MATH 318, Assignment 4

Due date: October 28

1. (2 points) Let B be the Lindenbaum–Tarski algebra with three variables p, q, r
   (1) Find all the atoms of B
   (2) How many elements does B have?

2. (2 points)
   (1) Does there exist an infinite Boolean algebra B such that for every nonzero $b \in B$ there is an atom $a \in B$ with $a \leq b$? Justify your answer.
   (2) Does there exist an infinite Boolean algebra which contains exactly one atom? Justify your answer.

A *preorder* is a binary relation that is reflexive and transitive.

3. (2 points) Suppose $\preceq$ is a preorder on a set X. Define the relation $\equiv$ on X by $x \equiv y$ if $x \preceq y$ and $y \preceq x$. Show that $\equiv$ is an equivalence relation. Define $\leq$ on $X/\equiv$ by $[x]_\equiv \leq [y]_\equiv$ if $x \preceq y$. Show that this definition is correct (does not depend on the choice of representatives of equivalence classes) and that $\leq$ is a partial order.

Below, $\mathbb{N}$ stands for the set of natural numbers $\{0, 1, 2, \ldots\}$.

4. (3 points) Consider the following binary relation $|$ on $\mathbb{N}$: $n|m$ if n divides m (i.e. there exists $k \in \mathbb{N}$ such that $m = n \cdot k$).
   (1) Is $|$ a partial order on $\mathbb{N}$?
   (2) Does $(\mathbb{N}, |)$ have the least element?
   (3) Does $(\mathbb{N}, |)$ have the greatest element?
   Justify your answers.

5. (2 points) Consider $P = \{1, 2, 3, 4\}$ and let $|$ be defined on P as in the previous problem: $i|j$ if there exists $k \in \mathbb{N}$ such that $j = k \cdot i$. Find the minimal number n such that there exists chains $C_1, \ldots, C_n$ in P with $P = \bigcup_{i=1}^{n} C_i$. Justify your answer.

6. (1 point) Consider the powerset $P(\{1, 2, 3\})$ with the relation of inclusion $\subseteq$. Find a linear order on $P(\{1, 2, 3\})$ that extends the inclusion relation $\subseteq$.

7. (1 point) Let $P = \{A \subseteq \mathbb{N} : A$ is nonempty, finite and has an even number of elements$\}$. Consider $X = \{A \in P : 1 \in A\}$. Does X have a lower bound in $(P, \subseteq)$? Justify your answer.

8. (2 points) Write $\{0, 1\}^*$ for $\bigcup_{n \in \mathbb{N}} \{0, 1\}^n$. Elements of $\{0, 1\}^*$ are called *words* and subsets of $\{0, 1\}^*$ are called *formal languages* (over the two-element alphabet $\{0, 1\}$). Write ϵ for the empty word (of length 0). Consider the function $f : \mathcal{P}(\{0, 1\}^*) \to \mathcal{P}(\{0, 1\}^*)$ defined as follows:
$$f(X) = X \cup \{w01 : w \in X\} \cup \{\epsilon\}$$
Find the least fixed point of f on $\mathcal{P}(\{0, 1\}^*)$ ordered by inclusion. Justify your answer.

The following problem is for extra credit.

9*. (4 points) Let $f : A \to B$ and $g : B \to A$ be arbitrary functions. Show that there are subsets $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$ such that $A_1 \cup A_2 = A$, $A_1 \cap A_2 = \emptyset$, $B_1 \cup B_2 = B$, $B_1 \cap B_2 = \emptyset$ and
$$f(A_1) = B_1, \quad g(B_2) = A_2.$$
Use this to give an alternative proof of the Schröder–Bernstein theorem.
hint: use the Knaster-Tarski fixed point theorem.