Math 54 Worksheet **#14**

1 Review T/F

In each of the following A is an $n \times n$ matrix. Write T or F next to each (if F, think of a counter example)

1. If B is a 2×2 matrix with determinant 0 then one row of B is a multiple of the other.
2. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$ then λ is an eigenvalue of A.
3. If 2 rows of a 3×3 matrix B are the same then $\det B = 0$.
4. if B is a 3×3 matrix then $\det 5B = 5 \det B$.
5. Matrix A is not invertible if and only if 0 is an eigenvalue of A.
6. A number c is an eigenvalue of A if and only if $(A - cI)\mathbf{x} = \mathbf{0}$ has a non-trivial solution.
7. If A, B are $n \times n$ matrices with $\det A = 2$, $\det B = 3$, then $\det(A + B) = 5$.
8. To find the eigenvalues of A reduce to echelon form.
9. If $\det A = 2$ then $\det A^3 = 6$.
10. If $A\mathbf{x} = \lambda\mathbf{x}$ for some vector $\mathbf{x}$ then $\mathbf{x}$ is an eigenvector of A.
11. If $\mathbf{v}, \mathbf{u}$ are linearly independent eigenvectors of A then they correspond to different eigenvalues.
12. If B is formed by adding to one row of A a linear combination of the other rows then $\det B = \det A$.
13. A steady-state vector for a stochastic matrix is actually an eigenvector.
14. If $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ satisfy $\det[\mathbf{u}\ \mathbf{v}] = 10$, then the triangle in the plane with vertices at $\mathbf{0}, \mathbf{u}$ and $\mathbf{v}$ has area 10.
15. The eigenvalues of a matrix are on its main diagonal.
16. $\det A^T A \geq 0$.
17. An eigenspace of A is the null space of a certain matrix.
18. Any system of n linear equations in n variables is solvable by Cramer's rule.
19. If $A^3 = 0$ then $\det A = 0$.
20. If A is invertible then $\det A^{-1} = \det A$.
21. Row reducing before subtracting λI can speed up eigenvalue calculations.

2 Cayley Hamilton

1. What is the statement of the Cayley Hamilton theorem again?

2. Use Cayley Hamilton to invert the matrix $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$.

3. In the following 2 cases A is a 3×3 matrix. If A is invertibe, write A^{-1} as a linear combination of positive powers of A. If not, explain why. Case 1: A has eigenvalues $i, -i$ and -1. Case 2: A has eigenvalues $i, -i$ and 0.

3 More Eigenvectors and Diagonalization

1. Based on a great question I got in office hours, this problem will try to explain what it really means for two matrices to be similar. In other words, why do we care if two matrices A, C are related by the odd-looking equation: $A = BCB^{-1}$?

   (a) Lets assume A, B, C are $n \times n$. Do the columns of B form a basis for $\mathbb{R}^n$?

   (b) Recall the text denoted the change of basis matrix from $\mathcal{B}$ to $\mathcal{S}$ by $\underset{\mathcal{S} \leftarrow \mathcal{B}}{P}$. Is B a change of basis matrix? (if so, between which bases?)

   (c) Is B^{-1} a change of basis matrix?

   (d) If you answered yes to the previous two parts: can you interpret the statement $A = BCB^{-1}$ in terms of a change of basis?

   (e) What is the meaning if C is a diagonal matrix?

2. Describe geometrically the eigenspaces of a linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ which reflects vectors across the xy-plane (do not compute the matrix!). Do the same for the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ which rotates each vector $\pi/4$ radians counter clockwise.

3. Let A be a 4×4 matrix. If the eigenvalues of A are $1, -2, 3, -3$, can you find $\det(A)$? What if the eigenvalues are $-1, 1, 2$?

4. Let A be an $n \times n$ matrix. Show that $\det(A^T - cI) = \det(A - cI)$ for any constant c. (Hint: $(A - cI)^T =?$). Do A, A^T have the same eigenvalues?

5. Suppose a matrix A is both diagonalizable and nilpotent. Prove A is the zero matrix.

6. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Show a is diagonalizable over $\mathbb{R}$ if $(a - d)^2 + 4bc > 0$.