MA 109 : Calculus-I
D4-T3, Tutorial 3

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Today we'll do...

- Sheet 2
  - 8 (ii), (iii) Finding Functions with given Conditions
  - 10 (i) Sketching curve with given properties
  - 11 Sketching curve with given properties
- Sheet 3
  - 1 (ii) Taylor Series for arctan x
  - 2 Taylor Series for a polynomial
  - 4 Convergence of Maclaurin Series of e^x
  - 5 Integration using Taylor Series

8. (ii). $f''(x) > 0$ for all $x \in \mathbb{R}$, $f'(0) = 1, f'(1) = 2$

$f''(x) > 0$ for all $x \in \mathbb{R}$

We know such a curve!!!

Can try quadratic $\implies$ Let $f(x) = ax^2 + bx + c$

$f'(x) = 2ax + b$ $\qquad$ $f''(x) = 2a$

$f''(x) > 0 \implies a > 0$

$f'(0) = 1 \implies b = 1$

$f'(1) = 2 \implies 2a + b = 2$

One such function is: $f(x) = \frac{x^2}{2} + x$

8. (iii). $f''(x) \geq 0$ for all $x \in \mathbb{R}$, $f'(0) = 1, f(x) < 100$ for all $x > 0$

Idea. Derivative at 0 is positive.
f'' non negative, means derivative not decreasing, always positive after 0.
f must be strictly increasing with derivative > 1, which means,
as x goes to $x + 1$, $f(x)$ goes to something $> f(x) + 1$.
(That's what derivative tells us, right?)
But $f(x)$ is bounded above for positive x. Contradiction!!!

Such a function cannot exist!
Let's write a formal argument.

8. (iii). **Proof.**
$f''(x) \geq 0 \quad \forall x \in \mathbb{R} \implies f'$ not decreasing, so $\forall x > 0, f'(x) \geq 1$
To prove: f must exceed 100 at some point.

Can write, using MVT $\dfrac{f(x) - f(0)}{x - 0} \geq f'(c) \geq 1 \because c \geq 0$

Thus, $f(x) \geq x + f(0)$

Just take $x = 101 - f(0) \implies f(x) \geq 101$.
Done!!!

10. (i). Graphing the polynomial $f(x) = 2x^3 + 2x^2 - 2x - 1$.

$f'(x) = 6x^2 + 4x - 2 = 2(x+1)(3x-1)$

$f'(x) > 0$ in $(-\infty, -1) \cup (1/3, \infty)$; so $f(x)$ is strictly increasing here.

and $f'(x) < 0$ in (-1, 1/3) so $f(x)$ is strictly decreasing here.

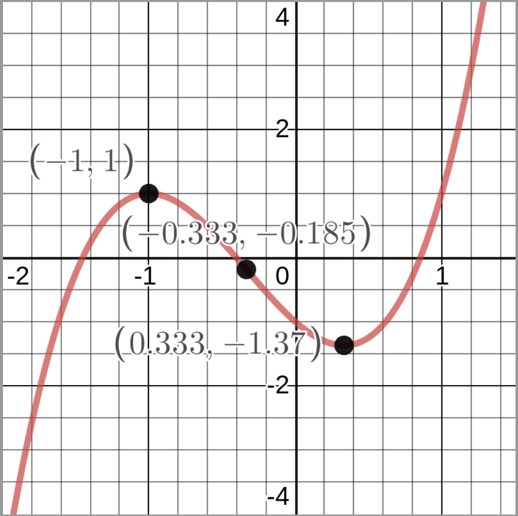
Thus, $x = -1$ is a local maximum, and $x = \frac{1}{3}$ is a local minimum.

$f''(x) = 12x + 4$

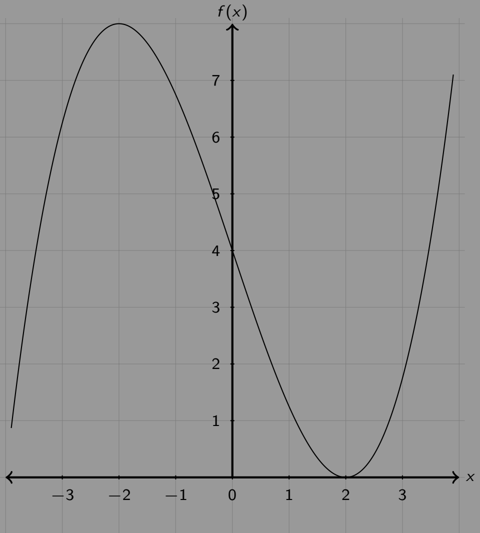
Thus, $f(x)$ is concave in $(-\infty, 1/3)$ and convex in $(-1/3, \infty)$,

with a point of inflection at $x = \dfrac{-1}{3}$.

Graph is on next slide.

10. graph. 

11.



11. (contd.)
I have actually graphed a polynomial that satisfies the given properties.
Can you come up with it?
Is there a unique such polynomial?
What's the minimum degree of such a polynomial?
Is there a unique polynomial with that degree?
Suppose you have two distinct polynomials f and g that satisfy the given conditions.
Can you come up with a distinct third polynomial such that it satisfies the conditions as well?

1. (ii). Taylor Series for arctan x at x=0.

$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} = g(x) \text{ (say)}$$

$$g(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k}$$

(This is only true for $|x| < 1$, so we restrict ourselves to that domain.)

$$g^{(2k+1)}(0) = 0 \qquad g^{(2k)}(0) = (-1)^k(2k)!\ ; \qquad k \geq 0 \quad \text{(Do verify this!)}$$

but these are n^{th} derivatives of $(arctan(x))'$, so are $(n+1)^{th}$ derivatives of $(arctan(x))$.

Thus the series for arctan x , using the formula $P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x)^n$, is :

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1} \qquad \text{(Do verify this!)}$$

Sheet 3

2. Taylor Series of $f(x) = x^3 - 3x^2 + 3x - 1 = (x-1)^3$

Can already guess!!! $\quad f(x) = x^3 - 3x^2 + 3x - 1 \implies f(1) = 0$

$f'(x) = 3x^2 - 6x + 3 \implies f'(1) = 0$

$f''(x) = 6x - 6 \implies f''(1) = 0$

$f'''(x) = 6 \implies f'''(1) = 6$

$f^{(n)}(x) = 0$ for all $n > 3$.

The Taylor series is $\quad P(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

Here, only the third derivative is non-zero! Only one term in the Taylor Series!

$$P(x) = \frac{f'''(0)}{3!}(x-1)^3 = \frac{6}{6}(x-1)^3 = (x-1)^3$$

4. Proving convergence of the series $\sum_{k=0}^{\infty} \frac{x^k}{k!}$.

We'll follow steps given in the question, i.e. prove cauchy.

Let us denote the partial sums of of the given series by $s_m(x)$.
We should show that for every $\epsilon > 0$, there is a $N \in \mathbb{N}$, such that for all $m, n > N$, $|s_m(x) - s_n(x)| < \epsilon$.

It can be shown that,

for $n > N_0 = \lceil 2x \rceil + 1 > 2x$, $\quad \frac{x^{n+1}}{(n+1)!} < \frac{1}{2} \cdot \frac{x^n}{n!}$

Observe that (assuming WLOG $m > n > N_0$),

$$|s_m(x) - s_n(x)| = \left| \sum_{k=n+1}^{m} \frac{x^k}{k!} \right| \leq \left| \frac{x^n}{n!} \right| \left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2^{m-n}} \right) \leq 2 \cdot \frac{|x^n|}{n!}$$

4 contd.

We have, $|s_m(x) - s_n(x)| \leq 2 \cdot \dfrac{|x^n|}{n!}$ for all $m > n$

Also, by induction, for all $k > 0$, we have, $\dfrac{|x|^{N_0+k}}{(N_0+k)!} < \dfrac{1}{2^k} \cdot \dfrac{|x|^{N_0}}{N_0!}$

For a given x and ϵ, for $N_0 = \lceil 2x \rceil + 1$ choose a k such that $\dfrac{1}{2^k} \cdot \dfrac{|x|^{N_0}}{N_0!} < \epsilon$.

Choose $N = N_0 + k$.

Hence this sequence of partial sums of the series $a_n = s_n(x)$ is cauchy for every $x \in \mathbb{R}$ and thus the series is convergent.

Contractive Sequences

Definition (Contractive Sequence)

A sequence $(a_n), n \in \mathbb{N}$ is contractive iff there exists a constant c, with $0 \leq c < 1$, such that:
$|a_{n+2} - a_{n+1}| \leqslant c\,|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$

Try to prove that a contractive sequence is always cauchy, and thus convergent.
A Proof is here-
`http://facstaff.cbu.edu/~wschrein/media/M414%20Notes/M414L68.pdf`

Is the sequence $s_n(x)$ as we defined in the previous question, contractive after N_0 terms? Does that make the proof easier? Do try this!

5. To integrate: $\displaystyle\int \frac{e^x}{x}\,dx$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies$$

$$\int \frac{e^x}{x}dx = \int \sum_{n=0}^{\infty} \frac{x^{n-1}}{n!}dx = \int \left(\frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}\right) dx = \int \frac{1}{x}dx + \int \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}dx$$

$$= \log(x) + \sum_{n=1}^{\infty} \int \frac{x^{n-1}}{n!}dx = \log(x) + \sum_{n=1}^{\infty} \frac{1}{n!}\frac{x^n}{n} = \log(x) + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$$