Recurrences

Forward Substitution

$$\begin{cases} T(n) = 2\cdot T(n-1) + 1 & \text{for } n>1 \\ T(1) = 1 \end{cases}$$

$T(2) = 2\cdot T(1) + 1 = 2\cdot 1 + 1 = 3 = 2^2 - 1$

$T(3) = 2\cdot 3 + 1 = 7 = 2^3 - 1$

$T(4) = 2\cdot 7 + 1 = 15 = 2^4 - 1$

...

$T(n) = 2^n - 1$

check validity

$T(n) = 2\cdot T(n-1) + 1$

$2^n - 1 \stackrel{?}{=} 2(2^{n-1} - 1) + 1$

$2^n - 1 \stackrel{?}{=} 2^n - 2 + 1$ ✓

$T(1) = 1$

$2^1 - 1 \stackrel{?}{=} 1$ ✓

Backward Substitution

$$\begin{cases} T(n) = T(n-1) + n & n>0 \\ T(0) = 0 \end{cases}$$

$T(n) = T(n-1) + n$

$= T(n-2) + (n-1) + n$

$= T(n-3) + (n-2) + (n-1) + n$

....

$= T(n-i) + (n-i+1) + (n-i+2) + \ldots + n$

$\stackrel{(i=n)}{=} \cancel{T(0)} + 1 + 2 + 3 + \ldots + n$

$\underset{\text{arithmetic series}}{=} \dfrac{n(n+1)}{2} = \Theta(n^2)$

$\boxed{T(n) = \Theta(n^2)}$

Master Theorem

$$T(n) = a \cdot T\left(\frac{n}{b}\right) + f(n)$$

$a \geq 1$

$b > 1$

case 1: $f(n) = O(n^{\log_b a - \varepsilon})$ for some const. $\varepsilon > 0$ $\Rightarrow$ $\boxed{T(n) = \Theta(n^{\log_b a})}$

case 2: $f(n) = \Theta(n^{\log_b a})$ $\Rightarrow$ $\boxed{T(n) = \Theta(n^{\log_b a} \cdot \lg n)}$

case 3: $f(n) = \Omega(n^{\log_b a + \varepsilon})$ for some const. $\varepsilon > 0$

regularity condition:

$a \cdot f\left(\frac{n}{b}\right) \leq c \cdot f(n)$ for some const. $c < 1$

$\Rightarrow$ $\boxed{T(n) = \Theta(f(n))}$

- Solve the following recurrences using the Master Theorem:

$$T(n) = 8T\left(\frac{n}{2}\right) + 5n^3$$

$$T(n) = 3T\left(\frac{n}{2}\right) + n^3 \lg n$$

$$T(n) = 12\,T\left(\frac{n}{3}\right) + \sqrt{n}$$

- $T(n) = 8 \cdot T\left(\frac{n}{2}\right) + 5n^3$

$a = 8$ $\quad f(n) = 5n^3$

$b = 2$

$f(n)$ vs. $n^{\log_b a}$

$5n^3$ vs $n^{\log_2 8} = n^3$

$5n^3 = \Theta(n^3)$

case 2 of the Master Thm $\Rightarrow$ $\boxed{T(n) = \Theta(n^3 \cdot \lg n)}$

- $T(n) = 3 \cdot T\left(\frac{n}{2}\right) + n^3 \lg n$

$f(n)$ vs $n^{\log_b a}$

$\underline{n^3 \lg n}$ vs $n^{\log_2 3} = \underline{n^{1.58}}$

$n^3 \lg n = \Omega\left(n^{1.58+\varepsilon}\right)$ for $\varepsilon = 1.42$

<u>regularity condition:</u>

$a f\left(\frac{n}{b}\right) \le c f(n)$ $\quad \underline{c < 1}$

$3 \cdot \left(\frac{n}{2}\right)^3 \lg\left(\frac{n}{2}\right) \le c \cdot n^3 \lg n$

$3 \cdot \frac{\cancel{n^3}}{8}\left(\lg n - \underset{1}{\lg 2}\right) \le c \cancel{n^3} \lg n$

$\frac{3}{8} \lg n - \frac{3}{8} \le c \lg n$

$\left(\frac{3}{8} - c\right) \lg n \le \frac{3}{8}$

true for $\underline{c = \frac{3}{8}}$ ✓

case 3 of the Master Thm $\Rightarrow$ $\boxed{T(n) = \Theta(n^3 \lg n)}$

- $T(n) = 12 T\left(\frac{n}{3}\right) + \sqrt{n}$

$f(n)$ vs $n^{\log_b a}$

$\underline{\sqrt{n}}$ vs. $n^{\log_3 12} = \underline{n^{2.26}}$

$\sqrt{n} = O\left(n^{2.26-\varepsilon}\right)$ for $\varepsilon = 1$

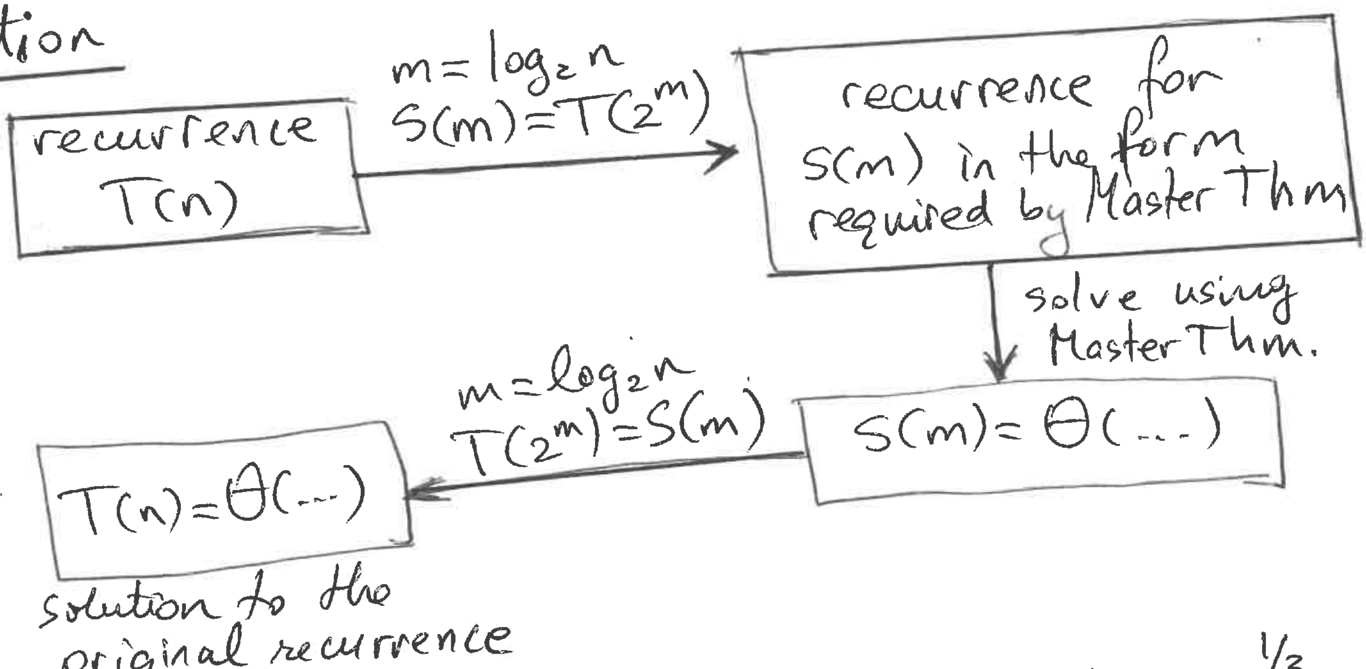
case 1 of the Master Thm $\Rightarrow$ $\boxed{T(n) = \Theta(n^{2.26})}$

Change of variable technique

- Solve the recurrence using the change of variable $m = \log_2 n$

$$T(n) = T(\sqrt[3]{n}) + (\log_2 n)^2$$

Solution



$$\sqrt[3]{n} = n^{1/3} = \left(2^m\right)^{1/3} = 2^{m/3}$$

$$m = \log_2 n \Rightarrow n = 2^m$$

$$T(2^m) = T(2^{m/3}) + m^2$$

$$S(m) = T(2^m)$$

$$S(m) = S(m/3) + m^2$$

m^2 vs. $m^{\log_3 1} = m^0 = 1$

$m^2 = \Omega(m^\varepsilon)$ for $\varepsilon = 2$

regularity condition

$$a \cdot f\left(\frac{m}{b}\right) \le c \cdot f(m) \qquad c < 1$$

$$\left(\frac{m}{3}\right)^2 \le c\, m^2$$

$$\frac{m^2}{9} \le cm^2$$

$$\frac{1}{9} \le c \quad \checkmark$$

case 3 of the Master Thm $\Rightarrow \boxed{S(m) = \Theta(m^2)}$

$T(2^m) = S(m)$

$T(2^m) = \Theta(m^2)$

$m = \log_2 n \Rightarrow n = 2^m$

$\boxed{T(n) = \Theta(\lg^2 n)}$ solution to the original recurrence.

- Solve the recurrence:

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n \lg n$$

Solution

- try to solve using Master Thm.

$$f(n) \text{ vs } n^{\log_b a}$$

$$\underline{n \lg n} \text{ vs } n^{\log_2 2} = \underline{n}$$

$$n \lg n = \Omega(n^{1+\varepsilon}) \text{ for some } \varepsilon > 0$$

we cannot find a const $\varepsilon > 0$ that works $\Rightarrow$ cannot use the master Theorem!

$$n \cdot \underline{\lg n} \text{ vs. } n \cdot \underline{n^{\varepsilon}}$$

- Backward substitution

- take $T(1) = \Theta(1)$
- assume that n is a power of 2 $\Rightarrow n = 2^k$

$k = \lg n$

$$T(n) = 2 \cdot T\left(\frac{n}{2}\right) + n \lg n$$

$$= 2 \cdot \left(2 \cdot T\left(\frac{n}{4}\right) + \frac{n}{2} \lg \frac{n}{2}\right) + n \lg n =$$

$$= 4 \cdot T\left(\frac{n}{4}\right) + n \lg \frac{n}{2} + n \lg n =$$

$$= 4 \cdot \left(2 \cdot T\left(\frac{n}{8}\right) + \frac{n}{4} \lg \frac{n}{4}\right) + n \lg \frac{n}{2} + n \lg n =$$

$$= 8 \cdot T\left(\frac{n}{8}\right) + n \lg \frac{n}{4} + n \lg \frac{n}{2} + n \lg n =$$

$$\ldots$$

$$= 2^k \cdot T\left(\frac{n}{2^k}\right) + n \lg \frac{n}{2^{k-1}} + n \lg \frac{n}{2^{k-2}} + \ldots + n \lg \frac{n}{2^0}$$

$$= 2^k \cdot T(1) + n \cdot \lg 2 + n \lg 2^2 + \ldots + n \lg 2^k =$$

$$= n \cdot \Theta(1) + n(\lg 2 + \lg 2^2 + \ldots + \lg 2^k) =$$

$$= \Theta(n) + n \lg 2^1 \cdot 2^2 \cdot 2^3 \cdot \ldots \cdot 2^k = \Theta(n) + n \cdot \lg 2^{1+2+3+\ldots+k} =$$

$$= \Theta(n) + n(\underbrace{1+2+3+\ldots+k}_{\text{arithmetic series}}) = \Theta(n) + n \cdot \frac{k \cdot (k+1)}{2}$$

$$= \Theta(n) + n \cdot \frac{\lg n \cdot (\lg n + 1)}{2} = \Theta(n \cdot \lg^2 n)$$

$$\boxed{T(n) = \Theta(n \cdot \lg^2 n)}$$