Math 54 Worksheet **#13**

1 More on Eigenvectors and Eigenvalues

Write T or F next to each of the following (if F, think of a counter example)

1. If A is a 4×4 matrix, then $\det(\lambda I - A) = 0$ has exactly four distinct roots.
2. If a matrix has one eigenvector, then it has an infinite number of eigenvectors.
3. The sum of two eigenvalues of a matrix A is also an eigenvalue of A.
4. The sum of two eigenvectors of a matrix A is also an eigenvector of A.
5. The characteristic polynomial of an $n \times n$ matrix has degree n.
6. There exists an $n \times n$ matrix with $n + 1$ distinct eigenvalues.

2 Attractors, Repellers and Saddle Points

Let A be a 2×2 matrix with eigenvalues $\lambda_1 = 0.4$, $\lambda_2 = 2$ and corresponding eigenvectors $\mathbf{v}_1 = [1, 1]^T$, $\mathbf{v}_2 = [-1, 1]^T$. Suppose $\mathbf{x}_0 = [2, 3]^T$ and that the other $\mathbf{x}_k$ are determined by the difference equation $\mathbf{x}_{k+1} = A\mathbf{x}_k$.

1. As in the webwork, write $\mathbf{x}_k$ in terms of λ_1, λ_2, $\mathbf{v}_1$, $\mathbf{v}_2$ and k.
2. Give an approximation of $\mathbf{x}_k$ as $k \to \infty$. (e.g. write $\mathbf{x}_k = a(b)^k\mathbf{u}$ for some real numbers a, b and a vector $\mathbf{u}$).
3. Draw (all on one graph) what happens to $\mathbf{x}_k$ for the following starting conditions: $\mathbf{x}_0 = [1, 1]^T$, $\mathbf{x}_0 = [-1, 1]^T$, $\mathbf{x}_0 = [-0.8, 1.2]^T$ and $\mathbf{x}_0 = [0, 0]^T$.
4. Does this system contain an attractor, repeller or saddle point?

3 Similar Matrices

Write T or F next to each of the following (if F, think of a counter example)

1. If A is similar to B then B must be similar to A.
2. Two similar matrices have the same eigenvalues (with the same multiplicities).
3. If two matrices have the same characteristic polynomial and the same eigenvalues then they are similar.

4 Diagonalization

1. This question begins to explain why diagonalization (and so eigenvectors and values) are so useful.

   (a) Suppose that $P^{-1}AP = D$ where D is diagonal. Show that $A^k = PD^kP^{-1}$ for any positive integer k.

   (b) Use part (a) to compute A^{10} where $A = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$.

2. Answer the following true or false. Explain your reasoning, or give a counterexample.

   (a) Any $n \times n$ matrix that has fewer than n distinct eigenvalues is not diagonalizable.

   (b) Eigenvectors corresponding to the same eigenvalue are always linearly dependent.

   (c) If A is diagonalizable, then it has at least one eigenvalue.

3. Suppose A is a 2×2 matrix with characteristic polynomial $(\lambda - 2)^2$. What can you conclude about the diagonalizability of A?

4. Let A be a 3×3 matrix with the following eigenvectors and corresponding eigenvalues: $[1, 1, 1]^T$ and $[1, -2, 0]^T$ are eigenvectors corresponding to the eigenvalue $\lambda = 3$. $[1, 1, -2]$ is an eigenvector corresponding to the eigenvalue $\lambda = -3$. Find A by finding an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

5. In this exercise we determine whether A is diagonalizable without doing any hard work. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

   (a) Find the eigenvalues of A. Call them λ_1 and λ_2.

   (b) Write down the matrix $\lambda_1 I - A$.

   (c) What is the rank of $\lambda_1 I - A$? What is its nullity? How many independent eigenvectors are there with eigenvalue λ_1? (Dont compute them.)

   (d) What is the rank of $\lambda_2 I - A$? What is its nullity? How many independent eigenvectors are there with eigenvalue λ_2?

   (e) Is A diagonalizable?