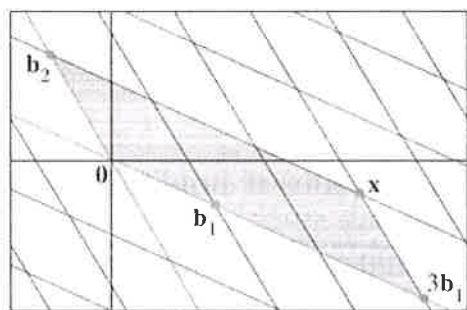


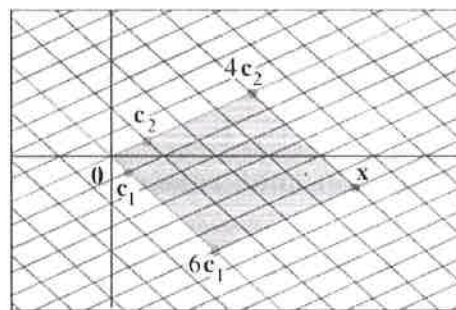
Recall last week we introduced coordinate mapping, and we found a matrix that transformed vectors from the standard basis into terms of a basis of our choice. If you don't remember, that's okay too, because we're basically going to do a more general version of it this week. First, let's try and get a better idea what we're actually doing by stealing an example from the textbook (page 241). They suppose  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$  is a basis for  $\mathbb{R}^2$  and that  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$  is another basis. Let  $\mathbf{x}$  be a vector in  $\mathbb{R}^2$  such that  $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$ . Then we said that  $\mathbf{x}$  in terms of  $\mathcal{B}$  is  $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ . If we also have  $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$  then  $\mathbf{x}$  in terms of  $\mathcal{C}$  is

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

The vector  $\mathbf{x}$  appears in both figures below, note that indeed in figure (a)  $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$  while in figure (b) we see  $\mathbf{x} = 6\mathbf{c}_1 + 4\mathbf{c}_2$ .



(a)



(b)

**FIGURE 1** Two coordinate systems for the same vector space.

So in the above example  $\mathbf{b}_1, \mathbf{b}_2$  are one basis for  $\mathbb{R}^2$  and  $\mathbf{c}_1, \mathbf{c}_2$  are another.

1. What do the standard basis  $\mathbf{e}_1, \mathbf{e}_2$  look like in these pictures? *roughly:*
2. If I told you  $\mathbf{b}_1 = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}$  and  $\mathbf{b}_2 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$  (in the standard basis), then can you write down  $\mathbf{x}$  in the standard basis? *then  $\bar{\mathbf{x}} = 3 \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$*

Note that the notation we've been using all along for  $\mathbf{x}$  as a column vector was secretly  $\mathbf{x}$  written in terms of the standard basis.

## Problems

1. We denote by  $P_{\mathcal{C} \leftarrow \mathcal{B}}$  the change of basis matrix (or "change of coordinates" matrix) transforming vectors in terms of basis  $\mathcal{B}$  to basis  $\mathcal{C}$ . So with the

notation  $[\mathbf{x}]_{\mathcal{B}}$ ,  $[\mathbf{x}]_{\mathcal{C}}$  as above we have  $[\mathbf{x}]_{\mathcal{C}} = {}_{\mathcal{C} \leftarrow \mathcal{B}}^P [\mathbf{x}]_{\mathcal{B}}$ . Find the following:

(a) Let  $\mathcal{E}$  be the standard basis of  $\mathbb{R}^3$ . Let  $\mathcal{B} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 8 \end{bmatrix} \right\}$ . What

is  ${}_{\mathcal{E} \leftarrow \mathcal{B}}^P$ ?  $\rightarrow$  it's just  $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

(b) Use  ${}_{\mathcal{E} \leftarrow \mathcal{B}}^P$  to find  $\mathbf{x}$  given that  $[\mathbf{x}]_{\mathcal{B}} = [4 \ -2 \ 1]^t$ .

calculate  ${}_{\mathcal{E} \leftarrow \mathcal{B}}^P [\bar{\mathbf{x}}]_{\mathcal{B}}$   
= I got  $\begin{bmatrix} 0 \\ 5 \\ 8 \end{bmatrix}$

(c) Recall that for any 2 bases  $\mathcal{B}$  and  $\mathcal{C}$ ,  ${}_{\mathcal{C} \leftarrow \mathcal{B}}^{P^{-1}} = {}_{\mathcal{B} \leftarrow \mathcal{C}}^P$  (or see the textbook page 242). What is  ${}_{\mathcal{B} \leftarrow \mathcal{E}}^P$ ?

we invert the above matrix to get:  $\begin{pmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{pmatrix}$

(d) Using the matrix you found in the previous part, find  $[\mathbf{y}]_{\mathcal{B}}$  for  $\mathbf{y} = [1 \ 0 \ 3]^t$ . I found  $[\mathbf{y}]_{\mathcal{B}} = \begin{pmatrix} -9 \\ -5 \\ 3 \end{pmatrix}$

2. Let  $\mathcal{D} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$  and  $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$  be bases for a vector space  $V$  and suppose  $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$ ,  $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$  and  $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{b}_3$ . Find the change of coordinates matrix from  $\mathcal{F}$  to  $\mathcal{D}$ ,  ${}_{\mathcal{D} \leftarrow \mathcal{F}}^P$ . Also find  $[\mathbf{x}]_{\mathcal{D}}$  for  $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$ .

3. Let  $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ ,  $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$  and  $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$  be bases for  $\mathbb{R}^2$ . Write an equation that relates  ${}_{\mathcal{C} \leftarrow \mathcal{B}}^P$ ,  ${}_{\mathcal{D} \leftarrow \mathcal{C}}^P$  and  ${}_{\mathcal{D} \leftarrow \mathcal{B}}^P$ . Why does it hold?

$P = P \cdot P$   
(think about what the matrices are doing)

4. Okay lets switch gears. Let  $A$  be an  $m \times n$  matrix. What are the definitions of the Row space of  $A$  and the Column space of  $A$ ? Which is a subspace of  $\mathbb{R}^n$  or  $\mathbb{R}^m$ ?

Row  $A$  = span of the rows and is inside  $\mathbb{R}^n$   
Col  $A$  = span of the columns and is inside  $\mathbb{R}^m$

5. Let  $A$  be an  $m \times n$  matrix and let  $U$  be an  $m \times n$  matrix in row echelon form which is obtained from  $A$  by row operations. Answer the following true or false. Explain your reasoning, or give a counterexample.

(a) Row  $A$  = Row  $U$  (b) Col  $A$  = Col  $U$  (c) dim Row  $A$  = dim Row  $U$  (d) dim Col  $A$  = dim Col  $U$  (e) dim Row  $A$  = dim Col  $A$ .

$\rightarrow$  for (b) and (d) remember our algorithm to find a basis of the column space.  
false, think of a non-square matrix.

6. Suppose  $A$  is an invertible  $n \times n$  matrix. What is the rank of  $A$ ? What is the dimension of the Null space?

$\rightarrow 0!$

7. Find the rank of these matrices:  $\begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix}$ ,  $\begin{bmatrix} t & -1 & 2 \\ t & t & 1 \\ t & t & 1 \end{bmatrix}$  (depending on  $t$ )

this became EF  $\begin{pmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 0 & (2-t)(1+t) \end{pmatrix}$

$\rightarrow$  rank = 3, unless  $t=1$  or  $2$ .

$\rightarrow n$   
the EF is  $\begin{pmatrix} t & -1 & 2 \\ 0 & t-1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

rank is always 2.

8. Bonus: What does it mean for a vector to be in the left nullspace of a matrix  $A$ ? Can you rewrite this condition to involve only  $A$ ? (and not  $A$  transpose).

see the posted reading for F20 Sept. The alternative condition is  $\bar{\mathbf{x}}$  in LNull  $A$  if  $\bar{\mathbf{x}}A = \bar{\mathbf{0}}$  (ie multiplication on the left!)