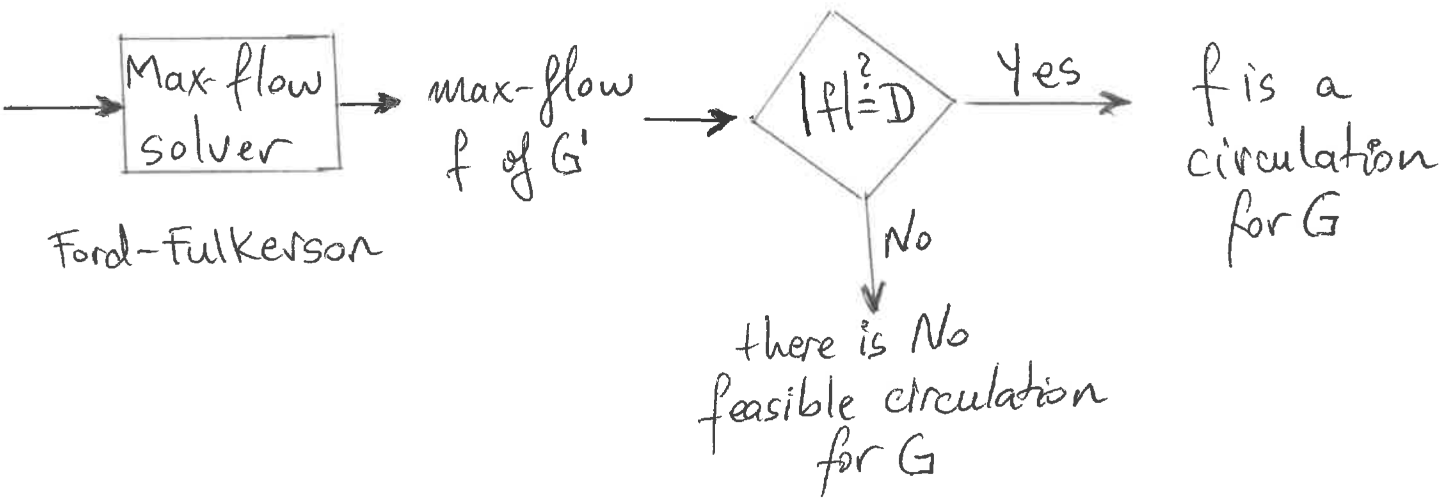
Circulations with demands

Input

- directed graph $G(V, E)$
- each vertex v has a demand d_v
- capacity c of each edge

Flow-network (standard form)

- directed graph $G'(V', E')$
- add super-source s^* and super-sink t^*
- Keep the edges in G and their capacities
- add edges from s^* to each source node v with capacity $-d_v$
- add edges from each sink node v to t^* with capacity d_v



RT analysis

Ford-Fulkerson $\Rightarrow RT = O(E' \cdot |f^*|)$

$$|f^*| \le D$$

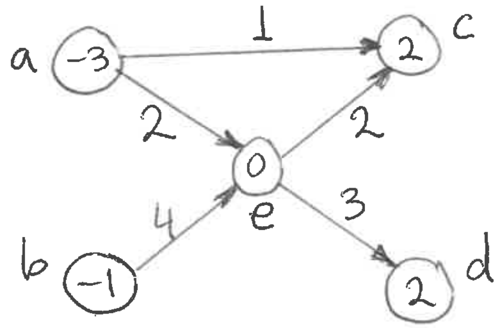
$$\left.\begin{array}{l} |E'| \le |E| + |V| \\ \text{graph } G \text{ is connected} \Rightarrow |E| \ge |V| - 1 \Rightarrow |V| = O(E) \end{array}\right\} \Rightarrow$$

$\Rightarrow |E'| = O(E)$

$$\boxed{RT = O(E \cdot D)}$$

Example

Consider the graph $G(V,E)$ with the following capacity and demand constraints:



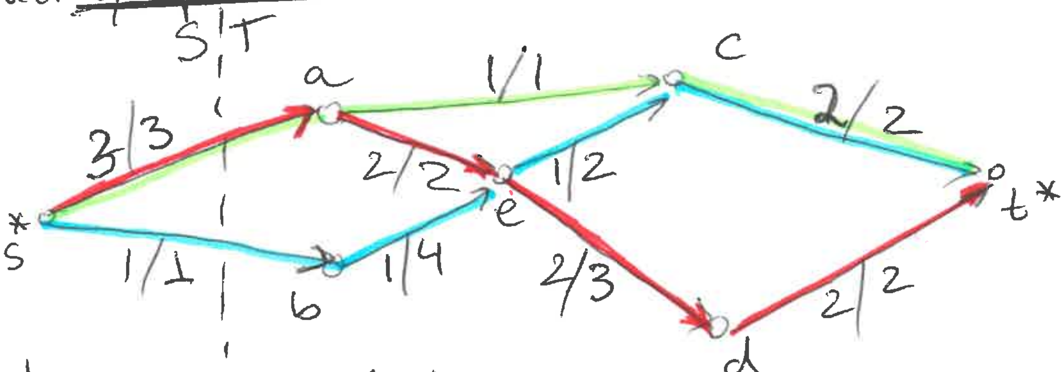
Find a feasible circulation.

Solution

a, b - source nodes $\quad D = 4$

c, d - sink nodes

construct graph G'



compute max-flow of G'

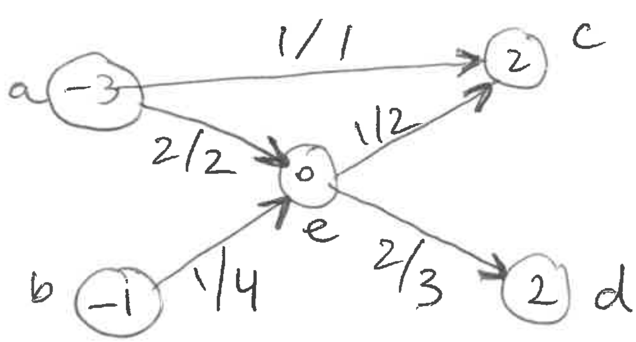
$p = \langle s^*, a, e, d, t^* \rangle \quad c_f(p) = 2$

$p = \langle s^*, b, e, c, t^* \rangle \quad c_f(p) = 1$

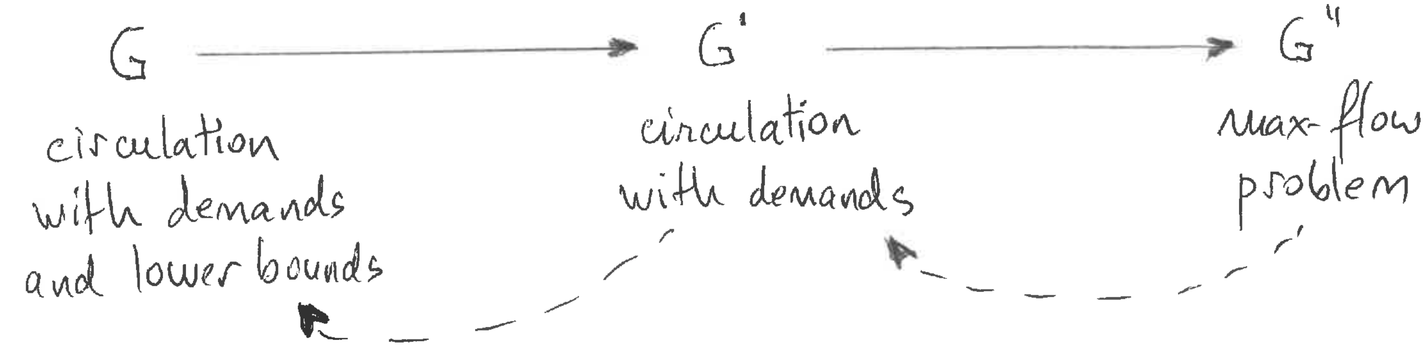
$p = \langle s^*, a, c, t^* \rangle \quad c_f(p) = 1$

$|f| = 4$. Since $|f| = D \Rightarrow G$ has a feasible circulation:

graph G



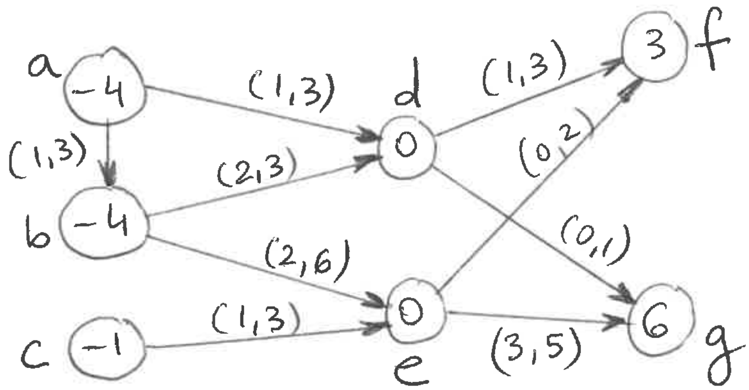
Circulations with demands and lower bounds



Example

Consider the graph $G(V, E)$ below, with the following constraints for demands, capacity and lower bounds:

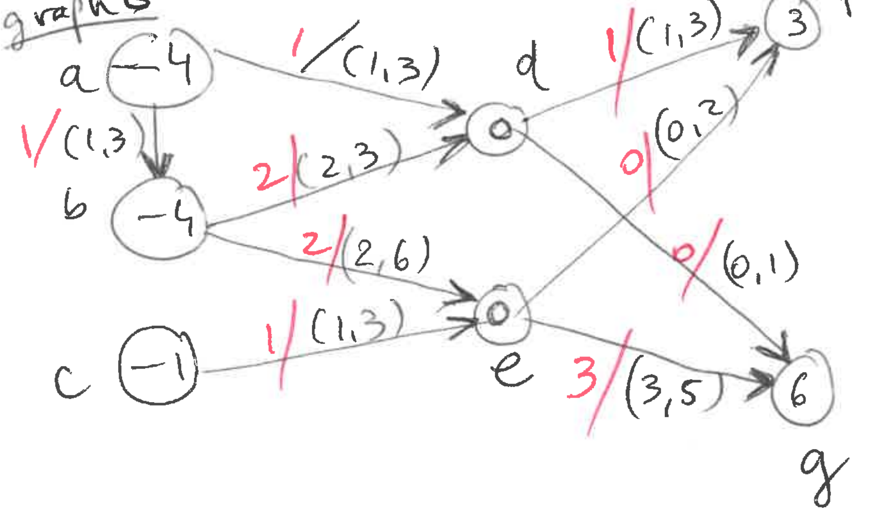
graph G



Find a feasible circulation.

Solution

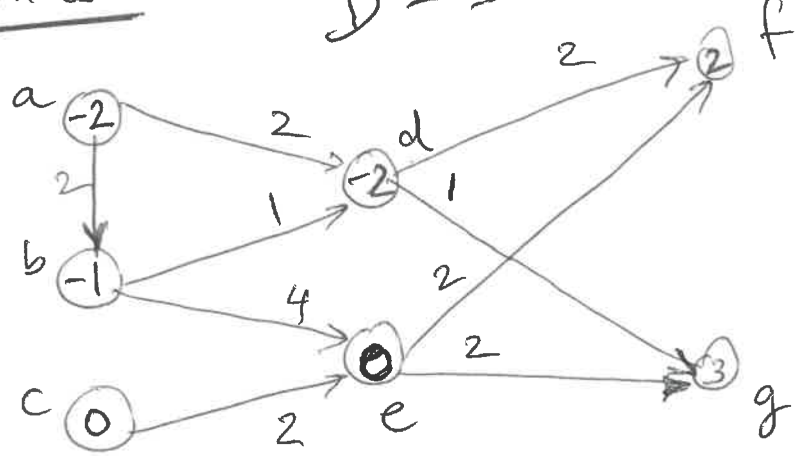
graph G



flow f_0

- started with a circulation $f_0(e) = \ell_e$

graph G' $D = 5$



construct G'

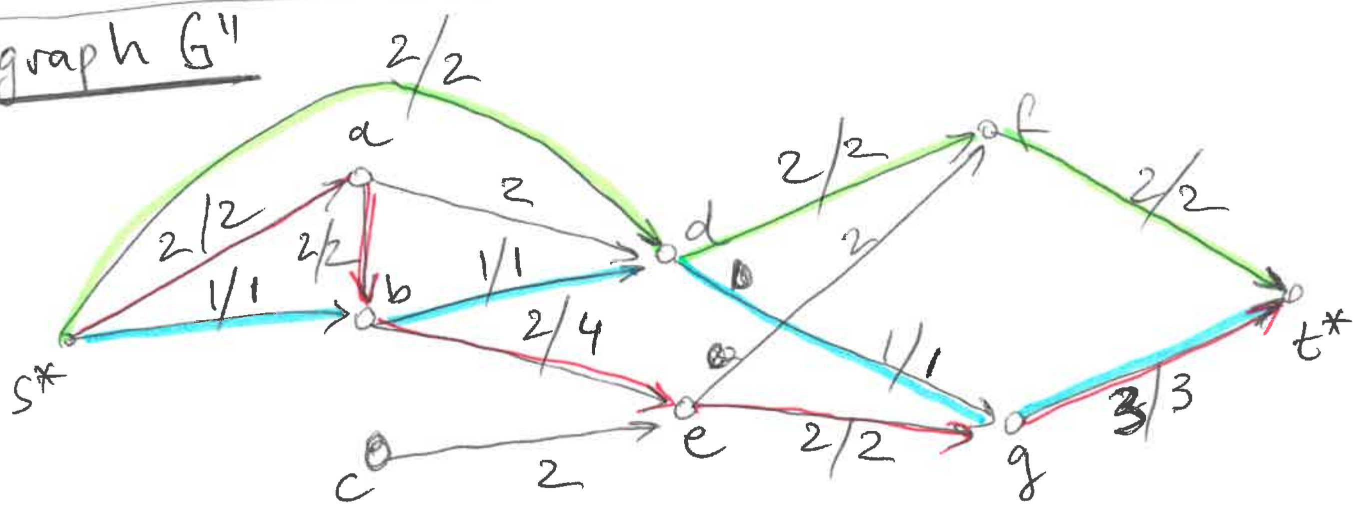
- capacity of each edge is computed as $c_e - \ell_e$
- compute the demands of each vertex

$$\left\{\begin{array}{l} L_v = f_0^{in}(v) - f_0^{out}(v) \\ \text{new demand} = d_v - L_v \end{array}\right.$$

vertex a	vertex b	vertex c
$L_a = 0 - 2 = -2$	$L_b = -3$	$L_c = -1$
new demand $= -4 - (-2) = -2$	new demand $= -4 - (-3) = -1$	new demand $= -1 - (-1) = 0$

vertex d	vertex e	vertex f	vertex g
$L_d = 2$	$L_e = 0$	$L_f = 1$	$L_g = 3$
new demand $= 0 - 2 = -2$	new demand $= 0 - 0 = 0$	new demand $= 3 - 1 = 2$	new demand $= 6 - 3 = 3$

graph G''



find a maximum flow in G''

$p = \langle s^*, a, b, e, g, t^* \rangle \quad c_f(p) = 2$

$p = \langle s^*, d, f, t^* \rangle \quad c_f(p) = 2$

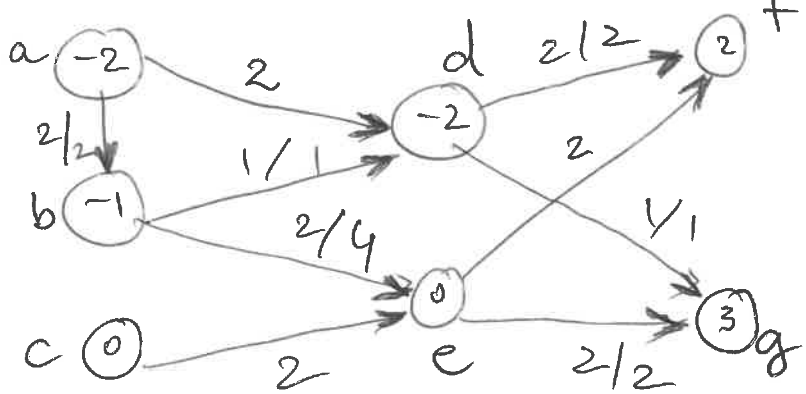
$p = \langle s^*, b, d, g, t^* \rangle \quad c_f(p) = 1$

$|f| = 5$

since $|f| = D \Rightarrow$ graph G' has a feasible circulation $\Rightarrow$

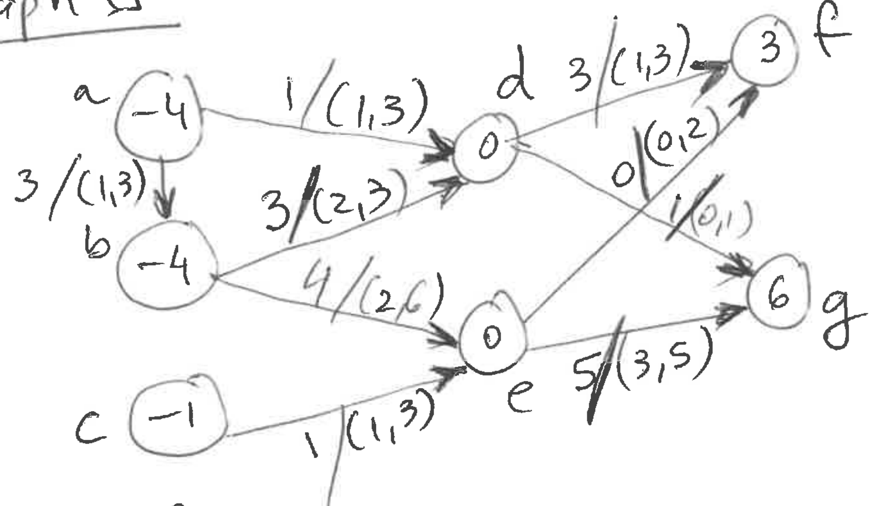
$\Rightarrow$ graph G has a feasible circulation!

graph G'



flow f'

graph G



f'+f0

final solution ← circulation for graph G