Mathematics Club
Contingent Problem Set - 3

Challenge posed on: 28/06/2024 **Challenge conquered by: 05/07/2024**

1 Overview

- **Topics focused:**
  - Calculus
  - Determinants
  - Series
- **Challengers:**
  - Abhinav
  - Saandeep
  - Pratyaksh
- Difficulty level is as follows:
  - Cyan :- Easy to moderate
  - Blue :- Moderate to Hard
  - Red :- Hard to Very Hard
- Problems are not as scary as they look. You just need to find the correct key to open the lock.
- Happy solving :)

2 Problems

1. **Vandermonde app!** Show that for $x_1, x_2, \cdots, x_n \in \mathbb{Z}$,

$$\prod_{1\leq i<j\leq n} \frac{x_i - x_j}{i-j} \in \mathbb{Z}$$

Hint: Calculate the determinant

$$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}$$

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2. **Scary Differential Equation!** Find the general solution to the DE:

$$\sin^2(2x)\frac{d^2y}{dx^2} + [x\sin(2x) - 1]\sin(2x)\frac{dy}{dx} + [\sin^2(2x) + 2\cos(2x)]y - \sin^{-\frac{5}{2}} x \cos^{\frac{13}{2}} x = 0$$

3. **Surely You're Joking, Mr. Feynman!** (a) Evaluate the integral:

$$\int_0^1 x^{n-1}(\ln x)^3\, dx \quad (n > 2)$$

(b) Define I_1 and I_2 as follows:

$$I_1 := \int_3^\infty \frac{1}{x^2(9+x^2)}\,dx,$$
$$I_2 := \int_3^\infty \frac{\cos(4x)}{x^2(9+x^2)}\,dx$$

Calculate (i) $I_1 + I_2$ and (ii) $I_1 - I_2$. How will your answers change if the lower limit is 0 instead of 3?

(c) Evaluate:

$$\int_0^\infty \frac{1}{e^x} \ln\left(\frac{1+2024e^x}{1+1012e^x}\right) dx$$

Hint: Consider calculating

$$f(a,b) = \int_0^\infty \frac{1}{e^x} \ln\left(\frac{1+be^x}{1+ae^x}\right) dx$$

4. **Shall we combine probab and calculus?** Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, and c are uniformly distributed random variables in the interval $[0,1]$. Find the probability that the roots of the equation are real, given that the product of the roots is less than 0.5.

5. **Permutation Peaks Puzzle** Consider a random permutation of $1, 2, 3, \ldots, n$, where $n \geq 2$. Find the expected number of local maxima in such a permutation.

   A local maximum in a permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ occurs at position i if $\pi_{i-1} < \pi_i > \pi_{i+1}$, where $2 \leq i \leq n-1$. Additionally, π_1 is a local maximum if $\pi_1 > \pi_2$, and π_n is a local maximum if $\pi_{n-1} < \pi_n$.

6. **Scary Comparision!** Which of the two numbers is larger?

$$\int_0^{\int_0^1 e^{-x^2}dx} e^{x^2}\,dx \quad \text{or} \quad \int_0^{\int_0^1 e^{x^2}dx} e^{-x^2}\,dx$$

7. **MA1102 vibes!** Prove that

$$S = \sum_{k=0}^{\infty} \frac{1}{f_{2k+1}+1} = \psi - \frac{1}{2}$$

   where $\psi = \frac{\sqrt{5}+1}{2}$, is the golden ratio ; $f_n = f_{n-1} + f_{n-2}$, and $f_0 = 0,\ f_1 = 1$.

8. **Fun Determinant Game** Saandeep and Pratyaksh are playing a game. In this variant of tic-tac-toe, Pratyaksh starts by placing a 1 anywhere in an $n \times n$ grid. Saandeep then follows by placing a 0 in any of the remaining spaces, and they continue to alternate turns. The goal of the game is determined by the determinant of the resulting $n \times n$ matrix: if the determinant is 0, Saandeep wins; if not, Pratyaksh wins. Assuming both players use optimal strategies, who will win and how?

9. **Tired? Enjoy the easy ones :)** Evaluate the integrals (a) and (b) and evaluate the series given in (c) if it converges.

   (a)

$$\int_{-1}^{1} \left[\frac{x^7 + 10x^5 + 6x^3 + x^4 + 6x^2 + 2x + 2}{x^2+1} + \cosh x \ln(1+x) - \cosh x \ln(1-x)\right] dx$$

(b)

$$\int \prod_{k=0}^{\infty} \frac{1}{1+x^{2^{k+3}}}\, dx$$

(c)

$$\sum_{n=1}^{\infty} \frac{1}{(4n+3)(4n+5)}$$