MA 109 : Calculus-I
D4-T3, Tutorial 4

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2 (i). (a).

For any partition P of [a,b], since $f(x) \geq 0$ everywhere, we have,

$\sigma(P) \geq 0$ for every partition P.

Taking limit on both sides as $||P|| \to 0$, we have

$$\lim_{||P|| \to 0} \sigma(P) \geq \lim_{||P|| \to 0} 0$$

As the function is Riemann integrable, the limit on LHS exists, and we have

$$\lim_{||P|| \to 0} \sigma(P) = \int_a^b f(x)dx \geq 0$$

2 (i) (b). We'll prove that if f is continuous,

$$\forall x \in [a,b], f(x) \geq 0 \text{ and } \int_a^b f(x)dx = 0 \implies \forall x \in [a,b], f(x) = 0$$

We proceed to prove using contradiction.

Therefore, let $k \in [a,b]$ be such that $f(k) \neq 0$ which in turn implies $M := f(k) > 0$. Note that if $k = a$ or $k = b$, we can find another $a < k' < b$ such that $f(k') > 0$, by virtue of intermediate value theorem. So, let $k \in (a,b)$.
By continuity of f, we note that there exists a $\delta > 0$, such that
$|x - k| < \delta \implies |f(x) - f(k)| < M/2$.
If δ is such that $k - \delta < a$ or $k + \delta > a$, we can always choose a smaller δ so that
$a < k - \delta < k < k + \delta < b$.
Thus, there exists a $\delta > 0$, such that $f(x) > M/2$ for all $x \in (k - \delta, k + \delta)$.

2 (i) (b) contd.
Let's partition $[a, b]$ as $P_k = \{a, k-\delta, k+\delta, b\}$.
Then, we have, $L(P_k) \geq (M/2)(k+\delta-(k-\delta)) = \delta M > 0$.
Now we know that for every upper sum is greater than every lower sum.
Thus, for all partitions P, $U(P) \geq \delta M > 0$.
Hence, the least bound on upper sums, that is

$$\inf_P U(P) \geq \delta M > 0$$

Now because the function f is Riemann integrable, we have

$$\int_a^b f(x)dx = \inf_P U(P) \geq \delta M > 0$$

which gives contradiction.
Hence, there is no $k \in [a, b]$ such that $f(k) > 0$, and thus, $\forall x \in [a, b], f(x) = 0$

2. (i). (b). Solution by Aneesh (TA for D2-T2)

Tutorial 4, Q2(a)

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and $f(x) \geq 0 \ \forall x \in [a, b]$. Show that $\int_a^b f(x)dx \geq 0$. Further, if f is continuous and $\int_a^b f(x)dx = 0$, show that $f(x) = 0$ for all $x \in [a, b]$.

- Suppose there is some $x_0 \in [a, b]$ such that $f(x_0) = M > 0$. Since f is continuous we have $|f(x) - f(x_0)| < M/2$ whenever $|x - x_0| < \delta$ for some $\delta > 0 \implies f(x) > M/2$ whenever $|x - x_0| < \delta$.
- Now consider any given partition P. There will exist $x_k, x_j \in P$ such that $x_k \leq \max\{a, x_0 - \delta\} \leq x_{k+1}$ and $x_{j+1} \geq \max\{b, x_0 + \delta\} \geq x_j$
- Thus, $U(f, P) = \sum M_i(x_{i+1} - x_i)$ but $M_i \geq 0$ since $f \geq 0$ and $M_i \geq M/2$ for $k \leq i \leq j$. (Note that w is independent of choice of partition.)
- So, $U(f, P) \geq \frac{M}{2}(x_{j+1} - x_k) \geq \frac{M}{2}w$ where $w = \min\{\delta, b - a\}$
- So, $U(f)$(greatest lower bound) $\geq \frac{Mw}{2}$ (a lower bound) > 0. So, if f is non-zero at one point, its integral cannot be zero.

2. (ii). If you take $f(a) = 1$ and $f(x) = 0, \forall x \in (a, b]$.
To prove $\int_a^b f(x)dx = 0$, consider any partition
P of $[a, b]$ using points $\{a = x_0, x_1, ..., x_n = b\}$
We have $x_0 = a$ and say, $x_1 = q > a$. Then we have, $||P|| \geq (q - a) \geq 0$
The Upper sum

$$U(P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = 1(q - a) + \sum_{i=2}^{n} M_i(x_i - x_{i-1}) = (q - a) + 0 = 0$$

The lower sum, $L(P) = \displaystyle\sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 0$

Then, for any Riemann sum $\sigma(P)$, we have, as $L(P) \leq \sigma(P) \leq U(P)$, we have,
$0 \leq \sigma(P) \leq (q - a) \leq ||P||$ (See above).

Hence by sandwich theorem, $\displaystyle\lim_{||P|| \to 0} \sigma(P) = 0 = \int_a^b f(x)dx$

3. (ii) Note that

$$S_n = \sum_{i=1}^{n} \frac{n}{i^2 + n^2} = \sum_{i=1}^{n} \frac{1}{\left(\frac{i}{n}\right)^2 + 1} \left(\frac{i}{n} - \frac{i-1}{n} \right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \tan^{-1} x$. Then, we have that $f'(x) = \frac{1}{x^2+1}$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $c_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = S(P_n, f')$. Since $||P_n|| = 1/n \to 0$, it follows that

$$S(P_n, f') \to \int_0^1 \frac{1}{x^2+1} dx = \int_0^1 f'(x) dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x) dx = f(1) - f(0) = \frac{\pi}{4}.$$

3. (iv) Note that

$$S_n = \frac{1}{n}\sum_{i=1}^{n}\cos\left(\frac{i\pi}{n}\right) = \sum_{i=1}^{n}\cos\left(\frac{i\pi}{n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right).$$

Define $f : [0,1] \to \mathbb{R}$ by $f(x) := \frac{1}{\pi}\sin(\pi x)$. Then, we have that $f'(x) = \cos(\pi x)$.
As f' is continuous and bounded, it is (Riemann) integrable.
For $n \in \mathbb{N}$, let $P_n := \{0, 1/n, \ldots, n/n\}$ and $c_i := i/n$ for $i = 1, 2, \ldots, n$.
Then, $S_n = S(P_n, f')$. Since $||P_n|| = 1/n \to 0$, it follows that

$$S(P_n, f') \to \int_0^1 \cos(\pi x)dx = \int_0^1 f'(x)dx.$$

By the Fundamental Theorem of Calculus (Part 2), we have it that

$$\lim_{n\to\infty} S_n = \int_0^1 f'(x)dx = f(1) - f(0) = 0.$$

4. (b). (i) Suppose $R(x)$ is a function such that $R'(x) = cos(x^2)$.
(For example, $R(p) = \int_0^p cos(t^2)dt$ works)
(Such a function exists because $cos(x^2)$ is a continuous function.)
Then by second fundamental Theorem of Calculus, we have, $F(x) = R(2x) - R(1)$.
Differentiating both sides wrt x, and using chain rule on RHS, we have,
$F'(x) = 2R'(2x) = 2cos((2x)^2) = 2cos(4x^2)$

4. (b). (ii) Suppose $R(x)$ is a function such that $R'(x) = cos(x)$.

(For example, $R(p) = \int_0^p cos(t)dt$ works)

(Such a function exists because $cos(t)$ is a continuous function.)

Then by second fundamental Theorem of Calculus, we have, $F(x) = R(x^2) - R(1)$.

Differentiating both sides wrt x, and using chain rule on RHS, we have,
$F'(x) = 2xR'(x^2) = 2xcos(x^2)$

6.

$$\begin{aligned}
g(x) &= \frac{1}{\lambda}\int_0^x f(t)\sin\lambda(x-t)dt \\
&= \frac{1}{\lambda}\int_0^x f(t)\left(\sin\lambda x\cos\lambda t - \cos\lambda x\sin\lambda t\right)dt \\
&= \frac{1}{\lambda}\sin\lambda x\int_0^x f(t)\cos\lambda t dt - \frac{1}{\lambda}\cos\lambda x\int_0^x f(t)\sin\lambda t dt
\end{aligned}$$

Now, we can differentiate g using product rule and Fundamental Theorem of Calculus (Part 1).

$$\therefore g'(x) = \cos\lambda x\int_0^x f(t)\cos\lambda t dt + \sin\lambda x\int_0^x f(t)\sin\lambda t dt$$

6. contd.
It is easy to verify that both $g(0)$ and $g'(0)$ are 0.
We can differentiate g' in a similar way and get,

$$\begin{aligned}
g''(x) &= -\lambda \sin \lambda x \int_0^x f(t) \cos \lambda t dt + f(x) \cos^2 \lambda x + \lambda \cos \lambda x \int_0^x f(t) \sin \lambda t dt \\
&+ f(x) \sin^2 \lambda x \\
&= f(x) - \lambda^2 \left(\frac{1}{\lambda} \int_0^x f(t) \left(\sin \lambda x \cos \lambda t - \cos \lambda x \sin \lambda t \right) dt \right) \\
&= f(x) - \lambda^2 g(x)
\end{aligned}$$

$$\implies g''(x) + \lambda^2 g(x) = f(x) \qquad \blacksquare$$

References

Lecture Slides by Prof. Ravi Raghunathan for MA 109 (Autumn 2020)
Lecture Slides by Prof. Manoj Kumar Keshari for MA 109 (Autumn 2020)
Tutorial slides prepared by Aryaman Maithani for MA 105 (Autumn 2019)
Solutions to tutorial problems for MA 105 (Autumn 2019)