MA109 Quiz: Solutions [Unofficial]

Krushnakant

December 2020

Contents

Note: a,b,A,B are constants dependent on Roll Numbers.

Disclaimer

These are unofficial solutions.
Correctness is not guaranteed.

1

Question.

Given $A > 0$, and $f(x) = \begin{cases} A & if\ x \leq 0 \\ x & if\ x > 0 \end{cases}$. Is $f(x)$ continuous at $x = 0$?

Solution.

Suppose $f(x)$ is continuous at $x = 0$. Then, if given $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - 0| < \delta \implies |f(x) - f(0)| < \epsilon$

Note that $f(0) = A$, and for $x > 0$, $f(x) = x$.

Let $\epsilon = A/2$, and consider the corresponding δ.

Then, $0 < x < \delta \implies |x - 0| < \delta \implies |x - A| < A/2$, i.e. $x \in (A/2, 3A/2)$

But, if $x = \min\{A/3, \delta/2\}$, then $0 < x < \delta$, but, $x \notin (A/2, 3A/2)$

This gives contradiction, hence $f(x)$ is not continuous at $x = 0$.

2

Question.
$f(x)$ is a twice differentiable function on $(0, 2B)$ such that $f''(x) > 0$ on $(0, B)$, $f''(B) = 0$. Is B an inflection point for $y = f(x)$?

Solution.
The given statement is **False**. We give an explicit counterexample.

Let $f(x) = (x - B)^4$. Then, $f'(x) = 4(x - B)^3$, and $f''(x) = 12(x - B)^2$.
It is easy to see that $f''(x)$ satisfies conditions given in the question.
As $f"(x) > 0$ for $x \in (0, B)$, f is convex on $(0, B)$.
As $f"(x) > 0$ for $x \in (B, 2B)$, f is convex on $(B, 2B)$.
Thus, $x = B$ cannot be an inflection point as convexity of the function does not change about the point $x = B$.

Caution.
Some may give an example for which $f''(x) > 0$ for $x \in (0, B)$, while $f''(x) = 0$ for $x \in [B, 2B)$. Note that this function is convex on $(0, B)$ but can be considered concave on $(B, 2B)$. Thus $x = B$ is an inflection point, hence technically this is not a counter-example.

3

Question.
Prove that for $A, B \geq 1$; $f(x) = x^3 + Ax + B$ has exactly one real root.

Solution.
(i). Existence of the root:
At $x = A$, $f(x) > 0$.
At $x = -A - B$, $f(x) = (-A - B)^3 + A(-A - B) + B < -A^3 - B(A - 1) < 0$ as $A \geq 1$.
Then, as $f(x)$ is continuous on R, by Intermediate value theorem, there exists a $c \in (-A - B, A)$ such that $f(c) = 0$
Thus, at least one root exists.
(ii). Uniqueness of root:
First, we observe that $f'(x) = 3x^2 + A > 0 \ \forall x \in \mathbb{R}$
As $f'(x) > 0 \ \forall x \in \mathbb{R}$, $f(x)$ is a strictly increasing function on $\mathbb{R}$, and hence it is one-to-one.
So if $f(c) = 0$ then for all $x < c, f(x) < 0$ and for all $x > c, f(x) > 0$.
Therefore only one c exists such that $f(c) = 0$.

Aliter. Suppose there are distinct c_1, c_2 such that $f(c_1) = f(c_2) = 0$, then, as f is differentiable (and thus continuous) on $\mathbb{R}$, using Rolle's theorem, there must be a $c_3 \in (c_1, c_2)$ such that $f'(c_3) = 0$ which contradicts $f'(x)$ being strictly positive. Hence, root must be unique.

4

Question.
Let (x_n) be a convergent sequence of non negative real numbers.
Then, prove that $x \geq b$ if $x = \lim_{n\to\infty}(b + x_n)$

Solution.
$x = \lim_{n\to\infty}(b + x_n)$

Claim 1. $\lim_{n\to\infty} b = b$.

This can be seen because, for any $\epsilon > 0$, $N = 1$ ensures that for all $n \in \mathbb{N}, n > N$
$|b - b| = 0 < \epsilon$

Claim 2.

Suppose $\lim_{n\to\infty}(x_n) = L$. Then, $L \geq 0$.

[Note : We can write that because L exists, as (x_n) is given to be convergent.]

Proof: Suppose $L < 0$. Let $L = -K$ for some $K > 0$

Then, if we take $\epsilon = K/2$,

for any $N \in \mathbb{N}$, for all $n \in \mathbb{N}, n > N$,

$|x_n - L| = |x_n + K| = x_n + K \geq K > K/2$

Thus, as no $N \in \mathbb{N}$ satisfies the $\epsilon - N$ condition for $\epsilon = K/2$; we cannot have $L < 0$.
Thus, we conclude that $L \geq 0$.

Now, we can use algebra of limits and say that
$x = \lim_{n\to\infty}(b + x_n) = \lim_{n\to\infty} b + \lim_{n\to\infty} x_n = b + L \geq b$, as $L \geq 0$

5

For simplicity we assume $a = 1$. For any other a, solution will be similar.
Question.
Evaluate $\lim_{n\to\infty} S_n$, where
$S_n = \sum_{k=1}^{n} \cos\left(\frac{2\pi}{3} \cdot \frac{k}{n}\right)$
Solution.
Let $f(x) = \cos(2\pi x/3)$.
We can observe that S_n is the Riemann sum of $f(x)$ on the interval $[0, 1]$, with respect to the partition $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$
The tags/markings corresponding to the partition are $\left\{\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}, 1\right\}$
As $f(x)$ is a continuous function on $\mathbb{R}$, hence it is Riemann integrable.
As $n \to \infty$, $||P|| = \frac{1}{n} \to 0$, hence, $\lim_{n\to\infty} S_n = \int_0^1 \cos(2\pi x/3)dx$
Let $F(x) = \frac{3}{2\pi}\sin\left(\frac{2\pi x}{3}\right)$, then $F'(x) = f(x)$, hence by FTC,
$\int_0^1 \cos(2\pi x/3)dx = F(1) - F(0) = \frac{3}{2\pi}[\sin(2\pi/3) - \sin(0)] = \frac{3\sqrt{3}}{4\pi}$
Therefore, $\lim_{n\to\infty} S_n = \frac{3\sqrt{3}}{4\pi}$

6

Question.
For $f(x) = \cos(x)$, calculate the Taylor polynomial of degree 3 around the point $x = \pi/2$, call it $P_3(x)$, and check if for $x \in [\pi/2, \pi]$; $|f(x) - P_3(x)| < 2C/3$, where $C = \frac{B}{B+1}$.

Solution.
$f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $f''(x) = -\cos(x)$, $f'''(x) = \sin(x)$, $f''''(x) = \cos(x)$
At $x = \pi/2$,
$f(x) = 0$, $f'(x) = -1$, $f''(x) = 0$, $f'''(x) = 1$.

$$P_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(\pi/2)}{n!} \left(x - \frac{\pi}{2}\right)^n$$

Therefore,

$$P_3(x) = -\left(x - \frac{\pi}{2}\right) + \frac{\left(x - \frac{\pi}{2}\right)^3}{3!}$$

The following holds for $x \in [\pi/2, \pi]$
We know from Taylor's theorem that there exists a $c \in [\pi/2, x)$ such that:

$$|f(x) - P_3(x)| = |R_3(x)| = \left|\frac{f''''(c)\left(x - \frac{\pi}{2}\right)^4}{4!}\right| = \left|\frac{\cos(c)\left(x - \frac{\pi}{2}\right)^4}{4!}\right|$$

We have $|\cos(c)| \leq 1$ and $|x - \pi/2| \leq \pi/2 < 1.6$ in the concerned domain.

Therefore, $|R_3(x)| < (1.6)^4/4! < 7/24 < 1/3$.

You can deduce the answer from here. For example, for $C = 2/3$, The Statement that $|f(x) - P_3(x)| < 2C/3$ for all $x \in [\pi/2, \pi]$ is **true**.