Mathematics Club
Contingent Problem Set - 4

Challenge posed on: 05/07/2024 **Challenge conquered by: 12/07/2024**

1 Overview

- **Topics focused:**
  - Probability
  - Linear Algebra
  - Geometry
- **Challengers:**
  - Shriram
  - Nandha
  - Madhav

2 Problems

1. **Probably Geometry** Let ABC be a triangle and P be any point on BC. Let $\angle APB = \theta$, calculate the expected value $\mathbb{E}[\cos(\theta)]$.

2. **Determinant Dilemma** Solve these interesting determinants ($n \in \mathbb{N}$).

   (a) Let $A_n \in \mathcal{M}_n(\mathbb{N})$ with $a_{ij} = \max(i, j)$. Calculate $\det(A_n)$.

   (b) Let $A_n \in \mathcal{M}_n(\mathbb{N})$ with $a_{ij} = \sum_{k=1}^{n} k^{i+j}$. Calculate

   $$\lim_{n\to\infty} \frac{(\det(A_n))^{\frac{1}{n^2}}}{n}$$

   (c) Let $A_n \in \mathcal{M}_n(\mathbb{N})$ with a_{ij} = number of common divisors of i and j. Calculate $\det(A_n)$.

3. **May the ODDS be with you :)** During vacation, Shriram got bored. So, he decided to play with n dice, say, $D_1, D_2, \ldots, D_n$, that is each die is numbered from 1 to n. He throws each of the dice twice (in order from 1 to n). Find the probability that the total sum obtained from the $2n$ throws is odd, given that the probability for the k^{th} die D_k to show an odd number is

   $$\frac{\sqrt{\dfrac{2^{k+1}-2}{2^{k+1}-1}}+1}{2}$$

4. **Escape the Matrix** Consider $A \in \mathcal{M}_{2020}(\mathbb{C})$ such that

   $$(1)\begin{cases} A + A^{\times} = I_{2020}, \\ A \cdot A^{\times} = I_{2020}, \end{cases}$$

   where $A^{\times}$ is the adjugate matrix of A, i.e., the matrix whose elements are $a_{ij} = (-1)^{i+j} d_{ji}$, where d_{ji} is the determinant obtained from A, eliminating the line j and the column i. Find the maximum number of matrices verifying (1) such that any two of them are not similar.

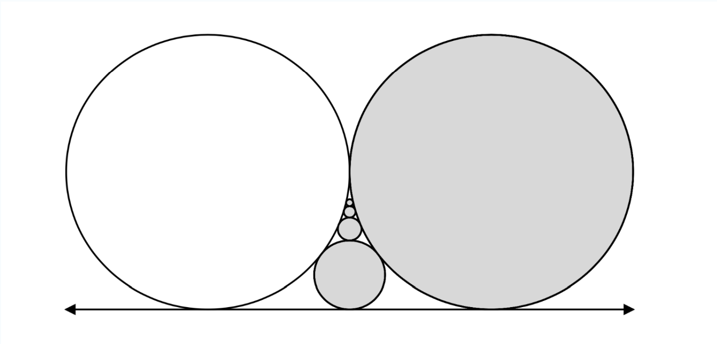
5. **What do you expect-double counting?** The Government has assigned 500 professors across the country to work with 100 different projects, with each project involving 50 professors. Prove that there exist two projects with at least 5 professors in common.

6. **Can you bound?** Consider the following determinant D_n

$$\begin{bmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{bmatrix}$$

Note that this is a $(n-1)\times(n-1)$ determinant. Is the set $\frac{D_n}{n!}$ bounded ?

7. **Circles all the way down** Two mutually tangent circles of radius 1 lie on a common tangent line as shown. The circle on the left is colored white and the circle on the right is colored gray. A third, smaller circle, is tangent to both of the larger circles and the line, and it is also colored gray. An infinite sequence of gray circles are inserted as follows: Each subsequent circle is tangent to the preceding circle, to the largest gray circle, and to the white circle. What is the total area bounded by the gray circles?



8. **Great Expectations** Let $p, n \in \mathbb{N}$. Let σ be a permutation of n objects chosen uniformly at random, and k be the number of cycles in σ. Find the expected value $\mathbb{E}[p^k]$ as a function of p and n.

9. **Enough with the Determinants** We consider two matrices $A, B \in \mathcal{M}_2(\mathbb{R})$, at least one of them is singular. Prove that $AB = BA = O_2$, if $A^2 + AB + B^2 = 2BA$.

10. **Lets end with some geogem** ABC is an inscribed triangle. CD is drawn parallel to AB and meets the tangent BD at D. AD cuts the circle at E. M is the midpoint of side BD. It is given that BCD is isosceles triangle with BC = CD. Prove that $\angle DEC + \angle BEM = 180$.

For the curious ones - BONUS

11. **Some more LinAl** Let $n \geq 1$ be an integer. Solve in $\mathcal{M}_2(\mathbb{Z})$ the equation $X^{2n+1} - X = I_2$

12. **Not again!** Let A, B and C be three points on a line (in the same order given) such that AC = 5 units and AB = 2 units. Let S be a circle passing through A and C such that AC is not a diameter of S. Suppose the tangents to the circle S at A and C meet at point R. RB meets the circle S at Q. Now,

    (a) Prove that the point of intersection of angular bisector of AQC and AC is same regardless of the choice of S.

    (b) Find the ratio of AQ and QC, if it's given that QB:QM = 1:3 for some S.