Section 3.4 – Arguments with Quantified Statements

Those of you with some previous Object-Oriented programming experience probably have encountered Class definitions and Objects.

For example, we could write code, maybe in Python or Java, to define a general class for Automobiles, and identify several attributes that are common for all autos, like:

- Number of wheels
- Number of doors
- Mileage on the odometer
- Color
- Etc…

Then, we ***instantiate*** a specific type of auto, like a Honda Civic, Toyota Highlander, Ford F-150, and so on.

In symbolic logic, particularly in deductive reasoning, universal instantiation is the fundamental tool.

Universal Instantiation

If a property is true of *everything* in a set, then it is true of *any particular* thing in the set.

Use of the words *universal instantiation* indicates that the truth of a property in a particular case follows as a special instance of its more general or universal truth. The validity of this argument form follows immediately from the definition of truth values for a universal statement. One of the most famous examples of universal instantiation is the following:

All men are mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

While most of the universal arguments won't be written as universal conditional statements, we know that they are equivalent:

All men are mortal is equivalent to *If a person is a man, then that man is mortal.*

The first universal argument is:

Universal Modus Ponens

	Formal Version	*Informal Version*
Major premise	$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
Minor premise	$P(a)$ for a particular a.	a makes $P(x)$ true.
Conclusion	$\therefore Q(a)$.	$\therefore$ a makes $Q(x)$ true.

Rewrite the following argument formally (using quantifiers, variables, and predicates):

If an integer is even, then its square is even.
k is a particular integer that is even.
$\therefore$ k^2 is even.

$\forall$ *integers* x, *if* x *is even then* x^2 *is even*

$\forall$ *integers* x, *if* $E(x)$ *then* $S(x)$
$E(k)$
$\therefore S(k)$

As you can see, this argument form is valid, since it is Modus Ponens, which from now on we will just abbreviate to ***MP***.

Universal Modus Tollens

	Formal Version	Informal Version
Major premise	$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
Minor premise	$\sim Q(a)$, for a particular a.	a does not make $Q(x)$ true.
Conclusion	$\therefore \sim P(a)$.	$\therefore$ a does not make $P(x)$ true.

Rewrite the following argument formally (using quantifiers, variables, and predicates):

All human beings are mortal.
Zeus is not mortal.
$\therefore$ Zeus is not human.

$\forall$ x, *if* x *is human then* x *is mortal*

$\forall$ x, *if* $H(x)$ *then* $M(x)$
$\sim M(Z)$
$\therefore \sim H(Z)$

MT is the heart of proof by contradiction.

But since a universal conditional statement is not logically equivalent to its converse, such a replacement cannot, in general, be made. We say that this argument exhibits the converse error.

Converse Error (Quantified Form)

Formal Version	Informal Version
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$Q(a)$ for a particular a.	a makes $Q(x)$ true.
$\therefore P(a)$. $\leftarrow$ invalid conclusion	$\therefore$ a makes $P(x)$ true. $\leftarrow$ invalid conclusion

The following form of argument would be valid if a conditional statement were logically equivalent to its inverse. But it is not, and the argument form is invalid. We say that it exhibits the inverse error. You are asked to show the invalidity of this argument form in the exercises at the end of this section.

Inverse Error (Quantified Form)

Formal Version	*Informal Version*
$\forall x$, if $P(x)$ then $Q(x)$.	If x makes $P(x)$ true, then x makes $Q(x)$ true.
$\sim P(a)$, for a particular a.	a does not make $P(x)$ true.
$\therefore \sim Q(a)$. $\leftarrow$ invalid conclusion	$\therefore$ a does not make $Q(x)$ true. $\leftarrow$ invalid conclusion

A few of the assigned homework problems have arguments with "No." Is the following a valid argument form?

No polynomial functions have horizontal asymptotes.
This function has a horizontal asymptote.
$\therefore$ This function is not a polynomial function.

An alternative way to solve Example 3.4.7 is to transform "No polynomial functions have horizontal asymptotes" into the equivalent statement "$\forall x$, if x is a polynomial function, then x does not have a horizontal asymptote." If this is done, the argument can be seen to have the form

$\forall x$, if $P(x)$ then $Q(x)$.
$\sim Q(a)$, for a particular a.
$\therefore \sim P(a)$.

where $P(x)$ is "x is a polynomial function" and $Q(x)$ is "x does not have a horizontal asymptote." This is valid by universal modus tollens.

We will restrict the universally quantified arguments we consider in Chapter 3 to ***just*** MP, MT, Converse error, and Inverse error. We will ***NOT*** use diagrams, or something like the algorithm from Chapter 2, with the truth tables, critical rows, etc, to prove validity of universally quantified arguments.

One reason why so many people make converse and inverse errors is that the forms of the resulting arguments would be valid if the major premise were a biconditional rather than a simple conditional. And, as we noted in Section 2.2, many people tend to conflate biconditionals and conditionals.

Consider, for example, the following argument:

All the town criminals frequent the Den of Iniquity bar.
John frequents the Den of Iniquity bar.
$\therefore$ John is one of the town criminals.

The conclusion of this argument is invalid—it results from making the converse error. Therefore, it may be false even when the premises of the argument are true. This type of argument attempts unfairly to establish guilt by association.

The closer, however, the major premise comes to being a biconditional, the more likely the conclusion is to be true. If hardly anyone but criminals frequent the bar and John also frequents the bar, then it is likely (though not certain) that John is a criminal. On the basis of the given premises, it might be sensible to be suspicious of John, but it would be wrong to convict him.

A variation of the converse error can be a useful reasoning tool, <u>provided it is used with caution</u>.

Partner up

Write the following universally quantified arguments symbolically, and determine their validity:

All healthy people eat an apple a day.

Jed doesn’t eat an apple a day.

∴ Jed isn’t healthy.

All freshmen must take FYC.

Hilario is a freshman.

∴ Hilario must take FYC.

All tall students sit in the back row.

Paul sits in the back row.

∴ Paul is tall.

All programs with compiler errors are not correct.

This program does not have a compiler error.

∴ Hilario must take FYC.

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All healthy people eat an apple a day. $\forall\ people\ x, if\ x\ is\ healthy\ then\ x\ eats\ an\ apple\ a\ day.$

$\forall\ x, H(x) \rightarrow A(x)$

Jed doesn't eat an apple a day. $\sim A(J)$

∴ Jed isn't healthy. $\therefore \sim H(J)$

Valid, MT

All freshmen must take FYC. $\forall\ students\ x, if\ x\ is\ freshman\ then\ x\ takes\ FYC.$

$\forall\ x, F(x) \rightarrow C(x)$

Hilario is a freshman. $F(H)$

∴ Hilario must take FYC. $\therefore C(H)$

Valid, MP

All tall students sit in the back row. $\forall\ students\ x, if\ x\ is\ tall\ then\ x\ sits\ in\ the\ back\ row.$

$\forall\ x, T(x) \rightarrow B(x)$

Paul sits in the back row. $B(P)$

∴ Paul is tall. $\therefore \sim T(P)$

Invalid, Converse Error

All programs with compiler errors are not correct. $\forall\ programs\ x, if\ x\ has\ compiler\ errors\ then\ x\ is\ not\ correct.$

$\forall\ x, E(x) \rightarrow C(x)$

This program does not have a compiler error. $\sim E(P)$

∴ Hilario must take FYC. $\therefore \sim C(P)$

Invalid, Inverse Error