# Mathematics Club
# Guild Selection Contest 2024-25

**Start time: 8:30PM, 11$^{th}$ Sept** **End time: 11:30PM, 11$^{th}$ Sept**

## 1 Instructions

- This contest consists of 6 problems from various domains of mathematics.
- You have 3 hours of time to try the problems and fair your solutions and submit.
- You are expected to give descriptive solutions to the problems. Which helps us to understand your approach in case of your answer is wrong.
- If you couldn't solve the problem completely, we encourage you to submit your approach.
- Happy solving :)

## 2 Problems

1. **Problem 1** Given 101 segments on a line, prove that there either exists a point contained in at least 11 of the segments or at least 11 segments that are pairwise disjoint.
2. **Problem 2** Let $ABC$ be a triangle, $I$ be its incenter. The incircle of $\triangle ABC$ is tangent to $AB$ at $D$ and $AC$ at $E$. Let $O$ denote the circumcenter of $\triangle BCI$. Prove that $\angle ODB = \angle OEC$.
3. **Problem 3**
   - Find the distinct number of ordered quadruples of non-negative integers $(a, b, c, d)$ such that
   $$a! \cdot b! \cdot c! \cdot d! = (6!)!$$
   Note that the ordered quadruples mean that $(1, 2, 3, 4)$ and $(2, 3, 4, 1)$ are considered different.
   - Prove that the expression
   $$\frac{m}{lcm(m,n)}\binom{n}{m}$$
   is an integer for all pairs of integers $n \geq m \geq 1$
4. **Problem 4** Prove the following identity
   $$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}k^{n+1} = \frac{n(n+1)!}{2}$$
   Brownie points for solving it intuitively.
5. **Problem 5** Let $n \geq 2$ and $a_1, a_2, \ldots a_n$ and $b_1, b_2, \ldots b_n$ are real numbers such that
   $$a_1^2 + a_2^2 + \ldots a_n^2 = b_1^2 + b_2^2 + \ldots b_n^2 = 1$$
   $$a_1b_1 + a_2b_2 + \ldots a_nb_n = 0$$
   Prove that $(\sum_{i=1}^{n} a_i)^2 + (\sum_{i=1}^{n} b_i)^2 \leq n$

6. **Problem 6** The Grand Integrators of 2024 Integration Bee Madhav and Arya wanted to start their journey in the field of Linear Algebra and got stuck while solving the below problems. Please help them solve these problems ~~not during the contest tho~~

   - Karthikeya writes the matrix $\begin{pmatrix} 2 & 3 \\ 2 & 4 \end{pmatrix}$ on the board. Then he performs the following operation on the matrix several times.
     - he chooses a row or a column of the matrix, and
     - he multiplies or divides the chosen row or column entry-wise by the other row or column, respectively.

     Can Karthikeya end up with the matrix $\begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ after finitely many steps?
   - Given that $A$ is a $3 \times 3$ matrix and $Tr(A) = 4, Tr(A^2) = 10$ and $Tr(A^3) = 28$ then find the value of the $det(A)$.

     [Note: $Tr(A)$ is the trace and $det(A)$ is the determinant of the matrix.]