Section 9.2 – Possibility Trees and the Multiplication Rule

Partner up

How many different outfits are there if you choose ***one*** of each item of clothing: (a) shirts, (b) pants, and (c) shoes

Shirts:

Home / Apparel / Supreme / T-Shirts / Supreme Box Logo L/S Tee White

Nike Tennessee Volunteers Peyton Manning #16 Game Football Jersey

Pants:

Levi's Men's 550 Relaxed Fit Jeans

Balenciaga Raver Denim

LOOSE FIT WASHED DUCK DOUBLE-FRONT UTILITY WORK PANT

Shoes:

Nike Air Force 1 '07

SAMBA SHOE

Do we add? 2 shirts + 3 pants + 2 shoes = 7 outfits

Or do we multiply? 2 shirts * 3 pants * 2 shoes = 12 outfits

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For each of the three pairs of pants, there are two pairs of shoes to choose from.

	Nike	Adidas
Levis	1	2
Balenciaga	3	4
Carhartt	5	6

So, we have 6 = 3*2 pants – shoes combinations.

Then, for both of the shirts, we can match them up with the 6 pants – shoes combinations, which gives us 2*3*2 = 12

All the outfits:

(1)	Supreme-Box-Logo	Levis	Nike
(2)	Supreme-Box-Logo	Levis	Adidas
(3)	Supreme-Box-Logo	Balenciaga	Nike
(4)	Supreme-Box-Logo	Balenciaga	Adidas
(5)	Supreme-Box-Logo	Carhartt	Nike
(6)	Supreme-Box-Logo	Carhartt	Adidas
(7)	Football Jersey	Levis	Nike
(8)	Football Jersey	Levis	Adidas
(9)	Football Jersey	Balenciaga	Nike
(10)	Football Jersey	Balenciaga	Adidas
(11)	Football Jersey	Carhartt	Nike
(12)	Football Jersey	Carhartt	Adidas

Theorem 9.2.1 The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in n_1 ways,
the second step can be performed in n_2 ways *[regardless of how the first step was performed]*,

$\vdots$

the kth step can be performed in n_k ways *[regardless of how the preceding steps were performed]*,
then the entire operation can be performed in $n_1 n_2 \cdots n_k$ ways.

To apply the multiplication rule, think of the objects you are trying to count as the output of a multistep operation. The possible ways to perform a step may depend on how preceding steps were performed, but the *number* of ways to perform each step must be constant regardless of the action taken in prior steps.

Independent Events occur when one event remains unaffected by the occurrence of the other event.

Independent Events imply the Multiplication rule.

Partner up

A certain personal identification number (PIN) is required to be a sequence of any four symbols chosen from the 26 uppercase letters in the Roman alphabet and the ten digits.

a. How many different PINs are possible if repetition of symbols is allowed?

b. How many different PINs are possible if repetition of symbols is not allowed?

c. What is the probability that a PIN does not have a repeated symbol assuming that all PINs are equally likely?

Note that "Picking the next character in an identifier doesn't depend on the previous character.*"
* - Identifiers with no repeated characters still are independent, there are just fewer choices.

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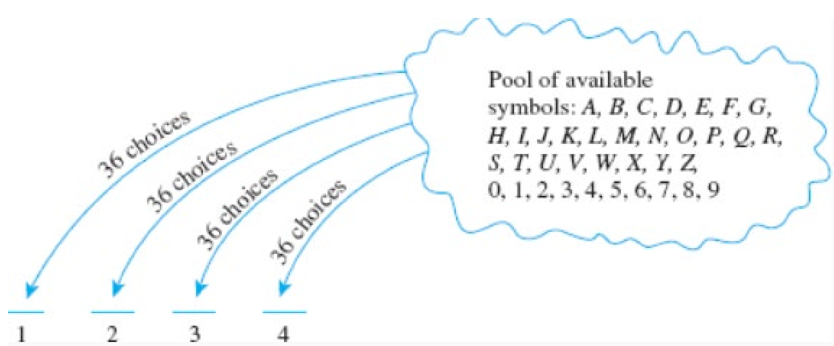
(a)



There is a fixed number of ways to perform each step, namely 36, regardless of how preceding steps were performed. And so, by the multiplication rule, there are $36 \cdot 36 \cdot 36 \cdot 36 = 36^4 = 1{,}679{,}616$ PINs in all.

(b) Forming the PIN is a four-step operation: choosing the first symbol, then the second, then the third, and finally the fourth.

There are 36 ways to choose the first symbol, 35 to choose the second symbol, 34 to choose the third, and 33 to choose the fourth, so applying the Multiplication Rule gives: $36 \cdot 35 \cdot 24 \cdot 33 = 1{,}413{,}720$.

(c) We've done all the hard work, so figuring out the Probability is straightforward, $\frac{N(E)}{N(S)} = \frac{1{,}413{,}720}{1{,}679{,}616} \cong 0.8417$. So about 84% of the PINs do not have a repeated symbol.

A **permutation** of a set of objects is an ordering of the objects in a row. For example, the set of elements a, b, and c has six permutations.

$$abc \quad acb \quad cba \quad bac \quad bca \quad cab$$

For any integer n with $n \geq 1$, the number of permutations of a set with n elements is $n!$.

And in the example above, with 3 elements, no repeats and order matters, there are $3! = 6$ permutations.

Another way to say that there are "no repeats and order matters" is to ask "how many ways can the elements be arranged in a row?" For example:

How many ways can the letters in the word *COMPUTER* be arranged in a row?

All eight letters in the word *COMPUTER* are distinct, so the number of ways in which you can arrange the letters equals the number of permutations of a set of eight elements. This equals $8! = 40{,}320$.

In our original example with the PINs, no repeats and order matters, the result looked like the beginning of a permutation: $36 \cdot 35 \cdot 24 \cdot 33 = \frac{36!}{32!} = \frac{36 \cdot 35 \cdot 34 \cdot 33 \cdot 32!}{32!}$. This is called an ***r-permutation***:

An **r-permutation** of a set of n elements is an ordered selection of r elements taken from the set of n elements. The number of r-permutations of a set of n elements is denoted $\boldsymbol{P(n, r)}$.

$$P(n, r) = \frac{n!}{(n-r)!}$$

Note, this is ***NOT*** a combination, $C(n, r) = \binom{n}{r}$, which we presented earlier and will discuss again.

So, $P(5,2) = \frac{5!}{(5-2)!} = \frac{5 \cdot 4 \cdot 3!}{3!} = 5 \cdot 4 = 20$

and the number of 4-permutations of a set of 7 objects is $P(7,4) = \frac{7!}{(7-4)!} = \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 7 \cdot 6 \cdot 5 \cdot 4 = 840$

As a recap, we have the following table of formulas:

	Order Matters
Repetition Is Allowed	n^k
Repetition Is Not Allowed	$P(n, k)$