Section 5.1 – Sequences and Series, Day 02

Previously we considered sequences. We are working toward you being comfortable translating back and forth between a sequence in roster notation and an explicit formula (or recursive) formula.

For example, consider the sequence: $1, 2, 4, 8, \cdots$

If you are determining the <u>recursive</u> formula, you look for the pattern in getting the next term from the previous term:

1 to 2   2 to 4   4 to 8   and so on…

If you are determining the <u>explicit/direct</u> formula, sometimes it helps to label the terms, and since there are infinitely many correct answers, you can choose the subscript to start.

a_0	a_1	a_2	a_3	$\cdots$
1	2	4	8	$\cdots$

Looking at the sequence this way helps show the relationship between the subscript and the term:

a_0	a_1	a_2	$\cdots$
1	2	4	$\cdots$
"function of 0 giving 1"	"function of 1 giving 2"	"function of 2 giving 4"	

Partner up

Give both a (1) recursive and an (2) explicit/direct formula for this sequence. ***Don't forget to quantify!***

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<u>Recursive</u>

$a_0 = 1$

$a_n = 2a_{n-1}, \forall\ int\ n \geq 1$

<u>Explicit/Direct</u>

$a_n = 2^n, \forall\ int\ n \geq 0$

Let's try another sequence: $2, -4, 6, -8, \cdots$

Note that every other term is negative. Remember that $(-1) \cdot (-1) = 1$ and

$(-1)^n = 1$ when n is even and $(-1)^n = -1$ when n is odd.

Partner up

Give both a (1) recursive and an (2) explicit/direct formula for this sequence. ***Don't forget to quantify!***

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<u>Recursive</u>

$a_0 = 2$

$a_n = (-1)^n (|a_{n-1}| + 2), \forall\ int\ n \geq 1$

<u>Explicit/Direct</u>

$a_n = (-1)^{n+1} \cdot 2n, \forall\ int\ n \geq 1$

Besides just considering sequences, we will also consider sums of numbers, which could defined by explicit/direct formulas, or given in roster format.

And in this section, we will just focus on shortcuts to writing these sums, ***NOT*** yet calculating the value of these sums.

Consider the sum $1 + 2 + \cdots + 10$. Without considering what this sum is (it's 55, btw, but we'll get to that next section), we can write it in ***sigma*** or summation notation, which is like a nickname ("Gerald" ←→"Jerry").

$$1 + 2 + \cdots + 10 \quad = \quad \sum_{k=1}^{10} k$$

We have a ***loop control variable*** (or "dummy" variable) which is as if we were writing a loop in a Python program, here it is k.

We start at 1

and

end at 10.

If we were writing a formula for the ***sequence*** $1, 2,3 \cdots ,10, a_k = k, for\ int\ 1 \le k \le 10$, and we use this explicit/direct formula in the sigma notation.

Again, $\sum_{k=1}^{10} k$ does not tell us anything about the what the sum of the numbers is, it's just a nickname for

$1 + 2 + \cdots + 10$

By the way, a loop in Python, using these variables, and introducing a variable to keep the running sum, would be:

```
sum = 0
for k in range (1,11):
    sum += k
```

Sometime we parameterize the final term in the summation, it goes to n rather than a particular number like 10 above.

This is actually just a simple change:

$$1 + 2 + \cdots + \mathbf{n} \quad = \quad \sum_{k=1}^{\mathbf{n}} k$$

This can be confusing, because now we have an expression with two variables, n and k. Sometimes it helps to refer back to this original example, and swap 10 for n. Note that the roster type notation is called ***expanded form***.

In general we have:

If m and n are integers and $m \le n$, the symbol $\sum_{k=m}^{n} a_k$, read the **summation from k equals m to n of a-sub-k**, is the sum of all the terms $a_m, a_{m+1}, a_{m+2}, \ldots, a_n$. We say that $a_m + a_{m+1} + a_{m+2} + \cdots + a_n$ is the **expanded form** of the sum, and we write

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + a_{m+2} + \cdots + a_n.$$

We call k the **index** of the summation, m the **lower limit** of the summation, and n the **upper limit** of the summation.

Analogous to summations, we can multiply terms for a ***product***:

$$1 \cdot 2 \cdot \cdots \cdot 10 \quad = \quad \prod_{k=1}^{10} k$$

And in general, multiplying to n rather than 10, we have:

$$1 \cdot 2 \cdot \cdots \cdot n \quad = \quad \prod_{k=1}^{n} k$$

For products, we have another shortcut/nickname, the ***factorial***:

For each positive integer n, the quantity $\boldsymbol{n}$ **factorial** denoted $\boldsymbol{n!}$, is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted $0!$, is defined to be 1:

$$0! = 1.$$

Partner up

Simplify the following expressions:

a. $\dfrac{8!}{7!}$

b. $\dfrac{5!}{2! \cdot 3!}$

c. $\dfrac{1}{2! \cdot 4!} + \dfrac{1}{3! \cdot 3!}$

d. $\dfrac{(n+1)!}{n!}$

e. $\dfrac{n!}{(n-3)!}$

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Solution

a. $\dfrac{8!}{7!} = \dfrac{8 \cdot \cancel{7!}}{\cancel{7!}} = 8$

b. $\dfrac{5!}{2! \cdot 3!} = \dfrac{5 \cdot 4 \cdot \cancel{3!}}{2! \cdot \cancel{3!}} = \dfrac{5 \cdot 4}{2 \cdot 1} = 10$

c. $\dfrac{1}{2! \cdot 4!} + \dfrac{1}{3! \cdot 3!} = \dfrac{1}{2! \cdot 4!} \cdot \dfrac{3}{3} + \dfrac{1}{3! \cdot 3!} \cdot \dfrac{4}{4}$ by multiplying each numerator and denominator by just what is necessary to obtain a common denominator

$= \dfrac{3}{3 \cdot 2! \cdot 4!} + \dfrac{4}{3! \cdot 4 \cdot 3!}$ by rearranging factors

$= \dfrac{3}{3! \cdot 4!} + \dfrac{4}{3! \cdot 4!}$ because $3 \cdot 2! = 3!$ and $4 \cdot 3! = 4!$

$= \dfrac{7}{144}$ by the rule for adding fractions with a common denominator

d. $\dfrac{(n+1)!}{n!} = \dfrac{(n+1) \cdot \cancel{n!}}{\cancel{n!}} = n+1$

e. $\dfrac{n!}{(n-3)!} = \dfrac{n \cdot (n-1) \cdot (n-2) \cdot \cancel{(n-3)!}}{\cancel{(n-3)!}} = n \cdot (n-1) \cdot (n-2)$

$= n^3 - 3n^2 + 2n$

How many freshmen are in class today?

How many ways can I pick ___ freshmen, if the order chosen doesn't matter and I can't repeat?

Let n and r be integers with $0 \leq r \leq n$. The symbol

$$\binom{n}{r}$$

is read "$\boldsymbol{n}$ **choose** $\boldsymbol{r}$" and represents the number of subsets of size r that can be chosen from a set with n elements.

For all integers n and r with $0 \leq r \leq n$,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

It is sometimes also written $C(n, r)$.

$$\binom{8}{5} = \frac{8!}{5!\,(8-5)!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (3 \cdot 2 \cdot 1)} = \frac{8 \cdot 7}{1} = 56$$