1 The Wronskian

The bonus question on the previous worksheet proved the following. Let $y'' + ay' + by = 0$ be a linear homogeneous ODE, with solutions $y_1(t)$, $y_2(t)$. Then if the Wronskian

$$W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = 0$$

at any point t, it follows that $W(t) = 0$ **for all** values of t, and in this case $y_1(t), y_2(t)$ are linearly dependent (e.g. $y_1(t) = cy_2(t)$ for some constant c). Alternatively, if $W(t)$ is non-zero for all t, then the two solutions are linearly independent.

Determine which of the following pairs of functions are linearly dependent or independent by computing the wronskian:

1. $y_1(t) = \cos t \sin t$, $y_2(t) = \sin(2t)$

$$\cancel{\begin{vmatrix} \cos t \sin t & \sin 2t \\ -\sin^2 t + \cos^2 t & 2\cos 2t \end{vmatrix} = 2\cos t \sin t \cos 2t + \sin^2 t \sin 2t - \sin 2t \cos^2 t}$$

Cancelled this question...

2. $y_1(t) = e^{3t}$, $y_2(t) = e^{-4t}$

$$\begin{vmatrix} e^{3t} & e^{-4t} \\ 3e^{3t} & -4e^{-4t} \end{vmatrix} = -4e^{-t} - 3e^{-t} = -7e^{-t} \neq 0$$ independent

3. $y_1(t) = te^{2t}$, $y_2(t) = e^{2t}$

$$\begin{vmatrix} te^{2t} & e^{2t} \\ e^{2t} + 2te^{2t} & 2e^{2t} \end{vmatrix} = \cancel{2te^{4t}} - e^{4t} - \cancel{2te^{4t}} = -e^{4t} \neq 0$$ independent

4. $y_1(t) = 0$, $y_2(t) = e^t$

$$\begin{vmatrix} 0 & e^t \\ 0 & e^t \end{vmatrix} = 0$$ dependent

2 First Order Variable Coefficient Equations

Based on the webwork, it looks like you're going to need to be be able to solve equations of this form:

$$y' + u(t)y = w(t).$$

Let's start with the homogeneous case, when $w(t) = 0$. In this case the general solution is

$$y(t) = ce^{-U(t)},$$

where $U(t) = \int u(t)dt$ and c is some constant to be determined by the initial values.

1. Solve the initial value problem $y' + \cos ty = 0$, $y(0) = 1/2$.

$U(t) = \int \cos t \, dt = \sin t$

so general soln: $y = Ce^{-\sin t}$

$\frac{1}{2} = y(0) = Ce^0 \rightarrow C = 1/2$

so $y = \frac{1}{2}e^{-\sin t}$

2. Solve the initial value problem $ty' + 3y = 0$, $y(1) = 2$. (assuming $t>0$)

rearrange equation to the form we know how to deal with: $y' + 3y/t = 0$

$U(t) = \int \frac{3}{t} dt = 3\ln|t| = 3\ln t$ b/c $t>0$

general soln: $y = ce^{-3\ln t} = ce^{\ln t^{-3}} = ct^{-3}$

$y(1) = c = 2$ so $\boxed{y = 2t^{-3}}$

3. Derive the formula for the general solution from the equation $y' + u(t)y = 0$ by moving $u(t)y$ over to the other side, dividing both sides by y, and integrating both sides.

① $y' = -u(t)y$

② $\frac{dy}{dt}\frac{1}{y} = -u(t)$

③ $\int \frac{dy}{dt}\frac{1}{y} \cancel{dt} = \int -u(t)dt$

④ $\ln|y| + c = -\int u(t)dt$

⑤ $\ln|y| = -c - \int u(t)dt$

⑥ $|y| = e^{-c - \int u(t)dt}$

⑦ $y = Ce^{-\int u(t)dt}$

Alright, now for the non-homogeneous case. The formula for the solution is just slightly more complicated...

$$y = e^{-U(t)}(c + \int e^{U(t)}w(t)dt),$$

as before $U(t) = \int u(t)dt$. ($e^{U(t)}$ is called the integrating factor).

1. Find the general solution to $ty' = -2y + 4t^2$. $\longrightarrow$ $y' + \frac{2}{t}y = 4t$

$U(t) = \int \frac{2}{t}dt = 2\ln|t|$

also: $\int e^{2\ln|t|} 4t\,dt = \int t^3 dt = \frac{1}{4}t^4$

so $y = t^{-2}(c_1 + t^4) = \boxed{\frac{c_1}{t^2} + t^2}$

2. Find the general solution to $y' - 2y = x$.

$u(t) = -\int 2dt = -2t$

so $\int e^{u(t)}w(t)dt = \int e^{-2t}t\,dt$

$= (t(-\frac{1}{2})e^{-2t} + \frac{1}{2}\int e^{-2t}dt)$

$= (-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t})$

$y = e^{2t}(c_1 + (-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}))$

$= \boxed{c_1e^{2t} + (-\frac{1}{2}t - \frac{1}{4})}$

3. Find the general solution to $y'/x - 2y/x^2 = x\cos x$ for $x > 0$.

$y' - \frac{2}{x}y = x^2\cos x$

$U(t) = -\int \frac{2}{x}dx = -2\ln|x|$

so $\int e^{U(t)}w(t)dt = \int x^{-2}x^2\cos x\,dx$

actually this was fine :)

$= \int \cos x\,dx = \sin x$

$\boxed{y = x^2(c + \sin x)}$