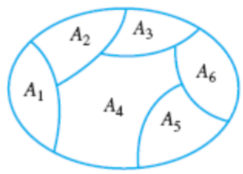
Section 8.3 – Equivalence Relations

Do you remember partitions of sets?

> A **partition** of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A. The diagram of Figure 8.3.1 illustrates a partition of a set A by subsets $A_1, A_2, \ldots, A_6$.

Figure 8.3.1

A Partition of a Set



$A_i \cap A_j = \emptyset$, whenever $i \neq j$
$A_1 \cup A_2 \cup \cdots \cup A_6 = A$

For example, if $A = \{0, 1, 2, 3, 4\}$, one possible partition for A would be:

$$A_1 = \{0, 3, 4\}$$

$$A_2 = \{1\}$$

$$A_3 = \{2\}$$

Note that the $A_i's$ are non-empty, don't overlap (null intersections), and their union gives the original set A.

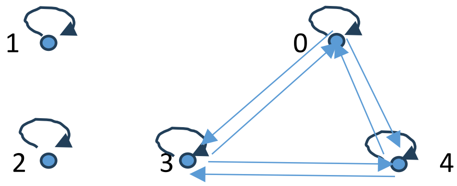
This partition can induce a relation:

> Given a partition of a set A, the **relation induced by the partition**, R, is defined on A as follows: For every $x, y \in A$,
>
> $$x \; R \; y \quad \Leftrightarrow \quad \text{there is a subset } A_i \text{ of the partition such that both } x \text{ and } y \text{ are in } A_i.$$

In our example we connect every element in the subsets to each other and them selves. For example,

> Since $\{0, 3, 4\}$ is a subset of the partition,
>
> $$\begin{aligned} 0 \; R \; 3 & \quad \text{because both 0 and 3 are in } \{0,3,4\} \\ 3 \; R \; 0 & \quad \text{because both 3 and 0 are in } \{0,3,4\} \\ 0 \; R \; 4 & \quad \text{because both 0 and 4 are in } \{0,3,4\} \\ 4 \; R \; 0 & \quad \text{because both 4 and 0 are in } \{0,3,4\} \\ 3 \; R \; 4 & \quad \text{because both 3 and 4 are in } \{0,3,4\} \end{aligned}$$

If we draw the arrow diagram it becomes clear that "everything in a subset is related":



What other properties of this relation induced by the partition on A do you notice?

It is ***Reflexive***, ***Symmetric***, and ***Transitive***. In fact, ANY relation induced by a partition will be Reflexive, Symmetric, and Transitive.

Theorem 8.3.1

> Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Our next idea concerns relations which are Reflexive, Symmetric, and Transitive. We've gotten then sense that the elements in the subsets in the partition are completely related to each other. Which is why we then call any relation that is Reflexive, Symmetric, and Transitive an ***equivalence relation***.

So, the relation induced by a partition is an equivalence relation.

Partner up

Let X be the set of all nonempty subsets of $\{1, 2, 3\}$. Then

$$X = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Define a relation $\mathbf{R}$ on X as follows: For every A and B in X,

$$A \mathbf{R} B \Leftrightarrow \text{the least element of } A \text{ equals the least element of } B.$$

(1) Draw an arrow diagram of the relation.

(2) Show that R is an equivalence relation.

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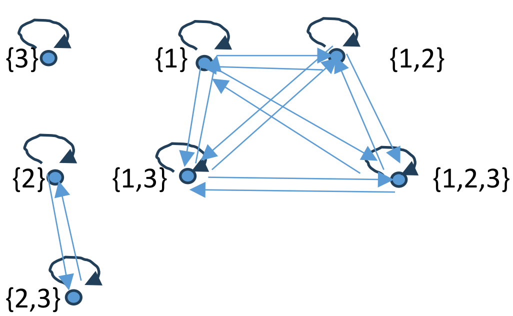

R is Reflexive: An element A in the relation, which in this case is a set, is related to itself.

R is Symmetric: If A $\boldsymbol{R}$ B because the least element of A is the same as the least element of B, then B $\boldsymbol{R}$ A

R is Transitive: If A $\boldsymbol{R}$ B because the least element of A is the same as the least element of B, and B $\boldsymbol{R}$ C then A $\boldsymbol{R}$ C, since the least element of A equals the least element of C.

Sometimes these seem obvious, but you still need to state them when asked to show Reflexivity, Symmetry, and Transitivity.

Suppose A is a set and R is an equivalence relation on A. For each element a in A, the **equivalence class of a**, denoted $[a]$ and called the **class of a** for short, is the set of all elements x in A such that x is related to a by R.

So in this example, the three (3) distinct equivalence classes are: $\{3\}$ $\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}$ $\{2\}, \{2,3\}$

We pick an element in an equivalence class to represent the entire class: $[\,\{2\}\,] = \{2\}$ and $[\,\{2\}\,] = \{2,3\}$

The "more fun" equivalence relations, and those more frequently occurring in CS, IT, MA, and DS, are like the following:

Certain implementations of computer languages do not place a limit on the allowable length of an identifier. This permits a programmer to be as precise as necessary in naming variables without having to worry about exceeding length limitations. However, compilers for such languages often ignore all but some specified number of initial characters: As far as the compiler is concerned, two identifiers are the same if they have the same initial characters, even though they may look different to a human reader of the program. For example, to a compiler that ignores all but the first eight characters of an identifier, the following identifiers would be the same:

NumberOfScrews NumberOfBolts.

Formally:

Let L be the set of all allowable identifiers in a certain computer language, and define a relation R on L as follows: For all strings s and t in L,

$$s\ R\ t \quad \Leftrightarrow \quad \text{the first eight characters of } s \text{ equal the first eight characters of } t.$$

How would we show this is an equivalence relation? ***We have to say something for each of the properties***:

R is reflexive: Let $s \in L$. *[We must show that $s\ R\ s$.]* Clearly s has the same first eight characters as itself. Thus, by definition of R, $s\ R\ s$ *[as was to be shown]*.

R is symmetric: Let s and t be in L and suppose that $s\ R\ t$. *[We must show that $t\ R\ s$.]* By definition of R, since $s\ R\ t$, the first eight characters of s equal the first eight characters of t. It follows that the first eight characters of t equal the first eight characters of s, and so, by definition of R, $t\ R\ s$ *[as was to be shown]*.

R is transitive: Let s, t, and u be in L and suppose that $s\ R\ t$ and $t\ R\ u$. *[We must show that $s\ R\ u$.]* By definition of R, since $s\ R\ t$ and $t\ R\ u$, the first eight characters of s equal the first eight characters of t, and the first eight characters of t equal the first eight characters of u. Hence the first eight characters of s equal the first eight characters of u. Hence, by definition of R, $s\ R\ u$ *[as was to be shown]*.

Since R is reflexive, symmetric, and transitive, R is an equivalence relation on L.