O. P. LOSSERS (Eindhoven University of Technology), ALLEN R. MILLER (George Washington University, Washington, DC), HELMUT PRODINGER (Technische Universität, Wien), MICHAEL RENARDY (Virginia Tech), ROLF RICHBERG (RWTH, Aachen, Germany), H.-J. SEIFFERT (Berlin), J. H. VAN LINT (Eindhoven University of Technology, the Netherlands), *and the proposers.*

An Impossible Pool Shot?

Problem 94-3*, *by* G. TOKARKSY (University of Alberta).

Figure 1 below denotes a nonconvex pool table $ABCD$, which is symmetric with respect to BC and with $\angle A = \angle D = 10°$ and $\angle B = 120°$. Does there exist a pool shot from A to D? If yes (no), are there other symmetric nonconvex quadrilaterals for which there aren't (are)?

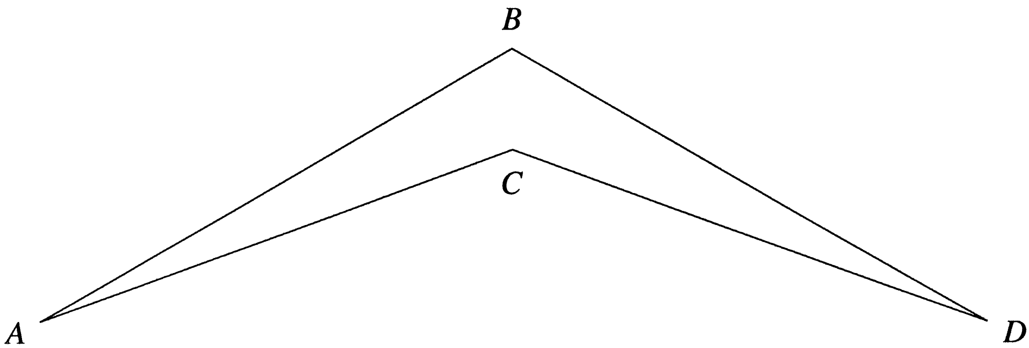

FIG. 1

Editorial note. The labeling of the vertices has been changed from that of the originally published version for consistency with the following solution.

Solution by the proposer.

We consider pool shots where reflection occurs only at the sides and such that the angle of incidence equals the angle of reflection. Pool shots which hit a vertex after leaving A are considered to end there.

THEOREM. *Let x divide $90°$. Then the triangular pool table ABC as shown in Fig. 2, where $\angle A = x$ and $\angle B = nx$, with n a positive integer, does not have a pool shot of nonzero length from A to A.*

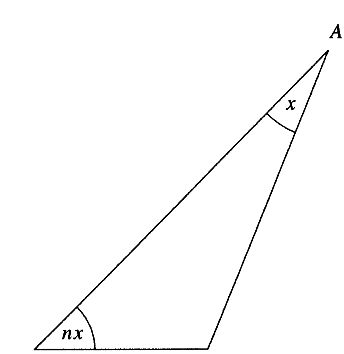

FIG. 2

Proof. We measure all angles (mod $2x$).

Case (i): n is even. Let $0 < \theta < x$ specify the direction of a pool shot leaving A as shown in Fig. 3. In this case, $\angle B \equiv 0 \pmod{2x}$ and $\angle C \equiv x \pmod{2x}$, and it is easy to establish inductively that each time the shot bounces off side AB or BC, it does so at an angle congruent to $\pm\theta$, and each time it bounces off side AC, it does so at an angle congruent to $x \pm \theta$. Hence, if the shot comes back to A, then it must re-enter at the angle $\pm\theta \pmod{2x}$. But $-\theta$ is impossible since $0 < \theta < x$. Hence the shot must re-enter at the same angle θ as it left. This can only happen if the pool shot hits one of the sides at 90°. But then $\pm\theta \equiv 90 \pmod{2x}$, which implies that $\pm\theta \equiv 0 \pmod{x}$ (since x divides 90°) or $x \pm \theta \equiv 90 \pmod{2x}$, which again implies that $\pm\theta \equiv 0 \pmod{x}$. This is impossible since $0 < \theta < x$.

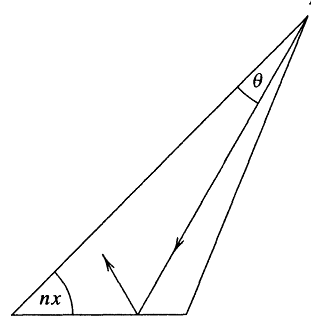

FIG. 3

Case (ii): n is odd. In this case $\angle B \equiv x \pmod{2x}$ and $\angle C \equiv 0 \pmod{2x}$. The analysis is similiar to that in Case (i). This time a pool shot leaving A at an angle $0 < \theta < x$ hits side AB at angles congruent to $\pm\theta$ and hits sides BC, AC at angles congruent to $x \pm \theta$. If it returns to A, it must return at the same angle θ at which it left, and, as before, this is impossible. □

COROLLARY. *On any symmetric pool table of the type shown in Fig.* 4 *where x divides* 90° ($\angle B, \angle C \neq 180°$), *there does not exist a pool shot from A to D.*

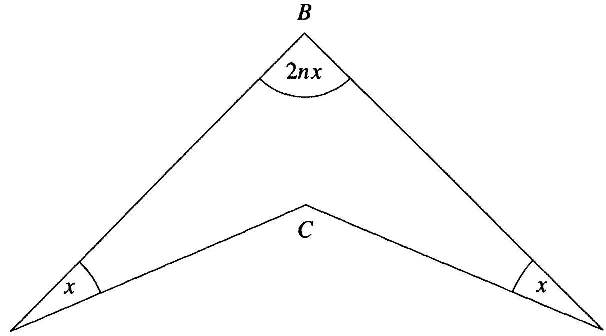

FIG. 4

Proof. By reflecting in BC, a pool shot from A to D would correspond to a pool shot of nonzero length from A to A in triangle ABC, which is impossible. □

There is a large group of symmetric quadrilateral pool tables for which the shot from A to D is possible. Consider any table of the type shown in Fig. 5, where $0 < y < 60°$ and $x > 90° - 3y/2$.

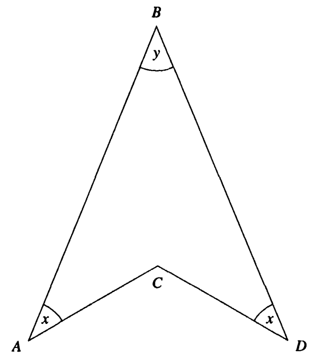

FIG. 5

If we shoot from A at an angle $90° - 3y/2$ measured clockwise from AB, then the shot will hit D after two bounces. This is illustrated below (Fig. 6) for the case of $x = y = 45°$, which also shows that $2nx$ cannot be replaced by nx in the corollary.

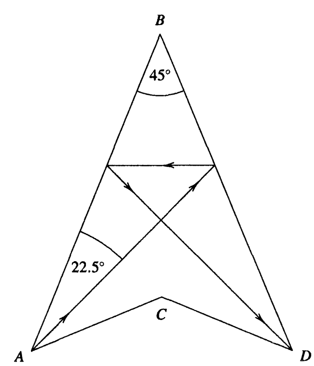

FIG. 6

Also, the shot is directly possible on the table in Fig. 5 if $x > 90° - y/2$, since then the table is convex.

I have used similar methods to settle an old illumination problem in the negative [3]. (See [1] and [2] for background on this problem.)

REFERENCES

[1] V. KLEE, *Is every polygonal region illuminable from some point?*, Amer. Math. Monthly, 76 (1969), p. 180.
[2] T. CROFT, K. J. FALCONER, AND R. GUY, *Unsolved Problems in Geometry*, Springer-Verlag, New York, 1991.
[3] G. TOKARSKY, *Polygonal rooms not illuminable from every point*, Amer. Math. Monthly, to appear.

Also solved by F. G. BOESE (Munich, Germany) *and* C. C. GROSJEAN (University of Ghent, Belgium).