See the text or previous worksheets for more detail on this, but in both cases you are just finding the pivots in the corresponding matrix. If all columns are pivots then all columns were linearly independent, if not we can throw out the dependent ones to get a basis.

Math 54 Worksheet #5

Review / Catch Up

1. In words, explain how one checks if a set of vectors is linearly independent. Compare this with the process of finding a basis for the column space of a matrix. (Is it the same process? Slightly different?)

2. Last day we saw how to find a basis for a column space (or equivalently, a subspace that's given to you as the Span of some vectors). Another *different* question is to **find the basis for a subspace given by a system of homogeneous equations** (this basis is that 'set of fundamental homogeneous solutions' thing that keeps getting mentioned in lecture).

   This was covered in detail in problem 4 of the previous worksheet, but in case you didn't get to it:

   The subspace is given to you as the solution set of a system of equations, i.e:

$$\begin{aligned} -3x_1 + 6x_2 - x_3 + x_4 - 7x_5 &= 0 \\ x_1 - 2x_2 + 2x_3 + 3x_4 - x_5 &= 0 \\ 2x_1 - 4x_2 + 5x_3 + 8x_4 - 4x_5 &= 0 \end{aligned}$$

   The steps to find a basis are: write down the augmented matrix, put it into reduced echelon form, write out the solutions in parametric form, and congratulations you're done! **The vectors showing up in the parametric form are your basis.** Give this a go on the above example if you didn't already do it last week. → See solutions to worksheet 4.

   → b/c it might not span the whole subspace it is supposed to be a basis of. Yes it can always be extended.

3. Recall that a linearly independent set is not always a basis (why?). Can it be extended to a basis? How? If so, extend $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ to a basis for $\mathbb{R}^3$. → in this case its enough to add e_3. In general the algorithm is too add a spanning set of vectors (eg e_1, e_2 and e_3) and then find a basis for the span of all the vectors together (your first vectors and the new ones)

4. What is the definition of the dimension of a subspace?
   (a) What is the dimension of $\mathbb{R}^3$ → 3
   (b) What is the dimension of *Span*$\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$ where the three vectors are all linearly independent. → 3

Problems

1. Definitions

   (a) What is the domain and range of a map? → see text page 64

   (b) When do we say a map is onto? → page 76

   (c) When do we say a map is one-to-one? (there are 2 equivalent definitions!) → page 76 & 77

   (d) What is a linear transformation? What 2 properties must a map satisfy to be a linear transformation? → page 66

2. What is the domain and range of the map $T(x, y) = (x^2, y)$? Is it a linear transformation? This question is a bit vague but one correct answer would be domain $= \mathbb{R}^2$ and range = the $\begin{pmatrix} a \\ b \end{pmatrix}$ in $\mathbb{R}^2$ such that $a \geq 0$

3. Describe geometrically what the following linear transformations do (if you're stuck, try plugging in some vectors, and if you're really stuck: these examples are just from page 66-68 of the text).

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1/10 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1/10 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

   projection ← (a); (b) Scaling; (c) Scaling; (d) rotation

4. Given an 'abstract' linear transformation T, is there always a matrix that represents it? Is that matrix unique or will multiple different matrices work? Can you write down that matrix in terms of T and the standard basis $e_1, e_2, ..., e_n$? → $A = [T(e_1) \cdots T(e_n)]$

   yes! ← And it's unique, see page 72

5. When can you say a linear transformation T is one-to-one? Hint: recall the second definition of one-to-one, that $T(\mathbf{x}) = \mathbf{0}$ has only one solution..

   When T's matrix has no free variables.

6. When can you say a linear transformation T is onto? Hint: write out the value of T on a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in terms of T's columns.

   → Actually maybe an easier way of seeing this than the hint: recall we said a system was consistent for all vectors $\bar{b}$ (eg in $A\bar{x} = \bar{b}$) if the coefficient matrix A had a pivot in every row. (b/c then you can't get 0 0 ... 0 1 in the augmented matrix) ↓ So pivot is every row is the answer!

7. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, can n be larger than m?

   No. if $n>m$ then the matrix for T has more columns than rows eg more variables than it can possibly have pivots ⇒ free variable ⇒ not one-to-one.

8. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto, can n be larger than m? Yup

9. For each of the following, determine if the linear transformation is one-to-one and onto. (a) $\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 2 \\ 11 & -8 & 26 \\ -1 & -2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

   (a) ↓ one-to-one but not onto; (b) ↓ one-to-one and onto; (c) ↓ one-to-one and onto.