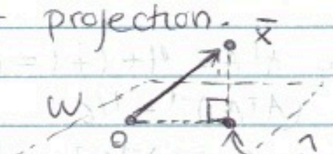
Projection Matrices

Q1. Generally, when we say projection we mean orthogonal projection. The text explains this better than I can on page 349, but the idea is to find the closest vector in the subspace W, to your original vector $\bar{x}$. That new vector is your projection.



$\hat{x}$ = projection of $\bar{x}$ onto W.

Q2 The 1st 4 rows of the RREF tell us that $Col(B)$ has basis: $[1\ 2\ 4]^T$, $[2\ 3\ 7]^T$.

so $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 4 & 7 \end{bmatrix}$.

Q3. $col(A)$ is 2 dimensional, so it would be enough to show $col(B)$ is as well, and that $col(A) \subseteq col(B)$.
(contained in)

~~This amounts to checking~~ Indeed from the RREF matrix we see $col(B)$ has only 2 linearly independent vectors, and it is certainly true that $col(A) \subseteq col(B)$, by definition of A.

(we want this b/c a projection matrix better at least have the right range)

Q4. From $RREF(B \mid I_3)$ we see $LNull(B)$ is gen by $[1\ \tfrac{1}{2}\ -\tfrac{1}{2}]^T$ or equivalently $[2\ 1\ -1]^T$. It's easy to check $P[2\ 1\ -1]^T = \bar{0}$.

(we want this b/c $LNull(B)$ is the orthogonal complement of the column space - exactly the set that a projection onto the column space should be sending to $\bar{0}$).

Q5 Yes, we can check this by taking the transpose:

$$(A(A^TA)^{-1}A^T)^T = (A^T)^T(A(A^TA)^{-1})^T$$
$$= A((A^TA)^{-1})^T A^T$$
$$= A((A^TA)^T)^{-1}A^T$$
$$= A((A^T)(A^{TT}))^{-1}A^T$$
$$= A(A^TA)^{-1}A^T = \text{the original matrix!}$$

Q6. so now $A' = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$ so $A^TA = 4+1+1 = 6$

$(A^TA)^{-1} = 1/6$

$$P' = A(A^TA)^{-1}A^T = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}\left(\tfrac{1}{6}\right)\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}$$
$$= \frac{1}{6}\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

Q7. $\frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \frac{1}{6}\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 2 & -2 & 2 \\ -2 & 5 & 1 \\ 2 & 1 & 5 \end{bmatrix}.$

Projecting onto W is the same as subtracting the projection on the orthogonal complement!

True / False

1. T (many ways to see this, for one the coordinate change matrix in the reverse direction must be an inverse)
2. T (by definition)
3. T (see page 336)

5. T → 4. F (same eigenvalues, different eigenvectors)

6. T (The characteristic poly is just det of $A - \lambda I$ so if it is zero for a particular λ then the determinant is zero so the nullspace of $A - \lambda I$ contains a non-trivial vector)
7. F (and when there aren't 3, the matrix is not diagonalizable)

Miscellaneous Problems

1. No. If the matrix only has 2 real eigenvalues with multiplicity 1 each then in order to have a full basis of eigenvectors it would need a 3rd eigenvector from a complex eigenvalue (b/c we said there were only 2 real eigenvalues). → but complex eigenvalues always come in conjugate pairs (eg if $a+ib$ is an eigenvalue then $a-ib$ is too).

2. Note $[-1/\sqrt{5}, 2/\sqrt{5}] \cdot [-1/\sqrt{5}, 2/\sqrt{5}] = \frac{1}{(\sqrt{5})} + \frac{4}{(\sqrt{5})} = 1$

$$[2/\sqrt{5}, 1/\sqrt{5}] \cdot [2/\sqrt{5}, 1/\sqrt{5}] = 1$$

and $[-1/\sqrt{5}, 2/\sqrt{5}] \cdot [2/\sqrt{5}, 1/\sqrt{5}] = -\frac{2}{5} + \frac{2}{5} = 0$, so yes.

$$U = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}, \quad U^T U = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} \frac{1}{5}$$

$$= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \frac{1}{5} = I \text{ as expected.}$$

3. Yes, since $A^T A = I$, we know A^T is at least a left-inverse, and recall for a square matrix this implies it is a right inverse as well.

4. Yes, it is enough to check $(UV)^T (UV) = I$

so: $(UV)^T UV = V^T U^T U V = V^T V$ (b/c $U^T U = I$)

$= I$ (b/c $V^T V = I$)

~~8.~~ $u_i = [u_{i1}, \dots, u_{ip}]$

5. Yes, let $u_1 \dots u_n$ be the colums of U, so U is orthogonal if and only if $u_i u_i = 1$ and $u_i u_j = 0$. But $u_i u_i = u_{i1}^2 + \dots + u_{ip}^2$ and changing the rows only reorders this sum, so $u_i u_i = 1$ and $u_i u_j = 0$ both still hold in V.