α Voronoi Diagrams

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Outline

Voronoi Diagrams

- A Voronoi diagram is a partition of a plane into regions closest to each of a given set of objects
- In a Euclidean Voronoi diagram, the distance measure is the l_2-norm between any two points

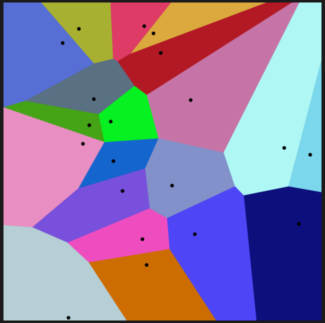

Figure: A Euclidean Voronoi Diagram

α Voronoi Diagrams

- In a standard Voronoi diagram, a point is said to belong to a region if and only if it is closest to the site forming the region
- What if we say that a point belongs to a region if and only if the ratio of its distances between two sites is less than some scalar $\alpha \in \left(0, \frac{1}{2}\right)$?
- We call such a partition of space into α regions as the α Voronoi diagram of a given set of sites

α Voronoi Diagrams

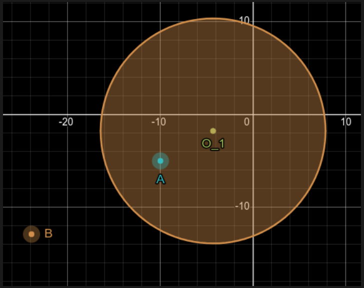

Figure: α-Boundary

- In Figure 2, we have two points A and B. We say that a point P belongs to the α-region of A if and only if

$$d(P, A) < \alpha \cdot d(P, B) \tag{1}$$

- Between any two points, for a given α and a fixed ordering, the α-region is a circle.
- The boundary of the region is called the α-boundary of the α-region

Formal Definitions

Definition (α-Closeness)

A point P is said to be α-closer to A than to B if and only if $d(P, A) < \alpha \cdot d(P, B)$ where $d(U, V)$ represents some measure of distance (l_2-norm unless specified otherwise) between two points U, V

Definition (α-Region)

The α-region of a point A with respect to another point B is defined to be the set of points S such that $P \in S$ is α-closer to A than to B

Two Variants

- When we want to construct the α Voronoi diagram of a set of points, a few choices have to be made
- Unlike a standard Voronoi diagram, the definition of an α Voronoi diagram makes it asymmetric. That is, the ordering of the points matters as the definition given in Equation 1 is asymmetric
- One can choose to fix α in one of two ways :-
  - By fixing one α for the entire diagram
  - By fixing a weight for each point and letting α vary between any two points

Outline

Fixed α Variant

- In this particular variant, we take a set of points, order them cyclically to form a polygon and construct the α Voronoi diagram for a specific α that is fixed for all points
- The ordering of the points in Figure 3 is A, B, C, D, E. The circles represent the α-region collection of the points

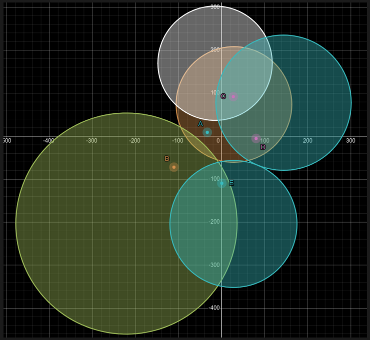

Figure: Fixed α Variant

Challenges Associated with the Fixed Variant

- The ordering of the points to solve Equation 1 is not trivial. Certain orders give consistent solutions, certain orderings are inconsistent
- The simplest way to ensure a consistent solution is by choosing a cyclic order of points. That is, using Equation 1 for A_i, A_{i+1} where A_i is a site. Recall that using Equation 1 on A_i, A_{i+1} is not the same as using it on A_{i+1}, A_i
- Even with a cyclic ordering, the part of the plane that doesn't belong to any of the α-region collection is inconsistent, as one can't find a unique α-closest site. However, this is not of much concern for sufficiently large α as all the points inside the hull are consistent in such cases

Properties

- The α-regions are all contained in a combination of nothing but circular arcs
- There can be no point that lies in the α-region of every pair of points A_i, A_{i+1}
- The radius of an α-circle depends only on the distance between two points and α itself
- Centres of the α-circles lie on the extensions of the line segments joining two adjacent points

Properties

- For a general n-sided polygon with vertices A_i, we denote the centre of the α-circle corresponding to A_i, A_{i+1} with O_i. In such cases,

$$(1-2\alpha)^2 \sum_{i=1}^{n} (O_i O_{i+1})^2 = \sum_{i=1}^{n} (A_i A_{i+1})^2 - \alpha(1-\alpha) \sum_{i=1}^{n} (A_i A_{i+2})^2$$

- We also have the following property

$$\frac{O_i A_i}{O_i A_{i+1}} = \left(\frac{\alpha}{1-\alpha}\right)^2$$

Algorithm to Determine Fixed α Voronoi diagram

- For this variant, the entire diagram can be found out in $\mathcal{O}(n)$
- The procedure is very simple. Given points A_i, A_{i+1}, O_i can be found using

$$O(A_i, A_{i+1}) = \frac{(1-\alpha)^2 A_i - \alpha^2 A_{i+1}}{1-2\alpha} \tag{2}$$

- Once O_i is found using Equation 2, the radius of the circle can be found out as

$$r = \frac{(A_i A_{i+1}) \cdot \alpha}{1-\alpha^2} \tag{3}$$

Outline

Pointwise α Variant

- In this variant, we assign every site a weight. Let w_i be the weight of the site A_i
- We then say that a point a belongs to site to the α-region of A_i over A_{i+1} if and only if

$$\frac{d(P, A_i)}{d(P, A_{i+1})} < \frac{w_i}{w_{i+1}} \tag{4}$$

- Note that for this variant, $d(U, V)$ can be any exponent of the Euclidean distance between U, V for the general idea to work. We will stick to l_2-norm for the rest of the presentation

Empirical Investigations

- We use Monte-Carlo simulations to estimate what the α-region will look like for this variant. The diagram for various grid resolutions is given in Figure 4
- All the region boundaries are circular arcs

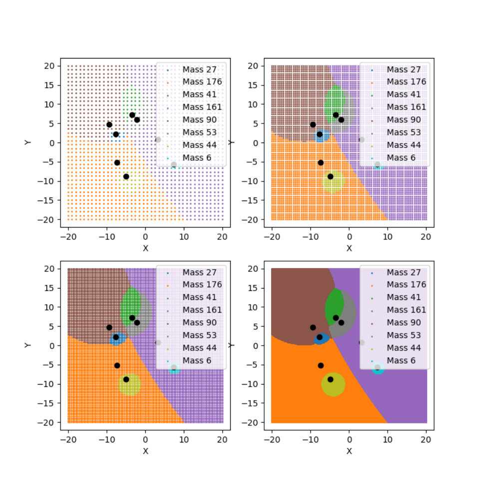

Figure: α-region Collection for this Variant

Empirical Investigations

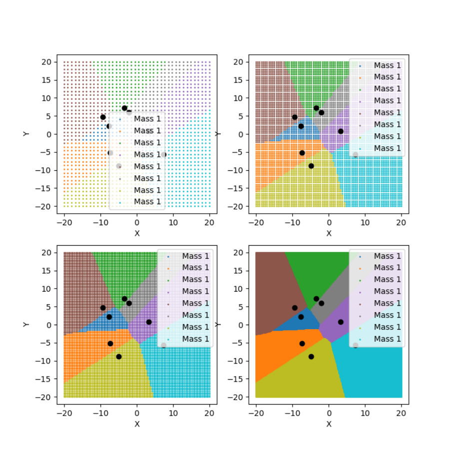

Figure: Voronoi Diagram from the α Voronoi diagram

- For the same sites, if we set their weights to be the same, Equation 4 reduces to the definition of the standard Voronoi diagram
- This is exactly what our Monte-Carlo simulation gives in Figure 5

Observations

- We can observe between Figures 4 and 5 that the α-boundary is pushed away from the site with the higher weight towards the point with the lower weight
- As a consequence, we can say that the α-region of the site with the smallest weight is a subset of its standard Voronoi region

Algorithm to Construct α Voronoi diagram

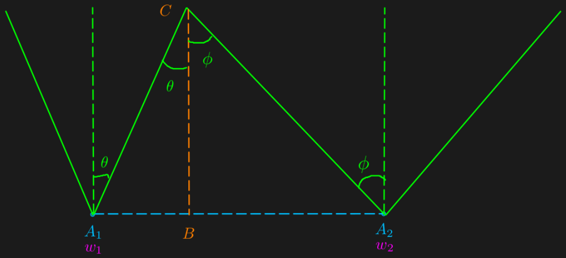

Figure: Plane Sweep Algorithm

- We want to pick cones with angles θ and ϕ such that $A_1B/A_2B = w_1/w_2$. The green lines represent the conical surface
- We have $A_1B/A_2B = tan(\theta)/tan(\phi)$

Algorithm to Construct α Voronoi diagram

- To ensure this, we can pick $\theta = tan^{-1}(w_1)$ and $\phi = tan^{-1}(w_2)$
- But doing so would require the sweeping plane to keep changing angles, which isn't convenient
- Instead we take $\theta_i = tan^{-1}(w_i/w_{max})$ to ensure that only the site with the highest weight gets a 45° cone angle
- We then sweep a plane inclined at 45° to the plane containing the sites

Algorithm to Construct α Voronoi diagram

- Store the intersection of the sweeping plane with the various cones
- Only the cone with the 45° sweep angle will have an unbounded parabolic front, the other cones will form bounded elliptic fronts as the plane would dissect the cone due to its higher angle
- Once the sweeping plane propagates through the entire plane, the projection of the intersections of the plane with the cones will given the Pointwise variant α Voronoi diagram
- The algorithm is similar to Fortune's algorithm, but we cleverly alter the cone angles of every site to give the α Voronoi diagram , yielding a worst case complexity of $\mathcal{O}(nlogn)$

Outline

Applications

- The α Voronoi diagram can be used in situations where not only the distance to the site matters, but also a property about the site itself is important
- A great application of the α Voronoi diagram would be in gravitation and electrostatics, where the field intensity at a point doesn't depend only on the distance, but also the mass/charge of the particle (weight of the site)
- This model effortlessly extends to any radial field variation, transcending into higher dimensions with ease

Application : Communication Towers

- In communication towers, the distance of the device from the tower is crucial, but so is the strength of the signal sent from the tower
- The strength decays as a power law with distance
- Moreover, due to the elevation of the towers, 3 dimensional effects should also be considered, which can be done easily with this algorithm
- Several other problems, such as long-range particle-liquid interactions (Stoke's law) can also be modeled using an α Voronoi diagram

Thank You