Section 6.1 – Set Theory

$$A = \underset{\text{the set of all}}{\{}x \in S \underset{\text{such that}}{\,|\,} P(x)\},$$

which is read "A is the set of all x in S such that P of x."

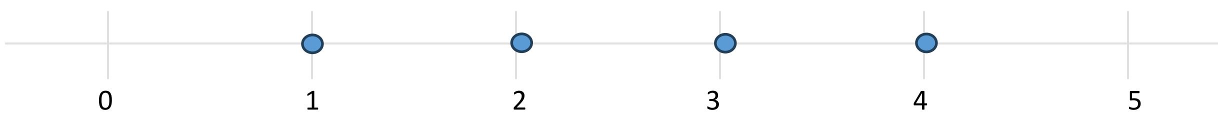
The ***Axiom of Extension*** states that a set is completely determined by its elements. Repeats don't matter, and neither does order.

$$\{\,1, 2, 3\,\} = \{\,3, 2, 1, 3, 3, 1, 2\}$$

Remember we have the ***set-roster notation***, as above, where we list each element, and ***set-builder notation*** where we give the instructions for determining elements in the set.

$$\{1, 2, 3, 4\} = \{n \in \mathbb{Z} \mid 1 \le n \le 4\}$$

We could represent the above on a number line as well:



We have subsets, the definition:

$$A \subseteq B \quad \Leftrightarrow \quad \forall x,\ \text{if } x \in A \text{ then } x \in B$$

and the negation of that definition:

$$A \nsubseteq B \quad \Leftrightarrow \quad \exists x \text{ such that } x \in A \text{ and } x \notin B$$

and ***proper subsets***:

Recall that a *proper subset* of a set is a subset that is not equal to its containing set. That is:

A is a **proper subset** of B $\quad \Leftrightarrow$

(1) $A \subseteq B$, and

(2) there is at least one element in B that is not in A.

Partner up

Let $A = \{1\}$ and $B = \{1, \{1\}\}$.

a. Is $A \subseteq B$?

b. If so, is A a proper subset of B?

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a. Because $A = \{1\}$, A has only one element—namely, the symbol 1. This element is also one of the elements in set B. Hence every element in A is in B, and so $A \subseteq B$.

b. B has two distinct elements, the symbol 1 and the set $\{1\}$ whose only element is 1. Since $1 \neq \{1\}$, the set $\{1\}$ is not an element of A, and so there is an element of B that is not an element of A. Hence A is a proper subset of B.

Recall that by the axiom of extension, sets A and B are equal if, and only if, they have exactly the same elements. We restate this as a definition that uses the language of subsets.

Definition

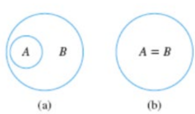
Given sets A and B, A **equals** B, written $\boldsymbol{A = B}$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically:

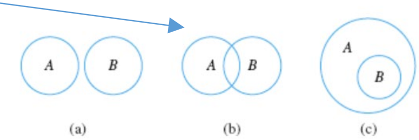
$$A = B \quad \Leftrightarrow \quad A \subseteq B \text{ and } B \subseteq A.$$

Venn Diagrams are helpful to visualize these definitions:

$A \subseteq B$



$A \nsubseteq B$

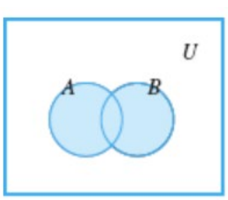


Let A and B be subsets of a universal set U.

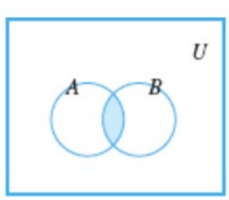
1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.
2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.
3. The **difference** of B minus A (or **relative complement** of A in B), denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.
4. The **complement** of A, denoted $\boldsymbol{A^c}$, is the set of all elements in U that are not in A.

Symbolically:

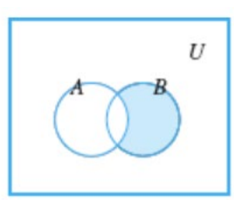
$$A \cup B = \{x \in U \mid x \in A \text{ or } x \in B\}$$
$$A \cap B = \{x \in U \mid x \in A \text{ and } x \in B\}$$
$$B - A = \{x \in U \mid x \in B \text{ and } x \notin A\}$$
$$A^c = \{x \in U \mid x \notin A\}.$$



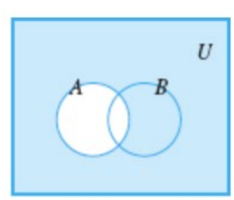
Shaded region represents $A \cup B$.



Shaded region represents $A \cap B$.



Shaded region represents $B - A$.



Shaded region represents A^c.

Consider the following set definitions:

$U = \{1,2,3,4,5,6\}$ $\quad A = \{1,2,3\}$ $\quad B = \{4,5\}$ $\quad C = \{1,2,5,6\}$

$A^c = \{4, 5, 6\}$ ***Note here that the superscript "c" is the complement (it's lowercase), not the set C (which is uppercase).***

$A \cup B = \{1, 2, 3, 4, 5\}$

$A \cap C = \{1, 2\}$

$A - C = A \cap C^c = \{3\}$

$C - A = C \cap A^c = \{5,6\}$

Partner up

Given the three sets: $A = \{1, 2, 3, 4, 5\}$ $\quad B = \{2, 4, 6, 8, 10\}$ $\quad C = \{3, 5, 9, 10\}$

with the Universe: $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

(1) Write a Venn Diagram representing these sets.

(2) Find expressions, if they exist, which are equal to the following sets using only intersection ∩, union ∪, and set difference, −

$\{6, 8\}$

$\{1, 2, 4\}$

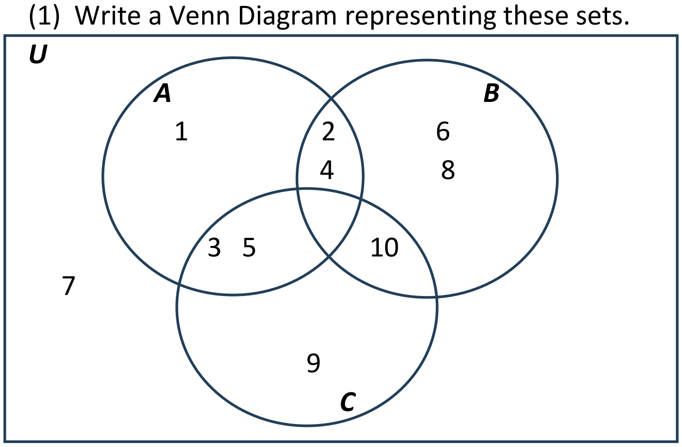
$\{1, 2, 3, 4, 5, 6, 8, 10\}$

$\{1, 3, 5, 7, 9\}$

$\{2, 3, 4, 5, 9, 10\}$

===

(1) Write a Venn Diagram representing these sets.



(2) Find expressions, if they exist, which are equal to the following sets using only intersection ∩, union ∪, and set difference, −

$\{6, 8\} = B - A - C = B \cap A^c \cap C^c$

$\{1, 2, 4\} = A - C$

$\{1, 2, 3, 4, 5, 6, 8, 10\} = A \cup B$

$\{1, 3, 5, 7, 9\} = B^c$

$\{2, 3, 4, 5, 9, 10\} = C \cup (A \cap B)$

Using our previous definitions of the set of Integers, $\mathbb{Z}$, the set of Rational numbers, $\mathbb{Q}$, and the set of Real numbers, $\mathbb{R}$:

The set with no elements is called the ***empty set***, and represented with the symbol $\emptyset$.

Describe the following sets.

a. $D = \{x \in \mathbf{R} \mid 3 < x < 2\}$.

b. $E = \{x \in \mathbf{Z} \mid 2 < x < 3\}$.

Solution:

a. Recall that $a < x < b$ means that $a < x$ and $x < b$. So D consists of all real numbers that are both greater than 3 and less than 2. Since there are no such numbers, D has no elements and thus $D = \emptyset$.

b. E is the set of all integers that are both greater than 2 and less than 3. Since no integers satisfy this condition, E has no elements, and so $E = \emptyset$.

Two sets are called **disjoint** if, and only if, they have no elements in common.

Symbolically:

The sets $A = \{1,3,5\}$ and $B = \{2,4,6\}$ are disjoint since $A \cap B = \emptyset$

$$A \text{ and } B \text{ are disjoint} \quad \Leftrightarrow \quad A \cap B = \emptyset.$$

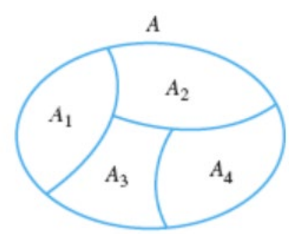
Sets $A_1, A_2, A_3, \ldots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) if, and only if, no two sets A_i and A_j with distinct subscripts have any elements in common. More precisely, for all integers i and $j = 1, 2, 3, \ldots$

$$A_i \cap A_j = \emptyset \quad \text{whenever } i \neq j.$$

A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3, \ldots\}$ is a **partition** of a set A if, and only if,

a. A is the union of all the A_i;

b. the sets $A_1, A_2, A_3, \ldots$ are mutually disjoint.



Given a set A, the **power set** of A, denoted $\boldsymbol{P(A)}$, is the set of all subsets of A.

So, the power set of $\{x, y\} = P(\{x, y\}) = \{\emptyset, \{x\}, \{y\}, \{x, y\}\}$