Sec 5.2- Mathematical Induction, Day 02

1. Work Sec 5.2 Problems 3, and 4

3. For each positive integer n, let $P(n)$ be the formula

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

a. Write $P(1)$. Is $P(1)$ true?

Answer

$P(1)$ is "$1^2 = \frac{1 \cdot (1+1) \cdot (2 \cdot 1 + 1)}{6}$." $P(1)$ is true because the left-hand side equals $1^2 = 1$ and the right-hand side equals $\frac{1 \cdot (1+1) \cdot (2+1)}{6} = \frac{2 \cdot 3}{6} = 1$ also.

b. Write $P(k)$.

c. Write $P(k+1)$.

d. In a proof by mathematical induction that the formula holds for every integer $n \geq 1$, what must be shown in the inductive step?

4. For each integer n with $n \geq 2$, let $P(n)$ be the formula

$$\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}.$$

a. Write $P(2)$. Is $P(2)$ true?

b. Write $P(k)$.

c. Write $P(k+1)$.

d. In a proof by mathematical induction that the formula holds for every integer $n \geq 2$, what must be shown in the inductive step?

2. Work Sec 5.2 Problem 2 and 5 (set up and proof of the same predicate):

2. For each positive integer n, let $P(n)$ be the formula

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2.$$

a. Write $P(1)$. Is $P(1)$ true?

Answer

$P(1)$ is the equation $1 = 1^2$, which is true.

b. Write $P(k)$.

Answer

$P(k)$ is the equation $1 + 3 + 5 + \cdots + (2k - 1) = k^2$.

c. Write $P(k + 1)$.

Answer

$P(k + 1)$ is the equation $1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$

d. In a proof by mathematical induction that the formula holds for every integer $n \geq 1$, what must be shown in the inductive step?

Answer

In the inductive step, show that if k is any integer for which $k \geq 1$ and $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ is true, then $1 + 3 + 5 + \cdots + (2(k + 1) - 1) = (k + 1)^2$ is also true.

Using Mathematical Induction, Prove: $P(n)$: $1 + 3 + 5 + \cdots + (2n - 1) = n^2, \forall$ int $n \geq 1$

Basis Step **[Show *P(1)* is true]**

Left-Hand Side (LHS) of $P(1)$ is just 1.

Right-Hand Side (RHS) of $P(1)$ is $1^2 = 1$.

So they are equal.

Inductive Step **[Assume *P(k)* is true and show *P(k+1)* is true]**

We assume $P(k)$: $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ is true.

We will show $P(k + 1)$: $1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = (k + 1)^2$

We start with the LHS of $P(k + 1)$, $1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1)$ and note that if we split off the last term, $(2k + 1)$, the first part of the series, $1 + 3 + 5 + \cdots + (2k - 1)$ is equal to kk^2 in our assumption.

Below, you will see the substitution, then FOIL factoring, to get the RHS of $P(k + 1)$:

So, $1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) = k^2 + (2k + 1) = (k + 1)(k + 1) = (k + 1)^2$.

We used a substitution and algebraic manipulation to turn the LHS of $P(k + 1)$ into the RHS of $P(k + 1)$, so our original assumption is true and our proof is complete.

Q.E.D.

3. Let's consider another predicate we can prove with induction, with quantification:

$$P(n): 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}, \forall \text{ int } n \geq 1$$

4. So, to prove ***P(n)*** in a proof by Mathematical Induction, we would have:

<u>Basis Step</u> [Show *P(1)* is true]
Left-Hand Side (LHS) of $P(1)$ is just 1.
Right-Hand Side (RHS) of $P(1)$ is $\frac{1(1+1)}{2} = \frac{2}{2} = 1$

So they are equal.

<u>Inductive Step</u> [Assume *P(k)* is true and show *P(k+1)* is true]
We assume $P(k): 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$ is true.
We will show $P(k+1): 1 + 2 + 3 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

We start with the LHS of $P(k+1)$, $1 + 2 + 3 + \cdots + k + (k+1)$ and note that if we split off the last term, $(k+1)$, the first part of the series, $1 + 2 + 3 + \cdots + k$ is equal to $\frac{k(k+1)}{2}$ in our assumption.
Below, you will see the substitution, then algebraic manipulation, to get the RHS of $P(k+1)$:

So, $1 + 2 + 3 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + (kk + 1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{k^2+3k+2}{2}$ $\frac{\quad}{2}$

$$= \frac{(k+1)(k+2)}{2}$$

We used a substitution and algebraic manipulation to turn the LHS of $P(k+1)$ into the RHS $P(k+1)$, so our original assumption is true and our proof is complete.

Q.E.D.