Section 9.4 – The Pigeonhole Principle

Partner up

Pittsburgh, PA is in Allegheny County. <u>Can you guarantee that at least two people have the same number of hairs on their head?</u>

Discuss (***DON'T GOOGLE, look up, etc…***) and report back to the class.

What would you like to know in order help to find this answer?

Population of Allegheny County ***Maximum number of hairs on a human head***

It turns out that in the 2020 Census, Allegheny County had a population of 1,250,578.

In the textbook, we are given that the maximum number of hairs on a human head is 300,000.

With this information, does that change your answer to <u>Can you guarantee that at least two people have the same number of hairs on their head?</u>

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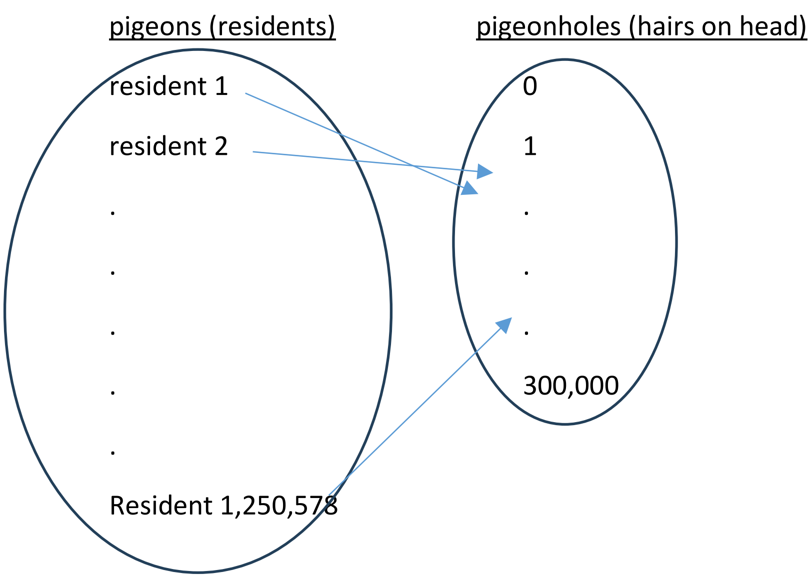
Think of this as a function $H(x)$, mapping the domain, a set containing each resident of Allegheny County (the ***pigeons***) to the co-domain, a set of the integer number of hairs on human heads, from 0 to 300,000 (the ***pigeonholes***).



The domain, which is the number of residents of Allegheny County, is finite, but much larger than the finite co-domain, which is the set of possible numbers of hairs on a head.

So $H(x)$ ***CANNOT*** be 1-1, mapping 1,250,578 elements to 300,000 means that at least one of those numbers of hairs on heads gets "pointed to" more than once. This gives us:

Pigeonhole Principle

> A function from one finite set to a smaller finite set cannot be one-to-one: There must be at least two elements in the domain that have the same image in the co-domain.

The Pigeonhole Principle is very useful for solving problems.

In a group of 6 people, must two have been born in the same month?

No, the key here is ***must***. Can you guarantee that two were born in the same month? The odds favor it, but they are not 100%.

So, how many people must you have to ***guarantee*** two were born in the same month?

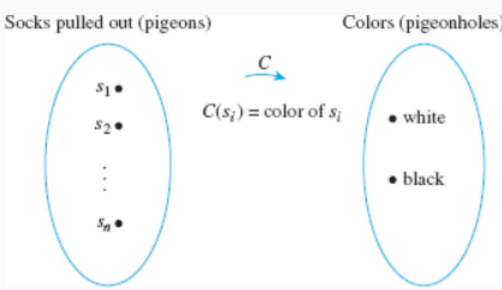
Since there are 12 months, you'd need 13 to guarantee.

A drawer contains ten black and ten white socks. You reach in and pull some out without looking at them. What is the *least* number of socks you must pull out to be sure to get a matched pair? Explain how the answer follows from the pigeonhole principle.

Solution

If you pick just two socks, they may have different colors. But when you pick a third sock, it must be the same color as one of the socks already chosen. Hence the answer is three.

This answer could be phrased more formally as follows: Let the socks pulled out be denoted $s_1, s_2, s_3, \ldots,$ s_n and consider the function C that sends each sock to its color, as shown below.



If $n = 2$, C could be a one-to-one correspondence (if the two socks pulled out were of different colors). But if $n > 2$, then the number of elements in the domain of C is larger than the number of elements in the co-domain of C. Thus by the pigeonhole principle, C is not one-to-one: $C(s_i) = C(s_j)$ for some $s_i \neq s_j$. This means that if at least three socks are pulled out, then at least two of them have the same color.

What if we changed the question above to "How many socks do you have to pull to guarantee getting a black sock?"

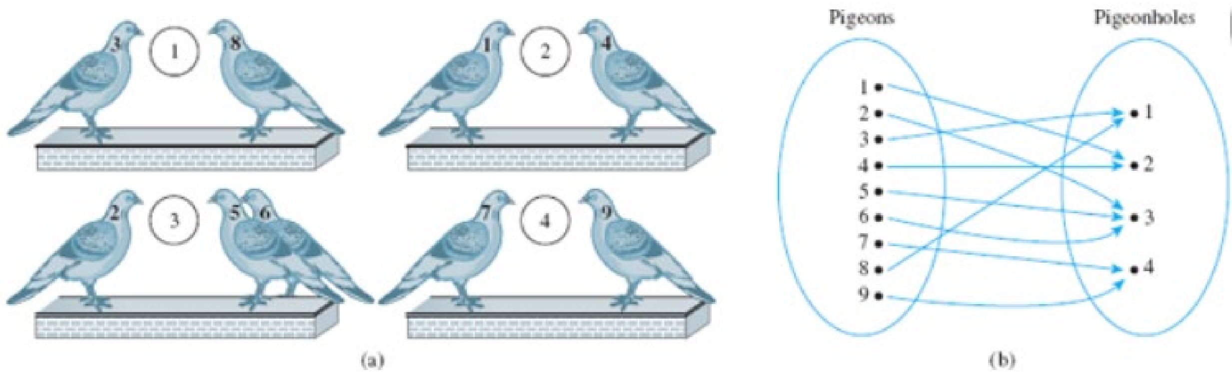
In this case, it is possible to pick all 10 white (or black) socks, before pulling one of a different color, so the answer is 11.

Generalized Pigeonhole Principle

For any function f from a finite set X with n elements to a finite set Y with m elements and for any positive integer k, if $km < n$, then there is some $y \in Y$ such that y is the image of at least $k + 1$ distinct elements of X.

We illustrate this with the following example. There are $n = 9$ pigeons (literally, ha) and $m = 4$ pigeonholes.

With $k = 2$, the Generalized Pigeonhole Principle says that there has to be a pigeonhole with more than 2 pigeons, since $k \cdot m = \quad 2 \cdot 4 = 8 \quad < 9 \quad = n$



Partner up

Show how the generalized pigeonhole principle implies that in a group of 85 people, at least 4 must have the same last initial.

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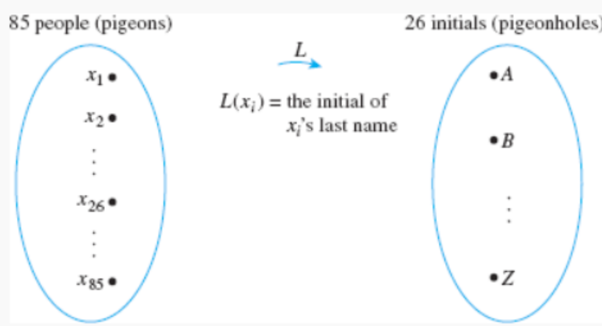
In this example the pigeons are the 85 people and the pigeonholes are the 26 possible last initials of their names.

Consider the function L from people to initials defined by the following arrow diagram.



Since $3 \cdot 26 = 78 < 85$, the generalized pigeonhole principle states that some initial must be the image of at least four $(3+1)$ people. Thus at least four people have the same last initial.