Algorithms in Computational Geometry
Assignment - 1

Assignment posed on: 02/08/2024 **Assignment submitted by: 07/08/2024**

Overview

- **Problems focused:**
  - On guarding the polygons with holes
  - Art Gallery Problem with Rook and Queen sight
  - Guards whose Range of Vision is 180°
  - A chromatic art gallery problem
  - The Dispersive Art Gallery Problem
- **Solved by:**
  - Karthikeya - CS22B026

General Assumptions

- **Polygon Type:** The polygon is simple, planar, has no holes and not curved.
- **Vertex Guards:** The guards are vertices and have a 360 degree field of view.
- **Stationary:** The guards are not allowed to move.
- **Infinite Visibility:** The guards have infinite visibility.

On guarding the polygons with holes[1]

Problem Statement For a given polygon $\mathcal{P}$ with n vertices and h holes, $\lfloor \frac{n+h}{3} \rfloor$ vertex guards are always sufficient to guard the vertices of $\mathcal{P}$ and also the entire boundary.

Problem Specific Assumptions

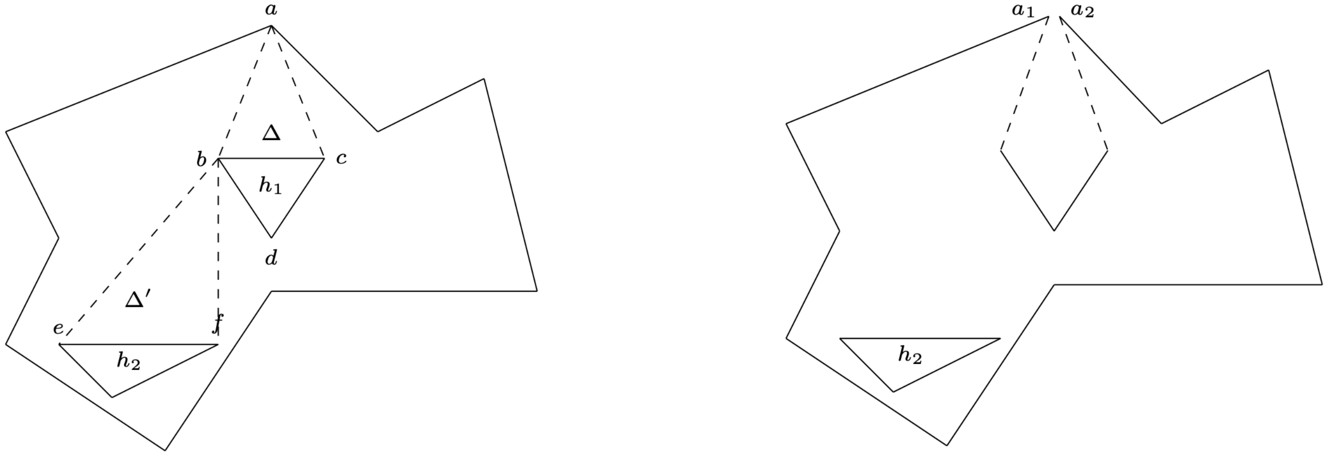
- **Existence of Holes:** The polygon $\mathcal{P}$ has h holes.

State of the solution

- The problem is solved for showing the sufficiency of $\lfloor \frac{n+h}{3} \rfloor$ vertex guards for guarding the vertices of $\mathcal{P}$ and the entire boundary.
- The Shermer's conjecture still remains unsolved.

Outline of the solution

- They have used the idea of special triangulation to prove the results.
- They have shown that every polygon with holes has a special triangulation(See Fig 1a).
- They have then used induction(See Fig 1b) to show that $\lfloor \frac{n+h}{3} \rfloor$ vertex guards are sufficient to guard the vertices.



(a) Special Triangulation (b) Induction Step

Figure 1: Images of Special Triangulation and Induction Step

Scope of further research

- It would be interesting to see if the condition shown was necessary.
- It would be interesting to see if the Shermer's conjecture can be solved using the same approach.

Art Gallery Problem with Rook and Queen sight[2]

Problem Statement How many chess rooks or queens does it take to guard all the squares of a given polyomino?

Problem Specific Assumptions

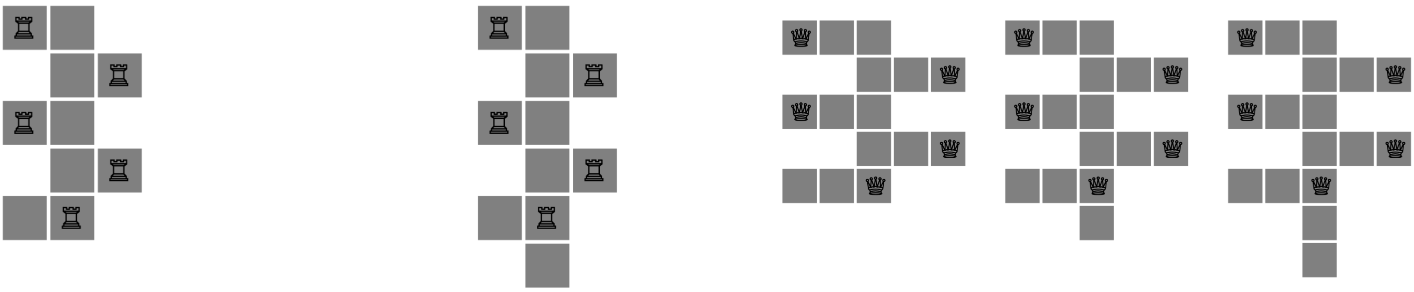
- **Queen and Rook Sight:** The guards have sight similar to that of a queen and a rook in chess.
- **Polygon Type:** The polygons considered are polyominoes.
- **Polyomino Type:** The polyominoes are simple.

State of the solution

- It is shown that for a given polyomino with n squares $\lfloor \frac{n}{2} \rfloor$ rooks or $\lfloor \frac{n}{3} \rfloor$ queens are sufficient and sometimes necessary.
- Finding the minimum number of rooks or queens required to guard a polyomino is NP-hard.
- The above results also apply for d dimensional case.

Outline of the solution

- They have first shown that $\lfloor \frac{n}{2} \rfloor$ rooks are sometimes necessary for guarding a polyomino(See Fig 2a).
- They have shown that $\lfloor \frac{n}{3} \rfloor$ queens are sometimes necessary for guarding a polyomino(See Fig 2b).
- They then proved that these are sufficient for guarding the polyomino and finding the optimal number is NP-hard.



(a) Maximum Rook Polyomino

(b) Maximum Queen Polyomino

Figure 2: Polyominoes requiring maximum number of rooks and queens

Scope of further research

- What are the kind of polyominoes that require the maximum number of rooks or queens?
- Would be interesting to see how many polyominoes require the maximum number of rooks or queens.
- Also there are 5 interesting open questions posed at the end of the paper.

Guards whose Range of Vision is 180°[3]

Problem Statement In any polygonal art gallery of n sides it is possible to place $\lfloor \frac{n}{3} \rfloor$ point guards whose range of vision is 180° so that every interior point of the gallery can be seen by at least one guard.

Problem Specific Assumptions

- **180° Sight:** The guards have a range of vision of 180°.
- **Point Guards:** The guards can be placed anywhere in the polygon.

State of the solution

- The problem is solved, which also solved one of the open problems posed by Urrutia.
- The article also shows that $\lfloor \frac{n}{3} \rfloor$ is the best possible upper bound for the problem.

Outline of the solution

- The proof is constructive in nature and naturally gives and $O(n^2)$ algorithm to find the guards.
- The proof is done by induction on cuts on the dual graph of a triangulation of the polygon.
- They have done exhaustive case analysis for the proof and have shown the results.

Scope of further research

- It would be interesting to see if the problem can be solved for polygons with holes.
- It would be interesting to see how the bounds of the problem change for human vision angle.
- It would be interesting if there exists an algorithm with better complexity than $O(n^2)$

A chromatic art gallery problem[4]

Problem Statement Suppose that two members of a finite point guard set $\mathcal{S} \subset \mathcal{P}$ must be given different colors if their visible regions overlap. What is the minimum number of colors required to color any guard set (not necessarily a minimal guard set) of a polygon $\mathcal{P}$? We call this number, $\chi_G(\mathcal{P})$, the chromatic guard number of $\mathcal{P}$.

Problem Specific Assumptions

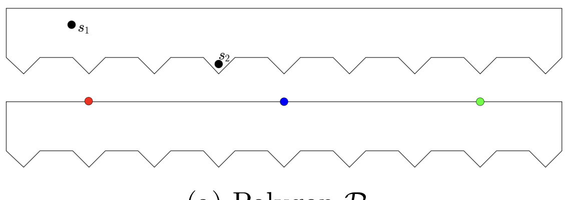
- **Point Guards:** The guards are any point in the polygon.

State of the solution

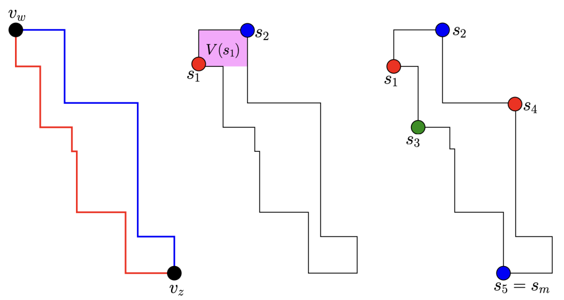
- It was proved that for any spiral polygon $\mathcal{P}_{spi} \leq 2$
- It was proved that for any staircase polygon $\mathcal{P}_{sta} \leq 3$
- It was proved for any $k \in \mathbb{Z}^+$ there exists $\mathcal{P}_k$ with $3k^2 + 2$ vertices such that $\chi_G(\mathcal{P}_k) \geq k$

Outline of the solution

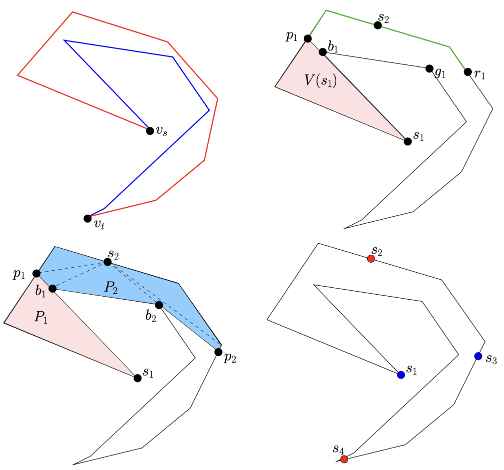
- Figure 3a shows the polygon $\mathcal{P}_k$ with $3k^2 + 2$ vertices such that $\chi_G(\mathcal{P}_k) \geq k$.
- The staircase polygons are solved by using the staircase polygons which are shown in Fig 3b.
- The spiral polygons are solved by using the reflex and convex subchains which are shown in Fig 3c.



(a) Polygon $\mathcal{P}_k$



(b) Staircase Polygon



(c) Spiral Polygon

Figure 3: Images of Key Polygons used in the paper

Scope of further research

- It would be interesting to see if the problem can be solved for all orthogonal polygons.
- Finding a bound better than $\lfloor \frac{n}{3} \rfloor$ for general polygons is an interesting problem to try.

The Dispersive Art Gallery Problem[5]

Problem Statement Given a polygon $\mathcal{P}$ and a real number l. Decide whether there exists a guard set $\mathcal{G}$ for $\mathcal{P}$ such that the pairwise geodesic distances between any two guards in $\mathcal{G}$ are at least l.

Problem Specific Assumptions

- **Polygon Type:** The polygons considered are polyominoes i.e., orthogonal polygons whose vertices have integer coordinates.
- **Simple Polyomino:** The polyominoes are simple i.e., they have no holes and are not curved.
- **Thin Polyomino:** The polyominoes are thin i.e., they don't have a 2×2 square as a subpolyomino.

State of the solution

- There are *(simple)* thin polyominoes such that every guard set has dispersion distance l at most 3.
- For every simple polyomino there exists a guard set that has dispersion distance at least 3.
- Deciding whether there exists a guard set with a dispersion distance of 5 for a given polyomino is NP-complete.

Outline of the solution

- Constructive algorithm to show the existence of guard set with dispersion distance 3 was given.
- Existence of a polyomino with dispersion distance ≤ 3 for every guard set was shown.(See Fig 4).
- They have used different gates and gadgets to construct the algorithm.

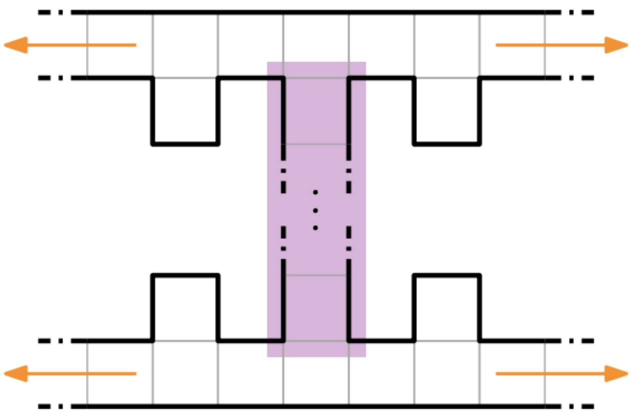


Figure 4: Polyomino with dispersion distance at most 3

Scope of further research

- It would be interesting to see if the problem is NP-complete for values of l other than 5.
- It would be interesting to see how the problem changes with different types of polygons.

References

[1] Sharareh Alipour. *On guarding polygons with holes*. 2021. arXiv: 2102.10317 [cs.CG]. URL: https://arxiv.org/abs/2102.10317.

[2] Hannah Alpert and Érika Roldán. "Art Gallery Problem with Rook Vision". In: *CoRR* abs/1810.10961 (2018). arXiv: 1810.10961. URL: http://arxiv.org/abs/1810.10961.

[3] Csaba D. Tóth. "Art gallery problem with guards whose range of vision is 180∘". In: *Computational Geometry* 17.3–4 (Dec. 2000), pp. 121–134. ISSN: 0925-7721. DOI: 10.1016/s0925-7721(00)00023-7. URL: http://dx.doi.org/10.1016/s0925-7721(00)00023-7.

[4] Lawrence Erickson and Steven M LaValle. *A chromatic art gallery problem*. 2010. URL: https://core.ac.uk/download/pdf/4825109.pdf.

[5] Christian Rieck and Christian Scheffer. *The Dispersive Art Gallery Problem*. 2023. arXiv: 2209.10291 [cs.CG]. URL: https://arxiv.org/abs/2209.10291.