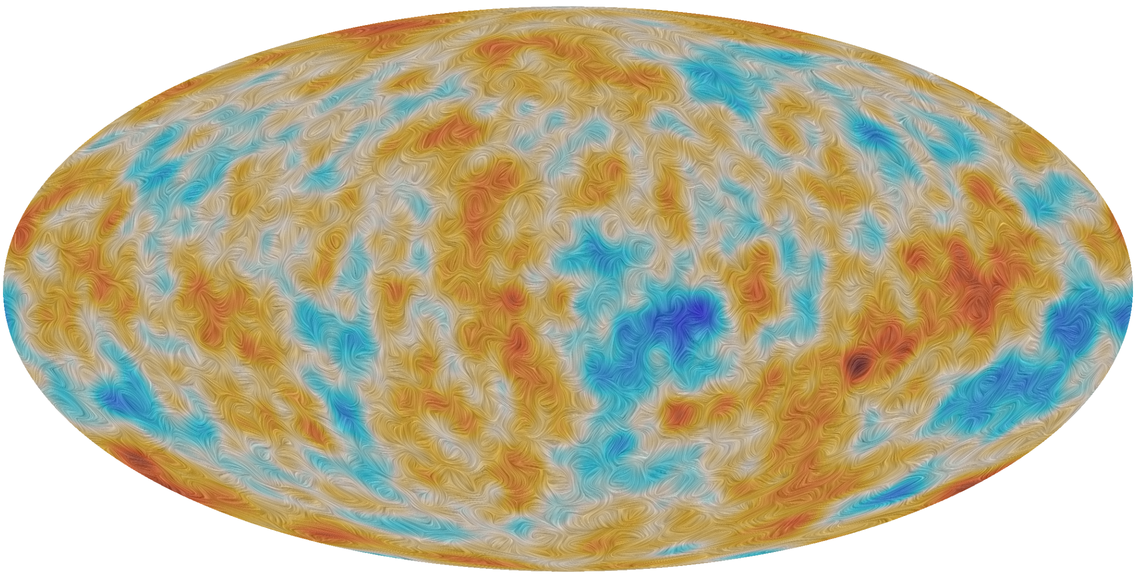
Université de Genève

Travail de fin de master

Leading and next-to-leading gravitational effects on Cosmic Microwave Background polarisation



Miguel Vanvlasselaer
Promo 2018

Superviseur : Mme Pr. Ruth Durrer
Superviseur : M. Dc. Pierre Fleury
Juré : Mme Pr. Ruth Durrer
Juré : M. Pr. Antonio Riotto

1^{st} October 2017 — 25^{th} Mai 2018

Contents

Abstract

We investigate the weak lensing corrections to the CMB with emphasis on the polarisation. Starting from fundamentals and working in Poisson gauge, we deduce the technology and the explicit expressions for the lensing and the transport of the polarisation for the leading order and beyond. We then compare the results found in the Poisson gauge with the ones found using the Geodesic Light-Cone gauge and underline a discrepancy involving the vorticity term.

The demonstration of the equivalence between the two points of view comes in two steps: we first show that the rotation angle worked out in Poisson gauge has the same mathematical expression than the angle obtained from Geodesic Light-cone gauge means. In the second step, we demonstrate, using only the formalism of $\mathcal{E}$- and $\mathcal{B}$-modes decomposition, that both methods lead to the same $\mathcal{B}$-modes power spectrum.

After this satisfactory reconciliation, we turn to the computation of the curl induced by the corrections up to second order, assuming vanishing primordial gravitational waves. Confronted with the problem of the inversion of the Laplacian operator and the lack of observational meaning of those quantities, we then complete the analysis by the computation of the corrections to the $\mathcal{B}$-modes power spectrum at the leading order and beyond.

Before to start

Notations and preliminary definitions

Generally, a dot will denote a derivative with respect to time. For example,

$$\frac{dx^\mu}{dt} = \dot{x}^\mu.$$

The Greek indices, for example μ, will run over the four space-time components: $\mu = 0, 1, 2, 3$, while the Latin letters will run over the three spatial components of space-time slices: $i = 1, 2, 3$. The first letters in the alphabet will be used from time to time to span 2-dimensional hypersurfaces (for example two angles): $a, b, c, d = 1, 2$. The vector k^μ will denote the components of the four-impulsion of the photon, defined by

$$k^\mu = \frac{dx^\mu}{d\lambda}. \tag{0.1}$$

The notation k, without indices, will denote the four-dimensional geometrical object, while a bold letter, $\mathbf{n}$, will denote a three-dimensional vector (with space-like components). We will mainly use ω for the frequency of the photon at the *emission*. A δ will denote a first order correction induced by perturbative gravitational effects, evaluated at the same value of the parameter λ: for example

$$\delta k^i(\lambda) = k^i(\lambda)\Big|_{\text{perturbed up to first order}} - k^i(\lambda)\Big|_{\text{unperturbed}}.$$

Similarly,

$$\delta^{(n)}$$

will denote the n^{th} correction. The gravitational (that we will call Weyl) field, in Poisson gauge, will be written $\phi(\lambda, \mathbf{n})$. To simplify a bit our notations in the following, we will also introduce the integrated fields

$$\Phi(\lambda, \mathbf{n}) = \int_\lambda^{\lambda_0} \frac{1}{\lambda'}\phi(\mathbf{x}(\lambda'))d\lambda', \tag{0.2}$$

and

$$\Psi(\lambda, \mathbf{n}) = \int_0^{\lambda} \frac{1}{\lambda'}\phi(\mathbf{x}(\lambda'))d\lambda', \tag{0.3}$$

and use often the fact that the derivatives commute with the integrations along λ, up to the appearance of a factor $\frac{1}{\lambda}$ which will be explained soon. This will prove very convenient in the following.

The parameter λ will be defined to be zero at the *observer* and to be λ_0 at the *emitter*, the last scattering surface. Therefore it runs in the opposite direction with respect to the *physical* photon. It does not really matter, because we can imagine a virtual photon starting at the observer and going back to the point of emission.

It will be useful to decompose the four-impulsion vector into its spatial and time parts

$$k^\mu = \omega(u^\mu + n^\mu), \tag{0.4}$$

where $u^\mu = (u^0, \mathbf{0})$ is the four velocity of a static observer and $n^\mu = (0, \mathbf{n})$ is the "unit direction" four-vector. Notice that the direction of the vector $\mathbf{n}$ is from the observer to the source. At this stage, it is also important to fix the dimension of the different quantities we will work with. The k vector has the dimension of the inverse of a length,

$$[k] = L^{-1}. \tag{0.5}$$

ω has the same dimension while u and n are defined to be dimensionless unit vector:

$$[\mathbf{n}] = 1, \qquad [u] = 1, \qquad [\omega] = L^{-1}, \tag{0.6}$$

and

$$[x^\mu] = 1. \tag{0.7}$$

As a consequence, (0.1) forces us to take

$$[\lambda] = L. \tag{0.8}$$

We will also decompose the spatial derivatives in transverse and parallel parts with respect to $\mathbf{n}$. Explicitly, we will use the notation

$$\partial_i = \nabla_{\perp i} + \nabla_{\parallel i}, \tag{0.9}$$

where the $\nabla_{\perp i}$, the angular derivative, is understood to be written in the spherical basis, (t, r, θ, ϕ), where it takes the form

$$\nabla_\perp^i = \left(0, 0, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right).$$

The factor $\frac{1}{r}$ will give rise to maybe surprising effects when we will exchange integrations and derivatives.

Indeed, if we take as an example the θ component and apply it on the integral $\int_0^R \phi(r,\theta,\phi)dr$, we have

$$\nabla_\perp^\theta \int_0^R \phi(r,\theta,\phi)dr = \frac{1}{R}\frac{\partial}{\partial\theta}\int_0^R \phi(r,\theta,\phi)dr = \frac{1}{R}\int_0^R \frac{\partial}{\partial\theta}\phi(r,\theta,\phi)dr = \frac{1}{R}\int_0^R r\nabla_\perp^\theta\phi(r,\theta,\phi)dr.$$

With exactly the same manipulations for the ϕ component, we have the general result

$$\nabla_\perp^i \int_0^\lambda \phi(\lambda')d\lambda' = \int_0^\lambda \frac{\lambda'}{\lambda}\nabla_\perp^i\phi(\lambda')d\lambda'. \tag{0.10}$$

We will have to follow this rule when we exchange transverse derivatives and integrals. Therefore, for simplicity, we will not exchange explicitly if not necessary.

To avoid those maybe annoying transformations, we will introduce the "tilde" angular derivative, with the definition

$$\tilde{\nabla}^i_\perp = \left(0, 0, \partial_\theta, \frac{1}{\sin\theta}\partial_\phi\right), \qquad \nabla^i_\perp = \frac{1}{r}\tilde{\nabla}^i_\perp. \tag{0.11}$$

For example,

$$\int_0^R \nabla^i_\perp \phi(r,\theta,\phi)dr = \int_0^R \frac{1}{r}\tilde{\nabla}^i_\perp \phi(r,\theta,\phi)dr = \tilde{\nabla}^i_\perp \int_0^R \frac{1}{r}\phi(r,\theta,\phi)dr. \tag{0.12}$$

Therefore we see that the tilde derivative is allowed to freely cross the integration, without paying the factor $\frac{r'}{r}$. With our definitions, we obtain also

$$\int_0^R \nabla^i_\perp \phi(r,\theta,\phi)dr = \tilde{\nabla}^i_\perp \Phi(0).$$

This justifies our previous definitions of Ψ and Φ.

Remerciements

Je commence cette section remerciements par remercier, bien entendu, mes parents qui auront fait montre d'un talent variable pour mimer leur intérêt pour mes pérégrinations scientifiques. Qu'ils se préparent (!) car il ne fait aucun doute que je continuerai de les abreuver de physique jusqu'à ... ce qu'ils aiment ça.

Ensuite, je pense que les gaillards de la résidence St.Boniface (et la résidence elle-même!) méritent leur part de louanges. Autant d'un point de vue académique, car on y trouve toutes les sortes de vocations, que d'un point humain, les liens qui s'y nouent, j'en suis persuadé, laisseront des marques, que finalement du point de vue intellectuel, car on y trouve moultes opinions (de celles qui m'insupportent, à celles qui m'amusent, en passant par celles avec lesquelles je ne peux être d'accord, et celles qui éveillent tout mon intérêt), les deux années ici m'auront transformé.

Je tiens aussi évidemment à remercier le Prof. Ruth Durrer, dont la disponibilité et la pédagogie m'ont sans nul doute encouragé à m'investir dans ce travail d'une année, mais tout autant à m'engager dans un futur doctorat. Je pense qu'une part de mon intérêt pour la cosmologie me vient d'elle. Bien d'autres professeurs que je ne saurais citer auront contribués à mon enrichissement (et éveil j'espère) intellectuel. Je les en remercie.

C'est également avec plaisir que je remercie le Docteur Pierre Fleury dont les bons conseils et les points de vue novateurs (en tous cas pour moi) n'auront pas manqués de me faire réfléchir, bousculant au passage quelques unes de mes certitudes, et de m'orienter tout au long de mon travail. Tout autant ses questions (gênantes!) que ses suggestions auront su me rester en tête. Il est tout autant certain que sa relecture aura beaucoup contribué à la netteté du travail.

Je n'oublie pas non plus Justine Gaberel, qui aura passé quelques heures autant à corriger mon anglais qu'à s'amuser de mon talent limité en langue étrangère. Quel trésor de patience elle a dû déployer pour ne pas (au moins une fois) m'arracher la tête lorsque je lui répondais de nouveau : "Pas aujourd'hui, je travaille".

Etant dans la section remerciements d'un travail de fin d'étude, la tradition m'impose également d'y insérer une référence absurde, de celles qui ne peuvent être imaginées que par des étudiants en mal d'occupations. J'honore ainsi tant bien que mal la tradition en remerciant mon ordinateur qui aura dû me supporter (et c'est littéralement de la fumée qui s'en échappait!) pendant plus d'heures que n'importe qui d'autre.

Introduction

Throughout its history, mankind has been questioning its very origin, building many cosmogonies and as much different philosophies. The Big Bang model is a long-standing theory to explain the origin of the Universe and its early evolution. On of the first outstanding experimental confirmation of this hypothesis was the prediction by (see [8] for the history) Alpher, Gamow and Herman of a residual black body radiation left over by the Big Bang, the so-called Cosmic Microwave Background (CMB), and its subsequent (rather lucky) observation by Wilson and Penzias in 1965. The formidable homogeneity of the temperature of this radiation in the sky is a direct counterpart of the almost perfect homogeneity of the matter distribution at cosmic scales. As time passed by, tiny temperature fluctuations, of order of 10^{-5}, in this radiation appeared, namely with the satellites COBE [11],

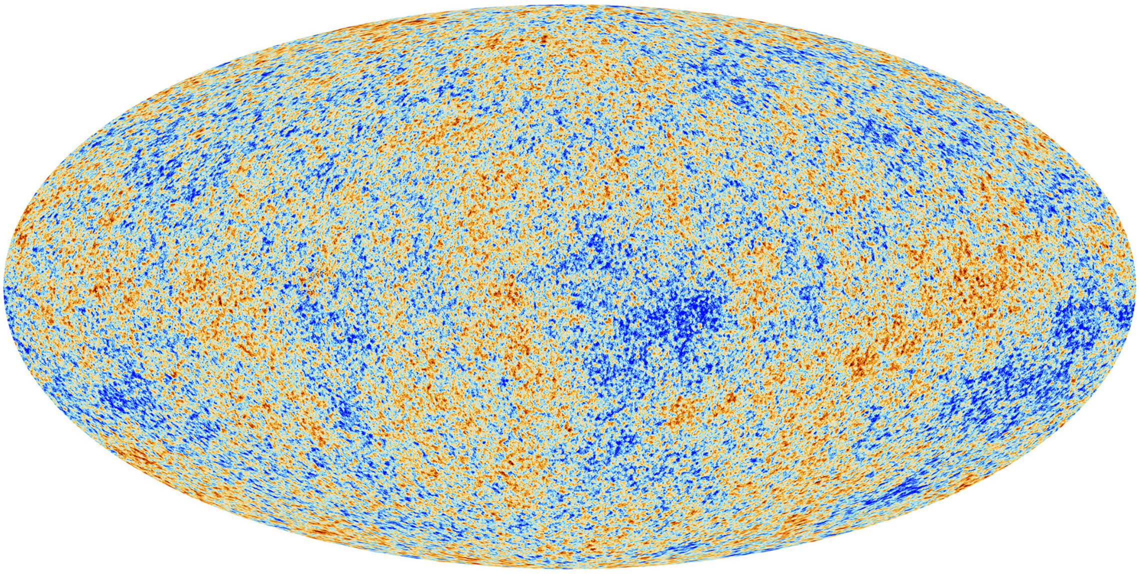


Fig. 0.1: CMB temperature fluctuations as seen by Planck. Credits : Collaboration ESA and Planck.

WMAP, and Planck (see Fig.0.1 for the temperature fluctuations in the CMB, as seen by Planck). After a long and intense effort from the theorists and the experimentalists, those fluctuations started to reveal their secrets, providing a wealth of information about the acoustic evolution of the Universe (see for example [33] or [32]), but also about its fundamental components (ratio baryonic matter, dark matter and dark energy, Hubble constant, [53]).

As the instruments developed to observe those fluctuations with increasing accuracy, the approximation of perfect matter homogeinity turned out to be insufficent. The field of high precision cosmology became more and more important, namely because a good understanding of the corrections coming from cosmic inhomogeneities was crucial for a correct interpretation of the information hidden in the CMB temperature fluctuations. Therefore, there is already an extensive literature on the so-called *lensing* of the CMB (see [41] or [56] for reviews).

On the other hand, in the recent years, the polarisation of the CMB has gained increasing interest (see Fig.0.2 for the polarisation pattern seen by Planck) among the community. Due to the very low power of this signal (compared with the power of the temperature fluctuations [34]), any experimental measurement has been out of reach during a long time. Nowadays, the wealth of information contained in the polarisation is accessible with the current instruments, provided that the measurements can be sufficiently cleaned up from the signal induced by the cosmic inhomogeneities. This is why a better understanding of all the types of cosmic perturbations and precise quantification of their effect is an inevitable step if we want to be able to extract relevant information from the CMB polarisation. In this thesis, we will mainly focus on the so-called $\mathcal{B}$*-modes* produced by those inhomogeneities. Indeed, primordial $\mathcal{B}$-modes are considered as a “smoking gun” for the *inflation* paradigm. An overview of this link between $\mathcal{B}$-modes and inflation, a mathematical characterisation of the polarisation and a quick exposition of how the polarisation arises at last scattering surface are provided, as a motivation for the following developments, in Chapter 1.

The purpose of Chapter 2 is to work out the deflection angle up to second order in the gravitational

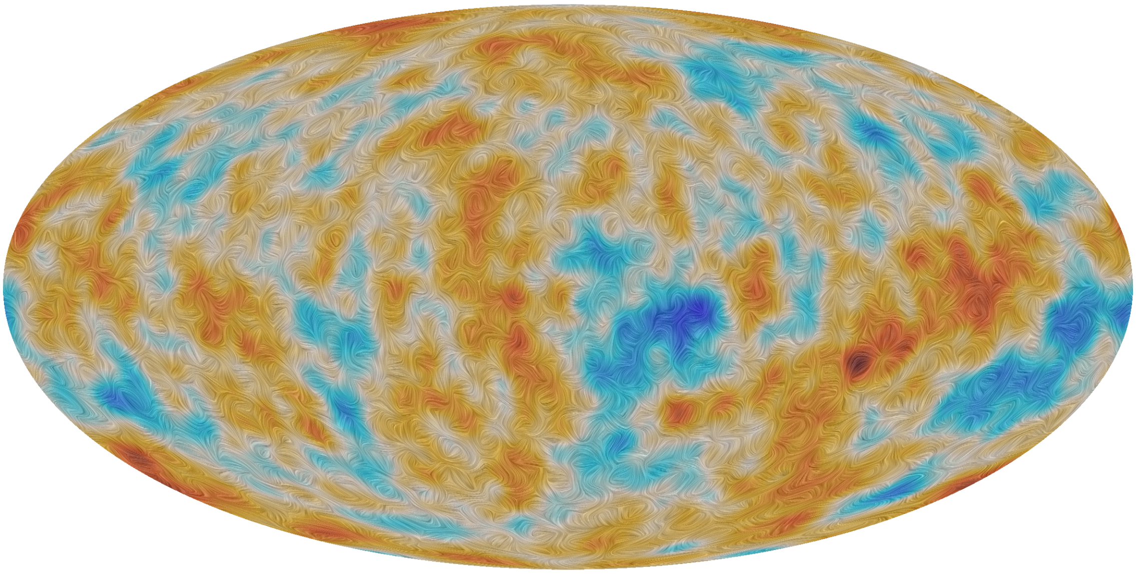


Fig. 0.2: Polarisation pattern of the CMB as seen by Planck. Credits : Collaboration ESA and Planck.

field. That is to say, we want to build the relation

$$\theta^a_O \xrightarrow{O\to E} \theta^a_E, \tag{0.13}$$

where θ^a_O is the line of sight of the observer, while θ^a_E is the direction of the emission of the photon at the last scattering. To do so, we will have to solve the geodesic equation up to second order in the Poisson gauge.

As claimed in [5] and [4], the lensing has also an important effect on the observed polarisation. Therefore, in Chapter 3, in order to account for the specificities of polarisation, we determine the equation of evolution of the polarisation (described by the direction of the electric vector $\mathbf{E}$) along the geodesic. Working in the Poisson gauge, we emphasize a possible clash with a calculation made in the "Geodesic Light-Cone" gauge and compare the two approaches.

In Chapter 4, we emphasize that typical "structures" of the CMB temperature undergo a Lie transport instead of a geodesic transport. We therefore argue, comparing computations in both GLC and Poisson gauges, that the real gauge-invariant quantity is the angle between those structures and the polarisation. We close the Chapter by showing that the $\mathcal{B}$-modes power spectrum computed in Poisson or in Geodesic Light-Cone gauge are indeed the same.

Chapter 5 is devoted to the enumeration and the classification of the different types of gravitational corrections to the polarisation, up to second order. We subsequently specialize in the curl, or $\mathcal{B}$-modes, possibly induced.

Finally, in Chapter 6, we give a more mathematical approach of the $\mathcal{B}$-modes by computing explicitly the correction to the $\mathcal{B}$-modes power spectrum P^B_l for the different sources of correction listed in Chapter 5.

1 Fundamentals

COSMOLOGY, as many other scientific branches, is a highly interdisciplinary field. It borrows practical tools from mathematics, new insights from astronomy, sound foundations from General Relativity and Quantum Field Theory, and models from fluid mechanics. Moreover, composing the vast majority of the signal we receive from outer space (even though today, we hear a lot about gravitational waves), light and photons are the companions of main of the current, past, and future investigations. We could not expose even a small fraction of all the pieces of knowledge involved in the study of cosmology, thus we just introduce few that are crucial for the presentation. This first chapter presents an overview of different useful mathematical representations of the light and its polarisation, the mechanism for its production at the last scattering surface, and finally the crucial link between $\mathcal{B}$-modes and the primordial gravitational waves.

Contents

1.1 Stokes parameters, E and B-modes

In this section, we present two different descriptions of the polarisation : The *Stokes parameters* and the $\mathcal{E}$ *and* $\mathcal{B}$*-modes*. To start, we define the polarisation matrix as

$$P_{ab} \equiv C\langle E_a E_b^\star \rangle_{\text{time}}. \tag{1.1}$$

E^a designates the components of the electric field as seen by the observer and $E^{a\star}$ is its complex conjugate. The constant C depends on the system of units we use and the brackets mean an average over a typical time of oscillation of the electromagnetic wave. This matrix can be decomposed into irreducible parts (see [36] for a general introduction to electrodynamics, or [41] or [19] for specific papers on CMB polarisation. We will also mainly follow the lines of the review by Lewis and Challinor [41]),

$$P_{ab} = P'_{ab} + \frac{1}{2}\delta_{ab} I + \frac{1}{2} V \epsilon_{ab}, \tag{1.2}$$

where P'_{ab} is the traceless symmetric part of P_{ab} and ϵ_{ab} is the totally antisymmetric tensor in two dimensions. The V component describes a circular polarisation, because antisymmetric part arises in (1.1) if there is a difference of phase between the two electric fields $\mathbf{E}$ in the brackets. The Thomson scattering does not induce circular polarisation and therefore we will not consider V-component in the following.

If we fix a basis in the plane perpendicular to the direction of propagation $\mathbf{n}$, described by the vectors $(\mathbf{e}_1, \mathbf{e}_2)$[1], we can introduce the two components U and Q. They are defined by

$$P_{ij} = \begin{pmatrix} Q & U \\ U & -Q \end{pmatrix} = Q\sigma_{3ij} + U\sigma_{1ij}, \tag{1.3}$$

where we used the Pauli matrices of the form $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. I, the total intensity, V, describing a possible circular polarisation, and U and Q describing linear polarisation are the four "Stokes parameters". They are often recast in the form of a formal vector

$$\mathbf{S} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}.$$

The Q and U components can be explicitly computed from the components of the electric vector $\mathbf{E}$.

$$Q = |E_1|^2 - |E_2|^2, \tag{1.4}$$

$$U = |E_x|^2 - |E_y|^2 = 2\mathcal{R}e(E_1 E_2^\star), \tag{1.5}$$

where we introduced the basis $\mathbf{e}_x = \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{e}_y = \mathbf{e}_1 - \mathbf{e}_2$. This is the same basis as $(\mathbf{e}_1, \mathbf{e}_2)$, but rotated by an angle of 45 degrees.

This gives us a beautiful interpretation of these two parameters. The Q component is the difference of intensity measured in the direction $\mathbf{e}_1$ and $\mathbf{e}_2$ while the U parameter is the difference of the intensity measured in the direction $\mathbf{e}_x$ and $\mathbf{e}_y$. If we diagonalize the matrix P_{ij}, we have the eigenvalues

$$\pm\sqrt{U^2 + Q^2},$$

with eigenvectors making an angle of

$$\alpha = \frac{1}{2}\tan^{-1}\left(\frac{U}{Q}\right)$$

with the direction of the north pole (in our case it is $\mathbf{e}_1$). We can interpret it as a polarisation of intensity $\sqrt{U^2 + Q^2}$ in the direction $\alpha = \frac{1}{2}\tan^{-1}\left(\frac{U}{Q}\right)$. Let us now study the behaviour of those parameters under a rotation of the basis.

To do so, we use a new basis composed of the vectors $\mathbf{e}_\pm = \mathbf{e}_1 \pm i\mathbf{e}_2$, defined in such a way that, under a clockwise rotation by an angle θ, they transform as

$$\mathbf{e}_\pm \to e^{\pm i\theta}\mathbf{e}_\pm.$$

[1] These vectors are defined by the measuring device and not on the CMB, as we could think

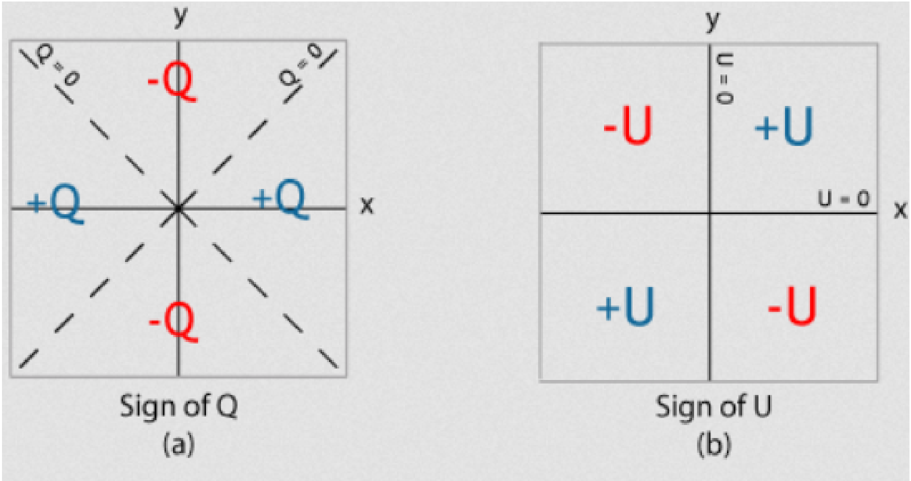


Fig. 1.1: Illustration of the Stokes parameters. From [55].

Let us define the $\mathcal{P}$ component, which contains only the symmetric part of (1.1), as

$$\mathcal{P} \equiv e_+^a e_+^b P_{ab} = Q + iU,$$

and, by simple complex conjugation, $\mathcal{P}^\star$ as

$$\mathcal{P}^\star = e_-^a e_-^b P_{ab} = Q - iU.$$

(Note that we have used the fact that Q and U are real quantities). Consequently, under a rotation of θ, those two quantities transform as

$$\mathcal{P} \rightarrow e^{2i\theta}\mathcal{P}, \qquad \mathcal{P}^\star \rightarrow e^{-2i\theta}\mathcal{P}^\star.$$

This is exactly the behaviour we expect for a spin-2 quantity. $\mathcal{P}$, Q and U are indeed spin-2 functions on the sphere. Our main concern now is to recover scalar quantities from the Stokes parameters Q and U. This is the how the $\mathcal{E}$- and $\mathcal{B}$-modes will appear. There are many equivalent characterisations, so we will provide some of them and try to relate them. Let us firstly extract those modes mathematically.

We start again from the matrix (1.1). We decompose it into the two following parts

$$P_{ab} = \nabla_{\langle a}\nabla_{b\rangle}P_E + \epsilon^c_{(a}\nabla_{c)}\nabla_b P_B, \tag{1.6}$$

where P_E and P_B are two scalar quantities. The angle brackets define a traceless symmetric sum and the round ones a symmetric sum. We introduced an antisymmetric tensor ϵ_{ab} for the definition of P_B. This will translate into an opposite parity for $\mathcal{E}$-modes, represented by P_E and $\mathcal{B}$-modes, represented by P_B. Of course, we could isolate each component by applying the right derivative operator on P_{ab}, and we will follow this method in Chapter 5, but this method is mainly local, while the separation in $\mathcal{E}$- and $\mathcal{B}$-modes is a non-local problem. It is therefore preferable to work in harmonic space, and decompose the scalar quantities P_E and P_B in spherical harmonics. We have

$$P_{ab} = \frac{N_l}{\sqrt{2}}\sum_l [\mathcal{E}_l \nabla_{\langle a}\nabla_{b\rangle}Y_l + \mathcal{B}_l \epsilon_{c(a}\nabla_{c)}\nabla_b Y_l],$$

where N_l is a normalisation constant and we defined the Fourier coefficients $\mathcal{E}_l$ and $\mathcal{B}_l$. Note that they are scalar functions. Contracting with $e^a_+ e^b_+$ and following the calculation of [41] for example, we can see that

$$\mathcal{P} = \sum_l [\mathcal{E}_l + i\mathcal{B}_l]_2 Y_l,$$

and

$$\mathcal{P}^\star = \sum_l [\mathcal{E}_l - i\mathcal{B}_l]_{-2} Y_l,$$

where, by definition,

$$_2Y_l \equiv \frac{N_l}{\sqrt{2}} e^a_+ e^b_+ \nabla_a \nabla_b Y_l, \tag{1.7}$$

and

$$_{-2}Y_l \equiv \frac{N_l}{\sqrt{2}} e^a_- e^b_- \nabla_a \nabla_b Y_l. \tag{1.8}$$

When we restrict to small angles, $l > 100$[2], the spherical harmonics simplify to simple exponentials (see for example [20] for an explanation and subsequent details)

$$Y_{lm}(\mathbf{n}) \to e^{i\mathbf{l}\cdot\mathbf{x}}, \tag{1.9}$$

where $\mathbf{l} = l(\cos\phi_l, \sin\phi_l)$[3] is now a vector in the "Fourier plane" and $\mathbf{x}$ is a small vector perpendicular to $\mathbf{n}$. The two vectors $\mathbf{x}$ and $\mathbf{n}$ point at the same position on the last scattering surface. The discrete labels l and m became two continuous variables l and ϕ_l. Using (1.7) and defining $N_l = \frac{\sqrt{2}}{l^2}$, we see that $_2Y_l$ becomes

$$_2Y_l \to l^{-2} e^a_+ e^b_+ \nabla_a \nabla_b e^{i\mathbf{l}\cdot\mathbf{x}} = l^{-2}(\partial_x + i\partial_y)^2 e^{i\mathbf{l}\cdot\mathbf{x}} = -e^{2i\phi_l} e^{i\mathbf{l}\cdot\mathbf{x}}. \tag{1.10}$$

See again [41]. This leads us directly to

$$\mathcal{P} = Q + iU = -\int \frac{d^2\mathbf{l}}{2\pi} [\mathcal{E}(\mathbf{l}) + i\mathcal{B}(\mathbf{l})] e^{2i\phi_l} e^{i\mathbf{l}\cdot\mathbf{x}}, \tag{1.11}$$

$$\mathcal{P}^\star = Q - iU = -\int \frac{d^2\mathbf{l}}{2\pi} [\mathcal{E}(\mathbf{l}) - i\mathcal{B}(\mathbf{l})] e^{-2i\phi_l} e^{i\mathbf{l}\cdot\mathbf{x}}. \tag{1.12}$$

So, upon restriction to flat-sky, we recover an almost Fourier transform (the factor $e^{2i\phi_l}$ appeared). This slightly modified Fourier transform can be easily inverted to single out the $\mathcal{E}$ and $\mathcal{B}$ functions.

$$\mathcal{E}(\mathbf{l}) + i\mathcal{B}(\mathbf{l}) = -\int \frac{d^2\mathbf{x}}{2\pi} \mathcal{P} e^{-2i\phi_l} e^{-i\mathbf{l}\cdot\mathbf{x}}, \tag{1.13}$$

$$\mathcal{E}(\mathbf{l}) - i\mathcal{B}(\mathbf{l}) = -\int \frac{d^2\mathbf{x}}{2\pi} \mathcal{P}^\star e^{2i\phi_l} e^{-i\mathbf{l}\cdot\mathbf{x}}. \tag{1.14}$$

We now turn to their respective properties. $\mathcal{E}$ and $\mathcal{B}$ stand, respectively, for electric and magnetic modes. The reason for this name, that we can already see in (1.6), is the behaviour of these modes under parity. Under such a symmetry, the $\mathcal{E}$-modes get a factor $(-1)^l$ while the $\mathcal{B}$-modes get a factor

[2] This is the so-called regime of the flat-sky approximation, for which we neglect the curvature of the sphere

[3] Therefore ϕ_l is defined as the angle between the vector $\mathbf{l}$ and the basis vector $\mathbf{e}_1$.

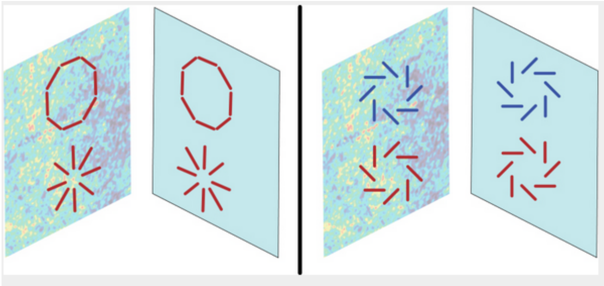


Fig. 1.2: $\mathcal{E}$- and $\mathcal{B}$-modes. On the left side, $\mathcal{E}$-modes are curl-free and under parity, they are unchanged. They are therefore even under parity. On the right side, $\mathcal{B}$-modes are divergence-free and are odd under parity.

$(-1)^{l+1}$. This reminds the behaviour, respectively, of electric and magnetic fields. The behaviour of $\mathcal{E}$- and $\mathcal{B}$-modes under parity is illustrated on Fig.1.2[4].

The second important property is a geometrical one. After drawing the headless electric vectors on a map, as can be seen on Fig.1.3, we can decompose them in $\mathcal{E}$ and $\mathcal{B}$ components. The $\mathcal{E}$-modes have a direction (defined by the direction of the headless vector) *parallel or perpendicular* to the main direction of variation of their own length (the direction of the Hessian curvature matrix), while a $\mathcal{B}$-modes have a direction *inclined by an angle of* 45 *degrees* with respect to the main direction of variation of their own length. The two cases are represented on Fig.1.3. We will illustrate this idea on examples in following sections.

Before closing this section, let us provide a last mathematical characterization (see [19] and [20]) of $\mathcal{E}$- and $\mathcal{B}$-modes, maybe a bit more intuitive than the previous ones. If we come back to the electric vector field we used to define the polarisation, we can split it into a gradient and a curl part[5]:

$$E^i = \nabla^i \Phi + \epsilon^i_k \nabla^k \Psi. \tag{1.15}$$

It can be shown that the two quantities

$$E = 2\nabla_i \nabla_j (\nabla_i \Phi \nabla_j \Phi^\star - \frac{1}{2}\delta_{ij}|\nabla\Phi|^2) \tag{1.16}$$

and

$$B = 2\nabla_i \nabla_j (\nabla_i \Psi \nabla_j \Psi^\star - \frac{1}{2}\delta_{ij}|\nabla\Psi|^2) \tag{1.17}$$

describe respectively the $\mathcal{E}$- and $\mathcal{B}$-modes (see [20]). Indeed, if we have a pure-gradient electric field, $E^i = \nabla^i\Phi$, B vanishes by definition, and if we have a pure curl electric field, $E^i = \epsilon^i_k \nabla^k \Psi$, the E component vanishes. Now, having at hand the vector E^i, it is really easy to pick up the gradient or the curl component. To isolate the gradient component, we contract with the operator ∇_i, giving

$$\nabla_i E^i = \nabla^2 \Phi.$$

On the other side, contracting with the operator $\epsilon_{ij}\nabla^j$, we are left with

$$\epsilon_{ij}\nabla^i\nabla^j E^i = \nabla^2\Psi.$$

[4] From the article Behind the Curtains of the Cosmos 3: Keys to the Cosmos, by Shane L. Larson on "https://writescience.wordpress.com/2014/04/11/behind-the-curtains-of-the-cosmos-3-keys-to-the-cosmos/"

[5] This decomposition is indeed very similar to the one of the polarisation tensor (1.6)

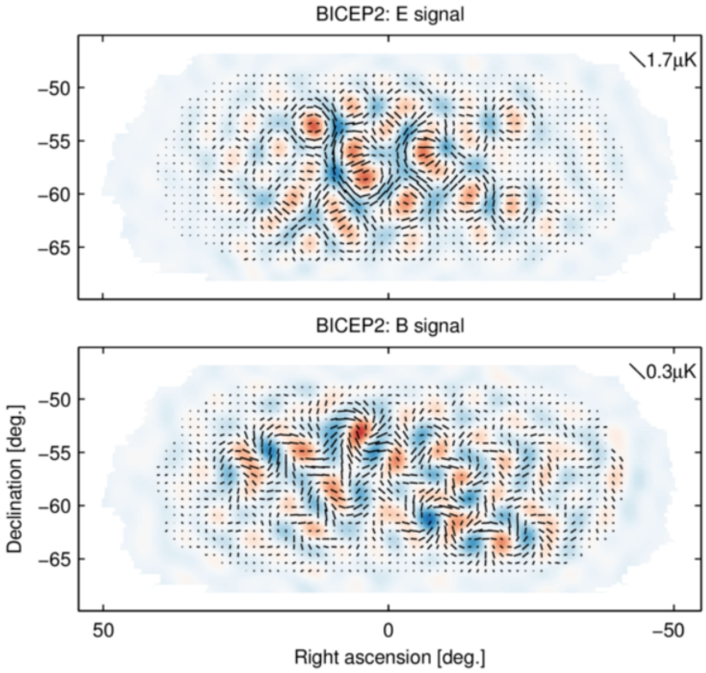


Fig. 1.3: From [3]. This is the decomposition in $\mathcal{E}$- and $\mathcal{B}$-modes of the Bicep2 results. Even if we today know that those $\mathcal{B}$-modes are not primordial, but come from foregrounds, it remains an interesting illustration.

Therefore the $\mathcal{B}$-modes originate from divergence-free part of $\mathbf{E}$ and $\mathcal{E}$-modes originate from curl-free part.

This mathematical characterization is particularly important, because it makes the bridge between Chapter 5, where we use directly the polarisation vector and trace out the appearance of a curl by applying $\epsilon_{ij}\nabla^i$ on P^j, and Chapter 6 where we compute the $\mathcal{B}$-modes from correlation functions. This mathematical characterization shows that both methods are the consistent (the first one being local, and the second being more global).

1.2 Temperature power versus Polarisation power

In this section, we quickly compare the CMB temperature power with the CMB polarisation power. On Fig.1.4, we can see that the fluctuations of temperature signal is roughly 10 thousands times stronger than the polarisation signal. This explains the difficulties in detecting purposefully the latter for a long time.

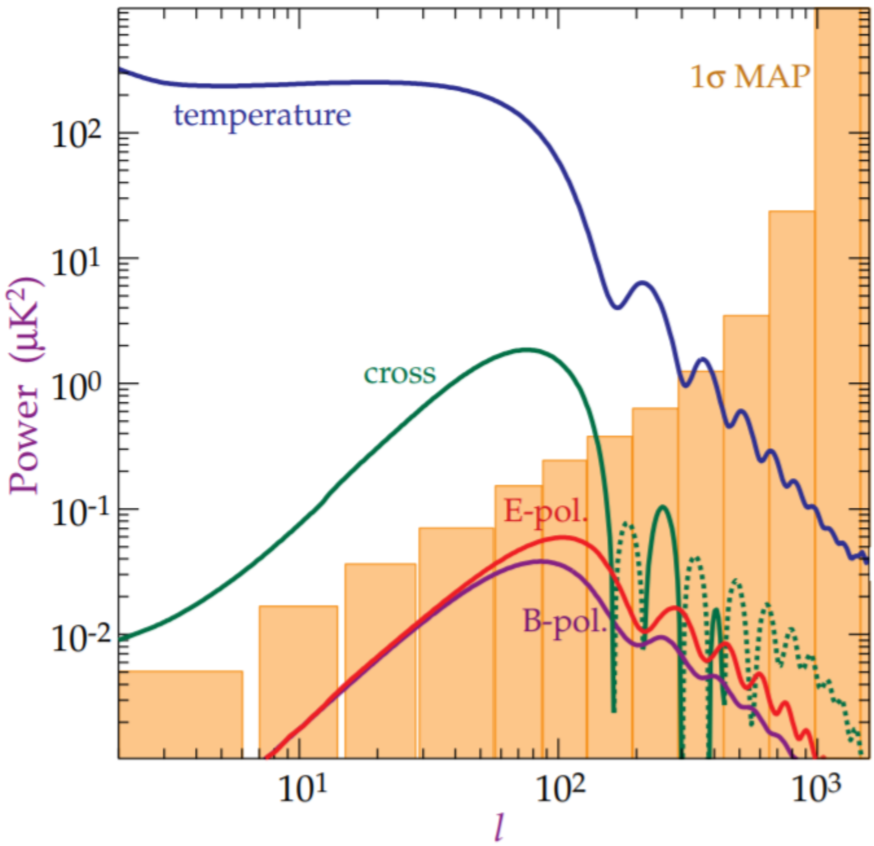


Fig. 1.4: From [34]. The purpose of this illustration is to provide an insight of the comparative power of CMB temperature fluctuations and polarisation fluctuations. At scales of interest, between 10^1 and 10^2, the ratio of temperature power (the blue curve) over polarisation power (the red curve for $\mathcal{E}$-modes and the purple curve for $\mathcal{B}$-modes) is roughly $10^3 - 10^4$.

1.3 Anisotropies and polarisation from Thomson scattering

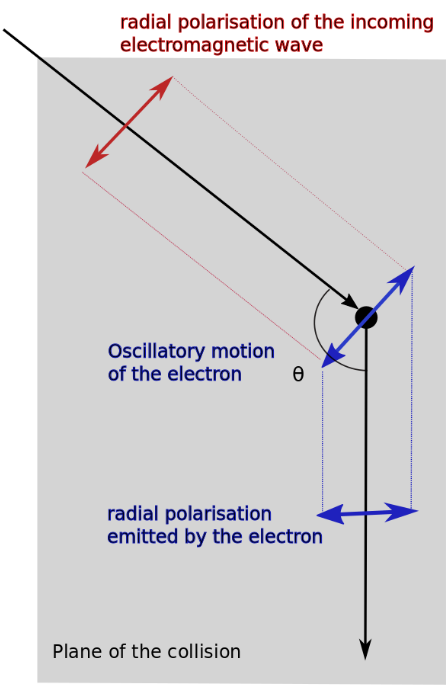


Fig. 1.5: The Thomson scattering for the radial polarisation component. We see the dependence with respect to the θ angle.

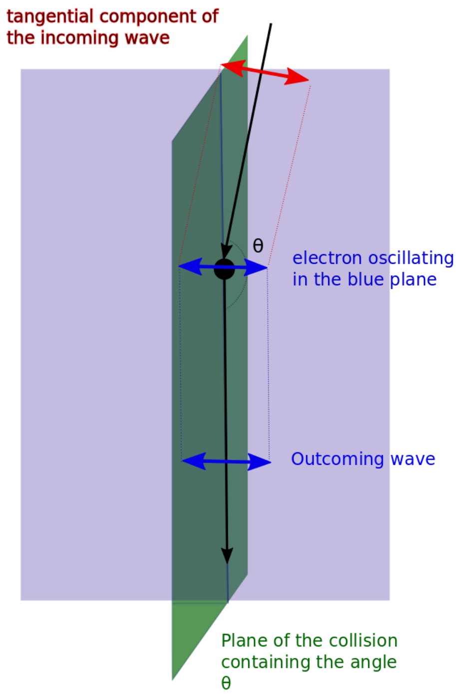


Fig. 1.6: The Thomson scattering for the tangential polarisation component. We see that there is no dependence with respect to the θ angle.

The polarisation comes from the joint action of the Thomson scattering, between an electromagnetic radiation and a free charged particle (an electron in our case), and the quadrupolar anisotropy present in the Universe at the time of Recombination. For an early presentation of the appearance of polarisation, see [14], and for more modern papers, see for example [34], [35],[19], [13], [16]. The goal of this section is to present these two phenomena and to explain their main consequence : The production of a *linear* polarisation in the CMB. First of all, let us write the Thomson cross-section

$$\sigma_t = \frac{8\pi}{3} r_e^2, \tag{1.18}$$

where r_e is the so-called "classical radius" of the electron and is defined as

$$r_e = \frac{q^2}{4\pi\epsilon_0 mc^2}$$

Actually, for the Thomson scattering, there are two characteristic behaviours, that we can capture using two coefficients. Those two emission coefficients ϵ_i are defined such that $\epsilon_i dV dt d\Omega d\lambda$ is the energy scattered by a volume element dV, in a time dt, in the direction $d\Omega$ with wave length comprised between λ and $\lambda + d\lambda$ for $i = t, r$. These labels correspond to

$$\epsilon_t = \frac{\pi \sigma_t}{2} In, \tag{1.19}$$

for *tangentially* polarized light[6], and

$$\epsilon_r = \frac{\pi \sigma_t}{2} In \cos^2 \theta, \tag{1.20}$$

for the *radially* polarized light[7]. I is the incident flux. Heuristically, the incoming electromagnetic wave sets up the oscillation of the electron in the direction of the polarisation, which emits an outgoing wave. In the previous formulas, we separated the polarisation in two orthogonal components: the radial component, being the component lying in the plane of the collision, and the tangential component being orthogonal to the previous one. The angle θ is illustrated on the Fig.1.5. On the Fig.1.6, we can see that the tangential part is not affected by the angle, while, on the Fig.1.5, we can see that the radial part is affected (and is actually proportional to $\cos^2 \theta$). The consequence is that *for* $\theta = \frac{\pi}{2}$*, only the tangential component is transmitted.* Let us memorize this result.

Now, we suspect that inflation left matter perturbations in the Universe. In principle there can be three types of matter anisotropies: scalar, vector, and tensor. But we do not expect inflation to produce vector perturbations, and if they were actually produced in some unknown way, they would be damped by the expansion of the Universe. Subsequently, joint action of the gravitational potential Ψ and the difference of temperature ΔT induced a flow of photons[8]. To describe both effects at the same time we can define the effective temperature

$$\left(\frac{\Delta T}{T}\right)_{\text{eff}} = \left(\frac{\Delta T}{T}\right) + \Psi. \tag{1.21}$$

The flow of photons is in the direction of the gradient of the effective temperature, from the hot regions to the cold ones. This flow is clearly not isotropic, and therefore, we refer to these fluctuations as *temperature anisotropies*. Then these temperature anisotropies, through Thomson scattering, can produce a polarisation. Actually, only the *quadrupolar* anisotropy (meaning that the hot and cold spots are separated by angles $\frac{\pi}{2}$) will produce a polarisation through Thomson scattering. An illustration and an explanation of this fact are provided on Fig.1.7. We can follow the same kind of reasoning to see that the other multipoles component do not provide polarisation after Thomson scattering. Furthermore, these anisotropies can be decomposed in the base of spherical harmonics with $l = 2$: $Y_2^0(\theta, \phi)$, $Y_2^{\pm 1}(\theta, \phi)$, $Y_2^{\pm 2}(\theta, \phi)$. Following the lines of [35], we can work out a remarkable one-to-one correspondence between the kind of matter perturbation (that causes the flow of photons) and the basis in which we decomposed the temperature anisotropies. More precisely,

- the scalar matter perturbations (compressional perturbations) correspond to

$$Y_2^0(\theta, \phi).$$

  This component does not produce any $\mathcal{B}$-modes even after plane wave modulation.

[6] The tangential component is the component of the polarisation perpendicular to the plane of the collision.

[7] The radial component is the component of the polarisation in the plane of the collision.

[8] Actually, before the recombination, the baryon and photon fluids were tightly coupled, and therefore the density fluctuations of photons and baryons followed a common movement. The baryon matter fluid felt the gravitational potential while the photon fluid exerted pressure, creating a gradient of pressure in the fluid. These effects set up the so-called *baryonic acoustic oscillations* (BAO). It is those two effects that we want to capture in the effective temperature.

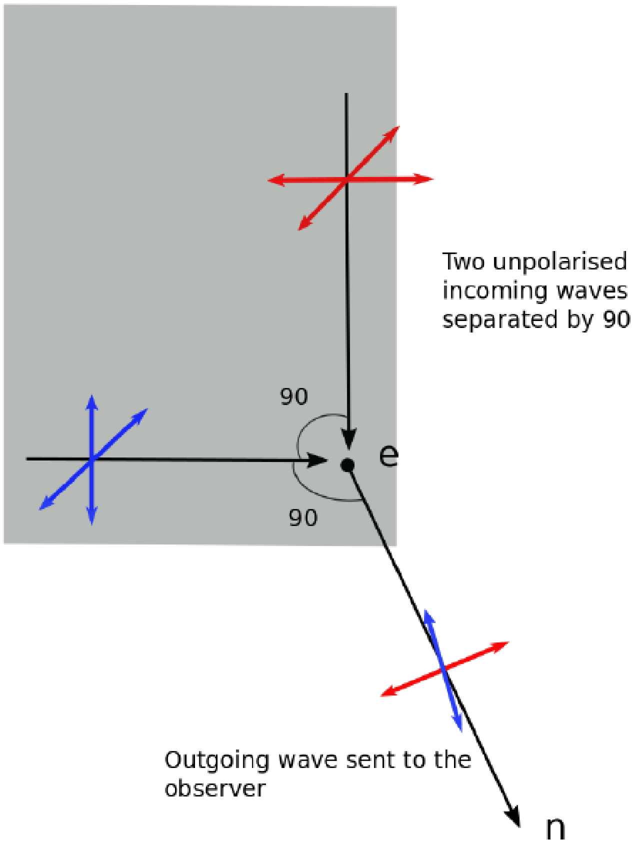


Fig. 1.7: Illustration of the selection induced by the Thomson scattering. We consider two incoming unpolarised waves separated by an angle of 90 degrees and assume a quadrupolar anisotropy *in the grey plane*. As a consequence, there are "more" photons coming from above than from the left. The direction of observation is the direction normal to the plane of the two waves (grey plane). In this case, $\theta = 90$ degrees and the radial component is totally suppressed while the tangential component is transmitted, as we explained above using the Thomson coefficients. But remember that we have more "red" photon than "blue" ones, therefore, the "red" polarisation is more important. This result in a *in a linear polarisation.* Thus, we conclude that: if there is an anisotropy in the distribution of the unpolarised incoming waves, this will translate into a *linear* polarisation of the emitted wave.

- the vector matter perturbations (vortical motion) correspond to

$$Y_2^{\pm 1}(\theta, \phi).$$

This type of anisotropies induce a large amount of $\mathcal{B}$-modes $\frac{B}{E} = 6$ (after plane wave modulation[9]).

- the tensor matter perturbations (gravitational waves emission) correspond to

$$Y_2^{\pm 2}(\theta, \phi).$$

This type of anisotropies induce $\mathcal{B}$-modes with $\frac{B}{E} = \frac{8}{13}$ (after plane wave modulation).

The thorough study of these three parts of the quadrupolar anisotropies is done in [35] and [34]. We will reproduce in the next subsection the simplest case, the scalar perturbation, even if this case does not give any $\mathcal{B}$-modes for the polarisation. The remaining can be found in [34].

At this point, it is important to note that, *locally*, the Thomson scattering cannot induce *any* $\mathcal{B}$-modes. For, under parity, the B-modes transform with $(-1)^{l+1}$ while the $\mathcal{E}$-modes transform with $(-1)^l$. On the other hand, the spherical harmonics describing temperature anisotropies transform with a factor $(-1)^l$ and then, locally, because the Thomson scattering conserves parity[10], $\mathcal{B}$-modes cannot be produced. We will see soon how they can be produced by the plane wave modulation.

To summarize, we can say that the polarisation induced is non-zero if there is a quadrupolar anisotropy of temperature in the plane perpendicular to the direction of sight, denoted $\mathbf{n}$. It is also worth to note that, considering a plane of emission in the sky, the oscillation of the electrons are in general not contained in this emission plane.

1.4 Scalar matter perturbations

In this subsection, we will follow the lines of [34] and give the complete reasoning leading from scalar matter perturbations to the polarisation and the possible presence of $\mathcal{B}$-modes in the spectrum. Here we show that these perturbations cannot provide any $\mathcal{B}$-modes. On the other hand, the vector and tensor perturbations *do give* $\mathcal{B}$-modes in the amount presented in the last section. As explained above, the matter perturbations produce, through gravitational potential and temperature gradient (both effects being considered in the effective temperature), a flow of photon from hot to cold regions. We will expand the scalar matter perturbations in Fourier series

$$\phi(\mathbf{x}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{1.22}$$

For now on, we focus on one specific (large-scale) Fourier component $\phi_{\mathbf{k}}$ where the north is defined as the $\mathbf{k}$-direction, therefore the matter perturbation is described by the plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}$. This matter perturbation induces a non-vanishing flow $\mathbf{v}$ of photons of the form from $\mathbf{k} \parallel \mathbf{v}$. This flow is illustrated on Fig.1.8. If we sit in the trough of the plane wave, in a cold region, we will see photon flowing inside from the crests, the hot regions. This kind of flow is non-rotational and is called "compressional" for

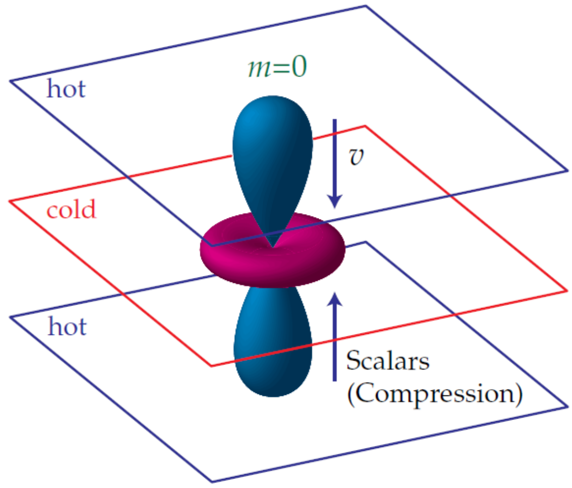


Fig. 1.8: The scalar quadrupole moment. The flow is from hot to cold areas, creating a dipole with $m = 0$ along the north pole. From [34].

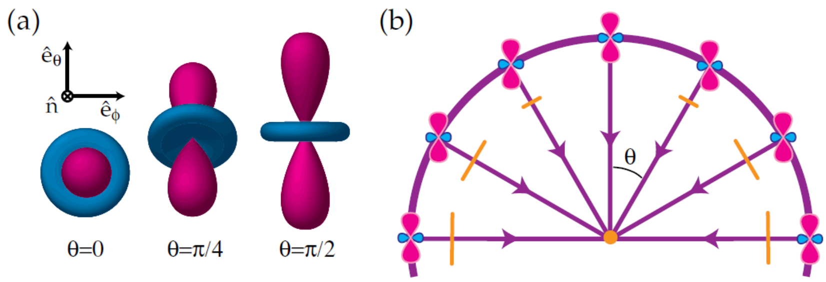


Fig. 1.9: Illustration of the transformation of quadrupolar anisotropies into linear polarisation. We see that, at the north pole, there is no variation of temperature in the plane tangent to the sphere, therefore, there is no polarisation produced. On the other hand, the polarisation is maximal at the equator because the lobes lie inside the tangent plane. From [34].

an obvious reason. Because at the trough we are surrounded by cold photons and that hot photons flow in from directions $\pm\mathbf{k}$, we can deduce that the mathematical flow pattern is

$$Y_2^0 \propto 3\cos^2\theta - 1, \qquad \cos\theta = \mathbf{k}\cdot\mathbf{n}.$$

This is the situation in the trough, of course. To obtain the result everywhere, we just multiply by the modulation $e^{i\mathbf{k}\cdot\mathbf{x}}$. From the photon flow, we can work out the polarisation pattern. As a reminder, the polarisation peaks when the variation of temperature lies in the plane perpendicular to the direction of observation, which is the plane tangent to the sphere of the sky. Thus the polarisation is maximal when the hot and cold lobes lie in the tangent plane. On Fig.1.9, we can see that this happens at the equator, while at the pole, there is no temperature variation.

The full polarisation pattern can be expressed in term of Q and U components. In the case of

[9] What we mean by plane wave modulation and the reason we can not have $\mathcal{B}$-modes from Thomson scattering is explained in the following.
[10] Quantum electrodynamics does conserve parity.

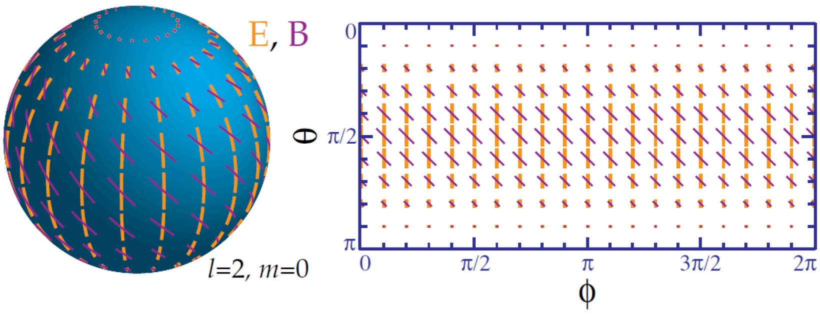


Fig. 1.10: Polarization pattern produced by the scalar modes. The $\mathcal{E}$-modes are displayed in yellow and the $\mathcal{B}$-modes are displayed in purple. From [34].

scalar matter perturbations, we have a pure Q-field

$$Q = \sin^2\theta \qquad U = 0.$$

The full pattern is represented on Fig.1.10. The yellow lines show the $\mathcal{E}$-modes and the purple lines, the other logical possibility, the $\mathcal{B}$-modes. The theory for $\mathcal{E}$- and $\mathcal{B}$-modes has been exposed in the previous sections. Their main properties were that, under parity $\mathbf{n} \to -\mathbf{n}$, $\mathcal{E}$-modes transform as $(-1)^l$ and $\mathcal{B}$-modes $(-1)^{l+1}$. $\mathcal{E}$-modes have amplitude variation in a direction parallel or perpendicular to their own direction, while $\mathcal{B}$-modes have amplitude variation in a direction inclined by an angle of 45 degrees with respect to their own variation. The property of parity has been used before to show that the Thomson scattering *is not able* to generate $\mathcal{B}$-modes. This is, of course, also true for the vector and tensor perturbations.

However, let us recall that we have considered just one Fourier component, the total density perturbation being the *superposition* of all the Fourier components. Actually, upon modulation, Q components are expected to produce $\mathcal{E}$-modes and U-components are expected to produce $\mathcal{B}$-modes. Because the scalar perturbations only produce Q-components (parallel or perpendicular to $\mathbf{k}$), the modulation cannot produce $\mathcal{B}$-components.

This is not true of the tensor and vector perturbations, because they present both U- and Q-components. They therefore produce $\mathcal{B}$-modes in the amounts exposed in the previous section. In the inflation paradigm, there is no vector modes, but gravitational waves are expected. That is why we see observation of primordial $\mathcal{B}$-modes in the CMB as a confirmation of inflation hypothesis. We make that a bit more precise in the next section.

1.5 Gravitational waves and inflation

One of the main purpose of the observation of the CMB polarisation is to verify and to test the so-called *inflation* paradigm. This paradigm states that, right after the Big Bang, the universe expanded exponentially by an extremely large amount in a tiny fraction of second. This was initially proposed to explain the apparent flatness of the Universe and the problem of homogeinity observed in the CMB temperature and in the matter distribution in the Universe. Eventually, it turned out that the inflation was also able to explain the different scales and structures in the matter distribution and CMB temperature. The scenario goes as follow: quantum fluctuations at the epoch of inflation are frozen by the expansion of space-time and become fluctuations in the matter density. Those perturbations are expected to be adiabatic density perturbations with an almost scale-invariant spectrum [53]. They subsequently imprint regular patterns in the temperature of the CMB and evolve into the structures we see today. The point is that this inflation must be accompanied by an emission of gravitational waves, see [43] and [2]. A heuristic explanation for the appearance of those *primordial* gravitational waves is provided in [39][11]. The prediction of those stochastic gravitational waves due to inflation appeared in the literature in the early 80' (see [1],[29],[21]). On the other hand, they are probably are far too weak to be directly observed with ground-based instruments.

As we have seen in the previous section, the density fluctuations, via the Thomson scattering, are not able to generate $\mathcal{B}$-modes, only the tensor and vector modes could produce a non-vanishing signal through modulation. Around 1996 and 1997, when the $\mathcal{B}$-modes were mentioned for the first time in the literature (see [37],[51],[38]), it was not clear at all if these $\mathcal{B}$-modes would be someday experimentally detectable. But surveys around 2010 suggested that they could be detected experimentally in the coming years (see for example [15] and [6]). Because vector modes are not expected in the inflation paradigm, only the gravitational waves are able to produce primordial $\mathcal{B}$-modes in the CMB polarisation pattern. As a consequence, detection of *primordial* $\mathcal{B}$-modes would be an unambiguous detection of *primordial* gravitational waves. Beside this qualitative point, there is a second good reason to be interested in inflation-induced gravitational waves.

The main difference between a tensor metric perturbation and a scalar metric perturbation is that the former depends only on the *scale* (the potential) of inflation, while the latter depends also on the slow-roll parameter. For an exposition of relations between inflation and CMB polarisation see [39], [2] or [27]. The argument goes roughly as follows: for the scalar perturbations, we choose the comoving gauge and we parametrize the perturbations in terms of the "curvature perturbation" $\mathcal{R}$. Therefore, $g_{0\mu} = 0$ and $g_{ij} = \delta_{ij}a^2(t)\text{exp}\,(2\mathcal{R}(\mathbf{x}, t))$. Inserting this parametrization in the Einstein-Hilbert action, expanding up to second order and Fourier transforming the field $\mathcal{R}(\mathbf{x}, t)$, we recognize the action of a harmonic oscillator with variance

$$\langle|\mathcal{R}_{\mathbf{k}}|^2\rangle = \frac{H(t)^2}{(4M_{PL}^2\epsilon k^3)}, \tag{1.23}$$

where $\mathcal{R}_{\mathbf{k}}$ are the Fourier components of the curvature perturbation with Fourier vector $\mathbf{k}$, $k = |\mathbf{k}|$, $H(t)$ is the Hubble parameter at the time t, M_{PL} is the Planck mass and ϵ is the slow-roll parameter of inflation defined by $\epsilon \equiv -\frac{\dot{H}}{H^2}$. Using the slow-roll equation, we can express this parameter as a function of the potential V of the inflaton ϕ (the scalar field responsible for the inflation):

$$\epsilon \approx \frac{M_{PL}^2}{2}\left(\frac{V'}{V}\right), \tag{1.24}$$

where $V' = \frac{V(\phi)}{d\phi}$. The curvature power, the observable quantity, is finally defined as

$$\Delta_{\mathcal{R}}^2(k) \equiv \frac{k^3}{2\pi^2}\langle|\mathcal{R}_{\mathbf{k}}|^2\rangle \approx \frac{1}{24\pi^2}\frac{V}{M_{PL}^4\epsilon}, \tag{1.25}$$

[11] The argument goes as follows: If we consider a black hole with classical horizon, quantum mechanics teach us that this black hole will radiate electromagnetic waves, see [28]. The same reasoning applies for any massless boson. Therefore gravitational waves are also emitted through Hawking-like process. In a universe with accelerated expansion, there is an horizon-like object: an object beyond which information disappears. In this case, the observer sees the horizon from inside. In other words, "we" are *inside* the horizon. As a consequence, this horizon must radiate gravitational waves. For a more rigorous argument and for a thorough review of the relation between inflation models and CMB polarisation, we refer to [39].

where we used in the last line $H^2 \propto V$. We therefore see that the curvature perturbation is sensitive to *two* parameters of the inflation: the potential and the slow-roll parameter. We can follow the same line of reasoning with the tensor perturbation. In this case, we use a metric parametrized by $g_{ij} = a^2(t)(\delta_{ij} + 2h_{ij})$ with $\partial^j h_{ij} = 0$ and $h^j_j = 0$ (transverse traceless). This gauge fixing leaves two independent degrees of freedom, that are conventionally called h_+ and $h_\times$. Inserting this parametrization in the Einstein-Hilbert action, Fourier transforming, we end up again with the harmonic oscillator-like action, with a non-vanishing variance. From this variance we can define the observational power spectrum of the gravitational waves by

$$\Delta_h^2(k) \equiv 2\frac{k^3}{2\pi^2}\langle|h_{p,\mathbf{k}}|^2\rangle = \frac{2}{\pi^2}\frac{H^2}{M_{PL}^2}. \tag{1.26}$$

Using again $V \propto H^2$, we see that the power spectrum of the gravitational waves does depend only on *one* parameter of the inflaton: its potential. Therefore a direct observation of this power spectrum can give an experimental value for this potential. Usually, we define also the ratio of the tensor power over the scalar power:

$$r = \frac{\Delta_h^2(k)}{\Delta_\mathcal{R}^2(k)}. \tag{1.27}$$

The current experimental upper bound on this ratio is

$$r < 0.1$$

(see for example [7]).

Therefore, an observation and quantitative measurement of $\mathcal{B}$-modes of the CMB should be able to rule out many inflation models (see for example [44] for a thorough review of inflation models) and turn interest toward more realistic ones. Note that most models of inflation predict a r ratio roughly between 0.1 and 0.001. however, some models using for example electroweak-symmetry breaking (for example [40]) predict a much weaker value (of the order $r \sim 10^{-52}$).

The main problem in this reasoning is that the parallel transport of the CMB photons, from the emission to the observer, could also induce a rotation and therefore $\mathcal{B}$-modes, blurring the primordial $\mathcal{B}$-modes present at the emission of the CMB. The goal of this work will be to try to predict the possible $\mathcal{B}$-modes induced by the parallel transport, assuming a pure $\mathcal{E}$-field at the emission [12].

1.6 Approximations

Before to go to the first calculations, we explain and justify the approximations we will use all along the work.

First of all, we will use the flat-sky approximation. This approximation is valid roughly for $l > 100$, which means angles smaller than 1 degree. As a justification, the physics we want to capture in this work lies in the small scales, below one degree, because this is the typical dimension of the

[12] Of course, this effect is not the only that could induce $\mathcal{B}$ of the polarisation, but this will be one of our main concern in this work.

Hubble horizon at the time of recombination.

We are only interested in photons path described by the geodesic equation. This geodesic equation being invariant with respect to space-time dilatation[13], the expansion of the universe does not enter the game. Therefore, we can safely neglect the dilatation factor.

To shed light in particular on the second-order calculations, we will also do an additional approximation: we assume the Weyl field $\phi(\mathbf{x})$ small with respect to its spatial variations $\partial\phi(\mathbf{x})$. At second order, we therefore neglect terms proportional to $H^2\phi(\mathbf{x})\phi(\mathbf{y})$ and $H\phi(\mathbf{x})\partial\phi(\mathbf{y})$ (where H is the Hubble parameter) with respect to $\partial\phi(\mathbf{x})\partial\phi(\mathbf{y})$. The latter will give the rotation we are searching for when inserted into the equations. The time variation of the field is also negligible with respect to the spatial variation of the field. We therefore also neglect $\dot{\phi}$, and then

$$\phi(\mathbf{x}, t) \equiv \phi(\mathbf{x}).$$

To explain this last approximation, we can do the following reasoning: in this work we are focusing on small scale perturbations (galactic or cluster dimension with respect to observable universe typical dimension). In the Fourier transform of the matter perturbations,

$$\phi(\mathbf{x}) \propto \int \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

rapidly varying field $\phi(\mathbf{x})$ corresponds to a plane wave $e^{i\mathbf{k}\cdot\mathbf{x}}$ with large $\mathbf{k}$. Now if we compare the Fourier transform of $H\phi(\mathbf{x})\partial\phi(\mathbf{y})$ and $\partial\phi(\mathbf{x})\partial\phi(\mathbf{y})$,

$$H\phi(\mathbf{x})\partial\phi(\mathbf{y}) \propto i\mathbf{k}, \tag{1.28}$$

and

$$\partial\phi(\mathbf{x})\partial\phi(\mathbf{y}) \propto -\mathbf{k}\cdot\mathbf{k}', \tag{1.29}$$

we note that the first term is negligible with respect to the second one when we consider large $\mathbf{k}$.

[13] Or conformal transformation.

2 Geodesic equation and deflection angle

When it was discovered, in 1964, the CMB was no more than an highly homogeneous, electromagnetic, with perfect black body spectrum signal coming from every direction in the sky. Despite being at first sight a quite simple physical effect, it had an immediate and tremendous consequence: it was the best confirmation of the Big Bang at that time. However, after more accurate measurements, some subtleties appeared. The CMB was no longer a perfectly homogeneous signal, but a complicated pattern of temperature fluctuations, of the order 10^{-5}. These inhomogeneities can be understood in the framework of the early Universe evolution and the CMB temperature fluctuations became immediately a rich source of information, and actually the oldest relic we are able to study. As the small scale observations improved, subtle effects suddenly appeared to matter. Indeed, the presence of matter distributed in the Universe is responsible for a well-know deflection of light rays, usually called *lensing*. This chapter is dedicated to a thorough presentation of the *delensing* and to the construction of the necessary formulas.

Contents

2.1 Preliminaries

2.1.1 Situation

In this chapter, we address the so-called "lensing" effect, which is due to a weak gravitational perturbation. This physical effect goes as follow: when looking at positions, say, in the CMB temperature, the correct identification of the emission point of the photon is crucial, because it subsequently serves to build the correlation functions. In a Universe without inhomogeneities, this would be straightforward. We could immediately identify the observed direction and the direction of emission of the photon. This is the case of the black dotted line on Fig.2.1. However, our Universe is filled with matter inhomogeneities that induce a deflection, and then a complicated path from emission to reception for the photon (red solid line on Fig.2.1). Because of this deflection, the observed direction (the red dotted line on Fig.2.1) is not the physical direction of emission (the blue dotted line on Fig.2.1). Our purpose is therefore to built a *remapping*, which actually corresponds to the green line, mapping the *observed* direction on the *emission* direction:

$$\theta^a_O \stackrel{O\to E}{\longrightarrow} \theta^a_E. \tag{2.1}$$

Hopefully, the gravitational field in the Universe is weak and thus we can attack the problem using *weak lensing*[1] and *perturbation series* technology. A discussion of the relevance of the weak lensing approximation is provided in [41]. Overall, the order of magnitude of the remapping angle is roughly one arcminute (see [57], [50]). An easy argument (see [41]) for the r.m.s order of magnitude goes as follow: during its path, the photon meets several density fluctuations. The typical depth of a potential well is $\sim 2\times 10^{-5}$. Then using that the deflection angle is given by $\frac{2\phi}{c^2}$ in Newtonian gravity, or twice this result in Einsteinian gravity, the expected deflection is of order $\sim 10^{-4}$. Now, using that the distance to last scattering surface (comoving) is $14,000$ Mpc and that the scale of potential wells is 300 Mpc, we expect the photon to meet 50 density fluctuations. The r.m.s deflection is therefore $\sim 50^{\frac{1}{2}}\times 10^{-4} \sim 7\times 10^{-4}$, or about 2 arcminutes. What is the effect on acoustic peaks ? Let us assume the primary peaks, with typical size of ~ 60 arcminutes. Lensing produces a broadening of primary acoustic peaks of order $\sim \frac{2}{60} = 3\%$. Strong lensing is however present, due to small scales lenses, and induces significant magnification ([41]). Dealing with this kind of effect would be much more difficult, hopefully strong lensing is present mainly for scales down to the arcminutes. At this scale, the spectra are really smooth due to damping and magnification of smooth surface has no effect on what we see.

For a long time, the Born-approximation was considered as sufficient (as claimed for example in [30] and [52]). However, more recently the next-to-leading effects, namely post-Born continuation and lens-lens coupling, have received more interest (see for example [47]). Therefore, it is now our goal to develop the usual lensing technology.

[1] The use of *weak* can be a bit ambiguous. In this context it means that the deflection angle (the angle between the blue and the red line on the figure) is small and that we exclude "strong" lensing, that is to say, when the line of sight cross and produce caustics. This strong lensing is not considered here.

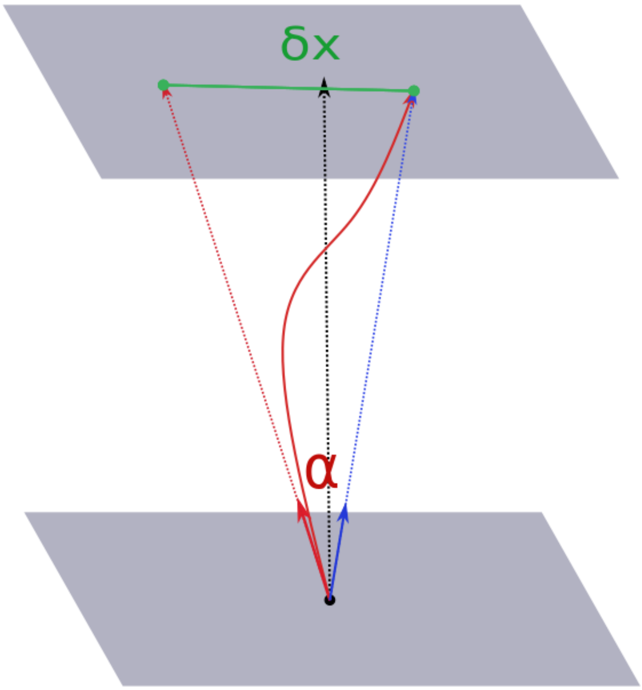


Fig. 2.1: Illustration of the lensing effect on apparent position. The black dotted line is the so-called unperturbed path. In a Universe without matter, the photon would follow a straight line from the source (the upper grey plane) to the observer (lower grey plane). The solid red line is the *real* photon path, due to presence of deflecting matter in the Universe. Now, the blue direction is the *physical direction* in which the photon has been emitted, while the red direction (the red dotted line) is the *observed* direction of emission. The difference between the two direction is sketched using a green line: this is the remapping induced by lensing.

2.1.2 Metric, canonical observers and Christoffel symbols

As a first approximation of the line element, we can use a parametrization without off-diagonal terms

$$ds^2 = -(1 + 2\phi_W(\mathbf{x}) + \phi_W^{(2)}(\mathbf{x}))dt^2 + (1 - 2\phi_W(\mathbf{x}) - \phi_W^{(2)}(\mathbf{x}))d\mathbf{x}^2. \tag{2.2}$$

This is the so-called *Poisson* gauge: the perturbative gravitational information is encoded in the Weyl potential ϕ_W and $\phi_W^{(2)}$ and the metric is diagonal. Here we assume that the Weyl field ϕ_W is time-independent. We define the Killing vector by $u = \frac{1}{(1+2\phi_W+\phi_W^{(2)})^{\frac{1}{2}}}\partial_t$, which is the canonically normalized four-velocity of a static observer.

We could also use the line element with the slightly modified form

$$ds^2 = -e^{2\phi_W(\mathbf{x})}dt^2 + e^{-2\phi_W(\mathbf{x})}d\mathbf{x}^2. \tag{2.3}$$

Because this simplifies the computations, we will mainly use this form in the following. In this case, we define the Killing vector u as follow:

$$u = e^{-\phi_W}(\mathbf{x})\partial_t. \tag{2.4}$$

From now on, we will drop the index W in ϕ_W, because there is no possible ambiguity with order quantities. The following expansions will prove very useful:

$$-e^{2\phi(\mathbf{x})} = -(1 + 2\phi + 2\phi^2) + O(3), \tag{2.5}$$

$$e^{-2\phi(\mathbf{x})} = (1 - 2\phi + 2\phi^2) + O(3), \tag{2.6}$$

and

$$e^{-\phi(\mathbf{x})} = 1 - \phi + \frac{1}{2}\phi^2 + O(3). \tag{2.7}$$

For example, the last one can be used to work out the correction to the observer four-velocity u.

$$u = u^{(0)} + \delta u + \delta^{(2)}u + O(3) = (1 - \phi + \frac{1}{2}\phi^2 + O(3))\partial_t, \tag{2.8}$$

This can be verified by using the normalisation condition for u, namely

$$\langle u, u\rangle = -1$$

where the brackets mean a scalar product with respect to the metric (2.3).

When computing the geodesic equation, we will make an extensive use of the Christoffel symbols. Let us compute them now. We know that they can be expressed in terms of the metric as follows:

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\gamma}(g_{\gamma\alpha,\beta} + g_{\beta\gamma,\alpha} - g_{\alpha\beta,\gamma}).$$

Those symbols are very trivial to work out with a diagonal metric. Therefore, we can use the rectangular $(t, \mathbf{x})$ or the spherical coordinates (t, r, θ, φ) [2]. In spherical coordinates, the line element takes the form

$$ds^2 = -e^{2\phi(\mathbf{x})}dt^2 + e^{-2\phi(\mathbf{x})}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2). \tag{2.9}$$

[2] We have to apologize for the confusing notation, φ is a coordinate angle and ϕ is the Weyl field.

From this line element, we can easily deduce the metric $g_{\mu\nu}$ and its inverse. If we write $U = e^{-2\phi(\mathbf{x})}$ and $K = e^{2\phi(\mathbf{x})}$, the matrices are

$$g_{\mu\nu} = \begin{pmatrix} -K & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & r^2U & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta U \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} -\frac{1}{K} & 0 & 0 & 0 \\ 0 & \frac{1}{U} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2U} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta U} \end{pmatrix}.$$

Using this form we can easily compute the Christoffel symbols. Because of our specific choice of metric (2.3), the result is actually non-perturbative and can therefore be used at every order without any additional correction.

$$\Gamma^0_{00} = \dot{\phi} \approx 0, \qquad \Gamma^0_{ij} = -\dot{\phi}\delta_{ij} \approx 0, \qquad \Gamma^0_{0i} = \partial_i\phi, \qquad \Gamma^i_{00} = \partial^i\phi, \tag{2.10}$$

$$\Gamma^j_{i0} = -\dot{\phi}\delta^j_i \approx 0, \qquad \Gamma^i_{jk} = -\delta^i_j\partial_k\phi - \delta^i_k\partial_j\phi + \delta_{jk}\partial^i\phi. \tag{2.11}$$

2.1.3 Redshift along the path

The redshift is defined such that

$$1 + z = \frac{\omega_E}{\omega_R},$$

where we know that the frequency ω of a photon is proportional to its energy. Therefore we deduce that

$$1 + z = \frac{E_E}{E_R} = \frac{k^\mu_E u_{\mu E}}{k^\mu_R u_{\mu R}},$$

where k^μ is the four-impulsion of the photon and we used the canonical observers (2.4) to define the frequency observed. There is a variation of the frequency along the parallel transported curve if and only if the scalar product of u and k, $\langle u, k\rangle$ varies along the photon path.

This variation can be captured by deriving (covariantly) the scalar product between u and k

$$\nabla_k\langle u, k\rangle = \langle \nabla_k u, k\rangle + \langle u, \nabla_k k\rangle = \langle \nabla_k u, k\rangle,$$

k being parallel transported, we get a redshift (or a blueshift) if the

$$\int_E^O \langle \nabla_k u, k\rangle$$

is non-vanishing. In other words, the observers notice a redshift if they are not themselves parallel transported. Let us separate the four-impulsion vector in spatial and time parts, $k = \omega(u + n)$ such that $\langle u, n\rangle = 0$ and $\langle n, n\rangle = 1$ at all orders. Inserting this decomposition in $\nabla_k u$ we obtain

$$\nabla_k u = \nabla_{\omega(u+n)} u = \omega\nabla_u u + \omega\nabla_{n^i\partial_i} u, \tag{2.12}$$

where in the last equality we used the linearity of the covariant derivative. With our definition of the observers $u = e^{-\phi}\partial_0$ (from now on, we will use the notation $\partial_0 = \partial_t$), we obtain

$$\nabla_k u = \omega(e^{-2\phi}\Gamma^\mu_{00}\partial_\mu + n^i e^{-\phi}\Gamma^\mu_{0i}\partial_\mu - e^{-\phi}n^i\partial_i\phi\partial_t), \tag{2.13}$$

or, written directly in terms of components,

$$(\nabla_k u)^\mu = \omega(e^{-2\phi}\Gamma^\mu_{00} + n^i e^{-\phi}\Gamma^\mu_{i0} - e^{-\phi}n^i\partial_i\phi\delta^\mu_0). \tag{2.14}$$

The scalar product is then easily found to be

$$\langle\nabla_k u, k\rangle = g_{\mu\nu}k^\nu(\nabla_k u)^\mu = \omega^2(e^{-2\phi}n_i\Gamma^i_{00} + g_{00}u^0 n^i e^{-\phi}\Gamma^0_{i0} - e^{-\phi}u^0 g_{00}n^i\partial_i\phi) = \omega^2 e^{-2\phi}n_i\partial^i\phi, \tag{2.15}$$

where we used $\Gamma^0_{00} = \Gamma^i_{j0} = 0$, $\Gamma^0_{i0} = \partial_i\phi$ and $\Gamma^i_{00} = \partial^i\phi$. Therefore, at first order, the redshift is given by

$$z \approx -\omega\int_O^E n_i\partial^i\phi(\lambda)d\lambda = -(\phi_E - \phi_O). \tag{2.16}$$

We can easily understand this result: If we assume that the emitter is in a potential well, with a lower potential, it will have to lose energy to reach the point of the observer, with higher potential. This thus induces a redshift.

2.2 First order deviation

2.2.1 First order calculation of the geodesic deviation

The usual equation describing the path of a photon is the geodesic equation, given by[3]

$$k^\nu\nabla_\nu k^\mu = 0, \tag{2.20}$$

or, written in vector notation,

$$\nabla_k k = 0. \tag{2.21}$$

We now expand k^μ up to first order: $k^\mu = k^{(0)\mu} + \delta k^\mu + O(2)$ (from now on, when there is no confusion possible, we call $k^{(0)\mu} = k^\mu$). The zeroth order corresponds to the case where all the Christoffel symbols vanish $\Gamma^\mu_{\alpha\beta} = 0$, because at this order, we consider an empty universe. We have thus

$$\frac{dk^\mu}{d\lambda} = 0,$$

[3] An interesting argument for this very common fact is given in [24] and goes as follow: Let us apply the covariant derivative on the photon dispersion relation $k^\mu k_\mu = 0$, we obtain

$$0 = 2k^\mu\nabla_\nu k_\mu. \tag{2.17}$$

Writing $k^\mu = \partial^\mu\phi$ and using the symmetry of the Christoffel symbols, we see easily that

$$\nabla_\nu k_\mu = \nabla_\mu k_\nu. \tag{2.18}$$

We can therefore rewrite (2.17) in the form

$$0 = 2k^\mu\nabla_\mu k_\nu, \tag{2.19}$$

where we recognize the equation for the propagation of light, the geodesic equation. So, we can conclude that the photon follows geodesics because it can be written as the gradient of a scalar function and because of it has zero mass.

from which we obtain that k^μ is a constant. The geodesic equation for k^μ at first order is

$$\frac{d\delta k^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta = 0. \tag{2.22}$$

Finally, integrating along the unperturbed trajectory (from the observer at $\lambda = 0$ to some λ), we obtain the full expression at first order:

$$\delta k^\mu(\lambda) = \delta k^\mu(0) - \int_0^\lambda \Gamma^\mu_{\alpha\beta}(x^\mu(\lambda')) k^\alpha k^\beta d\lambda'.$$

The constant $\delta k^\mu(0)$ will serve to fix the boundary conditions and will be discussed later. This expression is explicitly function of λ, where $x^\mu(\lambda_0)$ is the position of the emitter. To be consistent, we will define the $\lambda = 0$ as the position of the observer on earth (the Plank satellite, for example). Then in the following, an integration from $\lambda = 0$ to $\lambda = \lambda_0$ will be an integration on the full path. It seems also very physical to assume the observer at a fixed position. This enables us to impose the value of the field at the position of the observer

$$\phi(0) = 0.$$

We will now write explicitly this equation for the time and the spatial components.

2.2.2 Time component

The time component is given by (2.22) with $\mu = 0$. Replacing the Christoffel symbols by their explicit values, we obtain

$$\delta k^0(\lambda) = \delta k^0(0) - \int_0^\lambda \dot{\phi}(x^\mu(\lambda)) k^0 k^0 - \dot{\phi}(x^\mu(\lambda')) k^i k^i + 2\partial_i \phi k^i k^0 d\lambda'.$$

Because $0 = k^\mu k_\mu = k^0 k_0 + k^i k_i \approx -k^0 k^0 + k^i k^i$, the first two terms cancel within our approximation[4]. We are left with

$$\delta k^0(\lambda) = \delta k^0(0) - 2\int_0^\lambda \partial_i \phi(x^\mu(\lambda')) k^i k^0 d\lambda'. \tag{2.23}$$

This equation can be greatly simplified if we introduce the decomposition of k in terms of n and u, namely

$$k^\mu = \omega(u^\mu + n^\mu),$$

where ω is the frequency of the photon at the emission (by definition). The correction of the k vector at first order induces corrections in the frequency ω and in the vector n [5]:

$$\delta k^\mu = \omega \delta u^\mu + \delta\omega(u^\mu + n^\mu) + \omega \delta n^\mu. \tag{2.26}$$

[4] Remember that we are neglecting every $\phi\partial\phi$-terms and thus we can raise and lower indices $k^i \leftrightarrow k_i$ and $k^0 \leftrightarrow k_0$ with the flat metric δ_{ij}.

[5] The perturbations of those three quantities are submitted to strong constraints. The n and u have to keep the same normalisation, implying the condition $n^\mu n_\mu = 1$ and $u^\mu u_\mu = 1$. The last constraint is the orthogonality of u and n which is trivially verified by the fact that the vector u has only time component and the vector n has only spatial component.

We have already worked out the correction to u in (2.8). At first order we had

$$\delta u(\lambda) = -\phi(\lambda)u. \tag{2.27}$$

As a consequence, keeping only the time component, we end up with

$$\delta k^0(\lambda) = u\delta\omega(\lambda')\Big|_0^\lambda + \omega\delta u(\lambda')\Big|_0^\lambda, \tag{2.28}$$

from which we can isolate $\delta\omega(\lambda)$

$$\delta\omega(\lambda) - \delta\omega(0) = \frac{1}{u}\delta k^0(\lambda) + \omega(\phi(\lambda) - \phi(0)). \tag{2.29}$$

This last equation allows us to determine the redshift using only the correction to the time component k^0. Plugging (2.23), using the fact that $k^\mu\partial_\mu\phi = \frac{d\phi}{d\lambda}$, and neglecting the $\dot{\phi}$, we obtain

$$\delta\omega(\lambda) = \delta\omega(0) - 2\omega\int_0^\lambda \frac{d\phi}{d\lambda'}d\lambda' + \omega(\phi(\lambda) - \phi(0)) = \delta\omega(0) + \omega(\phi(0) - \phi(\lambda)). \tag{2.30}$$

Imposing the boundary condition, we can set $\delta\omega(0)$ to zero, leading to

$$\delta\omega(\lambda) = -\omega\phi(\lambda), \tag{2.31}$$

The redshift is then easily seen to be

$$1 + z = \frac{\omega_E}{\omega_O} = \frac{\omega + \delta\omega_E}{\omega + \delta\omega_O} = \frac{1 - \phi_E}{1 - \phi_O} \approx 1 + \phi_O - \phi_E, \tag{2.32}$$

which is consistent with (2.16).

2.2.3 Spatial components

The spatial components proceed in the same way. We put $\mu = i$ in eq (2.22) and use the Christoffel symbols computed before[6] to obtain

$$\begin{aligned}\delta k^i(\lambda) &= \delta k^i(0) - 2\int_0^\lambda \partial^i\phi(\lambda')k^0k^0 - k^ik^m\partial_m\phi(\lambda')d\lambda' \\ &= \delta k^i(0) - 2\int_0^\lambda \partial^i\phi(\lambda')\omega^2 + \omega n^i\frac{d\phi}{d\lambda'}d\lambda'.\end{aligned} \tag{2.33}$$

Therefore, the same must be true of their corrections:

$$\langle u, n\rangle|_{\text{unperturbed}} = 0 = \langle u + \delta u, n + \delta n\rangle. \tag{2.24}$$

implies

$$\langle \delta u, n\rangle = 0, \qquad \langle u, \delta n\rangle = 0. \tag{2.25}$$

Thus, the perturbations we are studying are only rotations in three dimensional space of the n vector and a redshift part coming from the variation of ω. Thus the variation of the four-impulsion k^μ induces a variation of the frequency of this photon and a variation of its direction.

[6] and using the relation $\mathbf{k}^2 = (k^0)^2$

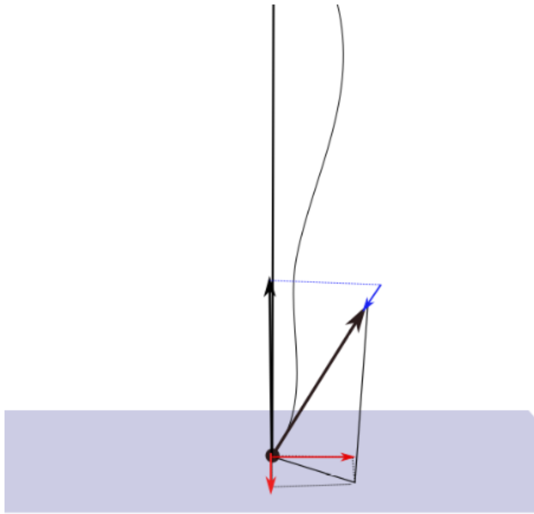


Fig. 2.2: Illustration of the correction at first order. The red correction comes from the third term in (2.35) (this will give us the deflection angle later) and the blue corrections correspond to the second term (the normalisation part, which ensures that $\langle n, n\rangle = 1$ at first order).

We can rewrite this expression with the handy decomposition in transverse and parallel derivatives described in the notations. In doing so, we note that the derivative along the line of sight (parallel component) cancels and we are left only with the component along the transverse directions, namely

$$\delta k^i(\lambda) = \delta k^i(0) - 2\omega^2 \int_0^\lambda \nabla^i_\perp \phi(\lambda')d\lambda'. \tag{2.34}$$

In rectangular coordinates, $\nabla^i_\perp = (0, 0, \partial_2, \partial_3)$ and in spherical coordinates, $\nabla^i_\perp = (0, 0, \frac{1}{r}\partial_\theta, \frac{1}{r\sin\theta}\partial_\phi)$. We keep explicitly the first term, the integration constant, because it will give us the possibility to make a change of boundary conditions.

Doing as in the previous section, we use the decomposition of k, $k^\mu = \omega(u^\mu + n^\mu)$, which, for the i components, reduces to

$$\delta k^i(\lambda) = n^i \delta\omega\Big|_0^\lambda + \omega\delta n^i\Big|_0^\lambda.$$

To factorize the variation of n, δn^i, we insert the variation of frequency at λ, (2.30). We obtain the correction to the "unit direction vector"

$$\boxed{\delta n^i(\lambda) = \delta n^i(0) + n^i\phi(\lambda) - 2\omega^2 \int_0^\lambda \nabla^i_\perp \phi(\lambda')d\lambda'.} \tag{2.35}$$

The second term gives the correction along the path itself (this insures the normalisation of n as we will see in the next subsection) and the third term is the correction perpendicular to the path. The figure (2.2) gives an physical illustration of this.

2.2.4 Normalisation of the n^i vector along the photon geodesic

We will pause here to check the normalisation of the vectors k and n. The spirit is as follows. Without perturbation (at zeroth order) we start with a field of vectors normalized all along the path. Switching on the Weyl field ϕ, we perturbed this normalisation. So in a first time, we will impose the normalisation condition at an arbitrary point of the path, for example $\lambda = \lambda_0$. To do so, we will use the freedom contained in the constant of integration (the remaining freedom will be use to enforce the boundary conditions). On the other hand, parallel transport is supposed to conserve the normalisation all along the path, thus, once we have imposed the normalisation somewhere, we must recover it for every point of the path. We will check it in this section.

We will start with k^μ submitted to the condition

$$g_{\mu\nu}k^\mu k^\nu = 0. \tag{2.36}$$

If we keep terms up to first order, we end up with the condition

$$\begin{aligned} 0 &= -2\delta_{ij}k^i\delta k^j + 2\phi k^i k^j + 2k^0\delta k^0 + 2\phi k^0 k^0 \\ &= -2\delta_{ij}k^i\delta k^j + 2k^0\delta k^0 + 4\phi(k^i k_i). \end{aligned} \tag{2.37}$$

Inserting the expressions computed above, the first term vanishes (because, as we said, the correction of k along n cancels) and we are left with

$$2k^0\delta k^0(0) + 4(\phi(0) - \phi(\lambda))k^0k^0 + 4\phi(\lambda)(k^0k^0) = 0, \tag{2.38}$$

and then, we see that if we impose

$$\delta k^0(0) = -2\phi(0)k^0, \tag{2.39}$$

the condition is satisfied.

Now that we imposed this at $\lambda = \lambda_0$, we can easily see, following the same calculation, that the condition (2.36) is respected for every λ, as imposed by the parallel transport.

Let us now turn to the $\mathbf{n}$ vector and impose the normalisation condition $\langle n, n\rangle = 1$. Writing the corrected "unit direction vector" in the form $\mathbf{n} + \delta\mathbf{n}$, the condition at first order is

$$\begin{aligned} 1 &= \langle \mathbf{n} + \delta\mathbf{n}, \mathbf{n} + \delta\mathbf{n}\rangle \\ &= g_{ij}n^in^j + 2g_{ij}n^i\delta n^j \\ &= \delta_{ij}(1-2\phi)n^in^j + 2g_{ij}n^i\delta n^j. \end{aligned}$$

The first term is obviously equal to 1, while in the third term we can replace g_{ij} by the flat metric δ_{ij}, the corrections being of higher order. We obtain

$$\begin{aligned} 0 &= -2\phi\delta_{ij}n^in^j + 2\delta_{ij}n^i\delta n^j \\ &= -2\phi + 2\delta_{ij}n^i\delta n^j \end{aligned}. \tag{2.40}$$

Taking the expression (2.35) for δn, we can compute $\delta_{ij}n^i\delta n^j$, namely

$$\delta_{ij}n^i\delta n^j = n_i\delta n^i(0) + \phi(\lambda).$$

Therefore, the condition above is satisfied if we take

$$n_i \delta n^i(0) = 0.$$

We could for example impose

$$\delta n^i(0) = 0, \tag{2.41}$$

but this is not the only possibility.

2.2.5 Different integration parameters: coordinate time and coordinate distance

We know that λ is an affine parameter which parametrizes the running of photon along its path. There are mainly two ways to rename this parameter[7]. First, we can call it the coordinate distance r and then the integral must be performed from $r = 0$, the place of the observer, to $r = R$[8], the place where the emission occured (last scattering surface). Then we can write the coordinate distance as a function of the affine parameter $r(\lambda)$, the integrals therefore read

$$\delta k^i(0) = \delta k^i(R) + \int_0^R \Gamma^i_{\alpha\beta}(r) k^\alpha k^\beta |J| dr, \tag{2.42}$$

where the Jacobian of the transformation $\lambda \to r(\lambda)$ is

$$J^{-1} = \frac{dr(\lambda)}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{dr}{dx^\mu}.$$

In the basis (t, r, θ, φ), we are led to $\frac{dx^\mu}{dr} = (0, 1, 0, 0) = \delta^\mu_r$. Thus

$$|J^{-1}| = |k^r| = |k^{(0)r} + \delta k^r + O(2)| = \omega |n^r| + O(2). \tag{2.43}$$

because $\delta k^r = 0$. Also we can easily see that $|n^r| = 1$, the unperturbed vector n points in the r-direction. Finally, inserting that in (2.42), we find

$$\delta k^i(0) = \delta k^i(R) + \frac{1}{\omega}\int_0^R \Gamma^i_{\alpha\beta}(r) k^\alpha k^\beta dr \tag{2.44}$$

The other possibility is to parametrize with the coordinate time and then to integrate from $t = 0$ to $t = t_0$[9]. Again, we can write the coordinate time as a function of the parameter $t(\lambda)$. The integrals become

$$\delta k^i(0) = \delta k^i(t_0) + \int_0^{t_0} \Gamma^i_{\alpha\beta}(t') k^\alpha k^\beta |J| dt', \tag{2.45}$$

where the Jacobian for this transformation $\lambda \to t(\lambda)$ is

$$J^{-1} = \frac{dt(\lambda)}{d\lambda} = \frac{dx^\mu}{d\lambda}\frac{dt}{dx^\mu}.$$

[7] Let us note that r, the coordinate distance, is not an affine parameter.
[8] For the unperturbed path, a deviation in θ or φ will never appear.
[9] This is maybe an awkward convention: t_0 is a negative quantity. Therefore $|t_0| = -t_0$.

Again, in the basis (t, r, θ, ϕ), we find

$$J^{-1} = k^t,$$

which at zeroth order (going to higher order would give a second-order contribution) leads to

$$|J^{-1}| = \omega|u| = \omega.$$

Inserting in the above relation, we have

$$\delta k^i(0) = \delta k^i(t_0) + \frac{1}{\omega}\int_0^{t_0} \Gamma^i_{\alpha\beta}(t')k^\alpha k^\beta dt'. \tag{2.46}$$

This case would correspond to the case where we fixed the time of the recombination in the universe at t_0 and perform all the integration until that time.

The last possibility would be simply to keep λ in the equation as we have done above. It is worth to emphasize that this could, in principle, give different interpretations and also different answers. The Fig.(2.3) and the next subsection give some insight about these two possibilities.

2.2.6 Work at fixed Recombination time or fixed affine parameter ? Time delay

Let us explain with more details the last statement, it may be not as obvious as it seems. The goal of this section is to show that, in our approximation, the solution will *not depend* on which parameter we choose to fix (the corrections will be negligible). Therefore, we are free to fix a parameter as we prefer.

First, if we suppose a running of the parameter λ from $\lambda = 0$ to $\lambda = \lambda_0$ and a running of the time coordinate from $t = 0$ to $t = t_0$, we see that the total running of λ is λ_0 and the total running of t is $|t_0|$. We can relate them using the unperturbed path

$$k^\mu = \frac{dx^\mu}{d\lambda},$$

which gives

$$\int_0^{\lambda_0} k^\mu d\lambda = d^\mu, \tag{2.47}$$

where d^i is the distance vector from the source to the observer, and $d^0 = |t_0|$. Then, if we just focus on the time component, we have

$$\int_0^{\lambda_0} k^0 d\lambda = \int_0^{\lambda_0} \omega d\lambda = |t_0|. \tag{2.48}$$

On the other hand, even though at zeroth order we would conclude that $|t_0| = \omega\lambda_0$, this is no longer true at higher orders, because the k^0 starts to get some corrections. Introducing the corrections, we have indeed

$$\int_0^{\lambda_0} k^0 d\lambda = \omega\int_0^{\lambda_0} 1 - 2\phi(\lambda) + O(2) d\lambda = |t_0|. \tag{2.49}$$

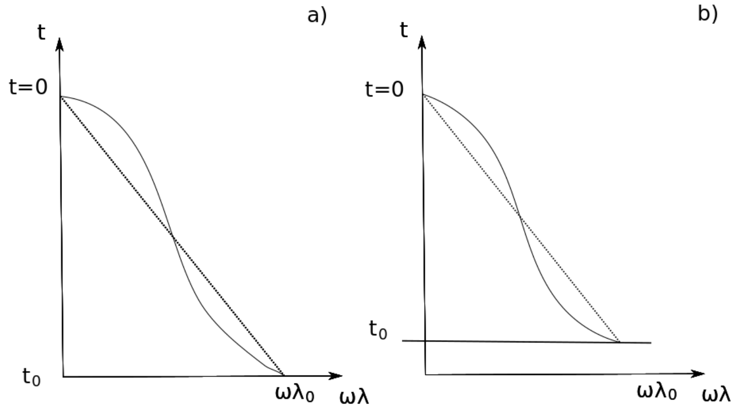


Fig. 2.3: Comparison of the two cases : λ_0 fixed and t_0 fixed. On a) we can see the case t_0 fixed. In that case, the length along the t axis is fixed. On b) we can see the case λ_0 fixed. The time of recombination arrives earlier as we switch on the perturbations. This can be intuitively understood considering the fact that if we keep fixed the length of integration and deform it in the perturbed path, thus the straight line joining the beginning and the end will be shorter than at zeroth order.

Up to first order, we have thus the correction

$$\lambda_0 - 2\int_0^{\lambda_0} \phi(\lambda)d\lambda = \frac{|t_0|}{\omega}. \tag{2.50}$$

In this equation, we have a functional relation between λ_0 and t_0, $t_0 = t_0[\lambda_0, \phi]$. Theoretically, we have different possibilities; we can work at fixed parameter λ_0, at fixed time t_0 or at fixed coordinate distance R. If we work with fixed parameter λ_0 and if we assume[10], for example, field ϕ to be positive definite, the total running of the time $|t_0|$ will be shorter. An illustration of this phenomenon is given on the Fig.(2.3). We can also work with a fixed time t_0 which we approximate as the time of the recombination. This is an approximation, because we know that the recombination didn't happen at fixed time, but instead at fixed *temperature*.

The correspondence can be written more systematically (for an illustrative purpose, we again assume a positive definite field):

- in the λ_0 fixed case, at zeroth order we have

  $$|t_0| = \lambda_0\omega.$$

  At first order we have

  $$|t_0'| = \omega\lambda_0 - 2\omega\int_0^{\lambda_0} \phi(\lambda)d\lambda < |t_0|.$$

  The total running of the time parameter got slightly shorter.

[10] Recall that in this case, we defined the Weyl to be zero at the observer. Thus the value of the field at any other point in now absolute.

- In the t_0 fixed case, we have just to reverse the previous equations. At zeroth order

$$\omega\lambda_0 = |t_0|.$$

At first order, we have the modified parameter:

$$\omega\lambda_0' = |t_0| + 2\int_0^{\lambda_0} \phi(\lambda)d\lambda > \omega\lambda_0.$$

The parameter λ_0' slightly grew to compensate for the longer path to run over.

Assume we have done all the work in the λ_0 fixed picture. If we want to change picture and work with t_0 fixed, things will go as follow. While an integral over t remains the same :

$$\int_0^{t_0} f(x^\mu)d\lambda \overset{\text{switch on }\phi}{\longrightarrow} \int_0^{t_0} f(x^\mu)d\lambda, \tag{2.51}$$

an integration over λ gets a correction.

$$\int_0^{\lambda_0} f(x^\mu)d\lambda \overset{\text{switch on }\phi}{\longrightarrow} \int_0^{\lambda_0+\frac{2}{\omega}\int_0^{\lambda_0}\phi(\lambda)d\lambda} f(x^\mu)d\lambda. \tag{2.52}$$

The part $2\int_0^{\lambda_0}\phi(\lambda)d\lambda$ has been add to "finish" the integration and go until t_0. Assuming this term small with respect to λ_0, we can approximate f as constant from λ_0 to $\lambda_0 + \frac{2}{\omega}\int_0^{\lambda_0}\phi(\lambda)d\lambda$, giving finally

$$\int_0^{\lambda_0} f(x^\mu)d\lambda \overset{\text{switch on }\phi}{\longrightarrow} \int_0^{\lambda_0} f(x^\mu)d\lambda + \frac{2}{\omega}f(\lambda_0)\int_0^{\lambda_0}\phi(\lambda)d\lambda. \tag{2.53}$$

Doing these replacements, we still work explicitly with the parameter λ, but with t_0 kept fixed during the expansion of perturbation series. Even if we will write most of our expressions with an explicit λ, here is the prescription to go to t_0 fixed. This result can be also explicitly deduced from the geodesic path for the photon as done in [41]. This effect gives a contributions to the so-called "time delay" effect. Other contributions are explained in [41], [31] and [42]. These effects are small enough to be discarded. We can deduce this fact directly from expression (2.53): if $f(x)$ is of first order in ϕ, thus the contribution is of the type $\phi\partial\phi$, which is negligible. Therefore, in our approximation, working with t_0, λ_0 or R fixed will give the same answer (up to a factor of ω).

2.2.7 The remapping function at first order

In this section we want to work out the lensing map at first order. First, let us define the vector $\delta\mathbf{x}$ as the integral of $k^i = \frac{dx^i}{d\lambda}$. We then need to express this vector as a function of the parameter at hand. We have seen in the previous sections that this parameter could be either λ, r, or t. In a first time, we will work with λ. Because $\delta\mathbf{x}$ would be zero at zeroth order, we could just now integrate δk^μ over λ

to recover the position x^μ at first order. In doing so, we will have two boundary conditions to impose. This will be the topic of the next section. In our case, we will start from the position of the observer $\lambda = 0$ and integrate to the source at $\lambda = \lambda_0$.

$$
\begin{aligned}
\delta x^i(\lambda) &= \delta x^i(0) + \int_0^\lambda \delta k^i(\lambda')d\lambda' \\
&= \delta x^i(0) + \int_0^\lambda [\delta k^i(0) - \int_0^\lambda \Gamma^i_{\alpha\beta}(\lambda'')k^\alpha k^\beta d\lambda'']d\lambda' \\
&= \delta x^i(0) + \lambda\delta k^i(0) - 2\omega^2\tilde{\nabla}^i_\perp \int_0^\lambda \Psi(\lambda')d\lambda' \\
&= \delta x^i(0) + \lambda\delta k^i(0) - 2\omega^2 \int_0^\lambda \frac{\lambda-\lambda'}{\lambda'}\tilde{\nabla}^i_\perp\phi(\lambda')d\lambda',
\end{aligned}
\tag{2.54}
$$

which is an explicit function of λ. $\delta x^i(0)$ is the constant of integration coming from the second integration. Here, we have again to fix the constants by some boundary conditions. We will soon study different cases but, before, let us study another way to recover the equation (2.54), using directly the geodesic equation written in terms of the coordinates:

$$
\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}. \tag{2.55}
$$

The unperturbed path can be parametrized by $x^\mu = (t, r, 0, 0) = \lambda\omega(1, 1, 0, 0)$. Introducing this in the geodesic equation above, we obtain

$$
\frac{d^2\delta x^i}{d\lambda^2} = -\Gamma^i_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = -\Gamma^i_{11}\frac{dx^1}{d\lambda}\frac{dx^1}{d\lambda} - \Gamma^i_{00}\frac{dx^0}{d\lambda}\frac{dx^0}{d\lambda}, \tag{2.56}
$$

where we discarded the time derivatives. In this way, we find

$$
\frac{d^2\delta x^i}{d\lambda^2} = -2\omega^2(\partial^i\phi - n^i n^m\partial_m\phi) = -2\omega^2\nabla^i_\perp\phi. \tag{2.57}
$$

If we integrate this expression twice, we end up finally with the remapping function as before, namely

$$
\frac{d\delta x^i}{d\lambda}(\lambda) - \frac{d\delta x^i}{d\lambda}(0) = \frac{d\delta x^i}{d\lambda}(\lambda) - \delta k(0) = -2\omega^2\int_0^\lambda \nabla^i_\perp\phi(\lambda')d\lambda', \tag{2.58}
$$

and integrating again

$$
\begin{aligned}
\delta x^i(\lambda) - \delta x^i(0) &= \lambda\frac{d\delta x^i}{d\lambda}(0) - 2\omega^2\int_0^\lambda\int_0^{\lambda'} \nabla^i_\perp\phi(\lambda')d\lambda' \\
&= \lambda\frac{d\delta x^i}{d\lambda}(0) - 2\omega^2\int_0^\lambda \frac{\lambda-\lambda'}{\lambda'}\tilde{\nabla}^i_\perp\phi(\lambda')d\lambda'.
\end{aligned}
\tag{2.59}
$$

We have recovered (2.54). Thus both methods are consistent.

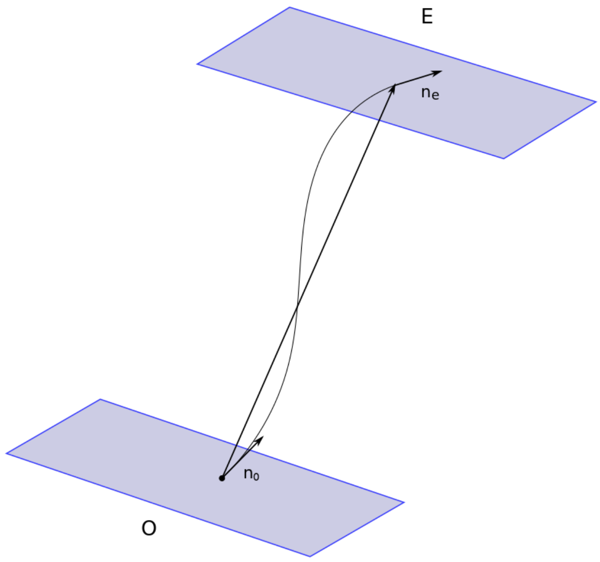


Fig. 2.4: Illustration of the "fixed points" boundary condition.

Note that another proof of this equation is provided in [41]. All of this can be rewritten in term of the spherical basis, $x = (t, r, \theta, \phi)$ which can seem more handy in many cases. The operator of derivative $\nabla_\perp$ is thus written in the form

$$\nabla_\perp = \left(\frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right).$$

We will introduce the deflection angle later in this chapter.

2.2.8 Choice of the boundary condition

In the last section, we have solved a second-order equation, thus we know that we have to fix two boundary conditions to fix definitely the δx function. In this section, we will discuss the different possibilities. We will do a theoretical enumeration and then just keep the most physical case.

Fixed points

The "fixed points" condition means that the starting and ending points of the path must remain the same for the unperturbed path and the perturbed path, and subsequently, all along the perturbation process. This implies that δx must vanish at the beginning and at the end of the path. We will start from (2.54).

$$\delta x^i(\lambda) = \delta x^i(0) + \lambda \delta k^i(0) - 2\omega^2 \tilde{\nabla}^i_\perp \int\limits_0^\lambda \Psi(\lambda')d\lambda'. \tag{2.60}$$

The vanishing of the expression for $\lambda = 0$ immediately gives that the integration constant, $\delta x^i(0)$, must be zero. For the position of the emitter $\lambda = \lambda_0$, we have

$$\delta k^i(\lambda_0) = \frac{2}{\lambda_0}\omega^2 \tilde{\nabla}^i_\perp \int\limits_0^{\lambda_0} \Psi(\lambda')d\lambda'. \tag{2.61}$$

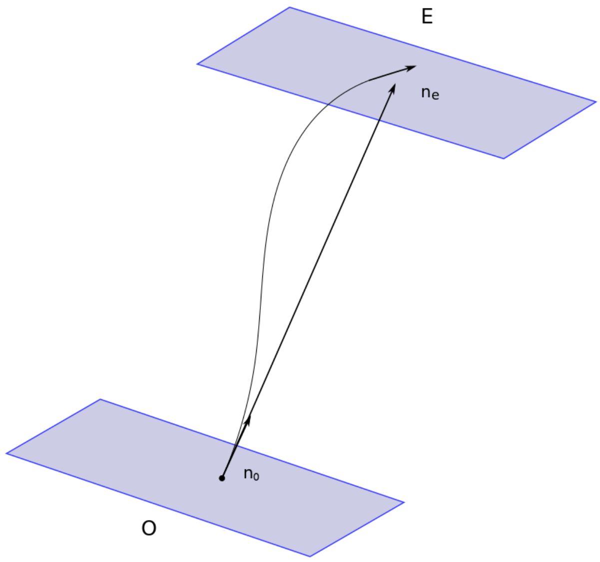


Fig. 2.5: Illustration of the "fixed direction" boundary condition.

In conclusion, for the conditions

$$\boxed{\delta x^i(0) = \delta x^i(\lambda_0) = 0,} \tag{2.62}$$

we obtain the remapping function

$$\boxed{\delta x^i(\lambda) = \frac{2\lambda}{\lambda_0}\omega^2\tilde{\nabla}^i_\perp \int\limits_0^{\lambda_0} \Psi(\lambda')d\lambda' - 2\omega^2\tilde{\nabla}^i_\perp \int\limits_0^{\lambda} \Psi(\lambda')d\lambda'.} \tag{2.63}$$

You can see an illustration of this type of boundary condition on the Fig.2.4. Of course, this way of fixing the freedom is far from being useful physically.

Fixed observer direction

The second possibility is to consider "fixed observer direction" boundary condition. This means that the starting point and the direction measured by the observer must remain the same at every order of the correction. We have therefore to impose the two following conditions: $\delta\mathbf{x}(0) = \delta\mathbf{n}(0) = 0$. Those conditions impose the vanishing of the two integration constant. Thus

$$\boxed{\delta x^i(0) = \delta k^i(0) = 0,} \tag{2.64}$$

leads to

$$\boxed{\delta x^i(\lambda) = -2\omega^2\tilde{\nabla}^i_\perp \int\limits_0^{\lambda} \Psi(\lambda')d\lambda'.} \tag{2.65}$$

This point of view is illustrated on the Fig.2.5. Being the only one physically relevant, this is the point of view we will adopt in the following.

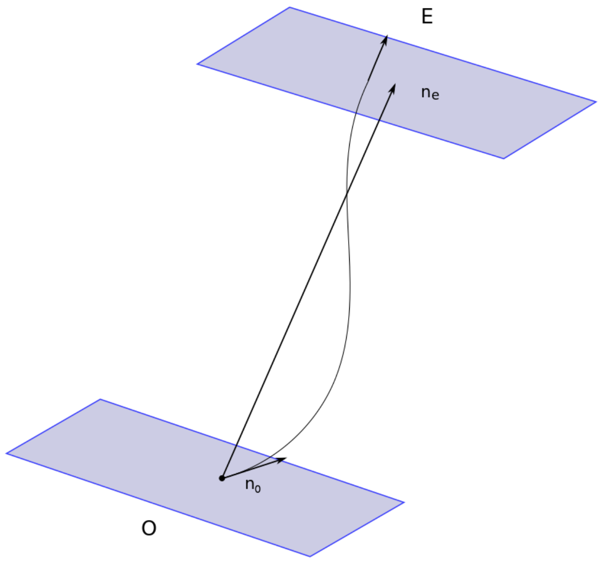


Fig. 2.6: Illustration of the "Emission direction fixed" boundary condition.

Emission direction fixed

The last point of view we could adopt is to impose the starting point and the emission direction to be the same for the perturbed and the unperturbed path. Mathematically, this is $\delta k^i(\lambda_0) = 0$ or $\delta n^i(\lambda_0) = 0$. Therefore

$$\boxed{\delta x^i(\lambda) = 2\omega^2 \tilde{\nabla}^i_\perp \int_0^\lambda \Phi(\lambda')d\lambda'} \tag{2.66}$$

comes as a consequence of

$$\boxed{\delta x^i(0) = \delta k^i(\lambda_0) = 0.} \tag{2.67}$$

We show an illustration of this point of view on Fig.2.6.

2.3 Second-order deviation

In this section we compute the second-order corrections to the path. We will again use the line element (2.3) to describe the gravitational perturbations. As we already mentioned, the Christoffel symbols we computed in (2.10) are exact, thus we will not consider correction at second order to them.

2.3.1 Second-order calculation

Inserting the expansions $k^\mu = k^{(0)\mu} + \delta k^\mu + \delta^{(2)} k^\mu$ and $\nabla_\nu = \nabla^{(0)}_\nu + \nabla^{(1)}_\nu + \nabla^{(2)}_\nu$ in the geodesic equation

$$k^\nu \nabla_\nu k^\mu = 0$$

and keeping along second-order terms, we end up with[11]

$$k^\nu \nabla_\nu^{(2)} k^\mu + k^\nu \nabla_\nu^{(0)} \delta^{(2)} k^\mu + \delta k^\nu \nabla_\nu^{(1)} k^\mu + k^\nu \nabla_\nu^{(1)} \delta k^\mu + \delta k^\nu \nabla_\nu^{(0)} \delta k^\mu = 0, \tag{2.68}$$

Upon reorganization and introduction of explicit expressions for the covariant derivatives, this translates into

$$\frac{d\delta^{(2)} k^\mu}{d\lambda} + 2\Gamma^\mu_{\alpha\beta} k^\alpha \delta k^\beta + \delta k^\nu \partial_\nu \delta k^\mu = 0. \tag{2.69}$$

The last term on LHS deserves some comments, because it has a specific interpretation. At second order, we have to keep track of fact that the path itself has been perturbed at first order: the integration should be performed along this perturbed path. Instead of doing this complicated integration, we can introduce one additional term and integrate along the unperturbed path: this is the role of the last term of the LHS in (2.69). Let us explain this claim. The derivative along the path can be written (formally)

$$\frac{d}{d\lambda} = k^\nu \partial_\nu.$$

Therefore, a variation of this path reads

$$\delta \frac{d}{d\lambda} = \delta k^\nu \partial_\nu.$$

We conclude that

$$\delta k^\nu \partial_\nu \delta k^\mu$$

is actually the so-called "post-Born term". There is another way to treat this "post-Born term", this is explained with more details in Appendix A.

We emphasize the fact that $\delta k^\alpha(\lambda)$ depends on λ while $k^{(0)\alpha}$ is constant all along the line and does not depend on the affine parameter. Integrating (2.69) and inserting the explicit expression for the Christoffel symbols[12],

$$\Gamma^0_{\alpha\beta}(x^\mu(\lambda')) k^\beta \delta k^\alpha = \partial_i \phi(x^\mu(\lambda'))(k^i \delta k^0 + k^0 \delta k^i) \tag{2.70}$$

and

$$\Gamma^i_{\alpha\beta}(x^\mu(\lambda')) k^\beta \delta k^\alpha = \partial^i \phi(x^\mu(\lambda'))(\delta k^0 k^0 + \delta k^j k_j) - \partial_j \phi(x^\mu(\lambda'))(\delta k^i k^j + k^i \delta k^j), \tag{2.71}$$

we obtain[13]

$$\begin{aligned} \delta^{(2)} k^i(\lambda_0) = \delta^{(2)} k^i(0) &- 2\int_0^\lambda \partial^i \phi(\lambda') \delta k^0(\lambda') k^0 d\lambda' + 2\int_0^\lambda \partial_j \phi(\lambda')(\delta k^i(\lambda') k^j + k^i \delta k^j(\lambda')) d\lambda' \\ &- 4\int_0^\lambda \Big[\phi(\lambda') \partial^i \phi(\lambda') k^0 k^0 d\lambda' + \delta k^j(\lambda') \partial_j \delta k^i(\lambda')\Big] d\lambda' \end{aligned} \tag{2.72}$$

[11] We used the fact that $\nabla^{(0)}$ is a derivative and that $k^{(0)\mu}$ is a constant
[12] The second term is zero because $n^i \delta k_i = 0$
[13] for simplicity, write $\Gamma^i_{\alpha\beta}(\lambda')$ instead of $\Gamma^i_{\alpha\beta}(x^\mu(\lambda'))$.

for the spatial components, and

$$\delta^{(2)}k^0(\lambda_0) = \delta^{(2)}k^0(0) - 2\int\limits_0^{\lambda_0} \Big[\partial_i\phi(\lambda')(k^i\delta k^0(\lambda') + k^0\delta k^i(\lambda')) + \delta k^i(\lambda')\partial_i\delta k^0(\lambda')\Big]d\lambda', \tag{2.73}$$

for the time component. As usual, we have discarded any derivative with respect to time. Integration is to be performed along the unperturbed path. Inserting the explicit form for the first-order correction to δk^μ and doing a bit of algebra[14] we conclude that there is no dominant contribution in $\delta^{(2)}k^0(\lambda_0)$:

$$\delta^{(2)}k^0(\lambda_0) = 0. \tag{2.74}$$

Let us be a bit more explicit with the spatial components. We can see that only the first, the third and the fifth terms in (2.72) are relevant within our approximation, because the second and the fourth are $\phi\partial\phi$-type terms. Thus we just need to compute

$$\partial_j\phi(\lambda')(\delta k^i(\lambda')k^j + k^i\delta k^j(\lambda'))$$

and

$$\delta k^i(\lambda')\partial_i\delta k^i(\lambda').$$

Plugging the explicit expressions for the first-order corrections to k^i, we can rewrite the term $2\omega\int\limits_0^{\lambda_0}\partial_j\phi n^i\delta k^j$ in the more convenient form

$$-4\omega^3\int\limits_0^{\lambda}\nabla_{\perp i}\phi(\lambda')d\lambda'\int\limits_0^{\lambda'}\nabla_\perp^i\phi(\lambda'')d\lambda''.$$

For the second part, there is however a slight subtlety, because we have a derivative applied directly on an integral. First of all, we can rewrite it in the form

$$-4\omega^4\int\limits_0^{\lambda}\int\limits_0^{\lambda'}\nabla_{\perp j}\phi(\lambda'')\partial^j\int\limits_0^{\lambda'}\nabla_\perp^i\phi(\lambda'') = -4\omega^4\int\limits_0^{\lambda}\int\limits_0^{\lambda'}\nabla_{\perp j}\phi(\lambda'')\nabla_\perp^j\int\limits_0^{\lambda'}\nabla_\perp^i\phi(\lambda''),$$

with a transverse derivative acting on the last derivative from 0 to λ. If we want to exchange to derivative and the integral, we have to remember the rule (0.10). For notational simplicity, we will not do it.

[14] we see that the term $2\int\limits_0^{\lambda_0}\partial_i\phi k^0\delta k^i d\lambda'$ and $\int\limits_0^{\lambda_0}\delta k^i\partial_i\delta k^0$ cancel each other (δk^0 being $-2\phi k^0$) except for a term

$$-4\omega\phi(0)(\phi(\lambda_0) - \phi(0)) + 2\omega(\phi^2(\lambda_0) - \phi^2(0)).$$

The term $k^i\delta k^0(\lambda')$ is of type $\phi\partial\phi$.

Summing everything together, the dominant contributions are

$$\begin{aligned}\delta^{(2)}k^i(\lambda) = \delta^{(2)}k^i(0) - 4n^i\omega^3 \int_0^{\lambda} \nabla_{\perp j}\phi(\lambda')d\lambda' \int_0^{\lambda'} \nabla_{\perp}^{j}\phi(\lambda'')d\lambda'' \\ - 4\omega^4 \int_0^{\lambda}\int_0^{\lambda'} \nabla_{\perp j}\phi(\lambda'')d\lambda''\nabla_{\perp}^{j}\int_0^{\lambda'} \nabla_{\perp}^{i}\phi(\lambda''')d\lambda'''d\lambda'\end{aligned} \tag{2.75}$$

(Notice that on the LHS, the dominant contribution comes from the last term, because it has more derivatives).

2.3.2 Normalisation at second order

In this section, we follow the same idea than the section 2.2.4. The normalisation has been perturbed by the corrections. Therefore, we impose the conditions of normalisation at some parameter λ and check that this is verified for every other point (as required by the parallel transport).

First of all, we will impose

$$g_{\mu\nu}k^{\mu}k^{\nu} = 0.$$

Expanding up to second order, we find an expression of the form

$$\begin{aligned}g_{\mu\nu}k^{\mu}k^{\nu} &= (1-2\phi+2\phi^2)\delta_{ij}k^ik^j + 2(1-2\phi)\delta_{ij}k^i\delta k^j + \delta_{ij}2k^i\delta^{(2)}k^j + \delta_{ij}\delta k^i\delta k^j \\ &\quad - (1+2\phi+2\phi^2)k^0k^0 - 2(1+2\phi)\delta k^0k^0 - 2\delta^2k^0k^0 - \delta k^0\delta k^0.\end{aligned} \tag{2.76}$$

The zeroth- and the first-order correction has been already checked and cancel. Focussing on second order, we are left with the condition

$$g_{\mu\nu}k^{\mu}k^{\nu} = 2\delta_{ij}k^i\delta^{(2)}k^j + \delta_{ij}\delta k^i\delta k^j - 2\delta^{(2)}k^0k^0 - \delta k^0\delta k^0 - 4\phi\delta k^0k^0. \tag{2.77}$$

We consider the point $\lambda = \lambda_0$. For the first term on the RHS of (2.77), we obtain

$$-8\omega^4 \int_0^{\lambda_0} \nabla_{\perp i}\phi \int_0^{\lambda} \nabla_{\perp}^{i}\phi, \tag{2.78}$$

for the second term, we obtain

$$4\omega^4 \int_0^{\lambda_0} \nabla_{\perp i}\phi \int_0^{\lambda_0} \nabla_{\perp}^{i}\phi. \tag{2.79}$$

All the other terms vanish. Satisfactory, these contributions sum up to zero. Indeed, for smooth functions, we have the identity[15]

$$\begin{aligned}\int_0^a f(x)dx\int_0^a f(x)dx &= \int_0^a f(x)\int_0^x f(x')dx'dx + \int_0^a f(x)\int_x^a f(x')dx'dx \\ &= 2\int_0^a f(x)\int_0^x f(x')dx'dx.\end{aligned} \tag{2.80}$$

This means that the normalisation condition is satisfied, at least for the leading terms.

The final expression for second-order corrections to the photon four-impulsion are[16]

$$\boxed{\delta^{(2)}k^0(\lambda) = 0,} \tag{2.81}$$

$$\boxed{\begin{aligned}\delta^{(2)}k^i(\lambda) = \delta^{(2)}k^i(0) - 2n^i\omega^3\int_0^\lambda \nabla_{\perp j}\phi(\lambda')d\lambda'\int_0^\lambda \nabla_\perp^j\phi(\lambda'')d\lambda'' \\ - 4\omega^4\int_0^\lambda\Big[\int_0^{\lambda'}\nabla_{\perp j}\phi(\lambda'')d\lambda''\nabla_\perp^j\int_0^{\lambda'}\nabla_\perp^i\phi(\lambda''')d\lambda'''\Big]d\lambda'.\end{aligned}} \tag{2.82}$$

2.3.3 Second-order correction to the unit direction vector

As we have done at first order, to make these corrections a bit more transparent, we can rewrite them in term of correction to the frequency ω and unit direction vector $\mathbf{n}$. First of all,

$$u\delta^{(2)}\omega = \delta^{(2)}k^0 - \underbrace{\omega\delta^{(2)}u}_{\frac{1}{2}\omega\phi^2 u} - \underbrace{\delta u\delta\omega}_{\omega\phi^2 u}. \tag{2.83}$$

Again, we neglect those terms and thus obtain

$$\boxed{\delta^{(2)}\omega(\lambda) = 0.} \tag{2.84}$$

Using a similar expansion for $\delta^{(2)}n(\lambda)$,

$$\delta^{(2)}n^i(\lambda) = \frac{1}{\omega}(\delta^{(2)}k^i(\lambda) - \underbrace{\delta^2\omega}_{\frac{1}{2}\omega\phi^2} - \underbrace{\delta n^i\delta\omega}_{-\omega\phi^2}), \tag{2.85}$$

[15] The proof of that fact goes as follow : If we call $\int_0^x f(x')dx' = F(x)$ and therefore $f(x) = F'(x)$, we can rewrite the first term in the form

$$\int_0^a f(x)\int_0^x f(x')dx'dx = \int_0^a f(x)F(x)dx = \int_0^a F'(x)F(x)dx = \frac{1}{2}F^2(a),$$

giving our proof.

[16] With equalities meaning "up to leading order", that is to say, neglecting terms of the form ϕ^2 and $\phi\partial\phi$ with respect to $\partial\phi\partial\phi$.

discarding sub-leading terms and inserting $\delta^{(2)}k^i$, we obtain

$$\boxed{\begin{aligned}\delta^{(2)}n^i(\lambda_0) = \delta^{(2)}n^i(0) - 2n^i\omega^2 \int_0^{\lambda_0} \nabla_{\perp j}\phi(\lambda')d\lambda' \int_0^{\lambda_0} \nabla_\perp^j \phi(\lambda)d\lambda \\ - 4\omega^3 \int_0^{\lambda_0} \Big[\int_0^{\lambda} \nabla_{\perp j}\phi(\lambda')d\lambda' \nabla_\perp^j \int_0^{\lambda} \nabla_\perp^i \phi(\lambda'')d\lambda''\Big] d\lambda.\end{aligned}} \tag{2.86}$$

To be sure, we can check the normalisation of the vector. The condition

$$\delta n^i(\lambda)\delta n^i(\lambda) + 2\delta^{(2)}n^i(\lambda)n_i + 2\phi(\lambda)^2 - 4\phi(\lambda)\delta n^i(\lambda)n_i = 0$$

is trivially verified. From the previous equation, it is really straightforward to impose the "fixed observer direction" boundary condition. We have just to impose $\delta^2 n^i(0) = 0$ and keep the rest of the equation. On the other hand, if we want to impose the "fixed emission direction", we have to slightly modify the previous form. Going through the calculations, doing the changes

$$\delta k^i \to -\delta k^i, \qquad \int_0^{\lambda} \to \int_{\lambda}^{\lambda_0},$$

and introducing an overall minus sign, we end up with

$$\boxed{\delta^{(2)}n^i(0) = -2n^i\omega^2 \int_0^{\lambda_0} \nabla_{\perp j}\phi \int_{\lambda}^{\lambda_0} \nabla_\perp^j \phi + 4\omega^3 \int_0^{\lambda_0}\int_{\lambda}^{\lambda_0} \nabla_{\perp j}\phi \nabla_\perp^j \int_{\lambda}^{\lambda_0} \nabla_\perp^i \phi.} \tag{2.87}$$

These results are consistent with the more general situation studied in [54], if we keep just the terms without derivatives with respect to time, and with [12]. We can do the change of variables $\lambda \to r$, to eliminate the ω factors while introducing non-relevant terms ($\phi\partial\phi$). The difference in the coefficients comes from a difference of convention for the expansion of the four-vector impulsion. We used $k^\mu \approx k^{(0)\mu} + \delta k^\mu + \delta^2 k^\mu$, while they used $k^\mu \approx k^{(0)\mu} + \delta k^\mu + \frac{1}{2}\delta^2 k^\mu$. This introduces the factor 2 which differs between the two solutions.

2.3.4 Change of variables

In this section, we perform the changes of variables $\lambda \to r$ and $\lambda \to t$ and discuss the modifications induced in the expressions. We will see that, in our approximation, the expressions remain the same up to a multiplicative factor.

Integration over coordinate distance

Let us first compute the change of variable $\lambda \to r$. We know that the Jacobian, up to first order, is

$$|J^{-1}| = |k^r| \simeq |k^{(0)r} + \delta k^r| = |k^{(0)r}|. \tag{2.88}$$

Therefore,

$$|J^{-1}| \simeq \omega |n^{(0)r}| = \omega. \tag{2.89}$$

This is true up to first order. Therefore, the expressions change only up to the multiplicative factor $\frac{1}{\omega}$. Explicitly, we have[17]

$$\delta n^i(r) = n^i\phi(r) - 2\int\limits_0^r \nabla^i_\perp \phi, \tag{2.90}$$

at first order, and

$$\delta^{(2)} n^i(R) = -2n^i \int\limits_0^R \nabla_{\perp j}\phi \int\limits_0^R \nabla^j_\perp \phi - 4\int\limits_0^R\int\limits_0^r \nabla_{\perp j}\phi\nabla^j_\perp \int\limits_0^r \nabla^i_\perp\phi. \tag{2.91}$$

at second order. Note that these equations are quite handy because all the factors ω factorized out.

Integration over coordinate time

Let us now express the integrals with a time parameter. The Jacobian up to first order is

$$|J| = |(k^0)^{-1}| \simeq \frac{1}{|k^{(0)0} + \delta k^0|} \simeq \frac{1}{\omega}(1 - \frac{\delta k^0}{\omega}), \tag{2.92}$$

Note that this time, the part δk^0 does not vanish in general. At first order we have

$$\delta n^i(t) = n^i\phi(r) - 2\int\limits_0^t \nabla^i_\perp \phi, \tag{2.93}$$

and, at second order,

$$\delta^{(2)} n^i(T) = -2n^i \int\limits_0^T \nabla_{\perp j}\phi \int\limits_0^T \nabla^j_\perp \phi + 4\int\limits_0^T\int\limits_0^t \nabla_{\perp j}\phi\nabla^j_\perp \int\limits_0^t \nabla^i_\perp\phi. \tag{2.94}$$

Actually, another term comes from the change in the first-order equation. Indeed, the Jacobian up to first order, when inserted in the first-order expression gives a second-order correction of the form

$$n^i(\phi(0)^2 - \phi(t)^2) + 2\int\limits_0^t \phi\partial^i\phi \quad . \tag{2.95}$$

This contribution can be safely neglected because it is of the form ϕ^2 and $\phi\partial\phi$.

2.3.5 Lensing potential and deflection angle

Before to close this chapter and to go to the polarisation transport, we would like to connect the results we have collected so far with an often used presentation (see for example [19] or [41]): the so-called *lensing potential*. We have already mentioned many times the CMB remapping: if we look at the direction $\mathbf{n}$, we don't really receive the CMB photons emitted at the position $\mathbf{n}R$[18] but actually the

[17] We are using a simplified notation for the integrals. In view of the previous results, this convention is obvious.
[18] Where R is of course the coordinate position of the last scattering surface

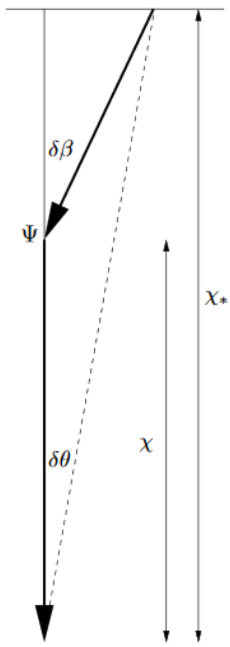


Fig. 2.7: Angles in the weak lensing case. From [41].

CMB photons coming from the position $(\mathbf{n}+\alpha)R$. Defined in this way, α is called the *deflection angle*. It can be shown (see again [19]) that the explicit form at first order is

$$\alpha_1 = -2\int_0^R \frac{R-r}{Rr}\tilde{\nabla}_\perp \phi(r,\theta,\phi)dr. \tag{2.96}$$

(We use the notation α_1 for the deflection angle at *first* order). From that we can define the *lensing potential*

$$\alpha_1 \equiv \tilde{\nabla}_\perp \Psi(\theta,\phi). \tag{2.97}$$

We present a rapid proof of this formula (following the lines of [41]). From the figure Fig.2.7, we can easily read the following formula for the angles (in the flat Universe case)

$$R\delta\theta = (R-r)\delta\beta = 2(R-r)dr\nabla_\perp\phi,$$

and we can then solve and integrate on the unperturbed path

$$\alpha_1 = \int_0^R \delta\theta(r).$$

Joining these two equations, we find (2.96). Exchanging derivatives and integration, we find the form

$$\Psi(\theta,\phi) = -2\int_0^R \frac{R-r}{Rr}\phi(r,\theta,\phi)dr \tag{2.98}$$

for the lensing potential[19].

[19] Actually, the result is valid for a universe with no spatial curvature.

On our side, at first order, we had

$$\delta x^i(\lambda_0) = -2\omega^2 \int_0^{\lambda_0} (\nabla_\perp \Psi(\lambda))^i d\lambda'. \tag{2.99}$$

Going from the parameter λ to the coordinate position r, we get the Jacobian, which introduces $\frac{1}{\omega}$ (at first order) for each integration. We obtain

$$\delta \mathbf{x}(\lambda_0) = -2 \int_0^R \int_0^r \nabla_\perp \phi(r) dr'. \tag{2.100}$$

that can be written as an angle:

$$\theta = \frac{\delta \mathbf{x}(\lambda_0)}{R} = -\frac{2}{R} \int_0^R \int_0^r \nabla_\perp \phi(r) dr' = -2\nabla_\perp \int_0^R \int_0^r \frac{\phi(r)}{r} dr'. \tag{2.101}$$

Let us show that $\theta = \alpha_{\mathbf{1}}$.

$$\alpha_{\mathbf{1}} = -2 \int_0^R \frac{R-r}{R} \nabla_\perp \phi(r,\theta,\phi) dr = -2\nabla_\perp \int_0^R \frac{R-r}{r} \phi(r,\theta,\phi) dr, \tag{2.102}$$

where the R transformed into a r because the derivative escaped the integral. We integrate $\frac{\phi}{r}$ and derive $R-r$ with respect to r. This gives

$$\int_0^R \frac{R-r}{r} \phi = \int_0^R \int_0^r \frac{\phi}{r} + (R-r) \int_0^r \frac{\phi}{r} \Big|_{r=0}^{r=R}. \tag{2.103}$$

If we choose ϕ to be zero at the observer (and we already said it was possible), the second term on the right hand side vanishes.

$$\alpha_{\mathbf{1}} = -2\nabla_\perp \int_0^R \int_0^r \frac{\phi}{r} = \theta. \tag{2.104}$$

This shows the equivalence between both formulations.

The same kind of rewriting exists for the second-order deviation. It has the form

$$\delta x^i = -4 \int_0^R dr \int_0^r dr_1 \int_0^{r_1} dr_2 \nabla_{\perp j} \phi(r_2) \nabla_\perp^j \int_0^{r_1} dr_3 \nabla_\perp^i \phi(r_3). \tag{2.105}$$

Going through two integrations by part, we obtain

$$\alpha_2^i = -2 \int_0^R dr \frac{(R-r)r}{R} \nabla_{\perp j} \nabla_\perp^i \phi(r) \alpha_1^j(r). \tag{2.106}$$

We can therefore summarize the two formulas for the deflection angle:

$$\boxed{\alpha_1 = -2\int_0^R \frac{R-r}{Rr}\tilde{\nabla}_\perp \phi(r,\theta,\phi)dr,} \tag{2.107}$$

$$\boxed{\alpha_2 = -2\int_0^R \frac{R-r}{Rr}\tilde{\nabla}_{\perp j}\tilde{\nabla}_\perp \phi(r,\theta,\phi)\alpha_1^j(r,\theta,\phi)dr.} \tag{2.108}$$

This is in agreement with result we can find in [54] and [46]. Finally, in the literature (for example in the articles just above), we can take an explicit expression for α_3:

$$\boxed{\alpha_3 = -2\int_0^R \frac{R-r}{Rr}\Big[\tilde{\nabla}_{\perp j}\tilde{\nabla}_\perp \phi(r,\theta,\phi)\alpha_2^j(r,\theta,\phi) + \frac{1}{2}\tilde{\nabla}_\perp\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp i}\alpha_1^j(r,\theta,\phi)\alpha_1^i(r,\theta,\phi)dr\Big].} \tag{2.109}$$

3 Transport of the photon polarisation

The CMB, as we know, contains two pieces of information, its temperature and its polarisation. The previous chapter was dedicated to the *lensing* of the CMB *temperature* and for a long time this has been very satisfactory. However, in the last two decades, despite being much more difficult to observe than the temperature fluctuations, fluctuations in the polarisation started to gain interest in the cosmologist community. The weakness of the signal and the fact that it can be easily confounded with noise demanded a new contribution from the theoretical side. Therefore, this chapter is dedicated to the delensing of the polarisation signal. We will see that the interplay between physical constraints and the parallel transport forces us to revise a bit our definition of the parallel transport. Once this is done, we will solve the transport equation up to second order and engage into the search for a rotation along the path. In this context, we emphasize the difference between the result obtained in GLC and in Poisson gauge.

Contents

3.1 The parallel transport of the polarisation

3.1.1 Situation

Our purpose in this chapter is to follow the evolution of the polarisation vector of a photon along its path. This polarisation vector can be defined, for example, as the electric vector of the electromagnetic wave. To describe the polarisation, we will need a two-dimensional purely spatial hypersurface normal to the four-impulsion of the photon, called the *Sachs screen*. We then define on this screen an orthonormal basis, the so-called *Sachs vectors* s_A, $A = 1, 2$. They are required to satisfy the following conditions: if we define a family of observers with four-velocity $u = e^{-\phi}\partial_t$ along the photon path, the first condition is

$$\langle u, s_A \rangle = 0. \tag{3.1}$$

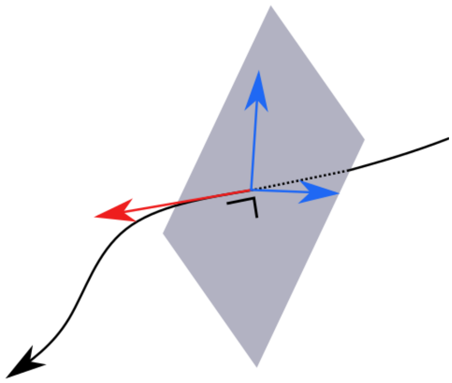


Fig. 3.1: Illustration of the Sachs basis. The black line is the photon path, the red arrow is the photon direction $\mathbf{n}$, and the blue arrows represents the Sachs basis vector, while the grey plane represents the Sachs screen.

This means that the Sachs basis never acquires a time component. The vectors of the basis are also supposed to remain normalized to one all along the path

$$\langle s_A, s_A \rangle = 1, \tag{3.2}$$

and they are defined in such a way to give an orthogonal basis

$$\langle s_1, s_2 \rangle = 0. \tag{3.3}$$

These two conditions are very natural to require for a basis in general. Finally, the Sachs screen is normal to the direction of propagation of the photon $\mathbf{n}$. This implies

$$\langle s_A, n \rangle = 0. \tag{3.4}$$

This basis is illustrated on Fig.3.1.

After the imposition of such constraints, we are left with one degree of freedom. Indeed we started with two vectors s_A for $A = 1, 2$ living in a four-dimensional space. Each of them has thus four degrees of freedom, for a total of 8 degrees of freedom. We have firstly the orthogonality with u, (3.1), and the orthogonality with n, (3.4) which together imply $\langle k, s_A \rangle = 0$. Because each of these conditions removes two degrees of freedom, we are left with four degrees of freedom. We have also the condition of normalisation and orthogonality of the two vectors (3.2) and (3.3) which kill three more degrees of freedom. This just leaves one degree of freedom for the whole system, that we are able to describe by one angle. Actually, this last degree of freedom is fixed by the evolution equation along the path. We would like to *impose* that the Sachs basis is parallel-transported

$$\nabla_k s_A = 0,$$

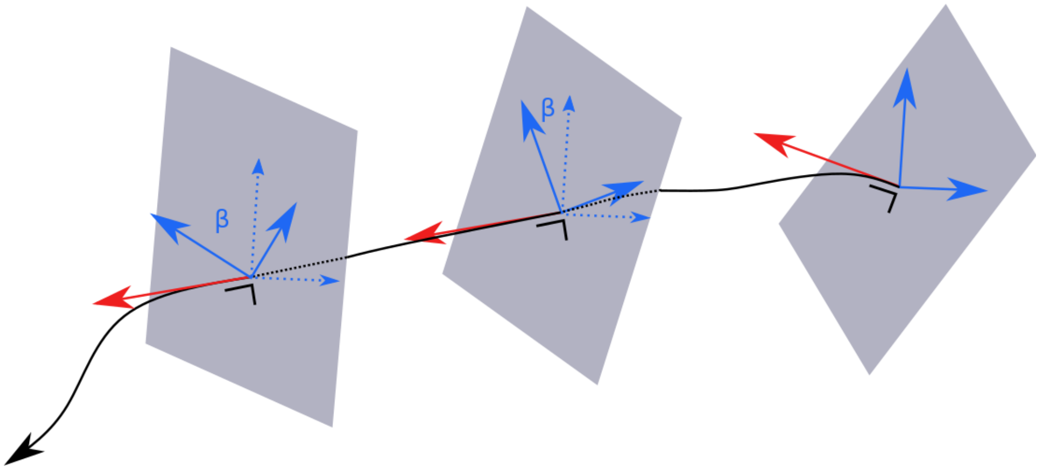


Fig. 3.2: Schematic evolution of the rotational degree of freedom along the path.

meaning that those basis vectors are parallel-transported. But it is not possible for the following reason: from (2.12), we know that

$$\nabla_k u \neq 0,$$

which means that the observers are not parallel-transported. Thus, if we impose the Sachs basis to be parallel-transported, this would spoil the orthogonality with u. Actually, it is impossible to enforce the previous conditions and to parallel transport the Sachs screen at the same time. The most intuitive way of solving the problem is to impose a slightly modified transport of the form

$$\Pi^\mu_\nu \nabla_k s^\nu_A = 0, \tag{3.5}$$

where $\Pi^\mu_\nu = \delta^\mu_\nu + u_\nu u^\mu - n_\nu n^\mu$ is the so called "projector on the Sachs screen". It actually eliminates the spurious components along the vector u or n that could appear during the parallel transport. The Sachs basis proves particularly relevant *because the unit electric field follows exactly this modified transport*, as we will show soon. The evolution of the rotational degree of freedom, provided by equation (3.5), is illustrated on Fig.3.2.

We can describe the polarisation using two different approaches. We could first use the *potential vector*, which present the advantage to be covariant. As we will see in the next subsection, the unit potential vector undergoes a parallel transport. A second way is to track directly the *electric field*, which is really the observable quantity at hand. However, the electric field introduces the difficulty that it is *observer-dependent*, as a consequence of its very definition using the Faraday tensor : $E^\mu = F^{\mu\nu} u_\nu$.

3.1.2 Transport of the potential vector and electric vector along a geodesic

In the previous section we stated a quasi-parallel transport equation for the Sachs basis, here we will prove that the electric field follows the same transport. To do that, we will follow the lines of [24]. Let us define the four-vector potential of an electromagnetic wave in some coordinate system as follows

$$A^\mu = a^\mu e^{i\phi} + c.c., \tag{3.6}$$

where a^μ and ϕ are respectively the amplitude and the phase of the wave. In the following we will assume that a^μ is real and restrict ourselves to linearly polarized waves[1]. In a curved space-time and in the Lorenz gauge $\nabla_\mu A^\mu = 0$, the free Maxwell equations take the form

$$\nabla_\nu \nabla^\nu A_\mu - R^\nu_\mu A_\nu = 0, \tag{3.7}$$

where R^ν_μ is the Ricci tensor. Actually, we can easily find this expression as an equation of motion of the Lagrangian

$$L = \frac{1}{4} F^{\mu\nu} F_{\mu\nu},$$

where we expressed the Faraday tensor in terms of the potential vector $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$. Plugging the form for the potential (3.6) into the equation of motion (3.7) and defining the four-impulsion of the photon by $k^\mu = \partial^\mu \phi$, we obtain

$$\nabla^\nu \nabla_\nu a^\mu - (k^\nu k_\nu) a^\mu - R^\mu_\nu a^\nu + i(2k^\nu \nabla_\nu a^\mu + a^\mu \nabla_\nu k^\nu) = 0.$$

Real and imaginary parts give respectively

$$k^\mu k_\mu = \nabla^\nu \nabla_\nu a^\mu + R^\mu_\nu a^\nu, \tag{3.8}$$

and

$$2k^\nu \nabla_\nu a^\mu + a^\mu \nabla_\nu k^\nu = 0. \tag{3.9}$$

On the other hand, the Eikonal approximation, that applies when the undulatory properties of an electromagnetic wave can be neglected and when light can be fairly described by a beam, is defined by

$$\partial \phi \gg \frac{1}{a} \partial a,$$

[2]and

$$\partial \phi \gg \frac{1}{l_c},$$

where l_c is the typical radius of the curvature[3]. This approximation decouples the right and left side of the equation (3.8). Therefore, we obtain two separate parts

$$k^\mu k_\mu = 0, \tag{3.10}$$

and

$$\nabla^\nu \nabla_\nu a^\mu + R^\mu_\nu a^\nu = 0. \tag{3.11}$$

(3.10) is the dispersion relation for the photon and (3.9) contains interesting information. If we introduce the unit polarisation vector, defined as

$$\alpha^\mu \equiv \frac{a^\mu}{\sqrt{a^2}}, \tag{3.12}$$

in (3.9), we get

$$\sqrt{a^2} k^\nu \nabla_\nu \alpha^\mu + \frac{\alpha^\mu}{2\sqrt{a^2}} \nabla_\nu (a^2 k^\nu) = 0. \tag{3.13}$$

[1] as explained in the introduction, the polarisation of the CMB, produced by the Thomson scattering, is expected to be linear.

[2] The phase varies much faster than the amplitude of the wave.

[3] the phase of the electromagnetic waves evolves much faster than the typical scale of the curvature

(See [24] for further developments). For our concern here, the second term vanishes because of the conservation of the photon density $\nabla_\nu j^\nu = 0$ where $j^\mu = \frac{a^2}{2\pi\hbar} k^\mu$ (whose conservation is shown in [24]). Therefore we are left with

$$k^\nu \nabla_\nu \alpha^\mu = 0. \tag{3.14}$$

We recognize immediately the parallel transport equation for the unit vector α^μ and we conclude that *the unit vector made from the potential vector is actually parallel-transported.*

As already mentioned in the previous subsections, the situation is not the same for the electric field. The point here is that we have to define a family of observers along the path of the photon to give a coherent definition of the electric field. We can indeed recover the electric vector directly from the Faraday tensor by taking the scalar product with u, labelling a family of observers along the path of the photon. Explicitly, we have

$$E^\mu = u_\nu F^{\nu\mu}. \tag{3.15}$$

This family of observers are not necessarily be parallel-transported along the line, and in our case, they are not. From (3.15) and $F^{\mu\nu}$ written in Fourier components, we can find an expression for the electric vector in terms of the potential vector

$$E^\mu = i u_\nu (k^\mu A^\nu - k^\nu A^\mu) + c.c.,$$

which holds in the Lorenz gauge. Taking the covariant derivative of this expression, we obtain

$$\nabla_k E^\mu = -\frac{1}{2}(\nabla_\nu k^\nu) E^\mu + i(\nabla_k u_\nu)(k^\mu A^\nu - k^\nu A^\mu) + c.c. \tag{3.16}$$

Let us note that $E^\mu u_\mu = E^\mu k_\mu = 0$ gives in turn $E^\mu n_\mu = 0$. Considering these identities, we see that

$$E^\sigma = i\Pi^\sigma_\mu E^\mu \tag{3.17}$$

and

$$E^\sigma = i\omega \Pi^\sigma_\mu A^\mu + c.c., \tag{3.18}$$

where, again, $\Pi^\sigma_\mu = \delta^\mu_\sigma + u^\mu u_\sigma - n^\mu n_\sigma$, is the projector on the screen normal to u and n and we used the decomposition $k = \omega(u + n)$. In the same way, the equation (3.16), when projected on the screen, simplifies to

$$\Pi^\sigma_\mu \nabla_k E^\mu = \left(-\frac{1}{2}\nabla_\nu k^\nu + \frac{1}{\omega}\nabla_k \omega\right) E^\sigma, \tag{3.19}$$

Now, if we contract it with E_σ, we obtain an equation for the evolution of E^2 along the path,

$$\Pi^\sigma_\mu \nabla_k E^\mu E_\sigma = \nabla_k(E^2) = \left(-\frac{1}{2}\nabla_\nu k^\nu + \frac{1}{\omega}\nabla_k \omega\right) E^2. \tag{3.20}$$

Using (3.19) and (3.20) and the natural definition

$$\epsilon^\mu = \frac{E^\mu}{\sqrt{E^2}}. \tag{3.21}$$

for the polarisation, we obtain

$$\Pi^\sigma_\mu \nabla_k \epsilon^\mu = \frac{\Pi^\sigma_\mu \nabla_k E^\mu}{\sqrt{E^2}} - \frac{\nabla_k(E^2) E^\sigma}{E^2\sqrt{E^2}} = 0. \tag{3.22}$$

Therefore the correct transport for the polarisation is

$$\boxed{\Pi^{\sigma}_{\mu}\nabla_k\epsilon^{\mu} = 0.} \tag{3.23}$$

We note immediately that this transport is the same than the transport of the Sachs basis (3.5). A very convenient choice will be to identify the electric field, and thus the polarisation, with one of the Sachs basis vector.

3.1.3 A more handy writing for the parallel transport equation

If we now define at the beginning, say, the unit electric vector to be s_1 and the unit magnetic vector to be s_2, it will suffice to follow the evolution of the vectors s_A for $A = 1, 2$ to follow the evolution of the polarisation. However, inspecting (3.5) we notice a problem. It is not obvious how to solve this equation perturbatively. On the other hand, if we rewrite the projected parallel transport (3.5) in the more explicit form

$$(\delta^{\mu}_{\nu} + u_{\nu}u^{\mu} - n_{\nu}n^{\mu})\nabla_k s^{\nu}_A = 0, \tag{3.24}$$

it can be shown that this is equivalent to the more handy equation

$$k^{\sigma}\nabla_{\sigma}s^{\mu}_A = \frac{k^{\mu}}{\omega}(k^{\nu}\nabla_{\nu}u_{\sigma})s^{\sigma}_A. \tag{3.25}$$

Let us immediately prove this assertion. If use the condition of orthogonality of the Sachs vectors and n and with u: $\langle u, s_A\rangle = \langle n, s_A\rangle = 0$, we obtain respectively

$$0 = \frac{d\langle u, s_A\rangle}{d\lambda} = \langle\nabla_k u, s_A\rangle + \langle u, \nabla_k s_A\rangle = \nabla_k u^{\mu}s_{A\mu} + u^{\mu}\nabla_k s_{A\mu},$$

and

$$0 = \frac{d\langle n, s_A\rangle}{d\lambda} = \langle\nabla_k n, s_A\rangle + \langle n, \nabla_k s_A\rangle = \nabla_k n^{\mu}s_{A\mu} + n^{\mu}\nabla_k s_{A\mu}.$$

Plugging these two relations into (3.24), we are left with

$$\nabla_k s^{\mu}_A - u^{\mu}s^{\nu}_A\nabla_k u_{\nu} + n^{\mu}s^{\nu}_A\nabla_k n_{\nu} = 0.$$

On the other and, if we expand the geodesic equation for k, we obtain

$$0 = \nabla_k k = \nabla_k\omega(u+n) = \frac{1}{\omega}(\nabla_k\omega)k + \omega\nabla_k(u+n).$$

Using this relation to eliminate the covariant derivative of n and using again the orthogonality of k and s_A, we obtain (3.25), as we wanted to prove.

Now that we have a handy form for the evolution equation in (3.25), we can solve it perturbatively. As in the last chapter, we will work with the metric (2.3) where the gravitational perturbation are encoded in a diagonal form and with $u = e^{-\phi}\partial_t$. We have already worked out the covariant derivative of u in (2.13). Plugging the result

$$\nabla_k u^{\sigma} = \omega(e^{-2\phi}\Gamma^{\sigma}_{00} + n^i e^{-\phi}\Gamma^{\sigma}_{0i} - e^{-\phi}n^i\partial_i\phi\delta_{\sigma 0}). \tag{3.26}$$

in the (3.25), we obtain the full non-perturbative expression

$$\frac{ds_A^\mu}{d\lambda} = k^\mu(e^{-2\phi}\Gamma^\sigma_{00}s_{A\sigma} + n^i e^{-\phi}\Gamma^\sigma_{0i}s_{A\sigma} - e^{-\phi}n^i\partial_i\phi s_{A0}) - \Gamma^\mu_{\alpha\beta}s_A^\alpha k^\beta. \tag{3.27}$$

Expanding (3.27) at zeroth order in ϕ, we obtain obviously

$$\frac{ds_A^{(0)\mu}}{d\lambda} = 0. \tag{3.28}$$

Thus, at zeroth order in ϕ, s_A is constant along the line. Let us work out the higher order corrections.

3.2 Parallel transport of the polarisation at first order

To compute the correction to s_A at first order, we take (3.27), expand it at first order in ϕ and integrate along the unperturbed path. We obtain

$$\delta s_A^\mu(\lambda) = \delta s_A^\mu(0) + \int_0^\lambda k^\mu\Big[\Gamma^\sigma_{00}(\lambda')s_{A\sigma} + n^i\Gamma^\sigma_{0i}(\lambda')s_{A\sigma} - n^i\partial_i\phi(\lambda')s_{A0}\Big] - \Gamma^\mu_{\alpha\beta}(\lambda')s_A^\alpha k^\beta d\lambda'. \tag{3.29}$$

We treat separately the time and the spatial components.

3.2.1 Time deviation of the polarisation

We first determine the deviation to the time component $\delta s_A^0(\lambda)$ of the polarisation. We know that this time component is zero at the beginning of the perturbative process, then $s_{A0}^{(0)} = 0$. Because of the orthogonality between u and s_A, we immediately conclude that this *must remain true at all higher orders*. Let us check that. Discarding immediately the time derivatives of the field and using (3.29), we find

$$\begin{aligned}\delta s_A^0(\lambda) &= \delta s_A^0(0)\\ &\quad - \int_0^\lambda \omega\Big[-n^i\partial_i\phi(\lambda')s_{A0} + n^i\partial_i\phi(\lambda')s_A^0 + \partial_i\phi(\lambda')s_A^i\Big] - \omega\partial_i\phi(\lambda')\Big[s_A^i + s_A^0 n^i\Big]d\lambda'.\end{aligned} \tag{3.30}$$

Taking account of the fact that $s_{A0} = 0$, all the terms in the integrand vanish except for the third and the fourth, which cancel each other. Then we find that the correction is zero also, as was to be shown:

$$\delta s_A^0(\lambda) = \delta s_A^0(0). \tag{3.31}$$

We can (and we must) impose the initial condition $\delta s_A^0(0)$. This result ensures that $\langle u, s_A\rangle$ remains zero (up to first order at least). This is an illustration of the effect of the projector in (3.5). It does well its job, by ensuring orthogonality with u.

3.2.2 Spatial deviation of the polarisation

Now let us compute the spatial component $\delta s^i_A(\lambda)$ of the polarisation. Again, we don't take care of the time derivatives of the field and we throw away the time component of s_A as soon as they appear. From (3.29) we obtain

$$\delta s^i_A(\lambda) = \delta s^i_A(0) + \int_0^\lambda \left[k^i \partial^j \phi(\lambda') s_{Aj} + \partial_j \phi(\lambda') k^j s^i_A + \partial_j \phi(\lambda') k^i s^j_A - s^j_A k_j \partial^i \phi(\lambda')\right] d\lambda', \qquad (3.32)$$

where the integral is meant to be taken along the unperturbed path. We can again simplify this solution by remembering that $\langle s_A, n\rangle = 0$, which eliminates the last term. Integrating the second term, we can rewrite in the form

$$\boxed{\delta s^i_A(\lambda) = \delta s^i_A(0) + 2\omega n^i \tilde{\nabla}^j \Psi(\lambda) s_{Aj} - (\phi(0) - \phi(\lambda)) s^i_A,} \qquad (3.33)$$

where $\tilde{\nabla} = (0, \partial_r, \partial_\theta, \frac{1}{\sin\theta}\partial_\varphi)$. Very general result have shown that the transport at first order should not induce rotation of the Sachs basis (see for example [46]). We will see in section 3.3.4 that it is actually the case with this correction. We can already feel it from the fact that the second term in (3.33) is a correction along $\mathbf{n}$ (to keep the orthogonality) and the third is a correction along s^i_A itself (to keep the normalisation). There is no term (rotation term) which would transform the vector $s^i_{A=1}$ into $s^i_{A=2}$ or the other way around:

$$\delta s^i_A(\lambda) = \delta s^i_A(0) + \underbrace{2\omega n^i \tilde{\nabla}^j \Psi(\lambda) s_{Aj}}_{\propto n^i} - \underbrace{(\phi(0) - \phi(\lambda)) s^i_A}_{\propto s^i_A}. \qquad (3.34)$$

We will discuss that in more details in section 3.2.3. We also give the formula for δs_{Ai}[4]

$$\delta s_{Ai}(\lambda) = \delta s_{Ai}(0) + 2\omega n_i \tilde{\nabla} \Psi(\lambda) s_{Aj} - (\phi(0) + \phi(\lambda)) s_{Ai}. \qquad (3.35)$$

3.2.3 Normalisation and orthogonality of the Sachs basis vectors

In this section we check if the constraints mentioned in 3.1.1 are still present after the computation of perturbations. As we did in Chapter 2, we impose these constraints at one point, and check whether they are indeed satisfied at every other λ.

We start with the condition of normalisation, which is

$$\langle s_A + \delta s_A, s_A + \delta s_A\rangle = 1,$$

which gives at first order

$$0 = 2\delta_{ij} s^j_A \delta s^i_A - 2\phi \delta_{ij} s^j_A s^i_A = 2\delta_{ij} s^i_A \delta s^j_A(\lambda) - 2\phi(\lambda).$$

Inserting the explicit expression of δs^j_A , we obtain

$$0 = 2\delta s^i(0) s_{Ai} - 2\phi(\lambda) - 2(\phi(0) - \phi(\lambda)) = -2\phi(\lambda) - 2\phi(0) + 2\phi(\lambda) + 2\delta s^i(0) s_{Ai}.$$

[4] using $\delta s_{Ai} = \delta(g_{ij} s^j_A) = \delta g_{ij} s^j_A + g_{ij} \delta s^j_A$

The condition if fulfilled as expected by taking

$$\delta s^i(0) = \phi(0) s^i_A.$$

Let us now treat the orthogonality. We have, at zeroth and first order,

$$0 = \langle n^{(0)}, s^{(0)}_A \rangle, \qquad 0 = \langle n + \delta n, s_A + \delta s_A \rangle = g_{ij} n^i s^j_A + g_{ij} \delta n^i s^j_A + g_{ij} n^i \delta s^j_A. \tag{3.36}$$

In the second equation, the first term on the RHS is immediately zero by the zeroth order condition and we can replace all the g_{ij} by δ_{ij}. We are left with the two conditions

$$\delta_{ij} \delta n^i s^j_A + \delta_{ij} n^i \delta s^j_A = 0 \qquad A = 1, 2. \tag{3.37}$$

Going through the calculations[5], we find

$$\langle \delta n, s^0_A \rangle = -2\omega \int_0^\lambda \partial^j \phi(\lambda') s_{Aj} d\lambda', \tag{3.38}$$

and

$$\langle n, \delta s_A \rangle = 2\omega \int_0^\lambda \partial^j \phi(\lambda') s_{Aj} d\lambda' + \delta s^i(0) n_i. \tag{3.39}$$

This gives us a condition on the integration constant:

$$n_j \delta s^j(0) = 0. \tag{3.40}$$

Thus, the final solution for the correction to the Sachs basis vectors at first order is

$$\boxed{\delta s^i_A(\lambda) = 2\omega n^i \tilde{\nabla}^j (\Psi(\lambda)) s_{Aj} + \phi(\lambda) s^i_A = 2\omega n^i \int_0^\lambda \partial^j \phi(\lambda') s_{Aj} d\lambda' + \phi(\lambda) s^i_A.} \tag{3.41}$$

The constant is totally fixed by the boundary conditions (“fixed observer direction” in our case).

However, there is another case of special interest. Assume we already know the “first-order corrected” polarisation field at the emission. Then in the formula of transport from the emitter to the observer, we need to impose the variation at first order of the polarisation to be zero at the *emission* . In this case, we end up with

$$\boxed{\delta s^i_A(\lambda) = -2\omega n^i \tilde{\nabla}^j \Phi(\lambda) s_{Aj} + (\phi(\lambda) - \phi(\lambda_0)) s^i_A = -2\omega n^i \int_\lambda^{\lambda_0} \partial^j \phi(\lambda') s_{Aj} d\lambda' + (\phi(\lambda) - \phi(\lambda_0)) s^i_A,} \tag{3.42}$$

where as expected $\delta s^i_A(\lambda_0) = 0$.

[5] Here, we are considering “fixed observer direction” boundary condition

3.3 Parallel transport of the polarisation at second order

In this section, we compute the correction of the Sachs basis at second order in the Weyl field ϕ. Like for the vector k, it will be a bit more tricky than the first-order calculation. Using (3.29), expanding up to second order and keeping only those terms[6], we find

$$\begin{aligned}\frac{d\delta^{(2)}s^\mu_A}{d\lambda} &= k^\mu(\Gamma^\sigma_{00}\delta s_{A\sigma} - 2\phi\Gamma^\sigma_{00}s_{A\sigma} + n^i\Gamma^\sigma_{0i}\delta s_{A\sigma} - \phi n^i\Gamma^\sigma_{0i}s_{A\sigma}) - \Gamma^\mu_{\alpha\beta}(s^\alpha_A\delta k^\beta + \delta s^\alpha_A k^\beta) \\ &\quad + \delta k^\mu(\Gamma^\nu_{00}s_{A\nu} + n^i\Gamma^\nu_{0i}s_{A\nu} - n^i\partial_i\phi s_{A0}) + \delta k^\nu\partial_\nu\delta s^\mu_A - 2k^\mu\phi\partial^i\phi s_{iA}.\end{aligned} \tag{3.43}$$

Let us now use this equation to compute as usual the time and then the spatial components.

3.3.1 Time component

For the time component, we only have to show that the correction to this component is zero, irrespective of the path of integration or anything else. First of all, we will discard any term containing s^0_A and insert the values for the Christoffel symbols in (3.43).

$$\begin{aligned}\frac{d\delta^{(2)}s^0_A}{d\lambda} &= k^0(\partial^j\phi\delta s_{Aj} - 2\phi\partial^j\phi s_{Aj}) + \delta k^0(\partial^j\phi s_{Aj}) \\ &\quad - \partial_j\phi(s^j_A\delta k^0 + \delta s^0_A k^j + \delta s^j_A k^0) + 2k^0\phi\partial^i\phi s_{iA}.\end{aligned} \tag{3.44}$$

The fact that no term comes from the perturbed path can be understood as a direct consequence of the vanishing of the first-order correction to the time component, δs^0_A.

In (3.44), we can immediately see the cancellations in $\delta k^0 s_{Aj} - s^j_A\delta k^0 - \delta s^0_A k^j - \delta s^j_A k^0$: the first and the second terms cancel each other while the third vanishes. If we replace by the explicit form $-2\phi k^0 = \delta k^0$, this leads to

$$\frac{d\delta^{(2)}s^0_A}{d\lambda} = \delta k^0\partial^j\phi s_{Aj} + \delta k^0(\partial^j\phi s_{Aj}) - \partial_j\phi(s^j_A\delta k^0) - \delta k^0\partial^i\phi s_{iA} = 0, \tag{3.45}$$

and, therefore,

$$\delta^{(2)}s^0_A(\lambda) = 0, \tag{3.46}$$

which agrees with our previous idea that the corrections to s^0_A are zero at every order in the perturbation series.

3.3.2 Spatial components

Now let us compute the spatial components, which are by far more meaningful. The situation gets more tricky because we have to face the problem that the integration is not performed along the unperturbed path. So far we have kept explicitly the $\phi\partial_i\phi$, but we have seen that they are not dominant terms because the field is supposed to be small, but not its variation. For simplicity and concision, we

[6] we already eliminated the time derivatives of the Weyl field.

will thus systematically discard them as soon as they appear. Therefore, if we keep only terms likely to give a contribution of the form $\nabla_{\perp i}\phi\nabla^i_\perp\phi$ (the transverse derivatives terms), (3.43) becomes

$$\begin{aligned}\frac{d\delta^{(2)}s^i_A}{d\lambda} &= \underbrace{k^i(\partial^j\phi\delta s_{Aj}) + \delta k^i(\partial^j\phi s_{Aj})}_{\text{modified parallel transport}} - \underbrace{\delta k^j\partial_j\delta s^i_A}_{\text{Post-Born term}} \\ &+ \underbrace{(\delta^i_j\partial_k\phi + \delta^i_k\partial_j\phi - \delta_{jk}\partial^i\phi)(s^j_A\delta k^k + \delta s^j_A k^k)}_{\text{Christoffel symbols}},\end{aligned} \tag{3.47}$$

where the first is the part ensures the orthogonality with $\mathbf{n}$, the second part is the deformation of the path or the so-called Post-Born term, and third part comes from the correction to the Christoffel symbols coupled with first-order corrected terms. Integrating, inserting the explicit necessary expressions and keeping only the dominant terms[7], we end up with

$$\begin{aligned}\delta^{(2)}s^i_A(\lambda) = \delta^{(2)}s^i_A(0) &+ 4n^i\omega^3\int_0^\lambda\Big[\int_0^{\lambda'}\nabla_{\perp j}\phi(\lambda'')d\lambda''\Big]\nabla^j_\perp\int_0^{\lambda'}\nabla_{\perp k}\phi(\lambda''')s^k_A d\lambda''' d\lambda' \\ &- 4\omega^2\int_0^\lambda s^j_A\nabla_{\perp j}\phi(\lambda')\int_0^{\lambda'}\nabla^i_\perp\phi(\lambda'')d\lambda'' d\lambda'\end{aligned} \tag{3.49}$$

where we have neglected the potential at the emission. We will see that, if we take the "fixed observer direction" boundary condition, involving $\delta n^i(0) = 0$ at all orders, the different constraints on the Sachs basis impose that $\delta^{(2)}s^i_A(0) = 0$. We check this assumption in the next subsection.

3.3.3 Normalisation and orthogonality of the basis

In this section we continue what we started in section 3.2.3. We fix the conditions at some λ and check the condition at other λ.

[7] we have

$$\begin{aligned}\delta^{(2)}s^i_A(\lambda) = \delta^{(2)}s^i_A(0) &- \int_0^\lambda \delta k^j(\lambda')\nabla_{\perp j}\delta s^i_A(\lambda')d\lambda' - \int_0^\lambda \partial^i\phi(\lambda')\big[s^j_A\delta k_j(\lambda') + \delta s^j_A(\lambda')k_j\big]d\lambda' \\ &+ \int_0^\lambda \big[s^i_A\delta k^k(\lambda') + \delta s^i_A k^k\big]\partial_k\phi(\lambda') + 2\delta k^i(\lambda')\partial^j\phi(\lambda')s_{Aj} + 2k^i\delta s^j_A(\lambda')\partial_j\phi(\lambda')d\lambda'\end{aligned} \tag{3.48}$$

We can easily see that the term of the form $\int_0^{\lambda_0}\partial^i\phi(\lambda')\big[s^j_A\delta k_j(\lambda') + \delta s^j_A(\lambda')k_j\big]d\lambda'$ is zero because $s^j_A\delta k_j + \delta s^j_A k_j$ vanishes (by condition). Inserting the explicit values (with "fixed observer direction" boundary conditions)

$$\delta k^i(\lambda) = -2\omega^2\int_0^\lambda \nabla^i_\perp\phi d\lambda',$$

$$\delta s^i_A(\lambda) = 2\omega n^i\int_0^\lambda \partial^j\phi(\lambda')s_{Aj}d\lambda' + \phi(\lambda)s^i_A.$$

We recover (3.49).

We will start with the orthogonality with $\mathbf{n}$, we have to impose the condition at $\lambda = \lambda_0$. The condition is

$$\langle n + \delta n + \delta^{(2)} n, s_A + \delta s_A + \delta^{(2)} s_A \rangle = 0.$$

The terms $g_{ij} n^i s_A^j$ and $\delta_{ij} n^i \delta s_A^j + \delta_{ij} \delta n^i s_A^j$ are zero by the results before. Thus this condition reduces to

$$-2\phi \langle n^i, \delta s_A \rangle - 2\phi \langle \delta n, s_A \rangle + \langle \delta n, \delta s_A \rangle + \langle \delta^{(2)} n, s_A \rangle + \langle n, \delta^{(2)} s_A \rangle = 0, \tag{3.50}$$

(where the brackets mean a scalar product with the flat metric δ_{ij}). The two first terms are negligible and we are just left with

$$\langle \delta n, \delta s_A \rangle + \langle \delta^{(2)} n, s_A \rangle + \langle n, \delta^{(2)} s_A \rangle = 0. \tag{3.51}$$

Going through the computation, keeping only the relevant terms, and using the previous formulas, we find that the first vanishes

$$\langle \delta n, \delta s_A \rangle = 0, \tag{3.52}$$

because there is no terms with two transverse derivatives. The second term is

$$\langle \delta^{(2)} n, s_A \rangle = -4 n^i \omega^3 \int_0^\lambda \Big[\int_0^{\lambda'} \nabla_{\perp j} \phi(\lambda'') d\lambda'' \Big] \nabla_\perp^j \int_0^{\lambda'} \nabla_{\perp k} \phi(\lambda''') s_A^k d\lambda''' d\lambda', \tag{3.53}$$

and the third one

$$\langle n, \delta^{(2)} s_A \rangle = 4 n^i \omega^3 \int_0^\lambda \Big[\int_0^{\lambda'} \nabla_{\perp j} \phi(\lambda'') d\lambda'' \Big] \nabla_\perp^j \int_0^{\lambda'} \nabla_{\perp k} \phi(\lambda''') s_A^k d\lambda''' d\lambda'. \tag{3.54}$$

This leads to a satisfactory cancellation.

The second part of the work is to check the normalisation of the vectors s_A at second order. Following the same procedure than at first order, we have to check that

$$2\phi(\lambda)^2 + \delta s_{Ai}(\lambda) \delta s_A^i(\lambda) + 2\delta^{(2)} s_{Ai}(\lambda) s_A^i - 2\phi(\lambda)(2\delta s_{Ai}(\lambda) s_A^i) = 0,$$

and as usual, we will neglect irrelevant terms. The first term is negligible, the second one is

$$4\omega^2 \int_0^{\lambda_0} \nabla_{\perp j} \phi s_A^j \int_0^{\lambda_0} \nabla_{\perp k} \phi s_A^k, \tag{3.55}$$

and the third one is

$$-8\omega^2 \int_0^{\lambda_0} s_A^j \nabla_{\perp j} \phi \int_0^{\lambda} \nabla_\perp^i \phi s_{Ai}. \tag{3.56}$$

The last term vanishes. Summing all together and using again (2.80), we see that the condition is fulfilled.

We have a last condition to fulfil: the orthogonality between the two s_A vectors. There was no deviation at first order, so we have only to take care of the second-order terms

$$\langle s_1, \delta^{(2)} s_2 \rangle + \langle s_2, \delta^{(2)} s_1 \rangle + \langle \delta s_1, \delta s_2 \rangle = 0, \tag{3.57}$$

(where the brackets mean a scalar product in flat space). We get, for the sum of the first two terms,

$$-2\omega^2 \int_0^{\lambda_0} s_1^j \nabla_{\perp j} \phi \int_0^{\lambda_0} \nabla_\perp^i \phi s_{i2} - 2\omega^2 \int_0^{\lambda_0} s_2^j \nabla_{\perp j} \phi \int_0^{\lambda_0} \nabla_\perp^i \phi s_{i1} \tag{3.58}$$

and, for the last one,

$$4\omega^2 \int_0^{\lambda_0} s_1^j \nabla_{\perp j} \phi \int_0^{\lambda_0} \nabla_\perp^i \phi s_{i2}. \tag{3.59}$$

Those terms, being completely symmetric, cancel each other, verifying the last condition.

To summarize, we have the following result for the deviation of the Sachs basis:

$$\boxed{\begin{aligned} \delta^{(2)} s_A^i(\lambda) = \delta^{(2)} s_A^i(0) + 4 n^i \omega^3 \int_0^{\lambda} \left[\int_0^{\lambda'} \nabla_{\perp j} \phi(\lambda'') d\lambda'' \right] \nabla_\perp^j \int_0^{\lambda'} \nabla_{\perp k} \phi(\lambda''') s_A^k d\lambda''' d\lambda' \\ - 2\omega^2 \int_0^{\lambda} s_A^j \nabla_{\perp j} \phi(\lambda') d\lambda' \int_0^{\lambda} \nabla_\perp^i \phi(\lambda'') d\lambda''. \end{aligned}} \tag{3.60}$$

The above analysis showed us that the first term is a correction along the $\mathbf{n}$-vector, to keep the orthogonality with it, this is the counterpart of the first term of (3.33). This term comes entirely from post-Born terms in (3.47). The second term, on the other hand, contributes to the normalisation of the Sachs basis and is the counterpart of the second term in (3.33). This term comes in half part part from the post-Born term and in half part from the Christoffel symbols in (3.47).

We also test the effect of lowering or raising the indices. Therefore, we compute $\delta^{(2)} s_{Ai}(\lambda)$, using $\delta^{(2)} s_{Ai}(\lambda) = \delta^{(2)}(g_{ij} s_A^j) = \delta^{(2)} s_A^j(\lambda) \delta_{ij} + \delta^{(2)} g_{ij} s_A^j(\lambda) + \delta g_{ij} \delta s_A^j(\lambda)$, as we did for the first-order corrections. The second term gives a contribution of the form $\propto \phi^2$ which we neglect and the third term gives a contribution of the form $\propto \phi^2 + \phi \partial \phi$ which we neglect too. As a consequence, we can raise and lower indices with the flat metric,

$$\boxed{\delta^{(2)} s_{Ai}(\lambda) = \delta^{(2)} s_A^i(\lambda) \delta_{ij},} \tag{3.61}$$

within our approximations.

3.3.4 Rotation in Poisson gauge and constraints

Before we go to the computation of the rotation itself, we would like to emphasize a remarkable fact. In section 3.1.1, we already noticed that there is only one degree of freedom left over after the imposition of constraints, and that this degree of freedom is an angle. The evolution of the angle is actually piloted by the evolution equation (3.5). In other words, the evolution equation could decide to rotate the angle all along the path. In this section, we want to show that it decided *not* to grab this opportunity! At the same time, we want to emphasize the interplay between evolution equation and constraints.

When we computed the correction to the $\mathbf{n}$ vector, we have seen that the normalisation condition completely fixes the correction of $\mathbf{n}$ in the $\mathbf{n}$ direction through the equation

$$\delta n^i n_i = \phi.$$

This is also true of the component of the Sachs basis along $\mathbf{n}$ through the orthogonality relation

$$\delta n^i s_{Ai} = -\delta s_{Ai} n^i.$$

For the first-order corrections, it gave

$$\delta s_{Ai}(\lambda) n^i = 2\omega \int_0^\lambda s_{Ai} \partial^i \phi. \tag{3.62}$$

In turn, the two normalisation conditions completely fix the component along s_{Ai}. The component along $\mathbf{s}_A$ and the component along $\mathbf{n}$ being fixed by the constraints, we are left with one degree of freedom perpendicular to them. If we build the vector

$$\mathbf{J} = \mathbf{n} \times \mathbf{s_1},$$

and normalize it, we find of course that

$$\mathbf{J} = \mathbf{s_2}.$$

Then the only third term that could appear in the first-order correction and which is not constrained by the conditions of normalisation and orthogonality is of the form

$$\delta s_1^i(\lambda) = s_2^i B,$$

and

$$\delta s_2^i(\lambda) = s_1^i A.$$

The last constraint we have not considered yet is the orthogonality between $\mathbf{s_1}$ and $\mathbf{s_2}$, which translates in

$$\delta s_{1i} s_2^i = -\delta s_{2i} s_1^i. \tag{3.63}$$

This in turn imposes that

$$B = -A.$$

Therefore, reconstructing we obtain

$$\delta s_1^i(\lambda) = \underbrace{2\omega n^i \int_0^\lambda s_{1j} \partial^j \phi - s_1^i \phi(\lambda)}_{\text{fixed by constraints}} + \underbrace{A s_2^i}_{\text{piloted by evolution equation}}, \tag{3.64}$$

and

$$\delta s_2^i(\lambda) = \underbrace{2\omega n^i \int_0^\lambda s_{2j} \partial^j \phi - s_2^i \phi(\lambda)}_{\text{fixed by constraints}} - \underbrace{A s_1^i}_{\text{piloted by evolution equation}}. \tag{3.65}$$

The first term being along n and the second along the Sachs vector itself, the third is a "flow of one vector in the other" or, in other words, a rotation. We can represent it perturbatively in the form

$$R_a^b = \begin{pmatrix} 1 & -A \\ A & 1, \end{pmatrix}$$

with A infinitesimal. Exponentiating, we have

$$R_a^b = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for $A = \theta$. The fact that the evolution equation provides a vanishing result for this term shows explicitly that, at first order, the Sachs basis does *not* rotate. This is a remarkable fact, but this does *not* mean immediately that the parallel transport of the polarisation will not provide $\mathcal{B}$-modes at first order. This question will be investigated in the next chapters.

What about the second-order parallel transport ? Just as in the first-order case, in

$$\delta^{(2)} s_A^i(\lambda) = \underbrace{4n^i\omega^3 \int_0^\lambda \int_0^{\lambda'} \nabla_{\perp j}\phi(\lambda'')\nabla_\perp^j \int_0^{\lambda'} \nabla_{\perp k}\phi(\lambda'') s_A^k}_{\text{fixed by orthogonality}} \underbrace{-2\omega^2 \int_0^\lambda s_A^j \nabla_{\perp j}\phi(\lambda') \int_0^\lambda \nabla_\perp^i \phi(\lambda'')}_{\text{fixed by ortho-normalisation}}, \tag{3.66}$$

the first term ensures the orthogonality with $\mathbf{n}$ up to second order, while the second one is necessary to ensure the normalisation and the orthogonality of the two vectors of the Sachs basis. Again, we want to investigate whether this contributes a rotation, that is whether this contains a contribution of the form:

$$\delta^{(2)} s_1^i(\lambda) = A^{(2)} s_2^i,$$
$$\delta^{(2)} s_2^i(\lambda) = -A^{(2)} s_1^i.$$

Contracting, we find that

$$A^{(2)} = \langle \delta^{(2)} s_1, s_2 \rangle = -\langle \delta^{(2)} s_2, s_1 \rangle.$$

The symmetry of $\int_0^\lambda \nabla_\perp^j \phi(\lambda') \int_0^\lambda \nabla_\perp^i \phi(\lambda'')$ in the indices i and j shows that $A^{(2)}$ must vanish.

This illustrates the fact that, in the Poisson gauge, there is no rotation along the path up to second order. Owing to the fact that the metric is *diagonal*, we could actually extrapolate this result at all orders. The claim is thus: *there is no rotation at all in the Poisson gauge.*

3.4 Rotation: GLC result versus Poisson gauge result

In the Appendix C of the paper [46] , "rotation angle using the Sachs formalism", we can find a calculation for the rotation of the vectors of the Sachs basis when parallel-transported. We have argued in the previous section that, in the Poisson gauge, the evolution equation does not provide rotation. In the following, we will try to trace out the origin of the second-order rotation in [46] and [45]. Understanding the difference and reconciling the two results will be the purpose of the end of this Chapter and the entire Chapter 4.

Let us firstly introduce the rotation matrix. We start with an unperturbed Sachs basis called $\bar{s}^i_A$ and compute order by order the corrected Sachs basis vectors. Because the only degree of freedom allowed by the constraints is rotational, we introduce a rotation matrix R^B_A, such that

$$\hat{s}^a_A = R^B_A s^a_B,$$

where $\hat{s}^b_A$ and s^a_A are two Sachs basis *fulfilling the constraints at some order in the perturbation series*. Therefore, the only possible difference between them is an angle that can come only from the evolution equation. We will then separate the perturbed Sachs basis into two parts, a symmetric part χ^{ba}, and a rotation part R^B_A[8]:

$$\hat{s}^a_A = \chi^{ba}\bar{s}_{bB}R^B_A. \tag{3.67}$$

3.4.1 In the Poisson gauge

Even though we already know the answer, we can apply this formalism, using a matrix to work out the rotation, in Poisson gauge. Expanding (3.67) on both side up to second order, we obtain

$$\bar{s}^a_A + \delta s^a_A + \delta^2 s^a_A = R^{B(0)}_A \bar{s}^a_B + [R^{B(1)}_A \bar{s}^a_B + \chi^{(1)ab}\delta^B_A \bar{s}_{Bb}] + [R^{B(2)}_A \bar{s}^a_B + \chi^{(1)ab}R^{(1)B}_A \bar{s}_{Bb} + \chi^{(2)ab}\delta^B_A \bar{s}_{Bb}],$$

where we clustered by order. Obviously the first term on each side cancels each other because at zeroth order the rotation matrix is δ_{AB}. Consequently $R^{B(1)}_A = A^{(1)}\epsilon^B_A$ for $A^{(1)}$ small[9]. Keeping only first-order terms and taking the scalar product with $\bar{s}^C_a\epsilon^A_C = g_{ac}\bar{s}^{Cc}\epsilon^A_C$ on both side, we obtain

$$\bar{s}^C_a \epsilon_{CA}\delta s^{Aa} = A^{(1)}\bar{s}^C_a \epsilon^A_C \epsilon^B_A \bar{s}^a_B + \underbrace{g_{ac}\chi^{(1)ab}\delta^B_A \bar{s}_{Bb}\bar{s}^{Cc}\epsilon^A_C}_{\propto \epsilon_{ab}\chi^{(1)ab}=0} = A^{(1)}\bar{s}^C_a \delta^A_C \bar{s}^a_A = A^{(1)}\bar{s}^C_a \bar{s}^a_C = 2A^{(1)}.$$

We used the normalisation of the Sachs basis; $\bar{s}^C_a\bar{s}^a_B = \delta^C_B$. That is how we isolated the rotation at first order, namely $A^{(1)}$. Now inserting the explicit form of δs^{Aa} we have found in the previous chapters, we obtain a value for this angle:

$$A^{(1)} = \frac{1}{2}\bar{s}^C_a \epsilon_{CA}\delta s^{Aa} = -\Delta\phi \frac{1}{2}\bar{s}^C_a \epsilon_{CA}\bar{s}^{Aa} = -\Delta\phi\frac{1}{2}\epsilon_{CA}\delta^{CA} = 0.$$

We can then repeat the same argument at second order. The first and the second terms under brackets can produce a rotation: The second term $\chi^{(1)ab}R^{B(1)}_A = 0$ because $R^{B(1)}_A$ has been shown to be zero. We are then left just with the first term under brackets. Inserting explicit values, we obtain

$$A^{(2)} = \frac{1}{2}s^C_a \epsilon_{CA}\delta^{(2)}s^{Aa} \propto \epsilon_{CA}\int \bar{s}^A_d \nabla_{\perp d}\phi \int \nabla^a_\perp \phi \bar{s}^C_a = 0.$$

Therefore, with a simple argument of symmetry, we have shown that there is no rotation in the corrections we have computed. This confirms what we stated in the previous section.

[8] In this decomposition, we treat formally the tetrad $\hat{s}^a_A$ as a two-two matrix in the indices A and a, and use the fact that a matrix can decomposed in a symmetric matrix and a rotation.

[9] This is because an infinitesimal rotation can be written

$$A^{(0)} + A^{(1)} = \begin{pmatrix} 1 & A^{(1)} \\ -A^{(1)} & 1 \end{pmatrix}. \tag{3.68}$$

3.4.2 In the geodesic light-cone gauge

In the publication [46], the authors use a rather different approach, with a different gauge[10]. The gauge is the GLC gauge[11], with the line element

$$ds^2_{\text{GLC}} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U dw^a)(d\tilde{\theta}^b - U dw^b), \tag{3.69}$$

where w is a null coordinate and the photon travels on surfaces with constant w and $\tilde{\theta}^a$. We can focus on the two-dimensional space spanned by $\tilde{\theta}^a$ because, as shown in [22], this two-dimensional space is actually the Sachs screen. Therefore, we can expect that only the γ_{ab} part of the GLC metric will enters the game. In this setting, the Sachs basis is determined by the normalisation conditions and the parallel transport. In other words,

$$s^a_A s^b_B \gamma_{ab} = \delta_{AB}, \qquad \nabla_k s_a = 0 \tag{3.70}$$

completely fixes the evolution. As we have noted in the previous sections, the normalisation constraints only fixes the lenght of the vectors, allowing for a rotational degree of freedom. The angle of rotation, called β, is actually piloted by the geodesic equation. One can show that (see again [22]) the equation for β takes the form

$$\partial_\tau \beta = \frac{1}{2}\epsilon^{AB}\partial_\tau s^a_A s_{Ba}, \tag{3.71}$$

where τ parametrizes the photon path. Expanding at first order in the Weyl field ϕ, one finds a vanishing result for the rotation angle: $\beta^{(1)} = 0$, as we do. But for second order, one actually finds a non-vanishing rotation. This does not come from the term combining $\delta^{(2)} s_a$ and s_A, but from the term containing a combination of two δs_a. Let us present these computations.

We first decompose in symmetric and rotation part, as in (3.67). Obviously, at zeroth order

$$s_{Aa} = \chi^{(0)}_{ba}\bar{s}^a_B = g_{ba}\bar{s}^a_B.$$

Thus $\chi^{(0)}_{ba} = g_{ba}$. Here g_{ab} is the angular part of the flat metric, which has the specific form

$$g_{ab} = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}.$$

The expansion of the evolution equation (3.71) at first order is

$$\partial_\tau \beta^{(1)} = \frac{1}{2}\epsilon^{AB}\partial_\tau \delta s^a_A s_{Ba} + \frac{1}{2}\epsilon^{AB}\partial_\tau s^a_A \delta s_{Ba}. \tag{3.72}$$

(We dropped the bars when talking about background values: $s_A = \bar{s}_A$). Now we have to deduce an expression for the correction of the Sachs basis vectors δs_{Ba}. We thus use the normalisation condition to determine it up to a rotation. This gives

$$\delta s^a_A g_{ab} s^b_B + s^a_A g_{ab}\delta s^b_B = -s^a_A \delta g_{ab} s^b_B.$$

[10] Strictly speaking, the GLC encoding can not be called a gauge, because we are not able to transform the Poisson gauge into GLC gauge by a local diffeomorphism. For simplicity, we will anyway still call it a gauge in the following.

[11] This specific gauge is presented with more details in Appendix C, here we just explain the basics

Using the zweibein relation $s^c_A s^{Aa} = g^{ca}$ and contracting with $s^c_A s_{Bc}$, we can rewrite in the form

$$s_a \delta s_b + s_b \delta s_a = s_{(a} \delta s_{b)} = -\delta g_{ab}, \tag{3.73}$$

where there is an implicit summation over A. We have thus a link between the correction to the Sachs vectors and the correction to the metric. Expanding (3.67) and keeping only first-order terms, we obtain

$$\delta s_{aA} = R_A^{B(1)} \bar{s}_{Ba} + \chi^{(1)}_{ab} \delta^B_A \bar{s}^b_B. \tag{3.74}$$

First of all, this relation permits us to conclude that

$$\chi^{(1)}_{ab} = -\frac{\delta g_{ab}}{2}. \tag{3.75}$$

Then inserting $\delta s^a_A = R_A^{B(1)} \bar{s}^a_B - \frac{1}{2}\delta g^{ab} \delta^B_A \bar{s}_{Bb}$ in the first-order truncated evolution equation (3.72) and distributing the derivative we find

$$\begin{aligned}\partial_\tau \beta^{(1)} &= \underbrace{\frac{1}{2}\epsilon^{AB}\partial_\tau [R_A^{C(1)} \bar{s}^a_C] \bar{s}_{Ba}}_{0} - \frac{1}{4}\epsilon^{AB}\partial_\tau[\delta g^{ab}\bar{s}_{Ab}]\bar{s}_{Ba} + \underbrace{\frac{1}{2}\epsilon^{AB}\partial_\tau s^a_A R_B^{C(1)} \bar{s}_{Ca}}_{0} - \frac{1}{4}\epsilon^{AB}\partial_\tau s^a_A \delta g_{ab} s_{Bb} \\ &= -\underbrace{\frac{1}{4}\epsilon^{AB}\partial_\tau[\delta g^{ab}]\bar{s}_{Ab}\bar{s}_{Ba}}_{\epsilon^{ab}\partial_\tau \delta g_{ab}} - \underbrace{\frac{1}{4}\epsilon^{AB}\partial_\tau[\bar{s}_{Ab}]\delta g^{ab}\bar{s}_{Ba}}_{\propto \epsilon_{ab}\delta g^{ab}} - \underbrace{\frac{1}{4}\epsilon^{AB}\partial_\tau s^a_A \delta g_{ab} s_{Bb}}_{\propto \epsilon_{ab}\delta g^{ab}},\end{aligned} \tag{3.76}$$

which is zero because of the contraction of the symmetric tensor with the antisymmetric tensor.

On the other hand, at second order, this method indeed provides a non-vanishing result for $\partial_\tau \beta$. This contribution comes in the equation (3.71) from the "cross term", namely

$$\partial_\tau \beta^{(2)} \propto \epsilon^{AB}[\partial_\tau \delta s^a_A]\delta s_{Ba}.$$

To work this out, we start again from the normalisation condition. It leads us to a relation between the symmetric part of the correction of s^a_A, χ_{ab}, and the correction of the metric. This yields

$$\chi^{(2)}_{ab} = \frac{1}{2}\delta^{(2)} g_{ab} - \frac{1}{8}\delta g_{ac} g^{cd} \delta g_{db}. \tag{3.77}$$

The next purpose is to solve the equation of evolution

$$\partial_\tau \beta^{(2)} = \frac{1}{2}\epsilon^{AB}\partial_\tau \delta^{(2)} s^a_A s_{Ba} + \frac{1}{2}\epsilon^{AB}\partial_\tau s^a_A \delta^{(2)} s_{Ba} + \frac{1}{2}\epsilon^{AB}\partial_\tau \delta s^a_A \delta s_{Ba} \tag{3.78}$$

expanded at second order. The first two terms cancel exactly for the same reasons than in the first-order calculation. In the end, we are left with the cross term:

$$\partial_\tau \beta^{(2)} = \frac{1}{2}\epsilon^{AB}\partial_\tau \delta s^a_A \delta s_{Ba}, \tag{3.79}$$

as we stated above.

We already have an expression for the first-order corrections to the Sachs basis: (3.74). Combining with (3.75) and inserting in (3.79), this yields

$$\partial_\tau \beta^{(2)} = \frac{1}{8}\epsilon^{AB}[\partial_\tau g^{ac}]\delta g_{cb} s^b_A \delta g_{ad} s^d_B + \frac{1}{8}\epsilon^{AB}[\partial_\tau \delta g_{cb}] g^{ac} s^b_A \delta g_{ad} s^d_B + \frac{1}{8}\epsilon^{AB}[\partial_\tau s_A] g^{ac}\delta g_{cb}\delta g_{ad} s^d_B. \tag{3.80}$$

Now everything is written as a function of corrections to the metric. At this point, one can use a result from [22] for the GLC gauge, namely

$$\delta g_{ab} = g_{ac}\partial_b \alpha_1^c + g_{bc}\partial_a \alpha_1^c, \tag{3.81}$$

where α_1 is the correction to the observed direction of emission due to lensing, the so-called deflection angle. This relation is proved in the Appendix C. The deflection angle, as we deduced in Chapter 2, has the explicit form

$$\alpha_1^a = -2\int_0^R \frac{R-r}{Rr}\tilde{\nabla}_\perp^a \phi. \tag{3.82}$$

Plugging these results in (3.80) leads to a term of the form $\partial_\tau \beta \propto \partial\partial\phi\partial\partial\phi$ (a term with four derivatives, thus highly dominant). In the GLC gauge, the expression for the corrections to the Sachs basis vectors is

$$\delta s_{Aa} = -\frac{1}{2}(\partial^b \alpha_{1a} + \partial_a \alpha_1^b)s_{Ab} = 2\int_0^R \frac{R-r}{R^2 r}\tilde{\nabla}_{\perp a}\tilde{\nabla}_{\perp b}\phi s_A^b \propto \int \partial\partial\phi. \tag{3.83}$$

By inserting (3.81) and the expression for the deflection angle (3.82) in the equation of evolution for $\beta^{(2)}$ (3.79), one finds

$$\boxed{\beta^{(2)}(R) = 2\epsilon_{ab}\int_0^R \frac{dr}{r^2}\int_0^r \frac{dr_1}{r_1^2}\int_0^{r_1} dr_2 \tilde{\nabla}_b\tilde{\nabla}_c\phi(r_2)\int_0^r dr_3 \tilde{\nabla}_a \tilde{\nabla}^c\phi(r_3),} \tag{3.84}$$

where $\tilde{\nabla}_\perp = (\partial_\theta, \frac{1}{\sin\theta}\partial_\phi)$.

The main problem is that we do not obtain this rotation term with the analysis in Poisson gauge. Providing an explanation of this discrepancy will the topic of the next Chapter. But, firstly we should note that the vectors components are, of course, gauge-dependant. The next step will be to find a convenient gauge-invariant quantity. Notice that so far, we have not really precisely defined the angle β. Doing so will lead us to a correct definition of the "rotation" of the Sachs basis and to a similar solution for the two gauges. We will also check that the computed power spectrum are indeed the same.

To conclude, we suggest an intuitive interpretation of the presence of rotation for the GLC gauge and the absence for the Poisson gauge. On the one hand, in the Poisson gauge, the perturbations are encoded in a purely diagonal metric, proportional to the identity matrix (at least for the spatial part). On the other hand, in the GLC gauge, the metric tensor has off-diagonal non trivial entries, see Appendix C. These non-trivial entries provide the rotation in the above formulas.

4 Reconciling Poisson and Geodesic Light-Cone gauge

In the early eighties, a quite audacious hypothesis, the inflation hypothesis, has been proposed to solve two important problems in cosmology, the famous *flatness* and *homogeneity* problems. It was soon recognized that inflation not only solve the problems it was engineered for, but also provides an unexpected explanation for the inhomogeneities in the CMB and in LSS. If the inflation is widely believed to be the correct scenario, the details of this event remain obscure. As we mentioned in the introduction, it appears that $\mathcal{B}$-modes in the CMB can be used as a tool to clarify many points. In so doing a correct interpretation of the $\mathcal{B}$-modes measurements is crucial, if we want to be able to read the primordial signal.

More recently, a slight contradiction showed up in the calculation of the $\mathcal{B}$-modes delensing. At first sight, it is not so obvious how to reconcile the result obtained in the GLC gauge ([46]) and the result obtained in the Poisson gauge ([47]). The purpose of this chapter is to shed some light on the relation between GLC and Poisson gauge and to reconcile the two results.

Contents

4.1 Introduction to the Lie transport

We start by studying the simultaneous propagation of a vector through *parallel transport* and through *Lie transport*. Let us assume an arbitrary vector $\mathbf{e}$ defined on the CMB[1]. The comparison between parallel transport and Lie transport will prove useful because we expect the polarisation to be parallel transported and the "typical structures" in the CMB temperature to be Lie transported (as shown in the Appendix B). Anyway, let us consider it as a simple exercise, the interpretation will become

[1] For example, this vector could describe the orientation of a typical pattern in the CMB temperature.

clearer in view of the result.

By definition, the Lie equation applied on the vector $\mathbf{e}$ is

$$L_k e \equiv \nabla_k e - \nabla_e k = 0, \tag{4.1}$$

and, thus, in this case, we say that the $\mathbf{e}$ *undergoes a Lie transport*. The specific expression of the Lie transport is demonstrated in Appendix B from more basic principles. For now, let us just mention that the Lie transport is the transport that extended structures undergo from the source to the observer. Using the natural basis ∂_i, we can write e in components $e^i\partial_i$. Inserting in (4.1), we end up with

$$\nabla_k e^i - e^j\nabla_j k = k^\nu\nabla_\nu e^i - e^j\nabla_j k = 0. \tag{4.2}$$

At zeroth order, of course, the equation is

$$\frac{de^i}{d\lambda} - e^j\partial_j k^i = 0. \tag{4.3}$$

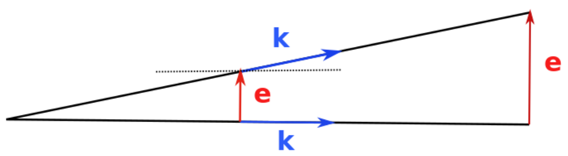


Fig. 4.1: Illustration of the "renormalisation" effect due to the different directions for the two $\mathbf{k}$.

At zeroth order, k^i is constant along its line and, therefore, the only effect is a normalisation of the length of e^i all along the path. Indeed, on Fig 4.1, we can see that $e^j\partial_j k^i$ is not vanishing, due to the different direction of the two $\mathbf{k}$. Because we are interested in rotation of the $\mathbf{e}$ and not in its length, we are not going to consider this term contribution in the following.

We also define the so-called *magnification matrix* as an application which takes a vector e at some point on the photon path for a universe *without* matter, to the same vector at the same point in a universe *with* matter. Explicitly, with e^i the components of the $\mathbf{e}$ in the universe with matter, and $\bar{e}^i$[2] the counterpart without matter,

$$e^i(\lambda) = A^i_j(\lambda)\bar{e}^j. \tag{4.4}$$

A^i_j is the magnification matrix. It can be written perturbatively $A^{i(0)}_j + A^{i(1)}_j + A^{i(2)}_j + \ldots$. Of course $A^{i(0)}_j = \delta^i_j$ and thus $A^{i(0)}_j$ as a vanishing antisymmetric part. In full generality, the A matrix is decomposed in shear, convergence and rotation[3]

$$A = \begin{pmatrix} 1+\kappa+\gamma_1 & \gamma_2-\omega \\ \gamma_2+\omega & 1+\kappa-\gamma_1 \end{pmatrix}. \tag{4.5}$$

See for example [19], [54], [9] or [41]. To extract the interesting ω (not to be confused with the frequency) from this formula, we have just to contract it with the antisymmetric tensor

$$\frac{1}{2}\epsilon^j_i.$$

We will now compute the correction at first and second order to this matrix and search for the antisymmetric part.

[2] This vector thus does not depend on the parameter.

[3] This interpretation however relies on the perturbative formulation.

4.2 First-order calculation of the Lie transport

Let us compute the transport at first order. Because we already did this kind of calculations twice in the previous chapters, we will be concise and concentrate only on the spatial components. We will use the "fixed observer direction" boundary condition as in the two previous cases, thus with $\delta e(0) = 0$. Expanding (4.1) at first order gives us

$$\delta k^j \nabla_j^{(0)} e^i + \Gamma^i_{jk} e^j k^k - \delta e^j \nabla_j^{(0)} k^i - \Gamma^i_{jk} e^k k^j + \frac{d\delta e^i}{d\lambda} - e^j \nabla_j^{(0)} \delta k^i = 0. \tag{4.6}$$

Using the symmetry of the Γ's and the fact that the unperturbed e and k are constant, we are left with

$$\begin{aligned} \frac{d\delta e^i}{d\lambda} &= e^j \nabla_j^{(0)} \delta k^i \\ &= -2\omega^2 e^j \nabla_{\perp j} \int_0^\lambda \nabla^i_\perp \phi. \end{aligned} \tag{4.7}$$

For the second equality we used that $\nabla^{(0)} = \partial$. We also go to angular coordinates for the vector e, therefore $e^j \to \lambda e^j$ and $e^j \nabla_{\perp j} \to e^j \tilde{\nabla}_{\perp j}$. Inserting these changes, we finally obtain the following first-order result:

$$\boxed{A_j^{i(1)}(\lambda_0) = -2\omega^2 \frac{1}{\lambda_0} \int_0^{\lambda_0} \tilde{\nabla}_{\perp j} \int_0^{\lambda} \frac{1}{\lambda'} \tilde{\nabla}^i_\perp \phi = -2\omega^2 \int_0^{\lambda_0} \frac{\lambda_0 - \lambda}{\lambda \lambda_0} \tilde{\nabla}_{\perp j} \tilde{\nabla}^i_\perp \phi.} \tag{4.8}$$

Again, obviously this is totally symmetric and

$$\epsilon_i^j A_j^{i(1)} = 0. \tag{4.9}$$

Therefore $A_j^{i(0)} + A_j^{i(1)}$ does not produce any contribution for the rotation ω, as it should.

4.3 Second-order calculation of the Lie transport

We now turn to the second-order corrections. The equation (4.1) fully expanded gives

$$\begin{aligned} 0 = {} & \delta^2 k^j \nabla_j^{(0)} e^i + \delta k^j \nabla_j^{(0)} \delta e^i + \frac{d\delta^2 e^i}{d\lambda} - \delta e^j \nabla_j^{(0)} \delta k^i - e^j \nabla_j^{(0)} \delta^2 k^i \\ & - \delta^2 e^j \nabla_j^{(0)} k^i + \Gamma^i_{jk}(\delta e^j k^k + \delta k^k e^j) - \Gamma^i_{jk}(\delta e^k k^j + \delta k^j e^k). \end{aligned} \tag{4.10}$$

Again, using the symmetries of the Γ's and the fact that the unperturbed vectors e and k are constant, we find the simpler relation

$$\frac{d\delta^2 e^i}{d\lambda} = -\delta k^j \nabla_j^{(0)} \delta e^i + \delta e^j \nabla_j^{(0)} \delta k^i + e^j \nabla_j^{(0)} \delta^2 k^i. \tag{4.11}$$

Integrating, inserting the specific values for the perturbations, discarding terms with less than four derivatives and going to angular coordinates for the vector e, therefore $e^j \to \lambda e^j$ and $e^j \nabla_{\perp j} \to e^j \tilde{\nabla}_{\perp j}$, we find the following expression for the second-order correction to the magnification matrix:

$$\begin{aligned} A^{i(2)}_j(\lambda_0) = 2\omega^4 \Big[&- \frac{2}{\lambda_0} \int_0^{\lambda_0} \int_0^{\lambda} \nabla^k_\perp \phi(\lambda'') d\lambda'' \nabla_{\perp k} \int_0^{\lambda} \frac{\lambda - \lambda'}{\lambda \lambda'} \nabla^i_\perp \nabla_{\perp j} \phi(\lambda') d\lambda' d\lambda \\ &- \frac{1}{\omega^2} \int_0^{\lambda_0} A^{k(1)}_j(\lambda) \nabla_{\perp k} \int_0^{\lambda} \nabla^i_\perp \phi(\lambda'') d\lambda'' d\lambda \\ &+ \frac{2}{\lambda_0} \int_0^{\lambda_0} \lambda \nabla_{\perp j} \Big(\int_0^{\lambda} \int_0^{\lambda'} \nabla_{\perp k} \phi(\lambda'') d\lambda'' \nabla^k_\perp \int_0^{\lambda'} \nabla^i_\perp \phi(\lambda''') d\lambda''' d\lambda' d\lambda \Big) \Big]. \end{aligned} \tag{4.12}$$

Now contracting with $\frac{1}{2}\epsilon^j_i$ to single out the antisymmetric part of this matrix, we obtain a non-vanishing result. The first term and the third term are totally symmetric, they thus vanish upon contraction with the antisymmetric tensor, but the second one is not. This term gives a contribution of the form

$$\begin{aligned} \epsilon^j_i \int_0^{\lambda_0} A^{k(1)}_j(\lambda) \nabla_{\perp k} \int_0^{\lambda} \nabla^i_\perp \phi(\lambda') d\lambda' d\lambda &= -2\epsilon^j_i \int_0^{\lambda_0} \frac{1}{\lambda} \int_0^{\lambda} \frac{\lambda - \lambda'}{\lambda \lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}^k_\perp \phi(\lambda') d\lambda' \int_0^{\lambda} \frac{1}{\lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}^i_\perp \phi(\lambda'') d\lambda'' d\lambda \\ &= -2\epsilon^j_i \int_0^{\lambda_0} \frac{1}{\lambda^2} \int_0^{\lambda} \frac{1}{\lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}^i_\perp \phi(\lambda'') d\lambda'' \int_0^{\lambda} \frac{\lambda - \lambda'}{\lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}^k_\perp \phi(\lambda') d\lambda' d\lambda. \end{aligned} \tag{4.13}$$

We can now integrate by part, with the formula

$$\int_0^R \frac{1}{r^2} f(r) dr = \int_0^R \frac{1}{r} \frac{df(r)}{dr} dr - \frac{1}{R} f(R)$$

where

$$\begin{aligned} \frac{df}{d\lambda}(\lambda) &= \epsilon^j_i \frac{d}{d\lambda} \left[\int_0^{\lambda} \frac{1}{\lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}^i_\perp \phi(\lambda'') d\lambda'' \int_0^{\lambda} \frac{\lambda - \lambda'}{\lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}^k_\perp \phi(\lambda') d\lambda' d\lambda \right] \\ &= \epsilon^j_i \frac{1}{\lambda} \tilde{\nabla}_{\perp k} \tilde{\nabla}^i_\perp \phi(\lambda) \int_0^{\lambda} \frac{\lambda - \lambda'}{\lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}^k_\perp \phi(\lambda') d\lambda' \\ &+ \epsilon^j_i \int_0^{\lambda} \frac{1}{\lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}^i_\perp \phi(\lambda'') d\lambda'' \left[\Big|_0^{\lambda} \frac{\lambda - \lambda'}{\lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}^k_\perp \phi(\lambda') \right] \\ &+ \epsilon^j_i \left[\int_0^{\lambda} \frac{1}{\lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}^i_\perp \phi(\lambda') d\lambda' \int_0^{\lambda} \frac{1}{\lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}^k_\perp \phi(\lambda'') d\lambda'' \right]. \end{aligned} \tag{4.14}$$

[4]In the last equality, the second term vanishes because we require the field ϕ to vanish at the observer, the third term vanishes because the part between brackets is totally symmetric and is contracted with the totally antisymmetric tensor. On the other hand,

$$
\begin{aligned}
f(\lambda_0) &= \epsilon_i^j \int_0^{\lambda_0} \frac{1}{\lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}_\perp^i \phi(\lambda'') d\lambda'' \int_0^{\lambda_0} \frac{\lambda_0 - \lambda'}{\lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(\lambda') d\lambda' \\
&= -\epsilon_i^j \int_0^{\lambda_0} \tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(\lambda') \int_0^{\lambda_0} \frac{1}{\lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}_\perp^i \phi(\lambda'') d\lambda'' d\lambda' \\
&= -\frac{1}{2} \epsilon_i^j \int_0^{\lambda_0} \tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(\lambda') d\lambda' \int_0^{\lambda_0} \frac{\lambda' - \lambda''}{\lambda' \lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}_\perp^i \phi(\lambda'') d\lambda'' d\lambda' \\
&= -\epsilon_i^j \int_0^{\lambda_0} \tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(\lambda') \int_0^{\lambda'} \frac{\lambda' - \lambda''}{\lambda' \lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}_\perp^i \phi(\lambda'') d\lambda'' d\lambda' \\
&= \epsilon_i^j \int_0^{\lambda_0} \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(\lambda') \int_0^{\lambda'} \frac{\lambda' - \lambda''}{\lambda' \lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}_{\perp j} \phi(\lambda'') d\lambda'' d\lambda',
\end{aligned}
\tag{4.15}
$$

where we mainly used the antisymmetry. To understand the fourth equality, it is enough to notice that

$$
\begin{aligned}
\epsilon_i^j \int_0^{\lambda_0} \tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(\lambda') \int_0^{\lambda_0} \frac{\lambda' - \lambda''}{\lambda' \lambda''} \tilde{\nabla}_{\perp k} \tilde{\nabla}_\perp^i \phi(\lambda'') d\lambda'' d\lambda' &= \int_0^{\lambda_0} \int_0^{\lambda_0} (\lambda' - \lambda'') \epsilon_i^j \frac{\tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(\lambda')}{\lambda'} \frac{\tilde{\nabla}_{\perp k} \tilde{\nabla}_\perp^i \phi(\lambda'')}{\lambda''} d\lambda'' d\lambda' \\
&= \int_0^{\lambda_0} \int_0^{\lambda_0} f(\lambda', \lambda'') d\lambda' d\lambda'',
\end{aligned}
$$

with $f(\lambda', \lambda'')$ perfectly symmetric under the interchange of λ' and λ''. We are integrating on a square of length λ_0 and the mapping $\lambda' \to \lambda''$, $\lambda'' \to \lambda'$, which maps the lower triangle in the upper triangle, leaves $f(\lambda', \lambda'')$ invariant.

Putting everything together, we find the following identity:

$$
\epsilon_i^j \int_0^{\lambda_0} A_j^{k(1)}(\lambda) \tilde{\nabla}_{\perp k} \int_0^{\lambda} \nabla_\perp^i \phi(\lambda'') d\lambda'' d\lambda = -2 \epsilon_i^j \int_0^{\lambda_0} \frac{\lambda_0 - \lambda}{\lambda_0 \lambda} \frac{1}{\lambda} \tilde{\nabla}_{\perp k} \tilde{\nabla}_\perp^i \phi(\lambda) \int_0^{\lambda} \frac{\lambda - \lambda'}{\lambda'} \tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(\lambda') d\lambda' d\lambda.
\tag{4.16}
$$

We can rewrite this equation more explicitly, by going from λ to r and by exchanging integrals and

[4] To determine the second equality, we made use of the identity

$$
\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, x_1) dx_1 = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} \frac{\partial f(x, x_1)}{\partial x} dx_1.
$$

derivatives. The $\omega^{(2)}$ at second order can then be written in the form

$$\begin{aligned}\omega^{(2)} &= 2\epsilon_i^j \int_0^R \frac{R-r}{Rr} \tilde{\nabla}_{\perp j} \tilde{\nabla}_\perp^k \phi(r) A_k^{(1)i}(r) dr \\ &= -2\epsilon_i^j \int_0^R \frac{R-r}{Rr} \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r) A_{jk}^{(1)}(r) dr\end{aligned} \tag{4.17}$$

where $\tilde{\nabla}_\perp = (\partial_\theta, \frac{1}{\sin\theta}\partial_\phi)$. Notice that this result agrees with the one found in [23] and [46] obtained from the alternative definition

$$A_j^i = \frac{\partial \theta_E^i}{\partial \theta_O^j} \tag{4.18}$$

for the magnification matrix (intuitively, this is the variation of the coordinate position of the sources with respect to the coordinate position of the images). We are now in position to show the equivalence between (4.17) and (3.84), following exactly the same lines as in [46].

$$\begin{aligned}\omega^{(2)} &= -\epsilon_i^j \int_0^R \frac{R-r}{Rr} \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r) A_{jk}^{(1)}(r) dr \\ &= 2\epsilon_i^j \int_0^R \frac{R-r}{Rr} \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r) \int_0^r \frac{r-r_1}{rr_1} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_1) dr_1 \\ &= 2\epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_1) dr_1 \int_0^{r_1} \frac{dr_2}{r_2^2} \int_0^{r_2} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3,\end{aligned} \tag{4.19}$$

where in the last line we used two integrations by parts. This last result can be further transformed

$$\begin{aligned}\omega^{(2)} &= 2 - \epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_1) dr_1 \int_0^{r_1} \frac{dr_2}{r_2^2} \int_0^{r_2} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3 \\ &= 2\epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r \frac{d}{dr_1} \Big(\int_0^{r_1} \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_4) dr_4 \Big) \int_0^{r_1} \frac{dr_2}{r_2^2} \int_0^{r_2} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3 \\ &= 2\epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_1) dr_1 \int_0^r \frac{dr_2}{r_2^2} \int_0^{r_2} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3 \\ &\quad - 2\epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r dr_1 \int_0^{r_1} dr_4 \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_4) \frac{d}{dr_1} \Big(\int_0^{r_1} \frac{dr_2}{r_2^2} \int_0^{r_2} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3 \Big) \\ &= 2\epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_1) dr_1 \int_0^r \frac{dr_2}{r_2^2} \int_0^{r_2} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3 \\ &\quad - 2\epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r \frac{dr_1}{r_1^2} \int_0^{r_1} dr_4 \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_4) \int_0^{r_1} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3 \\ &= \beta^{(2)} - 2\epsilon_i^j \int_0^R \frac{dr}{r^2} \int_0^r \frac{dr_1}{r_1^2} \int_0^{r_1} dr_4 \tilde{\nabla}_\perp^i \tilde{\nabla}_\perp^k \phi(r_4) \int_0^{r_1} \tilde{\nabla}_k \tilde{\nabla}_j \phi(r_3) dr_3.\end{aligned} \tag{4.20}$$

By antisymmetry, the second term vanishes and we are left with the very satisfactory result

$$\omega^{(2)} = \beta^{(2)}. \tag{4.21}$$

4.4 Lie transport in GLC gauge

Now, we would like to compute the Lie transport of an arbitrary vector in the Geodesic Light-Cone gauge. The form of the line element in GLC gauge is

$$ds^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U dw^a)(d\tilde{\theta}^b - U dw^b). \tag{4.22}$$

As explained in the Appendix C or in [26], the photon travels at w and $\tilde{\theta}^a$ constant and its four-impulsion is defined by

$$k^\mu = -\partial^\mu w = -g^{\mu w} = \Upsilon^{-1}\delta^\mu_\tau.$$

We want now to solve the Lie transport (4.1) in this gauge. Borrowing the result from the previous section, at first order we obtained

$$\frac{d\delta e^i}{d\lambda} = e^j \nabla^{(0)}_j \delta k^i = e^j \nabla_j \delta k^i. \tag{4.23}$$

Of course, by definition in GLC gauge, the deflection of the light ray by inhomogeneities is zero, because the photon path is *defined* in such a way that the direction of observed images corresponds to the direction of the sources. Therefore at all orders in the Weyl potential ϕ

$$\delta k^i \Big|_{\text{GLC}} = 0.$$

Therefore,

$$\frac{d\delta e^i}{d\lambda} = 0, \tag{4.24}$$

and we can safely conclude that

$$\delta e^i = 0.$$

We can now go to second-order corrections again by taking back the result of the previous section:

$$\frac{d\delta^2 e^i}{d\lambda} = -\delta k^j \nabla^{(0)}_j \delta e^i + \delta e^j \nabla^{(0)}_j \delta k^i + e^j \nabla^{(0)}_j \delta^2 k^i. \tag{4.25}$$

Following the same lines, the right hand side is trivially zero, giving

$$\delta^{(2)} e^i = 0.$$

In fact, this result can be extrapolated at all orders and we can conclude that

$$\delta^{(n)} e^i = 0,$$

for any positive integer n.

4.5 B-modes power spectrum in each point of view

There is already an extensive literature about the magnification matrix and there are different ways to find explicit expressions for this matrix. For example, in [48], [49] and [23], we can find a handy equation for the magnification matrix minus the identity part $\Psi^a_b \equiv A^a_b - \delta^a_b$. The claim is

$$\Psi^a_b = 2\int_0^R \frac{R-r}{Rr}\tilde{\nabla}^a_\perp \tilde{\nabla}_{\perp c}\phi(r,\theta)A^c_b dr. \tag{4.26}$$

This equation, when solved iteratively, permits us to find the expressions for Ψ^a_b at first and second order. Explicitly, we have

$$(\Psi^a_b)^{(1)} = 2\int_0^R \frac{R-r}{Rr}\tilde{\nabla}^a_\perp \tilde{\nabla}_{\perp b}\phi(r,\theta)dr \tag{4.27}$$

and

$$(\Psi^a_b)^{(2)} = 2\int_0^R \frac{R-r}{Rr}\left[\tilde{\nabla}_{\perp b}\tilde{\nabla}^a_\perp \tilde{\nabla}_{\perp c}\phi(r,\theta)\alpha_1(r)^c - \tilde{\nabla}^a_\perp \tilde{\nabla}_{\perp c}\phi(r,\theta)(\Psi^c_b)^{(1)}\right]dr. \tag{4.28}$$

One of the important claim of the two previous sections was that *the rotation at second order $\omega^{(2)}$ computed from the Lie transport or from* (4.28) *give the same result*. This can be seen at (4.17). A method using (4.28) was followed in [18], [30], [17] and [52]. For example, in [17], in the section "lens-lens coupling", the authors use an expansion of (4.26) to work out the rotation and subsequently the power spectrum induced by this rotation. The interpretation of the lens-lens coupling is that distant lenses interfere with closer lenses. But we have the problem that the $\mathcal{B}$-modes power spectrum induced by this vorticty $\omega^{(2)}$ does not seem, at first sight, to be the same than the one induced in the GLC gauge by the rotation angle $\beta^{(2)}$, for example [46] and [45] or in section 3.4 of this work. The purpose of this section is to give a fundamental argument to show that, actually, the two power spectra *must* be the same.

First of all, we present the relation between the magnification matrix and the Jacobian of the transformation GLC to PG. As exposed in the presentation of the GLC gauge in Appendix C, in this case, the information about the lensing is encoded in the gauge transformation itself

$$\tilde{\theta}^a_E \overset{GLC \to PG}{\longrightarrow} \theta^a_E \tag{4.29}$$

By definition, the angular coordinates of the GLC system $\tilde{\theta}^a$ are the same for the observed image and for the source itself. In other words, the coordinate position of the image and the source are the same. All the information about the remapping is encoded in (4.29), and therefore in the Jacobian matrix of the transformation

$$\frac{\partial \tilde{\theta}^a_E}{\partial \theta^b_E} = \frac{\partial \theta^a_O}{\partial \theta^b_E}. \tag{4.30}$$

But, comparing with (4.18), we notice that this is nothing but the magnification matrix expressed in Poisson gauge

$$\frac{\partial \theta^a_O}{\partial \theta^b_E}, \tag{4.31}$$

the Jacobian of the transformation source coordinates $\to$ image coordinates. This teaches us a really important lesson: *the matrix $A^a_b = \delta^a_b + \Psi^a_b$ is at the same time the magnification matrix and the Jacobian of the gauge transformation from the GLC to the PG.*

But to understand the equivalence between both point of view, it is preferable to go back to the spin-2 quantity

$$\mathcal{P} = e^a_+ e^b_+ P_{ab}. \tag{4.32}$$

If we work in GLC gauge, we know that

$$\mathcal{P}_{\text{GLC}}(\tilde{\theta}^a) \overset{\text{switch on } \phi}{\longrightarrow} e^{-2i\beta^{(2)}}\mathcal{P}_{\text{GLC}}(\tilde{\theta}^a) = e^{-2i\beta^{(2)}}\mathcal{P}_{\text{GLC}}(\theta^a + \alpha^a_1 + \alpha^a_2) \tag{4.33}$$

up to second order. On the other side, in the Poisson gauge, this rotation must be hidden in lensing, that is, the application of the magnification matrix on the direction

$$\theta^a \overset{\text{switch on } \phi}{\longrightarrow} A^a_b\theta^b \tag{4.34}$$

and therefore

$$\mathcal{P}_{\text{PG}}(\theta^a) \overset{\text{switch on } \phi}{\longrightarrow} \mathcal{P}_{\text{PG}}(A^a_b\theta^b) = \mathcal{P}_{PG}(\theta^a + \alpha^a_1 + \alpha^a_2 + (\Psi^a_b)^{(1)}_{\text{Anti}}\theta^b + (\Psi^a_b)^{(2)}_{\text{Anti}}\theta^b), \tag{4.35}$$

again up to second order[5]. Those two expression are not obviously equivalent, in the sense that $e^{-2i\beta(2)}\mathcal{P}_{\text{GLC}}(\theta^a + \alpha^a_1 + \alpha^a_2) \neq \mathcal{P}_{\text{PG}}(\theta^a + \alpha^a_1 + \alpha^a_2 + (\Psi^a_b)^{(1)}_{\text{Anti}}\theta^b + (\Psi^a_b)^{(2)}_{\text{Anti}}\theta^b)$. Here we would like to specialize in the rotation part, and to trace out its provenance:

- In GLC gauge, it comes from the multiplication by $e^{-2i\beta^{(2)}}$
- In Poisson gauge, it comes from the antisymmetric part of the magnification matrix $\omega^{(2)}$ (antisymmetric part of $(\Psi^a_b)^{(2)}\theta^b$).

We have already shown above that $\omega^{(2)} = \beta^{(2)}$ in (4.21). That is an important part of the equivalence. It remains to show that after integration, the computed $\mathcal{B}$-modes are the same. Using the formula from the introduction, we can isolate the $\mathcal{B}$-modes:

$$\mathcal{B}(\mathbf{l}) = -\int \frac{d^2\theta}{2\pi}\mathcal{I}\text{m}\Big[\mathcal{P}(\theta)e^{-2i\phi_l}\Big]e^{-i\mathbf{l}\cdot\theta}. \tag{4.36}$$

As we will do in Chapter 6, we can apply this formula to $\mathcal{P}$ in GLC gauge. With only the part coming from the rotation, we obtain (up to second order)

$$\mathcal{B}_{\text{GLC}}(\mathbf{l}) = -\int \frac{d^2\theta}{2\pi}\mathcal{I}\text{m}\Big[e^{-2i\beta^{(2)}}\mathcal{P}(\theta^a)e^{-2i\phi_l}\Big]e^{-i\mathbf{l}\cdot\theta}. \tag{4.37}$$

In the Poisson gauge, on the other hand, we obtain

$$\begin{aligned}\mathcal{B}_{\text{PG}}(\mathbf{l}) &= -\int \frac{d^2\theta}{2\pi}\mathcal{I}\text{m}\Big[\mathcal{P}(\theta^a + \omega^{(2)}\epsilon^a_b\theta^b)e^{-2i\phi_l}\Big]e^{-i\mathbf{l}\cdot\theta} \\ &= -\int \frac{d^2\theta}{2\pi}\mathcal{I}\text{m}\Big[\mathcal{P}(\theta^a)e^{-2i\phi_l}\Big]e^{-i\mathbf{l}\cdot\theta + il_a\omega^{(2)}\epsilon^a_b\theta^b}.\end{aligned} \tag{4.38}$$

[5] The subscript anti means that we have only to take the antisymmetric part of the magnification matrix

In the second equality, we have used a redefinition of the angle θ. This last line makes clear the fact that a rotation in one direction of the coordinates θ^a is equivalent to a rotation of l^a in the opposite direction. We can therefore redefine this l^a, by $\mathbf{l} \cdot \theta - l_a \omega^{(2)} \epsilon^a_b \theta^b \to \mathbf{l} \cdot \theta$. Actually, this amounts to rotate l^a by an angle $\omega^{(2)}$. But this modification of the vector $\mathbf{l}$ does have an incidence on the angle $\phi_{\mathbf{l}}$, which is the angle between between $\mathbf{l}$ and the reference $\mathbf{e}_x$ (defined on the telescope). Therefore the previous transformation is accompanied by $\phi_{\mathbf{l}} \to \phi_{\mathbf{l}} + \omega^{(2)}$. We obtain

$$\mathcal{B}_{\mathrm{PG}}(\mathbf{l}') = -\int \frac{d^2\theta}{2\pi} \mathcal{I}\mathrm{m}\Big[\mathcal{P}(\theta^a) e^{-2i\phi_l} e^{-2i\omega^{(2)}}\Big] e^{-i\mathbf{l}\cdot\theta}. \tag{4.39}$$

The equality (4.21) permits us to conclude that (4.38) and (4.39) give the same correction to the $\mathcal{B}$-modes power spectrum. Indeed the only difference between (4.38) and (4.39) is the that $\mathbf{l}'$ is slightly rotated with respect to $\mathbf{l}$, however, the power spectrum is only function of the norm of the vector $\mathbf{l}$. *Therefore this difference is irrelevant.*

4.6 Discussion and conclusion

This is a good place to pause and to summarize the results of all the previous chapters. We have indeed collected all the necessary results and are in position to reconcile the result in GLC gauge (from [46]) with the one found in Poisson gauge (for example [47]). The point is that a rotation in itself is meaningless, if we have no reference to compare with. What makes sense in the sky is the rotation of two objects (with intrinsic directions) with respect to each other.

Let us define the angle β_E as the observed *angle between the polarisation and a specific structure in the CMB* in a universe without matter and β_O as the same observed angle in the universe with matter. In our situation, the gauge-invariant quantity is the subtraction of them $\beta = \beta_O - \beta_E$. In other words, computing β in any gauge, we expect to find the same result. As shown in Appendix B, the temperature structures in the CMB are Lie transported from the source to the observer, while the polarisation follows the parallel transport we have defined for it in Chapter 3.

An intuitive way to test the gauge-invariance property of β is to define two parallel vectors at emission[6] (one describing a specific structure in the sky and one describing the polarisation of a photon) and to apply a Lie transport on one of them and a parallel transport on the other. The angle between these two vectors at the reception $\beta_O = \beta$ is computed perturbatively. Of course, at zeroth order, it is zero. At first order, there is no rotation induced neither by computing in GLC or in Poisson Gauge, thus we have also a vanishing resulg in both gauge. At second order, in the Poisson gauge, the parallel transport gives a vanishing contribution to the rotation while the Lie transport of the structure gives a dominant contribution. On the other hand, in the GLC gauge, parallel transport gives the dominant contribution while Lie transport vanishes. The result of (4.21)

$$\omega^{(2)} = \beta^{(2)} \tag{4.40}$$

shows the equality between the two dominant contributions. As a conclusion, we could summarize by *in the Poisson gauge, the image pattern rotates while, in the GLC gauge, the polarisation rotates*

[6] This amounts to take $\beta_E = 0$.

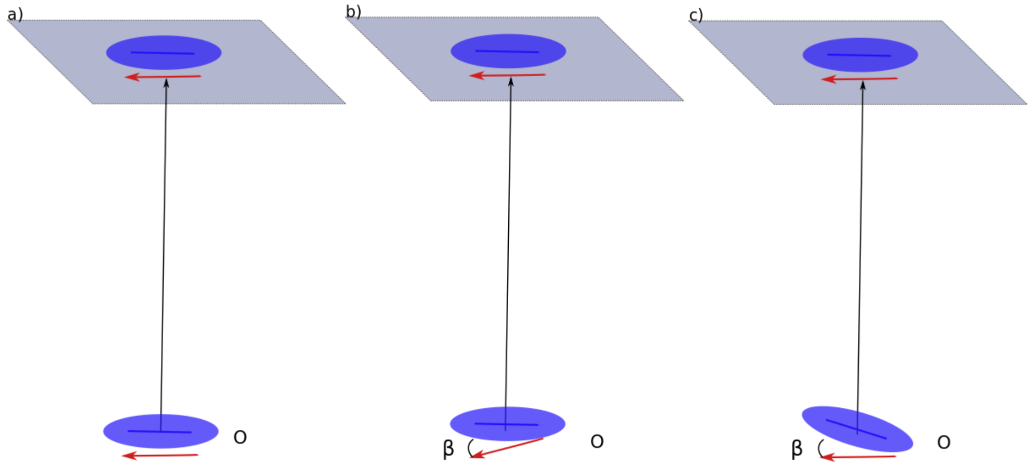


Fig. 4.2: Comparison between the two pictures. In a) we see an unperturbed image (assumed $\phi = 0$), therefore the observed image is the same as the one emitted. We chose a red arrow to describe the polarisation and a blue ellipse to describe a specific structure with the direction given by the blue headless arrow. b) shows the case when the polarisation rotates and the structure does not change, this is the case studied in section 3.4. The c) shows the case we studied in this section, with no rotation of the polarisation vector, but an opposite rotation of the structure given by the Lie transport, this was our point in the Poisson gauge.

in such a way that in both cases the angle between the two vectors remains the same. Figure (4.2) illustrates this fact.

The equality between the two angles is only half of the story. We still have to verify that the two power spectra obtained in the end are actually the same. In section 4.5, we have shown, using the equality between $\omega^{(2)}$ and $\beta^{(2)}$ that the power spectrum computed in GLC and in PG is indeed the same, as we expected!

5 The search for a curl in CMB polarisation

AFTER the realization of the relation between curl patterns in the CMB and primordial emission of gravitational waves, intensive experimental efforts have been made to observe unambiguously this very specific signature. However, as emphasized previously, this curl imprinted by gravitational waves in the CMB polarisation is most probably extremely weak, making its detection really challenging. Another source of confusion is the lensing and other effects causing *non primordial* curl in the polarisation. Discrimination of such sources and accurate cleaning is therefore a crucial step in the extraction of information from the CMB polarisation signal.

In previous Chapters, we have studied successively the parallel transport of the vector $\mathbf{k}$ and the Sachs basis $\mathbf{s_A}$ at first and at second order, and finally the Lie transport of a vector $\mathbf{e}$. In this process, we reconciled the GLC and the PG points of view. In the current chapter, we come back to the main purpose of this work, namely the computation of all possible "transport-induced" curls in the polarisation. We therefore exhibit and classify the different sources of curls and try to trace out (maybe intuitively) their origin.

Contents

5.1 Introduction of the polarisation

First, we introduce the field of "direction of oscillation of the electrons" at the last scattering surface

$$\mathbf{E}(x),$$

where x is a two-dimensional variable labelling the position *on* the last scattering surface. This direction of oscillation has no reason to be contained in the plane of the last surface[1]. It is thus a general direction in three dimensions.

As a result of this oscillation, the polarisation, called

$$\mathbf{P}(x),$$

will be emitted in the direction of the vector $\mathbf{n}$ and will be contained in the plane normal to it. We can then directly recover (at the surface of last scattering) this polarisation from a projection on the plane perpendicular to $\mathbf{n}$, that we called *screen plane* in previous chapters. To obtain the map of the sky we would observe on Earth, we then use the parallel transport of the projected vector $\mathbf{P}(x)$. We have seen that the possible emission of gravitational waves at earlier times, the so-called *primordial* gravitational waves, being a spin-2 object, are related to a curl in the distribution of the direction of oscillation. Indeed, as explained in [35], section 2.4, the tensor perturbations (transverse traceless perturbation of the metric) produce a stretching of the matter flow which, through Doppler effect, produces a temperature anisotropy. This temperature anisotropy in the *incoming* photons transfers into a curl in the direction of oscillation of the last scattering electrons. In other words gravitational waves, being of spin-2, will produce a non-vanishing curl in the direction of oscillations of the electrons on the last scattering surface.

The theorem of Helmholtz-Hodge ensures that any field can be decomposed into a "gradient part" and a "curl part". Explicitly, for any vector field $\mathbf{V}$ the following decomposition is allowed

$$\mathbf{V} = \mathbf{curl}\mathbf{A} - \mathbf{grad}\Psi,$$

with Ψ a scalar function and $\mathbf{A}$ a vector function. In our position, let us assume that the primordial field $\mathbf{E}$, that is to say before gravitational lensing and parallel transport, has no curl. We will therefore submit it to the condition

$$\epsilon_{ijk}\partial_j E^i = 0$$

for all k. The goal is now to trace out all the subsequent effects that could introduce a non vanishing curl in the received polarisation ($\mathcal{B}$-modes). We will mainly find three different types of effects :

1. The first one is the remapping coming from the lensing. This effect is already present at first order.

2. The second one is the projection of the direction of oscillation on the screen plane, the plane perpendicular to $\mathbf{n}$.

3. The third one is the parallel transport from the source to the observer, which can induce a non-vanishing rotation and thus a curl.

[1] Remember that we are using the flat-sky approximation, therefore the last scattering surface is a plane.

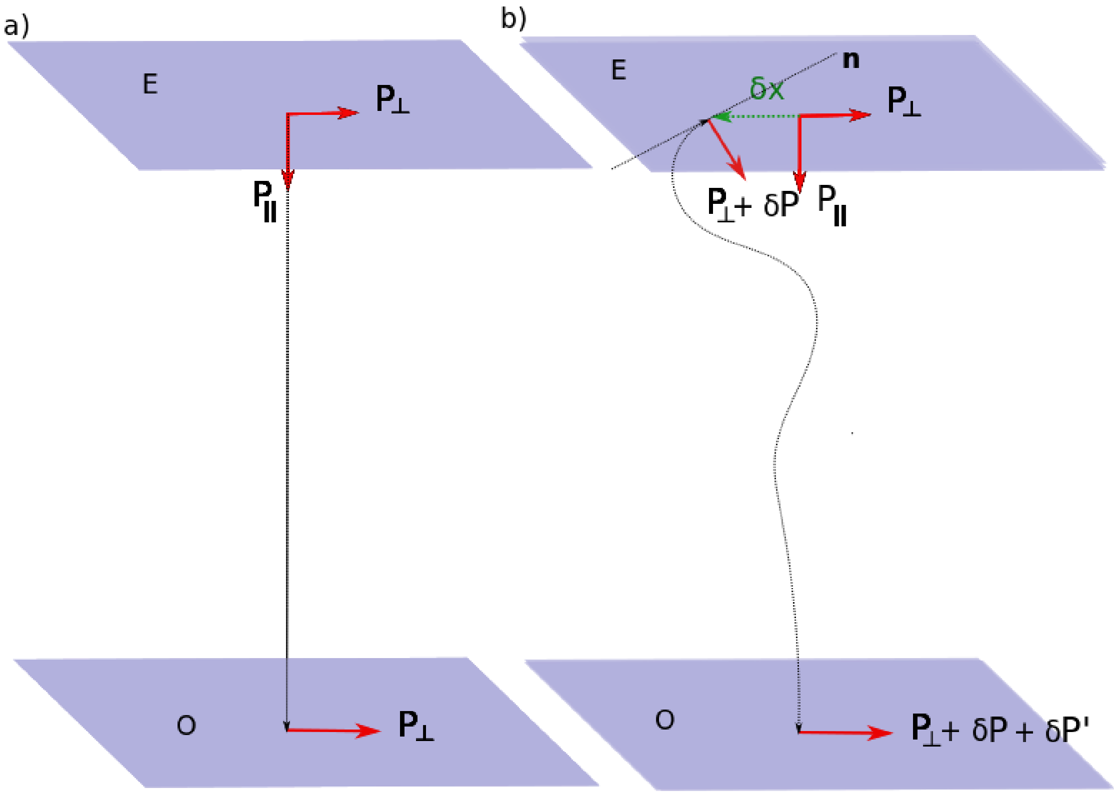


Fig. 5.1: Sources of correction. On a) we see an unperturbed ray of light going from the plane of emission to the plane of reception of the observer. The $\mathbf{n}$-vector being perpendicular to the plane of emission, only the $P_\perp$ component of the source oscillator is selected and then transported until the plane O of the observer. On b), we see a perturbed light ray, and we keep the same direction of observation than on a). Going through the geodesic in reverse direction (from the observer to the emitter), we see that the point of emission is different than on a), this is modelled by the green arrow on the plane of emission, and mathematically by the so-called "deflection angle" we computed in the previous chapters. We defined as on a) the component of the oscillator field by $P_\perp$ the component perpendicular to the unperturbed $\mathbf{n}$ and $P_{||}$ the component parallel to it. Before transporting these components, we have to project the polarisation on the plane perpendicular to the *perturbed* $\mathbf{n}$. The correction we get (from remapping and projection) is written in the form δP. We then transport it until the observer and get a new correction from transport, written $\delta P'$. We can then compare *at the same position* the two vectors $P_\perp$ and $P_\perp + \delta P + \delta P'$. Of course, as we will see in the following sections, both of these vectors are perpendicular to the direction of observation.

We have already computed the parallel transport of two orthogonal vectors, the vectors of the Sachs basis. We can choose the polarisation at the emission of the CMB to be one of these vectors, because, as we have proved in Chapter 3, the polarisation and the Sachs vectors follow exactly the same transport. Actually, we could even define one of the vector of the basis to be the electric vector field, and the other to be the magnetic vector field.

We will also separate the polarisation vector in two parts, the part parallel to the *unperturbed* line of sight $\mathbf{n}$, written $\mathbf{P}_{\|}$, and the part orthogonal to it, $\mathbf{P}_{\perp}$. Explicitly

$$\mathbf{P} = \mathbf{P}_{\|} + \mathbf{P}_{\perp}. \tag{5.1}$$

Because we chose the "fixed observer direction" boundary conditions, where the line of sight $\mathbf{n}$ is not corrected at the observer, this imposes that $\mathbf{P}_{\|}$ and its corrections must vanish at the observer. At this position, when we will compute the curl, we will use only the component coming from $\mathbf{P}_{\perp}$ defined in a plane, this is exactly the 2 and 3-components of our basis. The three types of correction we just discussed are illustrated on Fig. 5.1. Notice that our approach here is local. But we have seen in section 1.1 that a term of the form $\epsilon_{ij}\nabla^j\Psi$ in the polarisation will lead to $\mathcal{B}$-modes (see equation (1.17)). The spectrum of the $\mathcal{B}$-modes is a global object that will be studied in the next Chapter. Having at hand the polarisation $\mathbf{P}$, we can therefore chase the curl by applying the operator $\epsilon_{ij}\tilde{\nabla}^j_{\perp}$ on it. We obtain

$$\epsilon_{ij}\tilde{\nabla}^j_{\perp}P^i = \epsilon_{ij}\epsilon_{ik}\tilde{\nabla}^j_{\perp}\tilde{\nabla}^k_{\perp}\Phi = \tilde{\nabla}^2_{\perp}\Psi. \tag{5.2}$$

On the flat-sky, the inversion of the Laplace operator is *non-local*. Actually, we need the knowledge of the polarisation everywhere on the sphere to be able to invert the Laplace operator.

We will not address the problem of the inversion of the Laplace operator and we will just call the quantity $\tilde{\nabla}^2_{\perp}\Psi(\mathbf{x})$ "Curl($\mathbf{x}$)" and label the different contributions by its order and a subscript to mention its provenance:

$$\begin{aligned}\mathrm{Curl}(\mathbf{x}) = \mathrm{Curl}^{(1)}(\mathbf{x})\Big|_{\text{rem}} + \mathrm{Curl}^{(11)}(\mathbf{x})\Big|_{\text{rem}} + \mathrm{Curl}^{(2)}(\mathbf{n})\Big|_{\text{rem}} + \mathrm{Curl}^{(1)}(\mathbf{x})\Big|_{\text{prop}} \\ + \mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{prop}} + \mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{cross}} + \mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{rotation}}.\end{aligned} \tag{5.3}$$

$\mathrm{Curl}^{(1)}(\mathbf{x})\Big|_{\text{rem}}$ and $\mathrm{Curl}^{(11)}(\mathbf{x})\Big|_{\text{rem}}$ will be the correction induced by the expansion of $P^i(\mathbf{x}+\alpha_{\mathbf{1}})\Big|_{\text{rem}}$ up to second order, with α_1 the deflection angle at first order. $\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{rem}}$ will be the correction induced by the expansion of $P^i(\mathbf{x}+\alpha_{\mathbf{2}})\Big|_{\text{rem}}$ up to first order, with α_2 the deflection angle at second order. All those corrections are part of the so-called "remapping". $\mathrm{Curl}^{(1)}(\mathbf{x})\Big|_{\text{prop}}$ and $\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{prop}}$ will be the corrections induced by the parallel transport respectively at first and second order. $\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{cross}}$ will be the correction induced by the joined effect of parallel transport and remapping each at first order, the so-called *cross terms*. Finally, $\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{rotation}}$ will be the correction induced by the antisymmetric part of the magnification matrix at second order.

The last comment to do is about the gauge. We have seen in the previous chapter that the calculations can be dangerously gauge-dependent and that finding the right gauge-invariant quantity is crucial for the consistency. We will work in the Poisson gauge, and pay attention to the change in the Lie transported vector (via the magnification matrix, as we have done in Chapter 4).

5.2 The apparent rotation induced by the change of position

5.2.1 First-order deviation

Here we deduce the apparent rotation induced by the remapping. Remember that the point $\mathbf{x}$ (two-dimensional vector living on the emission plan) is sent to the point $\mathbf{x} + \delta\mathbf{x}$ by the remapping. We therefore have to investigate the curl induced by the correction

$$P^i(\mathbf{x}) \to P^i(\mathbf{x} + \delta\mathbf{x}), \tag{5.4}$$

where $\delta\mathbf{x}$ is the correction up to second order. The polarisation (5.4) can be expanded up to second order[2]

$$P^i(\mathbf{x}) \to P^i(\mathbf{x} + \alpha) = P^i(\mathbf{x}) + \tilde{\nabla}_{\perp j} P^i(\mathbf{x})\alpha^j + \frac{1}{2}\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp l} P^i(\mathbf{x})\alpha^j\alpha^l + O(3). \tag{5.6}$$

When computing the correlation functions, the second and the third terms will give a contribution of the same order of magnitude (actually, the two contributions are: second term with second term which combine to give a second-order contribution, and third term and first term which combine to give also a second-order term) therefore the third term in (5.6) can not be discarded if we want consistency. To work this out, we will use the deflection angle

The second term in (5.6) induces a curl of the form

$$\text{Curl}^{(1)}(\mathbf{x})\Big|_{\text{rem}} = -2\epsilon_{ik}\tilde{\nabla}_{\perp k}\Big[\tilde{\nabla}_{\perp j} P^i \alpha_1^j(\mathbf{x})\Big], \tag{5.7}$$

which splits into two parts,

$$-2\epsilon_{ik}\tilde{\nabla}_{\perp j} P^i(\mathbf{x})\Big[\tilde{\nabla}_{\perp k}\alpha_1^j(\mathbf{x})\Big],$$

and

$$-2\Big[\epsilon_{ik}\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp k} P^i(\mathbf{x})\Big]\alpha_1^j(\mathbf{x}).$$

The latter vanishes, because the factor between brackets is zero by the assumption that there is no curl at the emission.

As a consequence, the fully expanded expression is[3]

$$\boxed{\text{Curl}^{(1)}(\mathbf{x})\Big|_{\text{rem}} = -2\epsilon_{ik}\tilde{\nabla}_{\perp j} P^i(\mathbf{x})\Bigg[\int\limits_0^R \frac{(R-r)}{Rr}\tilde{\nabla}_\perp^k\tilde{\nabla}_\perp^j\phi(r,\theta,\phi)dr\Bigg].} \tag{5.9}$$

[2] notice that, because $\delta\mathbf{x} = R\alpha$ and $\nabla_\perp = \frac{1}{R}\tilde{\nabla}_\perp$, this is equivalent to the expansion

$$P^i(\mathbf{x}) \to P^i(\mathbf{x} + \delta\mathbf{x}) = P^i(\mathbf{x}) + \nabla_{\perp j} P^i(\mathbf{x})\delta x^j + \frac{1}{2}\nabla_{\perp j}\nabla_{\perp l} P^i(\mathbf{x})\delta x^j\delta x^l, \tag{5.5}$$

where j and l are two-dimensional labels which take the value 2 and 3 on the plane of emission

[3] We are using the expression for the deflection angle

$$\alpha = -2\int\limits_0^R \frac{R-r}{Rr}\tilde{\nabla}_\perp\phi(r,\theta,\phi)dr. \tag{5.8}$$

.

Similarly, the third term in (5.6), gives a curl term of the form

$$\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{rem}} = \frac{1}{2}\epsilon_{im}\tilde{\nabla}^{m}_{\perp}\Big[\nabla_{\perp j}\tilde{\nabla}_{\perp k}P^{i}\alpha_1^j(\mathbf{x})\alpha_1^k(\mathbf{x})\Big] \tag{5.10}$$

$$= \frac{1}{2}\epsilon_{im}\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp l}P^{i}(\mathbf{x})\tilde{\nabla}^{m}_{\perp}\Big[\alpha_1^j(\mathbf{x})\alpha_1^k(\mathbf{x})\Big] \tag{5.11}$$

$$= \epsilon_{im}\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp l}P^{i}(\mathbf{x})\alpha_1^j(\mathbf{x})\Big[\tilde{\nabla}^{m}_{\perp}\alpha_1^k(\mathbf{x})\Big] \tag{5.12}$$

where we used the same manipulations as before to show that the antisymmetric derivatives of $\mathbf{P}$ give a vanishing result. To obtain the last equality, we used the fact that $\nabla_{\perp j}\nabla_{\perp l}P^i$ is symmetric in the indices l and j.

5.2.2 Second-order deviation

Now that we have investigated the curl induced by first-order deflection angle, we would like to investigate the same curl, but induced by the second-order deflection angle. We thus borrow again the formula (now up to first order)[4]

$$P^i(\mathbf{x}) \to P^i(\mathbf{x}) + \tilde{\nabla}_{\perp j}P^i(\mathbf{x})\alpha_2^j. \tag{5.14}$$

As a consequence[5], the remapping at second order in (5.14) induces a curl of the form

$$\begin{aligned}\mathrm{Curl}^{(2')}(\mathbf{x})\Big|_{\text{rem}} &= -2\epsilon_{im}\tilde{\nabla}_{\perp m}\Big[\tilde{\nabla}_{\perp k}P^{i}(\mathbf{n})\alpha_2^k\Big] \\ &= -2\epsilon_{im}\Big[\tilde{\nabla}_{\perp m}\tilde{\nabla}_{\perp k}P^{i}(\mathbf{n})\Big]\alpha_2^k \\ &- 2\epsilon_{im}\tilde{\nabla}_{\perp k}P^{i}(\mathbf{n})\tilde{\nabla}_{\perp m}\Big[\int\limits_0^R \frac{(R-r)}{Rr}\tilde{\nabla}_{\perp j}\tilde{\nabla}^{k}_{\perp}\phi(r)\alpha_1^j(r)dr\Big].\end{aligned} \tag{5.15}$$

The first term vanishes by the following manipulations:

$$\begin{aligned}&\epsilon_{im}\Big[\tilde{\nabla}_{\perp m}\tilde{\nabla}_{\perp k}P^{i}(\mathbf{n})\Big]\alpha_2^k \\ &= \tilde{\nabla}_{\perp k}\underbrace{\Big[\epsilon_{im}\tilde{\nabla}_{\perp m}P^{i}(\mathbf{n})\Big]}_{=0}\alpha_2^k \\ &= 0.\end{aligned}$$

[4] The explicit formula for α_2^j has been worked out in 2.3.5, and has the form

$$\alpha_2^i = -2\int\limits_0^R \frac{(R-r)}{Rr}\tilde{\nabla}_{\perp j}\tilde{\nabla}^{i}_{\perp}\phi(r)\alpha_1^j(r)dr, \tag{5.13}$$

where i takes only the values 2 and 3. We have thrown away the component along the unperturbed vector n and focus on the correction perpendicular to it.

[5] Again we consider that the deviation at the point of the observer is zero ($\delta^{(2)}x^j(0) = 0$) and that there is no correction to the n vector at the observer ($\delta^{(2)}n^j(0) = 0$ for j taking values 2 and 3).

In the last equality, the factor under brackets vanishes. We are just left with the second term in (5.15). That is to say

$$\boxed{\mathrm{Curl}^{(2')}(\mathbf{x})\Big|_{\mathrm{rem}} = -2\epsilon_{im}\tilde{\nabla}_{\perp k}P^i(\mathbf{n})\int_0^R \frac{(R-r)}{Rr}\Big[\tilde{\nabla}_{\perp m}\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp}^k\phi(r)\alpha_1^j(r) + \tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp}^k\phi(r)\tilde{\nabla}_{\perp m}\alpha_1^j(r)\Big]dr.} \tag{5.16}$$

This term is far from being negligible because it has four derivatives inside the brackets (and then integrated over). We expect it to be of the same order of magnitude than the term collected in (3.84).

In this analysis, we missed the cross terms, coming from the first corrected $P^i(\mathbf{n}) \to P^i(\mathbf{n}) + \tilde{\nabla}_{\perp j}P^i(\mathbf{n})\alpha^j$ and then transported along the geodesic at first order. We will then resume this analysis at the end of the following section.

5.3 The apparent rotation induced by parallel transport

5.3.1 Procedure

In the previous chapters, we computed the correction to the Sachs basis under parallel transport in the Poisson gauge. At first order, if we identify the polarisation with some of the vector of the Sachs basis, we found (see (3.42))

$$\delta P^i(r) = -2n^i\int_r^R \partial^j\phi(r')P_j dr' + (\phi(r) - \phi(R))P^i, \tag{5.17}$$

where the second term can be interpreted as a normalisation of the vector, as we have seen above. The first term is a correction to keep the orthogonality with $\mathbf{n}$ at first order. We have written it assuming that we already know the correction at the point of the emission $r = R$, and we want to investigate the correction at $r = 0$. Therefore, at the point $r = R$, the correction vanishes.

In the decomposition we presented in the introduction, this separates into

$$\delta\mathbf{P}_{\parallel} = -2\mathbf{n}\int_r^R \partial^j\phi(r')P_j dr' + (\phi(r) - \phi(R))\mathbf{P}_{\parallel}, \tag{5.18}$$

and

$$\delta\mathbf{P}_{\perp} = (\phi(r) - \phi(R))\mathbf{P}_{\perp}. \tag{5.19}$$

Up to second order, and with the same constraints, we have obtained

$$\delta^{(2)}\mathbf{P}_{\parallel} = -4\mathbf{n}\int_0^R\int_0^r \nabla_{\perp j}\phi\nabla_{\perp}^j\int_0^r \nabla_{\perp k}\phi P^k, \tag{5.20}$$

and

$$\delta^{(2)}\mathbf{P}_{\perp} = -2\int_0^R P^j\nabla_{\perp j}\phi\int_0^R \nabla_{\perp}\phi. \tag{5.21}$$

We are confronted with the problem of finding a curl into these corrections. We work in the basis (u, r, θ, ϕ) and restrict to the following situation :

1. At the emission point, we also assume a curl-free polarisation generated by $P^i(\mathbf{x}) = \tilde{\nabla}^i_\perp \Psi(\mathbf{x})$ where Ψ is a scalar field in three dimensions.
2. We then project this field on the plane normal to the corrected vector $\mathbf{n}$ (the Sachs screen), using the projector
$$\Pi^i_j = \delta^i_j - n^i n_j.$$
and our formulas for the correction to $\mathbf{n}$.
3. We finally transport the resulting polarisation until the observer and compute the curl at this position.

5.3.2 Motivation for this procedure

The motivation for this specific procedure is that, physically, the emitters are the oscillating electrons. Therefore, our real purpose is to study a possible curl in the distribution of these oscillators. Moreover, the direction of oscillation of these electrons is not necessarily contained in the plane of emission and, in full generality, can have a non-vanishing component orthogonal to it. for definiteness, we will suppose that the basis vectors $\mathbf{z}$ and $\mathbf{x}$ span the *unperturbed* emission plane and that $\mathbf{y}$ can be identified with the unperturbed vector $\mathbf{n}$. This is illustrated on Fig.5.2 and 5.3. Again, we assume no curl in this distribution, and study the curl induced by the transport and by the projection along the corrected vector $\mathbf{n}$.

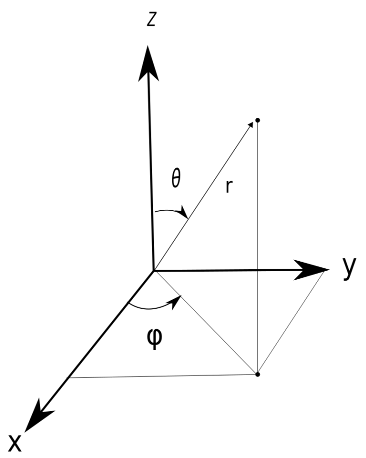


Fig. 5.2: Reference frame. Illustration of the angle θ and ϕ and their common expressions in rectangular coordinates. For definiteness, we assume that the basis vectors $\mathbf{z}$ and $\mathbf{x}$ span the emission plane and that $\mathbf{y}$ can be identified with the unperturbed vector $\mathbf{n}$.

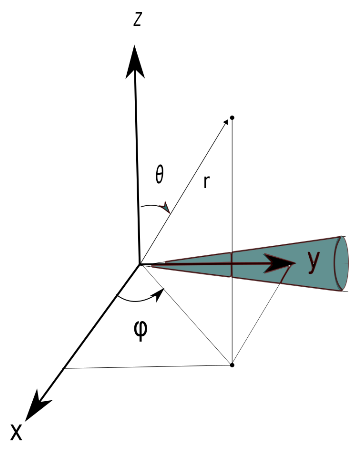


Fig. 5.3: The physical zone is around the y direction (blue-green cone). The cone is the direction of interest for us in the following. We will expand the cosine and the sine around this small angle. The plane x-z is the plane of the last scattering surface.

Of course, the electromagnetic radiation emitted by one oscillating electron has the form of a dipole, which, mathematically, is[6]

$$\mathbf{E} = -\frac{\mu_0 p_0 \omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t - \frac{r}{c}\right)\right]\theta, \tag{5.24}$$

where the angle θ used here is the angle between the axis $\mathbf{z}$ and the direction of observation, represented in the Fig. 5.2. We can also express θ in rectangular coordinates using the explicit formula:

$$\theta = \cos\theta\cos\phi\mathbf{x} + \cos\theta\sin\phi\mathbf{y} - \sin\theta\mathbf{z}.$$

If we keep only the angular dependence, the dipole formula simplifies to

$$\mathbf{E} \propto \sin\theta\theta \propto \sin\theta\cos\theta\cos\phi\mathbf{x} + \sin\theta\cos\theta\sin\phi\mathbf{y} - \sin\theta\sin\theta\mathbf{z} \tag{5.25}$$

This is valid only for the case where we have all the oscillators in the emission plane, all of them oscillating along the $\mathbf{z}$ direction. The unperturbed $\mathbf{n}$-direction coincide with $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{2}$, as explained on Fig 5.2. For the perturbed direction, we consider small perturbations around these angles, namely $\theta \approx \frac{\pi}{2} - \delta_1$ and $\phi \approx \frac{\pi}{2} - \delta_2$. This is illustrated in Fig. 5.3. Expanding (5.25) and keeping only first order terms, we obtain

$$\mathbf{E} \propto \delta_1\mathbf{y} - \mathbf{z}. \tag{5.26}$$

[6] We will just quickly show how to get the well-known formula (5.24). Assume a charge q_0 oscillating with a frequency ω in the direction $\mathbf{z}$ described by

$$q(t) = q_0\cos(\omega t).$$

We thus have an electric dipole of the form

$$\mathbf{p} = p_0\cos(\omega t)\mathbf{z}, \tag{5.22}$$

where $p_0 = q_0 d$. The retarded potential can found to be

$$V(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q_0\cos\left(t - \frac{r_+}{c}\right)}{r_+} - \frac{q_0\cos\left(t - \frac{r_-}{c}\right)}{r_-}\right],$$

where r_- and r_+ are given by

$$r_\pm = \sqrt{r^2 \mp rd\cos\theta + \left(\frac{d}{2}\right)^2}.$$

We will do the following approximations

$$d \ll r,$$
$$d \ll \frac{c}{\omega},$$
$$r \gg \frac{c}{\omega},$$

and, therefore, we end up with

$$V(r, \theta, t) = -\frac{p_0\omega}{4\pi\epsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t - \frac{r}{c}\right)\right]. \tag{5.23}$$

Using the relation between the potential and the electric field

$$\mathbf{E} = -\mathbf{grad}V(r, \theta, t) = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta},$$

and inserting the formula for V, we find the formula (5.24).

Now let us assume that the oscillation is not along $\mathbf{z}$ but is completely general. Instead of (5.22), we have

$$\mathbf{p} = p_0 \cos(\omega t)\mathbf{u}, \tag{5.27}$$

where $\mathbf{u}$ is normalized to one, but otherwise completely general. The direction of this $\mathbf{u}$ vector is determined by two angles, that we call a and b. We define those angles such as to simplify the computation and to have a straightforward physical meaning. a is defined like θ but running in the opposite direction. Therefore, the previous solution transforms as follows:

$$\theta \to a + \theta,$$

and the angle b is defined such that it induces a rotation in the plane z-x:

$$\mathbf{x} \to \cos b\mathbf{x} + \sin b\mathbf{z},$$

$$\mathbf{z} \to \cos b\mathbf{z} - \sin b\mathbf{x}.$$

With those two angles, we are able to describe the direction of $\mathbf{u}$. Applying those transformations to the dipole emission in (5.25), we obtain

$$\begin{aligned} \mathbf{E} \propto \sin(\theta + a)\theta \propto \sin(\theta + a)\cos(\theta + a)\cos(\phi)(\cos b\mathbf{x} + \sin b\mathbf{z}) \\ + \sin(\theta + a)\cos(\theta + a)\sin(\phi)\mathbf{y} - \sin(\theta + a)\sin(\theta + a)(\cos b\mathbf{z} - \sin b\mathbf{x}). \end{aligned} \tag{5.28}$$

Again the direction of the light ray, after being projected in the direction of the observer, is $\theta \approx \frac{\pi}{2} - \delta_1$ and $\phi \approx \frac{\pi}{2} - \delta_2$, with δ_1 and δ_2 computable corrections. Inserting these expressions in (5.28), we obtain finally

$$\begin{aligned} \mathbf{E} \propto \sin\left(\frac{\pi}{2} - \delta_1 + a\right)\cos\left(\frac{\pi}{2} - \delta_1 + a\right)\cos\left(\frac{\pi}{2} - \delta_2\right)(\cos b\mathbf{x} + \sin b\mathbf{z}) \\ + \sin\left(\frac{\pi}{2} - \delta_1 + a\right)\cos\left(\frac{\pi}{2} - \delta_1 + a\right)\sin\left(\frac{\pi}{2} - \delta_2\right)\mathbf{y} \\ - \sin\left(\frac{\pi}{2} - \delta_1 + a\right)\sin\left(\frac{\pi}{2} - \delta_1 + a\right)(\cos b\mathbf{z} - \sin b\mathbf{x}). \end{aligned} \tag{5.29}$$

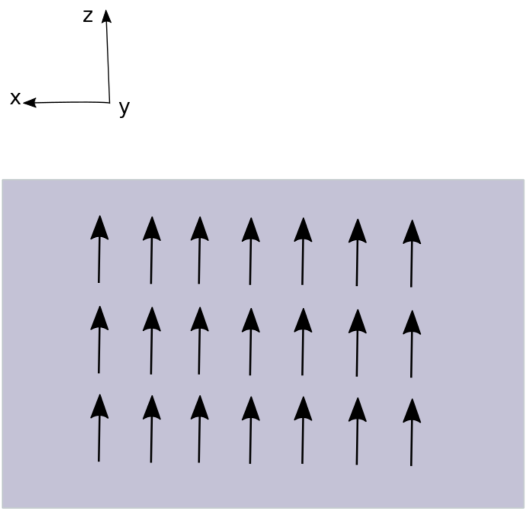


Fig. 5.4: The case where we take a and b to zero. We obtain a totally uniform distribution of oscillators in the sky.

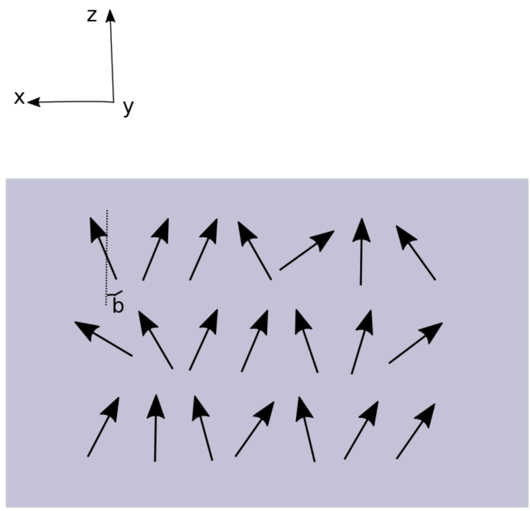


Fig. 5.5: An example of non-zero angle b. Because the b angle is a rotation in the plane z-x, it represents a rotation in the plane of the emission (last scattering surface).

If we set a and b to zero in these formulas, we recover the situation we had before, there is an illustration on Fig.5.4. If we set only a to zero, then we are in the case where the oscillators lie in the plane of emission, but inclined with an angle b with respect to the $\mathbf{z}$ direction. This situation is illustrated on Fig 5.5. With $a = 0$, (5.29) becomes

$$\begin{aligned}\mathbf{E} \propto \sin\left(\frac{\pi}{2} - \delta_1\right)\cos\left(\frac{\pi}{2} - \delta_1\right)\cos\left(\frac{\pi}{2} - \delta_2\right)(\cos b\mathbf{x} + \sin b\mathbf{z}) \\ + \sin\left(\frac{\pi}{2} - \delta_1\right)\cos\left(\frac{\pi}{2} - \delta_1\right)\sin\left(\frac{\pi}{2} - \delta_2\right)\mathbf{y} - \sin\left(\frac{\pi}{2} - \delta_1\right)\sin\left(\frac{\pi}{2} - \delta_1\right)(\cos b\mathbf{z} - \sin b\mathbf{x}).\end{aligned} \tag{5.30}$$

This is the case we would have studied if we had considered a 2-dimensional oscillators distribution. For small δ_1 and δ_2, the first term is of second order, the second one is of first order, and the third is of zeroth order. We are left, up to first order, with

$$\mathbf{E} \propto \delta_1\mathbf{y} - (\cos b\mathbf{z} - \sin b\mathbf{x}), \tag{5.31}$$

which, as expected, gives a correction along $\mathbf{y}$ and a mixing between the $\mathbf{z}$ and $\mathbf{x}$. On the other side, if we keep explicitly the a and b angle, the first term in (5.25) is of first order and the second and third one of zeroth order. Up to first order, this time we obtain

$$\begin{aligned}\mathbf{E} &\propto \sin(\delta_1 - a)\delta_2(\cos b\mathbf{x} + \sin b\mathbf{z}) + \sin(\delta_1 - a)\mathbf{y} - \cos(\delta_1 - a)(\cos b\mathbf{z} - \sin b\mathbf{x}) \\ &\approx -\delta_2 \sin a(\cos b\mathbf{x} + \sin b\mathbf{z}) + \sin(\delta_1 - a)\mathbf{y} - \cos a(\cos b\mathbf{z} - \sin b\mathbf{x}).\end{aligned} \tag{5.32}$$

Comparing (5.31) and (5.32), we conclude that the direction of emission is really perturbed by the a angle ! The most important difference is the appearance of the first-order term $-\delta_2 \sin a(\cos b\mathbf{x} + \sin b\mathbf{z})$ which perturbs the coefficients of the $\mathbf{z}$ and $\mathbf{x}$ directions. Rearranging terms we have finally

$$\mathbf{E} \propto \mathbf{x}(-\delta_2 \sin a \cos b + \cos a \sin b) - \mathbf{z}(\delta_2 \sin a \sin b + \cos a \cos b) + \mathbf{y}(\delta_1 \cos a - \sin a). \tag{5.33}$$

So, we can summarize the situation as follows: First we assume a plane sky of emission. In all point of this plane, we have electrons oscillating and then producing an electric field of the type previously shown (dipole type). We note that the direction of oscillation is not supposed a priori to be in the plane, but can be totally arbitrary.

Firstly, the question we want to address is: what would we see if we saw directly the polarisation field, without lensing ? The answer is simply the direction of oscillation of the electrons, projected on the plane orthogonal to $n_{E,\text{unperturbed}}$, namely the plane on the sky.

Now let us introduce the lensing. In this case, the vector n_E must be corrected, this correction of the direction unit vector induces a change in the screen plane, on which we project the polarisation. This new direction of projection "sees" the polarisation under a slightly different angle. In other words, we pick up a different part of the dipole emission.

5.3.3 First-order correction from projection and transport

In this section, we compute

$$\text{Curl}^{(1)}(\mathbf{x}, r)\Big|_{\text{prop}} = \epsilon_{ij}\tilde{\nabla}^i_\perp \delta P^j(\mathbf{x}, r), \tag{5.34}$$

where $\delta P^j(\mathbf{n}, r)$ is the correction induced by the projection and the transport at first order. The new plane under consideration is then determined by the fixed parameter r in the equation above. We can eventually impose $r = 0$ to obtain the polarisation at the observer.

Following the procedure described above[7], as we do in appendix D.1, we find the result

$$\text{Curl}^{(1)}(\mathbf{x}, 0)\Big|_{\text{prop}} = -\epsilon_{ij}P^j_\perp\tilde{\nabla}^i_\perp\phi(R) + \epsilon_{ij}\delta n^i(R)\tilde{\nabla}^j_\perp P_\parallel. \tag{5.35}$$

From this expression, we can conclude that most of the curl we receive comes from the $\mathbf{P}_\parallel$ at the emission. Then if the oscillators all oscillate in the plane of the emission, the curl at first order would be zero. The strange intrusion of the component $\mathbf{P}_\parallel$ (that would be absent at zeroth order) is illustrated in Fig. 5.6.

The calculation we have carried out here is totally local (we worked at one specific $\mathbf{n}$). The question we could ask is: globally, is this term still present ?

5.3.4 Cross terms between remapping and transport

The second "second-order" term that we study are the "cross" terms. We take the first-order deflection angle α^j_1, expand the polarisation at first order in it

$$P^i(\mathbf{x}) \to P^i(\mathbf{x}) + \tilde{\nabla}_{\perp j}P^i(\mathbf{x})\alpha^j_1. \tag{5.36}$$

[7] Computing the perturbed vector $\mathbf{n}$, projecting on the plane normal to it, transporting until the observer and finally computing the curl.

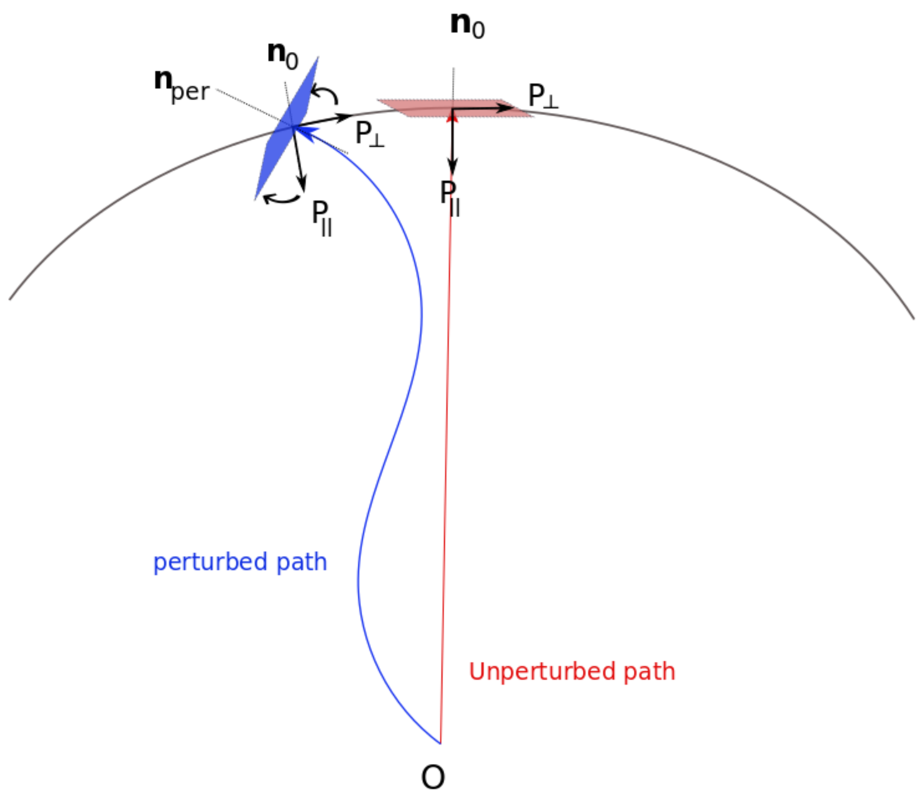


Fig. 5.6: Surprising intrusion of the component $P_{||}$ in the computation. The red path (unperturbed) exactly arrives perpendicularly to the last scattering surface. Therefore the projection on the plane perpendicular to $\mathbf{n}$ eliminates naturally the $P_{||}$ component. On the other hand, the blue (perturbed path) does not arrive perfectly perpendicularly to the last scattering surface. Therefore in the projection on the screen plane, we get both the perpendicular components (at zeroth order) and the parallel component (at first order). This explains the occurrence of the term $\propto P_{||}$ in eq. (5.35).

and then transport this expression for the polarisation along the geodesic (again at first order) until the observer.

The computation (the details are in the appendix D.2) gives a final result of the form:

$$\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\mathrm{cross}} = -4\epsilon_{ij}\tilde{\nabla}^{j}_{\perp}\Big[\tilde{\nabla}_{\perp k}P_{\parallel}(\mathbf{x})\int\limits_{0}^{R}\frac{R-r}{Rr}\tilde{\nabla}^{k}_{\perp}\phi(r)dr\Big]\int\limits_{0}^{R}\frac{1}{r}\tilde{\nabla}^{i}_{\perp}\phi(r)dr. \tag{5.37}$$

This term has three derivatives acting on the field ϕ and then being integrated, in that view, it is subleading with respect to (5.16). In the last section of this chapter, we will classify and sort out all the terms we collected. Let us continue now to the second-order parallel transport.

5.3.5 Second-order correction from projection and transport

Let us now generalize the problem to the second order. The calculations become now a bit more tricky. Before everything, we notice that, in the view of the comments we made in the last chapters, we don't expect to get leading corrections in this part, the leading correction coming actually not from the parallel transport, but from the the Lie transport of the "structures" in the CMB. This correction contains a dominant term with four derivatives, as we have seen in Chapter 4. Here we just confirm that the curl introduced at second order by the pure transport is sub-leading.

The final result is (with details in D.3)

$$\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\mathrm{prop}} = 2\epsilon^{ab}\tilde{\nabla}_{\perp b}P_{\parallel}(\mathbf{x})\tilde{\nabla}_{\perp a}\left[\int\limits_{0}^{R}\frac{dr}{r}\int\limits_{0}^{r}\frac{dr_2}{r_2}\tilde{\nabla}_{\perp j}\phi(r_2)\int\limits_{0}^{r}\frac{dr_3}{r_3}\tilde{\nabla}^{j}_{\perp}\phi(r_3)\right]. \tag{5.38}$$

This expression has just three derivatives acting on the field ϕ. As expected, this is a sub-leading correction. Note that this seems in contradiction with the conclusion of [18], which stated in section VI.B that curl could be induced by post Born-approximation in the case of a two-points lensing. The authors argued that after a first lensing, $\propto \partial_i\phi(\mathbf{x_1})$, the second lensing is proportional to $\propto \partial_k\partial_i\phi(r_1)\partial_i\phi(r_2)$, and that, taking the curl of this function we end up with, in general, a non-vanishing result of the form $\propto \epsilon_{lk}\partial_k\partial_i\phi(r_1)\partial_l\partial_i\phi(r_2)$. As it is, this expression does not vanish in general and we indeed found a term of this form in our derivation

$$2P_{\parallel}(\mathbf{x})\left[\epsilon^{ab}\int\limits_{0}^{R}\frac{dr}{r}\int\limits_{0}^{r}\frac{dr_2}{r_2}\tilde{\nabla}_{\perp b}\tilde{\nabla}_{\perp j}\phi(r_2)\int\limits_{0}^{r}\frac{dr_3}{r_3}\tilde{\nabla}_{\perp a}\tilde{\nabla}^{j}_{\perp}\phi(r_3)\right] = 0. \tag{5.39}$$

But the symmetries due to *the integrations* make it vanish in the end.

5.3.6 The curl coming from the rotation in the magnification matrix

In the previous Chapters, we have seen that there is a non-vanishing rotation part in the magnification matrix at second order. For the polarisation vector, this induces the following changes

$$P^{i}(x^{m}) \to P^{i}(x^{m} + \omega^{(2)}\epsilon^{m}_{k}x^{k}). \tag{5.40}$$

Expanding this correction up to first order, the second order correction to the polarisation tensor is

$$\delta P^i(\mathbf{x}) = \omega^{(2)}\epsilon_k^m x^k \nabla_m P^i(\mathbf{x}), \tag{5.41}$$

that we can use to compute the curl:

$$\begin{aligned}
\mathrm{Curl}^{(2)}(\mathbf{x})\Big|_{\text{rotation}} &= \epsilon_i^j \nabla_{\perp j}(\omega^{(2)}\epsilon_k^m x^k \nabla_m P^i(\mathbf{x})) \\
&= [\epsilon_i^j \nabla_{\perp j}\omega^{(2)}]\epsilon_k^m x^k \nabla_m P^i(\mathbf{x}) + \delta_{kj}\epsilon_i^j \omega^{(2)}\epsilon_k^m \nabla_m P^i(\mathbf{x}) \\
&= [\epsilon_i^j \nabla_{\perp j}\omega^{(2)}]\epsilon_k^m x^k \nabla_m P^i(\mathbf{x}) + \omega^{(2)}\nabla_i P^i(\mathbf{x})
\end{aligned} \tag{5.42}$$

This is a *leading* second order correction, because we have a total of five derivatives.

5.4 Summary

In this last section, we would like to gather all the pieces we have collected so far. We will classify the terms in first- and second-order contributions and mention the leading and the sub-leading parts. This will serve as a preliminary for the computation of the correlation functions we will do in the next chapter. Also, often, we have written the gravitational field ϕ, without mentioning its dependence with respect to r (or λ) and $\mathbf{x}$. To be clearer, we will reintroduce this dependence in the following formulas.

5.4.1 First order

The first-order contribution coming from the remapping is

$$-2\epsilon_{ik}\tilde{\nabla}_{\perp j}P^i(\mathbf{x})\Big[\int_0^R \frac{(R-r)}{Rr}\tilde{\nabla}_\perp^k \tilde{\nabla}_\perp^j \phi(r,\mathbf{x})dr\Big], \tag{5.43}$$

while the first-order contribution coming from the transport and the projection is

$$-\epsilon_{ij}P_\perp^j(\mathbf{x})\tilde{\nabla}_\perp^i \phi(R,\mathbf{x}) + \epsilon_{ij}\tilde{\nabla}_\perp^j P_\parallel(\mathbf{x})\int_0^R \frac{1}{r}\tilde{\nabla}_\perp^i \phi(r,\mathbf{x})dr. \tag{5.44}$$

The first term comes from the transport while the second comes from the projection. There is no obvious cancellation between all these first-order contributions.

5.4.2 Second order

In the case of second-order terms, we will just keep the terms with the higher number of derivatives acting on them and throw the other ones because they are sub-leading. The reason for this approximation is the same as usual and was explained in the introduction.

The contributions coming from the remapping at second order are

$$
\begin{aligned}
&8\epsilon_{im}\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp l}P^i(\mathbf{x})\int_0^R \frac{R-r}{Rr}\tilde{\nabla}^m_\perp\tilde{\nabla}^j_\perp\phi(r,\mathbf{x})dr\int_0^R \frac{R-r}{Rr}\tilde{\nabla}^l_\perp\phi(r,\mathbf{x})dr,\\
&= 4\epsilon_{im}\tilde{\nabla}_{\perp j}\tilde{\nabla}_{\perp l}P^i(\mathbf{x})\tilde{\nabla}^m_\perp\left[\int_0^R\int_0^R \frac{R-r}{Rr}\frac{R-r'}{Rr'}\tilde{\nabla}^l_\perp\phi(r,\mathbf{x})\tilde{\nabla}^j_\perp\phi(r',\mathbf{x})drdr'\right]
\end{aligned}
\tag{5.45}
$$

(coming from the composition of two first-order deflection angle) and

$$
\begin{aligned}
-2\epsilon_{im}\tilde{\nabla}_{\perp k}P^i(\mathbf{x})\int_0^R \frac{R-r}{Rr}\int_0^r \frac{r-r'}{rr'}\Big[&\tilde{\nabla}^j_\perp\phi(r',\mathbf{x})\tilde{\nabla}_{\perp m}\tilde{\nabla}_{\perp j}\tilde{\nabla}^k_\perp\phi(r,\mathbf{x})\\
&+\tilde{\nabla}_{\perp j}\tilde{\nabla}^k_\perp\phi(r,\mathbf{x})\tilde{\nabla}_{\perp m}\tilde{\nabla}^j_\perp\phi(r',\mathbf{x})\Big]dr'dr,
\end{aligned}
\tag{5.46}
$$

(coming from the second-order deflection angle). Those two terms are leading order contributions because they have five derivatives and we must therefore keep them. The cross term, being of the form

$$
-4\epsilon_{ij}\tilde{\nabla}^j_\perp\Big[\tilde{\nabla}_{\perp k}P_\parallel(\mathbf{x})\int_0^R \frac{R-r}{Rr}\tilde{\nabla}^k_\perp\phi(r,\mathbf{x})dr\Big]\int_0^R \frac{1}{r}\tilde{\nabla}^i_\perp\phi(r,\mathbf{x})dr,
\tag{5.47}
$$

is sub-leading and is then to be neglected.

The second-order contribution coming from the transport and the projection is

$$
2\epsilon^{ij}\tilde{\nabla}_{\perp j}P_\parallel(\mathbf{x})\tilde{\nabla}_{\perp i}\left[\int_0^R\frac{dr}{r}\int_0^r\int_0^r\frac{dr_2}{r_2}\frac{dr_3}{r_3}\tilde{\nabla}_{\perp k}\phi(r_2,\mathbf{x})\tilde{\nabla}^k_\perp\phi(r_3,\mathbf{x})\right].
\tag{5.48}
$$

As expected, it is also sub-leading and we neglect it.

Even if, at second order, there is no leading order contribution coming from the transport of the Sachs basis itself, we have seen in Chapter 3 and 4 that there is actually a rotation coming from the rotation $\omega^{(2)}$ in the magnification matrix. An explicit value for this rotation was worked out for example in (3.84). This rotation in turn induces a curl of the following form:

$$
[\epsilon_i^j\nabla_{\perp j}\omega^{(2)}]\epsilon_k^m x^k\nabla_m P^i + \omega^{(2)}\nabla_i P^i.
\tag{5.49}
$$

This term has a total of five derivatives and is thus a leading order term. We can then safely conclude that the total expression for the leading second-order correction is given by the addition of (5.45), (5.46) and (5.49).

5.4.3 Comments

If we now come back to the first-order contribution (5.44), we notice another problem. The projection part contains the component $P_\parallel$, which is the component of the electric field in the direction orthogonal to the plane of the sky. As we have seen, this part can not be discarded because it seems to have a strong effect, even at first order. On the other hand, there is no obvious way to measure directly the $P_{||}$ component. If our objective is to compare the result with and without inhomogeneities, the component $P_{||}$ is not present in the universe without inhomogeneities.

This probably means that our approach, using the corrections to the polarisation and applying on it the curl operator, is not really relevant physically and that we must change our point of view. In the next chapter, we use a characterization with $\mathcal{E}$- and $\mathcal{B}$-modes that proves way more useful and efficient in practice.

6 Extraction of the power spectrum of B-modes

In Chapter 5, we searched for a curl by applying the curl operator on the corrected polarisation, computing therefore the quantity $\nabla^2\Psi(\mathbf{x})$ at every position $\mathbf{x}$. This pointwise method, while interesting theoretically, proves very inconvenient in practice. The reason is that the Weyl field appearing in all our computations is not exactly the quantity predicted by inflation. Therefore using the naked Weyl field makes difficult the generation of simulated distributions and the comparison between prediction and observations. On the other hand, in the introduction 1.1, we have shown that the presence of curl in the polarisation[1] directly leads to presence of $\mathcal{B}$-modes in the power spectrum (see the relation between (1.15) and (1.17)). As a consequence, computing directly the $\mathcal{B}$-modes spectrum is another way of extracting the same information [2]. Moreover, as we will see, the perturbed power spectrum of $\mathcal{B}$-modes contains integration over the power spectrum of the unperturbed $\mathcal{E}$-modes and the power spectrum of the Weyl field (and also the power spectrum of the deflection angle, which can be expressed as a function of the Weyl field power spectrum). These two quantities are, at the same time, the observables we measure in practice and the theoretical predictions given by cosmic inflation.

Contents

[1] a term of the form $\epsilon_{ij}\tilde{\nabla}^i\Psi$ term.

[2] more precisely, this is a global method, involving Fourier transform of the correlation function.

6.1 Introduction

First of all, let us define the power spectrum. The power spectrum $P(|\mathbf{k}|)$ (written in the Fourier form) is defined using the Fourier transform of the correlation function:

$$\langle \delta^\star(\mathbf{k}_1)\delta(\mathbf{k}_2)\rangle = \delta_D(\mathbf{k}_1 - \mathbf{k}_2)P(|\mathbf{k_1}|). \tag{6.1}$$

The fact that there is no correlation between different $\mathbf{k}$ (we can say that the power spectrum is *diagonal*) comes from the requirement of *homogeneity* and the fact that $P(|\mathbf{k}|)$ depends only on the absolute value of $\mathbf{k}$ comes from the requirement of *isotropy*. These facts are shown in the Appendix E and are further elaborated. The definition (6.1) can be adapted to every kind of perturbations. For example, we can write $\delta(\mathbf{k}) = \mathcal{E}(\mathbf{k})$. In this case, the general power spectrum $P(|\mathbf{k}|)$ is called the "$\mathcal{E}$-modes power spectrum". We will note it $P^{\mathcal{E}}_{|\mathbf{l}|}$, but the convention $C^{\mathcal{E}}_{|\mathbf{l}|}$ is often used in the literature.

Before to start the computation of perturbed power spectra, let us collect some useful formula for $\mathcal{E}$- and $\mathcal{B}$-modes.

6.2 E and B-modes

We will be interested in the $\mathcal{B}$-modes specifically and will try to single out their power spectrum. To do so, the Fourier picture and the spin-2 polarisation $\mathcal{P}$ prove very convenient. Let us gather the required formula. As explained in [19], [41] and in our introduction, in the flat-sky approximation, the following identities hold:

$$\mathcal{P} = Q + iU = -\int \frac{d^2\mathbf{l}}{2\pi}\Big[\mathcal{E}(\mathbf{l}) + i\mathcal{B}(\mathbf{l})\Big]e^{2i\phi_\mathbf{l}}e^{i\mathbf{l}\cdot\mathbf{x}}, \tag{6.2}$$

and

$$\mathcal{P}^\star = Q - iU = -\int \frac{d^2\mathbf{l}}{2\pi}\Big[\mathcal{E}(\mathbf{l}) - i\mathcal{B}(\mathbf{l})\Big]e^{-2i\phi_\mathbf{l}}e^{i\mathbf{l}\cdot\mathbf{x}}, \tag{6.3}$$

where $\phi_\mathbf{l}$ is the angle between $\mathbf{l}$ and $\mathbf{x}$. We can easily invert these relations to get an expression for the $\mathcal{E}$ and $\mathcal{B}$ (by "modified" inverse Fourier transform):

$$\mathcal{E}(\mathbf{l}) + i\mathcal{B}(\mathbf{l}) = -\int \frac{d^2\mathbf{x}}{2\pi}\mathcal{P}e^{-2i\phi_\mathbf{l}}e^{-i\mathbf{l}\cdot\mathbf{x}}, \tag{6.4}$$

and

$$\mathcal{E}(\mathbf{l}) - i\mathcal{B}(\mathbf{l}) = -\int \frac{d^2\mathbf{x}}{2\pi}\mathcal{P}^\star e^{2i\phi_\mathbf{l}}e^{-i\mathbf{l}\cdot\mathbf{x}}. \tag{6.5}$$

Then, simple algebra allows us to single out specifically the $\mathcal{E}$ and the $\mathcal{B}$ part. Here $\mathcal{P}$ is defined as $\mathcal{P} = e^a_+ e^b_+ P_{ab}$, where e^a_+ is a basis defined by the two vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $\mathbf{e}_\pm = \mathbf{e}_1 \pm i\mathbf{e}_2$[3].

[3] This basis is defined on the measuring device, typically, the telescope. Therefore they will not be rotated when we switch on the perturbations.

The $\mathcal{E}$ and $\mathcal{B}$ power spectrum are defined by

$$\langle \mathcal{E}(\mathbf{l})\mathcal{E}^\star(\mathbf{l}')\rangle = \delta_D(\mathbf{l}-\mathbf{l}')P_l^{\mathcal{E}}, \qquad \langle \mathcal{B}(\mathbf{l})\mathcal{B}^\star(\mathbf{l}')\rangle = \delta_D(\mathbf{l}-\mathbf{l}')P_l^{\mathcal{B}}. \tag{6.6}$$

There is no term mixing $\mathcal{E}$ and $\mathcal{B}$, because of statistical parity (Remember that $\mathcal{E}$ is even under parity, while $\mathcal{B}$ is odd under parity). In the following we will consider that the unperturbed field P_{ab} (without transport or lensing) is a perfect E-field. As a consequence, we have $\mathcal{B}^{(0)}(\mathbf{l}) = 0$ and $P_l^{\mathcal{B}(0)} = 0$. Moreover, there is a correlation between temperature and $\mathcal{E}$-modes, of the form

$$\langle \mathcal{E}(\mathbf{l})\Theta^\star(\mathbf{l}')\rangle = \delta_D(\mathbf{l}-\mathbf{l}')P_l^{\mathcal{E}\Theta}. \tag{6.7}$$

For the same reason of statistical parity, there is no correlation between $\mathcal{B}$-modes and temperature[4]

$$\langle \mathcal{B}(\mathbf{l})\Theta^\star(\mathbf{l}')\rangle = \delta_D(\mathbf{l}-\mathbf{l}')P_l^{\mathcal{B}\Theta} = 0. \tag{6.8}$$

With those results, we have all the formulas to work out the corrections to the $\mathcal{B}$-modes spectrum. The computation will be perturbative in the Weyl field power spectrum $P_l^\phi(r,r')$. We will call "lowest order" a correction with one power of this power spectrum, $\sim P_l^\phi(r,r')$, and "higher order" a correction with two powers of it, $\sim (P_l^\phi(r,r'))^2$. This convention can be confusing, because a first-order correction in the *Weyl power spectrum* is actually a second-order correction in the *Weyl field*. Indeed, the power spectrum contains two powers of the field. Similarly, a second-order correction in the Weyl power spectrum is a fourth order correction in the Weyl field. To not get confused, we use two different conventions:

- "Lowest" and "higher" corrections will refer to expansion in the Weyl *power spectrum*.
- "First", "second", "third" ... corrections will refer to an expansion in the Weyl *field*.

The corrections to $\mathcal{E}$ and $\mathcal{B}$ functions can be expressed as corrections to $\mathcal{P}$:

$$\mathcal{E}(\mathbf{l}) + \delta\mathcal{E}(\mathbf{l}) + i\delta\mathcal{B}(\mathbf{l}) = \mathcal{E}(\mathbf{l}) - \int \frac{d^2\mathbf{x}}{2\pi}\delta\mathcal{P}e^{-2i\phi_l}e^{-i\mathbf{l}\cdot\mathbf{x}}, \tag{6.9}$$

and

$$\mathcal{E}(\mathbf{l}) + \delta\mathcal{E}(\mathbf{l}) - i\delta\mathcal{B}(\mathbf{l}) = \mathcal{E}(\mathbf{l}) - \int \frac{d^2\mathbf{x}}{2\pi}\delta\mathcal{P}^\star e^{2i\phi_l}e^{-i\mathbf{l}\cdot\mathbf{x}}. \tag{6.10}$$

The first goal is therefore to work out the correction $\delta\mathcal{P}$ up to second order. For that we can use the definition of the polarisation tensor $P_{ab} = \langle E_a E_b^\star\rangle_{\text{time}}$ (where the brackets denote an average over time). The corrections to the electric vector $\mathbf{E}$ comes from three different sources: the projection, the transport and the remapping (or lensing). We will keep emphasis on the correlation function of $\mathcal{B}$-modes, and leave away the correlation functions of $\mathcal{E}$-modes with itself or with temperature anisotropies. The important point is that there is no coupling between the $\mathcal{E}$ and $\mathcal{B}$ or between the temperature and $\mathcal{B}$. Therefore, the only correlation for B-mode is with itself. Consequently, we only have to compute $\langle\delta\mathcal{B}\mathcal{B}\rangle$, $\langle\delta\mathcal{B}\delta\mathcal{B}\rangle$, $\langle\mathcal{B}\delta^2\mathcal{B}\rangle$ to be able to build the complete correction at lowest order. We do that in the next section.

[4] Again, the temperature and the $\mathcal{B}$-modes have opposite parity. With more details, we can also explain it in this way: $\mathcal{B}$-modes cannot directly be produced by the Thomson scattering, but only by the summation on the different $\mathbf{k}$ of the perturbation in the sky. This summation, around hot spots, gives a correlation for the $\mathcal{E}$-modes, but the correlation for $\mathcal{B}$-modes cancels because of a cancellation of two opposite directed waves with $\mathbf{k}$ and $-\mathbf{k}$. This fact is explained more thoroughly in [34] and [35].

6.3 Lowest order corrections to B-modes power spectrum

In this section, we work out the corrections of lowest order to the correlation function of the $\mathcal{B}$-modes, namely $\langle \delta\mathcal{B}\delta\mathcal{B}\rangle$ and $\langle \delta^2\mathcal{B}\mathcal{B}\rangle$ (actually, this contains two powers of the gravitational field ϕ). We will see that the second ones, $\langle \delta^2\mathcal{B}\mathcal{B}\rangle$, vanish. As a consequence, the complete correction comes from the $\langle \delta\mathcal{B}\delta\mathcal{B}\rangle$ part.

6.3.1 Correction from the remapping

We will start the study by considering the remapping. The correction to the polarisation, up to second order, is

$$\delta\mathcal{P} = e_+^a e_+^b \tilde{\nabla}_c P_{ab}\alpha^c + \frac{1}{2}e_+^a e_+^b \tilde{\nabla}_d\tilde{\nabla}_c P_{ab}\alpha_1^c\alpha_1^d + e_+^a e_+^b \tilde{\nabla}_c P_{ab}\alpha_2^c. \tag{6.11}$$

Using the lensing potential $\alpha_1^c = \tilde{\nabla}_\perp^c \psi$, we can rewrite in the form

$$\delta\mathcal{P} = e_+^a e_+^b \tilde{\nabla}_c P_{ab}\tilde{\nabla}_\perp^c\psi + \frac{1}{2}e_+^a e_+^b \tilde{\nabla}_d\tilde{\nabla}_c P_{ab}\tilde{\nabla}_\perp^c\psi\tilde{\nabla}_\perp^d\psi + e_+^a e_+^b \tilde{\nabla}_c P_{ab}\alpha_2^b. \tag{6.12}$$

The derivatives we have factorized out are harmless, because after Fourier transform, they just become a $\mathbf{l}$ factor. Because they do not depend on the position, the vectors e_+^a and e_+^b can go through the derivatives and we are just left with $\mathcal{P}$, as defined before. The previous formula simplifies to

$$\delta\mathcal{P} = \tilde{\nabla}_c\mathcal{P}\tilde{\nabla}_\perp^c\psi + \frac{1}{2}\tilde{\nabla}_d\tilde{\nabla}_c\mathcal{P}\tilde{\nabla}_\perp^c\psi\tilde{\nabla}_\perp^d\psi + \tilde{\nabla}_c\mathcal{P}\alpha_2^c. \tag{6.13}$$

Applying the "modified Fourier transform" and using (6.9) and (6.10), we obtain

$$\begin{aligned}\mathcal{E}(\mathbf{l}) + \delta\mathcal{E}(\mathbf{l}) + i\delta\mathcal{B}(\mathbf{l}) = \mathcal{E}(\mathbf{l}) &- \int\frac{d^2\mathbf{x}}{2\pi}\tilde{\nabla}_c\mathcal{P}\tilde{\nabla}_\perp^c\psi e^{-2i\phi_\mathbf{l}}e^{-i\mathbf{l}\cdot\mathbf{x}}\\ &-\int\frac{d^2\mathbf{x}}{2\pi}\tilde{\nabla}_c\mathcal{P}\alpha_2^c e^{-2i\phi_\mathbf{l}}e^{-i\mathbf{l}\cdot\mathbf{x}} - \frac{1}{2}\int\frac{d^2\mathbf{x}}{2\pi}\tilde{\nabla}_d\tilde{\nabla}_c\mathcal{P}\tilde{\nabla}_\perp^c\psi\tilde{\nabla}_\perp^d\psi e^{-2i\phi_\mathbf{l}}e^{-i\mathbf{l}\cdot\mathbf{x}}.\end{aligned} \tag{6.14}$$

The next step is to insert the expansion for $\mathcal{P}$ in terms of $\mathcal{E}$ and $\mathcal{B}$, namely

$$\mathcal{P} = -\int\frac{d^2\mathbf{l}}{2\pi}\Big[\mathcal{E}(\mathbf{l}) + i\mathcal{B}(\mathbf{l})\Big]e^{2i\phi_\mathbf{l}}e^{i\mathbf{l}\cdot\mathbf{x}}, \tag{6.15}$$

where $\mathcal{B}(\mathbf{l}) = 0$. Introducing the Fourier decomposition of the lensing potential, written $\psi(\mathbf{x}) = \int\frac{d^2\mathbf{l}}{2\pi}\psi(\mathbf{l})e^{i\mathbf{l}\cdot\mathbf{x}}$, we see that the derivatives translate into simple $\mathbf{l}$ and $\mathbf{l}'$ factors. Then, integrating the $e^{i\mathbf{x}\cdot(\mathbf{l}-\mathbf{l}'-\mathbf{l}'')}$ over $\mathbf{x}$, we obtain a delta function, and finally[5]

$$\begin{aligned}\mathcal{E}(\mathbf{l}) + \delta\mathcal{E}(\mathbf{l}) + i\delta\mathcal{B}(\mathbf{l}) = \mathcal{E}(\mathbf{l}) &- \int\frac{d^2\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{2i(\phi_{\mathbf{l}'}-\phi_\mathbf{l})}\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\psi(\mathbf{l}-\mathbf{l}')\\ &-\frac{1}{2}\int\frac{d^2\mathbf{l}_1}{2\pi}\int\frac{d^2\mathbf{l}_2}{2\pi}e^{2i(\phi_{\mathbf{l}_1}-\phi_\mathbf{l})}\mathbf{l_1}\cdot\Big[\mathbf{l_1}+\mathbf{l_2}-\mathbf{l_c}\Big]\mathbf{l_1}\cdot\mathbf{l_2}\mathcal{E}(\mathbf{l}_1)\psi(\mathbf{l}_2)\psi^\star(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l})\\ &-\int\frac{d^2\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{2i(\phi_{\mathbf{l}'}-\phi_\mathbf{l})}\mathbf{l}'\cdot\alpha_\mathbf{2}(\mathbf{l}-\mathbf{l}'),\end{aligned} \tag{6.16}$$

[5] because $\psi(\mathbf{x})$ is real, $\psi(\mathbf{l}) = \psi^\star(-\mathbf{l})$)

where we used the change of variable $l_1 \to l_1 + l_2 - l$ in the second line. Of course, ψ is of first order in the Weyl field ϕ, while α_2 is of second order in this field. Therefore, the first line is of first order, while the second and third lines are of second order. As a consequence, a more transparent notation would be

$$\mathcal{E}(\mathbf{l}) + \delta^{(1)}\mathcal{E}(\mathbf{l}) \pm i\delta^{(1)}\mathcal{B}(\mathbf{l}) = \mathcal{E}(\mathbf{l}) - \int \frac{d^2\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}' \cdot (\mathbf{l}-\mathbf{l}')\psi(\mathbf{l}-\mathbf{l}'), \tag{6.17}$$

for the zeroth and first order, and

$$\begin{aligned}\delta^{(2)}\mathcal{E}(\mathbf{l}) \pm i\delta^{(2)}\mathcal{B}(\mathbf{l}) = &-\frac{1}{2}\int \frac{d^2\mathbf{l}_1}{2\pi}\int \frac{d^2\mathbf{l}_2}{2\pi}e^{\pm 2i(\phi_{\mathbf{l_1}}-\phi_{\mathbf{l}})}\mathbf{l_1}\cdot\Big[\mathbf{l_1}+\mathbf{l_2}-\mathbf{l}\Big]\mathbf{l_1}\cdot\mathbf{l_2}\mathcal{E}(\mathbf{l}_1)\psi(\mathbf{l}_2)\psi^\star(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l})\\ &-\int \frac{d^2\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}'\cdot\boldsymbol{\alpha_2}(\mathbf{l}-\mathbf{l}'),\end{aligned} \tag{6.18}$$

for the second order. We will not compute the α_2 further, because it will not give any contribution (at least at lowest order in the Weyl power spectrum). Actually, $\langle\mathcal{E}\delta^2\mathcal{E}\rangle$ gives a Dirac delta for l and l', this induces that the factor $e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}$ is one. We will see soon that the $\mathcal{B}$-modes comes with a factor $1 - e^{\pm 4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}$, which is zero if the exponential is one.

Now we can compute the correlation functions of the $\mathcal{E}$- and $\mathcal{B}$-modes. At zeroth order, the correlation for $\mathcal{B}$ was assumed to be zero. This is no longer the case when we add the higher order perturbations. The first thing to note is that the correlation of a first-order (in the Weyl field) correction with a background value, $\langle\mathcal{E}\delta\mathcal{E}\rangle$, vanishes. This is easily seen if we study, for example,

$$\begin{aligned}&\int \frac{d^2\mathbf{l}'}{2\pi}e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\psi(\mathbf{l}-\mathbf{l}')\langle\mathcal{E}(\mathbf{l}')\mathcal{E}(\mathbf{l})\rangle\\ &=\int \frac{d^2\mathbf{l}'}{2\pi}P_l^{\mathcal{E}}\delta_D(\mathbf{l}-\mathbf{l})e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\psi(\mathbf{l}-\mathbf{l}'),\end{aligned}$$

which vanishes because of the Dirac delta. Therefore we just have to consider $\langle\delta\mathcal{E}\delta\mathcal{E}\rangle$. We have to compute $\langle(\delta\mathcal{E}(\mathbf{l})+i\delta\mathcal{B}(\mathbf{l}))(\delta\mathcal{E}^\star(\mathbf{l})-i\delta\mathcal{B}^\star(\mathbf{l}))\rangle$ and $\langle(\delta\mathcal{E}(\mathbf{l})+i\delta\mathcal{B}(\mathbf{l}))(\delta\mathcal{E}^\star(\mathbf{l})+i\delta\mathcal{B}^\star(\mathbf{l}))\rangle$, which gives respectively $\delta^{(2)}P_l^{\mathcal{E}}+\delta^{(2)}P_l^{\mathcal{B}}$ and $\delta^{(2)}P_l^{\mathcal{E}}-\delta^{(2)}P_l^{\mathcal{B}}$. In doing so, we are confronted with quantities like

$$\langle\mathcal{E}(\mathbf{l})\psi(\mathbf{l})\mathcal{E}(\mathbf{l}')\psi(\mathbf{l}')\rangle.$$

To simplify them, we can apply the Wick theorem because we do not expect dominant non-Gaussianities (three points correlations of the form $\langle\Psi_1\Psi_2\Psi_3\rangle$ and higher). Therefore we obtain

$$\langle\mathcal{E}(\mathbf{l})\psi(\mathbf{l})\mathcal{E}(\mathbf{l}')\psi(\mathbf{l}')\rangle = \langle\mathcal{E}(\mathbf{l})\mathcal{E}(\mathbf{l}')\rangle\langle\psi(\mathbf{l})\psi(\mathbf{l}')\rangle,$$

where other terms containing factors of the form $\langle\mathcal{E}(\mathbf{l})\psi(\mathbf{l})\rangle$ vanish because we do not expect correlation between the lensing potential and $\mathcal{E}$-modes at the last scattering surface. Computing the power spectra and inserting the Wick decomposition, we obtain

$$\begin{aligned}\delta^{(2)}P_l^{\mathcal{E}}+\delta^{(2)}P_l^{\mathcal{B}} = &\int \frac{d^2\mathbf{l}'}{(2\pi)^2}[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')]^2\langle\psi(\mathbf{l}-\mathbf{l}')^2\rangle P_{l'}^{\mathcal{E}} - P_l^{\mathcal{E}}\int \frac{d^2\mathbf{l}'}{(2\pi)^2}[\mathbf{l}\cdot\mathbf{l}']^2\langle\psi(\mathbf{l}-\mathbf{l}')^2\rangle\\ &-\frac{1}{2\pi}P_l^{\mathcal{E}}\mathbf{l}\cdot\boldsymbol{\alpha_2}(0),\end{aligned} \tag{6.19}$$

and

$$\delta^{(2)}P_l^{\mathcal{E}} - \delta^{(2)}P_l^{\mathcal{B}} = \int \frac{d^2\mathbf{l}'}{(2\pi)^2} e^{4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')]^2\langle\psi(\mathbf{l}-\mathbf{l}')^2\rangle P_{l'}^{\mathcal{E}} - P_l^{\mathcal{E}}\int\frac{d^2\mathbf{l}'}{(2\pi)^2}[\mathbf{l}\cdot\mathbf{l}]^2\langle\psi(\mathbf{l}-\mathbf{l}')^2\rangle$$
$$- \frac{1}{2\pi}P_l^{\mathcal{E}}\mathbf{l}\cdot\alpha_{\mathbf{2}}(0). \tag{6.20}$$

The cancellation between the first and the second term in those expressions comes from the first and the second term in (6.13). We now introduce the correlation function between the potential: $\langle\psi(\mathbf{l})\psi(\mathbf{l}')\rangle = \delta_D(\mathbf{l}-\mathbf{l}')P_l^{\psi}$ (this quantity is explicitly computed in the Appendix F). Subtracting the two previous equations, we can single out the lowest order correction to the $\mathcal{B}$-modes power spectrum. We obtain

$$\boxed{\begin{aligned}\delta^{(2)}P_l^{\mathcal{B}} &= \frac{1}{2}\int\frac{d^2\mathbf{l}'}{(2\pi)^2}\Big[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\Big]^2 P^{\psi}_{|\mathbf{l}-\mathbf{l}'|}P^{\mathcal{E}}_{l'}\Big[1-e^{4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\Big]\\
&= \frac{1}{2}\int\frac{d^2\mathbf{l}'}{(2\pi)^2}\Big[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\Big]^2 P^{\psi}_{|\mathbf{l}-\mathbf{l}'|}P^{\mathcal{E}}_{l'}\Big[1-\cos[4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})]\Big]\\
&= \int\frac{d^2\mathbf{l}'}{(2\pi)^2}\Big[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\Big]^2 P^{\psi}_{|\mathbf{l}-\mathbf{l}'|}P^{\mathcal{E}}_{l'}\sin^2[2(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})]\\
&= 4\int_0^R dr\int_0^R dr'\frac{R-r}{Rr}\frac{R-r'}{Rr'}\int\frac{d^2\mathbf{l}'}{(2\pi)^2}\Big[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\Big]^2 P^{\phi}_{|\mathbf{l}-\mathbf{l}'|}(r,r')P^{\mathcal{E}}_{l'}\sin^2[2(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})].\end{aligned}} \tag{6.21}$$

This is the counterpart of (5.43). In the second equality, we decomposed the exponential in sine and cosine and noticed that the symmetry of the integral makes the sine part vanish. In the last equality, we used the expression for the power of the lensing potential in terms of the power of the field ϕ, given in the Appendix F equation (F.5). For a more thorough analysis, see [41]. Before to close the analysis, let us notice that, in the expansion

$$\delta P = \underbrace{e_+^a e_+^b\tilde{\nabla}_c P_{ab}\alpha^c}_{\mathcal{B}\text{-modes}} + \frac{1}{2}e_+^a e_+^b\tilde{\nabla}_d\tilde{\nabla}_c P_{ab}\alpha^c\alpha^d + e_+^a e_+^b\tilde{\nabla}_c P_{ab}\alpha_2^c, \tag{6.22}$$

the correction to the correlation function of the $\mathcal{B}$-modes at lowest order comes from the first term and from nothing else.

6.3.2 Correction from transport

In this part, we will redo the same analysis as in the previous subsection, but for the corrections coming from the transport. We make use of the fact that the polarisation is written, as a function of the electric field, in the form

$$\mathcal{P} = e_+^a e_+^b P_{ab} = e_+^a e_+^b\langle E^a E^{\star b}\rangle. \tag{6.23}$$

The corrections coming from the transport in Poisson gauge, up to second order, are

$$2e_+^a e_+^b\langle\delta E^a E^{\star b}\rangle + e_+^a e_+^b\langle\delta E^a\delta E^{\star b}\rangle + 2e_+^a e_+^b\langle\delta^2 E^a E^{\star b}\rangle,$$

with

$$\delta E^i = -2n^i\int_0^R\frac{dr}{r}\nabla_\perp^j\phi(r)E_j - (\phi(R)-\phi(0))E^i = -2n^i\int_0^R\frac{dr}{r}\nabla_\perp^j\phi(r)E_j - \Delta\phi E^i, \tag{6.24}$$

and we have seen that there is no leading contributions at second order, therefore we will not consider the $\delta^2 E^a$ here. Obviously, any part in the $\mathbf{n}$-direction will cancel with the e^a basis. Therefore

$$\delta\mathcal{P} = -2\mathcal{P}\Delta\phi + \mathcal{P}(\Delta\phi)^2.$$

We now define the Fourier transform of the difference of the Weyl potential: $\Delta\phi(\mathbf{l}) = \int \frac{d^2\mathbf{x}}{2\pi}\Delta\phi(\mathbf{x})e^{i\mathbf{l}\cdot\mathbf{x}}$. Inserting in the expression for the $\mathcal{E}$- and $\mathcal{B}$-modes, we obtain[6]

$$\mathcal{E}(\mathbf{l}) + \delta\mathcal{E}(\mathbf{l}) \pm i\delta\mathcal{B}(\mathbf{l}) = \mathcal{E}(\mathbf{l}) + 2\int \frac{d\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\Delta\phi(\mathbf{l}-\mathbf{l}'), \tag{6.25}$$

for the unperturbed result and the first-order correction, and

$$\delta^{(2)}\mathcal{E}(\mathbf{l}) \pm i\delta^{(2)}\mathcal{B}(\mathbf{l}) = -\int \frac{d^2\mathbf{l}_1}{2\pi}\int \frac{d^2\mathbf{l}_2}{2\pi}e^{\pm 2i(\phi_{\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2}-\phi_{\mathbf{l}})}\mathcal{E}(\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2)\Delta\phi(\mathbf{l}_2)\Delta\phi(\mathbf{l}_1), \tag{6.26}$$

for the second-order correction.

Now we can go to the computation of the first-order correction to the correlation functions $\langle\delta\mathcal{E}\mathcal{E}\rangle$. It is found to be

$$\delta P_l^{\mathcal{E}} = \frac{P_l^{\mathcal{E}}}{\pi}\Delta\phi(0). \tag{6.27}$$

But $\Delta\phi(0)$ is actually of the form $\int d^2\mathbf{x}\Delta\phi(\mathbf{x}) = \int d^2\mathbf{x}\phi(R,\mathbf{x})$[7] . This is the mean value of the Weyl field at the last scattering surface. It is quite surprising to find a correction of this form.

To second order, we have to compute $\langle\delta\mathcal{E}\delta\mathcal{E}\rangle$ and $2\langle\mathcal{E}\delta^2\mathcal{E}\rangle$. The perturbed power spectra are (using (6.34) and (6.35))

$$\delta^{(2)}P_l^{\mathcal{E}} + \delta^{(2)}P_l^{\mathcal{B}} = 4\int \frac{d^2\mathbf{l}'}{(2\pi)^2}P_{l'}^{\mathcal{E}}\langle(\Delta\phi(\mathbf{l}-\mathbf{l}'))^2\rangle - 2\int \frac{d^2\mathbf{l}'}{(2\pi)^2}P_{l'}^{\mathcal{E}}\langle\Delta\phi(\mathbf{l}')\Delta\phi^\star(\mathbf{l}')\rangle, \tag{6.28}$$

and

$$\delta^{(2)}P_l^{\mathcal{E}} - \delta^{(2)}P_l^{\mathcal{B}} = 4\int \frac{d^2\mathbf{l}'}{(2\pi)^2}P_{l'}^{\mathcal{E}}e^{4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\langle(\Delta\phi(\mathbf{l}-\mathbf{l}'))^2\rangle - 2\int \frac{d^2\mathbf{l}'}{(2\pi)^2}P_{l'}^{\mathcal{E}}\langle\Delta\phi(\mathbf{l}')\Delta\phi^\star(\mathbf{l}')\rangle. \tag{6.29}$$

Subtracting these two equations, only the first term survives. This gives

$$\boxed{\begin{aligned}\delta^{(2)}P_l^{\mathcal{B}} &= 2\int \frac{d^2\mathbf{l}'}{(2\pi)^2}P_{l'}^{\mathcal{E}}P_{|\mathbf{l}'-\mathbf{l}|}^{\Delta\phi}\left[1 - e^{4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\right]\\ &= 4\int \frac{d^2\mathbf{l}'}{(2\pi)^2}P_{l'}^{\mathcal{E}}P_{|\mathbf{l}'-\mathbf{l}|}^{\Delta\phi}\sin^2(2[\phi_{\mathbf{l}'}-\phi_{\mathbf{l}}])\\ &= 4\int \frac{d^2\mathbf{l}'}{(2\pi)^2}P_{l'}^{\mathcal{E}}P_{|\mathbf{l}'-\mathbf{l}|}^{\phi}(R,R)\sin^2(2[\phi_{\mathbf{l}'}-\phi_{\mathbf{l}}]),\end{aligned}} \tag{6.30}$$

[6] Here, of course, we refer to an expansion in Weyl potential.
[7] As usual, we assumed that the Weyl potential vanishes at the observer.

where, in the last line, we noticed that

$$P^{\Delta\phi}_{|\mathbf{l}'-\mathbf{l}|} \equiv \langle \Delta\phi(\mathbf{l}-\mathbf{l}')\Delta\phi(\mathbf{l}-\mathbf{l}')\rangle = \langle \phi(R,\mathbf{l}-\mathbf{l}')\phi(R,\mathbf{l}-\mathbf{l}')\rangle = P^{\phi}_{|\mathbf{l}-\mathbf{l}'|}(R,R)$$

(modulo the usual delta factor). This is the counterpart of the first term in (3.4). Actually, this correction comes from the first term in

$$\delta\mathcal{P} = \underbrace{-2\mathcal{P}\Delta\phi}_{\mathcal{B}\text{-modes}} + \mathcal{P}(\Delta\phi)^2, \tag{6.31}$$

because at first order the terms of the form $\langle \mathcal{E}\delta^2\mathcal{E}\rangle$ are not able to provide the factor $e^{4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}$ necessary to get $\mathcal{B}$-modes. Therefore, the only second-order correction comes from a $\langle \delta\mathcal{E}\delta\mathcal{E}\rangle$ term.
If we compare (6.30) with (6.21), we see that the main difference is the absence of a factor of the from

$$[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')]^2.$$

Therefore (6.21) is expected to be largely dominant with respect to (6.30).

6.3.3 Cross terms

In the previous subsections, we have seen that the corrections to $P^{\mathcal{B}}_l$ only come from $\langle \delta\mathcal{E}\delta\mathcal{E}\rangle$, and we have then found a generic formula for $\langle \delta\mathcal{E}_{\text{rem}}\delta\mathcal{E}_{\text{rem}}\rangle$ and $\langle \delta\mathcal{E}_{\text{prop}}\delta\mathcal{E}_{\text{prop}}\rangle$. Before to go to the higher order terms, we would like to finish the study by investigating the cross term $\langle \delta\mathcal{E}_{\text{rem}}\delta\mathcal{E}_{\text{prop}}\rangle$.

From the similarities between the formulas for the transport and the lensing, we immediately find

$$\boxed{\begin{aligned}\delta^{(2)}P^{\mathcal{B}}_l &= \int \frac{d^2\mathbf{l}'}{(2\pi)^2} P^{\mathcal{E}}_{l'}[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')]P^{\text{cross}}_{|\mathbf{l}'-\mathbf{l}|}\left[1-e^{4i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\right]\\ &= 2\int \frac{d^2\mathbf{l}'}{(2\pi)^2} P^{\mathcal{E}}_{l'}[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')]P^{\text{cross}}_{|\mathbf{l}'-\mathbf{l}|}\sin^2[2(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})]\\ &= -4\int_0^R dr\frac{R-r}{Rr}\int \frac{d^2\mathbf{l}'}{(2\pi)^2} P^{\mathcal{E}}_{l'}[\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')]P^{\phi}_{|\mathbf{l}'-\mathbf{l}|}(r,R)\sin^2[2(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})],\end{aligned}} \tag{6.32}$$

where we defined $P^{\text{cross}}_{|\mathbf{l}'-\mathbf{l}|} = \Delta\phi(\mathbf{l}'-\mathbf{l})\psi(\mathbf{l}'-\mathbf{l})$. We deduce an explicit expression for this power in terms of the power spectrum of the field ϕ in the Appendix, (F.6), giving the last equality. We can easily establish an expected hierarchy between those three terms by counting the $\mathbf{l}$ factors.

$$\langle \delta\mathcal{E}_{\text{rem}}\delta\mathcal{E}_{\text{rem}}\rangle \gg \langle \delta\mathcal{E}_{\text{rem}}\delta\mathcal{E}_{\text{prop}}\rangle \gg \langle \delta\mathcal{E}_{\text{prop}}\delta\mathcal{E}_{\text{prop}}\rangle. \tag{6.33}$$

The dominant term is $\langle \delta\mathcal{E}_{\text{rem}}\delta\mathcal{E}_{\text{rem}}\rangle$.

6.4 Higher order corrections to B-modes power spectrum

In the previous section, we have seen that the lowest order corrections of the $\mathcal{B}$-modes correlation function only come from $\langle \delta\mathcal{E}\delta\mathcal{E}\rangle$. The corrections coming from the terms of the form $\langle \mathcal{E}\delta^2\mathcal{E}\rangle$

cancel for the $\mathcal{B}$-modes. Of course we can not postulate that the same happens for the higher order corrections: therefore we will have to compute $\langle \delta^2\mathcal{E}\delta^2\mathcal{E}\rangle$, and $\langle \delta\mathcal{E}\delta^3\mathcal{E}\rangle$. The full correction is therefore given by $\langle \delta^2\mathcal{E}\delta^2\mathcal{E}\rangle + 2\langle \delta\mathcal{E}\delta^3\mathcal{E}\rangle$. Of course no correction comes from the part $\langle \mathcal{E}\delta^4\mathcal{E}\rangle$. The analysis is a bit more complicated and involves a lot of terms. To make it a bit more readable, we will use the following convention: we call $(2,2)$, $(1,3)$ the terms coming from $\langle \delta^2\mathcal{E}\delta^2\mathcal{E}\rangle$ and from $\langle \delta\mathcal{E}\delta^3\mathcal{E}\rangle$ respectively.

Before everything, we summarize all the corrections to $\mathcal{E}$ and $\mathcal{B}$ up to third order. For the remapping, the first-order correction is

$$\delta\mathcal{E}(\mathbf{l}) \pm i\delta\mathcal{B}(\mathbf{l}) = -\int \frac{d^2\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}'\cdot(\mathbf{l}-\mathbf{l}')\psi(\mathbf{l}-\mathbf{l}'), \tag{6.34}$$

the second-order correction is

$$\begin{aligned}\delta^{(2)}\mathcal{E}(\mathbf{l}) \pm i\delta^{(2)}\mathcal{B}(\mathbf{l}) = &-\int \frac{d^2\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}'\cdot\alpha_{\mathbf{2}}(\mathbf{l}-\mathbf{l}')\\ &-\frac{1}{2}\int\frac{d^2\mathbf{l}_1}{2\pi}\int\frac{d^2\mathbf{l}_2}{2\pi}e^{\pm 2i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})}\mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}]\mathbf{l}_1\cdot\mathbf{l}_2\mathcal{E}(\mathbf{l}_1)\psi(\mathbf{l}_2)\psi^\star(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}),\end{aligned} \tag{6.35}$$

and finally the third order correction, that can be computed from (2.109), is

$$\begin{aligned}\delta^{(3)}\mathcal{E}(\mathbf{l}) \pm i\delta^{(3)}\mathcal{B}(\mathbf{l}) = &-\int \frac{d^2\mathbf{l}'}{2\pi}\mathcal{E}(\mathbf{l}')e^{\pm 2i(\phi_{\mathbf{l}'}-\phi_{\mathbf{l}})}\mathbf{l}'\cdot\alpha_{\mathbf{3}}(\mathbf{l}-\mathbf{l}')\\ &-\int\frac{d^2\mathbf{l}_1}{2\pi}\int\frac{d^2\mathbf{l}_2}{2\pi}e^{\pm 2i(\phi_{\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2}-\phi_{\mathbf{l}})}\mathcal{E}(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\\ &\times\alpha_{\mathbf{1}}(\mathbf{l_1})\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\alpha_{\mathbf{2}}(\mathbf{l_2})\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\\ &-\frac{1}{6}\int\frac{d^2\mathbf{l}_1}{2\pi}\int\frac{d^2\mathbf{l}_2}{2\pi}\int\frac{d^2\mathbf{l_3}}{2\pi}e^{\pm 2i(\phi_{\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3}}-\phi_{\mathbf{l}})}\mathcal{E}(\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2-\mathbf{l}_3)\\ &\mathbf{l_1}\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3})\mathbf{l_2}\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3})\mathbf{l_3}\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3})\psi(\mathbf{l_2})\psi(\mathbf{l_3})\psi(\mathbf{l_1}).\end{aligned} \tag{6.36}$$

For the transport, we found the following corrections

$$\delta^{(2)}\mathcal{E}(\mathbf{l}) \pm i\delta^{(2)}\mathcal{B}(\mathbf{l}) = -\int\frac{d^2\mathbf{l}_1}{2\pi}\int\frac{d^2\mathbf{l}_2}{2\pi}e^{\pm 2i(\phi_{l-l_1-l_2}-\phi_l)}\mathcal{E}(\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2)\Delta\phi(\mathbf{l}_2)\Delta\phi(\mathbf{l}_1). \tag{6.37}$$

Counting the number of derivatives, or similarly, the number of $\mathbf{l}$ in the formula, we can see that the formula for the transport gives only sub-leading terms. As a consequence, we will not compute the $\mathcal{B}$-modes coming from them. In the formula for the remapping, we have used the expressions for the deflection angle at first and second order:

$$\alpha_2^a = -2\int_0^R \frac{(R-r)}{Rr}\tilde{\nabla}_{\perp b}\tilde{\nabla}_\perp^a\phi\alpha_1^b$$

which contains three derivatives and

$$\alpha_3^a = -2\int_0^R \frac{R-r}{Rr}\Big[\tilde{\nabla}_{\perp b}\tilde{\nabla}_\perp^a\phi(r,\theta,\phi)\alpha_2^b(r,\theta,\phi) + \frac{1}{2}\tilde{\nabla}_\perp^a\tilde{\nabla}_{\perp c}\tilde{\nabla}_{\perp b}\alpha_1^c(r,\theta,\phi)\alpha_1^b(r,\theta,\phi)dr\Big],$$

which contains five derivatives. We compute the Fourier transform and the power of these quantities in Appendix F.

To construct the corrected power spectrum, we proceed as before: for example, for the component $(2,2)$, we compute $\langle(\delta^{(2)}\mathcal{E}(\mathbf{l})+i\delta^{(2)}\mathcal{B}(\mathbf{l}))(\delta^{(2)}\mathcal{E}^\star(\mathbf{l})-i\delta^{(2)}\mathcal{B}^\star(\mathbf{l}))\rangle$ and $\langle(\delta^{(2)}\mathcal{E}(\mathbf{l})+i\delta^{(2)}\mathcal{B}(\mathbf{l}))(\delta^{(2)}\mathcal{E}^\star(\mathbf{l})+i\delta^{(2)}\mathcal{B}^\star(\mathbf{l}))\rangle$, which gives respectively $\delta^{(4)}P_l^{\mathcal{E}}+\delta^{(4)}P_l^{\mathcal{B}}$ and $\delta^{(4)}P_l^{\mathcal{E}}-\delta^{(4)}P_l^{\mathcal{B}}$. In so doing, we are confronted with quantities like

$$\langle abcd\rangle,$$

where the letters represent a power of the Weyl field $\phi(r,\mathbf{l})$. We can thus apply the Wick theorem to this correlation function because we do not expect dominant non-Gaussianities. We have therefore the usual factorisation. Going through computations, we obtain results proportional to

$$P^{\mathcal{E}}(P^{\phi})^2. \tag{6.38}$$

This is what we mean by *higher* order in Weyl power spectrum.

Let us say a last word about the organisation and the presentation of our terms: The expansion of $\mathcal{P}$ is given by

$$\mathcal{P}_{\text{GLC}}(\mathbf{x}) \to e^{2i\beta^{(2)}}\mathcal{P}(\mathbf{x}+\alpha_{\mathbf{1}}+\alpha_{\mathbf{2}}+\alpha_{\mathbf{3}}). \tag{6.39}$$

In section 6.4.1, we will study the corrections coming from

$$\mathcal{P}_{\text{GLC}}(\mathbf{x}) \to \mathcal{P}(\mathbf{x}+\alpha_{\mathbf{1}}+\alpha_{\mathbf{2}}+\alpha_{\mathbf{3}}),$$

and in section 6.4.2 we will study the correction coming from

$$\mathcal{P}_{\text{GLC}}(\mathbf{x}) \to e^{2i\beta^{(2)}}\mathcal{P}(\mathbf{x}).$$

We are not studying the correlation between the rotation and the remapping terms. Notice that the other possibility is to work in PG gauge, where we have

$$\mathcal{P}(x^a)_{\text{PG}} \to \mathcal{P}_{\text{PG}}(x^a+\alpha_1^a+\alpha_2^a+\alpha_3^a+(\Psi^a_b)^{(1)}_{\text{Anti}}\theta^b+(\Psi^a_b)^{(2)}_{\text{Anti}}\theta^b)). \tag{6.40}$$

Obviously, the correction studied in section 6.4.1 remains the same in this gauge. We will show in section 6.4.2 that the correction induced by the rotation is the same in both gauges.

6.4.1 Corrections from remapping

In this subsection, we compute the higher-order corrections coming from the remapping. The total expression for the correction to the power spectrum of the $\mathcal{B}$-modes is quite lengthy and complicated. Most of the necessary results are worked out in the Appendix F. To make the exposition clearer, we will split the $(2,2)$ part in three terms: $\mathcal{T}^{(2+2)}(2,2)$ which comes from the combination of two second-order deflection angles, $\mathcal{T}^{(2+11)}(2,2)$ which comes from the combination of two first-order deflection angles with one second-order deflection angle and finally $\mathcal{T}^{(11+11)}(2,2)$ which comes from the combination of two first-order deflection angles with two first-order deflection angles;

$$\mathcal{T}^{(2+2)}(2,2) = \frac{1}{2\pi^4}\int_0^R dr\frac{(R-r)}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^R dr_2\frac{(R-r_2)}{Rr_2}\int_0^{r_2} dr_3\frac{r_2-r_3}{r_2r_3}\int\int d^2\mathbf{l}_1 d^2\mathbf{l_2} P^{\mathcal{E}}_{l_1}$$
$$\times\left((1-e^{4i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})})([\mathbf{l}-\mathbf{l_2}-\mathbf{l}_1]\cdot\mathbf{l}_1)^2\right)\times\left(\Big[(\mathbf{l_2}\cdot\mathbf{l}_1)^2 P^{\phi}_{|\mathbf{l_1}|}(r,r_2)P^{\phi}_{|\mathbf{l}-\mathbf{l_2}-\mathbf{l_1}|}(r_1,r_3)\Big]\right.$$
$$\left.+\Big[\mathbf{l_2}\cdot\mathbf{l}_1\mathbf{l_2}\cdot(\mathbf{l}-\mathbf{l_2}-\mathbf{l}_1)P^{\phi}_{|\mathbf{l}_1|}(r,r_3)P^{\phi}_{|\mathbf{l}-\mathbf{l_2}-\mathbf{l}_1|}(r_1,r_2)\Big]\right) \tag{6.41}$$

$$\mathcal{T}^{(2+11)}(2,2) = \frac{1}{4\pi^4}\int_0^R dr\frac{R-r}{Rr}\int_0^R dr_1\frac{R-r_1}{Rr_1}\int_0^R dr_2\frac{(R-r_2)}{Rr_2}\int_0^{r_2} dr_3\frac{r_2-r_3}{r_2r_3}$$
$$\int\int d^2\mathbf{l}_1 d^2\mathbf{l}_2(1-e^{4i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})})$$
$$\times\left(\Big[\mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}](\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2)\cdot\mathbf{l}_2(\mathbf{l}_1\cdot\mathbf{l}_2)^2 P^{\mathcal{E}}_{l_1}P^{\phi}_{l_2}(r,r_2)P^{\phi}_{|\mathbf{l}_2+\mathbf{l}_1-\mathbf{l}|}(r_1,r_3)\Big]\right.$$
$$\left.+\Big[(\mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}])\mathbf{l}_1\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\mathbf{l}_1\cdot\mathbf{l}_2\mathbf{l}_2\cdot(\mathbf{l}-\mathbf{l}_2-\mathbf{l}_1)P^{\mathcal{E}}_{l_1}P^{\phi}_{l_2}(r,r_3)P^{\phi}_{|\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2|}(r_1,r_2)\Big]\right) \tag{6.42}$$

$$\mathcal{T}^{(11+11)}(2,2) = \frac{1}{8\pi^4}\int_0^R dr\frac{R-r}{Rr}\int_0^R dr_1\frac{R-r_1}{Rr_1}\int_0^R dr_2\frac{R-r_2}{Rr_2}\int_0^R dr_3\frac{R-r_3}{Rr_3}\int\int d^2\mathbf{l}_1 d^2\mathbf{l}_2$$
$$\times(1-e^{4i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})})\times\left(\Big[(\mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}])^2(\mathbf{l}_1\cdot\mathbf{l}_2)^2 P^{\mathcal{E}}_{l_1}P^{\phi}_{|\mathbf{l}_2|}(r,r_1)P^{\phi}_{|\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}|}(r_2,r_3)\Big]\right.$$
$$\left.+\Big[(\mathbf{l}_1\cdot[\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}])^2(\mathbf{l}_1\cdot\mathbf{l}_2)^2 P^{\mathcal{E}}_{l_1}P^{\phi}_{|\mathbf{l}_2|}(r,r_1)P^{\phi}_{|\mathbf{l_1}+\mathbf{l}_2-\mathbf{l}|}(r_2,r_3)\Big]\right). \tag{6.43}$$

For the classification of $(1,3)$, we use four terms. $\mathcal{T}_1^{(1+12)}(1,3)$ is the contribution given by the combination of one second-order deflection angle and one first-order deflection angle with one first-order deflection angle, $\mathcal{T}_1^{(1+3)}(1,3)$ and $\mathcal{T}_2^{(1+3)}(1,3)$ are the two contributions given by the combination of one third-order deflection angle with one first-order deflection angle. Finally, $\mathcal{T}^{(111+1)}(1,3)$ is the contribution given by the combination of three first-order deflection angles with one first-order deflection angle.

$$\begin{aligned}
\mathcal{T}_1^{(1+3)}(1,3) = \frac{1}{2\pi^4}\int_0^R dr\frac{R-r}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^{r_1} dr_2\frac{r_1-r_2}{r_1r_2}\int_0^R dr_3\frac{R-r_3}{Rr_3}\int d^2\mathbf{l_2} \\
\int d^2\mathbf{l}_1 P^{\mathcal{E}}_{l_1}[e^{4i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})}-1]\mathbf{l_1}\cdot(\mathbf{l}-\mathbf{l_1}) \\
\times\bigg(\Big[\mathbf{l_2}\cdot\mathbf{l_1}(\mathbf{l_2})^2\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}]P^{\phi}_{|\mathbf{l_2}|}(r,r_1)P^{\phi}_{|\mathbf{l}-\mathbf{l_1}|}(r_2,r_3)\Big] \\
-\mathbf{l_2}\cdot\mathbf{l_1}(\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}])^2P^{\phi}_{|\mathbf{l_2}|}(r,r_2)P^{\phi}_{|\mathbf{l}-\mathbf{l_1}|}(r_1,r_3)\Big] \\
-(\mathbf{l}-\mathbf{l_1})\cdot\mathbf{l_1}(\mathrm{l}_2\cdot[\mathbf{l}-\mathbf{l_1}])(\mathbf{l_2})^2P^{\phi}_{|\mathbf{l}-\mathbf{l_1}|}(r,r_3)P^{\phi}_{|\mathrm{l}_2|}(r_1,r_2)\Big]\bigg)
\end{aligned} \tag{6.44}$$

$$\begin{aligned}
\mathcal{T}_2^{(1+3)}(1,3) = -\frac{1}{4\pi^4}\int_0^R dr\frac{R-r}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^{r} dr_2\frac{r-r_2}{r_1r_2}\int_0^R dr_3\frac{R-r_3}{Rr_3}\int d^2\mathbf{l_2} \\
\int d^2\mathbf{l}_1 P^{\mathcal{E}}_{l_1}[e^{4i(\phi_{l_1}-\phi_{l})}-1]\mathbf{l_1}\cdot(\mathbf{l}-\mathbf{l_1}) \\
\times\bigg(\Big[\mathbf{l_2}\cdot\mathbf{l_1}(\mathbf{l_2})^2\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}]P^{\phi}_{|\mathbf{l_2}|}(r,r_1)P^{\phi}_{|\mathbf{l}-\mathbf{l_1}|}(r_2,r_3)\Big] \\
\mathbf{l_2}\cdot\mathbf{l_1}(\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}])(\mathbf{l_2})^2P^{\phi}_{|\mathbf{l_2}|}(r,r_2)P^{\phi}_{|\mathbf{l}-\mathbf{l_1}|}(r_1,r_3)\Big] \\
(\mathbf{l}-\mathbf{l_1})\cdot\mathbf{l_1}(\mathrm{l}_2\cdot[\mathbf{l}-\mathbf{l_1}])^2P^{\phi}_{|\mathbf{l}-\mathbf{l_1}|}(r,r_3)P^{\phi}_{|\mathrm{l}_2|}(r_1,r_2)\Big]\bigg).
\end{aligned} \tag{6.45}$$

$$\begin{aligned}
\mathcal{T}^{(111+1)}(1,3) = -\frac{1}{2\pi^4}\int_0^R dr\frac{R-r}{Rr}\int_0^R dr_1\frac{R-r_1}{Rr_1}\int_0^R dr_2\frac{R-r_2}{R_2}\int_0^R dr_3\frac{R-r_3}{Rr_3}\int d^2\mathbf{l}_1\int d^2\mathbf{l_2} \\
[e^{4i(\phi_{\mathbf{l}-\mathbf{l_1}}-\phi_l)}-1](\mathbf{l_2}\cdot(\mathbf{l}-\mathbf{l_2}))^2(\mathbf{l_1}\cdot(\mathbf{l}-\mathbf{l_1}))^2P^{\mathcal{E}}_{|\mathbf{l}-\mathrm{l}_1|} \\
\times\bigg(\Big[P^{\phi}_{\mathbf{l_1}}(r,r_1)P^{\phi}_{\mathbf{l_2}}(r_2,r_3)\Big]+\Big[P^{\phi}_{\mathbf{l_1}}(r,r_3)P^{\phi}_{\mathbf{l_2}}(r_1,r_2)\Big]+\Big[P^{\phi}_{\mathbf{l_1}}(r,r_2)P^{\phi}_{\mathbf{l_2}}(r_1,r_3)\Big]\bigg).
\end{aligned} \tag{6.46}$$

$$\begin{aligned}
\mathcal{T}^{(1+12)}(1,3) = \frac{1}{2\pi^4}\int_0^R dr\frac{R-r}{Rr}\int_0^R dr_1\frac{R-r_1}{Rr_1}\int_0^R dr_2\frac{R-r_2}{Rr_2}\int_0^{r_2} dr_3\frac{r_2-r_3}{r_2r_3}\int\int d^2\mathbf{l_1}d^2\mathbf{l_2} \\
\bigg(-\Big[(\mathbf{l_1}\cdot[\mathbf{l}-\mathbf{l_1}])^2(\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}])(\mathbf{l_2})^2P^{\phi}_{|\mathbf{l_1}|}(r,r_1)P^{\phi}_{|\mathbf{l_2}|}(r_2,r_3)P^{\mathcal{E}}_{|\mathbf{l}-\mathrm{l}_1|}\Big]\times[e^{4i(\phi_{\mathbf{l}-\mathbf{l_1}}-\phi_l)}-1] \\
+\Big[(\mathbf{l_1}\cdot[\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}])^2(\mathbf{l_1}+\mathbf{l_2})\cdot\mathbf{l_1}(\mathbf{l_1}+\mathbf{l_2})\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}) \\
P^{\phi}_{|\mathbf{l_2}+\mathbf{l_1}|}(r,r_2)P^{\phi}_{|\mathbf{l_1}|}(r_1,r_3)P^{\mathcal{E}}_{|\mathbf{l}-\mathrm{l}_1-\mathbf{l_2}|}\Big]\times[e^{4i(\phi_{\mathbf{l}-\mathrm{l}_1-\mathrm{l}_2}-\phi_l)}-1] \\
-\Big[(\mathbf{l_1}\cdot[\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}])(\mathbf{l_1}+\mathbf{l_2})\cdot\mathbf{l_1}((\mathbf{l_1}+\mathbf{l_2})\cdot(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}))^2 \\
P^{\phi}_{|\mathbf{l_1}+\mathbf{l_2}|}(r,r_3)P^{\phi}_{|\mathbf{l_1}|}(r_1,r_2)P^{\mathcal{E}}_{|\mathbf{l}-\mathrm{l}_1-\mathbf{l_2}|}\Big]\times[e^{4i(\phi_{\mathbf{l}-\mathrm{l}_1-\mathbf{l_2}}-\phi_l)}-1]\bigg)
\end{aligned} \tag{6.47}$$

(Notice that, for homogeneity, we have everywhere expressed the P^ψ in terms of P^ϕ). In this formula, all terms contain 8 factors $\mathbf{l}$, therefore they are roughly of the same order of magnitude. The total expression is given by

$$\delta^{(4)} P_l^{\mathcal{B}}\Big|_{\text{rem}} = \mathcal{T}_1(2,2)+\mathcal{T}_2(2,2)+\mathcal{T}_3(2,2)+2\mathcal{T}_1^{(112)}(1,3)+2\mathcal{T}_2^{(111)}(1,3)+2\mathcal{T}_2^{(1111)}(1,3)+2\mathcal{T}_2^{(121)}(1,3). \tag{6.48}$$

We could also mention again at this step that the term $\mathcal{T}(0,4)$ vanishes.

6.4.2 Correction from rotation

Up to fourth order in the Weyl field, we also have to compute the corrections coming from the *rotation*, because as we have seen, this provides a dominant contribution. Even if we know that the computations must give the same results in Poisson and in GLC gauge, we go through each computations and compare them.

GLC gauge computation

We first do the computation in the GLC gauge. In this gauge, the corrected polarisation quantity $\mathcal{P} = e_+^a e_+^b P_{ab}$ is rotated by and angle -2β (because this is a spin-2 quantity and under a rotation of angle θ, it rotates by an angle of 2θ). Therefore, the expansion of the polarisation up to second order is

$$\mathcal{P}(\mathbf{n}) \to e^{-i2\beta^{(2)}}\mathcal{P}(\mathbf{n}+\alpha_1+\alpha_2). \tag{6.49}$$

Expanding the exponential and keeping only the second-order quantities, we obtain

$$\delta\mathcal{P}(\mathbf{n})\Big|_{\text{second order}} = -2i\beta^{(2)}\mathcal{P}(\mathbf{n}) + \text{remapping at second order.} \tag{6.50}$$

At fourth order, we have also the contribution

$$\begin{aligned}\delta\mathcal{P}(\mathbf{n})\Big|_{\text{fourth order}} &= -4\frac{1}{2}(\beta^{(2)})^2\mathcal{P}(\mathbf{n}) + \text{remapping at fourth order.}\\ &= -2(\beta^{(2)})^2\mathcal{P}(\mathbf{n}) + \text{remapping at fourth order.}\end{aligned} \tag{6.51}$$

(The $\frac{1}{2}$ in the previous result comes from the expansion of the exponential). [46] argued that $\beta^{(3)}$ and $\beta^{(4)}$ do not contribute. We have therefore two sources for the correction coming from the rotation:

- From the correlation function of the form

$$-4\left\langle\beta^{(2)}\mathcal{P}(\mathbf{n})\beta^{(2')}\mathcal{P}(\mathbf{n}')\right\rangle \tag{6.52}$$

- and from the correlation function of the form

$$4\left\langle(\beta^{(2)})^2\mathcal{P}(\mathbf{n})\mathcal{P}(\mathbf{n}')\right\rangle. \tag{6.53}$$

As we did many times in the previous sections, we use the formulas for the $\mathcal{E}$- and $\mathcal{B}$-modes in term of correction to $\mathcal{P}$, (6.9) and (6.10), to write

$$\begin{aligned}\Big(\delta\mathcal{E}(\mathbf{l})+i\delta\mathcal{B}(\mathbf{l})\Big)_{\text{up to fourth order}} &= \mathcal{E}(\mathbf{l})+2i\int\frac{d^2\mathbf{x}}{2\pi}\beta^{(2)}(\mathbf{x})\mathcal{P}(\mathbf{x})e^{-2i\phi_l}e^{-i\mathbf{l}\cdot\mathbf{x}}\\ &+2\int\frac{d^2\mathbf{x}}{2\pi}(\beta^{(2)}(\mathbf{x}))^2\mathcal{P}(\mathbf{x})e^{-2i\phi_l}e^{-i\mathbf{l}\cdot\mathbf{x}},\end{aligned}\tag{6.54}$$

and

$$\begin{aligned}\Big(\delta\mathcal{E}^\star(\mathbf{l})-i\delta\mathcal{B}^\star(\mathbf{l})\Big)_{\text{up to fourth order}} &= \mathcal{E}(\mathbf{l})-2i\int\frac{d^2\mathbf{x}}{2\pi}\beta^{(2)\star}(\mathbf{x})\mathcal{P}^\star(\mathbf{x})e^{2i\phi_l}e^{-i\mathbf{l}\cdot\mathbf{x}}\\ &+2\int\frac{d^2\mathbf{x}}{2\pi}(\beta^{(2)}(\mathbf{x}))^2\mathcal{P}^\star(\mathbf{x})e^{2i\phi_l}e^{-i\mathbf{l}\cdot\mathbf{x}}.\end{aligned}\tag{6.55}$$

Introducing $\mathcal{P}(\mathbf{x})=-\int\frac{d^2\mathbf{l}}{2\pi}[\mathcal{E}(\mathbf{l})]e^{2i\phi_l}e^{i\mathbf{l}\cdot\mathbf{x}}$, we obtain

$$\begin{aligned}\Big(\delta\mathcal{E}(\mathbf{l})+i\delta\mathcal{B}(\mathbf{l})\Big)_{\text{up to fourth order}} &= \mathcal{E}(\mathbf{l})-2i\int\frac{d^2\mathbf{l_1}}{2\pi}\mathcal{E}(\mathbf{l}_1)e^{2i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})}\beta^{(2)}(\mathbf{l_1}-\mathbf{l})\\ &-2\int\int\frac{d^2\mathbf{l_1}}{2\pi}\frac{d^2\mathbf{l_2}}{2\pi}\beta^{(2)}(\mathbf{l_2})\beta^{(2)}(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\mathcal{E}(\mathbf{l_1})e^{2i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})},\end{aligned}\tag{6.56}$$

and

$$\begin{aligned}\Big(\delta\mathcal{E}^\star(\mathbf{l})-i\delta\mathcal{B}^\star(\mathbf{l})\Big)_{\text{up to fourth order}} &= \mathcal{E}(\mathbf{l})+2i\int\frac{d^2\mathbf{l_1}}{2\pi}\mathcal{E}^\star(\mathbf{l}_1)e^{2i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})}\beta^{(2)\star}(\mathbf{l_1}-\mathbf{l})\\ &-2\int\int\frac{d^2\mathbf{l_1}}{2\pi}\frac{d^2\mathbf{l_2}}{2\pi}\beta^{(2)}(\mathbf{l_2})\beta^{(2)}(\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\mathcal{E}(\mathbf{l_1})e^{2i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})}.\end{aligned}\tag{6.57}$$

From these expressions, we see immediately that there is no $\mathcal{B}$-modes produced at second order in the Weyl field (as usual the factor that produces $\mathcal{B}$-modes, $e^{2i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})}$, vanishes). We have to go to fourth order. The result at fourth order is

$$\begin{aligned}\delta^{(4)}P_l^{\mathcal{B}}\Big|_{\text{prop}} &= 2\int\frac{d^2\mathbf{l}_1}{(2\pi)^2}\Big[1+e^{4i(\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}})}\Big]P_{l_1}^{\mathcal{E}}\Big\langle(\beta^{(2)}(\mathbf{l_1}-\mathbf{l}))^2\Big\rangle.\\ &= 4\int\frac{d^2\mathbf{l}_1}{(2\pi)^2}\cos^2(2[\phi_{\mathbf{l}_1}-\phi_{\mathbf{l}}])P_{l_1}^{\mathcal{E}}\Big\langle(\beta^{(2)}(\mathbf{l_1}-\mathbf{l}))^2\Big\rangle.\end{aligned}\tag{6.58}$$

[8] Our next step is to compute the correlation function $\langle(\beta^{(2)}(\mathbf{l_1}-\mathbf{l}))^2\rangle$. (Note that there is also a contribution coming from the correlation between to angle $\beta^{(2)}$ and the remapping at second order).

Taking back our formula for this angle (3.84):

$$\beta^{(2)}(R)=2\epsilon^{ab}\int_0^R dr\frac{R-r}{Rr}\tilde{\nabla}_b\tilde{\nabla}_c\phi(r)\int_0^r dr_1\frac{r-r_1}{rr_1}\tilde{\nabla}_a\tilde{\nabla}^c\phi(r_1),$$

and introducing the Fourier transform of the potential,

$$\phi(r,\mathbf{l})=\int\frac{d^2\mathbf{x}}{2\pi}e^{i\mathbf{l}\cdot\mathbf{x}}\phi(r,\mathbf{x}),\qquad \phi(r,\mathbf{x})=\int\frac{d^2\mathbf{l}}{2\pi}e^{-i\mathbf{l}\cdot\mathbf{x}}\phi(r,\mathbf{l}),$$

[8] Notice that the plus sign between the 1 and the exponential, in contrast to previous results. This gives a cosine instead of a sine in the next equality.

we obtain

$$\beta(R,\mathbf{l}) = 2\epsilon^{ab}\int\frac{d^2\mathbf{x}}{2\pi}e^{i\mathbf{l}\cdot\mathbf{x}}\int_0^R dr\frac{R-r}{Rr}\int\frac{d^2\mathbf{l}_1}{2\pi}l_{1c}l_{1b}e^{-i\mathbf{l}_1\cdot\mathbf{x}}\phi(r,\mathbf{l}_1)\int_0^r dr_1\frac{r-r_1}{rr_1}\int\frac{d^2\mathbf{l}_2}{2\pi}l_2^c l_{2a}e^{-i\mathbf{l}_2\cdot\mathbf{x}}\phi(r_1,\mathbf{l}_2)$$
$$= \frac{1}{\pi}\int_0^R dr\frac{R-r}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int d^2\mathbf{l}_1\Big[\epsilon^{ab}(l^c-l_1^c)(l_a-l_{1a})l_{1c}l_{1b}\Big]\phi(r_1,\mathbf{l}-\mathbf{l}_1)\phi(r,\mathbf{l}_1). \tag{6.59}$$

The term between brackets $[\epsilon^{ab}(l^c-l_1^c)(l_a-l_{1a})l_{1c}l_{1b}]$ can be rewritten $\mathbf{n}\cdot(\mathbf{l}_1\times\mathbf{l})(\mathbf{l}_1\cdot[\mathbf{l}-\mathbf{l}_1])$. This gives our final form for the Fourier transform of the rotation angle. From it, we can now compute the correlation function $\langle\beta^{(2)}\beta^{(2)}\rangle$:

$$\langle\beta^{(2)}(\mathbf{l})\beta^{(2)\star}(\mathbf{l}')\rangle = \delta_D(\mathbf{l}-\mathbf{l}')\frac{1}{\pi^2}\int_0^R dr\frac{R-r}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^R dr_2\frac{R-r_2}{Rr_2}\int_0^{r_2} dr_3\frac{r_2-r_3}{r_2r_3}\int\int d^2\mathbf{l}_1 d^2\mathbf{l}_2$$
$$\times\,\mathbf{n}\cdot(\mathbf{l}_1\times\mathbf{l})(\mathbf{l}_1\cdot[\mathbf{l}-\mathbf{l}_1])(\mathbf{n}\cdot(\mathbf{l}'_2\times\mathbf{l}')(\mathbf{l}_2\cdot[\mathbf{l}'-\mathbf{l}_2])$$
$$\times\Big\langle\phi(r,\mathbf{l}-\mathbf{l}_1)\phi(r_1,\mathbf{l}_1)\phi^\star(r_2,\mathbf{l}'-\mathbf{l}_2)\phi^\star(r_3,\mathbf{l}_2)\Big\rangle. \tag{6.60}$$

This expression again contains 8 factors $\mathbf{l}$ and is therefore of the same order of magnitude than the correction we found for the remapping just in the previous section. The quantity $\langle\phi(r_1,\mathbf{l}-\mathbf{l}_1)\phi(r_1,\mathbf{l}_1)\phi^\star(r_2,\mathbf{l}'-\mathbf{l}_2)\phi^\star(r_3,\mathbf{l}_2)\rangle$ can be expanded using the Wick theorem:

$$\langle\phi(r_1,\mathbf{l}-\mathbf{l}_1)\phi(r_1,\mathbf{l}_1)\phi^\star(r_2,\mathbf{l}'-\mathbf{l}_2)\phi^\star(r_3,\mathbf{l}_2)\rangle = \delta_D(\mathbf{l})\delta_D(\mathbf{l}')P^\phi_{l_1}(r,r_1)P^\phi_{l_2}(r_2,r_3)$$
$$+\delta_D(\mathbf{l}-\mathbf{l}'-\mathbf{l}-\mathbf{l}_2)\delta_D(\mathbf{l}_1-\mathbf{l}_2)P^\phi_{l-l_1}(r,r_2)P^\phi_{l_1}(r_1,r_3)+\delta_D(\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2)\delta_D(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l}')P^\phi_{l_2}(r,r_3)P^\phi_{l_1}(r_2,r_1). \tag{6.61}$$

The first term on the right hand side of (6.61) does not give any contribution. Introducing this in (6.60), we finally find the power spectrum we were searching for:

$$P_l^{\beta^{(2)}} = \frac{1}{\pi^2}\int_0^R dr\frac{R-r}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^R dr\frac{R-r_2}{Rr_2}\int_0^{r_2} dr_3\frac{r_2-r_3}{r_2r_3}\int d^2\mathbf{l}_1$$
$$\times\left(\Big[\mathbf{n}\cdot(\mathbf{l}_1\times\mathbf{l})(\mathbf{l}_1\cdot[\mathbf{l}-\mathbf{l}_1])\Big]^2\times\Big[P^\phi_{l-l_1}(r,r_2)P^\phi_{l_1}(r_1,r_3)-P^\phi_{l-l_1}(r,r_3)P^\phi_{l_1}(r_2,r_1)\Big]\right). \tag{6.62}$$

Now, we have just to insert this form in (6.58) to deduce the correction to the $\mathcal{B}$-modes power spectrum produced by the rotation angle $\beta^{(2)}$.

$$
\begin{aligned}
\delta^{(4)} P_l^{\mathcal{B}}\Big|_{\text{prop}} &= \frac{1}{2\pi^4}\int_0^R dr \frac{R-r}{Rr}\int_0^r dr_1 \frac{r-r_1}{rr_1}\int_0^R dr \frac{R-r_2}{Rr_2}\int_0^{r_2} dr_3 \frac{r_2-r_3}{r_2 r_3}\int d^2\mathbf{l}_1 \int d^2\mathbf{l_2} \\
&\left(\cos^2(2[\phi_{l_1}-\phi_l])P^{\mathcal{E}}_{l_1}\Big[\mathbf{n}\cdot(\mathbf{l}_2\times[\mathbf{l}-\mathbf{l_1}])(\mathbf{l}_2\cdot[\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2])\Big]^2\right.\\
&\left.\times\Big[P^\phi_{|\mathbf{l}-\mathbf{l_1}-\mathbf{l}_2|}(r,r_2)P^\phi_{|\mathbf{l}_2|}(r_1,r_3)-P^\phi_{|\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2|}(r,r_3)P^\phi_{|\mathbf{l}_2|}(r_2,r_1)\Big]\right)\\
&=\frac{1}{2\pi^4}\int_0^R dr \frac{R-r}{Rr}\int_0^r dr_1 \frac{r-r_1}{rr_1}\int_0^R dr \frac{R-r_2}{Rr_2}\int_0^{r_2} dr_3 \frac{r_2-r_3}{r_2 r_3}\int d^2\mathbf{l}_1 \int d^2\mathbf{l_2} \\
&\left(\cos^2(2[\phi_{|\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2|}-\phi_l])P^{\mathcal{E}}_{|\mathbf{l}-\mathbf{l}_1-\mathbf{l}_2|}\Big[\mathbf{n}\cdot(\mathbf{l}_2\times\mathbf{l_1})(\mathbf{l}_2\cdot\mathbf{l}_1)\Big]^2\right.\\
&\left.\times\Big[P^\phi_{|\mathbf{l_1}|}(r,r_2)P^\phi_{|\mathbf{l}_2|}(r_1,r_3)-P^\phi_{|\mathbf{l}_1|}(r,r_3)P^\phi_{|\mathbf{l}_2|}(r_2,r_1)\Big]\right),
\end{aligned}
\tag{6.63}
$$

where in the second equality, we have made the change of variables $\mathbf{l}_1 \to \mathbf{l}-\mathbf{l}_1-\mathbf{l}_2$ to recover the result in [46] based on a GLC computation. Notice that their results include the case of a non-vanishing primordial $\mathcal{B}$-modes. If we set $P_l^{(0)\mathcal{B}} = 0$ in their results, we recover (6.63).

Poisson gauge computation

In the Poisson gauge, the corrected polarisation is

$$\mathcal{P}(x^a + \omega^{(2)}\epsilon^a_b x^b), \tag{6.64}$$

if we just consider the rotation of the line of sight induced by the magnification matrix. Therefore

$$
\begin{aligned}
\mathcal{B}(\mathbf{l}) &= -\int \frac{d^2\mathbf{x}}{2\pi}\mathcal{I}\text{m}\Big[\mathcal{P}(x^a+\omega^{(2)}\epsilon^a_b\theta^b)e^{-2i\phi_\mathbf{l}}\Big]e^{-i\mathbf{l}\cdot\mathbf{x}} \\
&= -\int \frac{d^2\mathbf{x}}{2\pi}\mathcal{I}\text{m}\Big[\mathcal{P}(x^a)e^{-2i\phi_\mathbf{l}}\Big]e^{-i\mathbf{l}\cdot\mathbf{x}+il_a x^{(2)}\epsilon^a_b x^b}.
\end{aligned}
\tag{6.65}
$$

The best way to proceed, if we do not want to meet difficulties in the computation, is to rotate the integration variable $\mathbf{x}$. As in Chapter 4, we obtain

$$\mathcal{B}(\mathbf{l}') = -\int \frac{d^2\mathbf{x}}{2\pi}\mathcal{I}\text{m}\Big[\mathcal{P}(x^a)e^{-2i\phi_l}e^{-2i\omega^{(2)}}\Big]e^{-i\mathbf{l}\cdot\mathbf{x}} \tag{6.66}$$

Expanding and keeping only second order terms, the correction to the $\mathcal{B}$-modes is

$$\delta^{(2)}\mathcal{B}(\mathbf{l}') = 2\int \frac{d^2\mathbf{x}}{2\pi}\mathcal{R}\text{e}\Big[\omega^{(2)}\mathcal{P}(x^a)e^{-2i\phi_l}\Big]e^{-i\mathbf{l}\cdot\mathbf{x}} \tag{6.67}$$

Computing the $\mathcal{B}$-modes power spectrum and owing to the equality between $\beta^{(2)}$ and $\omega^{(2)}$, we are thus led to (6.58) and eventually to (6.63).

More than probably, the slight difference between this computation and the result in [47] is due to a difference in the approximation made or in the method of computation.

Conclusions

THE main purpose of this work was to reconcile the corrections to the $\mathcal{B}$-modes obtained via the Poisson gauge and via the Geodesic Light-Cone gauge. At first sight, the conflict can be summarized as follows: computing the corrections to the power spectra in the both gauges gives two results that are difficult to match directly. Therefore we had to turn to a more fundamental demonstration of the equivalence of the two methods.

Chapter 2 and 3 were dedicated to a technical computation of all the necessary results at first and second order. In Chapter 2, we computed the formulas for the lensing, discussing the boundary conditions and the change of variables. We have namely concluded that the "fixed observer direction" boundary condition was the most physical choice and that a change of variable (r, λ, t) had no impact on the formulas, except for a multiplicative factor. Also, we have seen that "time delay" effect were negligible. In Chapter 3, from the laws of electromagnetism in vacuum, we determined the transport followed by the polarisation along the path. We solved this equation up to second order and insisted on the fact that only the equation of motion was controlling a possible rotation term. After the fixation of boundary conditions and the check of constraints, we went to the comparison of results in the GLC and in the PG gauge. We then identified the provenance of the second order rotation in GLC gauge and emphasized the difference with the Poisson gauge calculation.

Part of the contradiction was revealed when we understood in Chapter 4 that the rotation of a "structure" in the CMB temperature, represented by the Lie transport in Poisson gauge, has the same mathematical form than the rotation of the polarisation computed in the GLC gauge. This identification suggested the following interpretation: The polarisation, expressed in a coordinate system defined with respect to some pattern in the CMB temperature is indeed the *gauge-invariant quantity*. In other words, in the Poisson gauge, the coordinate system rotates while in the GLC gauge, this is the polarisation that rotates.

This immediately raised a question. Our explanation seemed to refer strongly on *a coordinate system defined using the CMB temperature*. What if we use a Galactic coordinate system, defined using the plane of the galaxy ? In this case, there is no rotation of the coordinate system induced by the Lie transport, and therefore, the curl induced by the rotation seems to disappear. To shed some light on this issue, we turned, in section 4.5 to the real observable, namely the $\mathcal{B}$-modes power spectrum. In this section, using only the definition of the fundamental quantities in the game, we have shown that the mathematical form of the $\mathcal{B}$-modes power spectrum *is* indeed the same, no matter the gauge. With this satisfactory result behind us, we were in position to work out the curl and the $\mathcal{B}$-modes power spectrum, choosing the gauge we prefer.

In the two following Chapters, we turned to a more systematic enumeration of all the possible sources of $\mathcal{B}$-modes. In Chapter 5, we used a local method to isolate curls and summarized all our results in section 5.4. There was still a slight uncertainty in the interpretation to give to term proportional to $P_{\|}$.

In Chapter 6 we computed the power spectrum of the corrections, specializing in the search for $\mathcal{B}$-modes. Our main purpose was to compare our results for the power spectrum of $\mathcal{B}$-modes with other ones found in the literature, keeping emphasis on the part coming from rotation.

Bibliography

[1] Starobinsky AA. Spectrum of relict gravitational radiation and the early state of the universe. *JETP Lett.*, 78, 1979.

[2] K. N. Abazajian et al. Inflation Physics from the Cosmic Microwave Background and Large Scale Structure. *Astropart. Phys.*, 63:55–65, 2015.

[3] P. A. R. Ade et al. Detection of B-Mode Polarization at Degree Angular Scales by BICEP2. *Phys. Rev. Lett.*, 112(24):241101, 2014.

[4] P. A. R. Ade et al. Evidence for Gravitational Lensing of the Cosmic Microwave Background Polarization from Cross-correlation with the Cosmic Infrared Background. *Phys. Rev. Lett.*, 112:131302, 2014.

[5] P. A. R. Ade et al. Measurement of the Cosmic Microwave Background Polarization Lensing Power Spectrum with the POLARBEAR experiment. *Phys. Rev. Lett.*, 113:021301, 2014.

[6] P. A. R. Ade et al. Planck 2013 results. XVI. Cosmological parameters. *Astron. Astrophys.*, 571:A16, 2014.

[7] P. A. R. Ade et al. Joint Analysis of BICEP2, Keck Array and Planck Data. *Phys. Rev. Lett.*, 114:101301, 2015.

[8] Victor S. Alpher. Ralph A. Alpher, George Antonovich Gamow, and the Prediction of the Cosmic Microwave Background Radiation. 2:17–26, 2014.

[9] Matthias Bartelmann and Peter Schneider. Weak gravitational lensing. *Phys. Rept.*, 340:291–472, 2001.

[10] I. Ben-Dayan, M. Gasperini, G. Marozzi, F. Nugier, and G. Veneziano. Backreaction on the luminosity-redshift relation from gauge invariant light-cone averaging. *JCAP*, 1204:036, 2012.

[11] C. L. Bennett, A. Banday, K. M. Gorski, G. Hinshaw, P. Jackson, P. Keegstra, A. Kogut, George F. Smoot, D. T. Wilkinson, and E. L. Wright. Four year COBE DMR cosmic microwave background observations: Maps and basic results. *Astrophys. J.*, 464:L1–L4, 1996.

[12] Camille Bonvin, Chris Clarkson, Ruth Durrer, Roy Maartens, and Obinna Umeh. Do we care about the distance to the CMB? Clarifying the impact of second-order lensing. *JCAP*, 1506(06):050, 2015.

[13] Paolo Cabella and Marc Kamionkowski. Theory of cosmic microwave background polarization. In *International School of Gravitation and Cosmology: The Polarization of the Cosmic Microwave Background Rome, Italy, September 6-11, 2003*, 2004.

[14] Z Cai and Y Zhang. Analytic spectra of cmb anisotropies and polarization generated by scalar perturbations in synchronous gauge. *Classical and Quantum Gravity*, 29(10):105009, 2012.

[15] Erminia Calabrese et al. Cosmological parameters from pre-planck cosmic microwave background measurements. *Phys. Rev.*, D87(10):103012, 2013.

[16] R. R. Caldwell, Marc Kamionkowski, and Leven Wadley. The First space based gravitational wave detectors. *Phys. Rev.*, D59:027101, 1999.

[17] Asantha Cooray and Wayne Hu. Second order corrections to weak lensing by large scale structure. *Astrophys. J.*, 574:19, 2002.

[18] Asantha Cooray, Marc Kamionkowski, and Robert R. Caldwell. Cosmic shear of the microwave background: The Curl diagnostic. *Phys. Rev.*, D71:123527, 2005.

[19] Ruth Durrer. The theory of CMB anisotropies. *J. Phys. Stud.*, 5:177–215, 2001.

[20] Ruth Durrer. *CMB polarization and the total angular momentum approach*, page 176–209. Cambridge University Press, 2008.

[21] R. Fabbri and M. d. Pollock. The Effect of Primordially Produced Gravitons upon the Anisotropy of the Cosmological Microwave Background Radiation. *Phys. Lett.*, 125B:445–448, 1983.

[22] G. Fanizza, M. Gasperini, G. Marozzi, and G. Veneziano. An exact Jacobi map in the geodesic light-cone gauge. *JCAP*, 1311:019, 2013.

[23] G. Fanizza, M. Gasperini, G. Marozzi, and G. Veneziano. A new approach to the propagation of light-like signals in perturbed cosmological backgrounds. *JCAP*, 1508(08):020, 2015.

[24] Pierre Fleury. *Light propagation in inhomogeneous and anisotropic cosmologies*. PhD thesis, Paris, Inst. Astrophys., 2015.

[25] Pierre Fleury, Fabien Nugier, and Giuseppe Fanizza. Geodesic-light-cone coordinates and the Bianchi I spacetime. *JCAP*, 1606(06):008, 2016.

[26] M. Gasperini, G. Marozzi, F. Nugier, and G. Veneziano. Light-cone averaging in cosmology: Formalism and applications. *JCAP*, 1107:008, 2011.

[27] Gabriel German, Graham G. Ross, and Subir Sarkar. Low scale inflation. *Nucl. Phys.*, B608:423–450, 2001.

[28] S. W. Hawking. Particle creation by black holes. *Comm. Math. Phys.*, 43(3):199–220, 1975.

[29] S.W. Hawking. The development of irregularities in a single bubble inflationary universe. *Physics Letters B*, 115(4):295 – 297, 1982.

[30] Christopher M. Hirata and Uros Seljak. Reconstruction of lensing from the cosmic microwave background polarization. *Phys. Rev.*, D68:083002, 2003.

[31] Wayne Hu and Asantha Cooray. Gravitational time delay effects on cosmic microwave background anisotropies. *Phys. Rev.*, D63:023504, 2001.

[32] Wayne Hu and Scott Dodelson. Cosmic microwave background anisotropies. *Ann. Rev. Astron. Astrophys.*, 40:171–216, 2002.

[33] Wayne Hu and Martin J. White. Acoustic signatures in the cosmic microwave background. *Astrophys. J.*, 471:30–51, 1996.

[34] Wayne Hu and Martin J. White. A CMB polarization primer. *New Astron.*, 2:323, 1997.

[35] Wayne Hu and Martin J. White. CMB anisotropies: Total angular momentum method. *Phys. Rev.*, D56:596–615, 1997.

[36] John David Jackson. *Classical electrodynamics*. Wiley, New York, NY, 3rd ed. edition, 1999.

[37] Marc Kamionkowski, Arthur Kosowsky, and Albert Stebbins. A probe of primordial gravity waves and vorticity. *Phys. Rev. Lett.*, 78:2058–2061, Mar 1997.

[38] Marc Kamionkowski, Arthur Kosowsky, and Albert Stebbins. Statistics of cosmic microwave background polarization. *Phys. Rev.*, D55:7368–7388, 1997.

[39] Marc Kamionkowski and Ely D. Kovetz. The Quest for B Modes from Inflationary Gravitational Waves. *Ann. Rev. Astron. Astrophys.*, 54:227–269, 2016.

[40] Lloyd Knox and Michael S. Turner. Inflation at the electroweak scale. *Phys. Rev. Lett.*, 70:371–374, 1993.

[41] Antony Lewis and Anthony Challinor. Weak gravitational lensing of the cmb. *Phys. Rept.*, 429:1–65, 2006.

[42] Antony Lewis, Alex Hall, and Anthony Challinor. Emission-angle and polarization-rotation effects in the lensed CMB. *JCAP*, 1708(08):023, 2017.

[43] David H. Lyth. What would we learn by detecting a gravitational wave signal in the cosmic microwave background anisotropy? *Phys. Rev. Lett.*, 78:1861–1863, 1997.

[44] David H. Lyth and Antonio Riotto. Particle physics models of inflation and the cosmological density perturbation. *Phys. Rept.*, 314:1–146, 1999.

[45] Giovanni Marozzi, Giuseppe Fanizza, Enea Di Dio, and Ruth Durrer. CMB-lensing beyond the Born approximation. *JCAP*, 1609(09):028, 2016.

[46] Giovanni Marozzi, Giuseppe Fanizza, Enea Di Dio, and Ruth Durrer. CMB-lensing beyond the leading order: temperature and polarization anisotropies. 2016.

[47] Geraint Pratten and Antony Lewis. Impact of post-Born lensing on the CMB. *JCAP*, 1608(08):047, 2016.

[48] Peter Schneider. Weak gravitational lensing. In *Proceedings, 33rd Advanced Saas Fee Course on Gravitational Lensing: Strong, Weak, and Micro: Les Diablerets, Switzerland, April 7-12, 2003*, pages 269–451, 2006.

[49] Peter Schneider, Ludovic Van Waerbeke, Bhuvnesh Jain, and Guido Kruse. A new measure for cosmic shear. *Monthly Notices of the Royal Astronomical Society*, 296(4):873–892, 1998.

[50] Uros Seljak. Gravitational lensing effect on cosmic microwave background anisotropies: A Power spectrum approach. *Astrophys. J.*, 463:1, 1996.

[51] Uros Seljak and Matias Zaldarriaga. Signature of gravity waves in polarization of the microwave background. *Phys. Rev. Lett.*, 78:2054–2057, 1997.

[52] Charles Shapiro and Asantha Cooray. The born and lens–lens corrections to weak gravitational lensing angular power spectra. *Journal of Cosmology and Astroparticle Physics*, 2006(03):007, 2006.

[53] D. N. Spergel et al. First year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Determination of cosmological parameters. *Astrophys. J. Suppl.*, 148:175–194, 2003.

[54] Obinna Umeh, Chris Clarkson, and Roy Maartens. Nonlinear relativistic corrections to cosmological distances, redshift and gravitational lensing magnification. II - Derivation. *Class. Quant. Grav.*, 31:205001, 2014.

[55] H.C. van de HULST. Light scattering by small particles. 10, 01 1981.

[56] Matias Zaldarriaga and Uro š Seljak. Gravitational lensing effect on cosmic microwave background polarization. *Phys. Rev. D*, 58:023003, Jun 1998.

[57] Matias Zaldarriaga and Uros Seljak. Gravitational lensing effect on cosmic microwave background polarization. *Phys. Rev.*, D58:023003, 1998.

APPENDIX

A Post-Born terms

In chapter 2, we addressed the post-Born term. In the section 2.3.1, we approached the problem by expanding, up to second order, the equation

$$k^\nu \nabla_\nu k^\mu = 0.$$

We subsequently identified the post-Born terms with $-\delta k^\nu \partial_\nu \delta k^\mu$. We obtain the post-Born correction by simple integration of this expression, namely

$$-\int_0^\lambda \delta k^\nu(\lambda')\partial_\nu \delta k^\mu(\lambda') d\lambda'. \tag{A.1}$$

Actually, there is another way to obtain those terms. Let us first come back to the first-order term, which is

$$\delta k^\mu(\lambda) = -\int_0^\lambda \Gamma^\mu_{\alpha\beta}(x^\nu(\lambda'))k^\alpha k^\beta d\lambda',$$

where the integration is to be performed along the unperturbed path. Now, computing the corrections at second order, there is a contribution coming from this expression, where the integration is performed along the first-order perturbed path. To handle this kind of correction we have to rewrite the Γ's at the corrected position and expand in series : $\Gamma(y^\nu(\lambda)) = \Gamma(x^\nu(\lambda) + \delta x^\nu(\lambda)) = \Gamma(x^\nu(\lambda)) + \partial_\rho \Gamma(x^\nu(\lambda))\delta x^\rho(\lambda)$, where the equalities are valid at first order. As a consequence, the post-Born correction in this framework is given by

$$-\int_0^\lambda \partial_\rho \Gamma^\mu_{\alpha\beta}(x^\nu(\lambda'))\delta x^\rho(\lambda')k^\alpha k^\beta d\lambda'. \tag{A.2}$$

To match (A.1) and (A.2), let us start with (A.1) and integrate by part.

$$\begin{aligned}
&-\int_0^\lambda \delta k^\nu(\lambda')\partial_\nu \delta k^\mu(\lambda') d\lambda' \\
&= -\int_0^\lambda \delta \frac{d}{d\lambda'} x^\nu(\lambda')\partial_\nu \delta k^\mu(\lambda') d\lambda' \\
&= -\Big[\delta x^\nu(\lambda')\partial_\nu \delta k^\mu(\lambda')\Big]_0^\lambda + \int_0^\lambda \delta x^\nu(\lambda')\frac{d}{d\lambda'}\partial_\nu \delta k^\mu(\lambda') d\lambda'
\end{aligned} \tag{A.3}$$

The second term in this expression can be further transformed

$$\int_0^\lambda \delta x^\nu(\lambda') \frac{d}{d\lambda'} \partial_\nu \delta k^\mu(\lambda') d\lambda'$$

$$= \int_0^\lambda \delta x^\nu(\lambda') \partial_\nu \underbrace{\frac{d}{d\lambda'} \delta k^\mu(\lambda')}_{-\Gamma^\mu_{\alpha\beta} k^\alpha k^\beta} d\lambda'$$

$$= -\int_0^\lambda \delta x^\nu(\lambda') \partial_\nu \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta d\lambda'$$

which is indeed the same expression than (A.2). Concerning the integrated term in (A.3), $-\delta x^\nu(\lambda)\partial_\nu \delta k^\mu(\lambda)$, we can provide an intuitive meaning: considering the four impulsion as a field over x^ν, we have that

$$k^\mu(x^\nu(\lambda) + \delta x^\nu(\lambda)) - k^\mu(x^\nu(\lambda)) = \delta x^\nu(\lambda) \partial_\nu \delta k^\mu(\lambda)$$

at first order. Therefore, we can understand it as the fact that, in (A.1),

$$k^\nu \partial_\nu \delta^{(2)} k^\mu = -\delta k^\nu(\lambda') \partial_\nu \delta k^\mu$$

(we kept only the post-Born term) the integration is to be performed along the unperturbed path.

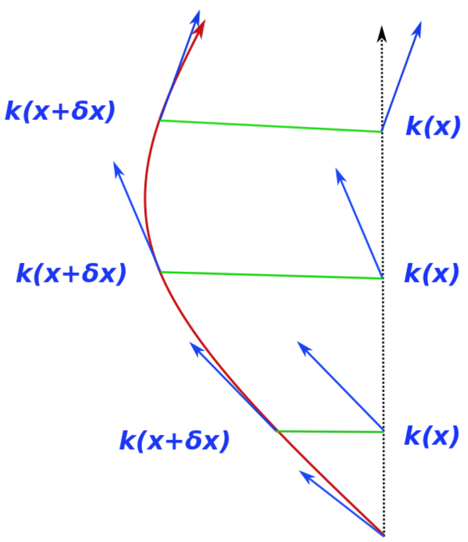


*Fig. A.*1: Here the dot line is the unperturbed path, on which we represented the **k** corrected up to second order. This is the point of view (A.1), where the perturbed impulsion vector are defined on the unperturbed path. The red path is the first-order perturbed path, on which we represented **k** corrected up to second order, but in the (A.2) point of view. The difference between both points of view is illustrated by the green lines.

Therefore, on the LHS,

$$\int_{0,\text{unperturbed}}^{\lambda} k^\nu \partial_\nu \delta^{(2)} k^\mu d\lambda' = \delta^{(2)} k^\mu(x^\nu(\lambda)).$$

On the other hand, in (A.2),

$$\frac{d\delta^{(2)} k^\mu}{d\lambda} = -\Gamma^\mu_{\alpha\beta}(x^\nu) k^\alpha k^\beta$$

the integration is previously performed along the first-order perturbed path, and subsequently transformed, namely

$$\int_{0,\text{perturbed}}^{\lambda} \frac{d\delta^{(2)} k^\mu}{d\lambda'} d\lambda' = -\int_{0,\text{perturbed}}^{\lambda} \Gamma^\mu_{\alpha\beta}(x^\nu) k^\alpha k^\beta d\lambda'$$

$$= -\int_{0,\text{unperturbed}}^{\lambda} \Gamma^\mu_{\alpha\beta}(x^\nu + \delta x^\nu) k^\alpha k^\beta d\lambda'.$$

On the LHS, we therefore have

$$\int_{0,\text{perturbed}}^{\lambda} \frac{d\delta^{(2)} k^\mu}{d\lambda'} d\lambda' = k^\mu(x^\nu(\lambda) + \delta x^\nu(\lambda)).$$

This is illustrated on Fig A.1.

B Lie transport and magnification matrix

In Chapter 4, we used the Lie transport to model the transport of a "structure" of the CMB. In this Appendix, we want to confirm that it is indeed the case directly from a rigorous treatment.

Let us assume a congruence of nearby geodesics. We will use a first parameter x which designates the depart point of the geodesic, on the plane of emission, and a second parameter λ to parametrize the path along this particular geodesic, and. So, each value of x generates an outgoing geodesic. $k(x)$ is the four-impulsion of the photons starting at the point x, normalized so that $\langle k, k\rangle = -1$. We also define the vector d which models the "distance" between the geodesic, orthogonally to k. With $l = \partial_x$, it is defined as

$$d = l + \langle l, k\rangle k. \tag{B.1}$$

We can indeed check that $\langle d, k\rangle = 0$. This is basically the structure we want to transport, because this captures the part of l of interest and discards the component along k. All those vectors are illustrated on Fig. B.1. The Lie derivative of a vector d along the vector k is therefore defined by

$$L_k d = [k, d] = [k, \langle l, k\rangle k] = k(\langle l, k\rangle)k. \tag{B.2}$$

On the other hand, the quantity $k(\langle l, k\rangle)$ can be easily computed. We have just to note that $[k, d] = 0$, which implies that $\nabla_k d = \nabla_d k$. Thus

$$k(\langle l, k\rangle) = \langle \nabla_k d, k\rangle = \langle \nabla_d k, k\rangle = \frac{1}{2}\nabla_d\langle k, k\rangle = 0. \tag{B.3}$$

As a consequence, this construction provides a convenient way to describe the evolution of a complex structure (for example, a typical shape in the CMB). In some ways, it is even more fundamental than the parallel transport, in the sense that we never really observe a unique photon, following a unique geodesic, but rather a complete picture in the sky with non-zero dimension. In this respect, we don't really observe parallel transport of photons, but Lie transport of "shapes". The final formula is

$$\boxed{L_k d \equiv \nabla_k d - \nabla_d k = 0,} \tag{B.4}$$

We solved this equation up to second order in Poisson gauge and GLC gauge in Chapter 4. Let us mention a last point before to close this appendix. Using the map provided by the Lie transport, we can build a matrix, the so-called *magnification matrix*

$$A = \begin{pmatrix} 1 + \kappa + \gamma_1 & \gamma_2 - \omega \\ \gamma_2 + \omega & 1 + \kappa - \gamma_1 \end{pmatrix}. \tag{B.5}$$

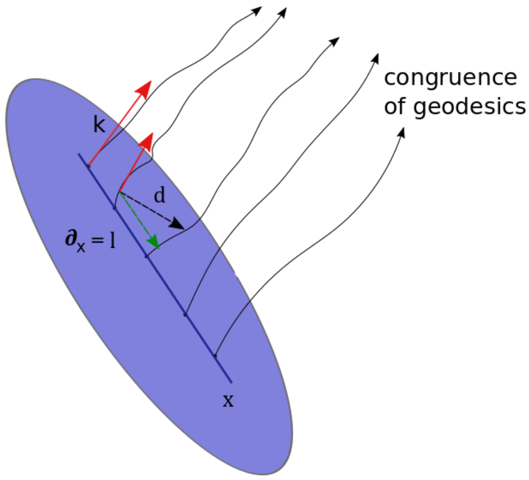


*Fig. B.*1: Congruence of geodesics. Here $\mathbf{x}$ spans the blue region and parametrizes the starting point of every geodesics. l is the vector defined by $l \equiv \partial_x$, displayed in green on the figure. k is the normalized four-impulsion, displayed in red on the figure. The vector, build from l but normal to k, $d = l + \langle l, k \rangle k$ is also explicitly displayed in black.

where $\gamma = \gamma_1 + i\gamma_2$ is the *shear* determining area distortions, κ is the *convergence*, that determines the magnification and ω that determines the rotation. For an intuitive presentation of these quantities, see [24]. We have seen in Chapter 4 that this definition of the magnification matrix provides the same second-order rotation (see (4.17)) than the definition (4.18):

$$A^i_j = \frac{\partial \theta^i_E}{\partial \theta^j_O}, \tag{B.6}$$

and using the lens equation.

C Geodesics light-cone gauge: a rapid exposition

In this Appendix, we quickly present the GLC gauge formalism. As we have seen, this formalism proves to be really convenient to compute light deflection and rotation on the Sachs basis. A more thorough exposition can be found for example in [26], in [25] or in [46].

We know that light signals, or photons, travel on null-geodesics of space-time. On the other hand, because the Christoffel symbols enter the definition of the covariant derivative, the photon path depends on the matter content of the universe. Specifically, the question we encounter in high precision cosmology is "if we receive a photon from the position θ^a_O on the sphere (a being a two-valued index which labels two angles), what is the *physical* direction of emission of this photon θ^a_E". So would like to build a map of the following sort

$$\theta^a_O \stackrel{O\to E}{\longrightarrow} \theta^a_E, \tag{C.1}$$

where the map is obviously a function of the matter content of the universe. In the first chapters of this work, we computed this relation in the Poisson gauge, and obtained an explicit expression for this map up to second order as a function of the Weyl potential ϕ. We have written it in the form of perturbative corrections to the four-vector impulsion k^μ and deflection angles α_1 and α_2.

However, there is also another approach, consisting in introducing new angular coordinates $\tilde{\theta}^a$, defined such that

$$\tilde{\theta}^a_O \equiv \tilde{\theta}^a_E. \tag{C.2}$$

This definition implies that the apparent position and the "actual" position of emission are the same. Therefore in this gauge the four-impulsion k is not corrected when we compute the path order by order in ϕ. In other words,

$$\delta^{(n)} k^\mu = 0, \tag{C.3}$$

for all positive integer n bigger than zero. We have seen in section 3.4.2 that this implies the same for the vector $\mathbf{e}$. We can also fix the equality between angular coordinates in both gauges at the observer:

$$\tilde{\theta}^a_O = \theta^a_O. \tag{C.4}$$

This actually means that the full information about the remapping (as we consider it usually in the Poisson gauge) is contained in the *gauge transformation*, namely

$$\theta^a_O = \tilde{\theta}^a_O = \tilde{\theta}^a_E \stackrel{GLC\to NG}{\longrightarrow} \theta^a_E. \tag{C.5}$$

Therefore the GLC gauge is particularly adapted to describe light signals travelling on the past-light cone of the observer, which is exactly our concern in the case of CMB temperature and polarisation lensing. As claimed in [46] and [26], in this gauge, the space-time metric can be described using one time-like coordinate τ, one null coordinate w and two angular (space-like) coordinates $\tilde{\theta}^a$. The four-velocity of a static observer can be written (see [10]) in the form

$$u_\mu = -\partial_\mu \tau,$$

and the line element is parametrized by

$$ds^2_{GLC} = \Upsilon^2 dw^2 - 2\Upsilon dw d\tau + \gamma_{ab}(d\tilde{\theta}^a - U^a dw)(d\tilde{\theta}^b - U^b dw). \tag{C.6}$$

There are a total of 6 parameters: the function Υ (1), the two-dimensional vector U^a (2) and finally the symmetric matrix γ_{ab} (3). The 6 parameters necessary to parametrize the Poisson 4×4 symmetric metric $g_{\mu\nu}$ have been transformed into the 6 parameters of the GLC gauge.

To understand the geometrical meaning of the 6 GLC parameters, let us work in the flat FLRW space. This space can be encoded using the GLC variables if we identify

$$w = r + \eta, \qquad \tau = t, \qquad \Upsilon = a(t), \qquad U^a = 0, \qquad \gamma_{ab} d\tilde{\theta}^a d\tilde{\theta}^b = a^2(t) r^2 (d\theta^2 + \sin^2\theta d\phi^2), \tag{C.7}$$

where η is defined by $d\eta = \frac{dt}{a(t)}$. In this limit the angles $\tilde{\theta}^a$ become the usual spherical angles θ and ϕ. From the line element, we can compute the metric and its inverse. We find the matrices

$$g_{\mu\nu} = \begin{pmatrix} \Upsilon^2 + U^2 & -\Upsilon & -U_b \\ -\Upsilon & 0 & \mathbf{0} \\ -U_a & \mathbf{0} & \gamma_{ab} \end{pmatrix}, \qquad g^{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{\Upsilon} & \mathbf{0} \\ -\frac{1}{\Upsilon} & -1 & -\frac{U^b}{\Upsilon} \\ \mathbf{0} & -\frac{U^a}{\Upsilon} & \gamma_{ab} \end{pmatrix}, \tag{C.8}$$

with $\gamma^{ab}\gamma_{ab} = 1$. We can see that w is a null coordinate (the vector normal to the plane with w constant is a null vector), because

$$\partial_\mu w \partial^\mu w = g^{\mu\nu}\partial_\mu w \partial_\nu w = g^{ww} = 0. \tag{C.9}$$

From $g^{\tau\tau} = -1$, we can obtain an interesting result for u_μ, namely

$$g^{\nu\sigma}\partial_\sigma \tau \nabla_\nu \partial_\mu \tau = 0,$$

meaning that the observer, as it was defined, is parallel transported. Using (C.9), we can define the four-impulsion of an arbitrary photon as

$$k^\mu = \partial^\mu w = -\frac{1}{\Upsilon}\delta^\mu_\tau. \tag{C.10}$$

We conclude that photons follow geodesics and travel on hypersurface of constant w and constant $\tilde{\theta}^a$, exactly as we claimed in the beginning of this Appendix.

We will finally prove the result

$$\delta\gamma_{ab} = \gamma_{ac}\partial_b \delta\theta^c + \gamma_{bc}\partial_a \delta\theta^c \tag{C.11}$$

we used in the section 3.4. The complete reasoning is done in [26]. The line element in geodesic light-cone gauge is given by (C.6). We have seen that the Sachs screen is actually spanned by the angles $\tilde{\theta}^a$. To describe polarisation, we can therefore restrict to the induced metric on this subspace: $ds^2_{GLC} = \gamma_{ab} d\tilde{\theta}^a d\tilde{\theta}^b$. In spherical coordinates (r, θ, ϕ), the Poisson gauge metric is

$$g^{\mu\nu} = a^{-2}\text{diag}(-1 + 2\phi, 1 + 2\phi, (1 + 2\phi)\gamma_0^{ab}).$$

Note that in the Poisson gauge, we use the unperturbed γ_0^{ab} and encode the perturbations in ϕ while in the GLC gauge, the angular metric γ_{ab} itself encodes the perturbations. To go from one representation to the other, we use the gauge transformation formula

$$g^{\mu\nu}_{GLC}(x) = \frac{\partial x^\mu}{\partial y^\rho}\frac{\partial x^\nu}{\partial y^\sigma} g^{\rho\sigma}_{PG}(y),$$

where $x = (\tau, w, \tilde{\theta}^a)$ and $y = (\eta, r, \theta^a)$. Restricting to the Sachs screen, we obtain

$$\gamma^{ab}(x) = \frac{\partial\tilde{\theta}^a}{\partial\theta^\rho}\frac{\partial\tilde{\theta}^b}{\partial\theta^\sigma}(1 + 2\phi)\gamma^{(0)\rho\sigma}_{PG}(y).$$

This is the equation for the corrections to the metric in GLC gauge. Expanding in series and keeping first order terms, we find

$$\delta\gamma^{ab}(x) = 2\phi\delta^{ab} + (\delta^b_\sigma \frac{\partial\delta\tilde{\theta}^a}{\partial\theta^\rho} + \delta^a_\rho \frac{\partial\delta\tilde{\theta}^b}{\partial\theta^\sigma})\gamma^{(0)\rho\sigma}_{PG}(y), \tag{C.12}$$

where we assumed that the unperturbed quantities $\tilde{\theta}^a$ and θ^a are the same. Noting that the first term is negligible with respect to the others, we recover our previous claim.

D Computation of the curl due to parallel transport and projection

In this Appendix, we compute the results presented in Chapter 5.

D.1 First order

In this first part, we compute the curl presented in 5.3.3. At first order, the reasoning is as follows. From the expressions for corrections of the vector $\mathbf{n}$ and the conditions of normalisation at the point $r = R$, we find that

$$\begin{aligned} n^1(R) &= 1 + \phi(R), \\ n^1(0) &= 1 + \phi(0), \\ n_1(R) &= 1 - \phi(R), \\ n_1(0) &= 1 - \phi(0), \end{aligned}$$

and

$$\delta n^a(R) = -2 \int_0^R \nabla_\perp^a \phi(r) dr, \qquad a = 2, 3. \tag{D.1}$$

We now project the polarisation on the plane perpendicular to this corrected $\mathbf{n}$. We denote the projected polarisation by an apostrophe "$'$":

$$P'_\parallel(R) = \Pi^1_j P^j = -\delta n_2(R) P^2 - \delta n_3(R) P^3,$$

$$P_\perp^{a\prime}(R) = \Pi^a_j P^j = P^a - \delta n^a(R) P_\parallel, \qquad a = 2, 3.$$

Transporting $P'_\parallel(R)$ and $P'_\parallel(R)$ from the emitter to the observer, we hope to find a polarisation perpendicular to the vector $n(r = 0)$ (at the reception). Because, with our boundary conditions, n is unperturbed at the reception, the condition of orthogonality reads

$$P^{i\prime}(r = 0) n_i = P'_\parallel(r = 0) = 0. \tag{D.2}$$

Thus $P'_\parallel(0)$ at the observer must vanish. We actually find

$$P'_\parallel(r) = -\delta n_2(R) P^2 - \delta n_3(R) P^3 - 2 P'_j \int_r^R \partial^j \phi(r') dr' + (\phi(r) - \phi(R)) P'_\parallel. \tag{D.3}$$

The last term is of second order, because $P'_\parallel$ is already of first order and we thus discard it. Putting everything together and using (D.1), we end up with

$$P'_\parallel(r) = -\delta n_2(r)P^2 - \delta n_3(r)P^3. \tag{D.4}$$

Continuing until $r = 0$, we satisfactory obtain

$$P'_\parallel(0) = 0. \tag{D.5}$$

We have also to transport the transverse components. The result is

$$P_\perp^{a'}(r=0) = P^a - \delta n^a(R)P_\parallel + (\phi(0) - \phi(R))P_\perp^a, \qquad a = 2,3. \tag{D.6}$$

Let us now compute the curl of the transverse components, $P_\perp^{2'}(r=0)$ and $P_\perp^{3'}(r=0)$. Obviously, the first term on the RHS of (D.6), $P^2 = \nabla^2\Psi$ and $P^3 = \nabla^3\Psi$, does not provide any curl. The second one gives a contribution of the form

$$2\tilde{\nabla}_\perp^2\tilde{\nabla}_\perp^3\Phi(0)P_\parallel - 2\tilde{\nabla}_\perp^3\tilde{\nabla}_\perp^2\Phi(0)P_\parallel = 2P_\parallel(\epsilon_{ij}\tilde{\nabla}_\perp^i\tilde{\nabla}_\perp^j)\Phi(0), \tag{D.7}$$

and

$$\epsilon_{ij}\delta n^i(R)\tilde{\nabla}_\perp^j P_\parallel. \tag{D.8}$$

Obviously, (D.7) vanishes by commuting the partial derivatives and only (D.8) survives. Finally, the full curl coming from the transport at first order is given by

$$\mathrm{Curl}^{(1)}(\mathbf{x},0)\Big|_{\text{prop}} = \epsilon_{ij}\tilde{\nabla}_\perp^i(\phi(0) - \phi(R))P_\perp^j + \epsilon_{ij}\delta n^i(R)\tilde{\nabla}_\perp^j P_\parallel. \tag{D.9}$$

As we have done before, we can use the gauge freedom to impose the field ϕ to vanish at the observer. The previous expression simplifies to

$$\mathrm{Curl}^{(1)}(\mathbf{x},0)\Big|_{\text{prop}} = -\epsilon_{ij}P_\perp^j\tilde{\nabla}_\perp^i\phi(R) + \epsilon_{ij}\delta n^i(R)\tilde{\nabla}_\perp^j P_\parallel. \tag{D.10}$$

as claimed in section 5.3.3.

D.2 Cross terms

In this part, we compute the curl presented in 5.3.4. We follow exactly the same procedure than in the previous section. We take the perturbation at the emitter

$$P^{i'}(\mathbf{n}) = \tilde{\nabla}_{\perp j}P^i(\mathbf{n})\alpha_1^j, \tag{D.11}$$

project it along the first order perturbed vector n and transport it until the observer. Projecting the correction (D.11) along the first order perturbed vector $\mathbf{n}$, we obtain

$$P_\parallel^{''}(R) = \Pi_j^{1'}P^j = -\delta n_2(R)\tilde{\nabla}_{\perp j}P_\perp^2(\mathbf{n})\alpha_1^j - \delta n_3(R)\tilde{\nabla}_{\perp j}P_\perp^3(\mathbf{n})\alpha_1^j,$$

$$P_\perp^{a''}(R) = \Pi_j^{a'}P^j = \tilde{\nabla}_{\perp j}P_\perp^a(\mathbf{n})\alpha_1^j - \delta n^a(R)\tilde{\nabla}_{\perp j}P_\parallel(\mathbf{n})\alpha_1^j, \qquad a = 2,3.$$

transporting those components until the observer and keeping only the second order terms, we find

$$\begin{aligned} P_{\|}''(0) &= -\delta n_2(R)\tilde{\nabla}_{\perp j}P^2(\mathbf{n})\alpha_1^j - \delta n_3(R)\tilde{\nabla}_{\perp j}P^3(\mathbf{n})\alpha_1^j \\ &\quad - 2\omega\tilde{\nabla}_{\perp j}\Phi(0)(P_{\perp}^{j''})\Big|_{\text{second}} + (\phi(0)-\phi(R))(P_{\|}'')\Big|_{\text{second}}, \end{aligned} \tag{D.12}$$

and

$$P_{\perp}^{a''}(0) = \underbrace{\tilde{\nabla}_{\perp j}P_{\perp}^a(\mathbf{n})\alpha_1^j}_{1} - \underbrace{\delta n^a(R)\tilde{\nabla}_{\perp j}P_{\|}(\mathbf{n})\alpha_1^j}_{2} + \underbrace{(\phi(0)-\phi(R))(P^{a''}+P^{a'})(R)\Big|_{\text{second}}}_{3}, \tag{D.13}$$

for $a = 2,3$. $f(\phi)\Big|_{\text{second}}$ means that we keep only the second order terms in the expression before. Going through the calculations, we see that the 'parallel" component vanishes with our approximations, exactly as it should do. For the "perpendicular" components, we will only keep the term relevant for our analysis. 1 is a first order term treated in the previous subsections, 3 is a contribution $(\phi(0)-\phi(\lambda_0))\delta n^2(\lambda_0)P_{\|}$ which is negligible within our approximation, and $(\phi(0)-\phi(\lambda_0))(P^{2''})(\lambda_0)$ is of third order.

We will therefore focus on

$$P^{a''}(0) = -\delta n^a(R)\tilde{\nabla}_{\perp j}P_{\|}(\mathbf{n})\alpha_1^j = 2\nabla_{\perp}^a\Phi(0)\tilde{\nabla}_{\perp k}P_{\|}(\mathbf{n})\alpha_1^k, \qquad a = 2,3. \tag{D.14}$$

We notice that no term comes from the transport itself, we have only contributions from the projection. Let us compute the curl of these corrections :

$$\text{Curl}^{(2)}(\mathbf{x})\Big|_{\text{cross}} = 2\epsilon_{ij}\tilde{\nabla}_{\perp}^j\Big[\tilde{\nabla}_{\perp}^i\Phi(0)\tilde{\nabla}_{\perp j}P_{\|}(\mathbf{n})\alpha_1^j\Big]. \tag{D.15}$$

Applied on $\Phi(0)$, we can see that the derivatives cancel each other. Therefore, the full curl, after inserting the expression for the deflection angle and Φ, is given by

$$\text{Curl}^{(2)}(\mathbf{x})\Big|_{\text{cross}} = -4\epsilon_{ij}\tilde{\nabla}_{\perp}^j\Big[\tilde{\nabla}_{\perp k}P_{\|}(\mathbf{n})\int_0^R \frac{R-r}{Rr}\tilde{\nabla}_{\perp}^k\phi(r)dr\Big]\int_0^R \frac{1}{r}\tilde{\nabla}_{\perp}^i\phi(r)dr. \tag{D.16}$$

as claimed in 5.3.4

D.3 Second order

In this last subsection we compute the transport at second order and prove the result presented in 5.3.5. First of all, we again take a polarisation field given by

$$P^i = \tilde{\nabla}^i\Psi(R),$$

at the plane of emission. We project it on the plane normal to the vector n corrected up to second order

$$\Pi_j^i P^j = \delta_j^i P^j - n^i n_j P^j,$$

where n is given by the explicit expression

$$\delta^2 n^i(R) = -2n^i\int_0^R \nabla_{\perp i}\phi\int_0^R \nabla_{\perp}^i\phi - 4\int_0^R\int_0^r \nabla_{\perp j}\phi\nabla_{\perp}^j\int_0^r \nabla_{\perp}^i\phi. \tag{D.17}$$

Using this expression, we can safely work out the components of $\mathbf{n}$. We end up with

$$\begin{aligned}
n^1(R) &= 1 + \phi(R) + \delta^{(2)} n^1(R), \\
n^1(0) &= 1 + \phi(0), \\
n_1(R) &= 1 - \phi(R) + \delta^{(2)} n^1(R), \\
n_1(0) &= 1 - \phi(0), \\
\delta^{(2)} n^a(R) &= -4 \int_0^R \int_0^r \nabla_{\perp j}\phi \nabla^j_{\perp} \int_0^r \nabla^a_{\perp}\phi, \qquad a = 2,3
\end{aligned}$$

where we have kept all the terms at first order and only the leading ones at second order. We have also defined

$$\delta^{(2)} n^1(R) = -2 \int_0^R \nabla_{\perp i}\phi \int_0^R \nabla^i_{\perp}\phi = -2 \int_0^R \nabla_{\perp 2}\phi \int_0^R \nabla^2_{\perp}\phi - 2 \int_0^R \nabla_{\perp 3}\phi \int_0^R \nabla^3_{\perp}\phi. \tag{D.18}$$

With these formulas, we can compute the projector along the vector n, corrected up to second order.

$$\Pi^1_j P^j = -2\delta^2 n^1(R) P_{\parallel} - (\delta n^2(R) + \delta^{(2)} n^2(R)) P^2_{\perp} - (\delta n^3(R) + \delta^{(2)} n^3(R)) P^3_{\perp}, \tag{D.19}$$

$$\Pi^2_j P^j = P^2_{\perp} - \delta^{(2)} n^2(R) P_{\parallel} - ((\delta n^2)(R))^2 P^2_{\perp} - (\delta n^2(R))(\delta n^3(R)) P^3_{\perp}, \tag{D.20}$$

$$\Pi^3_j P^j = P^3_{\perp} - \delta^{(2)} n^3(R) P_{\parallel} - (\delta n^3(R))^2 P^3_{\perp} - (\delta n^2(R))(\delta n^3(R)) P^2_{\perp}. \tag{D.21}$$

Again we have kept only the leading terms at second order.

We call these three new components respectively $P'_{\parallel}$, $P^{2'}_{\perp}$, $P^{3'}_{\perp}$ and transport them until the observer. Before to go further, we will check that the orthogonality condition is indeed satisfied at the emission and at the reception:

$$n^i(R) P'_i(R) = 0, \qquad n^i(0) P'_i(0) = P'_{\parallel}(0) = 0, \tag{D.22}$$

where $n^i(R)$ is the n-vector corrected up to second order. Doing so at the emission we end up with the following relation between $\delta^{(2)} n^1(R)$ and the $\delta^{(1)} n^a(R), \qquad a = 2,3,$

$$2\delta^{(2)} n^1(R) = -(\delta n^2(R))^2 - (\delta n^3(R))^2. \tag{D.23}$$

This relation is verified by the explicit form of each term.

If we now remember that $P'_{\parallel}$ is of first order while $P^{2'}_{\perp}$ and $P^{3'}_{\perp}$ contain zeroth order contributions, we obtain the following formula for the transport of $P'_{\parallel}$ at second order

$$\begin{aligned}
P'_{\parallel}(0)\Big|_{\text{second}} =& -2\delta^{(2)} n^1(R) P_{\parallel} + \sum_a^{2,3} \Big[-\delta^{(2)} n^a(R) P^a_{\perp} + 2\delta n^a(R) P_{\parallel} \int_0^R \nabla^a_{\perp}\phi(r')dr' \Big] \\
& - 4P^k_{\perp} \int_0^R \int_0^r \nabla_{\perp j}\phi \nabla^j_{\perp} \int_0^r \nabla_{\perp k}\phi,
\end{aligned} \tag{D.24}$$

where, of course, the j and k indices only run over the transverse components. We have only kept the second order terms, because we already treated the first order ones in previous subsections. The two terms $2\sum_a^{2,3}\delta n^a(R)P_\parallel \int\limits_0^R \nabla_\perp^a\phi$ leaves

$$-4P_\parallel \int\limits_0^R \nabla_{\perp i}\phi \int\limits_0^R \nabla_\perp^i\phi.$$

In pretty much the same way, $\delta^{(2)}n^a(R)P_\perp^a$ and $4P_\perp^a \int\limits_0^R\int\limits_0^r \nabla_{\perp j}\phi\nabla_\perp^j \int\limits_0^r \nabla_{\perp a}\phi$ combines to give

$$\delta^{(2)}n^a(0)P_\perp^a = 0, a = 2,3.$$

Therefore, we are left with

$$P_\parallel'(0) = -2\delta^{(2)}n^1(R)P_\parallel - 4P_\parallel \int\limits_0^R \nabla_{\perp i}\phi \int\limits_0^R \nabla_\perp^i\phi. \tag{D.25}$$

This vanishes by the relation (D.23);

$$P_\parallel'(0) = 0. \tag{D.26}$$

Now, with this result in hand, we would like to verify the condition of orthogonality $n^i(0)P_i(0) = 0$, up to second order. Expanded, this condition translates into

$$\begin{aligned} n^i(0)P_i'(0) &= \Big[1-\phi(0)+\delta^{(2)}n^1(0)\Big]P_\parallel'(0) + \Big[\delta^{(2)}n^2(0) - \phi(0)\delta n^2(0) + \delta n^2(0)\Big]P_\perp^{2'}(0) \\ &+ \Big[\delta^{(2)}n^3(0) + \delta n^3(0) - \phi(0)\delta n^3(0)\Big]P_\perp^{3'}(0) \\ &= \Big[1-\phi(0)\Big]P_\parallel'(0). \end{aligned} \tag{D.27}$$

This vanishes by (D.26), as it should.

This satisfactory result behind us, we can turn to the computation of the $P_\perp^{2'}(0)$ and $P_\perp^{3'}(0)$, and the computation of their curl. First of all, let us compute the two components $P^{2'}(0)$ and $P^{3'}(0)$. Transporting the projected polarisation, we obtain

$$\begin{aligned} P_\perp^{2'}(0) = &\underbrace{-\delta^{(2)}n^2(R)P_\parallel - (\delta n^2(R))^2P_\perp^2 - (\delta n^2(R))(\delta n^3(R))P_\perp^3}_{1} \\ &\underbrace{-(\phi(0)-\phi(R))\delta n^2(R)P_\parallel}_{2} \underbrace{-2P_\perp^j \int\limits_0^R \nabla_{\perp j}\phi \int\limits_0^R \nabla_\perp^2\phi}_{3}, \end{aligned} \tag{D.28}$$

where 1 comes from the projection at second order, 2 from the transport at first order of a projection at first order, and 3 comes from the transport at second order.

The part

$$P_\perp^{2'}(0) = P_\perp^2 - \delta n^2(R)P_\parallel - (\phi(R)-\phi(0))P_\perp^2 \tag{D.29}$$

is the first order part and has been already eliminated here. The term $(\phi(0) - \phi(R))\delta n^2(R)P_\parallel$ is out of our approximation and will be thrown away ($\phi\partial\phi$-type). We will then focus on the remaining part;

$$P_\perp^{2'}(0) = -\delta^{(2)}n^2(R)P_\parallel - (\delta n^2(R))^2 P_\perp^2 - (\delta n^2(R))(\delta n^3(R))P_\perp^3 - 2P_\perp^j \int_0^R \nabla_{\perp j}\phi \int_0^R \nabla_\perp^2 \phi. \quad \text{(D.30)}$$

We insert, in this expression, the explicit expressions

$$\delta n^2(R) = -2\tilde{\nabla}_\perp^2 \Psi(R),$$

$$\delta n^3(R) = -2\tilde{\nabla}_\perp^3 \Psi(R),$$

and obtain (generalizing to the two transverse components)

$$\begin{aligned} P_\perp^{a'}(0) &= 4P_\parallel \int_0^R \int_0^r \nabla_{\perp j}\phi \nabla_\perp^j \int_0^r \nabla_\perp^a \phi - 6P_\perp^j \int_0^R \nabla_{\perp j}\phi \int_0^R \nabla_\perp^a \phi \\ &= 4P_\parallel \int_0^R \frac{dr}{r} \int_0^r \frac{dr_2}{r_2} \tilde{\nabla}_{\perp j}\phi(r_2) \int_0^r \frac{dr_3}{r_3} \tilde{\nabla}_\perp^j \tilde{\nabla}_\perp^a \phi(r_3) - 6P_\perp^j \int_0^R \frac{dr_1}{r_1} \tilde{\nabla}_{\perp j}\phi(r_1) \int_0^R \frac{dr_2}{r_2} \tilde{\nabla}_\perp^a \phi(r_2) \end{aligned} \quad \text{(D.31)}$$

The second term is sub-leading with respect to the first one, therefore we will not consider it from now on. From this expression, we can compute the curl at second order. After distributing the derivative in the operator $\epsilon^{ab}\tilde{\nabla}_{\perp b}$, the only remaining term is

$$\begin{aligned} \text{Curl}^{(2)}(\mathbf{x})\Big|_{\text{prop}} &= 4\epsilon^{ab}\tilde{\nabla}_{\perp b} P_\parallel \int_0^R \frac{dr}{r} \int_0^r \frac{dr_2}{r_2} \tilde{\nabla}_{\perp j}\phi(r_2) \int_0^r \frac{dr_3}{r_3} \tilde{\nabla}_\perp^j \tilde{\nabla}_\perp^a \phi(r_3) \\ &= 2\epsilon^{ab}\tilde{\nabla}_{\perp b} P_\parallel \tilde{\nabla}_{\perp a} \left[\int_0^R \frac{dr}{r} \int_0^r \frac{dr_2}{r_2} \tilde{\nabla}_{\perp j}\phi(r_2) \int_0^r \frac{dr_3}{r_3} \tilde{\nabla}_\perp^j \phi(r_3) \right] \end{aligned} \quad \text{(D.32)}$$

as claimed in section 5.3.5.

E Power spectrum and correlation function

In this Appendix, we want to prove and elaborate some assertions we made in the beginning of Chapter 6 about the *power spectrum*. As we have seen the correlation function between the point $\mathbf{x}$ and the point $\mathbf{y}$ is defined by

$$\xi^{(ij)}(\mathbf{x},\mathbf{y}) = \langle \delta P^i(\mathbf{x})\delta P^{\star j}(\mathbf{y})\rangle, \tag{E.1}$$

where the brackets mean an average over some region of the sky.

The Fourier transform of a function ϕ of $\mathbf{x}$ is defined as

$$\phi(\mathbf{k}) = \int \frac{d^2x}{2\pi} e^{i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{x}),$$

while its inverse is defined as

$$\phi(\mathbf{x}) = \int \frac{d^2k}{2\pi} e^{-i\mathbf{k}\cdot\mathbf{x}}\phi(\mathbf{k}).$$

The *homogeneity* of the Universe implies that

$$\xi^{(ij)}(\mathbf{x},\mathbf{y}) = \xi^{(ij)}(\mathbf{x}-\mathbf{y}). \tag{E.2}$$

That is to say, the correlation function is only function of the difference of position. On the other hand, the *isotropy* of the Universe implies that (with $\mathcal{R}$ a rotation in the two-dimensional space)

$$\xi^{(ij)}(\mathcal{R}\mathbf{x},\mathcal{R}\mathbf{y}) = \xi^{(ij)}(\mathbf{x},\mathbf{y}) = \xi^{(ij)}(|\mathbf{x}-\mathbf{y}|). \tag{E.3}$$

That is to say, the invariance under rotation implies that the correlation function is only function of the norm of the difference vector $\mathbf{x}-\mathbf{y}$. Now, let us write the Fourier transform of the correlation function $\xi(|\mathbf{x}-\mathbf{y}|)$,

$$\xi^{(ij)}(|\mathbf{x}-\mathbf{y}|) = \langle \delta P^{i\star}(\mathbf{x})\delta P^j(\mathbf{y})\rangle = \frac{1}{(2\pi)^2}\int d^2k_1 \int d^2k_2 e^{i\mathbf{k_1}\cdot\mathbf{x}-i\mathbf{k_2}\cdot\mathbf{y}}\langle \delta P^{i\star}(\mathbf{k}_1)\delta P^j(\mathbf{k}_2)\rangle.$$

Now if we do the changes $\mathbf{x}\to\mathbf{x}+\mathbf{a}$ and $\mathbf{y}\to\mathbf{y}+\mathbf{a}$, we obtain

$$\xi^{(ij)}(|\mathbf{x}-\mathbf{y}|) = \frac{1}{(2\pi)^2}\int d^2k_1 \int d^2k_2 e^{i\mathbf{k_1}\cdot\mathbf{x}-i\mathbf{k_2}\cdot\mathbf{y}+i\mathbf{k_1}\cdot\mathbf{a}-i\mathbf{k_2}\cdot\mathbf{a}}\langle \delta P^{i\star}(\mathbf{k}_1)\delta P^j(\mathbf{k}_2)\rangle.$$

This implies the following identity:

$$\langle \delta P^{i\star}(\mathbf{k}_1)\delta P^j(\mathbf{k}_2)\rangle = e^{i\mathbf{k_1}\cdot\mathbf{a}-i\mathbf{k_2}\cdot\mathbf{a}}\langle \delta P^{i\star}(\mathbf{k}_1)\delta P^j(\mathbf{k}_2)\rangle. \tag{E.4}$$

On the other hand, this is only possible if

$$\langle \delta P^{i\star}(\mathbf{k}_1)\delta P^j(\mathbf{k}_2)\rangle = F^{(ij)}(\mathbf{k})\delta_D(\mathbf{k}_1-\mathbf{k}_2). \tag{E.5}$$

Going through the same manipulations for the isotropy, we find that F only depends on the modulus of $\mathbf{k}$. We then arrive at the conclusion that

$$\langle \delta P^{i\star}(\mathbf{k}_1)\delta P^j(\mathbf{k}_2)\rangle = P^{(ij)}(|\mathbf{k}|)\delta_D(\mathbf{k}_1 - \mathbf{k}_2), \tag{E.6}$$

where $P^{|\mathbf{k}|}$ is called the *power spectrum*. For convenience we will rewrite it in the form $P_k^{\delta(ij)}$.

The power spectrum can also be rewritten as a function of the correlation function, using the inverse Fourier transform.

$$P^{(ij)}(|\mathbf{k}|)\delta_D(\mathbf{k}_1 - \mathbf{k}_2) = \int \frac{d^2\mathbf{x}}{2\pi} \int \frac{d^2\mathbf{y}}{2\pi} e^{i\mathbf{k_1}\cdot\mathbf{x} - i\mathbf{k_2}\cdot\mathbf{y}} \xi^{(ij)}(|\mathbf{x} - \mathbf{y}|).$$

With $\mathbf{r} = \mathbf{x} - \mathbf{y}$ and $r = |\mathbf{r}|$, we get

$$P^{(ij)}(|\mathbf{k}|)\delta_D(\mathbf{k}_1 - \mathbf{k}_2) = \int \frac{d^2\mathbf{r}}{2\pi} \int \frac{d^2\mathbf{y}}{2\pi} e^{i\mathbf{k_1}\cdot\mathbf{r}} e^{i\mathbf{y}\cdot(\mathbf{k_1} - \mathbf{k_2})} \xi^{(ij)}(r).$$

Performing the integral over $\mathbf{y}$ gives us a $(2\pi)^2\delta_D(\mathbf{k_1} - \mathbf{k_2})$, leaving us with

$$P_k^{\delta(ij)} = \int d^2\mathbf{r} e^{i\mathbf{k}\cdot\mathbf{r}} \xi^{(ij)}(r). \tag{E.7}$$

This equation involves the Bessel functions and can be further simplified. We will not carry out those manipulations here.

F Correlation functions of the lensing potential in the flat sky approximation

In this last Appendix, we compute the correlation functions and the power spectrum P_l^ψ of the lensing potential (as a function of the power spectrum of the Weyl field ϕ), because we used it a lot in the previous main formulas. Other formulas needed in Chapter 6 are also explicitly computed.

The form of the lensing potential is

$$\psi(R, \mathbf{x}) = -2 \int_0^R \frac{R-r}{Rr} \phi(r, \mathbf{x}) dr.$$

Here we will focus on the flat-sky approximation (The full sky calculation can be found in [41]) and use the two-dimensional Fourier transformations

$$\phi(r, \mathbf{l}) = \int \frac{d^2\mathbf{x}}{2\pi} e^{i\mathbf{l}\cdot\mathbf{x}} \phi(r, \mathbf{x}), \qquad \phi(r, \mathbf{x}) = \int \frac{d^2\mathbf{l}}{2\pi} e^{-i\mathbf{l}\cdot\mathbf{x}} \phi(r, \mathbf{l}),$$

where r parametrizes the position along the line of integration[1] and $\mathbf{x}$ denotes the direction in the sky, and

$$\psi(R, \mathbf{l}) = \int \frac{d^2\mathbf{x}}{2\pi} e^{i\mathbf{l}\cdot\mathbf{x}} \psi(R, \mathbf{x}), \qquad \psi(R, \mathbf{x}) = \int \frac{d^2\mathbf{l}}{2\pi} e^{-i\mathbf{l}\cdot\mathbf{x}} \psi(R, \mathbf{l}).$$

We want to compute the power spectrum of the lensing potential P^ψ, defined by

$$\langle \psi(R, \mathbf{l_1}) \psi(R, \mathbf{l_2}) \rangle = \delta_D(\mathbf{l}_1 - \mathbf{l}_2) P_{l_1}^\psi. \tag{F.1}$$

The Fourier transform of the lensing potential can be written

$$\psi(R, \mathbf{l}) = -2 \int \frac{d^2\mathbf{x}}{2\pi} e^{i\mathbf{l}\cdot\mathbf{x}} \int_0^R \frac{R-r}{Rr} \phi(r, \mathbf{x}) = -2 \int_0^R \frac{R-r}{Rr} \int\int \frac{d^2\mathbf{x}}{2\pi} \frac{d^2\mathbf{l}_2}{2\pi} e^{i\mathbf{l}\cdot\mathbf{x}} e^{-i\mathbf{l}_2\cdot\mathbf{x}} \phi(r, \mathbf{l}_2). \tag{F.2}$$

In this expression, the integration over $\mathbf{x}$ gives a Dirac delta and the whole expression simplifies to

$$\psi(R, \mathbf{l}) = -2 \int_0^R \frac{R-r}{Rr} \phi(r, \mathbf{l}) dr. \tag{F.3}$$

[1] it can be also parametrized by the redshift $\phi(z(r), \mathbf{x})$, because the relation $r(z)$ can be easily inverted.

Using (F.3), $\langle\psi(R,\mathbf{l_1})\psi(R,\mathbf{l_2})\rangle$ can be straightforwardly expressed as a function of the power spectrum the matter of P^{ϕ}. If we define the matter power spectrum by

$$\langle\phi(r,\mathbf{l_1})\phi(r',\mathbf{l_2})\rangle = \delta_D(\mathbf{l}_1-\mathbf{l}_2)P^{\phi}_{l_1}(r,r'), \tag{F.4}$$

we can write the power spectrum of the lensing potential in the form

$$P^{\psi}_{l} = 4\int_0^R\int_0^R \frac{R-r}{Rr}\frac{R-r'}{Rr'}P^{\phi}_{l}(r,r')drdr'. \tag{F.5}$$

In the text, we also defined the quantity $P^{\rm cross}_{|\mathbf{l}'-\mathbf{l}|} = \langle\Delta\phi(\mathbf{l}'-\mathbf{l})\psi(\mathbf{l}'-\mathbf{l})\rangle$ when we came to the cross-terms contribution in Chapter 6. Again, we can easily express this quantity in terms of the power spectrum of the Weyl field ϕ.

$$\begin{aligned} P^{\rm cross}_{|\mathbf{l}'-\mathbf{l}|} &= -2\int_0^R \frac{R-r}{Rr}\langle\phi(r,\mathbf{l}-\mathbf{l}')\Delta\phi(\mathbf{l}-\mathbf{l}')\rangle dr \\ &= -2\int_0^R \frac{R-r}{Rr}\langle\phi(r,\mathbf{l}-\mathbf{l}')\phi(R,\mathbf{l}-\mathbf{l}')\rangle dr \\ &= -2\int_0^R \frac{R-r}{Rr}P^{\phi}_{|\mathbf{l}-\mathbf{l}'|}(r,R)dr. \end{aligned} \tag{F.6}$$

We now turn to the deflection angle at second order, our purpose is to write its power spectrum in terms of the matter power spectrum. The explicit expression for the second-order deflection angle is $\alpha^a_2(R,\mathbf{x}) = -2\int_0^R \frac{(R-r)}{Rr}\tilde{\nabla}_{\perp b}\tilde{\nabla}^a_{\perp}\phi(r,\mathbf{x})\alpha^b_1(r,\mathbf{x})dr$. As an example of the type of computations we have done in Chapter 6, we will present the whole calculation. Starting by inserting the Fourier transform expressions, we obtain

$$\alpha^a_2(R,\mathbf{l}_1) = 4\int_0^R\frac{(R-r)}{Rr}\int\frac{d^2\mathbf{x}}{2\pi}e^{i\mathbf{l}_1\cdot\mathbf{x}}\int\frac{d^2\mathbf{l}_2}{2\pi}l_{2b}l^a_2e^{-i\mathbf{l}_2\cdot\mathbf{x}}\phi(r,\mathbf{l_2})\int_0^r\frac{r-r'}{rr'}\int\frac{d^2\mathbf{l}_3}{2\pi}e^{-i\mathbf{l}_3\cdot\mathbf{x}}l^b_3\phi(r',\mathbf{l_3})drdr'. \tag{F.7}$$

Upon reorganization of terms and integrating over $\mathbf{x}$, we obtain

$$\begin{aligned} \alpha^a_2(R,\mathbf{l}_1) &= 4\int_0^R\frac{R-r}{Rr}\int_0^r\frac{r-r'}{rr'}\int\int\int\frac{d^2\mathbf{x}}{2\pi}\frac{d^2\mathbf{l}_2}{2\pi}\frac{d^2\mathbf{l}_3}{2\pi}e^{i(\mathbf{l}_1-\mathbf{l}_2-\mathbf{l}_3)\cdot\mathbf{x}}\mathbf{l}_3\cdot\mathbf{l}_2l^a_2\phi(r,\mathbf{l_2})\phi(r',\mathbf{l_3})drdr' \\ &= 8\pi\delta_D(\mathbf{l}_1-\mathbf{l}_2-\mathbf{l}_3)\int_0^R\frac{R-r}{Rr}\int_0^r\frac{r-r'}{rr'}\int\int\frac{d^2\mathbf{l}_2}{2\pi}\frac{d^2\mathbf{l}_3}{2\pi}\mathbf{l}_3\cdot\mathbf{l}_2l^a_2\phi(r,\mathbf{l_2})\phi(r',\mathbf{l_3})drdr' \\ &= 4\int_0^R\frac{(R-r)}{Rr}\int_0^r\frac{r-r'}{rr'}\int\frac{d^2\mathbf{l}_2}{2\pi}(\mathbf{l}_1-\mathbf{l}_2)\cdot\mathbf{l}_2l^a_2\phi(r,\mathbf{l_2})\phi(r',\mathbf{l}_1-\mathbf{l}_2)drdr'. \end{aligned} \tag{F.8}$$

From that we are now able to compute the power of the quantity $\alpha_2^a(R,\mathbf{l})$ in terms of the power of the Weyl field $\phi(r,\mathbf{l})$. Going through the computation, we obtain

$$\langle \alpha_2^a(R,\mathbf{l}')\alpha_2^{b\star}(R,\mathbf{l})\rangle = 16\int_0^R \frac{(R-r)}{Rr}\int_0^r \frac{r-r_1}{rr_1}\int_0^R \frac{(R-r_2)}{Rr_2}\int_0^{r_2}\frac{r_2-r_3}{r_2r_3}\int\frac{d^2\mathbf{l}_2}{2\pi}\int\frac{d^2\mathbf{l}_3}{2\pi}$$
$$\times(\mathbf{l}-\mathbf{l}_2)\cdot\mathbf{l}_2(\mathbf{l}'-\mathbf{l}_3)\cdot\mathbf{l}_3 l_2^a l_3^b\Big\langle\phi(r,\mathbf{l_2})\phi(r_1,\mathbf{l}-\mathbf{l}_2)\phi^\star(r_2,\mathbf{l_3})\phi^\star(r_3,\mathbf{l}'-\mathbf{l}_3)\Big\rangle. \tag{F.9}$$

We now aim to simplify the rather complicated quantity $\langle\phi(r,\mathbf{l_2})\phi(r_1,\mathbf{l}-\mathbf{l}_2)\phi^\star(r_2,\mathbf{l_3})\phi^\star(r_3,\mathbf{l}'-\mathbf{l}_3)\rangle$. To do so, we apply the Wick theorem which states, in the case of negligible non-Gaussianities, that

$$\langle abcd\rangle = \langle ab\rangle\langle cd\rangle + \langle ac\rangle\langle bd\rangle + \langle ad\rangle\langle bc\rangle.$$

Applied on $\langle\phi(r,\mathbf{l_2})\phi(r_1,\mathbf{l}-\mathbf{l}_2)\phi^\star(r_2,\mathbf{l_3})\phi^\star(r_3,\mathbf{l}'-\mathbf{l}_3)\rangle$, the Wick theorem reads

$$\langle\phi(r,\mathbf{l_2})\phi^\star(r_1,-\mathbf{l}+\mathbf{l}_2)\rangle\langle\phi(r_2,-\mathbf{l_3})\phi^\star(r_3,\mathbf{l}'-\mathbf{l}_3)\rangle + \langle\phi(r,\mathbf{l_2})\phi^\star(r_2,\mathbf{l_3})\rangle\langle\phi(r_1,\mathbf{l}-\mathbf{l}_2)\phi^\star(r_3,\mathbf{l}'-\mathbf{l}_3)\rangle$$
$$+\langle\phi(r,\mathbf{l_2})\phi^\star(r_3,\mathbf{l}'-\mathbf{l}_3)\rangle\langle\phi(r_2,\mathbf{l_3})\phi^\star(r_1,\mathbf{l}-\mathbf{l}_2)\rangle, \tag{F.10}$$

or, more simply,

$$\delta_D(\mathbf{l})\delta_D(\mathbf{l}')P^\phi_{|\mathbf{l}_2|}(r,r_1)P^\phi_{|\mathbf{l}_3|}(r_2,r_3) + \delta_D(\mathbf{l}_2-\mathbf{l}_3)\delta_D(\mathbf{l}-\mathbf{l}'+\mathbf{l}_2-\mathbf{l}_3)P^\phi_{|\mathbf{l_2}|}(r,r_2)P^\phi_{|\mathbf{l}-\mathbf{l}_2|}(r_1,r_3)$$
$$+\delta_D(\mathbf{l}_2+\mathbf{l}_3-\mathbf{l}')\delta_D(\mathbf{l}_2+\mathbf{l}_3-\mathbf{l})P^\phi_{|\mathbf{l}_2|}(r,r_3)P^\phi_{|\mathbf{l}_3|}(r_1,r_2). \tag{F.11}$$

where we used the fact that $\phi^\star(\mathbf{l}) = \phi(-\mathbf{l})$, by the reality of the Weyl field in real space.

Upon inserting this expression in (F.9), we finally obtain

$$\langle\alpha_2^a(R,\mathbf{l})\alpha_2^b(R,\mathbf{l}')\rangle = T_1 + T_2$$
$$= \frac{4}{\pi^2}\delta_D(\mathbf{l})\delta_D(\mathbf{l}')\int_0^R dr\frac{(R-r)}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^R dr_2\frac{(R-r_2)}{Rr_2}\int_0^{r_2}dr_3\frac{r_2-r_3}{r_2r_3}$$
$$\times\int d^2\mathbf{l}_2\int d^2\mathbf{l}_3|\mathbf{l}_2|^2|\mathbf{l}_3|^2 l_2^a l_3^b P^\phi_{l_2}(r,r_1)P^\phi_{l_3}(r_2,r_3)$$
$$+\delta_D(\mathbf{l}-\mathbf{l}')\frac{4}{\pi^2}\int_0^R dr\frac{(R-r)}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^R dr_2\frac{(R-r_2)}{Rr_2}\int_0^{r_2}dr_3\frac{r_2-r_3}{r_2r_3}\int d^2\mathbf{l}_1$$
$$\times\Big[(\mathbf{l}-\mathbf{l}_1)\cdot\mathbf{l}_1\Big]^2\times\Bigg(\Big[l_1^a l_1^b P^\phi_{|\mathbf{l}_1|}(r,r_2)P^\phi_{|\mathbf{l}-\mathbf{l}_1|}(r_1,r_3)$$
$$+ l_1^a(l-l_1)^b P^\phi_{|\mathbf{l}_1|}(r,r_3)P^\phi_{|\mathbf{l}-\mathbf{l}_1|}(r_1,r_2)\Big]\Bigg). \tag{F.12}$$

There is a total of six factors $\mathbf{l}$ in every terms, that means that for high l, this kind of term will be dominant. Notice however that the first term T_1, which is the term with $\delta_D(\mathbf{l})\delta_D(\mathbf{l}')$ will not contribute, except for the marginal part $l = l' = 0$.

In Chapter 6, we needed also the correlation function of two "lensing potential at first order" with one "lensing potential at second order": $\langle\psi(\mathbf{l}_2)\psi^\star(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l})\alpha_2^{\star a}(\mathbf{l}-\mathbf{l_1})\rangle$. We will follow the same steps as above, using the flat-sky approximation. Coming back to (F.8), we find

$$\langle\psi(\mathbf{l}_2)\psi^\star(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l})\alpha_2^{\star a}(\mathbf{l}-\mathbf{l_1})\rangle = 16\int_0^R \frac{R-r}{Rr}dr\int_0^R \frac{R-r_1}{Rr_1}dr_1\int_0^R \frac{(R-r_2)}{Rr_2}dr_2\int_0^{r_2}\frac{r_2-r_3}{r_2r_3}dr_3\int\frac{d^2\mathbf{l}_3}{2\pi}$$
$$\times(\mathbf{l}-\mathbf{l}_1-\mathbf{l}_3)\cdot\mathbf{l}_3 l_3^a\Big\langle\phi(r,\mathbf{l}_2)\phi^\star(r_1,\mathbf{l}_2+\mathbf{l}_1-\mathbf{l})\phi^\star(r_2,\mathbf{l_3})\phi^\star(r_3,\mathbf{l}-\mathbf{l}_1-\mathbf{l}_3)\Big\rangle \quad \text{(F.13)}$$

We use now, as usual, the Wick theorem to compute $\langle\phi(r,\mathbf{l}_2)\phi^\star(r_1,\mathbf{l}_2+\mathbf{l}_1-\mathbf{l})\phi^\star(r_2,\mathbf{l_3})\phi^\star(r_3,\mathbf{l}-\mathbf{l}_1-\mathbf{l}_3)\rangle$. Directly expressed as a function of the power spectra, the correlation function of the Fourier transformed field is

$$\begin{aligned}&\delta_D(\mathbf{l}_1-\mathbf{l})\delta_D(\mathbf{l}_1-\mathbf{l})P^\phi_{l_2}(r,r_1)P^\phi_{l_3}(r_2,r_3)+\delta_D(\mathbf{l}_2-\mathbf{l}_3)\delta_D(\mathbf{l}_2-\mathbf{l}_3)P^\phi_{l_2}(r,r_2)P^\phi_{|\mathbf{l}_2+\mathbf{l}_1-\mathbf{l}|}(r_1,r_3)+\\&\delta_D(\mathbf{l}_3+\mathbf{l}_2+\mathbf{l}_1-\mathbf{l})\delta_D(\mathbf{l}_3+\mathbf{l}_2+\mathbf{l}_1-\mathbf{l})P^\phi_{l_2}(r,r_3)P^\phi_{l_3}(r_1,r_2).\end{aligned} \quad \text{(F.14)}$$

Therefore, putting everything together, (F.13) becomes

$$\begin{aligned}&\langle\psi(\mathbf{l}_2)\psi^\star(\mathbf{l}_1+\mathbf{l}_2-\mathbf{l})\alpha_2^{\star a}(\mathbf{l}-\mathbf{l_1})\rangle =\\&16\int_0^R\frac{R-r}{Rr}dr\int_0^R\frac{R-r_1}{Rr_1}dr_1\int_0^R\frac{(R-r_2)}{Rr_2}dr_2\int_0^{r_2}\frac{r_2-r_3}{r_2r_3}dr_3\int\frac{d^2\mathbf{l}_3}{2\pi}(\mathbf{l}-\mathbf{l}_1-\mathbf{l}_3)\cdot\mathbf{l}_3 l_3^a\\&\times\Bigg(\Big[\delta_D^2(\mathbf{l}_1-\mathbf{l})P^\phi_{l_2}(r,r_1)P^\phi_{l_3}(r_2,r_3)+\delta_D^2(\mathbf{l}_2-\mathbf{l}_3)P^\phi_{l_2}(r,r_2)P^\phi_{|\mathbf{l}_2+\mathbf{l}_1-\mathbf{l}|}(r_1,r_3)\Big]\\&+\Big[\delta_D^2(\mathbf{l}_3+\mathbf{l}_2+\mathbf{l}_1-\mathbf{l})P^\phi_{l_2}(r,r_3)P^\phi_{l_3}(r_1,r_2)\Big]\Bigg).\end{aligned} \quad \text{(F.15)}$$

This is useful in Chapter 6. We finally still need to compute to correlation of the third-order deflection angle with the first-order lensing potential: $\langle{\alpha_3}^a(\mathbf{l}-\mathbf{l_1})\psi^\star(\mathbf{l}-\mathbf{l_1})\rangle$. Using (2.109), the Fourier transform of the third-order deflection angle is

$$\begin{aligned}\alpha_3^a(\mathbf{l}) = &-8\int_0^R dr\frac{R-r}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^{r_1}dr_2\frac{r_1-r_2}{r_1r_2}\int\int\frac{d^2\mathbf{l_1}}{2\pi}\frac{d^2\mathbf{l_2}}{2\pi}\\&\times l_1^a(\mathbf{l_1}\cdot\mathbf{l_2})(\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}])\phi(r,\mathbf{l_1})\phi(r_1,\mathbf{l_2})\phi(r_2,\mathbf{l}-\mathbf{l_1}-\mathbf{l_2})\\&-4\int_0^R dr\frac{R-r}{Rr}\int_0^r dr_1\frac{r-r_1}{rr_1}\int_0^r dr_2\frac{r-r_2}{r_1r_2}\int\int\frac{d^2\mathbf{l_1}}{2\pi}\frac{d^2\mathbf{l_2}}{2\pi}\\&\times l_1^a(\mathbf{l_1}\cdot\mathbf{l_2})(\mathbf{l_1}\cdot[\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}])\phi(r,\mathbf{l_1})\phi(r_1,\mathbf{l_2})\phi(r_2,\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}).\end{aligned} \quad \text{(F.16)}$$

With that result, we are now in position to compute the power spectrum. We have therefore

$$
\begin{aligned}
\langle {\alpha_3}^a(\mathbf{l}-\mathbf{l_1})\psi^\star(\mathbf{l}-\mathbf{l_1})\rangle &= 16\int_0^R dr\int_0^r dr_1\int_0^{r_1} dr_2\int_0^R dr_3 \frac{R-r}{Rr}\frac{r-r_1}{rr_1}\frac{r_1-r_2}{r_1r_2}\frac{R-r_3}{Rr_3}\int\int\frac{d^2\mathbf{l_2}}{2\pi}\frac{d^2\mathbf{l_3}}{2\pi}\\
&\times l_2^a(\mathbf{l_2}\cdot\mathbf{l_3})(\mathbf{l_3}\cdot[\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3}])\\
&\times \Big\langle \phi(r,\mathbf{l_2})\phi(r_1,\mathbf{l_3})\phi(r_2,\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3})\phi^\star(r_3,\mathbf{l}-\mathbf{l_1})\Big\rangle\\
&+8\int_0^R dr\int_0^r dr_1\int_0^{r} dr_2\int_0^R dr_3 \frac{R-r}{Rr}\frac{r-r_1}{rr_1}\frac{r-r_2}{r_1r_2}\frac{R-r_3}{Rr_3}\int\int\frac{d^2\mathbf{l_2}}{2\pi}\frac{d^2\mathbf{l_3}}{2\pi}\\
&\times l_2^a(\mathbf{l_2}\cdot\mathbf{l_3})(\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3}])\\
&\times \Big\langle \phi(r,\mathbf{l_2})\phi(r_1,\mathbf{l_3})\phi(r_2,\mathbf{l}-\mathbf{l_1}-\mathbf{l_2}-\mathbf{l_3})\phi^\star(r_3,\mathbf{l}-\mathbf{l_1})\Big\rangle\\
&=16\int_0^R dr\int_0^r dr_1\int_0^{r_1} dr_2\int_0^R dr_3 \frac{R-r}{Rr}\frac{r-r_1}{rr_1}\frac{r_1-r_2}{r_1r_2}\frac{R-r_3}{Rr_3}\int\frac{d^2\mathbf{l_2}}{(2\pi)^2}\\
&\times\Bigg(\Big[l_2^a(\mathbf{l_2})^2\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}]P^\phi_{|\mathbf{l_2}|}(r,r_1)P^\phi_{|\mathbf{l}-\mathbf{l_1}|}(r_2,r_3)\Big]\\
&-l_2^a(\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}])^2P^\phi_{|\mathbf{l_2}|}(r,r_2)P^\phi_{|\mathbf{l}-\mathbf{l_1}|}(r_1,r_3)\Big]\\
&-(l-l_1)^a(\mathrm{l}_2\cdot[\mathbf{l}-\mathbf{l_1}])(\mathbf{l_2})^2P^\phi_{|\mathbf{l}-\mathbf{l_1}|}(r,r_3)P^\phi_{|\mathrm{l}_2|}(r_1,r_2)\Big]\Bigg)\\
&+8\int_0^R dr\int_0^r dr_1\int_0^{r} dr_2\int_0^R dr_3 \frac{R-r}{Rr}\frac{r-r_1}{rr_1}\frac{r-r_2}{r_1r_2}\frac{R-r_3}{Rr_3}\int\frac{d^2\mathbf{l_2}}{(2\pi)^2}\\
&\times\Bigg(\Big[-l_2^a(\mathbf{l_2})^2\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}]P^\phi_{|\mathbf{l_2}|}(r,r_1)P^\phi_{|\mathbf{l}-\mathbf{l_1}|}(r_2,r_3)\Big]\\
&-l_2^a(\mathbf{l_2}\cdot[\mathbf{l}-\mathbf{l_1}])(\mathbf{l_2})^2P^\phi_{|\mathbf{l_2}|}(r,r_2)P^\phi_{|\mathbf{l}-\mathbf{l_1}|}(r_1,r_3)\Big]\\
&-(l-l_1)^a(\mathrm{l}_2\cdot[\mathbf{l}-\mathbf{l_1}])^2P^\phi_{|\mathbf{l}-\mathbf{l_1}|}(r,r_3)P^\phi_{|\mathrm{l}_2|}(r_1,r_2)\Big]\Bigg).
\end{aligned}
\tag{F.17}
$$

The other quantities needed in Chapter 6 are not explicitly computed here.