Find the general solution to the following differential equations:

1. $y' + 2y = 0$

$r+2=0 \rightarrow r=-2 \qquad y=Ce^{-2t}$

2. $y'' - 9y = 0.$

$r^2-9=0 \rightarrow r=\pm 3 \qquad y=C_1e^{3t}+C_2e^{-3t}$

3. $y'' + 4y' - 5y = 0.$

$r^2+4r-5=0 \rightarrow r=-5,1 \qquad y=C_1e^{-5t}+C_2e^{t}$
$(r+5)(r-1)$

4. $y'' - y' - 11y = 0.$

$r^2-r-11=0 \rightarrow r=\frac{1}{2}\pm\frac{3\sqrt{5}}{2} \qquad y=C_1e^{(\frac{1}{2}+\frac{3\sqrt{5}}{2})t}+C_2e^{(\frac{1}{2}-\frac{3\sqrt{5}}{2})t}$

5. $y'' - 4y' + 7y = 0.$

$r^2-4r+7=0 \rightarrow r=2\pm\sqrt{3}i \qquad y=C_1e^{(2+\sqrt{3}i)t}+C_2e^{(2-\sqrt{3}i)t}$
$=C_3e^{2t}\cos(\sqrt{3}t)+C_4e^{2t}\sin(\sqrt{3}t)$

6. $y'' + 7y = 0.$

$r^2+7=0 \rightarrow r=\pm\sqrt{7}i \qquad y=C_1e^{\sqrt{7}it}+C_2e^{-\sqrt{7}it}=C_3\cos(\sqrt{7}t)+C_4\sin(\sqrt{7}t)$

7. Solve the initial value problem: $y'' + y' = 0$, $y(0) = 2$, $y'(0) = 1$.

$r^2+r=0$
$\rightarrow r=0,-1$

so general soln:
$y=C_1+C_2e^{-t}$

$y(0)=C_1+C_2=2$
$y'(0)=-C_2=1$
$\rightarrow y=3-e^{-t}$

8. Solve the initial value problem: $y'' - 4y' + 4y = 0$, $y(1) = 1$, $y'(1) = 1$.

$r^2-4r+4=0$
$\rightarrow r=2,2$

so general soln:
$y=C_1e^{2t}+C_2te^{2t}$

$y(1)=C_1e^2+C_2e^2=1$
$y'(1)=2C_1e^2+C_2e^2+2C_2e^2=1$
$\rightarrow y(t)=\frac{2}{e^2}e^{2t}-\frac{1}{e^2}te^{2t}$

9. Solve the initial value problem: $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 1$.

$r^2 + 9 = 0$

$r = \pm i3$

general soln:

$y = C_1\cos(3t) + C_2\sin(3t)$

apply initial conditions:

$y = \frac{1}{3}\sin(3t) + \cos(3t)$

10. Find the general solution to the differential equation $y''' - y'' + y' + 3y = 0$. (try e^{rt} again)

$r^3 - r^2 + r + 3 = 0$

note -1 is a root

$\rightarrow r = -1, 1 \pm i\sqrt{2}$

general soln:

$y = C_1 e^{-t} + C_2 e^{t}\cos(\sqrt{2}t) + C_3 e^{t}\sin(\sqrt{2}t)$

11. Consider the initial value problem: $y'' + y' - 6y = 0$, $y(0) = a$, $y'(0) = 1$. For what values of a does the solution go to infinity as t goes to infinity? For what values of a does the solution go to zero as t goes to infinity? For what values of a does the solution go to negative infinity as t goes to infinity?

general soln:

$y = C_1 e^{-3t} + C_2 e^{2t}$

$y(0) = C_1 + C_2 = a$

$y'(0) = -3C_1 + 2C_2 = 1$

$\downarrow$

$5C_2 = 3a + 1$

$C_2 = \frac{3a+1}{5}$

$\rightarrow C_1 = \frac{2a-1}{5}$

general soln:

$y = \underbrace{\left(\frac{2a-1}{5}\right)e^{-3t}}_{\rightarrow 0} + \underbrace{\left(\frac{3a+1}{5}\right)e^{2t}}_{\rightarrow \infty}$

only C_2 matters (b/c $e^{-3t} \rightarrow 0$)

so if $\boxed{C_2 > 0 \ y \rightarrow \infty}$, $\boxed{C_2 < 0 \ y \rightarrow -\infty}$, $\boxed{C_2 = 0 \ y \rightarrow 0}$

12. (Bonus problem) For two function $y_1(t), y_2(t)$ we define the Wronskian as $W[y_1, y_2](t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix}$.

(a) If the Wronskian is nonzero at a point t_0, what does that tell you about the system of equations: $\begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$? has a unique solution.

(b) Explain why if the Wronskian is nonzero at a point t_0 then y_1 and y_2 are linearly independent. (From this we see that if the Wronskian is nonzero at a point the two functions are linealy independent.)

(c) Suppose y_1 and y_2 solve the differential equation $y'' + ay' + by = 0$. Show that the $y(t) = c_1y_1(t) + c_2y_2(t)$ also satisfies the differential equation.

(d) Following part (3), show that if the Wronskian is 0 at a point t_0 then $y(t)$ solves the initial value problem: $y'' + ay' + by = 0$, $y(t_0) = 0$, $y'(t_0) = 0$.

(e) Explain why this means that $y(t)$ is zero everywhere. Explain why this means that the Wronskian is zero everywhere.

(f) We have shown that if y_1 and y_2 solve the same ODE and the Wronskian is 0 at a point, then we have constants c_1 and c_2 not both zero such that $c_1y_1(t) + c_2y_2(t) = 0$ for all t, thus y_1 and y_2 are linearly dependent.

(b) if $y_2(t) = cy_1(t)$ then $y_2'(t) = cy_1'(t) \rightarrow W = 0$ everywhere.

(c) just plug it in

(d) I think this question meant that if $W = 0$ then the matrix in (a) has a non-trivial solution, so choosing that c_1, c_2 solves the initial value problem.

(e) Then all higher derivatives are zero too (differentiate the ODE) $\rightarrow y = 0$ everywhere