α Voronoi Diagrams

Karthikeya P, Pradyumnan R

December 7, 2024

Abstract

In this paper, we present a novel algorithm to compute α Voronoi Diagrams. We show that our algorithm is having a time complexity of $O(n \log n)$. We also present some applications of α Voronoi Diagrams in various fields.

1 Introduction

We will introduce the concept of α Voronoi Diagrams in this section. We will also discuss the motivation behind the study of α Voronoi Diagrams. Let us consider the following example to understand the concept of α Voronoi Diagrams. Suppose that you are in a city with lot of coffee shops. You want to find the nearest and **better** coffee shop from your current location. The α Voronoi Diagrams can be used to solve this problem. The α Voronoi Diagrams are a generalization of the Voronoi Diagrams. The Voronoi Diagrams are used to partition the space into regions based on the distance from a set of points. The α Voronoi Diagrams are used to partition the space into regions based on the distance from a set of points and a **interest** parameter α (which depends on the points under consideration).

2 Definitions

2.1 α-Closeness

A point is said to be α-close to point A compared to point B if the distance between the point and A is less than α times the distance between the point and B. Mathematically, a point P is α-close to point A compared to point B if $d(P, A) < \alpha_{A,B} \cdot d(P, B)$. Here, α is a parameter which depends on the points A and B.

2.2 α Voronoi Diagrams

The α Voronoi Diagram is a partition of the plane into regions based on the α-closeness of the points. The α Voronoi Diagram is a generalization of the Voronoi Diagram. The Voronoi Diagram is a special case of α Voronoi Diagram with $\alpha = 1$ (for all pairs of points).

2.3 α Boundary

The α Boundary of two points A and B is the set of all points which are α-close to A compared to B. The α Boundary is the set of all points P such that $d(P, A) = \alpha_{A,B} \cdot d(P, B)$.

3 α Voronoi Diagram of 2 points

The Figure 1 indicates the α Voronoi Diagram of two points A and B. The α Boundary is the circle centered at $(-8, 0)$ with radius 12. The points inside the circle are α-close to A compared to B. The points outside the circle are α-close to B compared to A. Here $\alpha = \frac{2}{3}$.

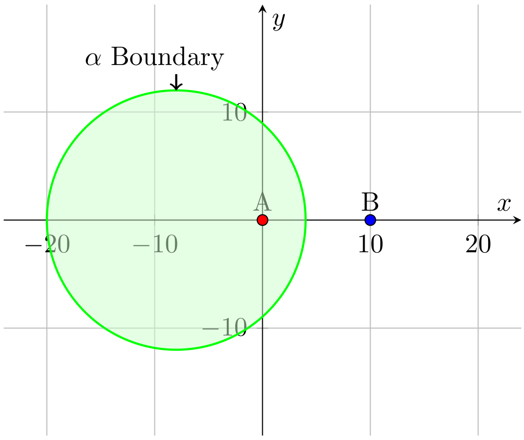


Figure 1: α Voronoi Diagram of 2 points

3.1 Observations and Properties

- The α Voronoi Diagram of 2 points is a circle.
- This tells us that the α-boundary is an arc of a circle.
- The center of the circle is collinear with the points A and B.
- The radius of the circle is $\frac{d(A,B)\cdot\alpha}{1-\alpha^2}$.

4 Experimentation

- We have implemented an algorithm which emphirically computes the α Voronoi Diagrams.
- Given a set of points and corresponding weights (which are used to compute the α parameter), the algorithm computes the α Voronoi Diagram.
- We take a grid and divide the grid into many number of points. We will compute and assign these grid points to the corresponding regions in the α Voronoi Diagram.
- If this is a $n \times n$ grid, then the time complexity of the algorithm is $O(n^2 \cdot m)$ where m is the number of points in the set.

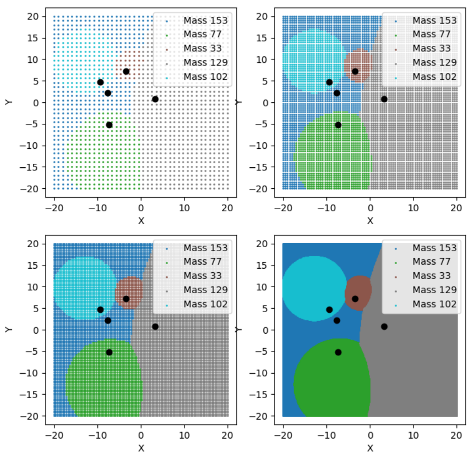


Figure 2: α Voronoi Regions

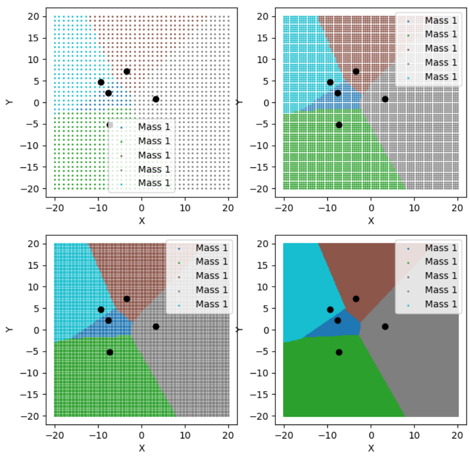


Figure 3: Voronoi Regions

Figure 4: Experimental α Voronoi Regions and Voronoi Regions

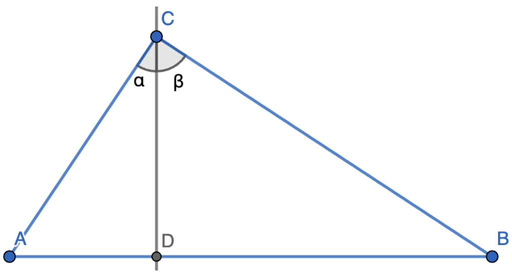
4.1 Observations and Conclusions

- When all the weights are equal (i.e., $\alpha = 1$), the α Voronoi Diagram is same as the Voronoi Diagram.
- The higher weight pushes the perpendicular bisector or the voronoi edge towards the point with lower weight.
- This implies that the least weight points α Voronoi region is a subset of the Voronoi region.
- Also the α Voronoi region of the highest weight point is a superset of the Voronoi region.

From the above discussion we know all the α boundaries, the task now is to compute where these boundaries intersect and how to construct the α voronoi diagram.

5 Methodology

- We will first observe some geometric implications of the below diagram.



- In the above figure, $CD \perp AB$ this implies that $AD = \tan\alpha$ and $BD = \tan\beta$.
- This gives us the following equation: $\frac{AD}{BD} = \frac{\tan\alpha}{\tan\beta}$.
- The following is crucial for our algorithm. The α boundary is the locus of points P such that $d(P, A) = \alpha \cdot d(P, B)$.

6 Algorithm

- Let $w_1, w_2, \ldots, w_n$ be the weights of the points $P_1, P_2, \ldots, P_n$.
- WLOG assume that $w_1 \leq w_2 \leq \ldots \leq w_n$.
- Now define the angle α_i as $\tan\alpha_i = \frac{w_i}{w_n}$.
- Now elevate cones of angles α_i from the points P_i.
- The intersection of these cones will give us the alpha voronoi diagram when projected on the plane.
- The time complexity of the algorithm is $O(n \log n)$.
- The algorithm is inspired from the Fortune's algorithm for Voronoi Diagrams.

7 Discussion

We would like to highlight few important points about the algorithm.

- Why are we dividing by the maximum weight?
  - This is done to normalize the weights. This is done to ensure that the weights are in the range $[0, 1]$.
  - So that the angles will be in the range $[0, \frac{\pi}{4}]$.

  - When we are sliding a plane, only the region of the largest weight should form an infinite voronoi region.
  - This implies that we want a parabola only at the region of the largest weight and all others we want to have closed regions (ellipses here).

- Then we can implement the algorithm similar to the Fortune's algorithm with plane at angle of $\frac{\pi}{4}$.

8 Conclusion

Your conclusion goes here.