1 Diagonalization True / False

In the following A, B are $n \times n$ matrices. Write T or F next to each (if true explain why, if false think of a counter example)

1. The determinant of A is the product of the diagonal entries of A.
2. An elementary row operation on A does not change the determinant.
3. $(\det A)(\det B) = \det(AB)$.
4. If $\lambda + 5$ is a factor of the characteristic polynomial of A (e.g. of $\chi_A(\lambda)$) then 5 is an eigenvalue of A.
5. If A is 3×3 with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, then $\det A$ equals the volume of the parallelepiped spanned by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$.
6. $\det A^T = (-1) \det A$ (dunno why the textbook has so many $\det A^T$ questions)
7. The multiplicity of a roof r of the characteristic equation of A is called the algebraic multiplicity of r as an eigenvalue of A.
8. A row replacement operation on A does not change the eigenvalues.
9. If A is diagonalizable over $\mathbb{R}$, then it has n linearly independent eigenvectors in $\mathbb{R}^n$.
10. The dimensions of the eigenspaces of A add up to n.
11. A is diagonalizable if $A = PDP^{-1}$ for some matrix D and invertible matrix P.
12. A is diagonalizable if and only if A has n eigenvalues, counting multiplicities.
13. If A is diagonalizable then A is invertible.
14. A is diagonalizable if A has n eigenvectors.
15. If A is diagonalizable then A has n distinct eigenvalues.
16. If $AP = PD$ with D diagonal, then the nonzero columns of P must be eigenvectors of A.
17. If A is invertible then A is diagonalizable.
18. If A has a factorization $A = PDP^{-1}$ then it is unique.

2 Diagonalization Problems

1. Suppose A is a diagonalizable 3×3 matrix. How many eigenspaces can it have? What are their possible dimensions? (e.g. can it have 4 eigenspaces of dimension 1?)

2. Suppose A is 3×3 and has eigenvalues $1, 2, 3$. Can you compute $Det(A)$? How about $Tr(A)$?

3. Suppose $n \times n$ real matrix A has complex eigenvalue λ. Show $\overline{\lambda}$ is also an eigenvalue of A. Suppose $[1 + 2i, -1 - i, 0]$ is an eigenvector corresponding to eigenvalue λ, give an eigenvector corresponding to $\overline{\lambda}$.

4. If a matrix A has complex eigenvalues, we can't diagonalize it over $\mathbb{R}$ (can we diagonalize it over $\mathbb{C}$?). Instead, if we want to stay in $\mathbb{R}$, we can write $A = PBP^{-1}$ where B is now a rotation matrix (technically it rotates and scales). So not quite as good (e.g. as simple) a representation as when B was diagonal, but now at least all the entries of B are real. Lets try this out.

   (a) Let $A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$, find the eigenvalues and a basis for each eigenspace of A.

   (b) Just for fun: diagonalize A over $\mathbb{C}$ (e.g. write $A = PDP^{-1}$ for D diagonal).

   (c) Hopefully you found two conjugate eigenvalues for A, e.g. λ and $\overline{\lambda}$. Let $\mathbf{v}$ be the eigenvector associated to $\lambda = a - bi$. Then theorem 9 in the text states: $A = PCP^{-1}$ where $P = [Re(\mathbf{v})\ Im(\mathbf{v})]$ and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. What are P and C in our case? Check $A = PCP^{-1}$.

3 Inner product and Orthogonality

This is one of those worksheets where you should read this front page and then flip it over and try to answer the questions without checking these notes (recalling improves retention!).

We're now moving on from diagonalization to Inner Products (also known as dot products) and Orthogonality. Inner products are really just matrix multiplication when both of your matrixes are size $1 \times n$ (e.g. when they are vectors). Formally, for $\mathbf{u} = [u_1, ..., u_n]^T$ and $\mathbf{v} = [v_1, ..., v_n]^T$, the inner product is:

$$\mathbf{u} \cdot \mathbf{v} = \begin{bmatrix} u_1 & & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \\ v_n \end{bmatrix} = u_1 v_2 + ... + u_n v_n.$$

We can use inner products to define the "length" of a vector. $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$. Using this in turn we can define the distance between two vectors to be the length of their difference, e.g. $dist(\mathbf{u}, \mathbf{v}) = ||\mathbf{u} - \mathbf{v}||$. We can use these definitions to calculate the angle between two vectors: it turns out

$$\mathbf{u} \cdot \mathbf{v} = ||\mathbf{u}||||\mathbf{v}|| \cos \theta$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ (ask me to draw a picture if this isn't clear).

The final concept that we need to define is orthogonality. We say that two vectors $\mathbf{u}$, $\mathbf{v}$ are orthogonal (or perpendicular) if $\mathbf{u} \cdot \mathbf{v} = 0$ (the textbook explains why on page 335). You can ask for the set of all vectors that are orthogonal/perpendicular to a vector. For example, in $\mathbb{R}^3$ the set of all vectors perpendicular to $[0, 0, 1]$ is the xy-plane (why?). More generally, given a subspace $W \subset \mathbb{R}^n$ we let $W^\perp$ be the subset of $\mathbb{R}^n$ of vectors that are orthogonal/perpendicular to all vectors in W. E.g. if W is the xy-plane, $W^\perp$ is the line $\{c[0, 0, 1] : c \in \mathbb{R}\}$. (why?)

4 Inner product and Orthogonality Problems

1. Which of the following are true: (a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, (b) $\mathbf{u} \cdot \mathbf{v} = 0$ implies $\mathbf{u}$ or $\mathbf{v} = \mathbf{0}$, (c) $\mathbf{u} \cdot \mathbf{u} \geq 0$.
2. Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ and $B^T = [\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3]$ be 3×3 matrices. Write BA in terms of inner products of the $\mathbf{a}_i$, $\mathbf{b}_j$.
3. Can you think of any intuitive reason why we might want the square root in our definition of the length of a vector? (Hint: think about pythagoras' theorem... $a^2 = b^2 + c^2$).
4. Compute the distance between the vectors/points $\mathbf{u} = [4, 1, 11]$, $\mathbf{v} = [1, -3, 6]$.
5. A unit vector is a vector of length 1. Find a unit vector pointing in the same direction as $[-30, 40]$.
6. Let $\mathbf{u} = [1, 0, -2]$, $\mathbf{v} = [1, 1, 1]$. Compute $||\mathbf{u} + \mathbf{v}||$ and $||\mathbf{u}|| + ||\mathbf{v}||$. Are they equal? When would they be equal? Compute $|\mathbf{u} \cdot \mathbf{v}|$ and $||\mathbf{u}||||\mathbf{v}||$. Are they equal?
7. Are $[8, -5]$, $[-2, -3]$ orthogonal?
8. Suppose $\mathbf{w}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{w}$ is orthogonal to $\mathbf{u} + \mathbf{v}$. In fact, show that $\mathbf{w}$ is orthogonal to every vector in $Span\{\mathbf{u}, \mathbf{v}\}$.
9. Show that if $\mathbf{x}$ is in both W and $W^\perp$ for some subspace W then $\mathbf{x} = \mathbf{0}$.
10. Consider the subspace $W = Span\{[1, 1, 1], [2, 0, -1]\}$. Find a vector spanning $W^\perp$. Write the vector $[2, 1, -3]$ in the form $\mathbf{w} + \mathbf{w}^\perp$ where $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$.