1 Subspaces of a matrix A

Lost my blue pen so this is going to be even harder to read than usual sorry :P

You may have seen that a matrix has 4 subspaces associated with it, the column space (denoted Col or A), the row space (or Row A), the Null space (Nul A) and the left null space or (L Nul A). This last one we haven't worked with so much so we'll talk about it more today. **Warning: All four subspaces are different! And you need to use (/memorize) different algorithms to find each one** (you may recall finding a basis for the column space and the null space were very different). Lets start with some warm up questions:

1. If A is a 7×5 matrix, what are the possible dimensions of Col A? → 0–5 Of Row A? → 0–5 Of Nul A? → 0–5

2. L Nul A is the null space of A^T, that is, the subspace of vectors $\mathbf{x}$ with $A^T\mathbf{x} = \mathbf{0}$. If A is a 3×6 matrix, what are the possible dimensions of L Nul A? A^T is 6×3 so dim LNul is 0–3

3. If A is a 5×6 matrix and the dimension of Col A is 3, what is the dimension of $Nul A$? recall dim Col A + Nul A = n for an $m \times n$ matrix. So 3.
   (also dim Col A = rank)

4. If A is a 7×5 matrix, what are the possible ranks of A? → 0–5 What about A^T? → always the same.

5. We saw a couple weeks ago how to find a basis for the nullspace of a matrix (how did that work again?). We could use this on A^T to find a basis for L Nul A, since it's just the null space of A^T. However there's another trick we can use, that will not only give us the left null space, but also the row space, the column space and the regular null space at once!

   Let $A = \begin{bmatrix} 3 & 5 & -4 \\ -3 & -4 & 0 \\ 6 & 1 & -8 \\ 0 & 1 & -4 \end{bmatrix}$. Step 1 is to augment A by sticking the 4×4 identity matrix on the right, e.g. write down $[A|I_4]$ (in general for a $m \times n$ matrix you add the $m \times m$ identity matrix). Now you row reduce $[A|I_4]$ until the A part is in reduced echelon form, so you have $[REF(A)|B]$ where B is whatever I_4 became.

   (a) Given $[REF(A)|B]$, write down a basis for the column space of A. What is the dimension of the column space?

   I got $\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & -8/27 & -1/3 & 4/27 & 0 \\ 0 & 1 & 0 & 2/9 & 0 & -1/9 & 0 \\ 0 & 0 & 1 & -7/36 & -1/4 & -1/36 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right]$ So dim Col A = 3 recall that unlike the column space you can read off the basis for the row space directly from the REF matrix! eg in this case $\{(1\ 0\ 0), (0\ 1\ 0), (0\ 0\ 1)\}$ works. dim=3.

(b) Write down a basis for the row space. What is the dimension of that?

(c) What about Nul A? Use $[REF(A)|B]$ to figure out a basis for the null space. What is its dimension? → we just use the REF(A) part of the matrix

REF(A) says $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\\0\end{pmatrix}$ so Nul $A=\{\vec{0}\}$.

(d) Finally, we come to the left null space. Here's the magic part: it turns out you can just read it off $[REF(A)|B]$. Look for any rows of $REF(A)$ that are entirely zero. Then the remainder of that row in B is a vector of the left null space. In fact, together those rows form a basis for the left null space! For example, if A was a 3×2 matrix and $[REF(A)|B]$ looked like: $\begin{bmatrix}1&0&4&-5&0\\0&0&1&11&-7\\0&0&-2&2&3\end{bmatrix}$ then a basis for L Nul A would be the vectors $[1, 11, -7]$ and $[-2, 2, 3]$. What is the basis in our example?

→ in our case the basis is just generated by 1 vector, $(-1\ -1\ 0\ 1)$

2 Change of Basis

1. Last week we saw the change of basis from a (weird) basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ to the standard basis $\mathcal{E}$. You may have also seen in the text (page 243) that you can find the change of basis matrix from $\mathcal{B}$ to any other basis $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$. To do that you write down the matrix $[\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3|\mathbf{b}_1\mathbf{b}_2\mathbf{b}_3]$ and row reduce to put the left half, $\mathbf{c}_1\mathbf{c}_2\mathbf{c}_3$, into reduced echelon form. You end up with $[I|\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}]$. Why does this work? (see page 243).
Try it on this example: $\mathbf{b}_1 = [7,5]$, $\mathbf{b}_2 = [-3,-1]$, $\mathbf{c}_1 = [1,-5]$, $\mathbf{c}_2 = [-2,2]$.
warning: the order of the vectors in your basis matters! Don't swap b_1 and b_2.
start with: $\left[\begin{array}{cc|cc}1&-2&7&-3\\-5&2&5&-1\end{array}\right]$ so $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is $\begin{bmatrix}17&-9\\5&-2\end{bmatrix}$

3 Rank & Dimension

1. Sort the possible forms a 2×2 reduced row echelon matrix can have by rank. What about 3×3 matrices? rank=1 $\begin{pmatrix}0&■\\0&0\end{pmatrix}$, $\begin{pmatrix}■&*\\0&0\end{pmatrix}$, rank=2 $\begin{pmatrix}■&*\\0&■\end{pmatrix}$

2. Consider the set $S = \left\{\begin{bmatrix}a-4b-2c\\2a+5b-4c\\-a+2c\\-3a+7b+6c\end{bmatrix} : a,b,c \text{ in } \mathbb{R}\right\}$ (ask me about it if this notation is unclear). Is S a subspace? What vector space is it a subspace of? Can you see a basis for S? What is the dimension of S?

So $S = \left\{a\begin{pmatrix}1\\2\\-1\\-3\end{pmatrix}+b\begin{pmatrix}-4\\5\\0\\7\end{pmatrix}+c\begin{pmatrix}-2\\-4\\2\\6\end{pmatrix}\right\}$ → found basis to be the 1st 2 vectors with row reduction. So dim = 2.

3. Consider $\mathbb{P}_5 = \{a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x^1 + a_0 : a_i \text{ in } \mathbb{R}\}$ (this is the set of all degree 5 polynomials, again ask me if this is unclear). Is this a subspace? Can you think of a basis for it? What is its dimension?

→ 6! (not 5!)

→ yes. check the 3 axioms for yourself!
a basis would be $1, x, x^2, x^3, x^4, x^5$ b/c if you take all linear combinations of these you get all polynomials of degree ≤ 5 (why are they linearly indep.?)