Efficient Full Waveform Inversion Subject To A Total Variation Constraint

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アブストラクト Full waveform inversion (FWI) aims to reconstruct subsurface properties from observed seismic data. Since FWI is an ill-posed inverse problem, appropriate regularizations or constraints are useful approaches to achieve accurate reconstruction. The total variation (TV) -type regularization or constraint is widely known as a powerful prior that models the piecewise smoothness of subsurface properties. However, the optimization problem of the TV-type regularized or constrained FWI is difficult to solve due to the non-linearity of the observation process and the non-smoothness of the TV-type regularization or constraint. Conventional approaches to solve the problem rely on an inner loop and/or approximations, resulting in high computational cost and/or inappropriate solutions. In this paper, we develop an efficient algorithm with neither an inner loop nor approximations to solve the problem based on a primal-dual splitting method. We also demonstrate the effectiveness of the proposed method through experiments using the SEG/EAGE Salt and Overthrust Models.

1 Introduction

Full waveform inversion (FWI)[1], [2] aims to reconstruct subsurface properties from observed seismic data. Since the observed seismic data are generated from the subsurface properties, FWI is formulated as an inverse problem. However, it is ill-posed, and the quality of the solution depends heavily on the initial model provided[2].

In general, FWI reconstruct the subsurface properties by solving an optimization problem that minimizes the squared error between the observed data and the modeled data. To stabilize the inversion and achieve more accurate reconstruction, other formulations have been proposed[3]–[8]. Adding regularizations or constraints to the objective function has also been proposed, such as Tikhonov regularization[9], Total Variation (TV)[10], and Total Generalized Variation (TGV)[11]. For example, studies have used regularization of Tikhonov[12], TV[13], directional TV[14], high-order TV[15], and TGV[16]. TV has also been used as a constraint[17]–[19].

Recently, neural networks (NNs) have also been proposed to estimate subsurface properties directly from observed data[20]–[23]. However, NNs require a large amount of observed data as training data, and since the dimensions of the observed data may vary depending on the observation method, training data must be prepared not only for each target prior but also for each observation method. Therefore, reconstruction using optimization is still useful.

Also, the value of the squared error between the observed data and the modeled data, which is the objective function of FWI, changes depending on the observation method. Therefore, The parameters of the regularization must be chosen accordingly. However, the TV constraint has the advantage that the parameters can be determined only from prior knowledge of the subsurface properties[24].

In conventional methods that apply the TV constraint to FWI[17], [18], when optimizing the usual FWI objective function using quasi-Newton methods such as the L-BFGS method, the parameter updates are computed to satisfy the constraints. Here, another optimization is required to compute the updates that satisfy the constraints, resulting in a double loop and high computational cost. Moreover, approximations are used in the process of introducing constraints, such as treating nonlinear transformations as linear or enforcing constraints outside the optimization method.

In this paper, we develop an efficient algorithm based on a primal-dual splitting method to solve the TV-constrained FWI problem with neither an inner loop nor approximations. We also demonstrate the effectiveness of the proposed method through experiments using the SEG/EAGE Salt and Overthrust Models.

2 PRELIMINARIES

2.1 Mathematical Tools

Throughout this paper, we denote vectors and matrices by boldface lowercase letters (e.g., $\boldsymbol{x}$) and boldface uppercase letters (e.g., $\boldsymbol{X}$), respectively. The operator norm of a vector and matrix is denoted by $\|\cdot\|_{\mathrm{op}}$.

For $\boldsymbol{x} \in \mathbb{R}^{\mathrm{N}}$, the mixed $l_{1,2}$ norm is defined as follows:

$$\|\boldsymbol{x}\|_{1,2} \coloneqq \sum_{\mathfrak{g}\in\mathfrak{G}} \|\boldsymbol{x}_{\mathfrak{g}}\|_2 \tag{1}$$

where $\mathfrak{G}$ is a set of groups, $\boldsymbol{x}_{\mathfrak{g}}$ is the $\mathfrak{g}$-th group of $\boldsymbol{x}$.

For $\boldsymbol{x} \in \mathbb{R}^{\mathrm{N}}$, the total variation (TV)[10] is defined as follows:

$$\mathrm{TV}(\boldsymbol{x}) \coloneqq \|\mathbf{D}\boldsymbol{x}\|_{1,2} = \sum_{i=1}^{\mathrm{N}} \sqrt{d_{h,i}^2 + d_{v,i}^2} \tag{2}$$

here, $d_{h,i}$ and $d_{v,i}$ are the horizontal and vertical differences of the i-th element of $\boldsymbol{x}$, respectively, when vector $\boldsymbol{x}$ is considered as a matrix.

For proper lower-semicontinuous convex function $f \in \mathbb{R}^N \to \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^N$, the convex conjugate function is defined as follows:

$$f^*(\boldsymbol{x}) \coloneqq \sup_{\boldsymbol{y}\in\mathbb{R}^N} \left\{\boldsymbol{y}^T\boldsymbol{x} - f(\boldsymbol{y})\right\} \tag{3}$$

For a set $C \subset \mathbb{R}^N$ and $\boldsymbol{x} \in \mathbb{R}^N$, the indicator function is defined as follows:

$$\iota_C(\boldsymbol{x}) \coloneqq \begin{cases} 0 & \text{if } \boldsymbol{x} \in C \\ \infty & \text{otherwise} \end{cases} \tag{4}$$

For $\gamma > 0$, $f \in \mathbb{R}^N \to \mathbb{R}$ and $\boldsymbol{x} \in \mathbb{R}^N$, the proximity operator is defined as follows:

$$\mathrm{prox}_{\gamma f}(\boldsymbol{x}) \coloneqq \underset{\boldsymbol{y}\in\mathbb{R}^N}{\mathrm{argmin}} \left\{ f(\boldsymbol{y}) + \frac{1}{2\gamma}\|\boldsymbol{y}-\boldsymbol{x}\|_2^2 \right\} \tag{5}$$

The proximity operator for the convex conjugate function is expressed as follows[25, Theorem 3.1 (ii)]:

$$\mathrm{prox}_{\gamma f^*}(\boldsymbol{x}) = \boldsymbol{x} - \gamma \mathrm{prox}_{f/\gamma}(\boldsymbol{x}/\gamma) \tag{6}$$

The proximity operator for the indicator function is expressed as follows.

$$\mathrm{prox}_{\gamma\iota_C(\cdot)}(\boldsymbol{x}) = P_C(\boldsymbol{x}) \coloneqq \underset{\boldsymbol{y}\in C}{\mathrm{argmin}} \|\boldsymbol{y}-\boldsymbol{x}\|_2 \tag{7}$$

The proximity operator for the box constraint is expressed as follows.

$$P_{[a,b]^N}(\boldsymbol{x}) = \min(\max(\boldsymbol{x}, a), b) \tag{8}$$

The proximity operator for the l_1 norm upper bound constraint is expressed as follows[26](faster algorithms are also proposed[27]):

$$P_{\{\boldsymbol{a}|\|\boldsymbol{a}\|_1\le\alpha\}}(\boldsymbol{x}) = \mathrm{SoftThrethold}(\boldsymbol{x}, \beta) \tag{9}$$

where

$$\boldsymbol{x}_{\mathrm{abs}} = \mathrm{abs}(\boldsymbol{x})$$
$$\boldsymbol{y} = \mathrm{sort}_{\mathrm{desc}}(\boldsymbol{x}_{\mathrm{abs}})$$
$$\beta' = \max\{\frac{1}{i}((\sum_{j=1}^{i}\boldsymbol{y}_j) - \alpha) \mid i = 1, \ldots, N\}$$
$$\beta = \max\{\beta', 0\}$$

The proximity operator for the $l_{1,2}$ norm upper bound constraint is expressed as follows[28]:

$$P_{\{\boldsymbol{a}|\|\boldsymbol{a}\|_{1,2}\le\alpha\}}(\boldsymbol{x}) = [\boldsymbol{p}_1^T, \ldots, \boldsymbol{p}_N^T]^T \tag{10}$$

where

$$\boldsymbol{p}_i = \begin{cases} 0 & \text{if } \|\boldsymbol{x}_i\|_2 = 0 \\ \boldsymbol{\beta}_i \frac{\boldsymbol{x}_i}{\|\boldsymbol{x}\|_2} & \text{otherwise} \end{cases}$$
$$\boldsymbol{\beta} = P_{\{\boldsymbol{a}|\|\boldsymbol{a}\|_1\le\alpha\}}([\|\boldsymbol{x}_1^T\|_2, \ldots, \|\boldsymbol{x}_N^T\|_2])$$

2.2 Primal-Dual Splitting Algorithm

The Primal-Dual Splitting (PDS) Algorithm[29]–[32] is applied to the following problem:

$$\min_{\boldsymbol{x}\in\mathbb{R}^N} \{f(\boldsymbol{x}) + g(\boldsymbol{x}) + h(\boldsymbol{L}\boldsymbol{x})\} \tag{11}$$

where $\boldsymbol{L} \in \mathbb{R}^{\mathrm{M}\times\mathrm{N}}$, f is differentiable convex function and g, h are convex functions whose proximity operator can be computed efficiently(proximable).

The PDS algorithm solve by iteratively updating the following:

$$\boldsymbol{x}^{(k+1)} = \mathrm{prox}_{\gamma_1 g}(\boldsymbol{x}^{(k)} - \gamma_1(\nabla f(\boldsymbol{x}^{(k)}) + \boldsymbol{L}^T\boldsymbol{y}^{(k)})) \tag{12}$$

$$\boldsymbol{y}^{(k+1)} = \mathrm{prox}_{\gamma_2 h^*}(\boldsymbol{y}^{(k)} + \gamma_2\boldsymbol{L}(2\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)})) \tag{13}$$

where γ_1, γ_2 are step sizes. for more details and convergence rates, please refer to[31].

2.3 Full Waveform Inversion

In general, an objective function of FWI is defined as follows:

$$E(\boldsymbol{m}) = \frac{1}{2}\|\boldsymbol{u}_{\mathrm{obs}} - \boldsymbol{u}_{\mathrm{cal}(\boldsymbol{m})}\|_2^2 \tag{14}$$

where $\boldsymbol{m}$ is velocity model, $\boldsymbol{u}_{\mathrm{obs}}$ is the observed seismic data, and $\boldsymbol{u}_{\mathrm{cal}(\boldsymbol{m})}$ is the calculated seismic data with the velocity model.

3 Proposed Method

制約付き FWI の目的関数を、PDS を用いて求解します TV 制約, box 制約付き FWI は以下の式で表されます

$$E(\boldsymbol{m}) \quad \text{s.t.} \quad \mathrm{TV}(\boldsymbol{m}) \le \alpha \ , \ \boldsymbol{m}_i \in [a,b]^N \tag{15}$$

PDS を適用するために、制約を indicator function として目的関数に組み込みます

$$E(\boldsymbol{m}) + \iota_{\|\cdot\|_{1,2}\le\alpha}(\mathbf{D}\boldsymbol{m}) + \iota_{[a,b]^N}(\boldsymbol{m}) \tag{16}$$

PDS を適用し、得られた step は以下の通りとなります

$$\begin{aligned}
\widetilde{\boldsymbol{m}}^{(k+1)} &= \boldsymbol{m}^{(k)} - \gamma_1(\nabla E(\boldsymbol{m}^{(k)}) + \boldsymbol{D}^T\boldsymbol{y}^{(k)}) \\
\boldsymbol{m}^{(k+1)} &= P_{[a,b]^N}(\widetilde{\boldsymbol{m}}^{(k+1)}) \\
\widetilde{\boldsymbol{y}}^{(k+1)} &= \boldsymbol{y}^{(k)} + \gamma_2\boldsymbol{D}(2\boldsymbol{m}^{(k+1)} - \boldsymbol{m}^{(k)}) \\
\boldsymbol{y}^{(k+1)} &= \widetilde{\boldsymbol{y}}^{(k+1)} - \gamma_2 P_{\{\boldsymbol{a}|\|\boldsymbol{a}\|_{1,2}\le\alpha\}}(\frac{1}{\gamma_2}\widetilde{\boldsymbol{y}}^{(k+1)})
\end{aligned}$$

ここで、$P_{[a,b]^N}(\cdot)$, $P_{\{\boldsymbol{a}|\|\boldsymbol{a}\|_{1,2}\le\alpha\}}(\cdot)$ はそれぞれ (8),(10) により, 勾配 $\nabla E(\cdot)$ は adjoint-state method[33] により計算可能です.

4 EXPERIMENTS

the SEG/EAGE Salt and Overthrust Models を用いて実験を行います.

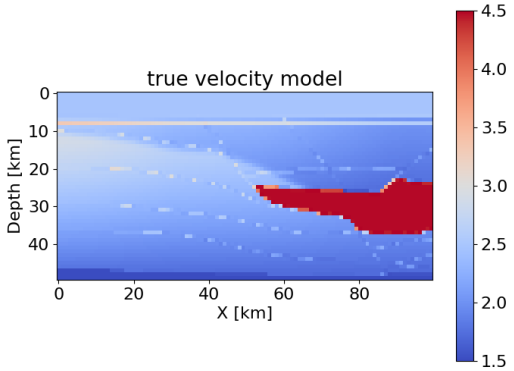

図 1: true

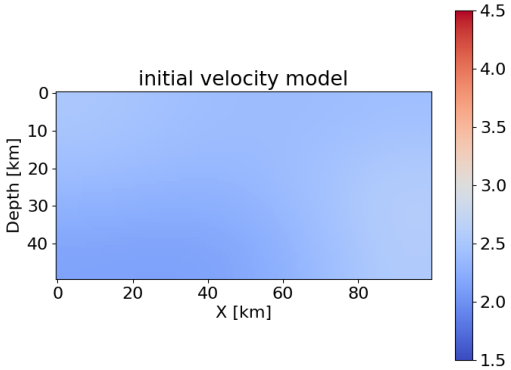

図 2: initial

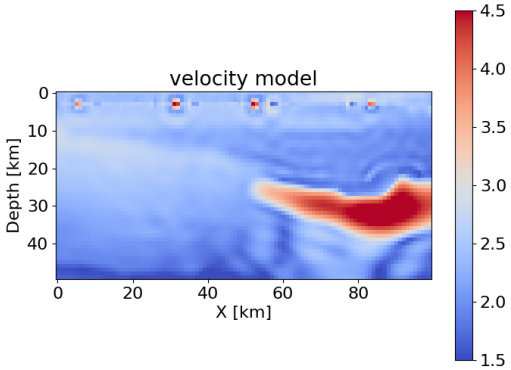

図 3: gradient

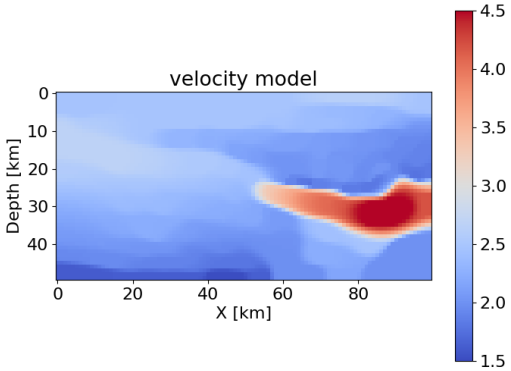

図 4: pds

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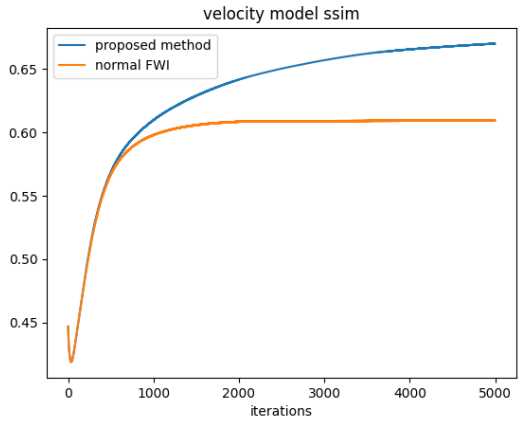

図 5: ssim

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