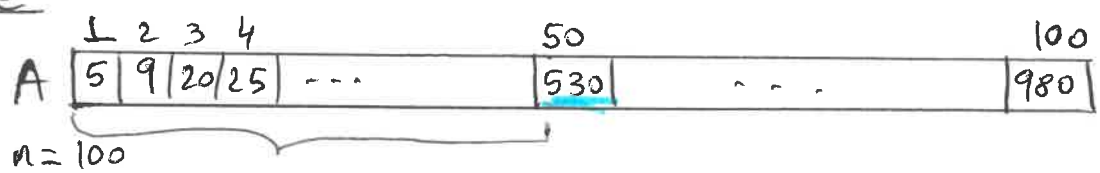
Sublinear time — Binary search

Given a sorted array A of n numbers, determine whether a given number x belongs to the array A.

example

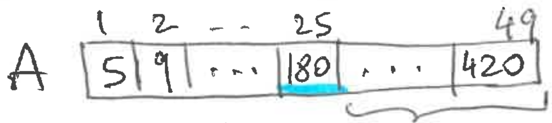


$n = 100$

$x = 250$

$q = \left\lfloor \frac{1+100}{2} \right\rfloor = 50$

$250 < A[50] \Rightarrow$ search x into $A[1..49]$



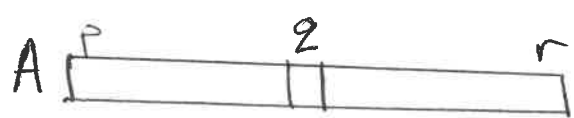
$q = \left\lfloor \frac{1+49}{2} \right\rfloor = 25$

$250 > A[25] \Rightarrow$ search x into $A[26...49]$

... so on

Binary search

search x into $A[p..r]$



- divide the array into two halves $q = \left\lfloor \frac{p+r}{2} \right\rfloor$
- compare x with the middle element $A[q]$
  - if $x = A[q]$, then return $A[q]$
  - if $x < A[q]$, then search x into $A[p..q-1]$
  - if $x > A[q]$, then search x into $A[q+1..r]$

$$\boxed{RT = \Theta(\log_2 n)}$$

Linear time $\Theta(n)$

Merge two sorted arrays into one sorted array.

example

$\cancel{i}$ $\cancel{i}$ $\cancel{i}$ $\cancel{i}$ $i=5$

	1	2	3	4
A	2	4	6	20

$\cancel{j}$ $\cancel{j}$ $\cancel{j}$ $\cancel{j}$ $j=5$

	1	2	3	4
B	5	10	14	18

$\cancel{k}$ $\cancel{k}$ $\cancel{k}$ $\cancel{k}$ $\cancel{k}$ $\cancel{k}$ $\cancel{k}$ $\cancel{k}$ $k=9$

	1	2	3	4	5	6	7	8
C	2	4	5	6	10	14	18	20

Merge (A[1..n], B[1..n])

allocate a new array C[1..2n]

$i = j = k = 1$

$\Theta(n)$

while $i \le n$ and $j \le n$
    if $A[i] \le B[j]$
        $C[k] = A[i]$
        $i = i+1$
    else
        $C[k] = B[j]$
        $j = j+1$
    $k = k+1$

→ max no. of iterations is $2n-1$

$O(n)$

while $i \le n$
    $C[k] = A[i]$
    $i = i+1$
    $k = k+1$

$O(n)$

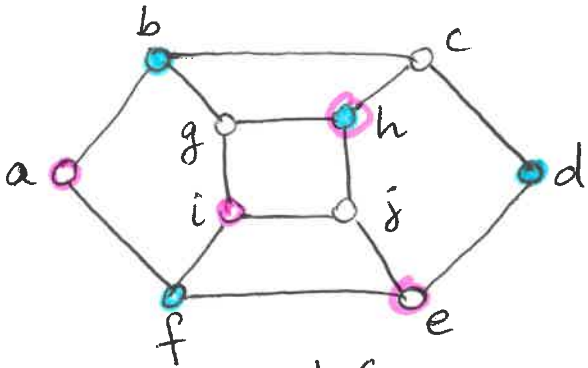
while $j \le n$
    $C[k] = B[j]$
    $j = j+1$
    $k = k+1$

return C

– total $\boxed{RT = \Theta(n)}$

Polynomial time $\Theta(n^k)$ where $k \geq 1$, k = constant

Given a graph G with n nodes, find whether G has an independent set of size k.



graph G

$k=4$

$\{b, h, d, f\}$ – independent sets of size 4

$\{a, i, h, e\}$

Solution

- take all groups (subsets) of k vertices
- for each group, check if it is an independent set. If yes, then stop and return the independent set found

$\{a, b, c, d\}$ ×
$\{a, b, c, e\}$ ×
$\{b, c, f, g\}$ ×
- - - - - - -

- How many groups of vertices of size k are in total?

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!} = \Theta(n^k) \quad k\text{-const}$$

$$\binom{n}{4} = \frac{n!}{(n-4)! \cdot 4!} = \frac{n(n-1)(n-2)(n-3)}{24} = \Theta(n^4)$$

- Check if a group (subset) of k vertices is independent or not

$$\binom{k}{2} = \frac{k!}{(k-2)!\,2!} = \frac{k(k-1)}{2} = \Theta(k^2)$$

if k – const $\Rightarrow \Theta(1)$

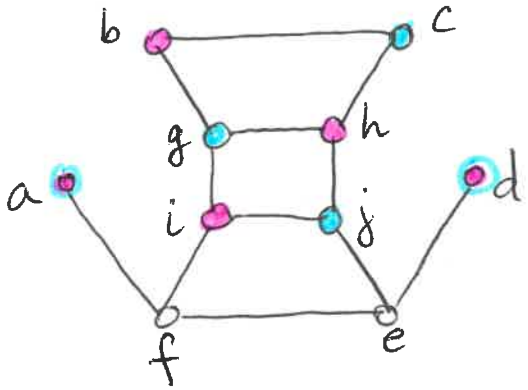
k vertices

$$\boxed{\text{Total RT} = \Theta(n^k)}$$

k – const

Exponential time $\Theta(c^n)$, c - constant

Given a graph G with n nodes, find an independent set of maximum size.



independent sets

$\{a, b, e\}$ size = 3

$\{a, d, g, c\}$ size = 4

$\{a, i, b, h, d\}$ size = 5

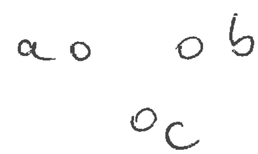
$\{a, d, g, c, j\}$ size = 5

Solution

- take all group (or subsets) of vertices in increasing order of the size $\Rightarrow \boxed{\Theta(2^n)}$

Power set - set of all subsets of a set

e.g. $\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}$

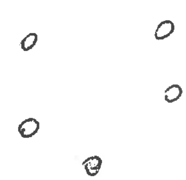


n=3

$\Rightarrow 2^n$ subsets of a set of n elements

- for each group of vertices, check if it is independent or not

$$\binom{k}{2} = \Theta(k^2)$$



k vertices

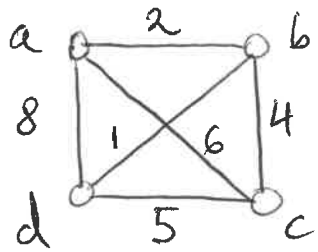
$\boxed{\Theta(n^2)}$

$$\boxed{\text{Total RT} = \Theta(2^n \cdot n^2)}$$

Factorial time $\Theta(n!)$

Given a set of n cities, with distances between all pairs of cities given, what is the shortest tour that visits all cities?

example



$n=4$

Solution

- without loss of generality, let us take all tours starting from the vertex "a"
- list all the possible tours

$$\left.\begin{array}{l} a\,b\,c\,d \\ a\,b\,d\,c \\ a\,c\,b\,d \\ a\,c\,d\,b \\ a\,d\,c\,b \\ a\,d\,b\,c \end{array}\right\} (n-1)! \text{ possible tours}$$

- for each tour, compute the cost:

$\langle a, b, c, d \rangle$ cost $= 2+4+5+8 = 19$

. . . .

The RT to compute a tour is $\Theta(n)$

- return the tour of minimum cost

Total RT $= \Theta((n-1)! \cdot n)$

$$\boxed{\text{Total RT} = \Theta(n!)}$$