Notes on Control of Quadrotors

Siamak Faal

Last updated on March 3, 2021

Chapter 1

Mathematical Model

A free-body diagram of the quadrotor is illustrated in Figure 1.1. As depicted, the coordinate system $C = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$ is attached to the body of the quadrotor and the fixed reference frame $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ serves as the inertial frame. We let $l > 0$ to denote the distance from each rotor to the center of mass and assume that all the rotors are located on the same plane as the center of mass that coincides with the origin of C. The external force and moment vectors $\mathbf{r} = r_1\,\mathbf{c}_1 + r_2\,\mathbf{c}_2 + r_3\,\mathbf{c}_3$ and $\mathbf{n} = n_1\,\mathbf{c}_1 + n_2\,\mathbf{c}_2 + n_3\,\mathbf{c}_3$ are directly applied to the center of mass. Moreover, we will ignore the rotor dynamics and assume the torque of each propeller is directly related to the input thrust with the proportionality constant $\sigma > 0$, that is $\tau_i = \sigma u_i$ for $i = 1, \ldots, 4$. The position of the quadrotor in the inertial frame E is denoted by $\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3$. Since E is an inertial frame, the velocity of the quadrotor in the reference frame E is

$$\mathbf{v} = \dot{\mathbf{x}} = \sum_{i=1}^{3} \dot{x}_i\,\mathbf{e}_i. \tag{1.1}$$

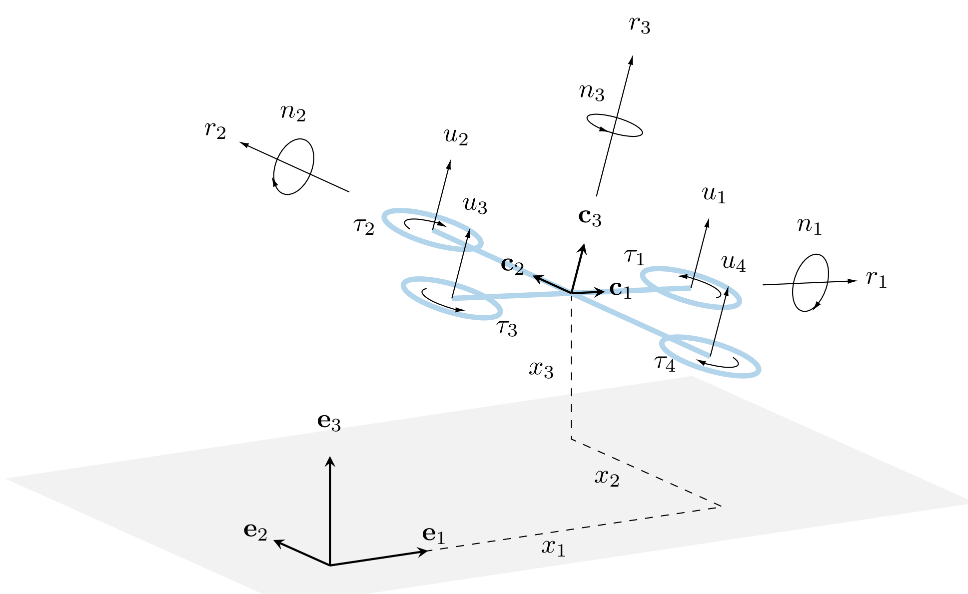

Figure 1.1: Free-body diagram of the quadrotor

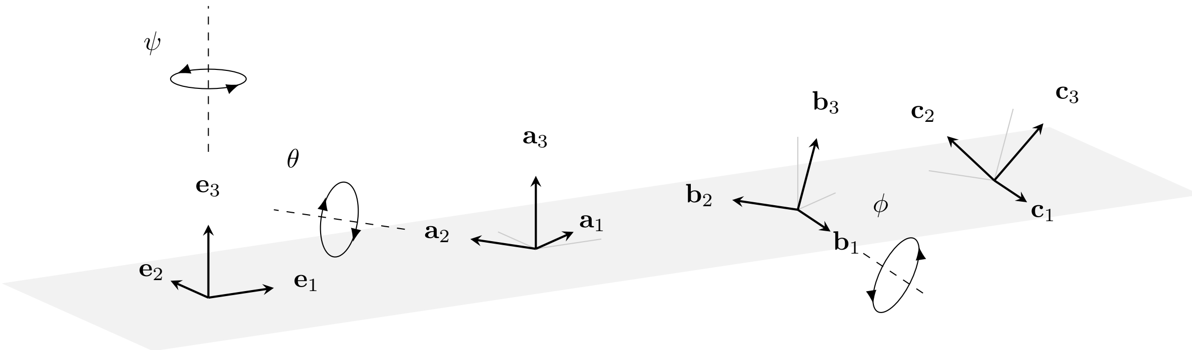

Figure 1.2: An illustration of 3-2-1 (z-y-x) Euler angle convention with the intermediate coordinate systems $A = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $B = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Based on this standard, the orientation of the coordinate frame C with respect to E is obtained by a rotation ψ about $\mathbf{e}_3$, followed by a rotation of θ about $\mathbf{a}_2$ and finally a rotation of ϕ about $\mathbf{b}_1$ axis.

1.1 Rotation Matrix Approach

In this approach, we will use the standard 3-2-1 (z-y-x) Euler angle convention to represent the orientation of the body-fixed frame C in the inertial reference frame E. As depicted in Figure 1.2, we start with a rotation about the $\mathbf{e}_3$ axis by angle ψ, followed by a rotation of θ about $\mathbf{a}_2$ and finally a rotation of ϕ about $\mathbf{b}_1$. The corresponding basic rotation (elemental rotation) matrices are

$$R_3(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}, \; R_1(\phi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}. \tag{1.2}$$

To eliminate ambiguities, we will use the notation $R_{A/B}$ to indicate the rotation matrix that maps a point defined in coordinate frame A to the corresponding point in coordinate frame B. Accordingly we have

$$R_{C/E} = R_3(\psi)R_2(\theta)R_1(\phi) \tag{1.3}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\sin\theta\cos\psi \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\theta\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{bmatrix}. \tag{1.4}$$

Since $R_{B/A} = R_{A/B}^{-1} = R_{A/B}^T$ (by orthogonality of the rotation matrices), we have

$$R_{E/C} = R_{C/E}^T = R_1^T(\phi)R_2^T(\theta)R_3^T(\psi). \tag{1.5}$$

As depicted in Figure 1.1 and equation (1.1), vectors $\mathbf{x}$ and $\mathbf{v} = \dot{\mathbf{x}}$ define the location and velocity of the center of mass of the quadrotor in the inertial frame E, respectively. Let $\boldsymbol{\omega} = \omega_1\,\mathbf{c}_1 + \omega_2\,\mathbf{c}_2 + \omega_3\,\mathbf{c}_3$ define the angular velocity of the quadrotor in the body fixed coordinate C. Then, based on Euler-Newton equations of motion, we have

$$m\,\dot{\mathbf{v}} = -m\,g\,\mathbf{e}_3 + R_{C/E}\,(u_1 + u_2 + u_3 + u_4)\,\mathbf{c}_3 + R_{C/E}\,\mathbf{r}, \tag{1.6a}$$

$$I\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\,\boldsymbol{\omega} = (u_2 - u_4)\,l\,\mathbf{c}_1 + (u_3 - u_1)\,l\,\mathbf{c}_2 + (\tau_1 - \tau_2 + \tau_3 - \tau_4)\,\mathbf{c}_3 + \mathbf{n}. \tag{1.6b}$$

Here $\times$ denotes the cross product, $m > 0$ is the total mass of the quadrotor, g is the gravitational acceleration and the positive definite matrix $I \in \mathbb{R}^{3\times3}$ represents the mass moment of inertia of the quadrotor with respect to coordinate frame C. Before formulating the state space form of the system, we need to write $\boldsymbol{\omega}$ in terms of Euler angle rates $\dot\psi$, $\dot\theta$ and $\dot\phi$. We can do so by mapping the angular rates $\dot\psi\,\mathbf{e}_3$, $\dot\theta\,\mathbf{a}_2$ and $\dot\phi\,\mathbf{b}_1$ to the frame C and set the sum equal to $\boldsymbol{\omega}$. Accordingly we have

$$\boldsymbol{\omega} = R_1^T(\phi)\,\dot\phi\,\mathbf{b}_1 + R_1^T(\phi)\,R_2^T(\theta)\,\dot\theta\,\mathbf{a}_2 + R_1^T(\phi)\,R_2^T(\theta)\,R_3^T(\psi)\,\dot\psi\,\mathbf{e}_3. \tag{1.7}$$

Since $R_1^T(\phi)\,\mathbf{b}_1 = \mathbf{c}_1$, $R_2^T(\theta)\,\mathbf{a}_2 = \mathbf{b}_2$ and $R_3^T(\psi)\,\mathbf{e}_3 = \mathbf{a}_3$, the above equation simplifies to

$$\boldsymbol{\omega} = \dot{\phi}\,\mathbf{c}_1 + R_1^T(\phi)\,\dot{\theta}\,\mathbf{b}_2 + R_1^T(\phi) R_2^T(\theta)\,\dot{\psi}\,\mathbf{a}_3, \tag{1.8}$$

or equivalently

$$\begin{cases} \omega_1 &= \dot{\phi} - \dot{\psi}\sin\gamma, \\ \omega_2 &= \dot{\gamma}\cos\phi + \dot{\psi}\sin\phi\cos\gamma, \\ \omega_3 &= \dot{\psi}\cos\phi\cos\gamma - \dot{\gamma}\sin\phi. \end{cases} \implies \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos\phi & \sin\phi\,\cos\theta \\ 0 & -\sin\phi & \cos\phi\,\cos\theta \end{bmatrix}}_{:=T} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix}. \tag{1.9}$$

That is, $\boldsymbol{\omega} = T\,\dot{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} := [\phi,\, \theta,\, \psi]^T$. Now we can find the rate of change of the Euler angles based on the angular velocity of the quadrotor using the inverse of T as $\dot{\boldsymbol{\alpha}} = T^{-1}\boldsymbol{\omega}$ where

$$T^{-1} = \begin{bmatrix} 1 & \sin\phi\,\tan\theta & \cos\phi\,\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix} \tag{1.10}$$

It is important to note that T^{-1} is defined for $\theta \in (-\pi/2,\, \pi/2)$. Thus, the formulation based on Euler angle standard only works if the motion of the quadrotor is restricted to $\theta \in (-\pi/2,\, \pi/2)$.

1.1.1 State Space Form

Let $\mathbf{z} := [\mathbf{x},\, \boldsymbol{\alpha},\, \mathbf{v},\, \boldsymbol{\omega}]^T$ be a state vector for the quadrotor system, using $\tau_i = \sigma u_i$ we have the corresponding state space form as

$$\dot{\mathbf{x}} = \mathbf{v}, \tag{1.11a}$$

$$\dot{\boldsymbol{\alpha}} = T^{-1}\boldsymbol{\omega}, \tag{1.11b}$$

$$\dot{\mathbf{v}} = -g\,\mathbf{e}_3 + \tfrac{1}{m}\,R_{C/E}\,(u_1 + u_2 + u_3 + u_4)\,\mathbf{c}_3 + \tfrac{1}{m}\,R_{C/E}\,\mathbf{r}, \tag{1.11c}$$

$$\dot{\boldsymbol{\omega}} = I^{-1}\Big((u_2 - u_4)\,l\,\mathbf{c}_1 + (u_3 - u_1)\,l\,\mathbf{c}_2 + (u_1 - u_2 + u_3 - u_4)\,\sigma\,\mathbf{c}_3 + \mathbf{n} - \boldsymbol{\omega} \times I\,\boldsymbol{\omega}\Big). \tag{1.11d}$$

Chapter 2

Linear Control

2.1 Linearization

Let $\mathbf{z}^* = [\mathbf{x}^*, \boldsymbol{\alpha}^*, \mathbf{v}^*, \boldsymbol{\omega}^*]^T = [\mathbf{x}^*, \mathbf{0}, \mathbf{0}, \mathbf{0}]^T$, for any arbitrary $\mathbf{x}^* \in \mathbb{R}^3$, and $\mathbf{u}^* = [1, 1, 1, 1]^T m\, g/4$. First, we show that the point $(\mathbf{z}^*, \mathbf{u}^*) \in \mathbb{R}^{12} \times \mathbb{R}^4$ is an equilibrium point of the quadrotor. Based on (1.11), for $\mathbf{r} = \mathbf{n} = \mathbf{0}$ we have

$$\dot{\mathbf{x}} = \mathbf{v}, \tag{2.1a}$$

$$\dot{\boldsymbol{\alpha}} = T^{-1}\boldsymbol{\omega}, \tag{2.1b}$$

$$\dot{\mathbf{v}} = -g\,\mathbf{e}_3 + \tfrac{1}{m}\,R_{C/E}\,(u_1 + u_2 + u_3 + u_4)\,\mathbf{c}_3 \tag{2.1c}$$

$$\dot{\boldsymbol{\omega}} = I^{-1}\Big((u_2 - u_4)\,l\,\mathbf{c}_1 + (u_3 - u_1)\,l\,\mathbf{c}_2 + (u_1 - u_2 + u_3 - u_4)\,\sigma\,\mathbf{c}_3 - \boldsymbol{\omega} \times I\,\boldsymbol{\omega}\Big). \tag{2.1d}$$

Substituting $\mathbf{v}^* = \mathbf{0}$ in (2.1a) immediately gives $\dot{\mathbf{x}}^* = \mathbf{0}$. Since $\cos(\theta^*) = \cos(0) = 1$, $T^{-1}(\boldsymbol{\alpha}^*)$ is well defined. Accordingly, based on (2.1b) we have $\boldsymbol{\omega}^* = \mathbf{0} \implies \dot{\boldsymbol{\alpha}}^* = \mathbf{0}$. Substituting $\boldsymbol{\alpha}^*$ in $R_{C/E}$ gives $R^*_{C/E} = \mathbf{I}_3$ (the 3×3 identity matrix). Thus, from (2.1c) we get $\dot{v}^*_1 = \dot{v}^*_2 = 0$ and

$$\dot{v}^*_3 = -g + \tfrac{1}{m}\,(u^*_1 + u^*_2 + u^*_3 + u^*_4) = -g + \tfrac{1}{m}(1 + 1 + 1 + 1)\tfrac{m\,g}{4} = 0. \tag{2.2}$$

Substituting $\boldsymbol{\omega}^* = \mathbf{0}$ in (2.1d) gives

$$\dot{\boldsymbol{\omega}}^* = I^{-1}\Big((u^*_2 - u^*_4)\,l\,\mathbf{c}_1 + (u^*_3 - u^*_1)\,l\,\mathbf{c}_2 + (u^*_1 - u^*_2 + u^*_3 - u^*_4)\,\sigma\,\mathbf{c}_3\Big) \tag{2.3}$$

$$= I^{-1}\Big((1-1)\,\tfrac{l\,m\,g}{4}\,\mathbf{c}_1 + (1-1)\,\tfrac{l\,m\,g}{4}\,\mathbf{c}_2 + (1 - 1 + 1 - 1)\,\tfrac{\sigma\,m\,g}{4}\,\mathbf{c}_3\Big) = \mathbf{0}. \tag{2.4}$$

Since $\dot{\mathbf{z}}^* = [\dot{\mathbf{x}}^*, \dot{\boldsymbol{\alpha}}^*, \dot{\mathbf{v}}^*, \dot{\boldsymbol{\omega}}^*]^T = \mathbf{0}$, we can conclude that the point $(\mathbf{z}^*, \mathbf{u}^*)$ is an equilibrium point of the quadrotor system.

In order to find a linear approximation of the quadrotor system, let us define the error, $\mathbf{e}$, and adjusted input, $\mathbf{w}$, as

$$\mathbf{e} := \mathbf{z}_d - \mathbf{z}, \tag{2.5}$$

$$\mathbf{w} := \mathbf{u}_d - \mathbf{u}. \tag{2.6}$$

where $\mathbf{z}_d$ denotes a desired state vector and $\mathbf{u}_d$ is the corresponding input for $\mathbf{z} = \mathbf{z}_d$. Based on Taylor's series, we can write the linear approximation of the state transition function about the point $(\mathbf{z}_d, u_d)$ as

$$\mathbf{f}(\mathbf{z}, \mathbf{u}) \approx \mathbf{f}(\mathbf{z}_d, \mathbf{u}_d) + \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)}(\mathbf{z} - \mathbf{z}_d) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)}(\mathbf{u} - \mathbf{u}_d). \tag{2.7}$$

Substituting (2.7) in the time derivative of the error vector gives

$$\dot{\mathbf{e}} = \dot{\mathbf{z}}_d - \dot{\mathbf{z}} \tag{2.8}$$

$$\approx \mathbf{f}(\mathbf{z}_d, \mathbf{u}_d) - \mathbf{f}(\mathbf{z}_d, \mathbf{u}_d) - \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)} (\mathbf{z} - \mathbf{z}_d) + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)} (\mathbf{u} - \mathbf{u}_d) \tag{2.9}$$

$$= \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)} \mathbf{e} + \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)} \mathbf{w}. \tag{2.10}$$

Let

$$A := \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)}, \quad \text{and} \quad B := \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}_d,\mathbf{u}_d)}, \tag{2.11}$$

we can write (2.10) concisely as

$$\dot{\mathbf{e}} = A\,\mathbf{e} + B\,\mathbf{w}. \tag{2.12}$$

If we take $(\mathbf{z}_d,\, \mathbf{u}_d) = (\mathbf{z}^*, \mathbf{u}^*)$, we find the linear approximation of the quadrotor system about the point $(\mathbf{z}^*, \mathbf{u}^*)$. Assuming $I = \mathrm{diag}(I_{11},\, I_{22},\, I_{33})$, the corresponding A and B matrices for $(\mathbf{z}^*, \mathbf{u}^*)$ are

$$A = \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}^*,\mathbf{u}^*)} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{0}_6 & \mathbf{I}_6 \\ A_{22} & \mathbf{0}_6 \end{bmatrix}, \tag{2.13}$$

$$B = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}\Big|_{(\mathbf{z},\mathbf{u})=(\mathbf{z}^*,\mathbf{u}^*)} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_6 \\ B_2 \end{bmatrix}, \tag{2.14}$$

$$\tag{2.15}$$

where $\mathbf{0}_6$ and $\mathbf{I}_6$ are respectively the 6×6 zero and identity matrices, and

$$A_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & g & 0 \\ 0 & 0 & 0 & -g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{2.16}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/m & 1/m & 1/m & 1/m \\ 0 & l/I_{11} & 0 & -l/I_{11} \\ -l/I_{22} & 0 & l/I_{22} & 0 \\ \sigma/I_{33} & -\sigma/I_{33} & \sigma/I_{33} & -\sigma/I_{33} \end{bmatrix}, \tag{2.17}$$

To check if the linear approximation of the system about $(\mathbf{z}^*, \mathbf{u}^*)$ is controllable, we need to check if the controllability matrix

$$\begin{bmatrix} B & A\,B & A^2\,B & \cdots A^{11}B \end{bmatrix} \in \mathbb{R}^{12\times 48} \tag{2.18}$$

has full row rank. Since $A^k = \mathbf{0}$ for $k > 3$, we only need to check for the first 12 columns of the controllable matrix, which corresponds to the columns formed by B, $A\,B$, $A^2\,B$ and $A^3\,B$. Since these 12 columns are linearly independent, the controllability matrix has full row rank of 12. Thus, we can conclude that the linear approximation of the quadrotor about equilibrium point $(\mathbf{z}^*, \mathbf{u}^*)$ is controllable.