Review / Catch Up

1. In words, explain how one checks if a set of vectors is linearly independent. Compare this with the process of finding a basis for the column space of a matrix. (Is it the same process? Slightly different?)

2. Last day we saw how to find a basis for a column space (or equivalently, a subspace that's given to you as the Span of some vectors). Another *different* question is to **find the basis for a subspace given by a system of homogeneous equations** (this basis is that 'set of fundamental homogeneous solutions' thing that keeps getting mentioned in lecture).

   This was covered in detail in problem 4 of the previous worksheet, but in case you didn't get to it:

   The subspace is given to you as the solution set of a system of equations, i.e:

$$\begin{aligned} -3x_1 + 6x_2 - x_3 + x_4 - 7x_5 &= 0 \\ x_1 - 2x_2 + 2x_3 + 3x_4 - x_5 &= 0 \\ 2x_1 - 4x_2 + 5x_3 + 8x_4 - 4x_5 &= 0 \end{aligned}$$

   The steps to find a basis are: write down the augmented matrix, put it into reduced echelon form, write out the solutions in parametric form, and congratulations you're done! **The vectors showing up in the parametric form are your basis**. Give this a go on the above example if you didn't already do it last week.

3. Recall that a linearly independent set is not always a basis (why?). Can it be extended to a basis? How? If so, extend $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ to a basis for $\mathbb{R}^3$.

4. What is the definition of the dimension of a subspace?
   (a) What is the dimension of $\mathbb{R}^3$
   (b) What is the dimension of $Span\{\mathbf{v}, \mathbf{u}, \mathbf{w}\}$ where the three vectors are all linearly independent.

Problems

1. Definitions
   (a) What is the domain and range of a map?
   (b) When do we say a map is onto?
   (c) When do we say a map is one-to-one? (there are 2 equivalent definitions!)
   (d) What is a linear transformation? What 2 properties must a map satisfy to be a linear transformation?

2. What is the domain and range of the map $T(x, y) = (x^2, y)$? Is it a linear transformation?

3. Describe geometrically what the following linear transformations do (if you're stuck, try plugging in some vectors, and if you're really stuck: these examples are just from page 66-68 of the text).
   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1/10 & 0 & 0 \\ 0 & 1/10 & 0 \\ 0 & 0 & 1/10 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

4. Given an 'abstract' linear transformation T, is there always a matrix that represents it? Is that matrix unique or will multiple different matrices work? Can you write down that matrix in terms of T and the standard basis $e_1, e_2, ..., e_n$?

5. When can you say a linear transformation T is one-to-one? Hint: recall the second definition of one-to-one, that $T(\mathbf{x}) = \mathbf{0}$ has only one solution..

6. When can you say a linear transformation T is onto? Hint: write out the value of T on a vector $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in terms of T's columns.

7. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one, can n be larger than m?

8. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto, can n be larger than m?

9. For each of the following, determine if the linear transformation is one-to-one and onto. (a) $\begin{bmatrix} 2 & 3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 0 & 2 \\ 11 & -8 & 26 \\ -1 & -2 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$