MA 109 : Calculus-I
D4-T3, Tutorial 2

Krushnakant

IIT Bombay

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Today we'll do...

- Sheet 1
  - 13 (ii) - Continuity
  - 15 - Differentiability
  - 18 - Functional Equation
- Sheet 2
  - 3 - Intermediate Value Property and Rolle's Theorem; Uniqueness and Existence.
  - 5 - Mean Value Theorem
- Optional Sheet 1
  - 7 - Increment Lemma
  - 10- Fixed Point

13. (ii) The given function is continuous everywhere.
Proof. For $x \neq 0$, it simply follows from the fact that product and composition of continuous functions is continuous.

To show continuity at $x = 0$: using $\epsilon - \delta$ method

Let $\epsilon > 0$ be given.
Then we have to find a $\delta > 0$ such that $0 < |x - 0| < \delta$ should imply $|f(x) - f(0)| < \epsilon$

$f(0) = 0$, so, we have to find a $\delta > 0$ such that $0 < |x| < \delta$ should imply $|f(x)| < \epsilon$

For $x \neq 0$ (which we can assume since we only worry about $0 < |x| < \delta$)
We have $|f(x)| = |x \sin(1/x)| < |x|$

So if we consider $\delta := \epsilon$, we have
$0 < |x| < \delta \implies 0 < |x| < \epsilon \implies |f(x)| < |x| < \epsilon$
And we are done.

13. (ii) The function is continuous everywhere.

Proof. For $x \neq 0$, it simply follows from the fact that product and composition of continuous functions is continuous.

To show continuity at $x = 0$:

Let (x_n) be any sequence of real numbers such that $x_n \to 0$. We must show that $f(x_n) \to 0$.

Let $\epsilon > 0$ be given.

Observe that $|f(x_n) - 0| = \left| x_n \sin\left(\frac{1}{x_n}\right) \right| \leq |x_n|$.

Now, we shall use the fact $x_n \to 0$.

By this hypothesis, there must exist $n_1 \in \mathbb{N}$ such that $|x_n| = |x_n - 0| < \epsilon \quad \forall n \geq n_1$.

Choosing $n_0 = n_1$, we have it that $|f(x_n) - 0| \leq |x_n| < \epsilon \quad \forall n \geq n_0$. ■

Sheet 1

15. To Show that $f(x)$ is differentiable where $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$

For $x \neq 0$, it simply follows from the fact that product and composition of differentiable functions is differentiable.

At $x = 0$, we use the definition:

$$f'(0)$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin(1/h) - 0)}{h} = \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0 \quad \text{(Proved earlier)}$$

15. contd.

By Chain Rule, the derivative of $f(x)$ is

$$f'(x) = \begin{cases} 2x\sin(1/x) - \cos(1/x) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

$\lim_{x \to 0} 2x\sin(1/x) = 0$ but $\cos(1/x)$ does not exist!

$\lim_{x \to 0} f(x)$ does not exist. f is discontinuous at 0.

This is a function which is differentiable, but not continuously differentiable.

18. Given: $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. (1)
To prove f is differentiable.
Let $x = y = 0$. This gives us that $f(0) = (f(0))^2$.
Thus, $f(0) = 0$ or $f(0) = 1$.

Case 1. $f(0) = 0$. Substitute $y = 0$ in (1). Thus, $f(x) = f(0)f(x) = 0$.
Thus, f is identically 0 which means it's differentiable everywhere with derivative 0.
Verify that $f'(c) = f'(0)f(c)$ does hold for all $x \in \mathbb{R}$. (We did not need to use the fact that f is differentiable at 0, it followed from definition.)

Case 2. $f(0) = 1$.
As f is differentiable at 0, we know that:

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = f'(0) \implies \lim_{h \to 0} \frac{f(h) - 1}{h} = f'(0). \tag{2}$$

18 contd.

Now, let us show that f is differentiable everywhere.

Let $c \in \mathbb{R}$. We must show that the following limit exists:

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

Using (1), we can write the above expression as:

$$\lim_{h \to 0} \frac{f(c)f(h) - f(c)}{h} = \lim_{h \to 0} \frac{f(c)(f(h) - 1)}{h} = f(c) \cdot \lim_{h \to 0} \frac{f(h) - 1}{h}.$$

By (2), we know that the above limit exists. Thus, we have it that f is differentiable at c for every $c \in \mathbb{R}$. Moreover, $f'(c) = f'(0)f(c)$.

(Optional) We have gotten that f' is a scalar multiple of f. Use this to conclude.

Note. Optional problems 7, 10 are covered last, after Sheet 2's question 5 is done.

Lagrange's Mean Value Theorem (MVT)

Theorem (MVT)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a function such that
(i) f is continuous on $[a, b]$, and
(ii) f is differentiable on (a, b).

Then there exists $c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Optional Exercise

Last Year's Quiz Question.

Let $f : [0,1] \to \mathbb{R}$ be continuous on $[0,1]$ and differentiable on $(0,1)$.
Further, Suppose $\lim\limits_{x \to 0^+} = L$ for some $L \in \mathbb{R}$.
Show that $f'(0)$ exists and $f'(0) = L$.

3. Part 1. We will first show the existence of such an $x_0 \in (a, b)$.

Proof. $I := [a, b]$ is an interval and f is continuous. Thus, f has the intermediate value property on I. Thus, the range $J := f(I)$ must be an interval. As $f(a)$ and $f(b)$ are of different signs, 0 lies between them. As $f(a), f(b) \in J$ and J is an interval, we have it that $0 \in J = f(I)$. Thus, $0 = f(x_0)$ for some $x_0 \in I = (a, b)$. ∎

Part 2. Now we will show the uniqueness of x_0. Assume that there exists $x_1 \in (a, b)$ such that $f(x_1) = 0$. We may assume that $x_0 < x_1$.

Now, we know the following:

(i) f is continuous on $[x_0, x_1]$,

(ii) f is differentiable on (x_0, x_1), and

(iii) $f(x_0) = f(x_1)$.

Thus, by Rolle's Theorem, there exists $x_2 \in (x_0, x_1)$ such that $f'(x_2) = 0$. But this contradicts the hypothesis that $f'(x) \neq 0$ for all $x \in (a, b)$. ∎

Sheet 2

5. To prove that $|\sin a - \sin b| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

Case 1. $a = b$. Trivial.

Case 2. $a \neq b$. Without loss of generality, we can assume that $a < b$.

As $f := \sin$ is continuous and differentiable on $\mathbb{R}$, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \text{(By MVT)}$$

Also, we know that $|f'(c)| = |\cos c| \leq 1$.

Thus, we have it that $\left|\dfrac{f(b) - f(a)}{b - a}\right| \leq 1$.

This is equivalent to what we wanted to prove. $\blacksquare$

Increment Lemma

Definition (Increment Lemma)

Let $f : D \to \mathbb{R}$, and c be an interior point of D. Then,
f is differentiable at c $\iff$ there is a function $f_1 : D \to \mathbb{R}$, which is continuous at c, and satisfies

$$f(x) - f(c) = (x - c)f_1(x) \text{ for all } x \in D$$

Here, f_1 is unique and $f'(c) = f_1(c)$.

Try. Prove the Chain Rule using the above definition of differentiability.

Optional Sheet 1: Problem 7

We'll prove $(i) \implies (ii)$, $(ii) \implies (i)$; $(i) \iff (iii)$. Equivalence follows.

1. For $(i) \implies (ii)$. Given: f is differentiable at c. Let $\alpha = f'(c)$. Then we have:

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

That means, for every $\epsilon > 0$, there is a $\delta > 0$, such that

$$0 < |h| < \delta \implies \left|\frac{f(c+h) - f(c)}{h} - \alpha\right| < \epsilon.$$

Then, choose some ϵ, say $\epsilon = 0.01$, take the corresponding δ and define $\epsilon_1(h)$ as follows:

$$\epsilon_1(h) = \begin{cases} \dfrac{f(c+h) - f(c) - \alpha h}{h} & \text{if} \quad h \in (-\delta, \delta) \setminus \{0\} \\ 0 & \text{if} \quad h = 0 \end{cases}$$

Easy to verify that, indeed, $\lim_{h\to 0} \epsilon_1(h) = 0$.

Thus we have shown the existence of a $\delta > 0$, and the function $\epsilon_1(h)$.

Optional Sheet 1: Problem 7

2. For $(ii) \implies (i)$.
We're given that there exists a $\delta > 0$ and a function $\epsilon_1 : (-\delta, \delta) \to \mathbb{R}$, with

$\lim_{h\to 0} \epsilon_1(h) = 0$, such that the following holds:

$f(c+h) = f(c) + \alpha h + h\epsilon_1(h)$.

We have to prove that f is differentiable at c.

For $h \in (-\delta, \delta) \setminus \{0\}$ (We don't have to worry about $h = 0$),
rewriting given equation as:

$$\frac{f(c+h) - f(c)}{h} = \alpha + \epsilon_1(h), \quad \text{and taking limit as } h \to 0\text{; we have:}$$

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \alpha$$

Optional Sheet 1: Problem 7

3. $(i) \iff (iii)$

Given: f is differentiable at c. Let $\alpha = f'(c)$.

$$\iff \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \alpha \qquad \text{[Step 1]}$$

$$\iff \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} - \alpha = 0 \qquad \text{[Step 2]}$$

$$\iff \lim_{h\to 0} \frac{f(c+h)-f(c)-\alpha h}{h} = 0 \qquad \text{[Step 3]}$$

$$\iff \lim_{h\to 0} \left|\frac{f(c+h)-f(c)-\alpha h}{h}\right| = 0 \qquad \text{[Step 4]}$$

$$\iff \lim_{h\to 0} \frac{|f(c+h)-f(c)-\alpha h|}{|h|} = 0 \qquad \text{[Step 5]}$$

Fixed Point

Definition (Fixed Point)

A **fixed point** (sometimes shortened to *fixpoint*, also known as an *invariant* point) of a function $f : \mathbb{A} \to \mathbb{B}$, is an element of the function's domain (some $c \in \mathbb{A}$) that is mapped to itself by the function, i.e. $f(c) = c$.

In simpler words, c is a fixed point of the function f if $f(c) = c$.

This means $f(f(\cdots f(c)\cdots)) = f^n(c) = c$.
It is an important terminating consideration when recursively computing f.
A set of fixed points is sometimes called a **fixed set**.

Optional Sheet 1: Problem 10

Q. Show that any (i.e. *every*) continuous function $f : [0, 1] \to [0, 1]$ has a fixed point.
We'll prove a more general result.
Q. For $a, b \in \mathbb{R}, a < b$; *every* continuous function $f : [a, b] \to [a, b]$ has a fixed point.

Proof.
First see that $f(a) \geq a$ and $f(b) \leq b$. Equivalently, $-f(a) \leq -a$ and $-f(b) \geq -b$
Consider the function $g(x) = x - f(x)$.
As g is a difference of two continuous functions, it is continuous.

$g(a) = a - f(a) \leq a - a = 0$ Also, $g(b) = b - f(b) \geq b - b = 0$

Thus, $g(a) \leq 0 \leq g(b)$, i.e. $0 \in [g(a), g(b)]$

By intermediate value theorem, there is some c for which $g(c) = 0.$, i.e. $f(c) = c$. ■