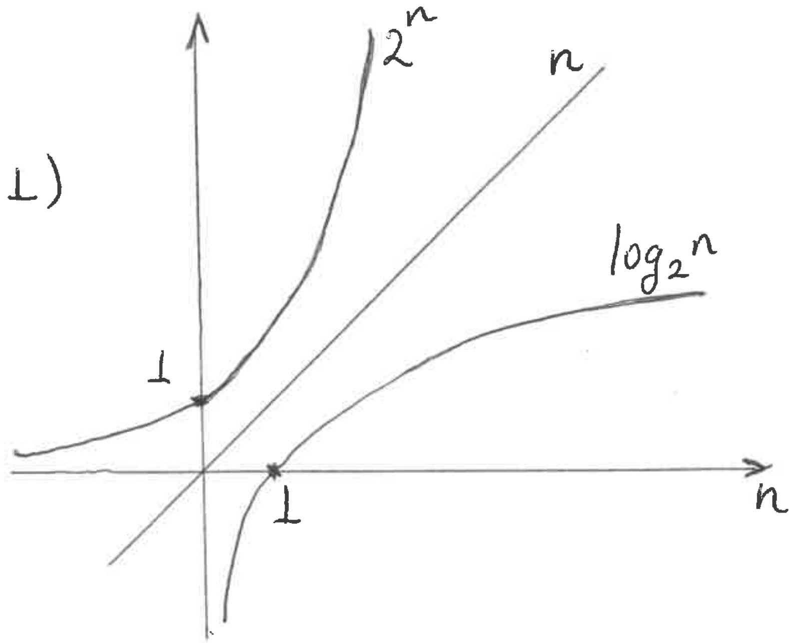
Rates of growth between polylogarithmic, polynomial, and exponential functions



Rule 1: asymptotically, any exponential function (with base > 1) grows faster than any polynomial function.

Rule 2: asymptotically, any positive polynomial function grows faster than any polylogarithmic function.

Log properties

$$\log_c(ab) = \log_c a + \log_c b$$

$$\log_c a/b = \log_c a - \log_c b$$

$$\log_c a^n = n \cdot \log_c a$$

$$a^{\log_c b} = b^{\log_c a}$$

proof: $\log_c(a^{\log_c b}) \overset{?}{=} \log_c(b^{\log_c a})$

$$\log_c b \cdot \log_c a \overset{?}{=} \log_c a \cdot \log_c b \checkmark$$

Notation

$$\lg x = \log_2 x$$

$$\boxed{a^{\lg b} = b^{\lg a}}$$

Notation

$$\log_3^7 x = (\log_3 x)^7$$

$$\log_2^7 x = \lg^7 x = (\lg x)^7$$

Observation: the base of the log is not relevant when we compute asymptotic notations.

For example: $\log_5 n = \dfrac{\log_2 n}{\log_2 5} = \dfrac{\log_2 n}{2.32} = \dfrac{1}{2.32} \cdot \lg n = \Theta(\lg n$

$$\log_5 n = \Theta(\lg n)$$

- For each of the expressions below, indicate whether it is TRUE or FALSE.

$2^n + 5\lg n = O(n) \rightarrow$ FALSE

$5n^3 + 2n - 100 = \Omega(\lg^7 n) \rightarrow$ TRUE

$n^5 + 7n + n^2\lg n = o(2^n) \rightarrow$ TRUE

$2n^3 - 7n + 100 = \Theta(2^n) \rightarrow$ FALSE

$n^5 + 7n^6 + n^2\lg n = \Omega(n) \rightarrow$ TRUE

$\lg^{100} n + 5000 = o(n^2) \rightarrow$ TRUE

$n^3\lg n + n^2 - 1000 = \omega(n^3\sqrt{n}) \rightarrow$ FALSE

- Find the Θ-notation for each of the following expressions:

$8n^{10} + 392\, n^2 \cdot \lg^{50} n + 500 = \Theta(n^{10})$

$n^{200} + 10^5 + 2^n = \Theta(2^n)$

$n^2 + n(\log_2 n)^{250} + 3^{100} = \Theta(n^2)$

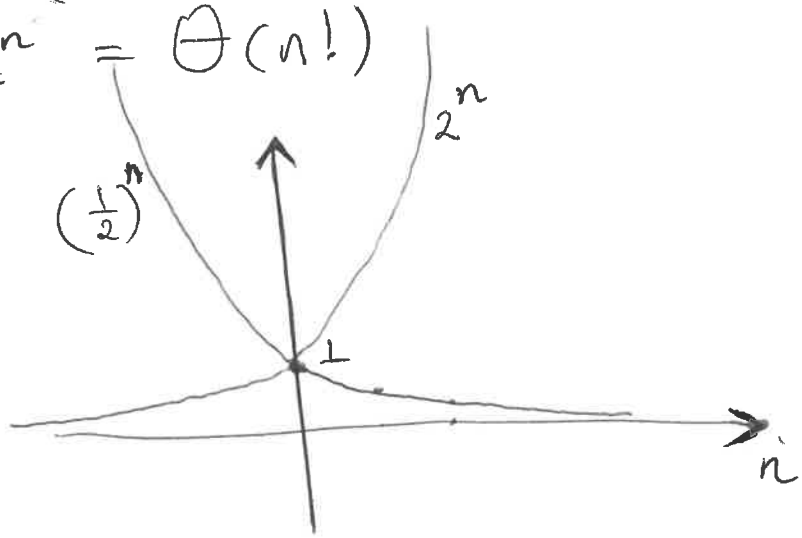
$6n^3 + 12\, n \cdot \lg^{10} n + \left(\frac{1}{2}\right)^n = \Theta(n^3)$

$n^{10}\lg n + n! + 2^n = \Theta(n!)$

$n^{10} = n^2 \cdot n^8$

$n \cdot n + n \cdot \lg^{250} n$

$6 \cdot n \cdot n^2 + 12 n \cdot \lg^{10} n + \cancel{\left(\frac{1}{2}\right)^n}$



• Use limits to compute asymptotic notations. Fill-out the table below with yes/no values.

$f(n)$	$g(n)$	$f(n)=O(g(n))$	$f(n)=o(g(n))$	$f(n)=\Omega(g(n))$	$f(n)=\omega(g(n))$	$f(n)=\Theta(g(n))$
$3^n+\lg n$	n^{100}	no	no	yes	yes	no
$3^{\lg n}+100$	$n^{\log_2 5}$	yes	yes	no	no	no
n^4+10	$n^3\lg^2 n+3n^4$	yes	no	yes	no	yes
n	$n^{1+\sin n}$	—	—	—	—	—
$\lg^2 n$	n^7	yes	yes	no	no	no

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{3^n+\lg n}{n^{100}} = \infty \text{ (Rule 1)} \Rightarrow \omega, \Omega$$

$$\lim_{n\to\infty}\frac{3^{\lg n}+100}{n^{\log_2 5}} = \lim_{n\to\infty}\frac{n^{\lg 3}+100}{n^{\lg 5}} = \lim_{n\to\infty}\frac{n^{1.58}+100}{n^{2.32}} = 0 \Rightarrow o, O$$

$$3^{\lg n} = n^{\lg 3} = n^{1.58}$$

$$\lim_{n\to\infty}\frac{n^4+10}{n^3\lg^2 n+3n^4} = \frac{1}{3} \Rightarrow O, \Omega, \Theta$$

$$n^3\lg^2 n + 3n^3\cdot n$$

$$\lim_{n\to\infty}\frac{n^1}{n^{1+\sin n}} = \text{undefined}$$

$$-1 \le \sin n \le 1$$

$$0 \le 1+\sin n \le 2$$

$$\lim_{n\to\infty}\frac{\lg^2 n}{n^7} = 0 \Rightarrow o, O$$

Using limits to compute order of growth between functions

Limit value	Asymptotic Notation
$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = 0$	f(n) = o(g(n))
$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = \infty$	f(n) = ω(g(n))
$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} < \infty$	f(n) = O(g(n))
$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} > 0$	f(n) = Ω(g(n))
$0 < \lim\limits_{n\to\infty} \frac{f(n)}{g(n)} < \infty$	f(n) = θ(g(n))
$\lim\limits_{n\to\infty} \frac{f(n)}{g(n)} = undefined$	cannot use

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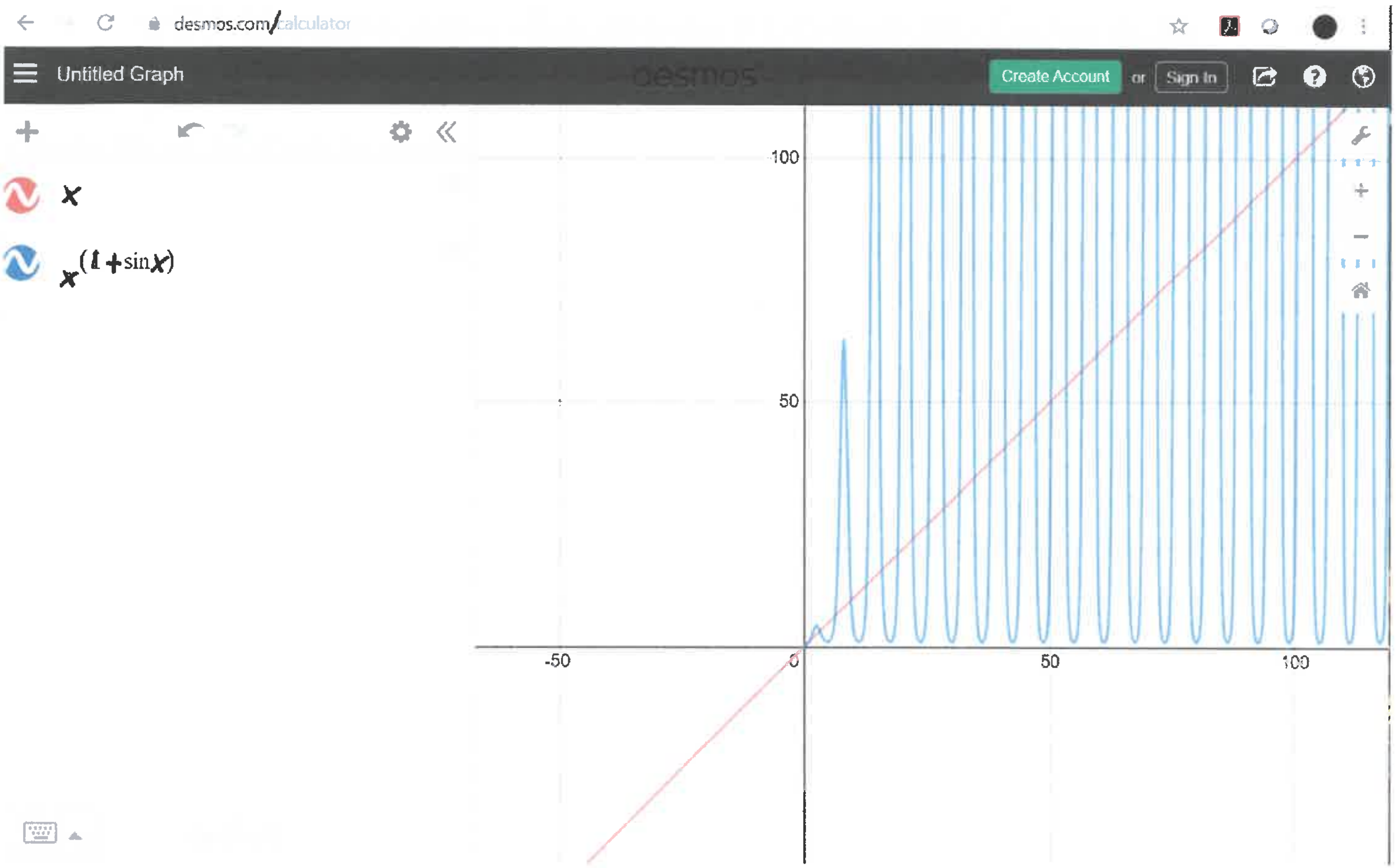

• Find Θ-notation for the following expressions:

$$(2+4+6+\ldots+2n)+n\lg n = 2\underbrace{(1+2+3+\ldots+n)}_{\text{arithmetic series}}+n\lg n =$$

$$= \cancel{2}\cdot\frac{n(n+1)}{\cancel{2}} + n\lg n = n^2+n+n\lg n = \Theta(n^2)$$

$$1+2+4+\ldots+2^n \underset{\uparrow}{=} \frac{2^{n+1}-1}{2-1} = 2^{n+1}-1 = 2\cdot 2^n - 1 = \Theta(2^n)$$

geometric series

$$1+\frac{1}{5}+\frac{1}{25}+\ldots+\left(\frac{1}{5}\right)^n \underset{\uparrow}{=} \frac{1}{1-\frac{1}{5}} = \frac{1}{\frac{4}{5}} = \frac{5}{4} = \Theta(1)$$

geometric series with $|x|<1$

• Arrange the following functions in ascending order of growth rate.

$f_1(n) = n^{2.5}$ $\qquad n^2 \cdot n^{0.5}$

$f_2(n) = \sqrt{2n}$

$f_3(n) = n + 10^9$ $\qquad e = 2.718$

$f_4(n) = 2^n + n^2 + \lg^5 n$

$f_5(n) = e^n$

$f_6(n) = n^2 \lg^{100} n$

$f_7(n) = n!$

<u>Solution</u>: $f_2, f_3, f_6, f_1, f_4, f_5, f_7$