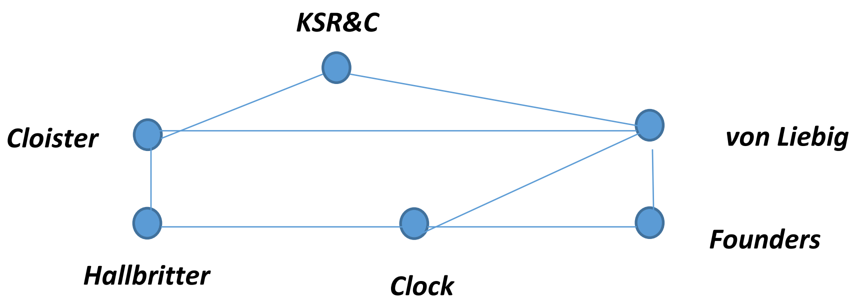
Section 10.1 – Trails, Paths, and Circuits, Day 01

Partner up

Here's a portion of a map of the Juniata Campus:

Let's represent with a graph, where the nodes are buildings/structures and the edges are walkways connecting buildings/structures.



Can you take a walk, starting and ending at the same node, using each edge (sidewalk) once?

Can you take a walk, starting and ending at a ***different*** node, using each edge (sidewalk) once?

===

Can you take a walk, starting and ending at the same node, using each edge (sidewalk) once?

No, but why not?

Can you take a walk, starting and ending at a ***different*** node, using each edge (sidewalk) once?

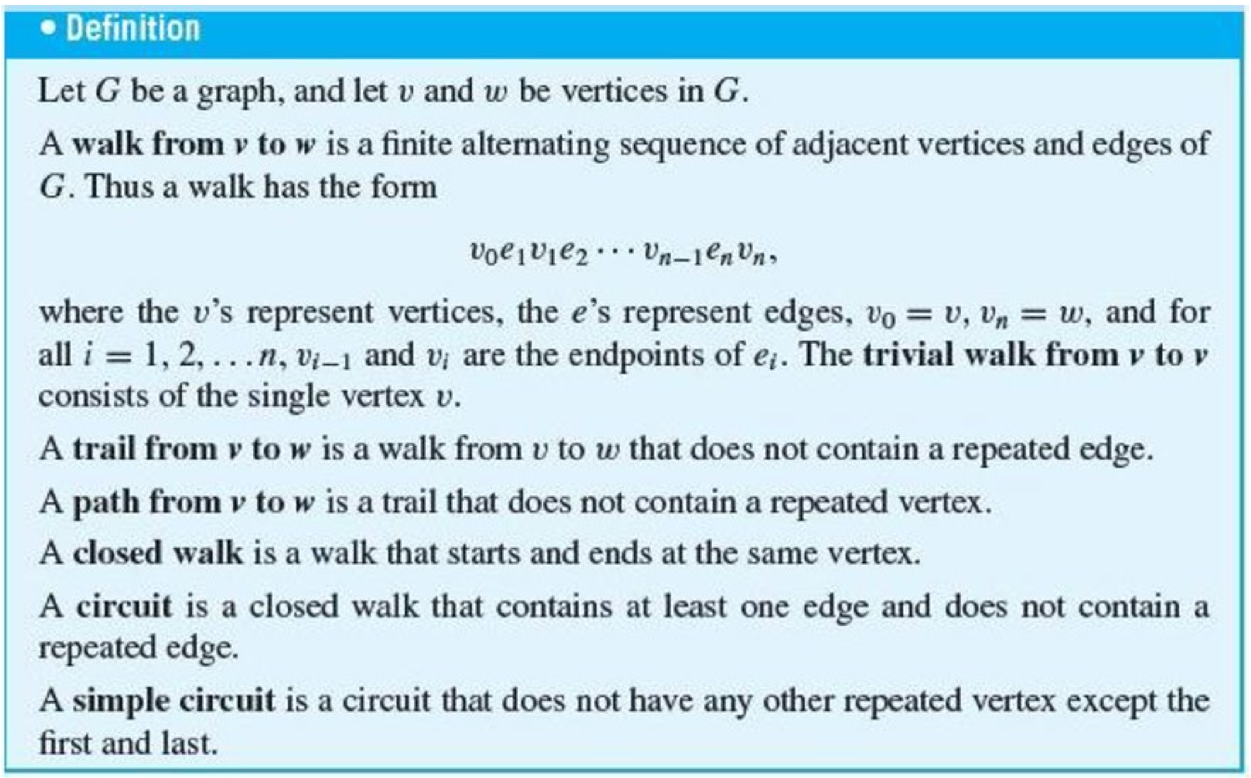
Yes, Clock → vLB → Founders → Clock → Halbritter → Cloister → vLB → KSR&C → Cloister

What is special about the Clock node and Cloister node compared to the other nodes?

It seems that the degree of the vertices matter. To get a circuit, stopping and starting at the same node, vertices of even degree are required. And the Clock and Cloister nodes have an odd degree, so they could be the starting and stopping nodes in a Trail (not a Circuit).

The problem here is a Juniata-specific version of the first published graph theory problem, from the great mathematician Euler, on why it wasn't possible for a person in Königsberg to take a walk around town, starting and ending at the same location and crossing each of the seven bridges in the town exactly once.

Let’s establish some definitions before we return to our campus circuit problem.



• Definition

Let G be a graph, and let v and w be vertices in G.

A **walk from v to w** is a finite alternating sequence of adjacent vertices and edges of G. Thus a walk has the form

$$v_0 e_1 v_1 e_2 \cdots v_{n-1} e_n v_n,$$

where the v’s represent vertices, the e’s represent edges, $v_0 = v$, $v_n = w$, and for all $i = 1, 2, \ldots n$, v_{i-1} and v_i are the endpoints of e_i. The **trivial walk from v to v** consists of the single vertex v.

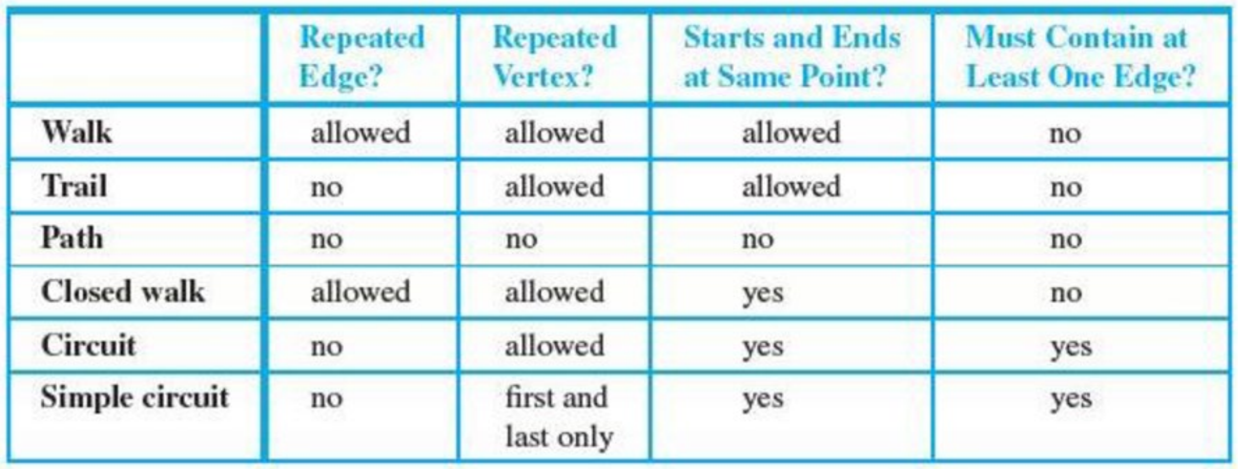
A **trail from v to w** is a walk from v to w that does not contain a repeated edge.

A **path from v to w** is a trail that does not contain a repeated vertex.

A **closed walk** is a walk that starts and ends at the same vertex.

A **circuit** is a closed walk that contains at least one edge and does not contain a repeated edge.

A **simple circuit** is a circuit that does not have any other repeated vertex except the first and last.



	Repeated Edge?	Repeated Vertex?	Starts and Ends at Same Point?	Must Contain at Least One Edge?
Walk	allowed	allowed	allowed	no
Trail	no	allowed	allowed	no
Path	no	no	no	no
Closed walk	allowed	allowed	yes	no
Circuit	no	allowed	yes	yes
Simple circuit	no	first and last only	yes	yes

In the graph below, determine which of the following walks are trails, paths, circuits, or simple circuits.

a. $v_1 e_1 v_2 e_3 v_3 e_4 v_3 e_5 v_4$

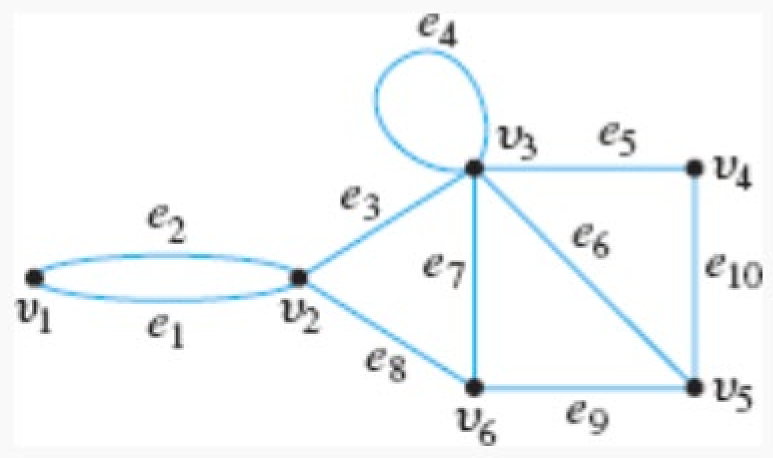
b. $e_1 e_3 e_5 e_5 e_6$

c. $v_2 v_3 v_4 v_5 v_3 v_6 v_2$

d. $v_2 v_3 v_4 v_5 v_6 v_2$

e. $v_1 e_1 v_2 e_1 v_1$

f. v_1



a. This walk has a repeated vertex but does not have a repeated edge, so it is a trail from v_1 to v_4 but not a path.

b. This is just a walk from v_1 to v_5. It is not a trail because it has a repeated edge.

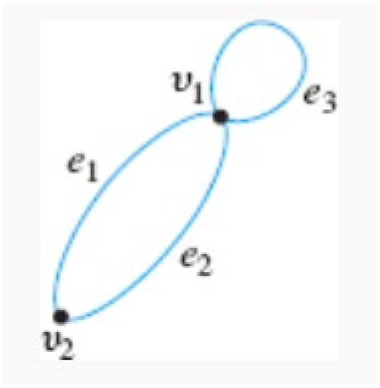
c. This walk starts and ends at v_2, contains at least one edge, and does not have a repeated edge, so it is a circuit. Since the vertex v_3 is repeated in the middle, it is not a simple circuit.

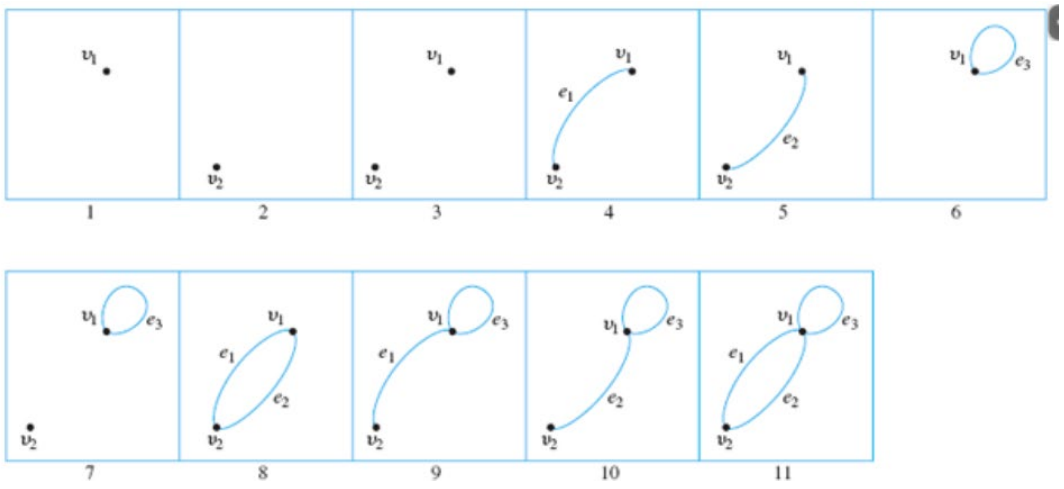
d. This walk starts and ends at v_2, contains at least one edge, does not have a repeated edge, and does not have a repeated vertex. Thus it is a simple circuit.

e. This is just a closed walk starting and ending at v_1. It is not a circuit because edge e_1 is repeated.

f. The first vertex of this walk is the same as its last vertex, but it does not contain an edge, and so it is not a circuit. It is a closed walk from v_1 to v_1. (It is also a trail from v_1 to v_1.)

Give all the subgraphs of the following graph:





Let G be a graph. Two **vertices v and w of G are connected** if, and only if, there is a walk from v to w. The **graph G is connected** if, and only if, given *any* two vertices v and w in G, there is a walk from v to w. Symbolically:

$$G \text{ is connected} \quad \Leftrightarrow \quad \forall \text{ vertices } v \text{ and } w \text{ in } G, \exists \text{ a walk from } v \text{ to } w.$$

If two vertices are connected by an edge, the edge is said to be ***incident*** on the vertices.

Now that we've defined some graph theory terms, we can return to our original problem.

Let G be a graph. An **Euler circuit** for G is a circuit that contains every vertex and every edge of G. That is, an Euler circuit for G is a sequence of adjacent vertices and edges in G that has at least one edge, starts and ends at the same vertex, uses every vertex of G at least once, and uses every edge of G exactly once.

And with the following theorem we can explain why we aren't able to make a circuit, using each edge exactly once.

If a graph has an Euler circuit, then every vertex of the graph has positive even degree.

The contrapositive of this statement is:

If every vertex of a graph does not have positive, even degree, then the graph does not have an Euler Circuit.

or

If some vertex of a graph has odd degree, then the graph does not have an Euler circuit.

And we can conclude:

If a graph G is connected and the degree of every vertex of G is a positive even integer, then G has an Euler circuit.

Finally,

Let G be a graph, and let v and w be two distinct vertices of G. An **Euler trail from v to w** is a sequence of adjacent edges and vertices that starts at v, ends at w, passes through every vertex of G at least once, and traverses every edge of G exactly once.

orollary 10.1.5

Let G be a graph, and let v and w be two distinct vertices of G. There is an Euler trail from v to w if, and only if, G is connected, v and w have odd degree, and all other vertices of G have positive even degree.

In our original problem, the Clock and Cloister nodes were the only two nodes with odd degree. Since the graph was connected, we use the above Corollary to show that there was an ***Euler Trail*** in that graph.