Mathematics Club
Contingent Problem Set - 5

Challenge posed on: 12/07/2024 **Challenge conquered by: 19/07/2024**

1 Overview

- **Topics focused:**
  - Combinatorics
  - Linear Algebra
  - Calculus
  - Number Theory
- **Challengers:**
  - Aprajithan
  - Arya
  - Aravind

2 Problems

1. **Warm Up**
   (a) Consider 3 real numbers x, y, z such that $x+2y+z=6$. Find the minimum value of $x^2+2y^2+z^2$.
   (b) Evaluate the following indefinite integral:
   $$\int \frac{x^2+n(n-1)}{(xsinx+ncosx)^2}dx$$
   where n is a natural number.
2. **A convergence test** Can there exist a convergent series $\sum a_n$ such that $\sum \frac{1}{n^2 a_n}$ is also convergent?
3. **The Cubic Polynomial** The polynomial ax^3+bx^2+cx+d has integral coefficients a, b, c, d with ad odd and bc even. Prove that all the roots of this polynomial cannot be rational.
4. **Time for an inequality** Find the minimum value of k such that the following holds
   $$2y(x-1)-x(y+1) \leq k$$
   Given that
   $$(x-1)y^2+4xy-4y+5x-16=0$$
   $$xy^2-6xy+10x-28=0$$
5. **Determinant ?!** Consider two matrices A and B with positive entries having sizes 3×2 and 2×3 respectively. Find the value of $det(BA)$, given that
   $$AB = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{bmatrix},\ det(BA) \neq 0$$
6. **Catalan Frenzy !**
   (a) We define an increasing lattice path on a 2×2 grid as a path where each step taken is of unit length and is either upwards or to the right. Find the number of increasing lattice paths from $(k,0)$ to $(n+k, n+k)$ such that you never cross the $x=y$ line where $k \geq 0$ and $n \geq 1$ are integers.

(b) Using the result from part (a), determine:

i. The number of rooted binary trees with $n \geq 2$ leaves such that each node has either both its left and right children or no child at all (i.e. It is a leaf).

ii. The number of rain trees with n nodes. Where a rain tree is defined as follows

i. A rain tree with 1 node is just the root.

ii. A rain tree with n nodes consists of a root node and some smaller rain trees joined to the root node such that the total sum of the number of nodes in the joined rain trees is $n-1$ and that the order of their connection to the root node matters.

7. **Can you solve these ?**

(a) Find all positive integer quadruples (x, y, z, w) satisfying $x^2 + 6y^2 = z^2$ and $6x^2 + y^2 = w^2$.

(b) Find all solutions in positive integers to $x^8 + y^8 = z^6$.

8. **Done anything like this before, have you ?** We call a subset of a set **neat** if the arithmetic mean of the elements in the subset is an integer. Let a_n denote the number of **neat subsets** of the set of the first n even numbers. Prove that $a_n - n$ is always even.

9. **NumberFizz** The fizz of a number is defined as follows:

- The fizz of a single digit number is the number itself.
- The fizz of any other number is the same as the fizz of the sum of the digits of the number.
- We denote the fizz of a number n by $f(n)$.

Find $\sum_{n=1}^{n=20} f_{p_n}(x)$ where p_n is the n^{th} prime number and $f_p(x)$ is defined as

$$f_p(x) = \sum_{i=0}^{\infty} f(p^i)x^i, \quad x \in [0, 1]$$