1 Variation of Parameters

Recall we previously saw how to solve non-homogeneous equations:

$$ay'' + by' + cy = f(t),$$

by guessing a solution (i.e. guessing something of the form $y(t) = t^s(A_m t^m + ... + A_0)e^{rt}$). Note however that this only worked on differential equations where $f(t)$ was a polynomial times an exponential e^{rt}, or of the form $a\sin(rt) + b\cos(rt)$. This technique **won't** work on something like $f(t) = 1/t$. It also wouldn't work if a, b, c were functions of t rather than constants.

Variation of parameters is a more powerful way to find a particular solution. It can be a bit more work, but works in more cases. We'll explain how to do this now, assuming your differential equation is in standard form: $y'' + p(t)y' + q(t)y = g(t)$, which you can always get by dividing through by any function or constant on y''.

1. Find two linearly independent solutions $\{y_1(t), y_2(t)\}$ to the homogeneous version of the equation. We'll guess a particular solution of the form $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$ (here $v_1(t), v_2(t)$ are unknown functions which we'll now solve for).

2. Now to find $v_1(t), v_2(t)$ we would normally take the derivates $y_p'(t)$ and $y_p''(t)$, plug this into the differential equation $y'' + p(t)y' + q(t)y = g(t)$ and try to solve for $v_1(t), v_2(t)$ with a bunch of algebraic manipulations. However all that work will end up just giving us the following formulas for v_1, v_2:

$$v_1(t) = \int \frac{-g(t)y_2(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} dt,$$

$$v_2(t) = \int \frac{g(t)y_1(t)}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} dt,$$

   so you can skip ahead by just using them.

   Note you can drop any $+C's$ because we're just looking for a particular solution, not all particular solutions. Question: Why is it safe to divide by $y_1(t)y_2'(t) - y_1'(t)y_2(t)$? I.e. why is this non-zero? Hint: can the Wronskian of y_1, y_2 be zero?

3. Substitute your newly found values for v_1, v_2 into $y_p = v_1y_1 + v_2y_2$ to obtain a particular solution to the differential equation!

With this, you can now do most of the questions in webwork 30. A full explanation of the formulas is in the text on pages 187-189. If you want more practice with variation of parameters a few problems are below.

1. Find a general solution (i.e. homogeneous and particular) to $2y'' + 18y = 6\tan(3t)$.

2. Find a general solution to $y'' - 2y' + y = \frac{e^t}{t^2+1}$.

3. Find a general solution to $ty'' - (t+1)y' + y = t^2$, given that $y_1(t) = e^t$ and $y_2(t) = t + 1$ are two linearly independent solutions to the homogeneous equation.

2 Reduction of Order Review

We did a bit of reduction of order before, but it looks like it's in the webwork so we'll look at it again.

Recall reduction of order is how you find a second solution to a homogeneous DE with variable coefficients, $y'' + p(t)y' + q(t)y = 0$, when you already know one solution. Previously we gave you a formula to do this, but it looks like in the webwork they want to you do it with the algorithm instead. (As with variation of parameters above, the algorithm and the formula do the same thing, the formula just skips some steps).

The steps to solve $y'' + p(t)y' + q(t) = 0$, given that we already know a solution $y_1(t)$ are as follows:

1. Guess a second solution of the form $y_2(t) = v(t)y_1(t)$ (we'll solve for $v(t)$).
2. Take both derivatives of $y_2(t)$ and plug into $y'' + p(t)y' + q(t) = 0$ to get (after grouping in terms of v, v', v''):
$$(y_1'' + py_1' + qy_1)v + y_1v'' + (2y_1' + py_1)v' = 0.$$
3. Notice the first term of this is just y_1 plugged into the differential equation. We assumed y_1 was a solution so this term is just zero! We now have:
$$y_1v'' + (2y_1' + py_1)v' = 0.$$
4. Substitute $w = v'$ to turn this into a first order differential equation in terms of t and w, which we can solve with separation of variables.
5. Integrate w to get v back. Then your final answer is $y_2(t) = v(t)y_1(t)$.

If you want more practice with reduction of order you can try the following problems, using either the algorithm or formula from before (the formula is also in the text on page 197).

1. Find a second linearly independent solution to: $t^2y'' + 6ty' + 6y = 0$, $t > 0$; $y_1(t) = t^{-2}$.

2. Find a second linearly independent solution to: $ty'' + (1 - 2t)y' + (t - 1)y = 0$, $t > 0$; $y_1(t) = e^t$.