Variation of Parameters

1. $2y''+18y=6\tan(3t)$ the homogeneous soln is $y_h(t)=c_1\cos(3t)+c_2\sin(3t)$

$\hookrightarrow y''+9y=3\tan(3t) \rightarrow \boxed{g(t)=3\tan(3t)}$

so take $y_1(t)=\cos(3t)$ $y_2(t)=\sin(3t)$

Compute the Wronskian: $W=\begin{vmatrix} \cos(3t) & \sin(3t) \\ -3\sin(3t) & 3\cos(3t) \end{vmatrix}$

$= 3\cos^2(3t)+3\sin^2(3t) = 3$

↑ trig identity!

Thus the particular soln is:

$$V_1=\int \frac{-3\tan(3t)\sin(3t)}{3}dt = -\int \frac{\sin^2(3t)}{\cos(3t)}dt$$

$$= -\int \frac{1-\cos^2(3t)}{\cos(3t)}dt = -\int \sec(3t)dt - \int \cos(3t)dt$$

$$= -\frac{\ln|\sec(3t)+\tan(3t)|}{3} + \sin(3t)$$

$$V_2=\int \frac{3\cos(3t)\tan(3t)}{3}dt = \int \sin(3t)dt = \frac{-\cos(3t)}{3}$$

$$\rightarrow \boxed{y_p(t) = -\frac{1}{3}\cos(3t)\ln|\sec(3t)+\tan(3t)|} + \cancel{\cos(3t)\sin(3t)}$$

$$\cancel{-\frac{1}{3}\sin(3t)\cos(3t)}$$ ← cancel.

gen soln = $y_h(t)+y_p(t)$.

2. $y''-2y'+y=\frac{e^t}{t^2+1}$ $\quad y_h(t)=c_1e^t+c_2te^t \rightarrow y_1(t)=e^t,\ y_2(t)=te^t$

$$W=e^{2t}$$

$$v_1(t)=\int\frac{te^te^t}{e^{2t}(t^2+1)}dt=\int\frac{t}{t^2+1}dt=\frac{1}{2}\ln(1+t^2)$$

$$v_2(t)=\int\frac{e^te^t}{e^{2t}(t^2+1)}dt=\int\frac{1}{t^2+1}dt=\tan^{-1}(t)$$

$$\rightarrow y_p(t)=-\frac{1}{2}e^t\ln(1+t^2)+te^t\tan^{-1}(t) \rightarrow y(t)=y_h(t)+y_p(t).$$

3. $ty''-(t+1)y'+y=t^2 \rightarrow$ divide by $t \rightarrow y''-\left(\frac{t+1}{t}\right)y'+\frac{1}{t}y=t$

$$W=\begin{vmatrix} e^t & t+1 \\ e^t & 1 \end{vmatrix}=-te^t.$$

$$v_1(t)=\int\frac{(t+1)t}{-te^t}dt=(-e^{-t}(t+2)) \qquad v_2(t)=\int\frac{e^tt}{te^t}dt=\int dt=t$$

$$\rightarrow y_p(t)=-t^2-2t-2 \rightarrow y(t)=y_h(t)+y_p(t).$$

Reduction of order

1. $t^2y''+6ty'+6y=0$, $y_1(t)=t^{-2}$ $\rightarrow y_1'(t)=-2t^{-3}$

$\Downarrow$

$y''+\frac{6}{t}y'+\frac{6}{t^2}y=0$

guess $y_2(t)=v(t)t^{-2}$

plug in and reduce to get $t^{-2}v''+(2(-2t^{-3})+\frac{6}{t}t^{-2})v'=0$

let $w=v'$, then have $t^{-2}w'+(-4t^{-3}+6t^{-3})w=0$

$$\rightarrow \frac{w'}{w}=\frac{4t^{-3}-6t^{-3}}{t^{-2}}=4t^{-1}-6t^{-1}$$

$$\rightarrow \ln|w|=4\ln t-6\ln t \rightarrow w=e^{4\ln t}e^{-6\ln t}$$

$$=t^4t^{-6}=t^{-2}$$

$u=\int w=-t^{-1} \rightarrow y_2(t)=t^{-3}$ (dropping the minus sign b/c all scalar multiples of this are already a soln and we might as well take the cleanest one).

2. $ty'' + (1-2t)y' + (t-1)y = 0$, $y_1(t) = e^t$

$\downarrow$

$$y'' + \frac{(1-2t)}{t}y' + \frac{(t-1)}{t}y = 0 \rightarrow p(t) = \frac{1}{t} - 2$$

$$\int p(t)dt = \ln t - 2t \quad \text{so} \quad w(t) = \frac{e^{-\ln t + 2t}}{e^{2t}} = \frac{t^{-1}\cancel{e^{2t}}}{\cancel{e^{2t}}} = \frac{1}{t}$$

thus $u(t) = \ln(t)$

and $\boxed{y_2(t) = \ln(t)e^t.}$

check: $y_2'(t) = \frac{1}{t}e^t + \ln t e^t$

$$y_2''(t) = -\frac{1}{t^2}e^t + 2\frac{1}{t}e^t + \ln t e^t$$

$$\rightarrow \left(-\frac{1}{t}e^t + 2\cancel{e^t} + \cancel{t\ln t e^t}\right) + \left(\frac{1}{t}e^t + \ln \boxed{t} e^t - 2\cancel{e^t} - \cancel{2\ln t e^t}\right)$$
$$+ \cancel{t\ln(t)e^t} - \ln\boxed{(t)}e^t = 0 \checkmark$$