COT 6405
ANALYSIS OF ALGORITHMS

Extensions to the Maximum-Flow Problem

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Outline

- Circulations with demands
- Circulations with demands and lower bounds

Reference: *Algorithm Design*, J. Kleinberg and E. Tardos, Addison-Wesley Publishing Company, 2006. Chapter 7.

Circulations with demands

In this problem:

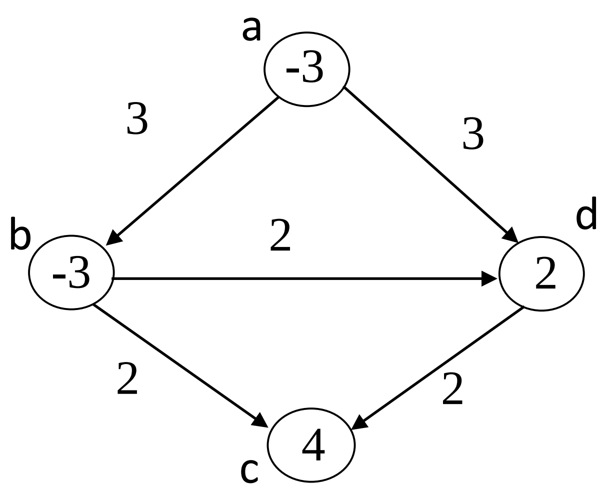
- Sources have fixed *supply* values
- Sinks have fixed *demands* value

Goal:

- Not to maximize the flow !
- Find a solution to satisfy all the demands using the available supply (e.g. circulate flow from the nodes with supply to the nodes with demands)

This is a ***feasibility problem***

Circulations with demands



Flow network G:

- Has capacities on the edges
- Each node $v \in V$ has a demand d_v
  - if $d_v > 0$, then the node is a **sink**: node v has a **demand** of d_v for flow, meaning it has to receive d_v more units of flow than it sends out
  - if $d_v < 0$, then the node is a **source**: node v has a **supply** of $-d_v$ for flow, meaning it wishes to send out $-d_v$ units more flow than it receives
  - if $d_v = 0$, then node v is neither a source nor a sink
- All capacities and demands are integers

Circulations with demands

A ***circulation*** with demands $\{d_v\}$ is a function f that assigns a nonnegative number to each edge such that:

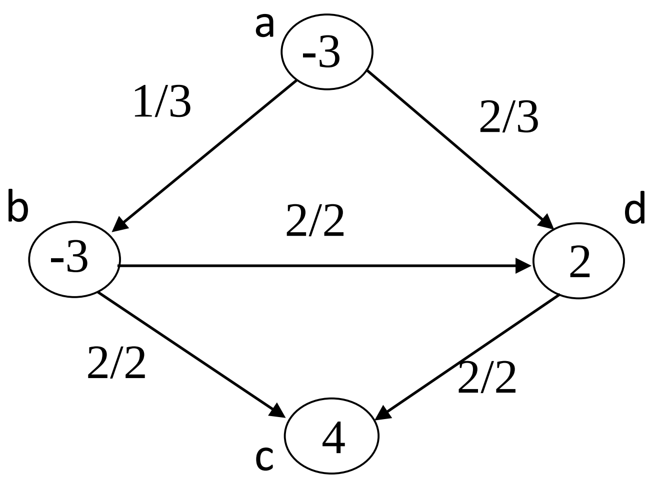
(i) *capacity condition*: for each $e \in E$, $0 \leq f(e) \leq c_e$

(ii) *demand conditions*: for each $v \in V$, $f^{in}(v) - f^{out}(v) = d_v$

We have a ***feasibility problem***: find whether there exists a circulation that meets conditions (i) and (ii).

Example:

- flow shows a feasible circulation



Circulations with demands

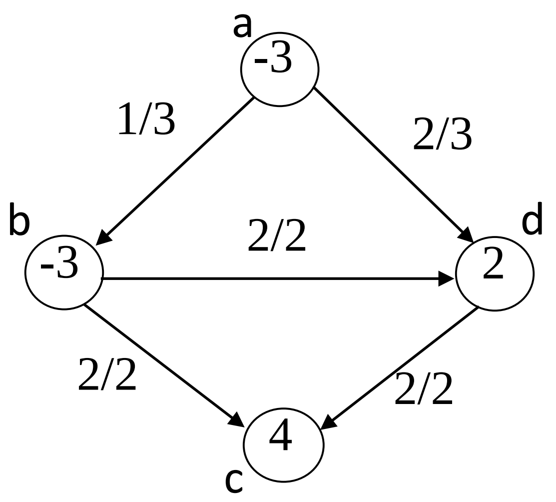
If there exists a feasible circulation with demands $\{d_v\}$, then $\Sigma_v\, d_v = 0$

Proof:

- suppose there is a feasible circulation f

$$\Sigma_v\, d_v = \Sigma_v\, (\, f^{in}(v) - f^{out}(v)\,)$$

- for each edge e = (u,v), f(e) is counted twice, as $f^{out}(u)$ and $f^{in}(v)$ $\Rightarrow$ these two terms cancel out
- it follows that $\Sigma_v\, d_v = 0$

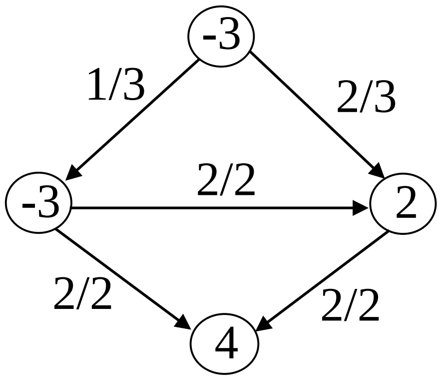


We can also write as: $\boxed{\displaystyle\sum_{v:d_v>0} d_v = \sum_{v:d_v<0} -d_v}$ = D

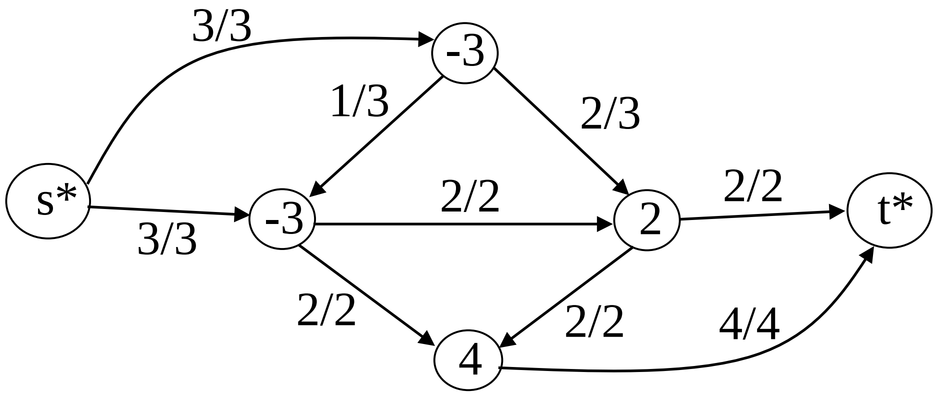
Algorithm for circulations

- Reduce the feasible circulation problem to a max-flow problem
- Let S – set with all the source nodes, T – set with all the sink nodes
- Create a graph G’ from G:
  - add a “super-source” s*
  - add a “super-sink” t*
  - for each node $v \in T$, add an edge (v, t*) with capacity d_v
  - for each node $u \in S$, add an edge (s*,u) with capacity $–d_u$
- Find a max-flow in G’
- If the max-flow s*-t* in G’ equals D, then G has a feasible demand circulation, given by the flow. Otherwise G does not have a feasible circulation that meets the demands.

Example



graph G



graph G’

- the flow in G’ cannot be larger than D
- take a cut (A,B) with A = {s*}, then c(A,B) = D

Analysis

There is a feasible circulation with demands $\{d_v\}$ in G iff the max-flow s*-t* in G' is D. If all capacities and demands in G are integers, and there is a feasible-circulation, then the feasible circulation is integer-valued.

Proof:

• If there is a feasible circulation with demands $\{d_v\}$ in G, by sending a flow $-d_v$ on each edge (s*,v) and a flow d_v on each edge (v,t*) we get a flow in G' of value D $\Rightarrow$ max-flow

• If G' has a max-flow with value D, then every edge out of s*, into t* must be saturated with flow. If we remove these edges $\Rightarrow$ circulation f in G with $f^{in}(v) - f^{out}(v) = d_v$ for each node v

Circulations with demands and lower bounds

- to force flow to make use of certain edges, we can place *lower bounds* on edges

Problem Definition Given:
- a flow network $G = (V,E)$ with capacity c_e and lower bound l_e on each edge, $0 \leq l_e \leq c_e$ for each $e \in E$
- each node v has a demand d_v (positive or negative)
- all demands, capacities and lower bounds are integers

A *circulation* f must satisfy the conditions:

(i) capacity condition: for each $e \in E$, $l_e \leq f(e) \leq c_e$

(ii) demand conditions: for each $v \in V$, $f^{in}(v)-f^{out}(v) = d_v$

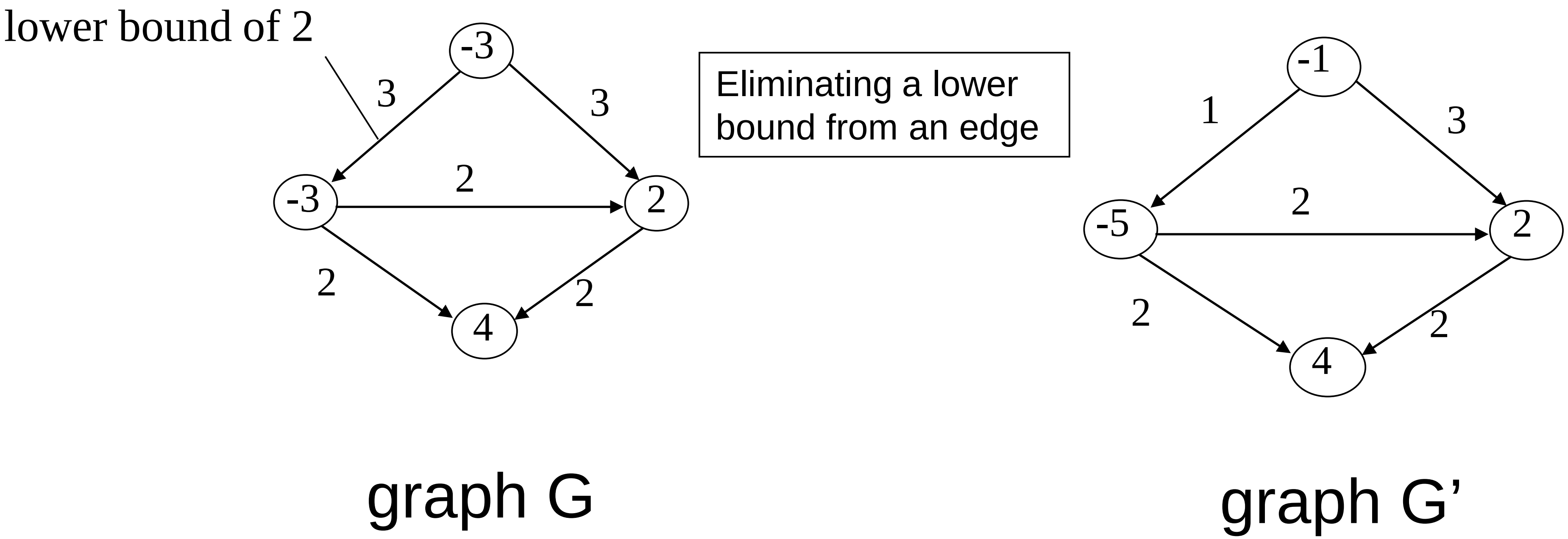
Find whether there is a *feasible circulation.*

Strategy

- reduce this problem to the problem of finding a circulation with demands but no lower bounds (which can be then reduced to the max-flow problem)
- start with an initial circulation f_0, such that $f_0(e) = \ell_e$
  - Capacity condition is satisfied
  - Demand condition may not be satisfied
- for any node v: $$L_v = f_0^{in}(v) - f_0^{out}(v) = \sum_{e\_into\_v} \ell_e - \sum_{e\_out\_of\_v} \ell_e$$
- if $L_v = d_v$, then the demand condition at v is satisfied
- otherwise, add a flow f' on top of f_0 to make the demand condition true: $(f')^{in}(v) - (f')^{out}(v) = d_v - L_v$
- capacity left is $c_e - \ell_e$

Construct a graph G'

- G' is a graph with capacities and demands, but *without lower bounds*
- G' has same nodes and edges as G
- for any edge e, capacity is $c_e - \ell_e$
- demand of a node v is $d_v - L_v$



Algorithm & Analysis

- The problem of finding a feasible circulation with demands & lower bounds in G reduces to the problem of finding a feasible circulation with demands (and *without* lower bounds) in G'.
- Can be solved using Ford-Fulkerson in RT = $O(|f^*|E) = O(DE)$

Analysis

There is a feasible circulation in G iff there is a feasible circulation in G'. If all demands, capacities, and lower bounds are integers, and there is a feasible circulation, then there is a feasible circulation that is integer-valued.

Proof:

- Let f' be a circulation in G'

  Define a circulation f in G by $f(e) = f'(e) + l_e$

  $f^{in}(v) - f^{out}(v) = \Sigma_{e \text{ into } v}(l_e + f'(e)) - \Sigma_{e \text{ out of } v}(l_e + f'(e)) = L_v + (d_v - L_v) = d_v$, thus it satisfies the demand condition

- Conversely, let f be a circulation in G

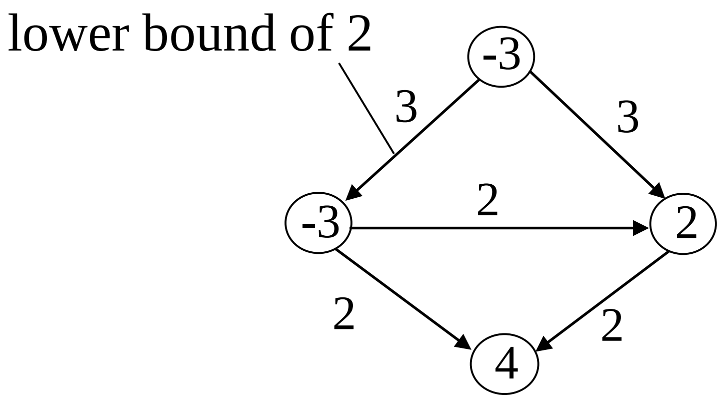
  Let f' in G' be defined by $f'(e) = f(e) – l_e$

  f' satisfies the capacity condition in G'

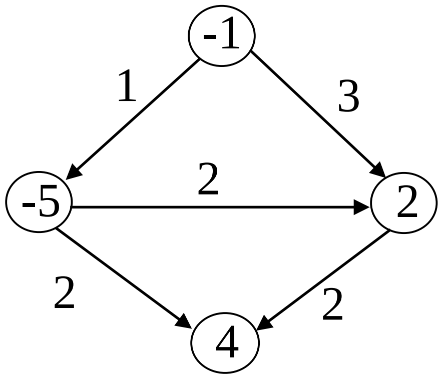
  f' satisfies the demand condition:

  $(f')^{in}(v) – (f')^{out}(v) = \Sigma_{e \text{ into } v}(f(e) – l_e) – \Sigma_{e \text{ out of } v}(f(e) – l_e) = d_v – L_v$

Discussion on the previous example



graph G



graph G'

- Graph G' does not have a feasible circulation
  - a node with supply of 5 and only 4 units of capacity on its outgoing edges
- Therefore G does not have a feasible circulation as well