

Section 3.3 – Statements with Multiple Quantifiers

Partner up

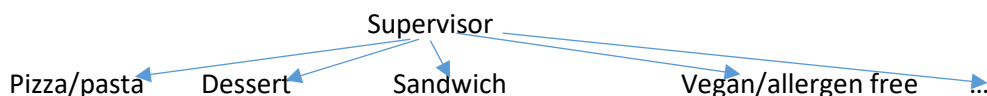
Write a list of the food stations in Baker Refectory.

Pizza/pasta	Dessert	Sandwich	Vegan/allergen free
Grill	Entrée	Mongolian Grill	Burrito

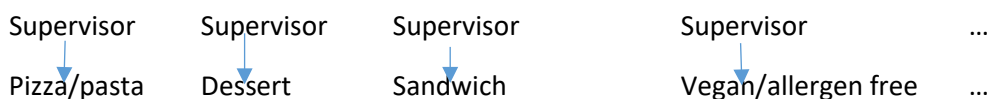
Now, consider the statement:

There is a person supervising every food station in Baker Refectory.

Does this mean there is a single person in charge of all food stations in Baker Refectory?



Or does it mean that each of the food stations has an individual in charge?



Well, it could mean either.

We'll come back to these two interpretations.

Informal language can be ambiguous. And in this case, the original statement has multiple quantifiers, which can contribute to the ambiguity. Usually, there are context cues that help determine what the intent is. **But when we write these statements formally, we remove the ambiguity.**

Because many important technical statements contain both \exists and \forall , a convention has developed for interpreting them uniformly. **When a statement contains more than one kind of quantifier, we imagine the actions suggested by the quantifiers as being performed in the order in which the quantifiers occur.** For instance, consider a statement of the form

$\forall x$ in set D , $\exists y$ in set E such that x and y satisfy property $P(x, y)$.

To show that such a statement is true, you must be able to meet the following challenge:

Note

The scope of $\forall x$ extends throughout the statement, whereas the scope of $\exists y$ starts in the middle. That is why the value of y depends on the value of x .

- Imagine that someone is allowed to choose any element whatsoever from the set D , and imagine that the person gives you that element. Call it x .
- The challenge for you is to find an element y in E so that the person's x and your y , taken together, satisfy property $P(x, y)$.

Because you do not have to specify the y until after the other person has specified the x , you are allowed to find a different value of y for each different x you are given.

\forall person x , \exists a person y such that x loves y .
 \exists a person y such that \forall person x , x loves y .

Note that except for the order of the quantifiers, these statements are identical. However, the first means that given any person, it is possible to find someone whom that person loves, whereas the second means that there is one amazing individual who is loved by all people. (Reread the statements carefully to verify these interpretations!) The two sentences illustrate an extremely important property about statements with two different quantifiers.

In a statement containing both \forall and \exists , changing the order of the quantifiers can significantly change the meaning of the statement.

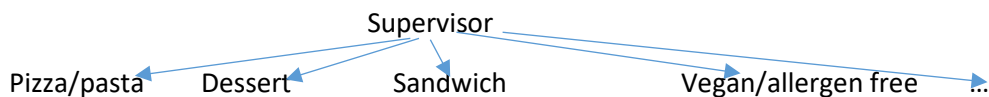
Everybody loves somebody.

Someone is loved by everybody.

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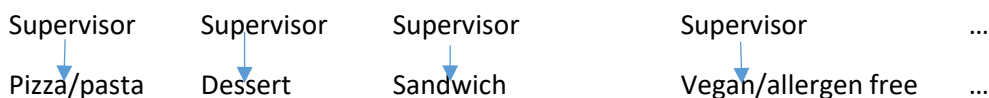
Write quantifications for each of the situations we discussed originally:

There is a single person in charge of all food stations in Baker Refectory:



\exists a supervisor s s. t. \forall food stations f , s supervises f .

Each of the food stations in Baker Refectory has an individual in charge:



\forall food stations f , \exists a supervisor s s. t. s supervises f .

We can use the same DeMorgan's rules to negate statements with multiple quantifiers.

Negations of Statements with Two Different Quantifiers

$$\sim(\forall x \text{ in } D, \exists y \text{ in } E \text{ such that } P(x, y)) \equiv \exists x \text{ in } D \text{ such that } \forall y \text{ in } E, \sim P(x, y)$$

$$\sim(\exists x \text{ in } D \text{ such that } \forall y \text{ in } E, P(x, y)) \equiv \forall x \text{ in } D, \exists y \text{ in } E \text{ such that } \sim P(x, y)$$

Sometimes, when given an informal statement to negate, it is easier to formally quantify first, negate, and then translate back.

Negate the following: *Any even integer equals twice some other integer.*

Formally quantify: $\forall \text{ even integers } x, \exists \text{ an integer } y \text{ s.t. } x = 2y.$

Negate the quantification: $\exists \text{ an even integer } x \text{ s.t. } \forall \text{ integers } y, x \neq 2y.$

Translate back to informal: *There is an even integer which does not equal twice some other integer.*