Section 10.1 – Trails, Paths, and Circuits, Day 02

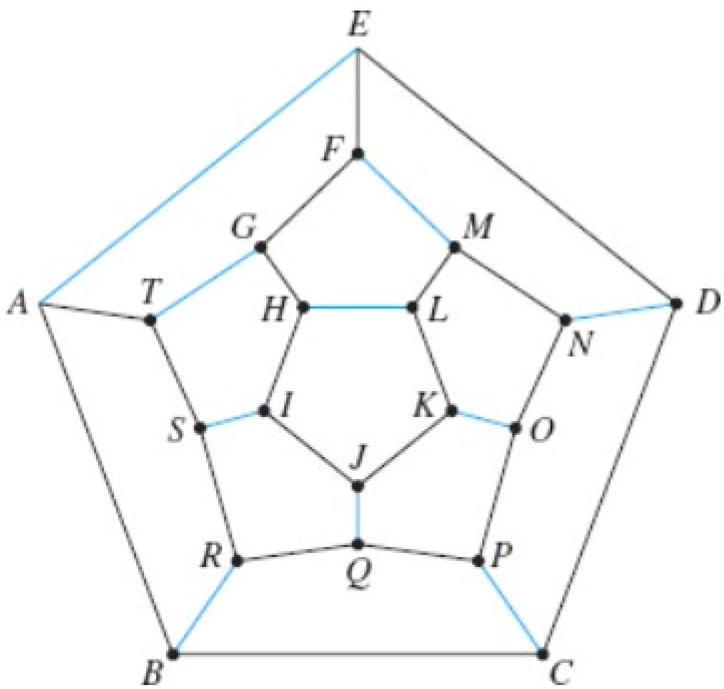
So, last class we spent a lot of time on determining whether graphs have a circuit in which all the ***edges*** appear exactly once. This was called an ***Euler Circuit*** (EC). It turns out this was fairly easy to determine existence, if the graph is connected and all the vertices have even degree, then the graph does have an EC. Actually giving an EC might be more challenging, but it's always easier to do if you know there must be one.

Today, we ask a related question: Is it possible to find a circuit in a graph where all the ***vertices*** (except the first and last) appear exactly once?

Partner up

(1) Can you find a circuit in the graph below where all the ***vertices*** (except the first and last) appear exactly once?

Note that you are not required to use all the edges.



(2) By the way, does this graph have an EC? If so, give the EC, if not, give a reason why not.

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(1) Yes, this does have a circuit satisfying the above conditions:

$A\ B\ C\ D\ E\ F\ G\ H\ I\ J\ K\ L\ M\ N\ O\ P\ Q\ R\ S\ T\ A$

We call this a ***Hamiltonian Circuit*** (HC).

(2) No, this graph does not have an EC, several vertices have odd degree, including A, B, and C.

Given a graph G, a **Hamiltonian circuit** for G is a simple circuit that includes every vertex of G. That is, a Hamiltonian circuit for G is a sequence of adjacent vertices and distinct edges in which every vertex of G appears exactly once, except for the first and the last, which are the same.

Note that although an Euler circuit for a graph G must include every vertex of G, it may visit some vertices more than once and hence may not be a Hamiltonian circuit. On the other hand, a Hamiltonian circuit for G does not need to include all the edges of G and hence may not be an Euler circuit.

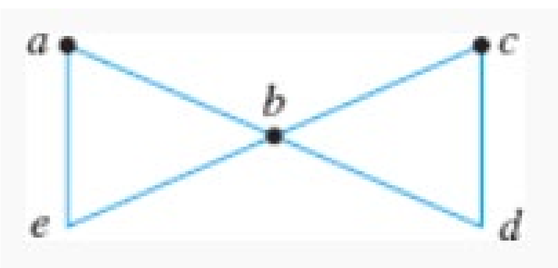
Despite the analogous-sounding definitions of Euler and Hamiltonian circuits, the mathematics of the two are very different. Theorem 10.1.4 gives a simple criterion for determining whether a given graph has an Euler circuit. Unfortunately, there is no analogous criterion for determining whether a given graph has a Hamiltonian circuit, nor is there even an efficient algorithm for finding such a circuit. There is, however, a simple technique that can be used in many cases to show that a graph does *not* have a Hamiltonian circuit. This follows from the following considerations:

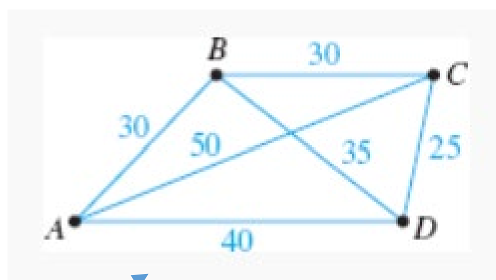
If a graph G has a Hamiltonian circuit, then G has a subgraph H with the following properties:

1. H contains every vertex of G.
2. H is connected.
3. H has the same number of edges as vertices.
4. Every vertex of H has degree 2.

Finding a HC is constructive. Start with just the vertices of the graph. In the HC, each of these vertices will have $deg = 2$, so include the edges that are required, those that are incident on a vertex of $deg = 2$.

So, do the following graphs have HC's? If so, give the HC, and if not, give a reason why:





No, it does not. The edges connecting a and b, a and e, e and b are required, as are the edges connecting c and b, c and d, d and b are also required. So in the HC, vertex b would have deg 4, which isn't allowed.

Yes, A, B, C, D, A is one.

For practical applications of this graph theory, let's consider how HC's help us with the Travelling Salesperson Problem.

In the above graph, the edges are weighted. Suppose a salesperson is to visit each city exactly once, starting and ending at A. Which route will minimize the total distance travelled?

This question is equivalent to asking what is the HC with minimum weight?

Since we are starting and ending at A, and the graph is complete (all vertices have edges to each other vertex), we have 3 choices for the next node (B, C, or D) , then two choices, then one, or $3! = 6$ choices. But we can disregard half of these, they would have the same weight but just travelled in reverse order, like A, B, C, D, A vs. A, D, C, B, A

Route	Total Distance (In Kilometers)
$A\ B\ C\ D\ A$	$30 + 30 + 25 + 40 = 125$
$A\ B\ D\ C\ A$	$30 + 35 + 25 + 50 = 140$
$A\ C\ B\ D\ A$	$50 + 30 + 35 + 40 = 155$