Questions

1.(a) In general Span is the set of all sums / "linear combinations" of the vectors, so Span$\{\bar{x}, \bar{y}, \bar{z}\}$ = the set of all vectors of the form $a\bar{x}+b\bar{y}+c\bar{z}$ for a, b, c in $\mathbb{R}$. In this case there is only 1 vector so you just get all its multiples.

(b) see ↗

(c) Well, $\bar{u}=\begin{bmatrix}0\\0\end{bmatrix}$, $\bar{v}=\begin{bmatrix}1\\0\end{bmatrix}$ wouldn't work, nor would $\bar{u}=\begin{bmatrix}1\\1\end{bmatrix}$, $\bar{v}=\begin{bmatrix}2\\2\end{bmatrix}$. The problem with these is that they are linearly dependent. If they were independent then we would know they generate all of $\mathbb{R}^2$ b/c any linearly independent set of n vectors, where n = the dimension * of the space, is a generating set for the space. (In fact this makes it a basis). Here I have also used that $\mathbb{R}^2$ has dimension = 2, which you can see from the fact that $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ is a basis for it. → dimension = size of its basis

2. point, line, plane, all of $\mathbb{R}^3$

3. (a) must be linearly dependent b/c any time the number of vectors exceeds the number of entries in the vectors you have a linear dependence.

(b) this is linearly dependent b/c it contains $\bar{0}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

(c) this would have been dependent if the 15 was −15 but currently it is independent. Recall you can always check if 2 vectors are linearly dependent b/c one would have to be a multiple of the other.

(d) the first + the second = the third, so dependent.

4. All three examples are. Span will always be a subspace. $\begin{bmatrix}0\\0\end{bmatrix}$ is kinda "trivially" a subspace. And you can convince yourself Nul A satisfies the 3 requirements of a subspace.

Problems

1\. (a) I found REF matrix = $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$ (though EF would have been good enough to find the pivot columns)

$\downarrow$

thus the vectors are independent.

(b) (c) found the same as part (a). Next time I'll get some dependent examples...

2\. The span of the given vectors is contained in $\mathbb{R}^4$ b/c each of the vectors is in $\mathbb{R}^4$ (recall $\mathbb{R}^4$ is just the space of all vectors with 4 components). It was pointed out in workshop that these vectors also happen to live in $\mathbb{R}^3$ since all have their last entry $= 0$).

The 5 vectors can't be a basis b/c they only have 4 components, so you already know the basis can't have more than 4 vectors in it (and it could be less!)

(a) Vectors 1, 2 and 5 form a basis b/c they generate the other 2 and are linearly independent.

(b) Pivot colums! Yes, there's no reason this trick couldn't work, except that the pivot columns aren't always obvious. (typo: changed the 2nd two A's in this question to B's!)

(c) Nope! The two matrices have the same solution set but there's no reason they should have the same column space. eg $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(d) Basically the point of this was: the colums are different but the linear relationships between them are the same (for A and B)
$\rightarrow$ so the same columns will work as bases for both!!

(e) All you really need to know: To find the basis put your matrix into echelon form, find the pivot columns, those are your basis.

3. Found REF :

(a)
$$\begin{bmatrix} 1 & 0 & -7 & 14 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

though EF would have been good enough to find the pivots as well.

$\uparrow \; \uparrow$

pivots $\longrightarrow$ so $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ -13 \end{bmatrix}$ are a basis.

(b) Found REF :

$$\begin{bmatrix} 1 & 0 & 1 & -3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\uparrow \; \uparrow$

pivots

So $\begin{bmatrix} 1 \\ 4 \\ -5 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \\ -5 \end{bmatrix}$ are a basis.

4. (a) b/c it contains $\bar{0}$ and you can convince your self that adding 2 solutions gives you another solution, as does scaling a solution.

$\rightarrow$ None of these properties need hold if the right hand side is non-ze

$\rightarrow$ Its also not a subspace if we allow non-linear equations eg $x_1^2 + x_2^2 = 0$ has solutions $(1,1)$, $(1,-1)$ but not $(2,0)$.

(b) $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

This doesn't change the basis b/c we are looking for a basis of the solution set, which is the same after putting the matrix into reduced echelon form.

(I dropped the constants column b/c it will always just be zero anyways.)

(d) the matrix turns into:

$$x_1 - 2x_2 - 1x_4 + 3x_5 = 0$$
$$x_3 + 2x_4 - 2x_5 = 0$$

free variables:

x_2, x_4, x_5

parametric form:

$$\mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

(e) Yes! the vectors in a parametric solution will always be a basis for the solution set they generate (that's the whole point of them). It might not be obvious at first that they are linearly independent, but this is true b/c each vector will have a 1 at the entry corresponding to the free variable it is associated with, and all other vectors will have a zero there, so there's no way to add them up to $\bar{0}$.

eg on the first vector, associated to x_2: $\begin{bmatrix} 2 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \leftarrow$ the x_2 entry is 1 and all other vectors have this being zero.

(f) So the steps to memorize are: put the matrix into REF form, find the parametric equations, those vectors are your basis!

→ note the number of free variables = size of the basis.

5. Unfortunately, I accidentally gave you a pretty gross matrix here, but the REF matrix is

$$\begin{bmatrix} 1 & 0 & 1/6 & 0 & 17/12 \\ 0 & 1 & -2/3 & 0 & -1/6 \\ 0 & 0 & 0 & 1 & -3/2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1/6 \\ 2/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -17/12 \\ 1/6 \\ 0 \\ 3/2 \\ 1 \end{bmatrix}$$

free variables:

x_3, x_5

So these are the basis vectors.