

Section 3.1 – Predicates and Quantified Statements

The sentence $x^2 + 2 = 11$ is not a propositional statement, because it may be either true or false, it all depends on the value of x . And the sentence $x + y > 0$ is not a propositional statement because its truth value depends on both the variables x and y .

Now, if you recall from your early studies in English... a **predicate** is the part of a sentence that gives information about the subject.

In the sentence *James is a student at Juniata College*, *James* is the subject and the phrase *is a student at Juniata College* is the predicate.

So we can focus on just the predicate, $P(x) = x \text{ is a student at Juniata College}$.

Then $P(\text{James}) = \text{James is a student at Juniata College}$ and $P(\text{Jane}) = \text{Jane is a student at Juniata College}$.

So we can turn a predicate into a propositional statement by substituting values for the variables.

Here's the official definition.

A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables. The **domain** of a predicate variable is the set of all values that may be substituted in place of the variable.

Example 3.1.1 Finding Truth Values of a Predicate

Let $P(x)$ be the predicate " $x^2 > x$ " with domain the set \mathbf{R} of all real numbers. Write $P(2)$, $P\left(\frac{1}{2}\right)$, and $P\left(-\frac{1}{2}\right)$, and indicate which of these statements are true and which are false.

Solution

$$\begin{aligned}P(2): \quad & 2^2 > 2, \quad \text{or} \quad 4 > 2. \quad \text{True.} \\P\left(\frac{1}{2}\right): \quad & \left(\frac{1}{2}\right)^2 > \frac{1}{2}, \quad \text{or} \quad \frac{1}{4} > \frac{1}{2}. \quad \text{False.} \\P\left(-\frac{1}{2}\right): \quad & \left(-\frac{1}{2}\right)^2 > -\frac{1}{2} \quad \text{or} \quad \frac{1}{4} > -\frac{1}{2}. \quad \text{True.}\end{aligned}$$

If $P(x)$ is a predicate and x has domain D , the **truth set** of $P(x)$ is the set of all elements of D that make $P(x)$ true when they are substituted for x . The truth set of $P(x)$ is denoted

Note

Recall that we read these symbols as "the set of all x in D such that $P(x)$."

$$\{x \in D \mid P(x)\}.$$

Partner up

Let $Q(n)$ be the predicate “ n is a factor of 8.” Find the truth set of $Q(n)$ if

- the domain of n is \mathbf{Z}^+ , the set of all positive integers
- the domain of n is \mathbf{Z} , the set of all integers.

Also, give the **set-builder** notation for the Truth Set of each of these, using the given variables and Predicate.

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- $\{n \in \mathbf{Z}^+ \mid Q(n)\} = \{1, 2, 4, 8\}$, since these positive integers are the only ones that divide 8 evenly (divide 8 without a remainder).
- $\{n \in \mathbf{Z} \mid Q(n)\} = \{-8, -4, -2, -1, 1, 2, 4, 8\}$, since these integers are the only ones that divide 8 evenly (divide 8 without a remainder).

We can add **quantifiers** to form statements from predicates. Quantifiers are words that refer to quantities such as “some” or “all” and tell for how many elements a given predicate is true.

The symbol \forall is called the **universal quantifier**. Depending on the context, it is read as “for every,” “for each,” “for any,” “given any,” or “for all.” For example, another way to express the sentence “Every human being is mortal” or “All human beings are mortal” is to write

$$\forall \text{ human beings } x, x \text{ is mortal},$$

which you would read as “For every human being x , x is mortal.” If you let H be the set of all human beings, then you can symbolize the statement more formally by writing

$$\forall x \in H, x \text{ is mortal}.$$

In a universally quantified sentence the domain of the predicate variable is generally indicated either between the \forall symbol and the variable name (as in \forall human being x) or immediately following the variable name (as in $\forall x \in H$). In sentences containing a mixture of symbols and words, the \forall symbol can refer to two or more variables. For instance, you could symbolize “For all real numbers x and y , $x + y = y + x$.” as “ \forall real numbers x and y , $x + y = y + x$.” *

Let $Q(x)$ be a predicate and D the domain of x . A **universal statement** is a statement of the form “ $\forall x \in D, Q(x)$.” It is defined to be true if, and only if, $Q(x)$ is true for each individual x in D . It is defined to be false if, and only if, $Q(x)$ is false for at least one x in D . A value for x for which $Q(x)$ is false is called a **counterexample** to the universal statement.

Partner up

Let $D = \{1, 2, 3, 4, 5\}$, and consider the statement

$$\forall x \in D, x^2 \geq x.$$

Write one way to read this statement out loud, and show that it is true.

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“For every x in the set D , x^2 is greater than or equal to x .” The inequalities below show that “ $x^2 \geq x$ ” is true for each individual x in D .

$$1^2 \geq 1, \quad 2^2 \geq 2, \quad 3^2 \geq 3, \quad 4^2 \geq 4, \quad 5^2 \geq 5.$$

Hence “ $\forall x \in D, x^2 \geq x$ ” is true.

When we have a finite domain, we can substitute every value into the predicate, creating several individual propositional statements. Doing this is called the **method of exhaustion**. And again, it only is possible when the domain is finite.

Partner up

. Consider the statement

$$\forall x \in \mathbf{R}, x^2 \geq x.$$

Find a counterexample to show that this statement is false.

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. *Counterexample:* The statement claims that $x^2 \geq x$ for every real number x . But when $x = \frac{1}{2}$, for example,

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4} \not\geq \frac{1}{2}.$$

Hence “ $\forall x \in \mathbf{R}, x^2 \geq x$ ” is false.

The symbol \exists denotes “there exists” and is called the **existential quantifier**. For example, the sentence “There is a student in Math 140” can be written as

\exists a person p such that p is a student in Math 140,

or

$\exists p \in P$ such that p is a student in Math 140,

where P is the set of all people. The domain of the predicate variable is generally indicated either between the \exists symbol and the variable name or immediately following the variable name, and the words *such that* are inserted just before the predicate. Some other expressions that can be used in place of *there exists* are *there is a*, *we can find a*, *there is at least one*, *for some*, and *for at least one*. In a sentence such as “ \exists integers m and n such that $m + n = m \cdot n$,” the \exists symbol is understood to refer to both m and n . *

Let $Q(x)$ be a predicate and D the domain of x . An **existential statement** is a statement of the form “ $\exists x \in D$ such that $Q(x)$.” It is defined to be true if, and only if, $Q(x)$ is true for at least one x in D . It is false if, and only if, $Q(x)$ is false for all x in D .

Consider the statement

$$\exists m \in \mathbf{Z}^+ \text{ such that } m^2 = m.$$

Write one way to read this statement out loud, and show that it is true.

. “There is at least one positive integer m such that $m^2 = m$.” Observe that $1^2 = 1$. Thus “ $m^2 = m$ ” is true for a positive integer m , and so “ $\exists m \in \mathbf{Z}^+$ such that $m^2 = m$ ” is true.

Let $E = \{5, 6, 7, 8\}$ and consider the statement

$$\exists m \in E \text{ such that } m^2 = m.$$

Show that this statement is false.

Note that $m^2 = m$ is not true for any integers m from 5 through 8:

$$5^2 = 25 \neq 5, \quad 6^2 = 36 \neq 6, \quad 7^2 = 49 \neq 7, \quad 8^2 = 64 \neq 8.$$

Thus “ $\exists m \in E$ such that $m^2 = m$ ” is false.

Partner up

Example 3.1.5 Translating from Formal to Informal Language

Rewrite the following formal statements in a variety of equivalent but more informal ways. Do not use the symbol \forall or \exists .

- a. $\forall x \in \mathbf{R}, x^2 \geq 0$.
- b. $\forall x \in \mathbf{R}, x^2 \neq -1$.
- c. $\exists m \in \mathbf{Z}^+$ such that $m^2 = m$.

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a. Every real number has a nonnegative square.

Or: All real numbers have nonnegative squares.

Or: Any real number has a nonnegative square.

Or: The square of each real number is nonnegative.

b. All real numbers have squares that do not equal -1 .

Or: No real numbers have squares equal to -1 .

(The words *none are* or *no ... are* are equivalent to the words *all are not*.)

c. There is a positive integer whose square is equal to itself.

Or: We can find at least one positive integer equal to its own square.

Or: Some positive integer equals its own square.

Or: Some positive integers equal their own squares.

Example 3.1.7 Translating from Informal to Formal Language

Rewrite each of the following statements formally. Use quantifiers and variables.

a. All triangles have three sides.

b. No dogs have wings.

c. Some programs are structured.

a. \forall triangle t , t has three sides.

Or: $\forall t \in T$, t has three sides (where T is the set of all triangles).

b. \forall dog d , d does not have wings.

Or: $\forall d \in D$, d does not have wings (where D is the set of all dogs).

c. \exists a program p such that p is structured.

Or: $\exists p \in P$ such that p is structured (where P is the set of all programs).

A reasonable argument can be made that the most important form of statement in mathematics is the **universal conditional statement**:

$$\forall x, \text{ if } P(x) \text{ then } Q(x).$$

Rewrite the following statement informally, without quantifiers or variables.

$$\forall x \in \mathbf{R}, \text{ if } x > 2 \text{ then } x^2 > 4.$$

Solution

If a real number is greater than 2, then its square is greater than 4.

Or: Whenever a real number is greater than 2, its square is greater than 4.

Or: The square of any real number greater than 2 is greater than 4.

Or: The squares of all real numbers greater than 2 are greater than 4.

Rewrite each of the following statements in the form

$$\forall \text{ ______}, \text{ if } \text{ ______} \text{ then } \text{ ______}.$$

- a. If a real number is an integer, then it is a rational number.
- b. All bytes have eight bits.
- c. No fire trucks are green.

Solution

- a. \forall real number x , if x is an integer, then x is a rational number.

Or: $\forall x \in \mathbf{R}$, if $x \in \mathbf{Z}$ then $x \in \mathbf{Q}$.

- b. $\forall x$, if x is a byte, then x has eight bits.

- c. $\forall x$, if x is a fire truck, then x is not green.

It is common, as in (b) and (c) above, to omit explicit identification of the domain of predicate variables in universal conditional statements.

Rewrite the following statement in the two forms " $\forall x$, if _____ then _____" and " \forall _____ x , _____": All squares are rectangles.

Solution

$\forall x$, if x is a square then x is a rectangle.

\forall square x , x is a rectangle.