Section 3.2 – Predicates and Quantified Statements II

Prof. Escuadro and Prof. Camenga don't wear glasses. Prof. Roth, Prof. Bukowski, and I wear glasses.

Consider the statement: *All mathematicians at Juniata College wear glasses*.

Is it true? What is its negation?

No mathematicians wear glasses. *or* ***There is at least one mathematician who doesn't wear glasses.***

The original statement is a universal statement. The negation of a universal statement should be existential.

Next, consider the statement: *Some computer programs are difficult to write.*

What is its negation?

There is a computer program that is difficult to write. *or* ***All computer programs are easy to write.***

The original statement is an existential statement. The negation of a universal statement should be existential.

The negation of a statement of the form

$$\forall x \text{ in } D, Q(x)$$

is logically equivalent to a statement of the form

$$\exists x \text{ in } D \text{ such that } \sim Q(x).$$

Symbolically,

$$\sim(\forall x \in D, Q(x)) \equiv \exists x \in D \text{ such that } \sim Q$$

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$$\sim(\exists x \in D \text{ such that } Q(x)) \equiv \forall x \in D, \sim Q(x).$$

Note here that DeMorgan's laws are applied to both the quantifiers and the predicates.

Partner up

Write formal negations for the following statements:

a. $\forall$ primes p, p is odd.

b. $\exists$ a triangle T such that the sum of the angles of T equals $200°$.

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a. By applying the rule for the negation of a $\forall$ statement, you can see that the answer is

$$\exists \text{ a prime } p \text{ such that } p \text{ is not odd.}$$

b. By applying the rule for the negation of a $\exists$ statement, you can see that the answer is

$$\forall \text{ triangles } T\text{, the sum of the angles of } T \text{ does not equal } 200°.$$

Rewrite the following statements formally. Then write formal and informal negations.

a. No politicians are honest.

b. The number 1,357 is not divisible by any integer between 1 and 37.

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a. *Formal version:* $\forall$ politicians x, x is not honest.

Formal negation: $\exists$ a politician x such that x is honest.

Informal negation: Some politicians are honest.

b. This statement has a trailing quantifier. Written formally it becomes:

$$\forall \text{ integer } n \text{ between 1 and 37, 1,357 is not divisible by } n.$$

Its negation is therefore

$$\exists \text{ an integer } n \text{ between 1 and 37 such that 1,357 is divisible by } n.$$

An informal version of the negation is

The number 1,357 is divisible by some integer between 1 and 37.

Write informal negations for the following statements:

(Hint: formally quantify first, negate the formal, and translate back to informal)

a. All computer programs are finite.

b. Some computer hackers are over 40.

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(a) Quantify: $\forall$ *computer programs p, p is finite.*

Negate quantification: $\exists$ *computer programs p s.t. p is not finite.*

Translate back to informal: There is a computer program that is not finite.

Some computer programs are infinite.

(b) Quantify: $\exists$ *computer hacker h s.t. h is over* 40.

Negate quantification: $\forall$ *computer hackers h, h is* 40 *or less.*

Translate back to informal: No computer hackers are over 40.

All computer hackers are 40 or under.

Negations of universal conditional statements are of special importance in mathematics. The form of such negations can be derived from facts that have already been established.

By definition of the negation of a *for all* statement,

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } \sim(P(x) \rightarrow Q(x)).$$

But the negation of an if-then statement is logically equivalent to an *and* statement. More precisely,

$$\sim(P(x) \rightarrow Q(x)) \equiv P(x) \wedge \sim Q(x).$$

Substituting (3.2.2) into (3.2.1) gives

$$\sim(\forall x, P(x) \rightarrow Q(x)) \equiv \exists x \text{ such that } (P(x) \wedge \sim Q(x)).$$

or:

$$\sim(\forall x, \text{if } P(x) \text{ then } Q(x)) \equiv \exists x \text{ such that } P(x) \text{ and } \sim Q(x).$$

Negate each of the following, providing the result in the same form, informal or formal, as the original statement:

$\forall$ *students s, if s has a CS POE then s must take MA* 116.

Answer: $\exists$ *student s s.t. s has a CS POE AND s does NOT need to take MA* 116.

If an exam is difficult, then no one will get a grade of “A.”

Quantify formally first: $\forall$ *exams e, if e is difficult then no one will earn a grade of* "*A.*"

Negate the formal quantification: $\exists$ *exams e s.t. e is difficult AND someone will earn a grade of* "*A.*"

Translate back to informal: There is a difficult exam that someone earned a grade of “A.”

Consider a statement of the form $\forall x \in D$, if $P(x)$ then $Q(x)$.

1. Its **contrapositive** is the statement $\forall x \in D$, if $\sim Q(x)$ then $\sim P(x)$.
2. Its **converse** is the statement $\forall x \in D$, if $Q(x)$ then $P(x)$.
3. Its **inverse** is the statement $\forall x \in D$, if $\sim P(x)$ then $\sim Q(x)$.

The formal version of this statement is $\forall x \in \mathbf{R}$, if $x > 2$ then $x^2 > 4$.

Contrapositive: $\forall x \in \mathbf{R}$, if $x^2 \leq 4$ then $x \leq 2$.

Or: If the square of a real number is less than or equal to 4, then the number is less than or equal to 2.

Converse: $\forall x \in \mathbf{R}$, if $x^2 > 4$ then $x > 2$.

Or: If the square of a real number is greater than 4, then the number is greater than 2.

Inverse: $\forall x \in \mathbf{R}$, if $x \leq 2$ then $x^2 \leq 4$.

Or: If a real number is less than or equal to 2, then the square of

Finally, consider the statement: *All the earrings in my ears have diamonds*.

Is this statement true or false? It would be false if it's negation is true…

Partner up

Quantify formally first: $\forall\ earrings\ x, if\ \ x\ is\ in\ my\ ear\ then\ x\ has\ diamonds.$

Negate the formal quantification: $\exists\ earring\ x\ s.t. x\ is\ in\ my\ ear\ AND\ x\ does\ NOT\ have\ diamonds.$

Translate back to informal: There is an earring in my ear that doesn't have a diamond.

The negation is false, so the original statement must be true. This is a universal statement that is ***vacuously true***.