Section 8.5 – Partial Order Relations

Recall that a relation R on a set A is:

Symmetric	$\Leftrightarrow$	$\forall\, x, y\ \in A, \boldsymbol{if}\ (x, y) \in R\ then\ (y, x) \in R$	We could also write $x\ R\ y$ and $y\ R\ x$
NOT Symmetric	$\Leftrightarrow$	$\exists\, x, y\ \in A\ s.t.\ (x, y) \in R\ and\ (y, x) \notin R$	We could also write $x\ R\ y$ and $y\ \cancel{R}\ x$

We introduce a new property for relations, which is similar to, ***but different from***, Symmetry.

a relation R on a set A is:

Antisymmetric	$\Leftrightarrow$	$\forall\, x, y\ \in A, \boldsymbol{if}\ (x, y) \in R\ and\ (y, x) \in R\ then\ x = y$
NOT Antisymmetric	$\Leftrightarrow$	$\exists\, x, y\ \in A\ s.t.\ (x, y) \in R\ and\ (y, x) \in R\ but\ x \neq y$

There are ***four*** (4) different combinations of Symmetric, Not Symmetric, Antisymmetric, and Not Antisymmetric.

Partner up

Give an arrow diagram of a relation for each of these:

Symmetric Not Antisymmetric	Not Symmetric Antisymmetric	Symmetric Antisymmetric	Not Symmetric Not Antisymmetric

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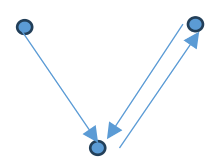
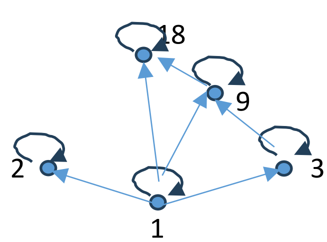
Give an arrow diagram of a relation for each of these:

			
Symmetric Not Antisymmetric	Not Symmetric Antisymmetric	Symmetric Antisymmetric	Not Symmetric Not Antisymmetric

A relation that is Reflexive, Antisymmetric, and Transitive is called a ***Partial Order***.

Two fundamental partial order relations are the "less than or equal to" relation on a set of real numbers and the "subset" relation on a set of sets. These can be thought of as models, or paradigms, for general partial order relations.

Let $A = \{\, 1, 2, 3, 9, 19 \,\}$ and define a relation: $\forall\, x, y\ \in A,\ \ x\ \boldsymbol{R}\ y \quad \Leftrightarrow \quad x\ divides\ y$



"3 divides 9" because $\frac{9}{3}$ has no remainder.

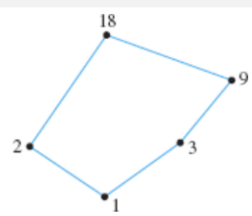
This relation is Reflexive, Antisymmetric, and Transitive, and so it is a Partial Order.

Note that there is a loop at every vertex, all other arrows point in the same direction (upward), and any time there is an arrow from one point to a second and from the second point to a third, there is an arrow from the first point to the third. Given any partial order relation defined on a finite set, it is possible to draw the directed graph in such a way that all of these properties are satisfied. This makes it possible to associate a somewhat simpler graph, called a **Hasse diagram** (after Helmut Hasse, a twentieth-century German number theorist), with a partial order relation defined on a finite set. To obtain a Hasse diagram, proceed as follows:

Start with a directed graph of the relation, placing vertices on the page so that all arrows point upward. Then eliminate

1. the loops at all the vertices,
2. all arrows whose existence is implied by the transitive property, and
3. the direction indicators on the arrows.

For the relation given previously, the Hasse diagram is as follows:



How would the following names be sorted in a dictionary?

Johnsonbaugh
Johns
Johnson
Johnsen

More generally, if A is any set with a partial order relation, then a *dictionary* or *lexicographic* order can be defined on a set of strings over A as indicated in the following theorem.

Johns

Johnsen

Johnson

Johnsonbaugh

Theorem 8.5.1 **Lexicographic Order**

Let A be a set with a partial order relation R, and let S be a set of strings over A. Define a relation $\preceq$ on S as follows:

Let s and t be any strings in S of lengths m and n, respectively, where m and n are positive integers, and let s_m and t_m be the characters in the mth position for s and t, respectively.

1. If $m \leq n$ and the first m characters of s and t are the same, then $s \preceq t$.
2. If the first $m-1$ characters in s and t are the same, $s_m \; R \; t_m$, and $s_m \neq t_m$, then $s \preceq t$.
3. If λ is the null string then $\lambda \preceq s$.

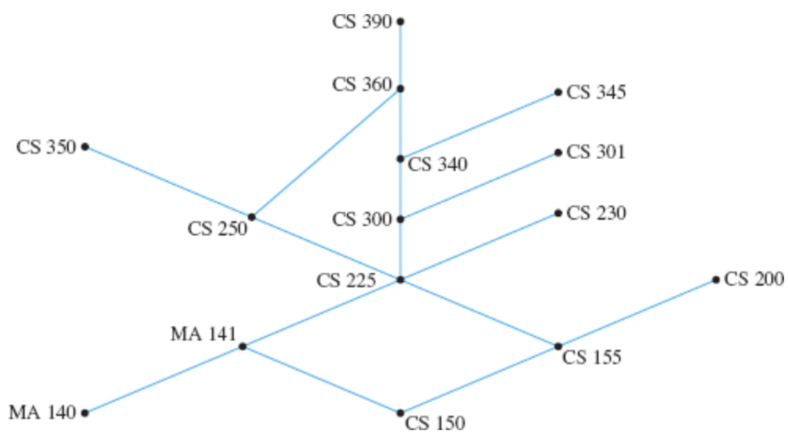
If no strings are related by $\preceq$ other than by these three conditions, then $\preceq$ is a partial order relation on S.

The underlying partial order, $a < b < c \cdots < y < z$ is what gives the rules for sorting.

Besides sorting, Partial Orders and Hasse Diagrams are used in scheduling and logistics. What's the minimum number of semesters required to complete the CS POE at Juniata, assuming all courses are offered each semester and there are no restrictions on the total number of credits per semester?

> To return to the example that introduced this section, note that the following defines a partial order relation on the set of courses required for a university degree: For all required courses x and y,
>
> $$x \preceq y \quad \Leftrightarrow \quad x = y \quad \text{or} \quad x \text{ is a prerequisite for } y.$$
>
> If the Hasse diagram for the relation is drawn, then the questions raised at the beginning of this section can be answered easily. For instance, consider the Hasse diagram for the requirements at a particular university, which is shown in Figure 8.5.1.

Figure 8.5.1



The minimum number of school terms needed to complete the requirements is the size of a longest chain, which is 7 (150, 155, 225, 300, 340, 360, 390, for example). The maximum number of courses that could be taken in the same term (assuming the university allows it) is the maximum number of noncomparable courses, which is 6 (350, 360, 345, 301, 230, 200, for example). A part-time student could take the courses in a sequence determined by constructing a topological sorting for the set. (One such sorting is 140, 150, 141, 155, 200, 225, 230, 300, 250, 301, 340, 345, 350, 360, 390. There are many others.)

Try this for the CS POE at Juniata!