Review T/F

1 T

2 F ($\bar{x}$ could be $\bar{0}$)

3 T

4 F (almost every matrix is a counter example :)

5 T (if it's not invertible then b/c it's square there's a non-zero vector in its nullspace)

6. T

7 F $\left(\begin{bmatrix}2&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&3\end{bmatrix}\right)$

8 F (again almost any matrix will be a counter example)

9 F ($\det A^3 = (\det A)^3 = 8$)

10 F (again $\bar{x}$ could be $\bar{0}$)

11 F $\begin{pmatrix}0\\1\end{pmatrix}$ and $\begin{pmatrix}1\\0\end{pmatrix}$ ~~both~~ correspond to $\lambda=2$ for $\begin{pmatrix}2&0\\0&2\end{pmatrix}$

12 T

13 T

14 F (it has area 5, the parallelogram has area 10)

15 F (true if it's a diagonal matrix)

16 T ($\det(A^T A) = \det(A^T)\det(A) = \det(A)^2 \geq 0$)

17 T

18 F (counter-example: any time the coefficient matrix is not invertible b/c then $\det(A)=0$ and you can't divide by it)

19 T ($0 = \det(A^3) = (\det(A))^3 \rightarrow \det(A) = 0$)

20 F ($\det(A^{-1}) = \det(A)^{-1}$ so just pick an A with $\det \neq \pm 1$)

21 F (don't do this!!)

Cayley Hamiton

1. That any square matrix satisfies its own characteristic polynomial (eg $\chi_A(A) = 0$)

2. $\begin{vmatrix}1-\lambda & -1\\ 2 & 3-\lambda\end{vmatrix} = \lambda^2 - 4\lambda + 5 \rightarrow$ So by C.H. $A^2 - 4A + 5I = 0$

$\rightarrow$ rearranging: $A\left(\frac{4-A}{5}\right) = I$

$\rightarrow$ so $A^{-1} = (4-A)/5$.

Cayley Hamilton cont.

3. The determinant of a matrix is the product of its eigenvalues (with multiplicities) so $\det(A) = (i)(-i)(-1) \neq 0$ so A is invertible. By C.H. $(A-i)(A+i)(A+1) = 0$ $\rightarrow A^3+A^2+A+I=0 \rightarrow A(-A^2-A-I) = I$ so $A^{-1} = -A^2-A-I$.

In the second case $\det(A)=0$ so A is not invertible.

More EV and Diagonalization.

the basis formed by the columns of matrix B.

1 (a) Yup (b) Yes, from vectors in terms of B into vectors in terms of the standard basis. This is denoted $\underset{S \leftarrow B}{P}$.

(c) Yup, $\underset{B \leftarrow S}{P}$

(d) So BCB^{-1} takes vectors in terms of the standard basis, rewrites them to be in terms of the basis $\mathcal{B}$, applies C to them in this form and then B translates the output back into the standard basis.

this is a lot to digest so give your-self time :)

$\rightarrow$ So $A = BCB^{-1}$ means they are secretly the same linear transformation, just A is written in terms of the standard basis and C in terms of the basis $\mathcal{B}$.

(e) If C is diagonal then this all means the linear transformation behind A can be written in terms of another basis as a diagonal matrix - this is great b/c diagonal matrices are really easy to understand. eg what does $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ do on an arbitrary vector $(x, y, z)^T$? It's just $(4x, 5y, 6z)^T$.

Eigenvectors / Diagonalization cont.

2. All vectors in the xy-plane are fixed so these belong to the eigenspace with $\lambda=1$. The line $x=y=0$ (eg the z-axis) is also an eigenspace with $\lambda=-1$.

This rotation has no non-trivial eigenvectors! (technically the eigenspace is $\{\bar{0}\}$ but remember that $\bar{0}$ still isn't an eigenvector!)

3. Yup $\det(A)=(1)(-2)(3)(-3)=18$. In the second case b/c there are only 3 eigenvalues, but the matrix is 4x4, we don't know their multiplicities (and remember the determinant is the product with multiplicies, think about the diagonal matrix case to remember this).

4. Since $(A-cI)^T = A^T-(cI)^T = A^T-cI$ we have $\det(A^T-cI)=\det((A-cI)^T)=\det(A-cI)$.
→ so yes A, A^T have the same eigenvalues.

5. Since A is diagonalizable $A=SDS^{-1}$ for diagonal D. Since A is nilpotent $0=A^n=SD^nS^{-1}$ (using that $(SDS^{-1})^n=SD^nS^{-1}$).

For $D=\begin{pmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{pmatrix}$, $D^n=\begin{pmatrix} d_1^n & & & 0 \\ & d_2^n & & \\ & & \ddots & \\ 0 & & & d_n^n \end{pmatrix}$

So, b/c $0=SD^nS^{-1} \rightarrow S^{-1}0S=D^n \rightarrow 0=D^n$ we can conclude $d_1=d_2=\dots=d_n=0$.
So $D=0$, so $A=S0S^{-1}=0$ too!

6. Char poly is $(a-\lambda)(d-\lambda)-bc$ → rearrange and apply the quadratic formula, roots $=\dfrac{(a+d)\pm\sqrt{(a-d)^2+4bc}}{2}$ ← after some rearranging
so get 2 distinct real e.v.'s if $(a-d)^2-4bc>0$!