Section 7.2 – One-to-One, Onto, and Inverse Functions

Partner up

When ***N*** eggs in a basket are removed 5 at a time, ultimately no eggs remain in the basket.

But when ***N*** eggs in a basket are removed 2 at a time, then 1 remains in the basket.

If with ***N*** eggs in a basket, 4 are removed at a time, then 3 remain.

What is the smallest possible value of ***N***?

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Mathematically, when we consider remainders, we use the ***mod*** function, which uses the % operator. It gives the remainder when ***N*** is divided by ____ .

In the problem, $N \% 5 = 0$, $N \% 2 = 1$, and $N \% 4 = 3$.

So from the first two statements, we know N is an odd multiple of 5.

5	15	25	35	…
$5 \% 4 = 1$	$15 \% 4 = 3$	$25 \% 4 = 1$	$35 \% 4 = 3$	

So the smallest value of ***N*** which works is 15. What is the next value of ***N*** which works? 35. It seems it repeats every 20.

We encounter the mod function frequently, and we will see some of its uses in class today.

Today we will introduce properties that some, but not all functions possess.

Let F be a function from a set X to a set Y. F is **one-to-one** (or **injective**) if, and only if, for all elements x_1 and x_2 in X,

$$\text{if } F(x_1) = F(x_2), \text{ then } x_1 = x_2,$$

or, equivalently,

$$\text{if } x_1 \neq x_2, \text{ then } F(x_1) \neq F(x_2).$$

Symbolically:

$$F: X \to Y \text{ is one-to-one} \Leftrightarrow \forall x_1, x_2 \in X, \text{if } F(x_1) = F(x_2) \text{ then } x_1 = x_2.$$

$$\text{A function } F: X \to Y \text{ is } \textit{not} \text{ one-to-one} \quad \Leftrightarrow \quad \exists \text{ elements } x_1 \text{ and } x_2 \text{ in } X \text{ with } F(x_1) = F(x_2) \text{ and } x_1 \neq x_2.$$

In other words, if elements x_1 and x_2 can be found that have the same function value but are not equal, then F is not one-to-one.

Figure 7.2.1(a)

A One-to-One Function Separates Points

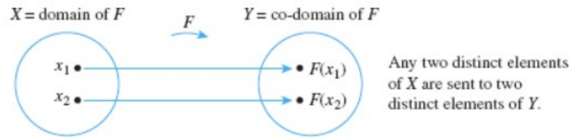
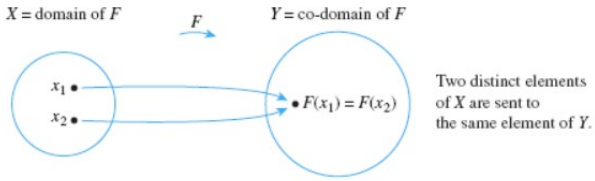


Figure 7.2.1(b)

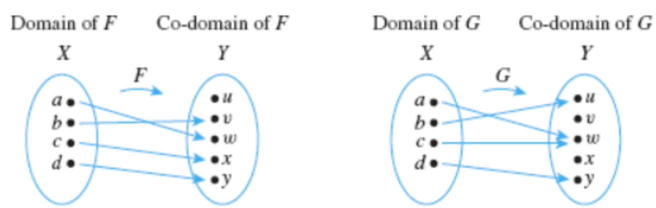
A Function That Is Not One-to-One Collapses Points Together



Partner up

a. Do either of the arrow diagrams in Figure 7.2.2 define one-to-one functions?

Figure 7.2.2



b. Let $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d\}$. Define $H: X \to Y$ as follows: $H(1) = c$, $H(2) = a$, and $H(3) = d$. Define $K: X \to Y$ as follows: $K(1) = d$, $K(2) = b$, and $K(3) = d$. Is either H or K one-to-one?

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a. F is one-to-one but G is not. F is one-to-one because no two different elements of X are sent by F to the same element of Y. G is not one-to-one because the elements a and c are both sent by G to the same element of Y: $G(a) = G(c) = w$ but $a \neq c$.

b. H is one-to-one but K is not. H is one-to-one because each of the three elements of the domain of H is sent by H to a different element of the co-domain: $H(1) \neq H(2)$, $H(1) \neq H(3)$, and $H(2) \neq H(3)$. K, however, is not one-to-one because $K(1) = K(3) = d$ but $1 \neq 3$.

Imagine a university with 10,000 students each with a nine-digit ID number, which the university plans to link to student records. Placing the record with ID number n in position n of an array would waste computer memory space because only a small fraction of the billion possible nine-digit ID numbers are needed for the 10,000 students.

Definition: Hash Function

A **hash function** is a function defined from a larger, possibly infinite, set of data to a smaller fixed-size set of integers.

To make it efficient for the university to store the records, a hash function is needed that

(1) is one-to-one and

(2) has a co-domain that is very much smaller than one billion.

Most hash functions are modifications of *mod* functions and are defined using prime numbers to increase the chance that their values will be scattered rather than clustered together. In addition, making their co-domains 50% to 100% larger than their domains makes it more likely that they will be one-to-one. Nonetheless, two input values may **collide**, that is, have the same output value, and various methods are used to avoid such a **collision**. One of these is illustrated in the following very much simplified example to address the university's situation.

Instead of 10,000 students, suppose there are only 6. Define a function H, from the set of student ID numbers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ as follows:

$$H(n) = n \text{ } mod \text{ } 11 \text{ for each ID number } n.$$

	0
356633102	1
223799061	2
	3
	4
	5
	6
513408716	7
328343419	8
	9
	10

Let F be a function from a set X to a set Y. F is **onto** (or **surjective**) if, and only if, given any element y in Y, it is possible to find an element x in X with the property that $y = F(x)$.

Symbolically:

$$F\colon X \to Y \text{ is onto} \quad \Leftrightarrow \quad \forall y \in Y, \exists x \in X \text{ such that } F(x) = y.$$

To obtain a precise statement of what it means for a function *not* to be onto, take the negation of the definition of onto:

$$F\colon X \to Y \text{ is } \textit{not} \text{ onto} \quad \Leftrightarrow \quad \exists y \text{ in } Y \text{ such that } \forall x \in X, F(x) \neq y.$$

That is, there is some element in Y that is *not* the image of *any* element in X.

In terms of arrow diagrams, a function is onto if each element of the co-domain has an arrow pointing to it from some element of the domain. A function is not onto if at least one element in its co-domain does not have an arrow pointing to it. This is illustrated in Figures 7.2.3(a) and 7.2.3(b).

Figure 7.2.3(a)

A Function That Is Onto

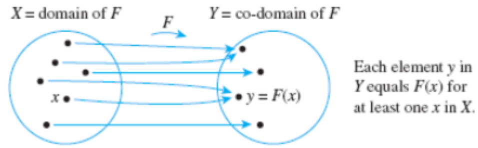
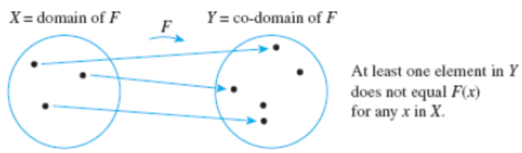


Figure 7.2.3(b)

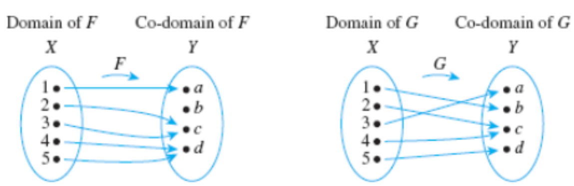
A Function That Is Not Onto



Partner up

a. Do either of the arrow diagrams in Figure 7.2.4 define onto functions?

Figure 7.2.4



b. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define $H: X \to Y$ as follows: $H(1) = c, H(2) = a, H(3) = c, H(4) = b$. Define $K: X \to Y$ as follows: $K(1) = c, K(2) = b, K(3) = b$, and $K(4) = c$. Is either H or K onto?

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a. F is not onto because $b \neq F(x)$ for any x in X. G is onto because each element of Y equals $G(x)$ for some x in X: $a = G(3), b = G(1), c = G(2) = G(4)$, and $d = G(5)$.

b. H is onto but K is not. H is onto because each of the three elements of the co-domain of H is the image of some element of the domain of H: $a = H(2), b = H(4)$, and $c = H(1) = H(3)$. K, however, is not onto because $a \neq K(x)$ for any x in $\{1, 2, 3, 4\}$.

Theorem 7.2.1 **Properties of Logarithms**

For any positive real numbers b, c, x, and y with $b \neq 1$ and $c \neq 1$ and for every real number a:

a. $\log_b(xy) = \log_b x + \log_b y$

b. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$

c. $\log_b(x^a) = a \log_b x$

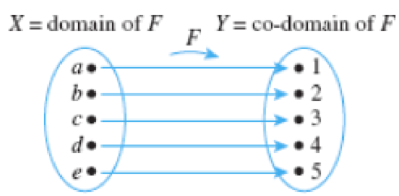
d. $\log_c x = \frac{\log_b x}{\log_b c}$

Typical calculators only have logarithms for base 10 or base ***e***. We can use property (d) to calculate logarithms on a calculator which have different bases. For example, $log_2 5 = \frac{log_{10} 5}{log_{10} 2} \cong 2.322$.

And $\quad log_8 27 = \frac{log_2 27}{log_2 8} = \frac{log_2 3^3}{log_2 2^3} = \frac{3 \cdot log_2 3}{3 \cdot log_2 2} = \frac{log_2 3}{1} = log_2 3$

A function that is both 1-1 and onto is called a ***1-1 correspondence*** or a ***bijection***. These functions are invertible.

An Arrow Diagram for a One-to-One Correspondence



If F is a one-to-one correspondence from a set X to a set Y, then there is a function from Y to X that "undoes" the action of F; that is, it sends each element of Y back to the element of X that it came from. This function is called the *inverse function* for F.

Theorem 7.2.2

Suppose $F: X \to Y$ is a one-to-one correspondence; in other words, suppose F is one-to-one and onto. Then there is a function $F^{-1}: Y \to X$ that is defined as follows: Given any element y in Y,

$$F^{-1}(y) = \text{that unique element } x \text{ in } X \text{ such that } F(x) \text{ equals } y.$$

Or, equivalently,

$$F^{-1}(y) = x \quad \Leftrightarrow \quad y = F(x).$$

F^{-1} is called the inverse function of F.