MATH 318, Assignment 5

Due date: November 13

1. (2 points)
   (1) Let $\Gamma = \{p \wedge q, (\neg p) \vee q, p \vee r\}$. Is it true that $\Gamma \models r$? Justify your answer.
   (2) Let $\Gamma = \{p \to q, q \to p, (\neg p) \wedge q\}$ Is it true that $\Gamma \models p$? Justify your answer.

In the problems below you are supposed to **find** formal deductions. This means it is not enough to show their existence. However, in every problem you can reuse formal deductions that were shown in class as well as the formal deductions found in previous problems. The variables α, β, γ denote formulas, you can use them in your deductions as in the axioms schemes.

We consider the system of Natural Deduction with the axiom

$$\Delta, \alpha \vdash \alpha$$

and the following inference rules:

$$\frac{\Delta, \alpha \vdash \beta}{\Delta \vdash \alpha \to \beta} \to\text{-intro} \qquad \frac{\Delta \vdash \alpha \to \beta,\ \Delta \vdash \alpha}{\Delta \vdash \beta} \to\text{-elim}$$

$$\frac{\Delta, \neg\alpha \vdash \bot}{\Delta \vdash \alpha} \text{RAA}$$

2. (1 point) Give a proof in the above Natural Deduction system of the following sequent: $\vdash p \to p$

3. (2 points) Give a proof in the above Natural Deduction system of the following sequent: $\vdash \alpha \to (\beta \to \alpha)$.

4. (2 points) Give a proof in the above Natural Deduction system of the sequent: $\vdash \neg\neg\alpha \to \alpha$.

5. (2 points) Give a proof in the above Natural Deduction system of the sequent: $\vdash (\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$.

6. (3 points) Give a proof in the above Natural Deduction system of the following sequent: $\vdash (\neg\beta \to \neg\alpha) \to (\alpha \to \beta)$.

7. (2 points) Find a proof by resolution of the empty clause from the following set of clauses: $\{\{p, q, \neg r\}, \{r, p\}, \{p, \neg q\}, \{\neg p\}\}$.

The problems below are for extra credit

We consider a Hilbert-style proof system with Modus Ponens as the inference rule and the following axiom schemes:

A1 $\varphi \to (\psi \to \varphi)$
A2 $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
A3 $((\neg\varphi) \to (\neg\psi)) \to (\psi \to \varphi)$

8*. (1 point) Find a formal deduction in the above Hilbert-style proof system of $(((\neg p) \to (\neg q)) \to q) \to (((\neg p) \to (\neg q)) \to p)$.

9*. Find a formal deduction in the above Hilbert-style proof showing that:
(a) (3 points) $\{\alpha \to \beta, \beta \to \gamma\} \vdash \alpha \to \gamma$,
(b) (3 points) $\{\alpha \to \beta\} \vdash (\beta \to \gamma) \to (\alpha \to \gamma)$,
(c) (3 points) $\vdash (\alpha \to \beta) \to ((\beta \to \gamma) \to (\alpha \to \gamma))$.
(hint: in finding deductions in parts (b) and (c) you can use the deductions from the previous parts and mimic the proof of the deduction theorem)

10*. (3 points) Find a formal deduction in the above Hilbert-style proof system of the following formula $\neg\neg\alpha \to \alpha$.

11*. (3 points) Find a formal deduction in the above Hilbert-style proof system of the following formula $\alpha \to \neg\neg\alpha$.