In this last section of the course we're going to use our linearly algebra knowledge to solve systems of differential equations. There are basically 2 things you'll need to know for this: (1) How to convert a scalar ODE into a linear system (in normal form):

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where A's a matrix and $\mathbf{x}, \mathbf{x'}$ are vectors. And (2) How to solve a solve differential equation system in normal form. This last step is easier if A is diagonalizable, so that's the case we'll stick to today. (When A is not diagonalizable we have to use matrix exponentials, which will be covered in lecture on Wednesday).

1 Turning Scalar Equations into Linear Systems

We can turn a (scalar) differential equation of arbitrary degree into a degree one equation by replacing the derivatives of y with new variables. For example, given:

$$y^{(4)} + 3y'' - \sin(t)y' + 8y = t^2$$

we would introduce the new variabales x_1, x_2, x_3, x_4 and let $x_1 = y$, $x_2 = y'$, $x_3 = y''$ and $x_4 = y'''$.
We're looking for a system of the form $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x_1, x_2, x_3, x_4)$, so we need to be able to write the derivatives of the x_i in terms of the other (non-differentiated x_i). Note that $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_4$.

What is x_4'? (hint: it's more complicated, use the differential equation).

Given these formula for the x_i', what is the matrix A?

Rewrite the following scalar equations as linear systems in normal form.

1. $x''(t) + x(t) = t^2$.

2. $\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \cos t$.

3. $y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + ... + p_0(t)y(t) = g(t)$.

4. What does question 3 tell you about the kinds of problems you can solve with linear systems of ODEs vs scalar ODEs?

2 The Wronskian

Just like with scalar ODEs we want to write down the general solution to a linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$. But what does the general solution look like? It turns out, just like the scalar case, it's a linear combination of linearly independent solutions (we're *not* going to prove this). How many solutions? If A is an $n \times n$ matrix then there should be n. So our solutions look like:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + .. + c_n\mathbf{x}_n(t).$$

As in the scalar case, the c_1 are unknown coefficients which would be determined by an initial condition. The $\mathbf{x}_1, ..., \mathbf{x}_n$ form the set of **fundamental solutions**. And if you put them in a matrix together, $[\mathbf{x}_1, ..., \mathbf{x}_n]$, we call that the **fundamental matrix**.

As before, we can use the Wronskian to check if a set of functions vectors are actually linearly independent. Let

$$W(t) = \det[\mathbf{x}_1(t), ..., \mathbf{x}_n(t)].$$

Then W will be a function of t, and the rule is: it'll be zero everywhere if the functions are dependent. If it's not zero everywhere the functions are linearly independent. Further: if the functions are independent solutions to an ODE, W will **never** be zero (for any value t)!!

Compute the Wronskian of the following functions. Decide if they are independent, and if they could be the solution set to some ODE system.

1. $\mathbf{x}_1 = \begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$

2. $\mathbf{x}_1 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

3. Without using the Wronskian (because you can't) check if the following function vectors are linearly independent. $\mathbf{x}_1 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} \sin t \\ \sin t \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} \cos t \\ \cos t \end{bmatrix}$

4. Suppose the following $\mathbf{x}_i$ are solutions to the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$. Determine if they form a fundamental solution set, and if so write down the fundamental matrix and a general solution. $\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}$

3 Homogeneous Linear Systems with Constant Coefficients

Now we're going to solve equations of the form $\mathbf{x}'(t) = A\mathbf{x}(t)$. The underlying idea is the same as in the scalar case: guess a solution made out of ce^{rt}'s, because we're looking for functions $x_i(t)$ which are equal to their derivative (or at least, linearly combine to form their derivative).

It turns out guessing solutions of the form $x_i(t) = ce^{rt}$ plugging them into $\mathbf{x}'(t) = A\mathbf{x}(t)$ and solving for c and r amounts to finding the eigenvalues and eigenvectors of A. Specifically if A has n linearly independent eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$ with associated eigenvalues $r_1, ..., r_n$ (not all necessarily distinct), then a set of fundamental solutions is:

$$\mathbf{x}_1(t) = e^{r_1 t}\mathbf{v}_1, \; .... \; , \; \mathbf{x}_n(t) = e^{r_n t}\mathbf{v}_n.$$

Let's try an example:

$$\mathbf{x}' = A\mathbf{x}, \text{ where } A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

First we find the eigenvalues and eigenvectors of A (do this).

Then we write down the fundamental set of solutions as described above. For example if one of the eigenvalues was -1, and it had eigenvector $(-1, 1)^t$, one of our fundamental solutions would be $e^{-t}(-1, 1)^t$. Find both and write down the general solution.

Finally, we could be asked to solve an initial value problem. What if we were told $\mathbf{x}(0) = (0, -4)^t$? Can you solve for the coefficients in your general solution found above? What is the final solution?

Would this method work if A was not a constant matrix? (e.g. if it had functions like t^2 in it?). Does this method depend on A being diagonalizable?

1. Find a general solution to $\mathbf{x}' = A\mathbf{x}$ for $A = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix}$.

2. Find a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ for $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$.

3. Solve the initial value problem $\mathbf{x}' = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$.

4. Solve the initial value problem $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$.

One final case we'll handle are systems with complex eigenvalues. In this case, as always, we'll want to replace any expressions of the form $e^{\alpha + i\beta}$ with sines and cosines. The rule is as follows:

If A has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$, then the corresponding two linearly independent solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$ are

$$\mathbf{x}_1(t) = e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b},$$

$$\mathbf{x}_2(t) = e^{\alpha t}\sin(\beta t)\mathbf{a} + e^{\alpha t}\cos(\beta t)\mathbf{b}.$$

Let's do an example. Consider the system $\mathbf{x}'(t) = \begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix} \mathbf{x}(t)$. It has eigenvalues $\pm 3\sqrt{3}i$, so $\alpha = 0$ and $\beta = 3\sqrt{3}$. The corresponding eigenvectors are $\begin{bmatrix} 3 \\ -1 \pm \sqrt{3}i \end{bmatrix}$, so $\mathbf{a} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}$. Thus our two solutions are:

$$\mathbf{x}_1(t) = \begin{bmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{bmatrix}$$

$$\mathbf{x}_2(t) = \begin{bmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{bmatrix}.$$

1. Find a general solution of the system $\mathbf{x}' = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} \mathbf{x}$.

2. Find a general solution of the system $\mathbf{x}' = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} \mathbf{x}$.

3. Find a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -13 & 4 \end{bmatrix}$.