Computational Astrophysics and Geophysics

Exercise 4

Solve the non-relativistic barotropic stellar structure equations

$$\frac{dP(r)}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)$$

$$P = P(\rho)$$

To find the Mass-radius relation of white dwarfs.

For low-mass white dwarfs, one may use the low-density form of the equation of state

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e}\left(\frac{\rho}{\mu_e m_p}\right)^{5/3}$$

At higher densities, special relativity becomes important and the equation of state asymptotically goes over to

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hc\left(\frac{\rho}{\mu_e m_p}\right)^{4/3}$$

The full expression of the cold electron degenerate EoS valid at all densities is

$$P = \frac{\pi m_e^4 c^5}{h^3}\left\{x(1+x^2)^{1/2}\left(\frac{2x^2}{3}-1\right) + \ln\left[x + (1+x^2)^{1/2}\right]\right\}$$

$$x = \left(\frac{h}{m_e c}\right)\left(\frac{3n_e}{8\pi}\right)^{1/3}$$

$$n_e = \frac{\rho}{\mu_e m_p}$$

In the above, h=Planck's constant, m_e=Mass of the electron, m_p=Mass of the proton, c=Speed of light in vacuum, μ_e=Mean no. of AMU per electron (=2 for White Dwarfs), n_e=number density of electrons.

Numerically, the full EoS expression becomes

$$P = 1.80 \times 10^{22} \left\{ x(1+x^2)^{1/2}\left(\frac{2x^2}{3} - 1\right) + \ln\left[x + (1+x^2)^{1/2}\right] \right\} \quad \text{Pa} \qquad \text{(A)}$$

where

$$x = 0.80\left(\frac{\rho}{10^9 \text{ kg/m}^3}\right)^{1/3}$$

Which for $x << 1$ ($\rho << 10^9$ kg/m^3) reduces to

$$P = 3.16 \times 10^{21}\left(\frac{\rho}{10^9 \text{ kg/m}^3}\right)^{5/3} \quad \text{Pa} \qquad \text{(B)}$$

And at high density (x >> 1) to

$$P = 4.935 \times 10^{21}\left(\frac{\rho}{10^9 \text{ kg/m}^3}\right)^{4/3} \quad \text{Pa} \qquad \text{(C)}$$

In all the above expressions, $\mu_e = 2$ has been assumed.

Obtain Mass-Radius relation of white dwarfs for central densities in the range 10^7 kg/m^3 to 10^{12} kg/m^3 using first the approximate EoS relation (B) and then the full EoS (A). Take logarithmic steps in assumed central density. For each assumed central density, solve the structure equations from the centre to the surface where the pressure falls to zero (or below a pre-set small value). The r value corresponding to this is the stellar radius R and the mass contained within, $M(R)$, is the total mass of the star for that central density. By varying the central density, stars of different masses can be constructed. The relation between $M(R)$ and the corresponding R for different central densities is the desired Mass-Radius relation. Tabulate and plot them in both cases of the assumed EoS. Express the masses in units of the solar mass (1.988×10^{30} kg) and the radii in units of km.

How does the M-R relation differ in the two cases? What special property can you infer in case (A)?

Use a numerical scheme that is higher than second order in stepsize h. Submit your tabular results, plots and a listing of your code.