Math 54 Worksheet #19

1 Midterm Review!

the number of pivots, the dimension of the column space, the dimension of the row space.

1. There are many equivalent definitions of rank. Give 3. What is the rank of 3×3 matrix with a nullspace spanned by $[1, 2, -2]^T$? What are the possible ranks of an $n \times n$ projection matrix?

   rank(A) = dim col = n - dim Nul = 2 (this is true for any m x n matrix)

   rank(A) = dim col(A) = dimension of the space you're projecting onto so 0-n are all possible.

2. Diagonalization. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$.

   note symmetric so the spectral theorem applies!

   (a) Suppose I tell you $\lambda = 1$ is an eigenvalue for A. Find all other eigenvalues (or show that 1 is the only eigenvalue). → char poly = $(1-\lambda)(\lambda^2 - 2\lambda - 1)$ so $\lambda_2 = 1+\sqrt{2}$, $\lambda_3 = 1-\sqrt{2}$

   (b) Find a basis ~~for~~ of eigenvectors for each eigenspace.

   $\lambda_1 \Rightarrow v_1 = (-1,0,1)^T$, $\lambda_2 \Rightarrow v_2 = (1,\sqrt{2},1)^T$, $\lambda_3 \Rightarrow v_3 = (1,-\sqrt{2},1)^T$

   (c) Diagonalize A.

   $A = PDP^{-1}$ for $P = \begin{pmatrix} -1 & 1 & 1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -\sqrt{2} \end{pmatrix}$

3. Kristina's favourite example to try on T/F questions: Can you diagonalize this matrix: $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$? → no!! Can you invert it? → yes!! Bonus: write down a T/F question this matrix would be a counterexample for. "Is an invertible 2x2 matrix also diagonalizable"

4. Suppose A is a 2×2 matrix with eigenvalue $\lambda_1 = 2$ corresponding to eigenvector $\mathbf{v}_1 = [-1, 1]^T$ and $\lambda_2 = -2$ corresponding to $\mathbf{v}_2 = [2, -3]^T$.

   (a) For $\mathbf{x}_0 = [0, 1]^T$, find $\mathbf{x}_k = A^k \mathbf{x}_0$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \lambda_1$ and λ_2.

   (b) Calculate A^n.

   $x_0 = -2v_1 - v_2$ so $x_k = -2(2)^k v_1 - (-2)^k v_2$

   $A = PDP^{-1}$ for $P = \begin{pmatrix} -1 & 2 \\ 1 & -3 \end{pmatrix}$, $D = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ so A^k is $P\begin{pmatrix} 2^k & 0 \\ 0 & (-2)^k \end{pmatrix}P^{-1}$

5. Prove "complex eigenvalues always come in conjugate pairs" (e.g. if λ in $\mathbb{C}$ is an eigenvalue of A, then $\overline{\lambda}$ is too). → $Av = \lambda v$ implies $A\overline{v} = \overline{Av} = \overline{\lambda v} = \overline{\lambda}\overline{v}$.

6. This is not really review, but let's learn the spectral theorem. I'll give a statement of the theorem, read it and then try to answer the last question as best you can without checking back here.

   The Spectral Theorem. An $n \times n$ symmetric matrix has the following properties:

   (a) A has n REAL eigenvalues, counting multiplicities (e.g. no complex eigenvalues, ever. we love symmetric matrices).

   (b) A is orthogonally diagonalizable.

It isn't b/c the 2 real eigenvalues only supply 2 linearly indep. vectors between them (we need 3 for a basis of eigenvectors) and we can't have a 3rd complex eigenvalue b/c there's no room for its complex conj.

7. Suppose I told you a 3×3 matrix has exactly 2 distinct real eigenvalues with one eigenvector each (but I don't tell you anything about the complex eigenvlaues). Can you tell me if it's diagonalizable or not?

the columns are orthonormal, $A^TA = I$, and $\|Ax\| = \|x\|$ for all x.

8. Orthogonal $n \times n$ matrices have lots of nice properties. Write down 3 equivalent definitions of a square matrix being orthogonal. (hint: one defintion has to do with what the matrix does to vectors you multiply it with).

9. Let V be the subset of $\mathbb{R}^4$ satisfying the equations $\mathbf{x}_1 + 2(\mathbf{x}_2 + \mathbf{x}_3) + \mathbf{x}_4 = 0$, $\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_4$ and $10\mathbf{x}_1 = 0$. Is this a subspace (hint: is it the nullspace of a matrix)? Find a basis for its orthogonal complement.

a yes!

it's the null space of $\begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & -1 & 0 & 1 \\ 10 & 0 & 0 & 0 \end{pmatrix}$ ← b/c Row $A \perp$ Null A, the rows are a basis for the orthogonal complement

10. Let $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 2 & 2 \\ 2 & 0 & 4 \\ -3 & 1 & -5 \end{bmatrix}$. What are the steps to find the projection matrix P onto A's column space? (hint: is A^TA invertible?)

Find a basis for A's column space (can check the first 2 columns work) → let this new 4x2 matrix be B and then $P = B(B^TB)^{-1}B^T$

(a) Carry out those steps to find P. ← $P = \begin{pmatrix} 1 & -3 \\ 0 & 2 \\ 2 & 0 \\ -3 & 1 \end{pmatrix} \frac{1}{14^2-6^2} \begin{pmatrix} 14 & 6 \\ 6 & 14 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 & 3 \\ -3 & 2 & 0 & 1 \end{pmatrix}$

(b) Find the projection matrix onto A's left nullspace (hint: this shouldn't require much computation). ← b/c Col(A) $\perp$ LNull(A), the projection matr onto it is just $I - P$ ← proj onto Col(A).

11. Suppose we are trying to fit the points $(x, y) = (1, 3)$, $(2, 0)$, $(-1, 1)$ with a cubic of the form $y = c_1x + c_2x^3$.

seems unlikely since there are more equations (eg rows) than variables (eg columns) in our system.

(a) Do we expect to find a solution for c_1, c_2 satisfying all three points?

$A^TA\hat{x} = A^Ty$ ← (b) If not, set up the least squares problem that would find us the best possible solution (e.g. what matrix equations do we need to solve?).

where $A = \begin{pmatrix} 6 & 18 \\ 18 & 66 \end{pmatrix} \hat{x} = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 8 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

(c) Will we have a unique solution? (why?) Solve the least squares problem (it's ok to leave the answer as a product of matrices).

yes b/c ATA is invertible b/c the columns of A were LI. Get $\hat{x} = \frac{1}{6\cdot 66-18^2} \begin{pmatrix} 66 & -18 \\ -18 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 8 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

12. Spectral Theorem T/F:

(a) There are symmetric matrices which are not orthogonally diagonalizable (i.e. diagonalizable with $A = PDP^{-1}$ where P is an orthogonal matrix). ← F

(b) If $A^T = A$ and $A = PDP^{-1}$ with D diagonal, then all entries of D must be real numbers. → T

(c) The dimension of an eigenspace of a symmetric matrix (i.e. the geometric multiplicity) is sometimes less than the multiplicity of that eigenvalue as a root of the characteristic polynomial (i.e. the algebraic multiplicity). ← F (then it wouldn't be diagonalizable)

(d) If $A^T = A$ and if $\mathbf{u}, v$ satisfy $A\mathbf{u} = 3\mathbf{u}$, $A\mathbf{v} = 2\mathbf{v}$, then $\mathbf{u} \cdot \mathbf{v} = 0$.

↳ T b/c then they come from different eigenspaces, which are orthogonal on a symmetric matrix.