1 Inner Products, Orthogonality and Projection True / False

Write T or F next to each (if true explain why, if false think of a counter example)

1. $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$.
2. The orthogonal projection $\widehat{\mathbf{y}}$ of $\mathbf{y}$ onto a subspace W can sometimes depend on the orthogonal basis for W used to compute $\widehat{\mathbf{y}}$.
3. For any scalar c, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.
4. If an $n \times p$ matrix U has orthonormal columns then $UU^T\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.
5. For a square matrix A, vectors in Col A are orthogonal to vectors in Nul A.
6. If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is an orthogonal basis for W, then multiplying $\mathbf{v}_3$ by a scalar c gives a new orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2, c\mathbf{v}_3\}$.
7. If the vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n$ span a subspace W and $\mathbf{x}$ is orthogonal to each $\mathbf{v}_j$ then $\mathbf{x}$ is in $W^\perp$.
8. The Gram-Schmidt process produces from a linearly independent set $\{\mathbf{x}_1, ..., \mathbf{x}_p\}$ an orthogonal set $\{\mathbf{v}_1, ..., \mathbf{v}_p\}$ such that for each k, the vectors $\mathbf{v}_1, ..., \mathbf{v}_k$ span the same subspace as $\mathbf{x}_1, .., \mathbf{x}_k$.
9. Not every linearly independent set it in $\mathbb{R}^n$ is an orthogonal set.
10. If $A = QR$, where Q has orthonormal columns, then $R = Q^TA$.
11. If $\mathbf{y}$ is a combination of nonzero vectros from an orthogonal set, then the weights in the linear combination can be computed without row operations on a matrix.
12. If you normalize the vectors in an orthogonal set of nonzero vectors then some of the new vectors may not be orthogonal.
13. A matrix with orthonormal columns is an orthogonal matrix.
14. If L is a line through $\mathbf{0}$ and $\widehat{\mathbf{y}}$ is the orthogonal projection of $\mathbf{y}$ onto L, then $||\widehat{\mathbf{y}}||$ is the distance from $\mathbf{y}$ to L.
15. If P is a projection matrix, then $P^2 = P$.

2 Problems

1. Find the orthogonal projection of $[3, -2, 10]^T$ onto **one** of the two following subspaces: $W_1 = Span\{[2, 2, -4]^T, [1, 0, 3]^T\}$ or $W_2 = Span\{[5, -2, 1]^T, [1, 2, -1]^T\}$. (Hint: one of these is much easier than the other, why is that?)

2. Find the equation of a line $y = mx + b$ that best fits the points $(-1, -1)$, $(1, 0)$ and $(2, 4)$ in the least-squares sense by following these steps:
   (a) Write down the (inconsistent) system of three equations in two unknowns for this problem.
   (b) Rewrite this system as a matrix equation $A\mathbf{x} = \mathbf{y}$, where $\mathbf{x} = [m, b]^T$. What are A and $\mathbf{y}$?
   (c) Find A^T and form the equation $A^T A\mathbf{x} = A^T\mathbf{y}$.
   (d) Since the columns of A are linearly independent, $A^T A$ is invertible. Find $(A^T A)^{-1}$ and use it to solve the equation in part (c).
   (e) What is the equation of the line of best fit?

3. Let $A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 5 & -2 \\ 3 & -7 & 8 \end{bmatrix}$. We're going to project some vectors onto A's column space.
   (a) Use Gram-Schmidt to find an orthogonal basis for A's column space.
   (b) Project $\mathbf{x} = [1, 2, 3, 4]^T$ onto A's column space and write $\mathbf{x}$ in terms of the basis you found in part (a) (e.g. $[\mathbf{x}]_{\mathcal{B}}$ where $\mathcal{B}$ is that basis).
   (c) What if we want to project a lot of vectors? Does there exist a matrix projecting vectors onto A's columns space? What is it in terms of A? What would its dimensions be?

4. Let $A = \begin{bmatrix} 2 & 1 \\ -3 & -4 \\ 3 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix}$, $\mathbf{u} = [4, 5]^T$ and $\mathbf{v} = [6, -5]^T$. Compute $A\mathbf{u}$ and $A\mathbf{v}$, and compare them with $\mathbf{b}$. Is it possible that at least one of $\mathbf{u}$, $\mathbf{v}$ could be a least-squares solution of $A\mathbf{x} = \mathbf{b}$?

5. Let $A = \begin{bmatrix} 1 & -2 & -4 \\ 2 & -5 & -3 \\ 3 & -7 & -7 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 20 \\ -16 \\ 14 \end{bmatrix}$ and express $\mathbf{x}$ as the sum of a vector in the row space of A and a vector in the nullspace of A.