Math 54 Worksheet **#12**

Instructions: Read one of the sections on the other sheet. Then flip that sheet over and with a partner / group answer as many of the associated questions on this sheet as you can, without looking at the notes (think of it as a mini zero-stakes, team quiz :).

1 Cramer's Rule Questions

1. How many determinants do you have to compute to solve $A\mathbf{x} = \mathbf{b}$ for an $n \times n$ matrix A with Cramer's rule?
2. What if $A\mathbf{x} = \mathbf{b}$ has two solutions? Can you apply Cramer's rule?
3. Use Cramer's rule to find $\mathbf{x}$ in $A\mathbf{x} = \mathbf{b}$ for $A = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$.

2 Determinants as Area Questions

1. Can the determinant formula for an area or volume ever give you a negative area? Why or why not?
2. Does the formula still make sense if the matrix is not invertible? What does it say? What is happening geometrically?
3. Draw the vectors $[1, 2]^T$ and $[0, 3]^T$ in $\mathbb{R}^2$, shade in the parallelogram they form and compute its area.
4. Suppose S is the portion of $\mathbb{R}^2$ inside a circle, and you know it has area 9π. Let T be the linear transformation defined by matrix $A = \begin{bmatrix} -2 & 3 \\ 0 & 2 \end{bmatrix}$. What is $Area(T(S))$?
5. Suppose S is a subset of $\mathbb{R}^{100}$ with volume 14. Suppose A is a 100×100 diagonal matrix with -2 down the diagonal. What is the volume of $A(S)$?

3 Eigenvector Questions

1. Is zero always an eigenvector?
2. Is $\mathbf{v}$ an eigenvector of A if $\mathbf{v}$ is in the null space of A?

3. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. I claim it has eigenvectors $[6, -5]^T$ and $[1, 1]^T$. Check this by drawing these vectors in the plane, as well as $A[6, -5]^T$ and $A[1, 1]^T$.

4. Suppose I tell you that the matrix $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$ has eigenvalue $\lambda = 4$. Can you find the eigenvectors associated to that eigenvalue?

4 Eigenvalue Questions

1. Find the eigenvalues for $A = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & -3 \\ -1 & 0 & 1 \end{bmatrix}$. Then, for each eigenvalue, find the associated eigenvectors.

2. Let A be an $n \times n$ matrix. Explain why A is invertible if and only if 0 is not an eigenvalue of A.

3. For $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ find one eigenvector and eigenvalue (hint: no computation needed).

4. (a) Show that the eigenvalues of an upper triangular $n \times n$ matrix are the entries on the main diagonal.
   (b) Show that if λ is an eigenvalue of an $n \times n$ matrix A then λ^2 is an eigenvalue of A^2. More generally, show that λ^k is an eigenvalue of A^k if k is a positive integer.
   (c) Use (a) and (b) to find the eigenvalues of A^9, where

$$A = \begin{bmatrix} 1 & 3 & 7 & 11 \\ 0 & -1 & 3 & 8 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$