_3 + 5x_4 - 6x_5 &=0 \end{aligned}$$