Algebraic Geometry - Homework 1

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Given a circle of radius r, the area is $A(r) = \pi r^2$ and the perimeter is $p(r) = 2\pi r$. Note that $\frac{dA}{dr} = p(r)$.

Let's look at polygons. Pick $n \in \mathbb{Z}^+ \geq 3$.

Problem 1. *Is the set $\{(t, \sin t) | t \in \mathbb{R}\}$ algebraic?*

Proof. Suppose that $V = \{(t, \sin t) | t \in \mathbb{R}\}$ is algebraic. Now let $F \in I(V)$. Now, of course, $F(x, 0)$ is a polynomial in x. But $F(x, 0)$ has zeros at $n\pi$ for all $n \in \mathbb{Z}$. Hence $F(x, 0)$ has infinitely many zeros and so $F(x, 0) = 0$. Similarly, for any value $y_0 \in [-1, 1]$, we have infinitely many zeros for $F(x, y_0)$ and so $F(x, y_0) = 0$. Therefore F is zero on $\mathbb{R} \times [-1, 1]$. Continuing the argument, we see that for any x_0 we have $F(x_0, y)$ is zero on $[-1, 1]$ and so is the zero polynomial. Hence $F(x_0, y) = 0$ for all $y \in \mathbb{R}$. Therefore $F(x, y)$ is the zero polynomial. This is true for all $F \in I(V)$ and so $I(V) = 0$ and hence $V = \mathbb{R}$. This is a contradiction and so our original V is not an algebraic set. $\square$

Problem 2. *Let V be an affine algebraic set, $V \subset k^n$, and consider $x \notin V$. Show that there is an $F \in k[X_1, \ldots, X_n]$ such that $F(x) = 1$ and $F|V = 0$.*

Proof. Since V is affine algebraic then it is an intersection of basic closed sets. That is, $V = \cap V(f_i)$ for some set of functions $\{f_i\}$. Now if $x \notin V$ then there must be some f_i such that $x \notin V(f_i)$. Thus $f_i(x) \neq 0$ and we can define $G(X) = f_i(X)/f_i(x)$. Of course, this means that $G(x) = 1$. Finally $G|V = 0$ since, by construction, $V(f_i) = V(G)$ and $V(f_i) \subset V$. $\square$

Problem 3. *Let $F \in k[X, Y]$ be an irreducible polynomial. Assume that $V(F)$ is infinite. Prove that $I(V(F)) = (F)$. Let F be of the form $F_1^{\alpha_1} \cdots F_r^{\alpha_r}$, where the polynomials F_i are irreducible and the sets $V(F_i)$ are infinite. Find the irreducible components of $V(F)$.*

Proof. Since F is irreducible then (F) is a prime ideal, since $k[X, Y]$ is a UFD. Also we know that $I(V(F)) = \sqrt{(F)}$. Now since (F) is prime then $\sqrt{(F)} = (F)$. That is, if $\mathfrak{p}$ is prime and $f^{n-1}f = f^n \in \mathfrak{p}$ then $f \in \mathfrak{p}$ or $f^{n-1} \in \mathfrak{p}$, by primality. Coninuing inductively gives $f \in \mathfrak{p}$. Hence $I(V(F)) = (F)$. Furthermore $V(F)$ is irreducible since it is the variety of a prime ideal (F).

Now suppose that $F = F_1^{\alpha_1} \cdots F_r^{\alpha_r}$. Now I claim that $V(F) = V(F_1^{\alpha_1}) \cup V(F_r^{\alpha_r})$. This is easy to see. If $F(x) = 0$ then $0 = F_1(0) \cdots F_r(0)$ and at least one of $F_i(x) = 0$ because k is an integral domain. Thus we have $V(F) \subset V(F_1^{\alpha_1}) \cup V(F_r^{\alpha_r})$. Furthermore if $F_i(x) = 0$ then of course $F(x) = 0$ because F_i is a factor of F. Hence $V(F) = V(F_1^{\alpha_1}) \cup V(F_r^{\alpha_r})$. These $V(F_i^{\alpha_i})$ are irreducible by the last sentence of the previous part of this question, hence they are the irreducible components. $\square$

Problem 4. *Let X be any topological space.*

a *If X is irreducible and U is an open subset of X, show that U is irreducible.*

b *If X is of the form $U_1 \cup U_2$, where the sets U_i are open and irreducible, and $U_1 \cap U_2 \neq \varnothing$, show that X is irreducible.*

c *If $Y \subset X$ and Y is irreducible, show that $\overline{Y}$ is irreducible.*

Proof of a. I will prove this by contrapositive. Suppose that $U \subset X$ is open and not irreducible. This means there are non trivial relatively closed sets U_1 and U_2 of U such that $U_1 \cup U_2 = U$. Now by definition of the relative topology there are sets, closed in X, such that $Y_1 \cap U = U_1$ and $Y_2 \cap U = U_2$. Let $C = X \setminus U$. Now I claim that $X_1 := Y_1 \cup C$ and $X_2 := Y_2 \cup C$ are nontrivial closed sets in X whose union is X. That is, X_1 and X_2 show that X is not irreducible. First these sets are unions of closed sets and hence closed. Of course, X_1 and X_2 are neither equal to X because each of their intersections with U is not all of U. Finally, we see that

$$\begin{aligned} X_1 \cup X_2 &= (Y_1 \cap U) \cup (Y_2 \cap U) \cup C \\ &= ((Y_1 \cup Y_2) \cap U) \cup C \\ &= U \cup C = X. \end{aligned}$$

Hence X is not irreducible. □

Proof of b. Suppose that $X = U_1 \cup U_2$ with $U_1 \cap U_2 \neq \varnothing$. Suppose that X is not irreducible. We will again use the contrapositive to prove this. So we will show that either U_1 or U_2 is not irreducible. By definition, there are nontrivial closed sets X_1 and X_2 such that $X = X_1 \cup X_2$. First of all, if $U_i \subset X_j$ for some i, j then either U_1 or U_2 is not irreducible and we are done. Without loss of generality we can assume that $U_1 \subset X_1$ and $U_2 \subset X_2$. Since X_1 and X_2 are nontrivial then this means that $U_1 \not\subset X_2$ and $U_2 \not\subset X_1$. But we do have $U_1 \subset X_2 \cup (X \setminus U_2)$. This is because $U_2 \subset X_2$. Now $U_1 \not\subset X_2$ by assumption and $U_1 \not\subset X \setminus U_2$, because $U_1 \cap U_2 \neq \varnothing$. Hence U_1 is not irreducible. □

Proof of c. Suppose that Y is an open irreducible subset of some space reducible space A. If we show that there is a closed irreducible subset of A containing Y then this will prove that the closure of Y must be a proper subset of A. Since the only assumption we have on A is reducibility then this will show that the closure of Y is irreducible. Now since A is reducible then we can nontrivially represent A as $A = A_1 \cup A_2$, where A_1 and A_2 are nontrivial closed subsets of A. Now since Y is irreducible then either $Y = Y \cap A_1$ or $Y = Y \cap A_2$, because these are closed subsets of Y whose union is all of Y. Suppose that $Y = Y \cap A_1$. Hence $\overline{Y} \subset A_1$ and hence $\overline{Y} \neq A$. Thus A cannot be the closure of Y. Therefore by contrapositive we know that $\overline{Y}$ must be irreducible. □

Problem 5 (Irreducibility)**.** *A ring A is said to be connected if every idempotent in A is trivial.*

a *Prove that every integral domain is connected.*

b *If A is the direct product of two non-trivial rings, prove that A is not connected.*

c *Conversely, if A possesses a non-trivial idempotent e, prove that $A \cong A/(e) \times A/(1-e)$.*

d *Let V be an affine algebraic set over an algebraically closed field k. Prove that V is connected (in the Zariski topology) if and only if $\Gamma(V)$ is connected. (If V has two connected components, start by finding a function which is* 0 *on one and 1 on the other.) Is this still the case if K is not algebraically closed?*

Proof of a. Of course, every idempotent is a root of the polynomials $x^2 - x$. Now in an integral domain a polynomial f has at most $\deg f$ roots. Hence we can have no more that 2 idempotents. Thus every integral domain is connected. $\square$

Proof of b. Suppose that $A \cong B \times C$ where B and C are nontrivial. Then the unit of A is $1_B \times 1_C$ and the zero is $0_B \times 0_C$ (where $1_B, 0_B$ and $1_C, 0_C$ are the units and zeros of B and C respectively). Then the element $1_B \times 0_C$ is, of course, an idempotent. Since B and C are not trivial then this is a nontrivial idempotent. $\square$

Proof of c. We look at a map $\varphi : A \to A/(e) \times A/(1-e)$ given by $a \mapsto (\tilde{a}, \hat{a})$, where $\tilde{a}$ is reduction modulo (e) and $\hat{a}$ modulo $(1-e)$. Since the maps to each of the modulo rings are homomorphism then this map is too. Now of course, the kernel of the map is given by $a \in (e) \cap (1-e)$. Thus our map is injective if $(e) \cap (1-e) = \{0\}$. Now suppose $a \in (e) \cap (1-e)$. This means $a = \alpha e$ and $a = \beta(1-e)$ for some $\alpha, \beta \in A$. Then we see that $ea = \alpha e^2 = \alpha e = a$ and $ea = \beta e(1-e) = \beta(e - e^2) = 0$. Thus $a = 0$, and our map is injective. To show surjectivity, suppose that $(\tilde{a}, \hat{b}) \in A/(e) \times A/(1-e)$. Now pick any representatives a and b of $\tilde{a}$ and $\hat{b}$ respectively. Then we have

$$\varphi(a(1-e) + be) = (\tilde{\varphi}(a - ae + be), \hat{\varphi}(a(1-e) + be + b(1-e))) = (\tilde{a}, \hat{\varphi}(a(1-e) + b)) = (\tilde{a}, \hat{b}).$$

Thus our map is surjective. Hence we have the desired isomorphism. $\square$

Proof of d. Suppose that V is disconnected. This means V can be written as the disjoint union of two nontrivial relatively open sets X and Y. Since V is closed (affine algebraic) and X and Y are closed in the relative topology then X and Y are closed. This means that X and Y are both finite sets. Thus suppose $X = \{a_0, \ldots, a_n\}$ and $Y = \{b_0, \ldots, b_m\}$. Then if we let $p = \prod(x - a_i)$ and

$$f = p(x) \sum_{i=0}^{m} \frac{1}{p(b_i)} \prod_{j=0, j\neq i}^{m} (x - b_i),$$

it is clear that $f(b_i) = 1$ for all $i = 0, \ldots, n$ and $p(a_i) = 0$ for all $i = 0, \ldots, n$. Hence $f^2(x) = f(x)$ for all $x \in V$.

Suppose that $\Gamma(V)$ is disconnected. This means that $\Gamma(V)$ has a nontrivial idempotent e. Now $V(e) \neq \varnothing$ because e is not constant, since it is a nontrivial idempotent and fields only have 0 and 1 as idempotents. Furthermore for any $x \in V$ we see that $e^2(x) = e(x)$ so $e(x)$ must be either 0 or 1. Hence $V(e) \cup V(1-e) = V$. Thus V is disconnected.

We need k to be algebraically closed for the second half of the proof. That is, to assure that e and $1-e$ have roots for our idempotent, e. $\square$

Problem 6. *Assume that k is infinite. Determine the function rings A_i ($i = 1, 2, 3$) of the plane curves whose equations are $F_1 = Y - X^2$, $F_2 = XY - 1$, $F_3 = X^2 + Y^2 - 1$. Show that A_1 is isomorphic to the ring of polynomials $k[T]$ and that A_2 is isomorphic to its localised ring $k[T, T^{-1}]$. Show that A_1 and A_2 are not isomorphic (consider their invertible elements). What can we say about A_3 relative to the two other rings? (Treat separately the cases where -1 is or is not a square in k, and pay special attention to the characteristic 2 case.)*

Proof. We have the map $k[x,y] \to k[T]$ given by $x \mapsto T$ and $y \mapsto T^2$. This of course, maps onto $k[T]$. I claim that the kernel of this map is $(y - x^2)$. Of course this is in the kernel. Now given an item $f(x,y)$ in the kernel, we can reduce modulo $(y - x^2)$ which gives us

$$f(x,y) = g(x,y)(y - x^2) + h(x).$$

Now clearly the only way for this to be zero is for $h(x) = 0$. Hence $f(x,y) \in (y - x^2)$. Thus we have $A_1 = k[x,y]/(y - x^2) \cong k[T]$.

For A_2 we have the map $XY - 1$. We similarly construct a map $\varphi : k[x,y] \to k[T, T^{-1}]$ given by $x \mapsto T$ and $y \mapsto T^{-1}$, which is surjective. Also $(XY - 1) \subset \ker \varphi$. Now suppose that $f(x,y) \in \ker \varphi$. This gives us

$$f(x,y) = g(x,y)(XY - 1) + h(x) + \ell(y).$$

Thus in the image we have $h(T) + \ell(T^{-1})$. In $k[T, T^{-1}]$, T and T^{-1} are algebraically independent. Hence to have this in the kernel we just have $h(T) = 0 = \ell(T^{-1})$, and so $A_2 \cong k[T^{-1}, T]$, as desired.

It is easy to see that A_1 and A_2 are not isomorphic. Suppose we have a map $\varphi : k[T, T^{-1}] \to k[T]$. I claim this cannot be an isomorphism. If it were then $\varphi(-T)$ must be an invertible element. That is $\varphi(-T) \in k$. Now $\varphi(-T) \neq -1$, because $\varphi(-1)$ must be -1. Hence $\varphi(1 - T) = \varphi(1) + \varphi(-T) \in k$ since both $\varphi(1)$ and $\varphi(-T) \in k$. Also $\varphi(1-T) \neq 0$. Thus $\varphi(1-T)$ is invertible. But this is a problem since $1 - T$ is not invertible. In particular, the inverse of $1 - T$ would need to be $1 + T + T^2 + \cdots$ which is not in $k[T, T^{-1}]$.

Now we look at A_3. If k is characteristic 2 then

$$Y^2 + X^2 - 1 = Y^2 + X^2 + 1 = (Y + X + 1)^2.$$

Then we can easily map x to T and Y to $T + 1$. This is easily seen to give an isomorphism between A_3 and $k[T]$. □

Problem 7. *Let $f : k \to k^3$ be the map which associates (t, t^2, t^3) to t and let C be the image of f (the space cubic). Show that C is an affine algebraic set and calculate $I(C)$. Show that $\Gamma(C)$ is isomorphic to the ring of polynomials $k[T]$.*

Proof. I claim that $C = V(y - x^2, z - x^3)$. This is easy to see. If we need to satisfy $y - x^2$ then $y = x^2$ and $z - x^3$ gives us $z = x^3$. Hence a point must be (x, x^2, x^3) for any x. Thus $C = V(y - x^2, z - x^3)$.

Now $I(C) = \sqrt{y - x^2, z - x^3}$. I claim that $I(C) = (y - x^2, z - x^3)$. That is, the original ideal is radical. Suppose that $p \in I(C)$. We can successively devide by $y - x^2$ and $z - x^3$ with respect to y and z and we get

$$p = h(x,y,z)(y - x^2) + g(x,y,z)(y - x^2) + j(x).$$

Now we know that a point in C is of the form (t, t^2, t^3). If we then replace (x, y, z) with these values we get $p(t, t^2, t^3) = h(t)$. For this to be in C we must have $h(t) = 0$ for all t. Hence $h(t) = 0$ and we have $I(C) = (y - x^2, z - x^3)$.

Now of course the map $t \mapsto (t, t^2, t^3)$ is an isomorphism of algebraic varieties. It is a polynomial map on way and projection the other way. Hence $\Gamma(C) \equiv \Gamma(k) = k[T]$. □

Problem 8. *Assume that k is algebraically closed. Determine the ideals $I(V)$ of the following sets.*

$$V(XY^3 + X^3Y - X^2 + Y), V(X^2Y, (X - 1)(Y + 1)^2), V(Z - XY, Y^2 + XZ - X^2).$$

Proof. Let $F(X,Y) = XY^3 + X^3Y - X^2 + Y$. Since k is algebraically closed then $V = V(F)$ has infinitely many points. This is because for every x_0 we get a polynomial in Y, which must have a root. Thus we have infinitely many points in V. A quick maple command shows that F is irreducible. (One could also show irreducibility by assuming that $F = GH$ where $\deg G = 1$ and $\deg H = 3$ or $\deg G = \deg H = 2$. Then finding values for various coefficients until a contradiction is met.) Hence $I(V(F)) = (F)$ by problem 3.

We see that $Y * X^2Y = (XY)^2$ so we can replace X^2Y with XY. Furthermore $(X-1)*(X-1)(Y+1)^2 = ((X-1)(Y+1))^2$, so we can replace $(X-1)(Y+1)^2$ with $(X-1)(Y+1)$. So then we have $V(XY, (X-1)(Y+1))$, but this is $V(XY, XY - Y + X - 1)$, which can be replace with $V(XY, X - Y - 1)$. Finally we can replace the Y in the first equation with $X - 1$ and we have $V(X(X-1), X-Y-1)$. Then it is clear to see that $V(X(X-1), X-Y-1) = \{(0,-1),(1,0)\}$. Now suppose $p \in I(V(X(X-1), X-Y-1))$. We can reduce modulo $X-Y-1$ with respect to Y and we get

$$p = h(X,Y)(X-Y-1) + j(X).$$

This needs to be zero on $(0,-1)$ and $(1,0)$. In the former case we have $p(0,-1) = j(0)$ and in the latter we have $p(1,0) = j(1)$. Hence $j(0) = 0$ and $j(1) = 0$. Hence $j(X) = X(X-1) * \ell(X)$ for some polynomial $\ell(X)$. Thus $p \in (X(X-1), X-Y-1)$ and so $I(V(X^2Y, (X-1)(Y+1)^2)) = (X(X-1), X-Y-1)$. $\square$