Section 8.2 – Reflexivity, Symmetry, and Transitivity

Let $A = \{1, 2, 4, 6\}$ and define a relation R on A as follows:

$\forall\, x, y \in A,\ x\, R\, y \iff x - y$ is even.

Partner up

Draw an arrow diagram of this relation, and the set of ordered-pairs for this relation.

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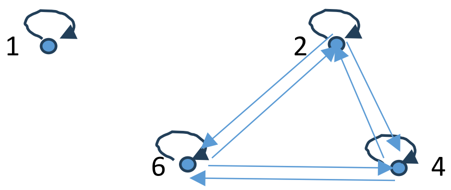


$$R = \{\, (1,1), (2,2), (4,4), (6,6), (2,6), (6,2), (2,4), (4,2), (4,6), (6,4)\, \}$$

What properties do you notice about this graph?

1. Each point of the graph has an arrow looping around from it and going back to it.
2. In each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first.
3. In each case where there is an arrow going from one point to a second and from the second point to a third, there is an arrow going from the first point to the third. That is, there are no "incomplete directed triangles" in the graph.

Properties (1), (2), and (3) correspond to properties of general relations called *reflexivity, symmetry*, and *transitivity*.

Which leads to the formal mathematical definitions:

Let R be a relation on a set A.

1. R is **reflexive** if, and only if, for every $x \in A$, $x\, R\, x$.
2. R is **symmetric** if, and only if, for every $x, y \in A$, if $x\, R\, y$ then $y\, R\, x$.
3. R is **transitive** if, and only if, for every $x, y, z \in A$, if $x\, R\, y$ and $y\, R\, z$ then $x\, R\, z$.

or equivalently,

1. R is reflexive $\Leftrightarrow$ for every x in A, $(x, x) \in R$.
2. R is symmetric $\Leftrightarrow$ for every x and y in A, ***if*** $(x, y) \in R$ then $(y, x) \in R$.
3. R is transitive $\Leftrightarrow$ for every x, y, and z in A, ***if*** $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

In class transitivity example with shirt (or shoe) colors.

It’s important to note that the definitions for symmetry and transitivity are conditional statements, so they can be vacuously true.

The “first,” “second,” and “third” elements in the informal versions need not all be distinct. This is a disadvantage of informality: It may mask nuances that a formal definition makes clear.

Each of these definitions is a universal statement.

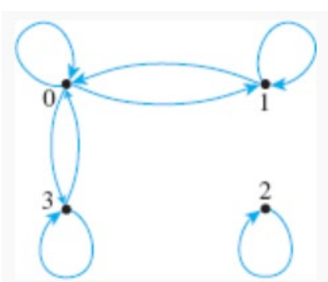
1. **Reflexive:** Each element is related to itself.
2. **Symmetric:** If any one element is related to any other element, then the second element is related to the first.
3. **Transitive:** If any one element is related to a second and that second element is related to a third, then the first element is related to the third.

Now consider what it means for a relation *not* to have one of the properties defined previously. Recall that the negation of a universal statement is existential. Hence if R is a relation on a set A, then

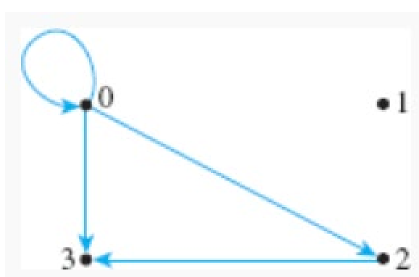
1. R is **not reflexive** $\Leftrightarrow$ there is an element x in A such that $x \not R\, x$ *[that is, such that* $(x, x) \notin R$*].*
2. R is **not symmetric** $\Leftrightarrow$ there are elements x and y in A such that $x \, R \, y$ but $y \not R\, x$ *[that is, such that* $(x, y) \in R$ *but* $(y, x) \notin R$*].*
3. R is **not transitive** $\Leftrightarrow$ there are elements x, y, and z in A such that $x \, R \, y$ and $y \, R \, z$ but $x \not R\, z$ *[that is, such that* $(x, y) \in R$ *and* $(y, z) \in R$ *but* $(x, z) \notin R$*].*

Partner up

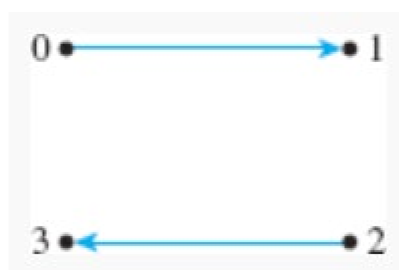
Consider the following three relations:



Relation R



Relation S



Relation T

Determine whether each is: ***Reflexive*** ***Symmetric*** ***Transitive*** If not, give a counter-example.

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R is reflexive: There is a loop at each point of the directed graph. This means that each element of A is related to itself, so R is reflexive.

R is symmetric: In each case where there is an arrow going from one point of the graph to a second, there is an arrow going from the second point back to the first. This means that whenever one element of A is related by R to a second, then the second is related to the first. Hence R is symmetric.

R is not transitive: There is an arrow going from 1 to 0 and an arrow going from 0 to 3, but there is no arrow going from 1 to 3. This means that there are elements of A—0, 1, and 3—such that $1\ R\ 0$ and $0\ R\ 3$ but $1 \not R\ 3$. Hence R is not transitive.

S is not reflexive: There is no loop at 1, for example. Thus $(1,1) \notin S$, and so S is not reflexive.

S is not symmetric: There is an arrow from 0 to 2 but not from 2 to 0. Hence $(0,2) \in S$ but $(2,0) \notin S$, and so S is not symmetric.

S is transitive: There are three cases for which there is an arrow going from one point of the graph to a second and from the second point to a third. In particular, there are arrows going from 0 to 2 and from 2 to 3; there are arrows going from 0 to 0 and from 0 to 2; and there are arrows going from 0 to 0 and from 0 to 3. In each case there is an arrow going from the first point to the third. (Note again that the "first," "second," and "third" points need not be distinct.) This means that whenever $(x,y) \in S$ and $(y,z) \in S$, then $(x,z) \in S$, for every $x, y, z \in \{0,1,2,3\}$, and so S is transitive.

T is not reflexive: There is no loop at 0, for example. Thus $(0,0) \notin T$, so T is not reflexive.

T is not symmetric: There is an arrow from 0 to 1 but not from 1 to 0. Thus $(0,1) \in T$ but $(1,0) \notin T$, and so T is not symmetric.

T is transitive: The transitivity condition is vacuously true for T. To see

Note that T is transitive by default because it is *not* ***not*** transitive…. That's a classic example of "vacuously true."

We also have relations on infinitely large sets. While we can't show them with an arrow diagram, we can still determine if they are Reflexive, Symmetric, or Transitive.

Define a relation R on the set of real numbers $\mathbb{R}$ as follows: For all real numbers x and y$\mathbb{R}$,

$$x\ R\ y \iff x < y$$

Is R: ***Reflexive*** ***Symmetric*** ***Transitive***

a. ***R is not reflexive***: R is reflexive if, and only if, $\forall x \in \mathbf{R}$, $x\ R\ x$. By definition of R, this means that $\forall x \in \mathbf{R}$, $x < x$. But this is false: $\exists x \in \mathbf{R}$ such that $x \not< x$. As a counterexample, let $x = 0$ and note that $0 \not< 0$. Hence R is not reflexive.

b. ***R is not symmetric***: R is symmetric if, and only if, $\forall x, y \in \mathbf{R}$, if $x\ R\ y$ then $y\ R\ x$. By definition of R, this means that $\forall x, y \in \mathbf{R}$, if $x < y$ then $y < x$. But this is false: $\exists x, y \in \mathbf{R}$ such that $x < y$ and $y \not< x$. As a counterexample, let $x = 0$ and $y = 1$ and note that $0 < 1$ but $1 \not< 0$. Hence R is not symmetric.

c. ***R is transitive***: R is transitive if, and only if, $\forall x, y, z \in \mathbf{R}$, if $x\ R\ y$ and $y\ R\ z$ then $x\ R\ z$. By definition of R, this means that $\forall x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$, then $x < z$. But this statement is true by the transitive law of order for real numbers (Appendix A, T18). Hence R is transitive.

Generally speaking, a relation fails to be transitive because it fails to contain certain ordered pairs. For example, if $(1, 3)$ and $(3, 4)$ are in a relation R, then the pair $(1, 4)$ *must* be in R if R is to be transitive. To obtain a transitive relation from one that is not transitive, it is necessary to add ordered pairs. Roughly speaking, the relation obtained by adding the least number of ordered pairs to ensure transitivity is called the *transitive closure* of the relation. More precisely, the transitive closure of a relation is the smallest transitive relation that contains the relation.

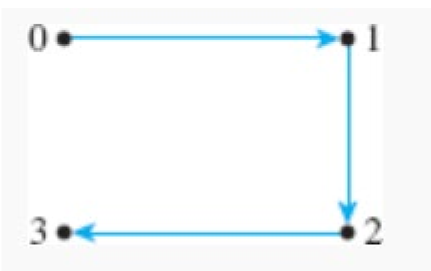
So, we can add ordered-pairs to make a relation transitive.

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.

Partner up

Find the transitive closure of the graph below:



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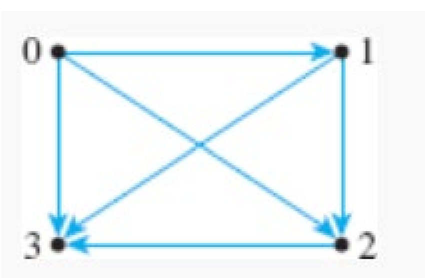

The transitive closure contains all the edges/ordered-pairs of the original relation.

$$R^t = \{\ (0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\ \}$$

Note that a  b is ***NOT*** transitive. The "shortcut" from a to a, given a to b and b to a (and similarly from b to b)

But this a  b ***IS*** transitive.