CS 315 - Day 06, Induction

Example predicates:

$$P(x): x\ is\ in\ CS\ 315$$

$$R(y): y = \frac{3 \cdot y}{3}$$

What are $R(1)$, $R(2)$, $R(3)$, $R(k)$, and $R(k+1)$?

What are $P(Joe)$, $P(Jen)$, and $P(Jer)$?

It's on us to prove whether these predicates are correct! Just stating a predicate doesn't mean it's true!

Discuss how Gauss might have derived his formula, $1 + 2 + \cdots + n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

$$\underbrace{1 + \overbrace{2 + 3}^{5} + 4}_{5} = 5 \cdot 2$$

$$(4+1) \cdot \frac{4}{2} = \frac{4 \cdot (4+1)}{2}$$

$$\overbrace{1 + \underbrace{2 + \underbrace{3 + 4}_{7} + 5}_{7} + 6}^{7} = 7 \cdot 3$$

$$= (6+1) \cdot \frac{6}{2}$$

$$= \frac{6(6+1)}{2}$$

Guess

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

n odd?

$$\overbrace{1 + \underbrace{2 + 3 + 4}_{6} + 5}^{6} = 6 \cdot 2 + 3$$

$$= 2 \cdot 3 \cdot 2 + 3$$

$$= 4 \cdot 3 + 1 \cdot 3$$

$$= 3(5)$$

$$= \frac{6}{2}(5)$$

$$= \frac{5 \cdot 6}{2}$$

$$1 + 2 + 3 = 4 \cdot 1 + 2 \cdot 1$$

$$= 2 \cdot 2 \cdot 1 + 2 \cdot 1$$

$$= 2(2+1)$$

$$= \frac{4}{2}(2+1)$$

$$= \frac{3 \cdot 4}{2}$$

Review the Inductive Proof of Gauss' formula, and note the quantified predicate:

$P(n):\quad 1+2+\cdots+n=\frac{n(n+1)}{2}$, for all integers $n\geq 1$.

<u>Basis Step [Show ***P(1)*** is true]</u>

LHS of ***P(1)*** is simply 1 RHS of ***P(1)*** is $\frac{1(1+1)}{2}=\frac{1(2)}{2}=\frac{2}{2}=1$

So the LHS and RHS of ***P(1)*** are the same, so the Basis Step is true.

<u>Inductive Step [Assume ***P(k)*** is true, and use it to show ***P(k+1)*** is true]</u>

Assume $P(k):\quad 1+2+\cdots+k=\frac{k(k+1)}{2}$ is true.

Show $P(k+1):\quad 1+2+\cdots+k+\ (k+1)=\frac{(k+1)(k+2)}{2}$ is true.

Start with the LHS of $P(k+1)$, $1+2+\cdots+k+\ (k+1)$.

We can substitute what we assumed from of $P(k)$,

$$1+2+\cdots+k+\ (k+1)=\frac{k(k+1)}{2}+\ (k+1)$$

We can factor out a $\frac{(k+1)}{2}$, which gives $\frac{(k+1)}{2}(k+2)=\frac{(k+1)(k+2)}{2}$.

This is the RHS of of $P(k+1)$, so we manipulated the LHS of $P(k+1)$ into the RHS.

Q.E.D.

<u>In-class w/***assigned groups of four (4)***, write 0. and 1. on board as ground rule(s):</u>

0. No devices.
1. Intro: Name, POE, Icebreaker.
2. Pick someone on the group who will report out.
3. Determine a formula for the sum of odd integers:

$$1=1$$
$$1+3=4$$
$$1+3+5=9$$
$$1+3+5+7=16$$

Do you notice a pattern?

What do you think $1+3+5+\cdots+(2n-1)$ would be equal to?

Why did we use $2n-1$?

Use Induction to prove

$P(n)$: $1 + 3 + 5 + \cdots + (2n - 1) = n^2$, for all integers $n \geq 1$.

<u>Basis Step [Show ***P(1)*** is true]</u>
LHS of ***P(1)*** is simply 1   RHS of ***P(1)*** is $1^2 = 1$
So the LHS and RHS of ***P(1)*** are the same, so the Basis Step is true.

<u>Inductive Step [Assume ***P(k)*** is true, and use it to show ***P(k+1)*** is true]</u>
Assume $P(k)$: $1 + 2 + \cdots + (2k - 1) = k^2$ is true.

Show $P(k + 1)$: $1 + 2 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$ is true.

Start with the LHS of $P(k + 1)$, $1 + 2 + \cdots + (2k - 1) + (2k + 1)$.
We can substitute what we assumed from of $P(k)$,

$$1 + 2 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$$

Which if we factor (think FOIL) gives $(k + 1)(k + 1) = (k + 1)^2$.
This is the RHS of of $P(k + 1)$, so we manipulated the LHS of $P(k + 1)$ into the RHS.
Q.E.D.

So far, the Predicates and their associated Inductive proofs have been "mechanical." In this class, our applications of Induction will be more intuitive.

Identify the predicate to prove
A group of students are in a classroom, and each shakes the hand of all the other students present.

$\boldsymbol{P(n)}$: ***n students in class will result in*** $\boldsymbol{\frac{n(n-1)}{2}}$ ***handshakes, for at least*** $\boldsymbol{n = 2}$ ***students.***

Inductive proof

Basis Step [***<u>Show P(2) is true</u>***]

If two people come to a meeting, there will only be one handshake.
<u>LHS</u>   ***<u>RHS</u>***
The formula is $\frac{n(n-1)}{2} = \frac{2(2-1)}{2} = \quad 1$
So the formula agrees with what actually happens.

Inductive Step [***<u>Assume P(k) is true, use it to show P(k+1) is true</u>***]
Assume: if *k* people come to the meeting, then $\frac{k(k-1)}{2}$ handshakes occur is true.
We will show: if *k+1* people come to the meeting, then $\frac{(k+1)((k+1)-1)}{2} = \frac{(k+1)k}{2}$ handshakes occur. ***(subst)***
We know that if *k* people come to the meeting, then $\frac{k(k-1)}{2}$ handshakes occur. If the *k+1*st person comes to the meeting, there will be *k* more handshakes. So we have:

$$\frac{k(k-1)}{2} + k = \frac{k(k-1)}{2} + \frac{2k}{2} = \frac{k(k-1)+2k}{2} = \frac{k(k-1)+2k}{2} = \frac{k^2-k+2k}{2} = \frac{k^2+k}{2} = \frac{(k+1)k}{2} \qquad \textit{\textbf{(adding k)}}$$

Q.E.D.