Application for Contingent Guild Member

Mathematics Club
Center for Innovation

Instructions

- Join the Mathematics Club Aspiring Team WhatsApp group for further updates: Mathematics Club Aspiring Team.
- You are free to discuss any of your doubts with the Contingent Strategists.
- Once you are done with your application, convert it into a PDF and submit here. The file must be named as "⟨your name⟩_⟨roll number⟩_Contingent_Member_MC_2024-25".
- Attach your grade card (screenshot) along with the application in the same PDF.
- Even if you don't get the answer, explain your approach so that we can understand your thought process.
- Don't google solutions and submit answers ~~(We have intentionally set such questions)~~. You will get caught in the interview.
- Don't get intimidated by the questions and the length of the application. Even if you can't solve completely, it is fine.
- Clear and rigorous solutions are more likely to fetch more points.
- Focus more on the technical part of the app.

Contacts

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1 General Questions

- Why do you want to be part of Mathematics Contingent and what is your motivation for the same? Mention your strengths and weaknesses. Also, mention what makes you suitable for being a contingent member.
- What do you think you will be doing as a contingent member and How much time can you dedicate to the contingent (hours per week)?
- Have you attended any Mathematics Competitions/Olympiads before? If so, mention all of them. What are all the Mathematics Competitions do you think we can participate in a Contingent?
- Would becoming a contingent member leave you with less time for academics? If you do become a contingent member and face a situation where you find it difficult to cope with academics, what would you do?
- Mention all topics in Mathematics (broadly) that you feel you are strong in.
- Feel free to add anything related that you want us to know.

2 Warm Up

Let us get started with some basic questions. You should submit the answers to the following questions with solutions.

- Let S be the set of all possible symmetric matrices of order 3, formed by rearranging the elements of the matrix
$$\begin{bmatrix} 10 & 10 & 10 \\ 10 & 20 & 20 \\ 20 & 0 & 0 \end{bmatrix}$$
If $n(S)$ denotes the cardinality of the set S and $tr(B)$ denotes the trace of B (where B is a matrix), find the value of $10n(S) + \sum_{B \in S} tr(B)$.
- Consider the determinant $\triangle = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$, where $\begin{pmatrix} a_i \in \{-1, 1\} \\ b_i \in \{-2, 2\} \\ c_i \in \{-3, 3\} \end{pmatrix} \forall i \in \{1, 2, 3\}$.
Find the number of determinants ($\triangle$) whose value is *zero*.
- a, b, c, d, e, f, g, h are real numbers and
$$S = \{ac + bd, ae + bf, ag + bh, ce + df, cg + dh, eg + fh\}$$
Find the minimum number of non-negative numbers in set S.
- This question contains 3 sub-problems. In all three $f : A \to A, A = \{1, 2, \ldots, n\}$ and a_n denote number of functions f that satisfy the condition given in that sub-problem. Find a_5 in all the three sub-problems.
  - Sub-problem 1: $f(f(x)) = x$
    (Hint + Bonus) Give the relation between a_{n+1}, a_n, a_{n-1}.
  - Sub-problem 2: $f(f(x)) = f(x)$
  - Sub-problem 3: $f(f(f(x))) = f(x)$

  Even if you couldn't solve all three sub-problems, explain the ones that you did.

3 Core Boredom

The core team of the Mathematics Club got bored on a Sunday and decided to have a competition amongst themselves.
For those who don't know the current core team comprises of

- Aditya
- Atreya
- Ganesh
- Kailash
- Karthikeya
- Nikhil
- Pradyumnan

Karthikeya has assumed the role of a question maker and therefore doesn't participate in the competition.

- **Math Race**
  As the name suggests, this is a race where they are given a set of questions. The score for this round depends on the time taken to solve **all the questions** and is given by $score = \frac{1}{time}$. At the end of this round how many possible outcomes or rankings are possible? Solve for the two cases independently.

Case 1. Assume that no two of them finish solving at the same time.

Case 2. Assume that few of them finish solving all the questions at the same time. Include all possibilities of participants finishing in your total count.

- **Math Knock**
  In this round Karthikeya asked everyone to go through the solutions of all the others. Each of them are asked to vote for one of the remaining members based on how elegant they felt the solutions were. Each of them vote independently without the knowledge of other votes. Assuming that at most one of them doesn't get any votes, how many different votings are possible?

- **Math Loop**
  2C (a human) became interested in this fun competition and participated in this round. This is a strange round containing n interesting questions and weird rules.

  Rules

  – Either all of them win and get qualified the to next round or all of them lose and get eliminated.
  – *Losing Criteria:* If any two people have same outcome (both have solved or both couldn't) for two or more questions then entire core team loses.
  – *Winning Criteria:* If any two people have at most one outcome same with each other.

  - Decide whether all of them qualify to the next round or not for the cases where $n = 3, 4$. Present appropriate justification.
  - Will there be any change in the answer if 2C did not participate? Give justification.

- **Math Polygon**
  In this round Sreejaa also joined (2C continues participating). Karthikeya wanted to make it easier for the participants so he prepared 24 questions and placed the question numbers in a circle (in order 1, 2,. . . , 24, 1). The participants can choose one of the 24 numbers and start attempting them based on the below rules:

  Rules

  – No two people can choose the same question.

– No two chosen questions are seperated by 3 or 8 questions.

Find the number of ways in which they can choose the questions and attempt.

4 Calculus love

Aditya and Karthikeya are challenged by their friends. They have set some problems in Calculus. However, these guys are lazy and want your help in solving them, so that they can get back to enjoying their summer vacation peacefully.

- **The one with the Euler number**

$$\int \left(e^x \sqrt{\frac{4-x}{x}} + e^x \sqrt{4x - x^2} - e^x \sqrt{\frac{x}{4-x}} \right) dx$$

- **The one with the polynomials**

$$\int \left(\frac{x^6 - 5x^5 + 4x^4 - 3x^3 + 2x^2 + x + 1}{x^4 + 2x^2 + 4} \right) dx$$

- **The one with the function**
  Aditya and Karthikeya are given the description of a function f(x), and are asked to evaluate the following integral

$$\int_0^2 f(x)\, dx$$

  f(x) is a continuous function in $[0, 2]$ satisfying the equation:

$$f(x) + f(1 + x) = |2^x - 1| + |x - 1|$$

- **The one where the problem is more important than the caption**
  Consider the following function

$$f(x) = \frac{4x^2 + 1}{\sqrt{x^4 + x^2}} + \frac{2x^4 + 2x^2 + 3}{(x^2 + 1)\sqrt{x^4 + 2}}; x > 0$$

  If $f(x)$ attains its minimum value at $x = \alpha$ then find the value of α.

5 A Bit of LinAl

"Matrices act. They don't just sit there." ~ Gilbert Strang

- Atreya has 3 matrices P, Q, R which satisfy the following equations.

$$P^2 = Q^2 = R^2 \ \ \& \ \ Q^3 = PQR + 2I$$

  Atreya wants the value of the $PQP \ \ \& \ \ P^6$. Help him (solve the entire question for him).

- Let A be a real square matrix such that A^3 is a null matrix. Find the number of solutions of the following equations in X.

1. $X + AX + A^2X = A$
2. $X + AX + XA^2 = A$

- A real symmetric $n \times n$ matrix $A = (a_{ij})$ satisfies $|a_{ij} - n| \leq 1$ for every $1 \leq i, j \leq n$. Denote the largest eigen value of A by $\mu(A)$. Then find the minimum and maximum values of $\mu(A)$.

6 A bit of NT

"Mathematics is the queen of the sciences and number theory is the queen of mathematics." ∼ Carl Friedrich Gauss

- Find the number of positive integers a, b, c satisfying the following equation:

$$a^{2024} - (b!)^{2016} + 13(c!)^{2020} = 56^{2041}$$

- Evaluate:

$$\left[\frac{2024!}{2023! + 2022! + ... + 2! + 1!}\right]$$

where $[x]$ is the greatest integer lesser than or equal to x.

7 Let us switch roles

"In mathematics the art of proposing a question must be held of higher value than solving it" ∼ Georg Cantor

In your journey of solving this application you have tried problems of various difficulty levels. Have you ever wondered how people come up with such interesting questions? Now it's your turn to amaze us with the questions you set.

- Make 3 questions from any domain of your interest.
- You are free to take inspiration from existing questions in the internet. But do not directly copy questions from Olympiads or any other standard source of questions.
- Provide answers and solutions for the questions you make.
- In short, assume that you are preparing a problem set for the contingent and make the questions, i.e., you don't want them to Google the questions and come up with a solution.

8 Hungry for More Problems

This is a **bonus section**. Once you are done with the above problems, you can try these out. These problems are just an addition to your application. Try to focus more on the previous sections.

- Find the number of values of $\alpha \in [0, 1]$ satisfying

$$4\alpha(1 - \alpha) = \beta, \quad 4\beta(1 - \beta) = \gamma, \quad 4\gamma(1 - \gamma) = \frac{8\alpha}{9}$$

- Given that $f(x) = x^2 - 2x + 3$ & $p(x) = f(f(x)) - x$, find the roots of $p(x)$.
- a, b, c, k are positive real numbers, such that

$$(a+b)^2 + (a+b+4c)^2 \geq \frac{kabc}{a+b+c} \forall a, b, c > 0$$

  Find the maximum possible value of k.