Inter IIT Tech Fest Mathematical Competition 2023

Problems and Solutions

Problem 1

A real symmetric 2023×2023 matrix $A = (a_{ij})$ satisfies $|a_{ij} - 2023| \leq 1$ for every $1 \leq i, j \leq 2023$. Denote the largest eigenvalue of A by $\lambda(A)$. Find maximum and minumum value of $\lambda(A)$.

2018 South Korea USCM P5

Solution. Since, the matrix is real symmetric, we know that all of it's eigenvalues are real.

Now, let $\mathbf{1} = \frac{1}{\sqrt{2023}}[1\ 1\ \ldots 1] \in \mathbb{R}^{2023}$. The scaling is just so that $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. Thus,

$$2022 \times 2023 \leq \frac{\sum\limits_{1 \leq i,j \leq n} a_{ij}}{2023} = \langle A\mathbf{1}, \mathbf{1} \rangle \leq \|A\| = \lambda(A)$$

Thus, $\lambda A \geq 2023 \times 2022$. Now, for the other side, we observe that $\|Av\| \leq 2024 \times 2023 \|v\|$ since the sum of all entries in the matrix is at most 2024×2023.

Both of these bounds are achievable by setting the matrix as $2022J$ or $2024J$ respectively where J is the all-one matrix in $\mathbb{R}^{2023 \times 2023}$ as we have eigenvector $\mathbf{1}$. $\square$

Problem 2

Prove that for positive real numbers a, b, c such that $a+b+c=1$,

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1} \leq \sqrt{2-(a^2+b^2+c^2)}.$$

Serbian National Mathematical Olympiad 2017

Solution. Let $x_1, x_2, x_3 = \sqrt{a}, \sqrt{b}, \sqrt{c}$ respectively and $y_1 = \sqrt{a(2b+1)}, y_2 = \sqrt{b(2c+1)}, y_3 = \sqrt{c(2a+1)}$ then by Cauchy Schwarz inequality, we have

$$x_1y_1 + x_2y_2 + x_3y_3 \leq \sqrt{(x_1^2+x_2^2+x_3^2)(y_1^2+y_2^2+y_3^2)}$$

Thus, we get

$$a\sqrt{2b+1}+b\sqrt{2c+1}+c\sqrt{2a+1} \leq \sqrt{(a+b+c)(2ab+2bc+2ab+a+b+c)} = \sqrt{1+2(ab+bc+ac)}$$

But, $1+2(ab+bc+ac) = 1+(a+b+c)^2-a^2-b^2-c^2 = 2-a^2-b^2-c^2$ as desired! $\square$

Problem 3

1. Show that for each function $f : \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$, there exists a function $g : \mathbb{Q} \to \mathbb{R}$ with $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{Q}$.
2. Find a function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, for which there is no function $g : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) \leq g(x) + g(y)$ for all $x, y \in \mathbb{R}$.

IMC 2003

Solution. The main point of the problem is $\mathbb{Q}$ is countable and cardinality of $\mathbb{R}$ is more, in particular it is the same as $2^{\mathbb{N}}$.

We begin with part **a)**

Let $q_1, q_2, \ldots$ be an enumeration of the rationals so that each rational appears exactly once in this list. Also, define $S_i = \{q_1, \ldots q_i\}$.

Now, given $f(x, y)$, we define $g(q_1) = 0$ and recursively define $g(q_i) = \max(0, r_i)$ where r_i is the maximum value taken by f restriced to $S_i \times S_i$ for any $i \geq 2$. This function is clearly well defined and works. If $i \geq j$ then

$$f(q_i, q_j) \leq g(q_i) \leq g(q_i) + g(q_j)$$

The argument follows for the other direction as well. $\square$

Moving onto part **b)**.

We clearly cannot induct as before. Now, to construct such an f, observe that

$$|\mathbb{R}| = |2^{\mathbb{N}}| \leq |\mathbb{N}^{\mathbb{N}}| \leq |2^{\mathbb{N} \times \mathbb{N}}| = |2^{\mathbb{N}}| = |\mathbb{R}|$$

Thus, let $\sigma : \mathbb{R} \mapsto \mathbb{N}^{\mathbb{N}}$ be a bijection between $\mathbb{R}$ and sequences of naturals.
Now, define f as follows:

$$f(x, y) = \begin{cases} \sigma(x)(y) & ; \text{if } y \in \mathbb{N} \\ 0 & ; \text{otherwise} \end{cases}$$

where $\sigma(x)(y)$ is the yth entry of the sequence $\sigma(x)$.
We prove that this function works.

Now, given any $g : \mathbb{R} \mapsto \mathbb{R}$, define $h : \mathbb{N} \mapsto \mathbb{N}$ as $h(n) = \lceil g(n) + n \rceil$. Thus, $\sigma^{-1}(h) = r$ for some real r.

Now, if $m > g(r)$ is some natural then observe that

$$f(r, m) = h(r)(m) = \lceil g(m) + m \rceil > g(m) + g(r)$$

Thus, for any function $g : \mathbb{R} \mapsto \mathbb{R}$, we have found a point x, y such that $f(x, y) > g(x) + g(y)$! $\square$

Remark. g is a function that kind of determines how fast growing f is. Now, when our domain for f is countable then we just recurse and define a function f that grows faster. This is easier to see when we replace $\mathbb{Q}$ with $\mathbb{N}$. For reals, we notice that the set of growth rate of functions is at most the same as reals and thus we can index f's first coordinate to give arbitrarily fast growing functions. To think about this slightly more easily, try to keep x fixed and vary y.

Problem 4

Let $G = (V, E)$ be a simple graph which chromatic number k. Now, all the edges of G are coloured either red or blue. Prove that there exist a monochromatic tree with k vertices.

Miklos Schweitzer 2012

Solution. We proceed by contradiction. Let $R_1, \ldots R_m$ be the connected components of the graph restricted to red edges and $B_1, \ldots B_n$ be the connected components of the graph restricted to blue edges.

Now, we can assume that $|R_i|, |B_j| \le k-1$ as else we will have a k sized monochromatic connected component and any of its spanning trees will satisfy the problem conditions. We can also assume $m = n$ by possibly adding empty components and that $|R_i| = |B_j| = k-1$ by adding more edges and vertices to V. (This cannot decrease the chromatic number.)

We will now argue that if G is as above then it is $k-1$ colourable which would be a contradiction.

To show a $k-1$ colouring, we will prove the following claim:

Claim. If $R_1, \ldots R_m$ and $B_1, \ldots B_m$ are two partitions of $[m\ell]$ with $|R_i| = |B_j| = \ell$. There exists a way to color the elements of $[m\ell]$ with ℓ colors such that each R_i and B_i contains all ℓ-distinct colors.

Proof. Fixing $\ell = k-1$, we get the colouring as required. Thus, we can focus only on this claim.

We prove it via induction on ℓ. If $\ell = 1$, then we directly have a colouring by giving every element the same colour.

Now, for $\ell > 1$, say we have proved the result for all $1 \le \ell < t$ with $t \ge 2$.

Now, for $\ell = t$, create a bipartite graph on $R_1, \ldots R_m$ and $B_1, \ldots B_m$ with an edge between R_i and B_j if they share some vertex.

Observe that there is a perfect matching in this graph due to Hall's theorem. (j red components have jt different vertices and thus must intersect atleast t blue components.) Now, each edge in the matching corresponds to some element in V, give them all the same colour (they are non-adjacent vertices since if they were adjacent, then their edge would be either red or blue and thus in the same component.) We can now remove these vertices from each component and proceed by induction. Thus, we are done! $\square$

Remark. The above idea is a fairly standard application of Hall's marriage theorem.

Problem 5

A natural number n is called *uwu* if sum of all divisors of n is less than $2n$. Does there exist an infinite set M such that for all $a, b \in M$, $a + b$ is *uwu*.

Korea winter practice test 2018

Solution. Yes, there does exist such a set and we construct it recursively!

We call a set S *uwu* if $s_1 + s_2$ is *uwu* for all $s_1, s_2 \in S$. We also use $\sigma(n)$ for the sum of divisors of n.

Thus, we define a chain of *uwu* sets under inclusion. We begin by defining $M_1 = \{1\}$. Now, recursively define $M_i = M_{i-1} \cup m_i$ where $m_i = p_1 p_2 \dots p_N + 1$ where $p_1, \dots p_N$ are the N smallest primes. We now show that there is a choice of N that keeps M_i is *uwu*. We will pick N such that $N > m_1 \dots m_{i-1}$.

Since M_{i-1} is *uwu*, we just need to show that $m_i + m$ is *uwu* for any $m \in M_i$.

If $m = m_i$ then,

$$\frac{\sigma(2m_i)}{2m_i} = \frac{3}{2}\sigma(m_i) \leq \frac{3}{2} \cdot \prod_{p|m_i} \frac{p}{p-1} \leq \frac{3}{2} \cdot \left(1 + \frac{1}{p_N}\right)^N$$

If $m \neq m_i$ then

$$\frac{\sigma(m_i + m)}{m_i + m} = \frac{\sigma(1 + m + p_1 p_2 \cdots p_N)}{\sigma(1 + m + p_1 p_2 \cdots p_N)} \leq \frac{\sigma(1+m)}{1+m} \cdot \left(1 + \frac{1}{p_N}\right)^N$$

Now, pick $\epsilon < 4/3$ such that $(1+\epsilon) \cdot \frac{\sigma(1+m)}{1+m} < 2$ for all $m \in M_{i-1}$. This is possible since $1, m \in M_{i-1}$ and thus $\frac{\sigma(m+1)}{1+m} < 2$.

Thus, it is sufficient to show that $\left(1 + \frac{1}{p_N}\right)^N \leq 1 + \epsilon$ for some large enough N. This is true since $\frac{p_N}{N}$ tends to infinity and $\left(1 + \frac{1}{p_N}\right)^{N \cdot p_N / N} \leq e \implies \left(1 + \frac{1}{p_N}\right)^N \leq e^{\frac{N}{p_N}}$ and $1 + \epsilon >^{\frac{N}{p_N}}$ for large enough N.

Thus, we can define M_i as required and setting $M = \bigcup_{i \geq 1} M_i$ works!