Assignment 1: Sample Solution

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Claim. For $a, b > 0 : \lim_{n\to\infty} \frac{an+1}{bn+2} = \frac{a}{b}$

What do we have to do?

We have to show that, for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$, such that for all naturals $n > N$, the following holds :

$$\left| \frac{an+1}{bn+2} - \frac{a}{b} \right| < \epsilon$$

In more technical language, we have to prove that

$$n \in \mathbb{N}, n > N \implies \left| \frac{an+1}{bn+2} - \frac{a}{b} \right| < \epsilon$$

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Proof.
To show this, we consider an arbitrary $\epsilon > 0$ and give an appropriate $N \in \mathbb{N}$ for it.
Consider any $\epsilon > 0$ and define $N = \left\lfloor \frac{|b-2a|}{b^2\epsilon} \right\rfloor + 1$

Then, for all $n > N$, the following holds:

$$\begin{aligned}
\left| \frac{an+1}{bn+2} - \frac{a}{b} \right| &= \left| \frac{b(an+1) - a(bn+2)}{b \cdot (bn+2)} \right| \\
&= \left| \frac{b-2a}{b \cdot (bn+2)} \right| \\
&< \left| \frac{b-2a}{b^2 n} \right| \\
&< \left| \frac{b-2a}{b^2 N} \right| \\
&< \epsilon
\end{aligned}$$

And thus, we have proved that for every $\epsilon > 0$, there exists a $N \in \mathbb{N}$, for which:
$n \in \mathbb{N}, n > N \implies \left| \frac{an+1}{bn+2} - \frac{a}{b} \right| < \epsilon$

Thus our claim is proved and we have it that-
For $a, b > 0 : \lim_{n \to \infty} \frac{an+1}{bn+2} = \frac{a}{b}$

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Common Mistakes:

1. You have to prove that **for every** ϵ, there exists some $N \in \mathbb{N}$, after which that inequality holds. You *cannot* "choose" epsilon.
2. Most people derived "N", i.e. basically proved

   $$\left|\frac{an+1}{bn+2} - \frac{a}{b}\right| < \epsilon \implies n > \frac{|b-2a|}{b^2\epsilon}$$

   You needed to show the other direction.

   $$\text{(i.e. } n \in \mathbb{N}, n > N \implies \left|\frac{an+1}{bn+2} - \frac{a}{b}\right| < \epsilon)$$

3. Almost everyone ignored the fact that N was supposed to be a Natural Number.

   To assert its existence, you should have written,

   $$\text{"Choose some natural number } N > \frac{|b-2a|}{b^2\epsilon} \text{" OR}$$

   used the ceiling function or integer part or something like that.

   Some people mentioned "Archimedian Property" or "$\mathbb{N}$ is not bounded above" to prove existence of N, but if you used the ceiling function or integer part, this was not necessary.

4. Many people confused between n and N (or as some have used n_0). The difference is that, for a given ϵ, N is a fixed natural number for which we have to prove that inequality for all $n > N$. So basically, N is fixed, and n can take any value greater than N, say $N+1$ or $N+14$.