Section 5.1 – Sequences and Series, Day 03

Consider the following series given in expanded form:

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{10}$$

Partner up

Write this series in sigma/summation notation, two different ways:

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$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{10}=\sum_{k=1}^{10}\frac{1}{k}$ or $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{10}=\sum_{j=0}^{9}\frac{1}{j+1}$

(1) (2)

This leads us to the idea that we can transform a summation/sigma notation with a change of variables. Note, the value of the sum wouldn't change, just how we write it.

The transform from (1) to (2) above would be with $j=k-1$, which then gives $j+1=k$.

The symbol used to represent an index of a summation is an example of a local variable, often called a **dummy variable**, because, as illustrated above, it can be replaced by any other symbol as long as the replacement is made in each location where it occurs. Outside of that context (both before and after), the symbol may have another meaning entirely. In the same way, a symbol used to represent a

Transform the following summation by making the specified change of variable:

$$summation\colon \sum_{k=0}^{6}\frac{1}{k+1} \qquad change\ of\ variable\colon j=k+1$$

First calculate the lower and upper limits of the new summation:

$$\text{When } k=0,\quad j=k+1=0+1=1.$$
$$\text{When } k=6,\quad j=k+1=6+1=7.$$

Thus the new sum goes from $j=1$ to $j=7$.

Next calculate the general term of the new summation. You will need to replace each occurrence of k by an expression in j:

$$\text{Since } j=k+1, \text{ then } k=j-1.$$
$$\text{Hence } \frac{1}{k+1}=\frac{1}{(j-1)+1}=\frac{1}{j}.$$

Finally, put the steps together to obtain

$$\sum_{k=0}^{6} \frac{1}{k+1} = \sum_{j=1}^{7} \frac{1}{j}.$$

Another thing we often do with series is splitting off the last term:

$$1 + 2 + \cdots + 99 + 100 = \sum_{k=1}^{100} k$$

Can also be written as:

$$(1 + 2 + \cdots + 99) + 100 = \left(\sum_{k=1}^{99} k\right) + 100$$

And we generalize, ending at n rather than 100, we have:

$$\big(1 + 2 + \cdots + (n-1)\big) + n = \left(\sum_{k=1}^{n} k\right) + n$$

We can also simply two series in summation/sigma notation, and it is fairly straightforward if the summations have the same limits. We combine expressions in the same sigma, collect like terms, and simplify:

$$\sum_{k=1}^{n} (3k-2) + \sum_{k=1}^{n} 2k + 1 = \sum_{k=1}^{n} (3k-2) + (2k+1) = \sum_{k=1}^{n} 5k - 1$$

Consider the equation $2^n = y$, so y is a function of n , and it is exponential, since the independent variable n is in the exponent. If we represent values in a table, we have:

n	y
0	1
1	2
2	4
3	8
3	16

Now, when we undo, or mathematically find the inverse of the function, it's like considering given a value of y, what value of n would produce it? This is called the ***logarithm***, which is the inverse function of an exponential function. And in this example, since we are considering powers of 2, we use the base of 2 in the logarithm (typically if the base isn't given, it's assumed to be 10). $log_2 n = y$

n	y
1	0
2	1
4	2
8	3
16	4

$log_2 1 = 0$, since 2 raised to the 0 gives 1.

What is $log_2 3$?   We don't know exactly without a calculator, but it would be between 1 and 2.

We can say $\lfloor log_2 3 \rfloor = 1$.

Partner up

Calculate (or simplify) the following:

$\frac{10!}{8!}$ $\binom{10}{8}$ "10 choose 8" $\frac{(n+1)!}{(n-1)!}$ $\binom{n+1}{n-1}$ "(n+1) choose (n-1)"

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$$\frac{10!}{8!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 10 \cdot 9 = 90$$

$$\binom{10}{8} = \frac{10!}{8! \cdot (10-8)!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (2 \cdot 1)} = \frac{10 \cdot 9}{2 \cdot 1} = 45$$

$$\frac{(n+1)!}{(n-1)!} = \frac{(n+1) \cdot n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{(n-1) \cdot (n-2) \cdots 2 \cdot 1} = (n+1) \cdot n$$

$$\binom{n+1}{n-1} = \frac{(n+1)!}{(n-1)! \cdot ((n+1)-(n-1))!} = \frac{(n+1)!}{(n-1)! \cdot 2!} \frac{(n+1) \cdot n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1}{((n-1) \cdot (n-2) \cdots 2 \cdot 1) \cdot 2!} = \frac{(n+1) \cdot n}{2 \cdot 1} = \frac{(n+1) \cdot n}{2}$$

What homework questions do you have?

Work on the homework as there is time remaining.