1 Midterm Review!

1. There are many equivalent definitions of rank. Give 3. What is the rank of 3×3 matrix with a nullspace spanned by $[1, 2, -2]^T$? What are the possible ranks of an $n \times n$ projection matrix?

2. Diagonalization. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.
   (a) Suppose I tell you $\lambda = 1$ is an eigenvalue for A. Find all other eigenvalues (or show that 1 is the only eigenvalue).
   (b) Find a basis of eigenvectors for each eigenspace.
   (c) Diagonalize A.

3. Kristina's favourite example to try on T/F questions: Can you diagonalize this matrix: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$? Can you invert it? Bonus: write down a T/F question this matrix would be a counterexample for.

4. Suppose A is a 2×2 matrix with eigenvalue $\lambda_1 = 2$ corresponding to eigenvector $\mathbf{v}_1 = [-1, 1]^T$ and $\lambda_2 = -2$ corresponding to $\mathbf{v}_2 = [2, -3]^T$.
   (a) For $\mathbf{x}_0 = [0, 1]^T$, find $\mathbf{x}_k = A^k \mathbf{x}_0$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \lambda_1$ and λ_2.
   (b) Calculate A^n.

5. Prove "complex eigenvalues always come in conjugate pairs" (e.g. if λ in $\mathbb{C}$ is an eigenvalue of A, then $\overline{\lambda}$ is too).

6. This is not really review, but let's learn the spectral theorem. I'll give a statement of the theorem, read it and then try to answer the last question as best you can without checking back here.
   The Spectral Theorem. An $n \times n$ symmetric matrix has the following properties:
   (a) A has n REAL eigenvalues, counting multiplicities (e.g. no complex eigenvalues, ever. we love symmetric matrices).
   (b) A is orthogonally diagonalizable.

7. Suppose I told you a 3×3 matrix has exactly 2 distinct real eigenvalues with one eigenvector each (but I don't tell you anything about the complex eigenvlaues). Can you tell me if it's diagonalizable or not?

8. Orthogonal $n \times n$ matrices have lots of nice properties. Write down 3 equivalent definitions of a square matrix being orthogonal. (hint: one defintion has to do with what the matrix does to vectors you multiply it with).

9. Let V be the subset of $\mathbb{R}^4$ satisfying the equations $\mathbf{x}_1 + 2(\mathbf{x}_2 + \mathbf{x}_3) + \mathbf{x}_4 = 0$, $\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_4$ and $10\mathbf{x}_1 = 0$. Is this a subspace (hint: is it the nullspace of a matrix)? Find a basis for its orthogonal complement.

10. Let $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 2 & 2 \\ 2 & 0 & 4 \\ -3 & 1 & -5 \end{bmatrix}$. What are the steps to find the projection matrix P onto A's column space? (hint: is $A^T A$ invertible?)

    (a) Carry out those steps to find P (can leave P as a product of matrices).

    (b) Find the projection matrix onto A's left nullspace (hint: this shouldn't require much computation).

11. Suppose we are trying to fit the points $(x, y) = (1, 3)$, $(2, 0)$, $(-1, 1)$ with a cubic of the form $y = c_1 x + c_2 x^3$.

    (a) Do we expect to find a solution for c_1, c_2 satisfying all three points?

    (b) If not, set up the least squares problem that would find us the best possible solution (e.g. what matrix equations do we need to solve?).

    (c) Will we have a unique solution? (why?) Solve the least squares problem (it's ok to leave the answer as a product of matrices).

12. Spectral Theorem T/F:

    (a) There are symmetric matrices which are not orthogonally diagonalizable (i.e. diagonalizable with $A = PDP^{-1}$ where P is an orthogonal matrix).

    (b) If $A^T = A$ and $A = PDP^{-1}$ with D diagonal, then all entries of D must be real numbers.

    (c) The dimension of an eigenspace of a symmetric matrix (i.e. the geometric multiplicity) is sometimes less than the multiplicity of that eigenvalue as a root of the characteristic polynomial (i.e. the algebraic multiplicity).

    (d) If $A^T = A$ and if $\mathbf{u}, v$ satisfy $A\mathbf{u} = 3\mathbf{u}$, $A\mathbf{v} = 2\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.