Introduction

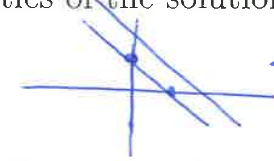
A linear equation in n variables is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b,$$

where a_1, a_2, ..., a_n and b are real numbers (constants). Notice that a linear equation doesnt involve any roots, products, or powers greater than 1 of the variables, and that there are no logarithmic, exponential, or trigonometric functions of the variables. Solving a linear equation means finding numbers r_1, r_2, ..., r_n such that the equation is satisfied when we make the substitution $x_1 = r_1, x_2 = r_2, \dots, x_n = r_n$. In this course we will be concerned with solving systems of linear equations, that is, finding a sequence of numbers $r_1, r_2, \dots, r_n$ which simultaneously satisfy a given set of equations in n variables. No doubt you have solved systems of equations before. In this course we will not only learn techniques for solving more complicated systems, but we will also be concerning ourselves with important properties of the solution sets of systems of equations.

Problems

$x+y=1$
$x+y=11$



← not to scale :)

1. Write down a system of two linear equations in two unknowns which has no solution. Draw a picture of the situation.

2. Solve the following system of equations and describe in words each step you use (explain both in terms of manipulations of the equations and row operations if possible).

$$x + 3y - z = 1$$
$$3x + 4y - 4z = 7$$
$$3x + 6y + 2z = -3$$

$$\begin{bmatrix} 1 & 3 & -1 & 1 \\ 3 & 4 & -4 & 7 \\ 3 & 6 & 2 & -3 \end{bmatrix}$$ augmented matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & -3/2 \end{bmatrix}$$ REF

How many solutions are there, and what does the solution set look like geometrically?

3. What condition on a, b, c, and d will guarantee that there will be exactly one solution to the following system?

$$ax + by = 1$$
$$cx + dy = 0$$

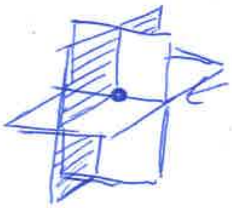
These are just 2 lines, and the problem wants you to make them intersect once.

Could solve as a matrix but probably easiest is to just remember that any 2 non-parallel lines will cross exactly once.

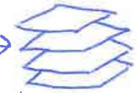
assuming $b \neq 0$, $d \neq 0$ → $y = -\frac{a}{b}x + \frac{1}{b}$, $y = -\frac{c}{d}x$ so just want $a/b \neq c/d$

an intersection of 3 planes, can make the 4th plane equal to one of the previous 3.

4 equations of a plane



4. Consider a system of four equations in three variables. Describe in geometric terms conditions that would correspond to a solution set that
   (a) is empty. → 4 parallel planes
   (b) contains a unique point.
   (c) contains an infinite number of points.



make all 4 planes the same plane

Additional Problems

1. Set up a system of linear equations for the following problem and then solve it: The three-digit number N is equal to 15 times the sum of its digits. If you reverse the digits of N, the resulting number is larger by 396. Also, the units (ones) digit of N is one more than the sum of the other two digits. Find N.

2. Consider the system of equations

$$ax + by = k$$
$$cx + dy = l$$
$$ex + fy = m$$

Show that if this system has a solution, then at least one equation can be thrown out without altering the solution set.

① We have the equations:

(1) $N = 100x + 10y + z$
(2) $N = 15x + 15y + 15z$
(3) $N + 396 = 100z + 10y + x$
(4) $z = 1 + x + y$

setting (1) = (3) gives you $x - z = -4$

subbing that into (4) gives you $y = 3$

the subbing this value for y and the result $x = z - 4$ into (1) gives you $z = 5$

so $x = 1$ and $N = 135$.

② If this system has a solution then there is a sequence of row operations turning

$$\left(\begin{array}{cc|c} a & b & k \\ c & d & l \\ e & f & m \end{array}\right) \text{ into } \left(\begin{array}{cc|c} 1 & 0 & s \\ 0 & 1 & t \\ 0 & 0 & 0 \end{array}\right)$$

for some s, t

or $\left(\begin{array}{cc|c} 1 & 0 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ because these are the only 3 REF matrices of this shape where you'll get a solution.

or $\left(\begin{array}{cc|c} 0 & 1 & s \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$

In any case, we can make the bottom row 0 with the other 2 rows. This is equivalent to saying we can write the last equation by adding combinations of the other 2!

(and therefore it is clearly redundant).