$\begin{pmatrix} 13 & 11 \\ -3 & -3 \\ 0 & 4 \end{pmatrix}$

Math 54 Worksheet #6

Definitions and Notation

1. if $A = (a_{ij})$ is the matrix $\begin{pmatrix} 1 & 3 \\ 12 & -3 \end{pmatrix}$ then what is the value $a_{1,2}$? → 3
2. What is $\begin{bmatrix} 13 & 4 \\ 0 & -2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 7 \\ -3 & -1 \\ 1 & 1 \end{bmatrix}$? How about $\begin{bmatrix} 13 & 4 \\ 0 & -2 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 7 & -9 \\ -3 & -1 & 2 \end{bmatrix}$?
   → not defined
3. Give an example of a diagonal matrix. Why is multiplying diagonal matrices easy?
   → $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$
   → b/c you can just multiply them entry by entry, eg $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 8 \end{pmatrix}$.
4. Is it true that $A + B = B + A$ for $n \times n$ matrices A and B? Is it true that $AB = BA$? If $AC = BC$, it it always true that $A = B$?
   yes, $A+B=B+A$. No, AB does not always $= BA$ and no $AC = BC$ does not imply $A=B$ (eg. cancellation does NOT work with matrices!).
   Consider $A = \begin{pmatrix} 0 & 4 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ $C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Problems

1. Suppose that Math 54 is being taught by two different professors. Prof. As lecture is more popular than Prof. Bs lecture. In fact, each week 90% of As students remain in the lecture, while only 10% switch into Bs lecture. On the other hand, 20% of Bs students switch into As lecture, with 80% remaining in Bs section.

   So the idea in this problem is to write the number of students in each lecture as entries in a vector, eg $\begin{bmatrix} 200 \\ 200 \end{bmatrix}$ at the beginning, and then convince yourself multiplying by this matrix is the same as moving the students around the way the paragraph above describes.

   This situation is described in the following table:

	from A	from B
into A	90%	20%
into B	10%	80%

   which can be represented by the matrix $\begin{bmatrix} 0.90 & 0.20 \\ 0.10 & 0.80 \end{bmatrix}$.

   Supposing that at the start of the semester each professor had 200 students, use matrix multiplication to answer the following:

   (a) How many students are there in each professors section after the 1^{st} week? (Hint: represent the number of students in each section by a 2×1 column matrix.)

   $\begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}\begin{pmatrix} 200 \\ 200 \end{pmatrix} = \begin{pmatrix} 220 \\ 180 \end{pmatrix}$

   (b) How many students are there in each professors section after the second week of classes?

   $\begin{pmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{pmatrix}\begin{pmatrix} 220 \\ 180 \end{pmatrix} = \begin{pmatrix} 234 \\ 166 \end{pmatrix}$

   *note, equivalently you could times the matrix by itself, and multiply this new matrix by $\begin{bmatrix} 200 \\ 200 \end{bmatrix}$.

for AB i got $\begin{pmatrix}15 & -7\\ -7 & 4\\ -25 & 10\end{pmatrix}$, for BC $\begin{pmatrix}7 & -19\\ -4 & 11\\ 2 & -5\end{pmatrix}$, CA isn't defined, for ABC i got $\begin{pmatrix}-22 & 59\\ 11 & -29\\ 35 & -95\end{pmatrix}$

2. Let $A = \begin{bmatrix}-2 & 1 & -2\\ 1 & 0 & 2\\ 3 & -3 & 1\end{bmatrix}$, $B = \begin{bmatrix}-5 & 2\\ 3 & -1\\ -1 & 1\end{bmatrix}$, $C = \begin{bmatrix}-1 & 3\\ 1 & -2\end{bmatrix}$ Which of the following matrix multiplications are defined? Compute those which are defined.
(a) AB (b) BC (c) CA (d) ABC

3. Let A and B be $n \times n$ matrices. Under what conditions is it true that $(A+B)(A-B) = A^2 - B^2$?

So $(A+B)(A-B) = A^2 + BA - AB - B^2$ so we need $BA = AB$ (eg we need them to commute)

$IA = AI = A$ for all other matrices A. Also, it's linear transformation does nothing (eg output equals input).

4. (a) What special property does the matrix $I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ possess?

(b) Given a 2×2 matrix A, can you always find another matrix B so that $AB = I$? → not if A is not invertible, eg $A = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$

(c) Given two 2×2 matrices A and B such that $AB = I$, is there anything noteworthy about BA? → BA also equals I (eg this is one example where A, B commute, also note this means you can check something is an inverse by just multiplying one side!). You can convince yourself this is true by noting $AB = I \to A$ is a bijection $\to A$ has an inverse

$A^{-1}(AB) = A^{-1}(I)$
↓
$B = A^{-1}$ ✓

5. Compute the inverse of $\begin{bmatrix}1 & 2 & 3\\ 2 & 5 & 3\\ 1 & 0 & 8\end{bmatrix}$.

$\begin{pmatrix}-40 & 16 & 9\\ 13 & -5 & -3\\ 5 & -2 & -1\end{pmatrix}$

6. Without doing any row reduction, determine if the following matrices are invertible: (a) $\begin{bmatrix}2 & 1 & -3 & 1\\ 0 & 5 & 4 & 3\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 3\end{bmatrix}$ (b) $\begin{bmatrix}5 & 1 & 4 & 1\\ 0 & 0 & 2 & -1\\ 0 & 0 & 1 & 1\\ 0 & 0 & 0 & 7\end{bmatrix}$

(a) yes (b) nope → an $n \times n$ matrix is invertible if it has n pivots.

7. There are many equivalent conditions for when a matrix is invertible. One you have probably already seen is that a matrix is invertible whenever you can row reduce it to the identity matrix. But there are others. Give a condition for when an $n \times n$ matrix A is invertible in terms of: (a) the pivots of A, (b) the linear transformation (e.g. is it onto?), (c) the columns of A.

(a) when it has all n pivots (should have put this question before 6)

(b) ↳ either onto or one-to-one will do since each implies the other for an $n \times n$ matrix.

(c) the columns must span the range.

8. Compute the inverse of the following matrices, if they exist.

(a) $\begin{bmatrix}12 & 0 & 0 & 0\\ 0 & 7 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$ (b) $\begin{bmatrix}3 & 0 & 1 & 1\\ 20 & 5 & 4 & 1\\ -7 & -2 & -1 & 0\\ -1 & -1 & 0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}-2 & -1 & -4\\ 5 & 2 & 10\\ 3 & 1 & 6\end{bmatrix}$.

(a) $\begin{pmatrix}1/12 & 0 & 0 & 0\\ 0 & 1/7 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1/3\end{pmatrix}$

(b) $\begin{pmatrix}2 & -1 & -2 & -1\\ -7 & 3 & 5 & 4\\ 0 & 1 & 3 & -1\\ -5 & 2 & 3 & 4\end{pmatrix}$

(c) this matrix isn't invertible! (you can tell b/c you shouldn't have been able to row reduce it to the identity)