MA 109 : Calculus-I
D4 T3, Tutorial 5

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Questions to be Discussed

- Sheet 5
  - 2 (ii), (iii) - Contours and Level sets
  - 4 - Continuity of operations on f(x), g(y)
  - 6 (ii) - Partial derivatives
  - 8 - Partial derivatives again
  - 10 - Directional derivatives

2. (i) Given any c from the options, the level curve is the line $x - y = c$ in the XY plane, that is, the set of points $\{(x,\ y) \in \mathbb{R}^2 : x - y = c\}$ in $\mathbb{R}^2$.
The contour line for that c is the line in $\mathbb{R}^3$ which consists of the set of points $\{(x,\ y,\ z) \in \mathbb{R}^3 : x - y = c,\ z = c\}$. That is, it is the contour line just shifted parallel-y in the $z-$direction.

(ii) For $c < 0$, the contour lines and level curves are empty sets.
For $c = 0$, the level curve is just the point $(0,\ 0) \in \mathbb{R}^2$ and the counter line is $(0,\ 0,\ 0) \in \mathbb{R}^3$.
For $c > 0$, the level curve L is the circle $\{(x,\ y) \in \mathbb{R}^2 : x^2 + y^2 = c\}$ and the contour line is the "same curve, just shifted c units upwards" in $z-$direction. More precisely, the contour line is the set $L \times \{c\} = \{(x,\ y,\ c) \in \mathbb{R}^3 : x^2 + y^2 = c\}$

(iii) You can work this out similarly.

Note: It is technically not correct to say that the contour lines are just the "level curves shifted upwards" because the two curves are not lying in the same space. More precisely, $\mathbb{R}^2 \not\subset \mathbb{R}^3$. However, we do have a natural "embedding" of $\mathbb{R}^2$ into $\mathbb{R}^3$ which is what we were referring to.

Sequential continuity

A 2-D sequence (x_n, y_n) is said to be convergent to (x_0, y_0) iff $x_n \to x_0$ and $y_n \to y_0$.

Definition (Sequential Continuity for Multivariate functions)

A multivariate function $f(x, y)$ is said to be continuous at a point (x_0, y_0) iff for all sequences (x_n, y_n) in domain, such that $(x_n, y_n) \to (x_0, y_0)$, we should have $f(x_n, y_n) \to f(x_0, y_0)$

This definition is again, equivalent to the $\epsilon - \delta$ definition of continuity for Multivariate functions, just like its univariate analogue.

(4) (i), (ii), (iii), (iv)
Let (x_0, y_0) be any point in $\mathbb{R}^2$. We show that the function is continuous at this point.
Let (x_n, y_n) be any sequence in $\mathbb{R}^2$ such that $(x_n, y_n) \to (x_0, y_0)$. This gives us that $x_n \to x_0$ and $y_n \to y_0$. (Why?)
Hence, $f(x_n) \to f(x_0)$ and $g(y_n) \to g(y_0)$. (Definition of continuity of real functions.)
Now, we can use properties of sum and difference of real sequences to get our answers.

For (iii), use the fact that $\max\{a, b\} = \frac{|a+b|+|a-b|}{2}$ and that modulus is a continuous function. Similar considerations apply for (iv).

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I purposefully left the proof for you. There are several ways to write and I don't want to influence on your practice.
You all know how the quiz went and some of you might have realized the lack of practice of writing proofs.
This is a question you can write properly for practice and I will help improve.

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6. (ii) Let $f : \mathbb{R}^2 \to \mathbb{R}$ denote the function given.
Then,

$$f_x(0,\ 0) = \lim_{h\to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h}$$
$$= \lim_{h\to 0} \left(\frac{\sin^2(h)}{h|h|} \right)$$

The above limit does not exist. (Why?) (Hint: Take a strictly positive sequence and a strictly negative sequence, both of which converge to 0.)
It can be verified that $f_y(0,\ 0)$ also does not exist in a similar manner.

8. The continuity of f is immediate, from the fact that $|f(x,y)| \leq |x| + |y|$
Let us show that the partial derivatives don't exist.
The partial derivative of f at $(0,\ 0)$ with respect to the first variable (x) is given by

$$\lim_{h \to 0} \frac{f(0+h,\ 0) - f(0,\ 0)}{h} = \lim_{h \to 0} \sin\left(\frac{1}{h}\right),$$

which we know does not exist.
Similar considerations apply for the other partial derivative.

10. The continuity of f at $(0,0)$ is easy to show using the $\epsilon - \delta$ condition.
Indeed, observe that $|f(x,y) - f(0,0)| = \left|\sqrt{x^2+y^2}\right|$ for $y \neq 0$ and $|f(x,y) - f(0,0)| = 0$ for $y = 0$.
Thus, in general, we have that $|f(x,y) - f(0,0)| \leq \left|\sqrt{x^2+y^2}\right|$.
Let $\delta := \epsilon$ and call it a day.

For a unit vector $\mathbf{u} := (u_1, u_2)$ and $t \neq 0$,

$$\frac{f\left(0+tu_1, 0+tu_2\right) - f(0,0)}{t} = \begin{cases} 0 & u_2 = 0 \\ \frac{u_2}{|u_2|} & u_2 \neq 0 \end{cases}$$

Hence, $\left(\mathrm{D_u} f\right)(0,0)$ exists for all $\mathbf{u}$. Thus, all directional derivatives exist.

If f is differentiable, then the total derivative *must* be $(f_x(0,0), f_y(0,0)) = (0,0)$. Let us now see whether this does indeed satisfy the condition for being the total derivative. For that, we must check whether

$$\lim_{(h,k)\to(0,0)} \frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = 0.$$

For $(h,k) \neq (0,0)$, we have it that

$$\frac{f(0+h, 0+k) - f(0,0) - 0h - 0k}{\sqrt{h^2+k^2}} = \frac{k}{|k|}.$$

It is clear that the limit of the above expression as $(h,k) \to (0,0)$ does not exist. Hence, f is not differentiable at $(0,0)$.

An Interesting Example

Define $f(x,y) = 0$ at origin and $f(x,y) = \dfrac{x^2 y}{x^4 + y^2}$ otherwise.

Exercises:

1. Find the partial derivatives of f at origin.
2. Find the directional derivatives of f at origin.
3. Check whether f is differentiable at origin.
4. Check whether f is continuous at origin.

References

Lecture Slides by Prof. Manoj Kumar Keshari for MA 109 (Autumn 2020)
Lecture Slides by Prof. Sudhir R Ghorpade for MA 105 (Autumn 2019)
Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)