1. Use the method of undetermined coefficients to find a general solution to the system $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f}(t)$ for the following:

   (a) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$

   $$c_1 \begin{bmatrix} e^{3t} \\ 2e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ -2e^{-t} \end{bmatrix} + \begin{bmatrix} t \\ 2 \end{bmatrix}.$$

   (b) $A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} -4\cos t \\ -\sin t \end{bmatrix}$

   $$c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} + \begin{bmatrix} -2\sin t \\ 2\sin t + \cos t \end{bmatrix}.$$

2. Find the solution to $\mathbf{x}'(t) = \begin{bmatrix} 0 & 2 \\ 4 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 4t \\ -4t-2 \end{bmatrix}$ satisfying each of the following initial conditions:

   (a) $\mathbf{x}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

   $$\begin{bmatrix} 3e^{-4t} + e^{2t} \\ -6e^{-4t} + e^{2t} - 2t \end{bmatrix}$$

   (b) $\mathbf{x}(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $$\begin{bmatrix} -(4/3)\,e^{-4(t-2)} + (7/3)\,e^{2(t-2)} \\ (8/3)\,e^{-4(t-2)} + (7/3)\,e^{2(t-2)} - 2t \end{bmatrix}$$

3. If $A^2 = 0$, prove that $e^A = I + A$.

We have $e^A = I + A + A^2/2! + A^3/3! + ... = I + A$.

4. Let $A = \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$. Find the eigenvalues and vectors of A and use them to compute e^{At}.

The reader can easily verify that 4 and 3 are eigenvalues of A, with corresponding eigenvectors $w_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. It follows that

$$A = PDP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

so that

$$e^A = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^4 & 0 \\ 0 & e^3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 2e^4 - e^3 & e^4 - e^3 \\ 2e^3 - 2e^4 & 2e^3 - e^4 \end{bmatrix}$$

5. Find the general solution to the following systems using matrix exponentiation.

(a) $\mathbf{x}'(t) = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} \mathbf{x}(t)$

One has $\begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$. Thus

$$\begin{aligned} e^A &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \exp\left(\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \right) \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{5t} + e^{-t} & e^{5t} - e^{-t} \\ e^{5t} - e^{-t} & e^{5t} + e^{-t} \end{bmatrix} \end{aligned}$$

(b) $\mathbf{x}'(t) = \begin{bmatrix} 5 & -3 \\ 1 & 1 \end{bmatrix} \mathbf{x}(t)$

$$e^{tA} = \begin{bmatrix} \frac{3}{2}e^{4t} - \frac{1}{2}e^{2t} & -\frac{3}{2}e^{4t} + \frac{3}{2}e^{2t} \\ \frac{1}{2}e^{4t} - \frac{1}{2}e^{2t} & -\frac{1}{2}e^{4t} + \frac{3}{2}e^{2t} \end{bmatrix}$$

6. Bonus: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show by direct computation that $e^{A+B} \neq e^A e^B$.

Check that $e^A = \begin{bmatrix} e & 1 \\ 1 & 0 \end{bmatrix}$ and $e^B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ so $e^A e^B = \begin{bmatrix} e & 1+e \\ 1 & 1 \end{bmatrix}$.

Meanwhile $e^{A+B} = \begin{bmatrix} e & e-1 \\ 0 & 0 \end{bmatrix}$ because $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ so

$$\exp \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = I + \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{2!} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}^2 + ... = \begin{bmatrix} e & e-1 \\ 0 & 0 \end{bmatrix}.$$