1 The Method of Undetermined Coefficients

Today we add another type of ODE to the collection of problems we know how to solve. Recall we previously did homogenous linear equations $ay'' + by' + cy = 0$ and variable coefficient first order equations $y' + u(t)y = w(t)$.

The method of undetermined coefficients helps us solve non-homogenous differential equations (with constant coefficients). The textbook gives a good introduction (chapters 4.4, 4.5) which everyone should read sometime, but the statement you ultimately have to memorize is:

To find a particular solution to the differential equation:

$$ay'' + by' + cy = P_m(t)e^{rt}$$

where $P_m(t)$ is a polynomial of degree m, guess a solution of the form:

$$y(t) = t^s(A_m t^m + \cdots + A_1 t + A_0)e^{rt}$$

if r is **not** a root of the associated auxiliary equation (i.e. of $ar^2 + br + c = 0$) then take $s = 0$. If r is a simple root (i.e. not a double root) then take $s = 1$. And if r is a double root, take $s = 2$.

To find a particular solution to the differential equation:

$$ay'' + by' + cy = P_m(t)e^{\alpha t}\cos(\beta t) + Q_n(t)e^{\alpha t}\sin(\beta t)$$

where $P_m(t)$ is a polynomial of degree m, $Q_n(t)$ is one of degree n and $\beta \neq 0$, guess a solution of the form:

$$y(t) = t^s(A_k t^k + \cdots + A_1 t + A_0)e^{\alpha t}\cos(\beta t) + t^s(B_k t^k + \cdots B_1 t + B_0)e^{\alpha t}\sin(\beta t),$$

where k is the larger of m and n. If $\alpha + i\beta$ is not a root of the associated auxiliary equation take $s = 0$, if $\alpha + i\beta$ is a root take $s = 1$.

1. Find a particular solution to $y'' + y' - 6y = 5t$.

2. Find a particular solution to $y'' - 4y' - 12y = 3e^{5t}$.

3. Find a particular solution to $y'' - 4y' - 12y = \sin(2t)$.

4. Find a particular solution to $y'' - 4y' - 12y = 2t^3 - t + 3$.

5. Find a particular solution to $y'' - 4y' - 12y = te^{4t}$.

6. Find a particular solution to $y'' - 4t' - 12y = e^{6t}$.

7. Find a particular solution to $y'' + y' + y/4 = te^{-t/2}$.

8. Write down the general form of the particular solution to $y'' + ay' + by = g(t)$ for:

   (a) $g(t) = 16e^{7t}\sin(10t)$.

   (b) $g(t) = (9t^2 - 103t)\cos t$.

   (c) $g(t) = -e^{-2t}(3 - 5t)\cos(9t)$.

9. Find the particular solution to $y''-4y'-12y = 3e^{5t}+2\sin(2t)+4te^{4t}$. (Note the right hand side is the linear combination of 3 functions we know how to handle. If you haven't seen this yet: it turns out you can just linearly combine the particular solutions in the same way - why?).

10. Find the particular solution to $y'' + y' - 6y = 5t + 4e^{2t}$.

11. Solve the following initial value problem $y'' - 4y' - 12y = 3e^{5t}$, $y(0) = 18/7$, $y'(0) = -1/7$. (You do this by combining the homogenous solution with the particular solution and solving for the coefficients, ask me if you haven't seen this before and want an example).

12. Solve the following initial value problem $y'' + y' - 6y = 5e^{4t}$, $y(0) = 1/2$, $y'(0) = \sqrt{2}\pi/2$.