Review

1. What is a basis (e.g. what are the two requirements).
   Is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$ a basis for $\mathbb{R}^2$ inside $\mathbb{R}^3$?

2. Given a matrix A, explain why both Nul A and the column space of A are subspaces (e.g. go through the 3 requirements of a subspace and explain why each holds. Hopefully this is easier now that you can think about things in terms of matrix multiplication). Are they "abstract vector spaces" as defined on page 192?

3. Compute $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^n$. What is $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}^n \begin{bmatrix}1\\1\end{bmatrix}$?

4. Suppose x is a real number satisfying $x^2 = 1$. To solve for x, we factor $x^2 - 1 = (x-1)(x+1) = 0$, and conclude that $x = \pm 1$. What if X is a 2×2 matrix satisfying $X^2 = I$?
   (a) Show that $(X - I)(X + I) = 0$.
   (b) Are those the only solutions? What about $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$?
   (c) Let a be any real number, and let $A = \pm\begin{bmatrix}1 & 0\\a & -1\end{bmatrix}$. Show that $A^2 = I$.
   (d) Let $0 \leq \theta \leq 2\pi$, and let $R = \begin{bmatrix}\cos\theta & -\sin\theta\\-\sin\theta & -\cos\theta\end{bmatrix}$. Show $R^2 = I$.
   (e) Explain why there can be so many solutions. What is different about matrices vs numbers that allows this to happen? (hint: what happens if you plug one of the matrices in (b) into the factored equation of (a)?)

Problems

1. Let V be a vector space, and suppose that L and M are two subsets of V that happen to also be vector spaces (e.g. they are subspaces). Is it true that $L \cup M$ is a vector space ($L \cup M$ is the set of vectors in either L or M). How about $L \cap M$? ($L \cap M$ is the set of vectors that are in both L and M).

2. This question is going to try and explain "coordinate mapping" which is covered in chapter 4.4 of the text. First we need the following fact: Let $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x} \in V$ there exists a unique set of real numbers $c_1, ..., c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + ... + c_n\mathbf{b}_n$.

   (a) What are the coordinates of $\mathbf{x}$ in terms of the basis B? (answer in case you haven't seen this before: the coordinates are the real numbers $c_1, ..., c_n$).

   (b) What is the coordinate vector of $\mathbf{x}$ relative to B? (answer: it's just the vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. The textbook denotes this by saying $[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$).

   (c) Let $\mathbf{b}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Do $\mathbf{b}_1$, $\mathbf{b}_2$ form a basis for $\mathbb{R}^2$?

   (d) Ok, spoiler, they do. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Suppose $[\mathbf{y}]_B = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$. What was $\mathbf{y}$?

   (e) We can also got the other way. Find the coordinate vector for $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in terms of B, e.g. find $[\mathbf{x}]_B$. Hint: you know $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{x}$. Turn this into a matrix equation and solve it.

   (f) Note that the matrix you found in (e) can be reused to put any vector into B-coordinates! Use it to write $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ in terms of B.

   (g) The above matrix is the "change of coordinates" matrix, lets call it P_B like the textbook does. The hardest thing about the change of coordinates matrix is remembering which direction it goes. Write down the general formula for how P_B relates a vector $\mathbf{x}$ to $[\mathbf{x}]_B$.

3. Let $\mathbf{b}_1 = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Show $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ form a basis.

   (a) Find the change of basis matrix to go from B to the standard basis.

   (b) Use the matrix from part (a) to put $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ into B-coordinates.

   (Note: be careful, do you multiply $\mathbf{x}$ by the matrix or by it's inverse?)