1 The Wronskian

The bonus question on the previous worksheet proved the following. Let $y'' + ay' + by = 0$ be a linear homogeneous ODE, with solutions $y_1(t)$, $y_2(t)$. Then if the Wronskian

$$W(t) = \det \begin{bmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{bmatrix} = 0$$

at any point t, it follows that $W(t) = 0$ **for all** values of t, and in this case $y_1(t), y_2(t)$ are linearly dependent (e.g. $y_1(t) = cy_2(t)$ for some constant c). Alternatively, if $W(t)$ is non-zero for all t, then the two solutions are linearly independent.

Determine which of the following pairs of functions are linearly dependent or independent by computing the wronskian:

1. $y_1(t) = e^{3t}$, $y_2(t) = e^{-4t}$

2. $y_1(t) = te^{2t}$, $y_2(t) = e^{2t}$

3. $y_1(t) = 0$, $y_2(t) = e^t$

2 First Order Variable Coefficient Equations

Based on the webwork, it looks like you're going to need to be be able to solve equations of this form:

$$y' + u(t)y = w(t).$$

Let's start with the homogeneous case, when $w(t) = 0$. In this case the general solution is

$$y(t) = ce^{-U(t)},$$

where $U(t) = \int u(t)dt$ and c is some constant to be determined by the initial values.

1. Solve the initial value problem $y' + \cos t y = 0$, $y(0) = 1/2$.

2. Solve the initial value problem $ty' + 3y = 0$, $y(1) = 2$ assuming $t > 0$.

3. Derive the formula for the general solution from the equation $y' + u(t)y = 0$ by moving $u(t)y$ over to the other side, dividing both sides by y, and integrating both sides.

Alright, now for the non-homogeneous case. The formula for the solution is just slightly more complicated...

$$y = e^{-U(t)}(c + \int e^{U(t)}w(t)dt),$$

as before $U(t) = \int u(t)dt$. ($e^{U(t)}$ is called the integrating factor). In practice people don't memorize the formula, they memorize these steps:
Write in standard form $y' + u(t)y = w(t)$
Multiply by the integrating factor $e^{U(t)}(y' + u(t)y) = e^{U(t)}w(t)$
Rewrite left hand side as a derivative $(ye^{U(t)})' = e^{U(t)}w(t)$
Integrate both sides: $ye^{U(t)} = \int e^{U(t)}w(t) + c$.

1. Find the general solution to $ty' = -2y + 4t^2$ for $t > 0$.

2. Find the general solution to $y' - 2y = t$ for $t > 0$.

3. Find the general solution to $y'/t - 2y/t^2 = t\cos t$ for $t > 0$.

4. The Wronskian theorem is super subtle (see last worksheet for a proof of it).
   (a) Show the functions x, x^2 are linearly independent (without the Wronskian).
   (b) Show the Wronskain $W(x, x^2) = 0$ at $x = 0$.
   (c) What can you conclude about the possibility that x, x^2 are solutions of a differential equation $y'' + ay' + by = 0$?
   (d) Verify that x, x^2 are solutions of the equation $x^2y'' - 2xy' + 2y = 0$.

5. If a, b, c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach 0 as $x \to \infty$