Section 7.1 – Functions Defined on General Sets

In everyday language, we use the term function to indicate that one varying quantity depends on another.

Here's a fun example:

We discussed functions earlier in the class, but now take the opportunity to define them more completely:

A **function f from a set X to a set Y**, denoted $f: X \to Y$, is a relation from X, the **domain** of f, to Y, the **co-domain** of f, that satisfies two properties:

(1) every element in X is related to some element in Y, and

(2) no element in X is related to more than one element in Y.

Thus, given any element x in X, there is a unique element in Y that is related to x by f. If we call this element y, then we say that "f sends x to y" or "f maps x to y" and write $x \xrightarrow{f} y$ or $f: x \to y$. The unique element to which f sends x is denoted

$f(x)$ and is called **f of x**, or
the output of f for the input x, or
the value of f at x, or
the image of x under f.

The set of all values of f taken together is called the *range of f* or the *image of X under f*. Symbolically:

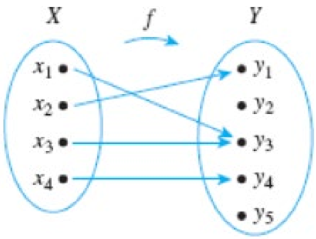
$$\textbf{range of } f = \textbf{image of } X \textbf{ under } f = \{y \in Y \mid y = f(x), \text{ for some } x \text{ in } X\}.$$

Note that the ***range*** of a function is the set of attained values, and it is a subset (they could be equal) of the ***co-domain***. The range is sometimes referred to as the ***image of X*** (the domain) ***under f***.

We use $f(x)$ to refer to the values of the function named f at x.

f is the name of the function, $f(x)$ is the function value.

Where ***X*** is the domain, and ***Y*** is the co-domain, we have an arrow diagram:



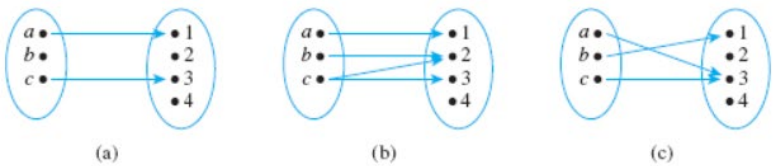
This arrow diagram does define a function because:

1. Every element of X has an arrow that points to an element in Y.
2. No element of X has two arrows that point to two different elements of Y.

Partner up

Which of the arrow diagrams in Figure 7.1.2 define functions from $X = \{a, b, c\}$ to $Y = \{1, 2, 3, 4\}$?

Figure 7.1.2



For the arrow diagram(s) above which define functions, give the:

a. Write the domain and co-domain of f.

b. Find $f(a)$, $f(b)$, and $f(c)$.

c. What is the range of f?

d. Is c an inverse image of 2? Is b an inverse image of 3?

e. Find the inverse images of 2, 4, and 1.

f. Represent f as a set of ordered pairs.

===

Only (c) defines a function. In (a) the element b in X is not related to any element of Y because there is no arrow that points from b to an element in Y. And in (b) the element c is not related to a *unique* element of Y because from c there are two arrows that point to two different elements of Y—one toward 2 and the other toward 3.

a. domain of $f = \{a, b, c\}$, co-domain of $f = \{1, 2, 3, 4\}$

b. $f(a) = 2$, $f(b) = 4$, $f(c) = 2$

c. range of $f = \{2, 4\}$

d. yes, no

e. inverse image of $2 = \{a, c\}$

inverse image of $4 = \{b\}$

inverse image of $1 = \emptyset$ (*since no arrows point to* 1)

f. $\{(a, 2), (b, 4), (c, 2)\}$

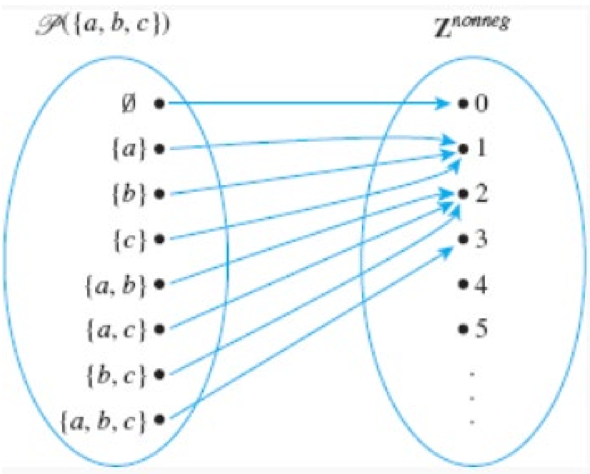
Partner up

Recall from Section 6.1 that $P(A)$ denotes the set of all subsets of the set A. Define a function $F: P(\{a, b, c\}) \to \mathbf{Z}^{nonneg}$ as follows: For each $X \in P(\{a, b, c\})$,

$$F(X) = \text{the number of elements in } X.$$

Draw an arrow diagram for F.

==



Let b be a positive real number with $b \neq 1$. For each positive real number x, the **logarithm with base b of x**, written $\log_b x$, is the exponent to which b must be raised to obtain x. Symbolically:

The **logarithmic function with base b** is the function from $\mathbf{R}^+$ to $\mathbf{R}$ that takes each positive real number x to $\log_b x$.

It is written $log_b x = y \Leftrightarrow b^y = x$

Partner up

Find the following:

a. $\log_3 9$

b. $\log_2 \left(\frac{1}{2}\right)$

c. $\log_{10}(1)$

d. $\log_2(2^m)$ (m is any real number)

e. $2^{\log_2(m)}$ $(m > 0)$

==

a. $\log_3 9 = 2$ because $3^2 = 9$.

b. $\log_2 \left(\frac{1}{2}\right) = -1$ because $2^{-1} = \frac{1}{2}$.

c. $\log_{10}(1) = 0$ because $10^0 = 1$.

d. $\log_2(2^m) = m$ because the exponent to which 2 must be raised to obtain 2^m is m.

e. $2^{\log_2(m)} = m$ because $\log_2(m)$ is the exponent to which 2 must be raised to obtain m.