State whether the following statement is true or false. Justify your answer.

Q-2. If S is a nonempty subset of $\mathbb{R}$ such that S is bounded above and if $c := \sup S$, then there exists a sequence (x_n) of elements of S such that (x_n) is convergent and $x_n \to c$.

Meaning of the Question.

For True:

We need to show that "for all" sets S such that $S \subset \mathbb{R}$, $S \neq \phi$,
There exists "at least one" such sequence (x_n).

For False:

We'll have to find a counterexample, i.e. we'll have to find a set S such that $S \subset \mathbb{R}$, $S \neq \phi$, for which, we'll have to show that there exists no such sequence (x_n).

"Such" sequence means, the sequence (x_n) must have the following properties :

1. Every term x_i of the sequence must be in S, i.e. $\forall i \in \mathbb{N},\ x_i \in S$
2. (x_n) must converge to $c = \sup S$.

Solution to Q-2

Claim. Given Statement is true.

Proof.
Since $c := \sup S$, by definition, for every $\epsilon > 0$,
there exists an element $a \in S$, such that $c - \epsilon < a \leq c$

Using this fact, we define the sequence $\{x_n\}$ as follows:

We can use $\epsilon = 1/n$, and thus we know that for every $n \in \mathbb{N}$
there exists an element $a \in S$,
such that $c - 1/n < a \leq c$

We define the n^{th} term of the sequence to be this a.

Thus for every $n \in \mathbb{N}$, we have, $c - 1/n < x_n \leq c$
As we let $n \to \infty$, using sandwich theorem, we get $x_n \to c$.
Since, for every such set S, we have explicitly constructed a sequence (x_n) of elements of S such that (x_n) is convergent and $x_n \to c$, hence the statement is true.

Same idea is used to prove that there exists a "rational" sequence convergent to any given real number.