Polygonal Rooms Not Illuminable from Every Point

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1. INTRODUCTION. Imagine two people in a dark room with many turns and cul-de-sacs. Assuming that the walls, floors and ceilings are constructed of reflective material, can one person strike a match and be seen by the other after repeated reflections, no matter where the two are located?

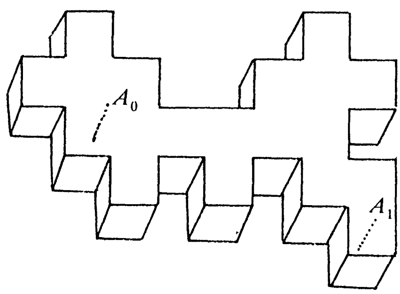

Figure 1. Can a match lit somewhere along the dotted line at A_0 say be seen somewhere along the dotted line at A_1 say?

This problem has been attributed to Ernst Straus in the early 1950's, and has remained open for over forty years. It was first published by Victor Klee in 1969 [**1**]. It has since reappeared on various lists of unsolved problems, notably Klee again in 1979 [**2**] and in two recent books on unsolved problems, one by Klee and Wagon in 1991 [**3**] and one by Croft, Falconer and Guy, also in 1991 [**4**].

In this article, we will settle the above problem in the negative. We will as well give elementary techniques for constructing rooms, both in the plane and in three-space, which are not illuminable from every point. In particular, we will show that if the two people are located in a two-dimensional planar room as shown in Figure 2, then they cannot see each other.

2. THE PLANAR PROBLEM. If G is a bounded simple polygonal region in the plane, is G illuminable from every point? In other words, if we view the sides of G as mirrors, can a single light source placed at any point, illuminate or be seen at every other point of the room? The problem can equivalently be posed in terms of a billiard ball bouncing around a pool table. Is there a "pool shot" between any two points on a polygonal pool table?

A light ray or pool ball reflects only at the sides of the room in such a way that the angle of incidence equals the angle of reflection. A light ray or pool ball that strikes a vertex is considered to end or be absorbed there.

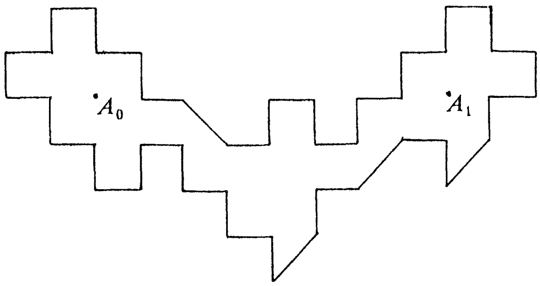

Figure 2

All paths or pool shots will be taken to be of non-zero length.

The main idea to solving this problem is that any path in a polygon unfolds to a path in another polygon constructed from mirror images of the first. Conversely, the second path can be considered to fold up to the first.

Example 1. Path $ABCD$ in 3(a) corresponds to the straight line path $ABCD$ in 3(b).

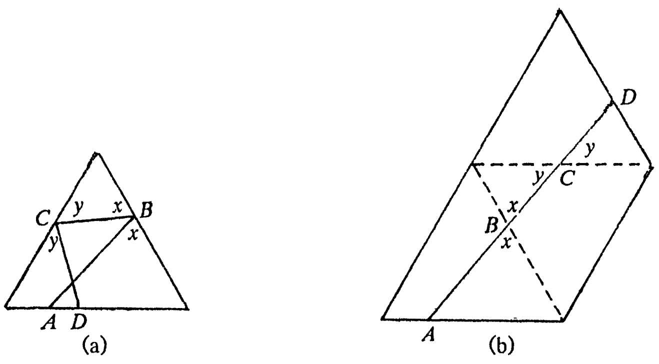

Figure 3

Example 2. Path $ABCDEF$ in 4(a) corresponds to the path $ABCDEF$ in 4(b).

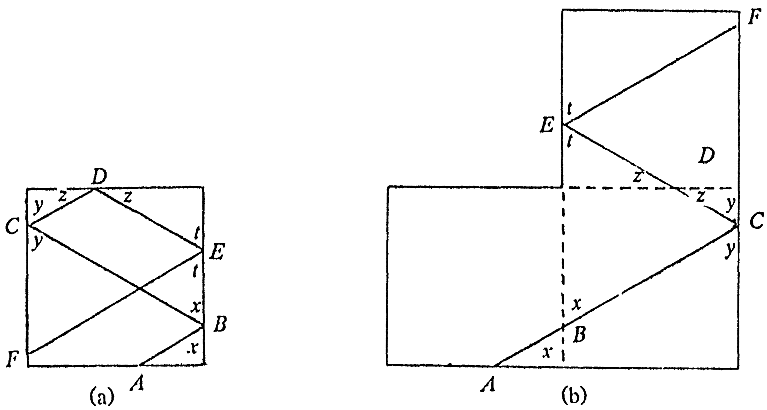

Figure 4

3. SOLUTION. We will first need the following lemma.

Lemma 3.1. *In an isosceles right triangle ABC (with right angle at C), there do not exist any pool shots from A coming back to A.*

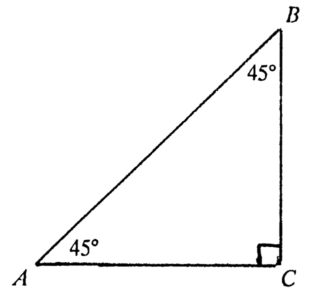

Figure 5

Proof: We start by taking a lattice of mirror images of this triangular table and assigning integer coordinates to the vertices as shown below, with A at the origin. Vertices labelled A have even coordinates $(2m, 2n)$ and vertices labelled B or C all have at least one odd coordinate. A pool shot from A to A on the original table would unfold or correspond to a straight line segment joining $A(0,0)$ to say $A(2m, 2n)$ in the lattice. This segment then must pass through the point (m, n) [or $(m/2, n/2)$ if both m and n are even, etc.] and thus must pass through a point labelled B or C. This means the pool shot would hit a vertex and be absorbed before returning to A. ■

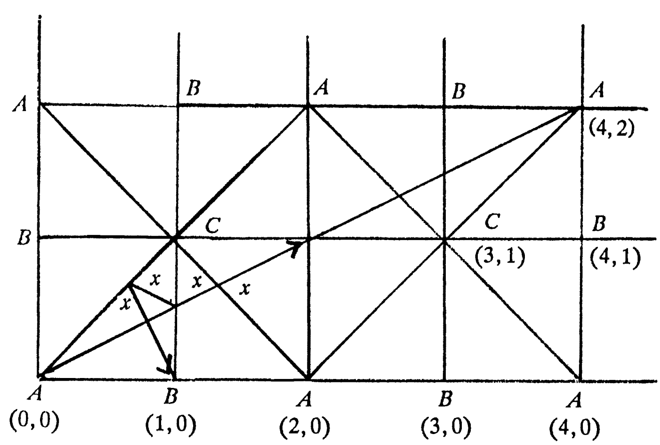

Figure 6

Theorem 3.2. *There do not exist any pool shots from A_0 to A_1 on the table shown in Figure 2.*

Proof: This table is constructed by taking mirror images of a right angled isosceles triangle as shown in Figure 7. The key to the diagram and the proof is that any point labelled B or C must be a vertex of this table, while points labelled A do not have to be.

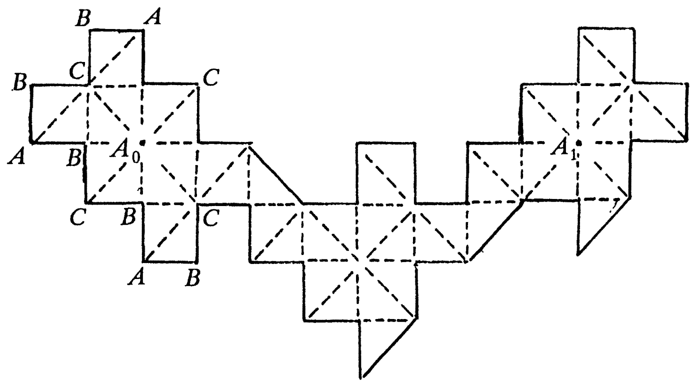

Figure 7

If there were a pool shot from A_0 to A_1, the initial path must pass through the interior of one of the eight triangles surrounding A_0. Let us call this triangle T. As in the lemma, a pool shot from A_0 to A_1 would correspond or fold up to a pool shot from A_0 to A_0 in triangle T, which is impossible. ■

Incidentally, it should be clear from the proof that there does not exist a pool shot between any two points labelled A on this table.

4. OTHER TABLES

Example 3. It would be interesting to find the table with the least number of sides which is not illuminable from every interior point. The example below has 26 sides.

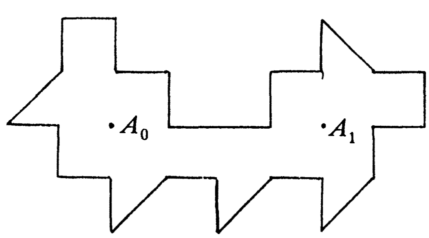

Figure 8

Example 4. By using the same kind of lattice argument given for the isosceles right triangle, there do not exist any pool shots from a corner of a square pool table $ABCD$ coming back to itself. We can also get this result by observing that a square is the mirror image of a right isosceles triangle in its hypotenuse and that a path in the square folds to a path in the triangle. A square then can be used to construct tables with only right angles, one of which is shown below. Again, we must follow the rule that points labelled B, C or D must remain vertices, while there isn't any restriction on points labelled A.

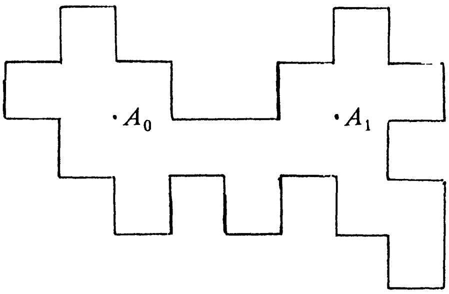

Figure 9

To construct tables using other kinds of triangles, we need a different type of argument.

Lemma 4.1. *If x divides 90 and $< A$ has size $x°$ and $< B$ has size $nx°$ where n is a positive integer, then the triangular pool table ABC does not have a pool shot from A coming back to A.*

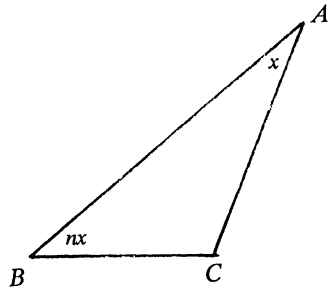

Figure 10

Proof: We measure all angles $\bmod 2x$.

Case I. n is even.

Let $0 < \theta < x$ be the angle of a pool shot leaving A as in Figure 11(a), then inductively it bounces off sides AB and BC at angles $\pm\theta$ and side AC at angles $x \pm \theta$ as shown in Figure 11(b)(c)(d).

If it comes back to A then it must re-enter at the angle $\pm\theta \bmod 2x$, but since $0 < \theta < x$, $-\theta$ is impossible. Hence, it must re-enter at the same angle θ that it left. This can only happen if the pool shot hits one of the sides at 90°. But, then $\pm\theta \equiv 90 \bmod 2x$ which implies that $\pm\theta \equiv 0 \bmod x$ (since x divides 90) or $x \pm \theta \equiv 90 \bmod 2x$ which again implies that $\pm\theta \equiv 0 \bmod x$. This is impossible since $0 < \theta < x$.

Case II. n is odd.

Similar to the first case, a pool shot leaving A at an angle $0 < \theta < x$ hits side AB at angles $\pm\theta$, and sides BC and AC at angles $x \pm \theta$ as shown in Figure 12.

If it returns to A, then as before it must return at the same angle θ that it left. This is impossible for the same reason given above. ■

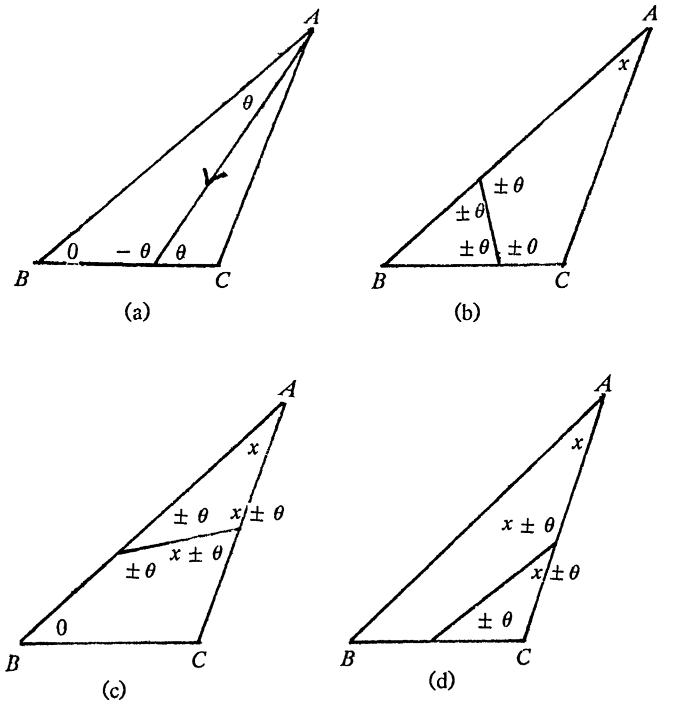

Figure 11

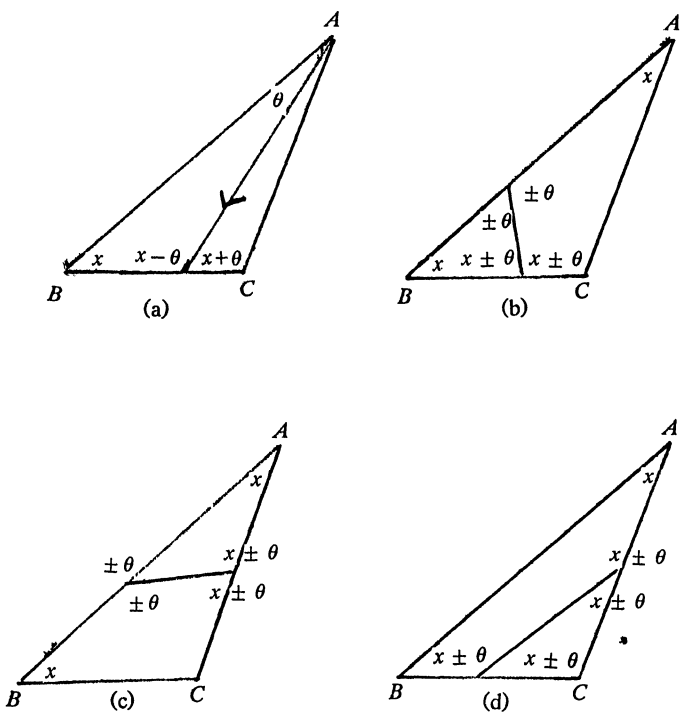

Figure 12

Example 5. On any symmetric pool table of the type shown below where x divides 90 and with angles B and C having size different from 180° (which guarantees that B and C remain vertices), there does not exist a pool shot from A to D.

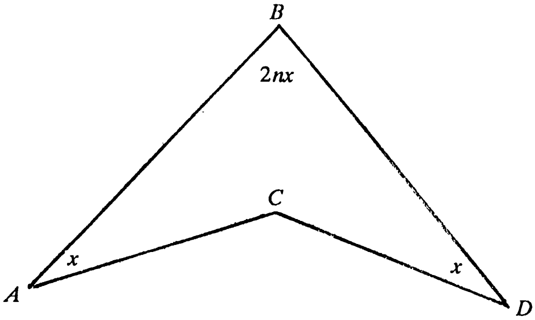

Figure 13

Proof: A pool shot from A to D would fold up to a pool shot from A to A in triangle ABC which is impossible. ■

This is an example of a quadrilateral pool table in which it is not possible to make a pool shot between two distinct boundary points.

Example 6. By the lemma, there do not exist any pool shots from A to A on the triangle ABC shown below with $m(< A) = 9°$ and $n = 8$.

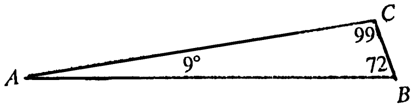

Figure 14

By taking mirror images of this triangle and following the usual rule that B and C must remain vertices, we can construct a pool table without right angles which does not have a pool shot from A_0 to A_1.

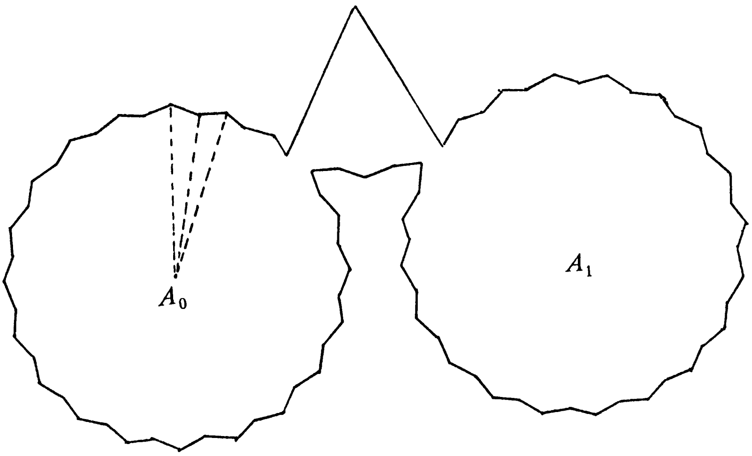

Figure 15

This example can be extended to construct pool tables with any finite number of pool shots that cannot be made.

5. GENERAL CONSTRUCTION THEOREM

Theorem 5.1. *Let G be a pool table built from a triangle ABC of the type shown in Figure* 10 *and which is constructed using only successive mirror images of this triangle. If G is constructed following the rule that every occurrence of B or C is a vertex, then there does not exist a pool shot between any two points labelled A.*

Proof: The pool shot is impossible by Lemma 4.1, since a path between any two points labelled A corresponds to a pool shot from A to A in triangle ABC. ■

This is the general construction result used to form the various polygonal tables.

6. THREE DIMENSIONAL EXAMPLES. In three space, reflection occurs only at points which have tangent planes, and rays bounce off the surface such that the angle between the incoming ray and the normal equals the angle between the outgoing ray and the normal. The incoming ray, the outgoing ray and the normal must be coplanar. Any ray which hits a vertex or an edge does not reflect.

If P and Q are parallel planes, it is known that a parallel projection between P and Q will preserve angles and hence reflections. This is not so if the planes are not parallel.

However, if a reflection occurs off a face whose normal $\vec{n}$ is either perpendicular or parallel to a plane P, and Q is the plane formed by the two reflecting rays, then an orthogonal projection taking Q to P will preserve the reflection. (If $\vec{n}$ is perpendicular to P, the projected image is a straight line).

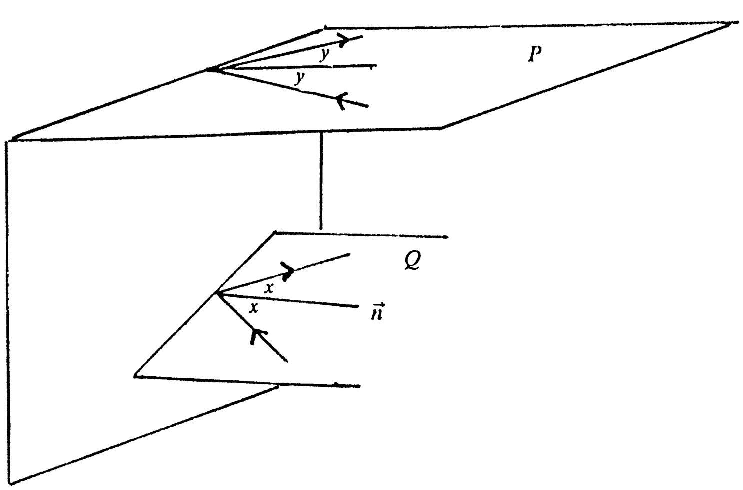

Figure 16

This means that if we form a cylinder on any of the polygonal rooms R already constructed to form a polytopal room $R \times I$, then a pool shot in the polytopal room would project orthogonally to a pool shot in R.

Example 7. The following polytopal room is not illuminable from every point. In particular there does not exist a pool shot from any point on $A_0 \times I$ to any point on $A_1 \times I$.

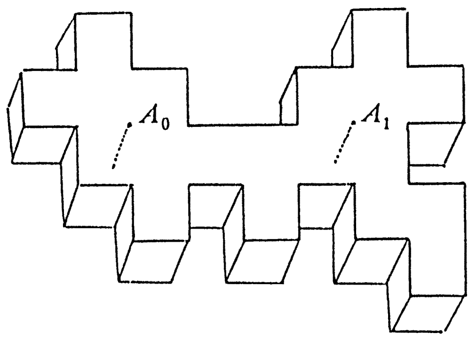

Figure 17

Proof: This would correspond to a pool shot from A_0 to A_1 in the room of Figure 9, which is impossible.

7. NON-CYLINDRICAL EXAMPLES

Lemma 7.1. *Given a cube with one corner labelled A, there do not exist any pool shots from A coming back to A.*

Proof: Let us take a lattice of mirror images of the cube with A at the origin and the vertices having integer coordinates. The A's appear at even coordinates $(2m, 2n, 2p)$ and every other vertex has at least one odd coordinate. As before a pool shot from A to A in the original cube corresponds to a straight line segment from $A(0, 0, 0)$ to $A(2m, 2n, 2p)$ which must pass through a vertex other than A. It follows that the pool shot is impossible. ∎

By virtually the same lattice argument, there does not exist a pool shot from A to any point on any edge attached to A. Alternately, we can use the projection argument with a given cube *ABCDEFGH*. If there were a reflecting path from A to X where X is on AH say, then using a suitable orthogonal projection, this path projects onto another path from A to A in the square *ABCD* which is impossible.

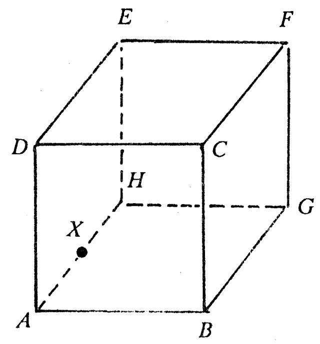

Figure 18

Example 8. It is now easy to construct a polytopal room with two interior points which are not illuminable from each other. We need only take mirror images of the cube in Figure 18 following the rule that any edge not attached to say vertex A must remain an edge. The following example was constructed in this way.

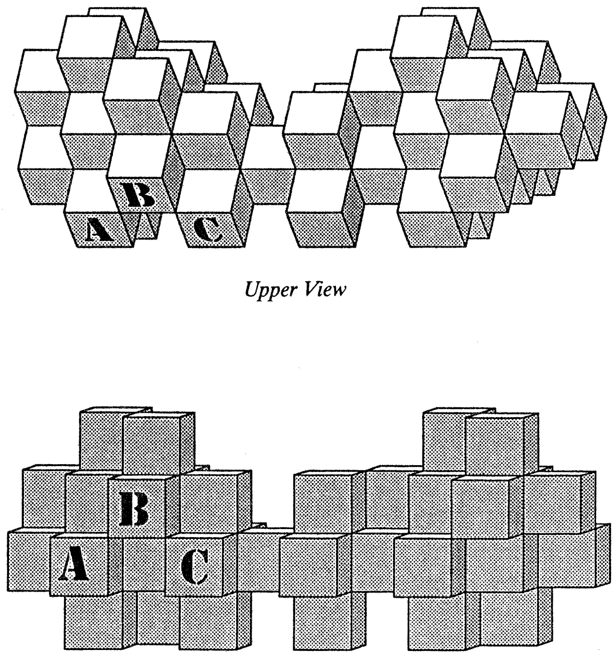

Figure 19

Proof: The above rule guarantees that a pool shot leaving A and hitting another point labelled A must pass through the interior of one of the cubes surrounding it. By the comment to Lemma 7.1 and the unfolding argument, it could never hit the second A. ∎

More generally, we can use cylindrical triangular building blocks by making use of the following lemma.

Lemma 7.2. *Let T be a cylinder built on a triangle of the type shown in Figure* 10, *then there does not exist a pool shot between any two points on edge AD.*

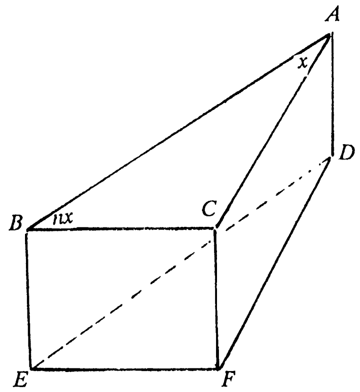

Figure 20

Proof: If we orthogonally project the cylinder onto triangle ABC, then a path in the cylinder between two points on AD corresponds to a path from A to A in triangle ABC which is impossible. ■

We immediately obtain the following result.

Three Dimensional Construction Theorem 7.3. *Let G be a polytopal room built from cylindrical triangles T of the type shown in Figure* 20 *and which is constructed using only successive mirror images of T. If G is constructed following the rule that every occurrence of an edge different from AD remains an edge, then*

(a) *there does not exist a pool shot between any two points labelled A,*
(b) *there does not exist a pool shot between A and any D not immediately attached to A,*
(c) *there does not exist a pool shot between A and any interior point of a segment labelled AD which is not attached to the original A, and*
(d) *there does not exist a pool shot between any interior point X of AD and any interior point Y of a different segment labelled AD.*

Proof: The above rule guarantees that a pool shot leaving X and hitting Y must pass through the interior of one of the cylindrical triangles surrounding X. By Lemma 7.2 and the unfolding argument, it can never hit Y. Similar proofs can be given for the other statements. ■

By symmetry, the result also holds if we interchange A and D.

8. A NON-POLYTOPAL EXAMPLE. We give a three dimensional example which is not polytopal and non-cylindrical but is a simple solid of revolution.

Example 9. If we take any symmetric quadrilateral of the type shown in Figure 13 and rotate it about the axis AD, then there does not exist a pool shot from vertex A to D, or A back to A or D back to D.

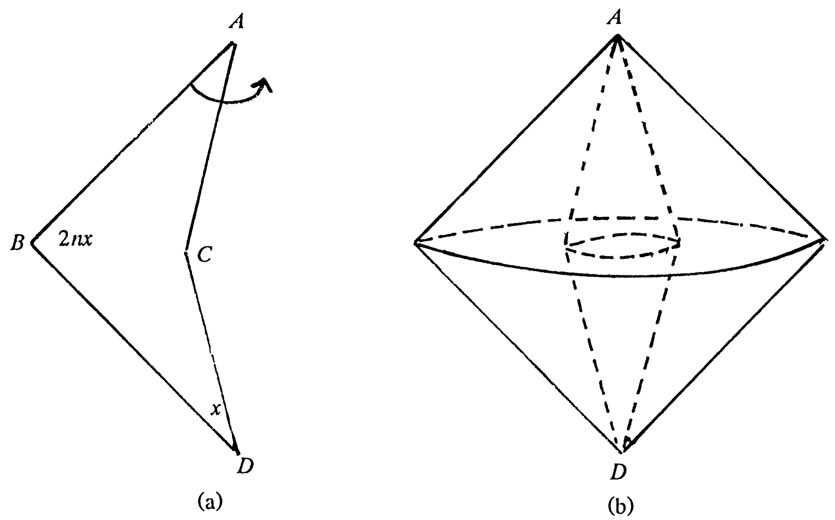

Figure 21

Proof: Rays emitted from A (say) stay in the plane determined by AD and the ray and hence correspond to a pool shot in the planar table of Figure 21(a). ∎

It is possible that such a device would have physical applications, in acoustics or thermodynamics. Rays generated at A or D never reach the opposite vertex and never come back on themselves.

9. HISTORY. The illumination problem has been tentatively traced back to Ernst Straus in the early 1950's. Two questions were posed.

(1) Is a polygonal region illuminable from every point in the region? and
(2) Is a polygonal region illuminable from at least one point in the region?

Penrose and Penrose, 1958 **[5]**, in an entertaining article constructed a smooth region based on properties of the ellipse which is not illuminable from various points. Other authors in written communications, then modified this example to construct a smooth region not illuminable from any point. Thus, both questions were answered negatively for smooth regions. Rauch, 1978 **[6]**, gave an example of a smooth region not illuminable from any finite set of points.

On the other hand, the solution for polygonal regions was not forthcoming and no significant progress appeared in the literature. The nature of these problems, being easily stated and easily understood together with their apparent intractability had an obvious appeal. Thus, they started appearing on various lists of unsolved problems. Klee's paper, 1969 **[1]**, seems to be the first published version. This was followed by a survey article of Klee and Guy, 1971 **[7]**. Klee again, 1979 **[2]**, in an excellent exposition provided a list of the ten most appealing unsolved problems in plane geometry of which the illumination problem was his fifth. Recently, in 1991, two texts **[3]** and **[4]** of unsolved problems have been published both of which give excellent discussions of the two illumination problems.

I think that Klee **[2]** best captured the spirit of these problems in his 1979 paper subtitled, "A collection of simply stated problems that deserve equally simple solutions".

He eloquently says, "In considering the problems of this paper, it is natural to wonder whether anyone has a reasonable chance of solving them. I can't answer that, except to say that problems of this sort are great equalizers among mathematicians, for solutions usually depend on clever ideas rather then extensive knowledge or development of complicated mathematical machinery."

The second problem is still open.

REFERENCES

1. V. Klee, Is every polygonal region illuminable from some point? *Amer. Math. Monthly* 76 (1969), 180.
2. V. Klee, Some unsolved problems in plane geometry, *Math. Mag.* 52 (1979), 131–145.
3. V. Klee and S. Wagon, *Old and New Unsolved Problems in Plane Geometry and Number Theory*, The Math. Assoc. of America, 1991.
4. H. T. Croft, K. J. Falconer, R. K. Guy, *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991.
5. L. Penrose and R. Penrose, Puzzles for Christmas, *New Scientist*, 25 December (1958), 1580–1581, 1597.
6. J. Rauch, Illuminations of bounded domains, *Amer. Math. Monthly* 85 (1978), 359–361.
7. R. Guy and V. Klee, Monthly research problems, *Amer. Math. Monthly* 78 (1971), 1114.

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Neither you nor I nor anybody else knows what makes a mathematician tick. It is not a question of cleverness. I know many mathematicians who are far abler than I am, but they have not been so lucky. An illustration may be given by considering two miners. One may be an expert geologist, but he does not find the golden nuggets that the ignorant miner does.

—*L. J. Mordell*

Mathematical Circles Adieu. Howard W. Eves,
Boston: Prindle, Weber and Schmidt, 1977.