Variants of TM

Multi-track, multi-tape, NTM

https://www.andrew.cmu.edu/user/ko/pdfs/lecture-14.pdf
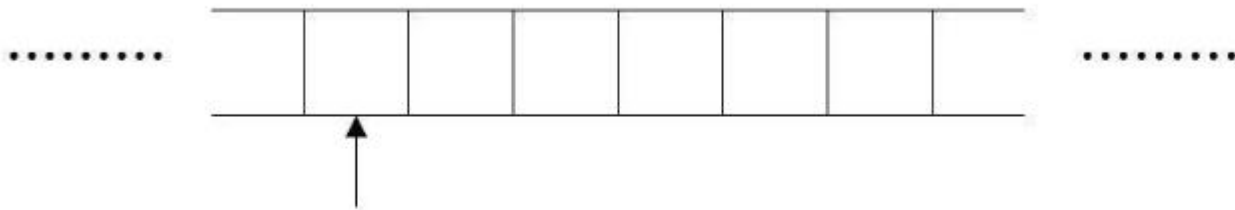
Multi-track
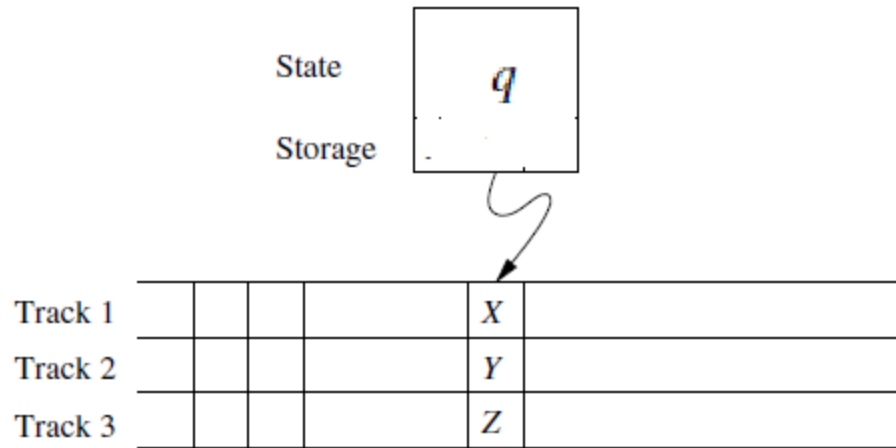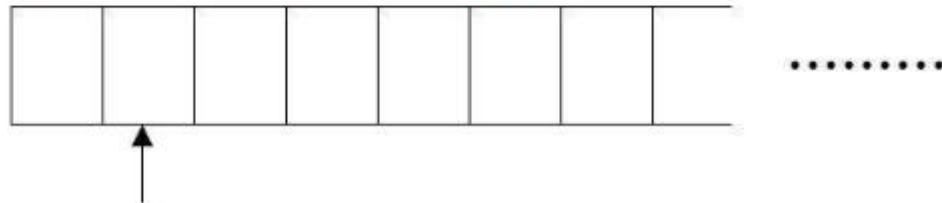

- In this example, the tape symbol is the triplet (X, Y, Z) and we can see the tape as a single-track one.
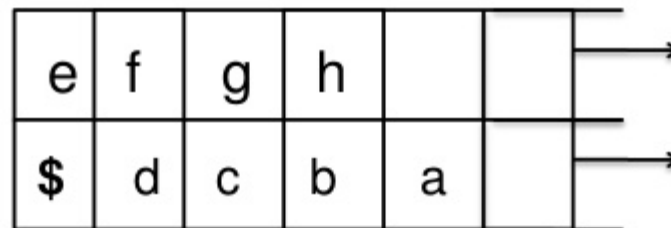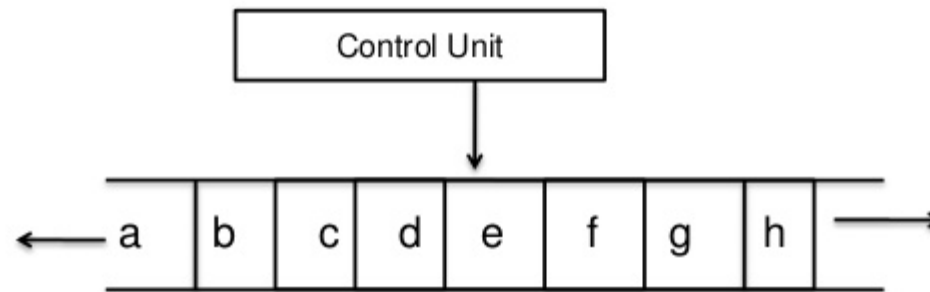
Semi-infinite tape

Simulation of two way infinite by semi-infinite tape

➢ Two way infinite tape simulated by semi -infinite tape

Multi-tape
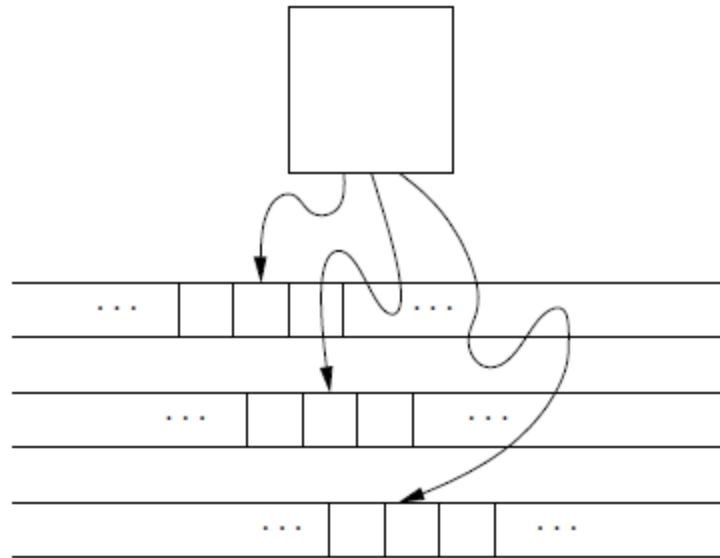
Figure 8.16: A multitape Turing machine

Simulation of multi-tape by one-tape
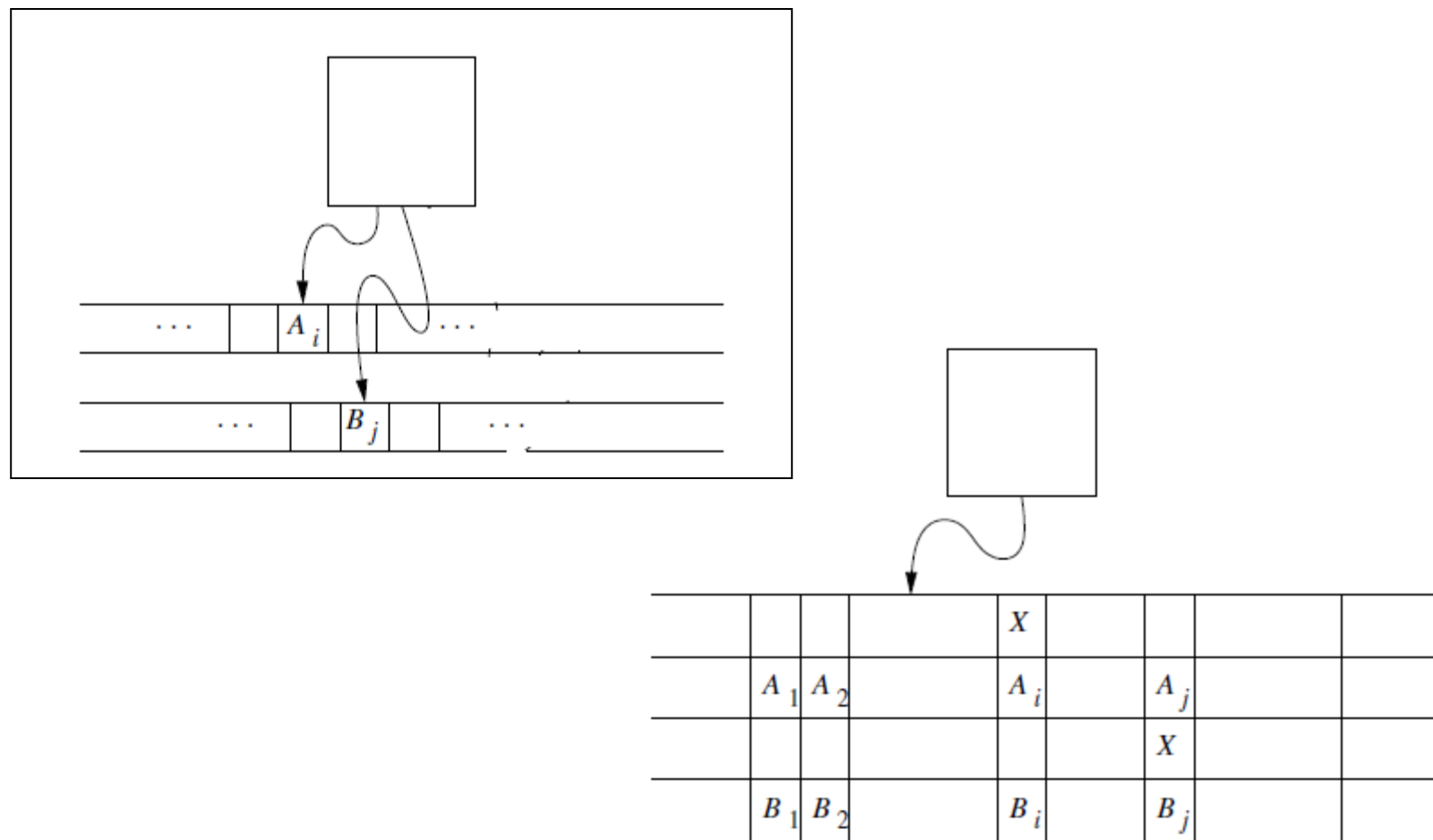
Figure 8.17: Simulation of a two-tape Turing machine by a one-tape Turing machine

NTM

NONDETERMINISTIC TM

- There is a choice in the next move.

$$\delta(q, X) = \{(q_1, Y_1, D_1),\ (q_2, Y_2, D_2), \ldots, (q_k, Y_k, D_k)\}$$

- Here, Y_i is a tape symbol, and D_i is one from $\{L, R\}$, the direction of movement of the head.

The transition function for a nondeterministic Turing machine has the form

$$\delta : Q \times \Gamma \longrightarrow \mathcal{P}(Q \times \Gamma \times \{L, R\}).$$

NONDETERMINISTIC COMPUTATION

Configurations of the nondeterministic computation

$q_0 w_1 w_2 ..... w_n$ ← Initial Configuration

Nondeterministic choices available from C4

NONDETERMINISTIC COMPUTATION

Configurations of the nondeterministic computation

$q_0 w_1 w_2 \ldots w_n$

Initial Configuration

C1

Nondeterministic choices available from C4

A rejecting branch

NONDETERMINISTIC COMPUTATION

Configurations of the nondeterministic computation

Initial Configuration

$q_0 w_1 w_2 \ldots w_n$

Nondeterministic choices available from C4

An accepting branch

Accepting Configuration

$u \, q_{accept} \, v$

NONDETERMINISTIC TURING MACHINES

- A computation of a Nondeterministic TM is a tree, where each branch of the tree is looks like a computation of an ordinary TM.
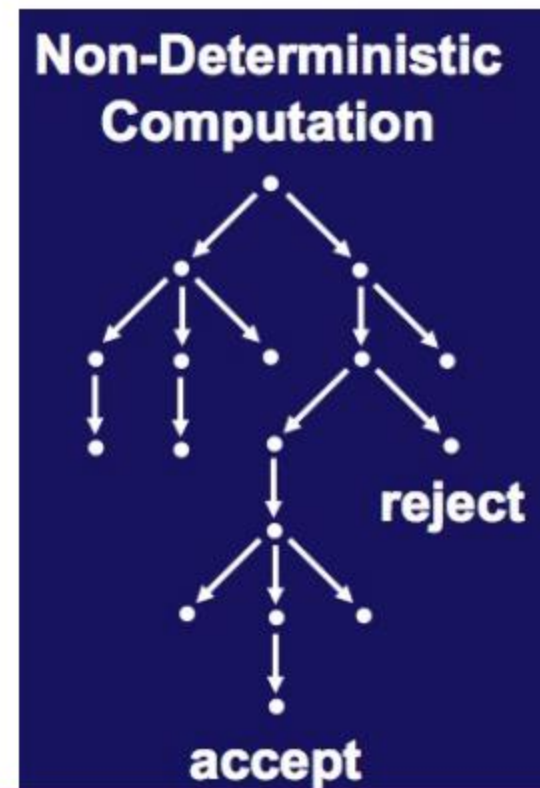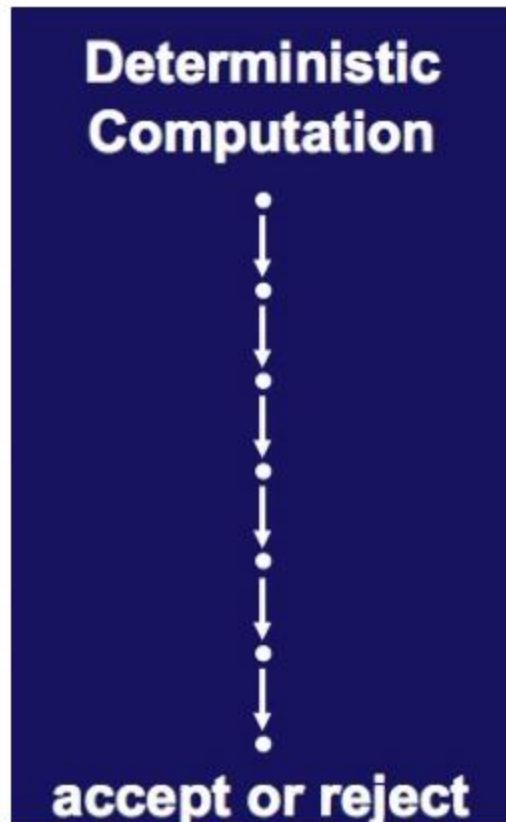
NONDETERMINISTIC TURING MACHINES

- If a single branch reaches the accepting state, the Nondeterministic TM accepts, even if other branches reach the rejecting state.
- What is the power of Nondeterministic TMs?
  - Is there a language that a Nondeterministic TM can accept but no deterministic TM can accept?

- Note, NTM rejects means
  - Each of the branch either explicitly rejects (by getting stuck in a non-final state), or goes in to an infinite loop.

Nondeterministic Turing Machines

Theorem

Every nondeterministic Turing machine has an equivalent deterministic Turing Machine.

Proof Idea

- Timeshare a deterministic TM to different branches of the nondeterministic computation!
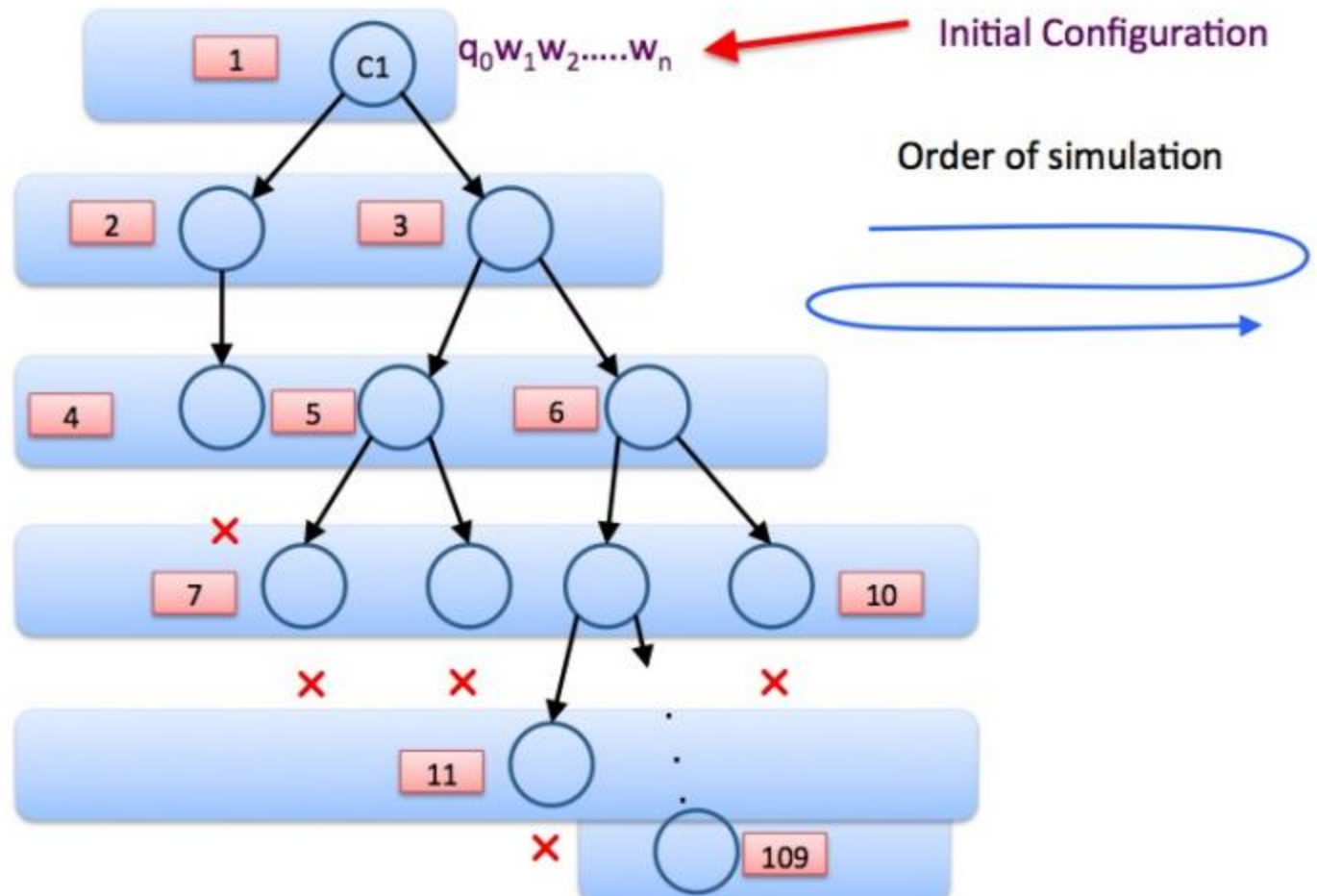- Try out all branches of the nondeterministic computation until an accepting configuration is reached on one branch.
- Otherwise the TM goes on forever.

NONDETERMINISTIC TURING MACHINES

- Deterministic TM D simulates the Nondeterministic TM N.
- Some of branches of the N's computations may be infinite, hence its computation tree has some infinite branches.
- If D starts its simulation by following an infinite branch, D may loop forever even though N's computation may have a different branch on which it accepts.
- This is a very similar problem to processor scheduling in operating systems.
  - If you give the CPU to a (buggy) process in an infinite loop, other processes "starve".
- In order to avoid this unwanted situation, we want D to execute all of N's computations concurrently.

Initial Configuration

$q_0 w_1 w_2 \ldots w_n$

Order of simulation

SIMULATING NONDETERMINISTIC COMPUTATION

Initial Configuration

$q_0 w_1 w_2 \ldots w_n$

Order of simulation
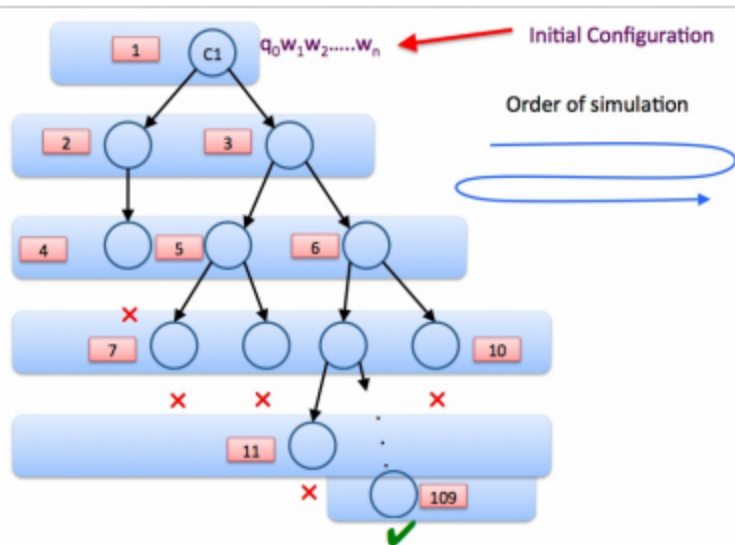
- During simulation, D processes the configurations of N in a breadth-first fashion.

- Thus D needs to maintain a queue of N's configurations (Remember queues?)

- D gets the next configuration from the head of the queue.

- D creates copies of this configuration (as many as needed)

- On each copy, D simulates one of the nondeterministic moves of N.

- D places the resulting configurations to the back of the queue.

STRUCTURE OF THE SIMULATING DTM

- *N* is simulated with 2-tape DTM, *D*
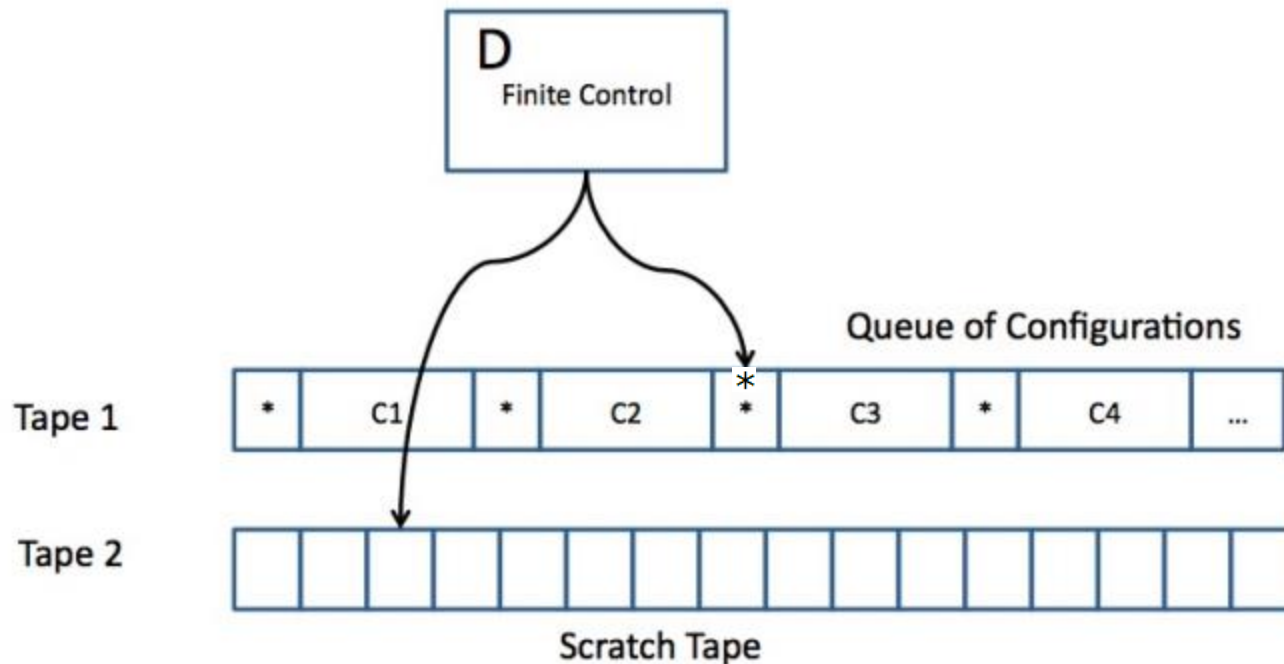
STRUCTURE OF THE SIMULATING DTM

- N is simulated with 2-tape DTM, D
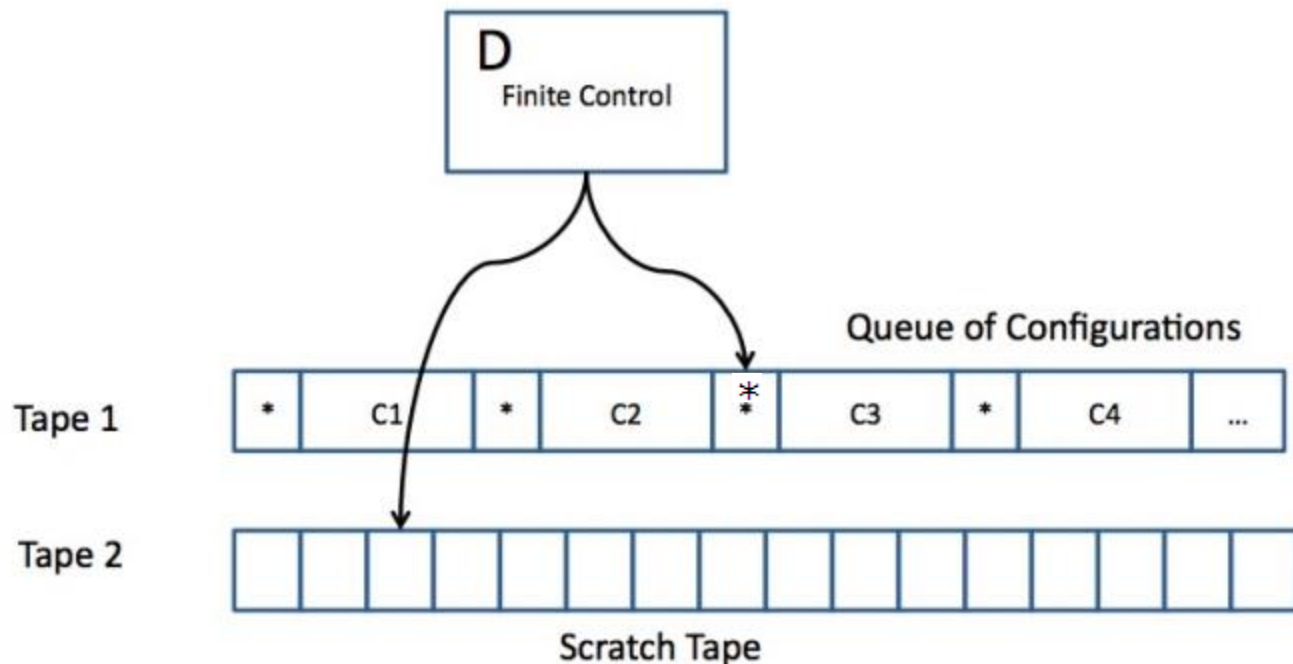
- Built into the finite control of D is the knowledge of what choices of moves N has for each state and input.

STRUCTURE OF THE SIMULATING DTM

- *N* is simulated with 2-tape DTM, *D*

D
Finite Control

Queue of Configurations

| Tape 1 | * | C1 | * | C2 | * | C3 | * | C4 | ... |

Tape 2

Scratch Tape
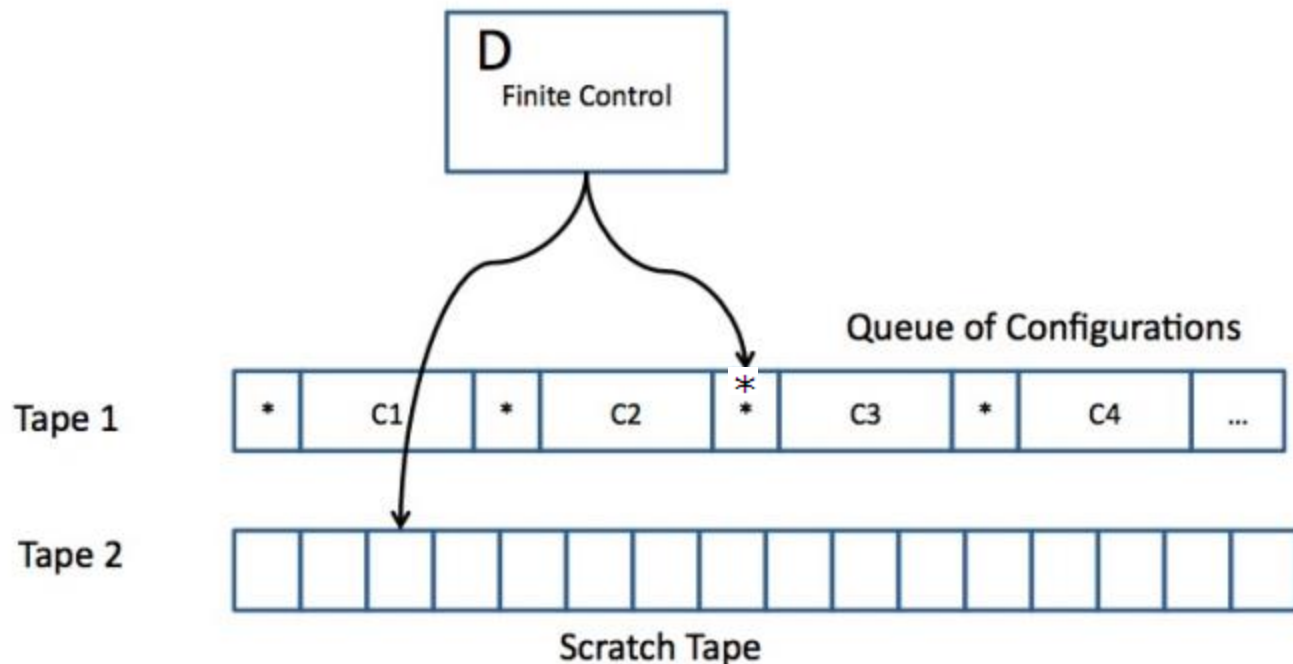
1. *D* examines the state and the input symbol of the current configuration (right after the dotted separator)

2. If the state of the current configuration is the accept state of *N*, then *D* accepts the input and stops simulating *N*.

HOW D SIMULATES N

- Let m be the maximum number of choices N has for any of its states.

- Then, after n steps, N can reach at most $1 + m + m^2 + \cdots + m^n$ configurations (which is at most nm^n)

- Thus D has to process at most this many configurations to simulate n steps of N.

- Thus the simulation can take exponentially more time than the nondeterministic TM.

- It is not known whether or not this exponential slowdown is necessary.

IMPLICATIONS

COROLLARY

A language is Turing-recognizable if and only if some nondeterministic TM recognizes it.

COROLLARY

A language is decidable if and only of some nondeterministic TM decides it.

ENUMERATORS

- Remember we noted that some books used the term recursively enumerable for Turing-recognizable.
- This term arises from a variant of a TM called an enumerator.
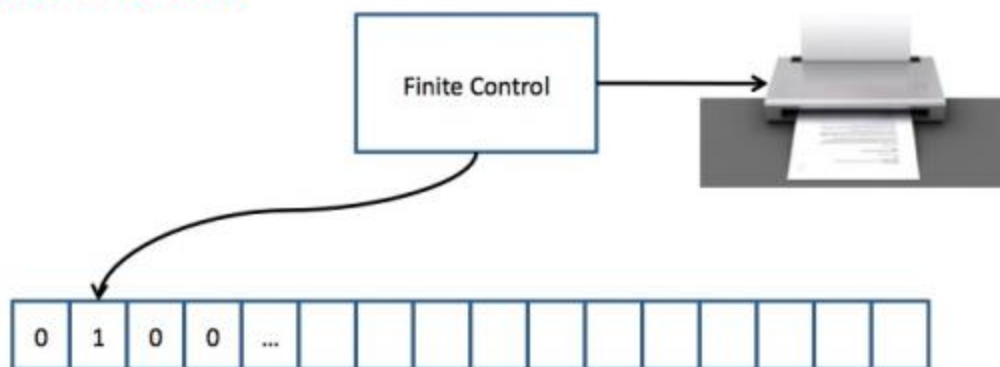
- TM generates strings one by one.
- Everytime the TM wants to add a string to the list, it sends it to the printer.

ENUMERATORS

- The enumerator E starts with a blank input tape.
- If it does not halt, it may print an infinite list of strings.
- The strings can be enumerated in any order; repetitions are possible.
- The language of the enumerator is the collection of strings it eventually prints out.

- We can assume that the enumerator E writes one string at a time over a tape (it can use a tape symbol # to separate strings).

ENUMERATORS

THEOREM

A language is Turing recognizable if and only if some enumerator enumerates it.

PROOF.

The If-part: If an enumerator E enumerates the language A then a TM M recognizes A.

$M = $ "On input w

1. Run E. Everytime E outputs a string, compare it with w.
2. If w ever appears in the output of E, *accept*."

Clearly M accepts only those strings that appear on E's list.

□

The TM M accepts w only when E produces w as one of its output strings.

ENUMERATORS

THEOREM

A language is Turing recognizable if and only if some enumerator enumerates it.

PROOF.

The Only-If-part: If a TM M recognizes a language A, we can construct the following enumerator for A.

- For each possible string $s \in \Sigma^*$ we can verify whether M accepts s, if so output s on to the tape .

Following attempt, does not work.

- Assume that there is an enumerator which enumerates Σ^* in a standard order.

- For each possible string $s \in \Sigma^*$ we can verify whether M accepts s, if so output s on to the tape .

- The problem with this is, for some s the TM M can enter in to an infinite loop and never returns.
  - There may be other strings which are accepted by M but will never gets a chance to be verified (and thus never is outputted).

- A feasible way of doing this (without falling in to an infinite loop) is given in the next slide.

- Basic idea:

- For example if there are two strings w1 and w2 and one of them makes the TM to loop infinitely. Do the following.

  1. k = 1
  2. Run TM for k steps on w1 and if accept occurs then output "accept" and stop.
  3. Run TM for k steps on w2 and if accept occurs then output "accept" and stop.
  4. k++; goto step 2.

ENUMERATORS

THEOREM

A language is Turing recognizable if and only if some enumerator enumerates it.

PROOF.

The Only-If-part: If a TM M recognizes a language A, we can construct the following enumerator for A. Assume $s_1, s_2, s_3, \ldots$ is a list of possible strings in Σ^*.

$E = $ "Ignore the input

1. Repeat the following for $i = 1, 2, 3, \ldots$
2.   Run M for i steps on each input $s_1, s_2, s_3, \ldots s_i$.
3.   If any computations accept, print out corresponding s_j."

If M accepts a particular string, it will appear on the list generated by E (in fact infinitely many times)

THE DEFINITION OF ALGORITHM - HISTORY

- in 1900, Hilbert posed the following problem:

  "Given a polynomial of several variables with integer coefficients, does it have an integer root – an assignment of integers to variables, that make the polynomial evaluate to 0"

- For example, $6x^3yz^2 + 3xy^2 - x^3 - 10$ has a root at $x = 5, y = 3, z = 0$.
- Hilbert explicitly asked that an algorithm/procedure to be "devised". He assumed it existed; somebody needed to find it!
- 70 years later it was shown that no algorithm exists.
- The intuitive notion of an algorithm may be adequate for giving algorithms for certain tasks, but was useless for showing no algorithm exists for a particular task.

This is known as Hilbert's Tenth Problem.

THE DEFINITION OF ALGORITHM - HISTORY

- In early 20th century, there was no formal definition of an algorithm.

- In 1936, Alonzo Church and Alan Turing came up with formalisms to define algorithms. These were shown to be equivalent, leading to the

CHURCH-TURING THESIS

Intutitive notion of algorithms $\equiv$ Turing Machine Algorithms

THE DEFINITION OF AN ALGORITHM

- Let $D = \{p \mid p$ is a polynomial with integral roots$\}$
- Hilbert's 10th problem in TM terminology is "Is D decidable?" (No!)
- However D is Turing-recognizable!
- Consider a simpler version
  $D_1 = \{p \mid p$ is a polynomial over x with integral roots$\}$
- $M_1 =$ "The input is polynomial p over x.
  1. Evaluate p with x successively set to 0, 1, -1, 2, -2, 3, -3, ...
  2. If at any point, p evaluates to 0, *accept*."
- D_1 is actually decidable since only a finite number of x values need to be tested (math!)
- D is also recognizable: just try systematically all integer combinations for all variables.

- For D_1 (polynomial of single variable) there is a bound and we can abandon the search beyond that and declare "No".

- For D such a bound cannot exist (proof is given by Matijasevich (1971)).

  - So, if answer is "No" we enter in to an infinite loop.

In 1971, Yuri Matijasevich gave a resounding negative answer to Hilbert's tenth problem.

- This is called undecidability.
- Hilbert's tenth problem is undecidable.

- We will see the theory behind this in the next …