linear independence & spanning

Review

1. What is a basis (e.g. what are the two requirements).

   Is $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$ a basis for $\mathbb{R}^2$ inside $\mathbb{R}^3$? → to be more precise I should say is this a basis for a copy of $\mathbb{R}^2$ in $\mathbb{R}^3$? But it's not anyways b/c it contains $\bar{0}$.

2. Given a matrix A, explain why both Nul A and the column space of A are subspaces (e.g. go through the 3 requirements of a subspace and explain why each holds. Hopefully this is easier now that you can think about things in terms of matrix multiplication). Are they "abstract vector spaces" as defined on page 192?

   col A = span of the columns and Nul A = solutions to $A\bar{x}=\bar{0}$ note if $\bar{x},\bar{y}$ ~~are~~ satisfy $A\bar{x}=\bar{0}$, $A\bar{y}=\bar{0}$ then $A(\bar{x}+\bar{y})=A\bar{x}+A\bar{y}=0$.

$\begin{bmatrix}1&1\\0&1\end{bmatrix}^n = \begin{bmatrix}1&n\\0&1\end{bmatrix}$ so $\begin{bmatrix}1&1\\0&1\end{bmatrix}^n\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}n+1\\1\end{bmatrix}$

3. Compute $\begin{bmatrix}1&1\\0&1\end{bmatrix}^n$. What is $\begin{bmatrix}1&1\\0&1\end{bmatrix}^n\begin{bmatrix}1\\1\end{bmatrix}$?

4. Suppose x is a real number satisfying $x^2 = 1$. To solve for x, we factor $x^2 - 1 = (x-1)(x+1) = 0$, and conclude that $x = \pm 1$. What if X is a 2×2 matrix satisfying $X^2 = I$?

   (a) Show that $(X-I)(X+I) = 0$. do this by expanding: $X^2+XI-IX-I^2 = X^2+X-X-I = X^2-I = 0$ ✓ (b/c $X^2=I$)

   (b) Are those the only solutions? What about $\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ and $\begin{bmatrix}1&0\\0&-1\end{bmatrix}$? you can check these matrices also square to I

   (c) Let a be any real number, and let $A = \pm\begin{bmatrix}1&0\\a&-1\end{bmatrix}$. Show that $A^2 = I$.

   (d) Let $0 \le \theta \le 2\pi$, and let $R = \begin{bmatrix}\cos\theta & -\sin\theta\\ -\sin\theta & -\cos\theta\end{bmatrix}$. Show $R^2 = I$. this one squares to I too!

   (e) Explain why there can be so many solutions. What is different about matrices vs numbers that allows this to happen? (hint: what happens if you plug one of the matrices in (b) into the factored equation of (a)?)

   so if you plug these matrices into $(X-I)(X+I)$ as X you'll notice neither "X−I" nor "X+I" is zero yet when you multiply them you get zero! That's what's different about matrices compared to numbers and why $X^2=I$ can have more than 2 solutions with matrices.

Problems

1. Let V be a vector space, and suppose that L and M are two subsets of V that happen to also be vector spaces (e.g. they are subspaces). Is it true that $L\cup M$ is a vector space ($L\cup M$ is the set of vectors in either L or M). How about $L\cap M$? ($L\cap M$ is the set of vectors that are in both L and M).

$L\cup M$ is not a vector space. consider Span$\{\begin{pmatrix}1\\0\end{pmatrix}\}\cup$ Span$\{\begin{pmatrix}0\\1\end{pmatrix}\}$ and try adding vectors.

$L\cap M$ is a vector space (for example $\bar{v},\bar{u}\in L\cap M$ implies $\bar{v}+\bar{u}\in L$, and $\bar{v}+\bar{u}\in M$ therefore $\bar{v}+\bar{u}\in L\cap M$.

Chapters 4.4 and 4.7 cover this better than I can here so check them out if you're confuzed by my scrawled answers!

2. This question is going to try and explain "coordinate mapping" which is covered in chapter 4.4 of the text. First we need the following fact: Let $B = \{\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x} \in V$ there exists a unique set of real numbers $c_1, ..., c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + ... + c_n\mathbf{b}_n$.

   (a) What are the coordinates of $\mathbf{x}$ in terms of the basis B? (answer in case you haven't seen this before: the coordinates are the real numbers $c_1, ..., c_n$).

   (b) What is the coordinate vector of $\mathbf{x}$ relative to B? (answer: it's just the vector $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. The textbook denotes this by saying $[\mathbf{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$).

   (c) Let $\mathbf{b}_1 = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Do $\mathbf{b}_1$, $\mathbf{b}_2$ form a basis for $\mathbb{R}^2$? → yup, recall 2 vectors are linearly independ. if one's not a multiple of the other.

   (d) Ok, spoiler, they do. Let $B = \{\mathbf{b}_1, \mathbf{b}_2\}$. Suppose $[\mathbf{y}]_B = \begin{bmatrix} 1/2 \\ 3 \end{bmatrix}$. What was $\mathbf{y}$? $\bar{y} = \frac{1}{2}\bar{b}_1 + 3\bar{b}_2 = \frac{1}{2}\begin{bmatrix} 6 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

   (e) We can also got the other way. Find the coordinate vector for $\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ in terms of B, e.g. find $[\mathbf{x}]_B$. Hint: you know $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 = \mathbf{x}$. Turn this into a matrix equation and solve it.

   $c_1\bar{b}_1 + c_2\bar{b}_2 = \bar{x}$ becomes $\begin{bmatrix} 6 & -1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and I found $c_1 = 1$ $c_2 = 3$ so $[x]_B = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

   (f) Note that the matrix you found in (e) can be reused to put any vector into B-coordinates! Use it to write $\mathbf{y} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ in terms of B. I found $[y]_B = \begin{bmatrix} 3/2 \\ 7 \end{bmatrix}$ → note instead of doing row reduction twice you could have inverted the matrix.

   (g) The above matrix is the "change of coordinates" matrix, lets call it P_B like the textbook does. The hardest thing about the change of coordinates matrix is remembering which direction it goes. Write down the general formula for how P_B relates a vector $\mathbf{x}$ to $[\mathbf{x}]_B$. → $P_B[\bar{x}]_B = \bar{x}$

   in next weeks' notation this is $P_{\mathcal{E} \leftarrow B}$

3. Let $\mathbf{b}_1 = \begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Show $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ form a basis. → row reduce the matrix and confirm there are 3 pivots.

   (a) Find the change of basis matrix to go from B to the standard basis.

   b/c we're going to the standard basis this is just: $\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 0 \\ -1 & -3 & 0 \end{bmatrix} = A$.

   (b) Use the matrix from part (a) to put $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ into B-coordinates.

   (Note: be careful, do you multiply $\mathbf{x}$ by the matrix or by it's inverse?)

   → you can either multiply with the inverse or do row reduction to solve the system $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$. I found $A^{-1} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1/3 & -2/3 \\ 1 & -4 & -4 \end{bmatrix}$ and $[x]_B = \begin{bmatrix} -1 \\ 4/3 \\ 5 \end{bmatrix}$