In this last section of the course we're going to use our linearly algebra knowledge to solve systems of differential equations. There are basically 2 things you'll need to know for this: (1) How to convert a scalar ODE into a linear system (in normal form):

$$\mathbf{x}'(t) = A\mathbf{x}(t),$$

where A's a matrix and $\mathbf{x}, \mathbf{x}'$ are vectors. And (2) How to solve a solve differential equation system in normal form. This last step is easier if A is diagonalizable, so that's the case we'll stick to today. (When A is not diagonalizable we have to use matrix exponentials, which will be covered in lecture on Wednesday).

1 Turning Scalar Equations into Linear Systems

We can turn a (scalar) differential equation of arbitrary degree into a degree one equation by replacing the derivatives of y with new variables. For example, given:

$$y^{(4)} + 3y'' - \sin(t)y' + 8y = t^2$$

we would introduce the new variabales x_1, x_2, x_3, x_4 and let $x_1 = y$, $x_2 = y'$, $x_3 = y''$ and $x_4 = y'''$.
We're looking for a system of the form $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $\mathbf{x}(t) = (x_1, x_2, x_3, x_4)$, so we need to be able to write the derivatives of the x_i in terms of the other (non-differentiated x_i). Note that $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_4$.

What is x_4'? (hint: it's more complicated, use the differential equation).

$$x_4' = y^{(4)} = -3y'' + \sin(t)y' - 8y + t^2 = -3x_3 + \sin(t)x_2 - 8x_1 + t^2.$$

Given these formula for the x_i', what is the matrix A?

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -8 & \sin(t) & -3 & 0 \end{bmatrix} \quad \text{b/c then} \quad \begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

Rewrite the following scalar equations as linear systems in normal form.

1. $x''(t) + x(t) = t^2$.

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t^2 \end{bmatrix}$$

2. $\frac{d^3y}{dt^3} - \frac{dy}{dt} + y = \cos t$.

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cos t \end{bmatrix}$$

3. $y^{(n)}(t) + p_{n-1}(t)y^{(n-1)}(t) + ... + p_0(t)y(t) = g(t)$. (Note: in case you haven't seen it yet, $y^{(n)}$ means the n^{th} derivative of y).

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ & & 1 & \ddots & \vdots \\ & & & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & & -p_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

4. What does question 3 tell you about the kinds of problems you can solve with linear systems of ODEs vs scalar ODEs?

all scalar ODEs (of any order) can be rewritten as a first degree linear system. ⇒ linear systems of ODEs are more general.

2 The Wronskian

Just like with scalar ODEs we want to write down the general solution to a linear system $\mathbf{x}'(t) = A\mathbf{x}(t)$. But what does the general solution look like? It turns out, just like the scalar case, it's a linear combination of linearly independent solutions (we're *not* going to prove this). How many solutions? If A is an $n \times n$ matrix then there should be n. So our solutions look like:

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + .. + c_n\mathbf{x}_n(t).$$

As in the scalar case, the c_1 are unknown coefficients which would be determined by an initial condition. The $\mathbf{x}_1, ..., \mathbf{x}_n$ form the set of **fundamental solutions**. And if you put them in a matrix together, $[\mathbf{x}_1, ..., \mathbf{x}_n]$, we call that the **fundamental matrix**.

As before we can use the Wronskian to check if a proposed set of solutions are actually linearly independent. Let

$$W(t) = \det[\mathbf{x}_1(t), ..., \mathbf{x}_n(t)].$$

Then W will be a function of t, and the rule is: it'll be zero if the functions are dependent, and never zero (for any value of t) if the functions are independent.

Compute the Wronskian of the following sets of solutions.

1. $\mathbf{x}_1 = \begin{bmatrix} te^{-t} \\ e^{-t} \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix}$

$$\begin{vmatrix} te^{-t} & e^{-t} \\ e^{-t} & e^{-t} \end{vmatrix} = te^{-2t} - e^{-2t} = (t-1)e^{-2t}$$ independent.

Also, these can't be solns to an ODE!

2. $\mathbf{x}_1 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$

double angle formulas

$$\begin{vmatrix} \sin t & \sin(2t) \\ \cos t & \cos(2t) \end{vmatrix} = \sin t(\cos^2 t - \sin^2 t) - \cos t(2\sin t \cos t)$$
$$= -\sin^3 t - \sin t \cos^2 t \neq 0$$ independent

3. $\mathbf{x}_1 = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} \sin t \\ \sin t \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} \cos t \\ \cos t \end{bmatrix}$

can't plug them into the Wronskian so just try to create a dependence

try: $c_1\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \sin t \end{bmatrix} + c_3\begin{bmatrix} \cos t \\ \cos t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ at $t=0 \to c_1\begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

$\to c_1 = c_3 = 0$

$\to c_2 = 0$.

4. Suppose the following $\mathbf{x}_i$ are solutions to the equation $\mathbf{x}'(t) = A\mathbf{x}(t)$. Determine if they form a fundamental solution set, and if so write down the fundamental matrix and a general solution.

$\mathbf{x}_1 = \begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix}$

Check LI: $c_1\begin{bmatrix} e^t \\ e^t \\ e^t \end{bmatrix} + c_2\begin{bmatrix} \sin t \\ \cos t \\ -\sin t \end{bmatrix} + c_3\begin{bmatrix} -\cos t \\ \sin t \\ \cos t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

at $t=0 \to c_1\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\to c_1 = c_3$, $c_1 = -c_2$, $c_1 = -c_3$, so $c_1 = c_3 = 0$

so $c_2 = 0$ too.

fund matrix is

$$\begin{bmatrix} e^t & \sin t & -\cos t \\ e^t & \cos t & \sin t \\ e^t & -\sin t & \cos t \end{bmatrix}$$

gen soln is $\bar{X}(t) = c_1\bar{X}_1(t) + c_2\bar{X}_2(t) + c_3\bar{X}_3(t)$.

3 Homogeneous Linear Systems with Constant Coefficients

Now we're going to solve equations of the form $\mathbf{x}'(t) = A\mathbf{x}(t)$. The underlying idea is the same as in the scalar case: guess a solution made out of ce^{rt}'s, because we're looking for functions $x_i(t)$ which are equal to their derivative (or at least, linearly combine to form their derivative).

It turns out guessing solutions of the form $x_i(t) = ce^{rt}$ plugging them into $\mathbf{x}'(t) = A\mathbf{x}(t)$ and solving for c and r amounts to finding the eigenvalues and eigenvectors of A. Specifically if A has n linearly independent eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$ with associated eigenvalues $r_1, ..., r_n$ (not all necessarily distinct), then a set of fundamental solutions is:

$$\mathbf{x}_1(t) = e^{r_1 t}\mathbf{v}_1,\ \dots\ ,\ \mathbf{x}_n(t) = e^{r_n t}\mathbf{v}_n.$$

Let's try an example:

$$\mathbf{x}' = A\mathbf{x}, \text{ where } A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}.$$

First we find the eigenvalues and eigenvectors of A (do this).

$\lambda = -1, 4$

$\begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Then we write down the fundamental set of solutions as described above. For example if one of the eigenvalues was -1, and it had eigenvector $(-1, 1)^t$, one of our fundamental solutions would be $e^{-t}(-1, 1)^t$. Find both and write down the general solution.

$\bar{x}(t) = c_1 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

Finally, we could be asked to solve an initial value problem. What if we were told $\mathbf{x}(0) = (0, -4)^t$? Can you solve for the coefficients in your general solution found above? What is the final solution?

$c_1\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \end{pmatrix} \longrightarrow c_1 = -\frac{8}{5},\ c_2 = -\frac{4}{5}$

Would this method work if A was not a constant matrix? (e.g. if it had functions like t^2 in it?). Does this method depend on A being diagonalizable?

nope, we don't know how to find eigenvalues of those.

yes!! otherwise we wont have a full gen. soln.

1. Find a general solution to $\mathbf{x}' = A\mathbf{x}$ for $A = \begin{bmatrix} -7 & 0 & 6 \\ 0 & 5 & 0 \\ 6 & 0 & 2 \end{bmatrix}$.

we find $\lambda_1 = -10 \quad v_1 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_2 = 5 \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 5 \quad v_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

So gen soln:

$\bar{x}(t) = c_1 e^{-10t}\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{5t}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

2. Find a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$ for $A = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ 3 & 3 & -1 \end{bmatrix}$.

$\lambda_1 = 2,\ v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \lambda_3 = 1,\ v_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

$\lambda_2 = 2,\ v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} e^{2t} & -e^{2t} & e^t \\ 0 & e^{2t} & e^t \\ e^{2t} & 0 & 3e^t \end{pmatrix}$

3. Solve the initial value problem $\mathbf{x}' = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} -10 \\ -6 \end{bmatrix}$.

$\lambda_1 = 3,\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\rightarrow$ gen soln $\bar{x}(t) = c_1 e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\lambda_2 = 4,\ v_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$ IC's: $\begin{pmatrix} -10 \\ -6 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\rightarrow c_1 = 2,\ c_2 = -4$

4. Solve the initial value problem $\mathbf{x}' = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \mathbf{x}$, $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$.

$\lambda_1 = 2$ $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ gen soln: $\bar{x}(t) = c_1 e^{2t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_2 = -1$ $v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ apply ICs and get $c_1 = 1,\ c_2 = 3,\ c_3 = -1$

$\lambda_2 = -1$ $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

One final case we'll handle are systems with complex eigenvalues. In this case, as always, we'll want to replace any expressions of the form $e^{\alpha + i\beta}$ with sines and cosines. The rule is as follows:

If A has complex conjugate eigenvalues $\alpha \pm i\beta$ with corresponding eigenvectors $\mathbf{a} \pm i\mathbf{b}$, then the corresponding two linearly independent solutions to $\mathbf{x}'(t) = A\mathbf{x}(t)$ are

$$\mathbf{x}_1(t) = e^{\alpha t} \cos(\beta t)\mathbf{a} - e^{\alpha t} \sin(\beta t)\mathbf{b},$$

$$\mathbf{x}_2(t) = e^{\alpha t} \sin(\beta t)\mathbf{a} + e^{\alpha t} \cos(\beta t)\mathbf{b}.$$

Let's do an example. Consider the system $\mathbf{x}'(t) = \begin{bmatrix} 3 & 9 \\ -4 & -3 \end{bmatrix} \mathbf{x}(t)$. It has eigenvalues $\pm 3\sqrt{3}i$, so $\alpha = 0$ and $\beta = 3\sqrt{3}$. The corresponding eigenvectors are $\begin{bmatrix} 3 \\ -1 \pm \sqrt{3}i \end{bmatrix}$, so $\mathbf{a} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ \sqrt{3} \end{bmatrix}$. Thus our two solutions are:

$$\mathbf{x}_1(t) = \begin{bmatrix} 3\cos(3\sqrt{3}t) \\ -\cos(3\sqrt{3}t) - \sqrt{3}\sin(3\sqrt{3}t) \end{bmatrix}$$

$$\mathbf{x}_2(t) = \begin{bmatrix} 3\sin(3\sqrt{3}t) \\ -\sin(3\sqrt{3}t) + \sqrt{3}\cos(3\sqrt{3}t) \end{bmatrix}.$$

1. Find a general of the system $\mathbf{x}' = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} \mathbf{x}$.

$\lambda_1 = i$ $v_1 = (-2+i, 1)$

$\alpha = 0,\ \beta = 1$ $a = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ $b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$x_1(t) = \cos(t) \begin{pmatrix} -2 \\ 1 \end{pmatrix} - \sin(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2\cos t - \sin t \\ \cos t \end{pmatrix}$

$x_2(t) = \sin(t) \begin{pmatrix} -2 \\ 1 \end{pmatrix} + \cos(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} -2\sin t + \cos t \\ \sin t \end{pmatrix}$

(various answers possible depending on choice of v_1)

2. Find a general of the system $\mathbf{x}' = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} \mathbf{x}$.

$\rightarrow c_1 \begin{bmatrix} 5e^{2t}\cos t \\ -2e^{2t}\cos t + e^{2t}\sin t \\ 5e^{2t}\cos t \end{bmatrix} + c_2 \begin{bmatrix} 5e^{2t}\sin t \\ -2e^{2t}\sin t - e^{2t}\cos t \\ 5e^{2t}\sin t \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ e^{2t} \\ e^{-2t} \end{bmatrix}.$

eg $v_1 = (5, 2+i)$ also possible.

3. Find a fundamental matrix for the system $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -13 & 0 \end{bmatrix}$. $\rightarrow$ didn't leave enough room for the whole matrix :) but the solns are

$x_1 = e^t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $x_2 = e^{-t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $x_3 = (0, 0, e^{2t}\cos 3t, e^{2t}(2\cos 3t - 3\sin 3t))$

$x_4 = (0, 0, e^{2t}\sin 3t, e^{2t}(2\sin 3t + 3\cos 3t))$