

# Efficient Full Waveform Inversion Subject To A Total Variation Constraint

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**Abstract** Full waveform inversion (FWI) aims to reconstruct subsurface properties from observed seismic data. Since FWI is an ill-posed inverse problem, appropriate regularizations or constraints are useful approaches to achieve accurate reconstruction. The total variation (TV) -type regularization or constraint is widely known as a powerful prior that models the piecewise smoothness of subsurface properties. However, the optimization problem of the TV-type regularized or constrained FWI is difficult to solve due to the non-linearity of the observation process and the non-smoothness of the TV-type regularization or constraint. Conventional approaches to solve the problem rely on an inner loop and/or approximations, resulting in high computational cost and/or inappropriate solutions. In this paper, we develop an efficient algorithm with neither an inner loop nor approximations to solve the problem based on a primal-dual splitting method. We also demonstrate the effectiveness of the proposed method through experiments using the SEG/EAGE Salt and Overthrust Models.

## 1 Introduction

Full waveform inversion (FWI) [1], [2] aims to reconstruct subsurface properties from observed seismic data. These properties are used for geological research and resource exploration, including gas, oil, mineral deposits and groundwater [2]–[4]. FWI has also been applied to non-destructive testing [5], [6].

Since the observed seismic data are generated by subsurface properties, FWI is formulated as an inverse problem. However, it is ill-posed, and the quality of the solution depends significantly on the initial model [2]. To achieve accurate reconstruction, several formulations have been proposed [1], [7]–[12]. Typically, FWI is treated as an optimization problem, where the objective is to minimize the squared error between observed and modeled data.

To enhance stability and accuracy, regularization terms are often added to the objective function, such as

Tikhonov regularization [13], Total Variation (TV) [14], and Total Generalized Variation (TGV) [15]. For example, studies have used regularization of Tikhonov [16], TV [17], directional TV [18], high-order TV [19], and TGV [20].

The value of the objective function of FWI depends on the observation method such as input signal type and number of observation equipment, because it contains the squared error between the observed data and the modeled data. Consequently, the regularization parameters must be adapted to the observation method. While, adding constraints to the objective function is advantageous because their parameters can be derived only from prior knowledge of the subsurface properties [21]. Therefore, it has been proposed to add the TV constraint to the objective function [22]–[24].

In conventional methods that apply the TV constraint to FWI [22], [23], parameter updates amount in one step of optimization algorithms are adjusted to satisfy the constraints. This often requires an additional optimization, resulting in an inner loop and increased computational cost. In addition, approximations are introduced to incorporate constraints, such as treating non-linear transformations as linear or imposing constraints outside the optimization method.

In this paper, we develop an efficient algorithm to solve the optimization problem of the constrained FWI. Our contributions are as follows:

1. We propose a novel algorithm based on the primal-dual splitting (PDS) method to solve the TV and box constrained FWI problem. This method eliminates the need for an inner optimization loop and approximations, which are common in conventional approaches.
2. We demonstrate that our algorithm efficiently handles constraints while maintaining computational feasibility, with no significant increase in computation time, and that it achieves accurate reconstruction.

3. The PDS framework is flexible and allows for the integration of more complex constraints and regularizations, making it a valuable tool for further development in FWI.

## 2 PRELIMINARIES

### 2.1 Mathematical Tools

Throughout this paper, we denote vector and matrix by boldface lowercase letter (e.g.,  $\mathbf{x}$ ) and boldface uppercase letter (e.g.,  $\mathbf{X}$ ), respectively. The operator  $l_X$  norm of a vector and matrix is denoted by  $\|\cdot\|_X$ .

For  $\mathbf{x} \in \mathbb{R}^N$ , the mixed  $l_{1,2}$  norm is defined as follows:

$$\|\mathbf{x}\|_{1,2} := \sum_{\mathbf{g} \in \mathfrak{G}} \|\mathbf{x}_{\mathbf{g}}\|_2, \quad (1)$$

where  $\mathfrak{G}$  is a set of disjoint index sets, and  $\mathbf{x}_{\mathbf{g}}$  is the subvector of  $\mathbf{x}$  indexed by  $\mathbf{g}$ .

For  $\mathbf{x} \in \mathbb{R}^N$ , the total variation (TV) [14] is defined as follows:

$$\text{TV}(\mathbf{x}) := \|\mathbf{D}\mathbf{x}\|_{1,2} = \sum_{i=1}^N \sqrt{d_{h,i}^2 + d_{v,i}^2}, \quad (2)$$

where  $d_{h,i}$  and  $d_{v,i}$  are the horizontal and vertical differences of the  $i$ -th element of  $\mathbf{x}$ , respectively, when vector  $\mathbf{x}$  is considered as a matrix.

For proper lower-semicontinuous convex function  $f \in \mathbb{R}^N \rightarrow \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^N$ , the convex conjugate function is defined as follows:

$$f^*(\mathbf{x}) := \sup_{\mathbf{y} \in \mathbb{R}^N} \{\mathbf{y}^T \mathbf{x} - f(\mathbf{y})\}. \quad (3)$$

For a set  $C \subset \mathbb{R}^N$  and  $\mathbf{x} \in \mathbb{R}^N$ , the indicator function is defined as follows:

$$\iota_C(\mathbf{x}) := \begin{cases} 0 & \text{if } \mathbf{x} \in C, \\ \infty & \text{otherwise.} \end{cases} \quad (4)$$

For  $\gamma > 0$ ,  $f \in \mathbb{R}^N \rightarrow \mathbb{R}$  and  $\mathbf{x} \in \mathbb{R}^N$ , the proximity operator is defined as follows:

$$\text{prox}_{\gamma f}(\mathbf{x}) := \underset{\mathbf{y} \in \mathbb{R}^N}{\text{argmin}} \left\{ f(\mathbf{y}) + \frac{1}{2\gamma} \|\mathbf{y} - \mathbf{x}\|_2^2 \right\}. \quad (5)$$

Define the proximity operator for the indicator function as  $P_C$  as follows.

$$\text{prox}_{\gamma \iota_C(\cdot)}(\mathbf{x}) = P_C(\mathbf{x}) := \underset{\mathbf{y} \in C}{\text{argmin}} \|\mathbf{y} - \mathbf{x}\|_2. \quad (6)$$

The proximity operator for the specific function used in this paper is given below.

$$\text{prox}_{\gamma f^*}(\mathbf{x}) = \mathbf{x} - \gamma \text{prox}_{f/\gamma}(\mathbf{x}/\gamma). \quad (7)$$

$$P_{[a,b]^N}(\mathbf{x}) = \min(\max(\mathbf{x}, a), b). \quad (8)$$

$$P_{\{\mathbf{a} \mid \|\mathbf{a}\|_1 \leq \alpha\}}(\mathbf{x}) = \text{SoftThreshold}(\mathbf{x}, \beta), \quad (9)$$

where

$$\begin{aligned} \mathbf{x}_{\text{abs}} &= \text{abs}(\mathbf{x}), \\ \mathbf{y} &= \text{sort}_{\text{desc}}(\mathbf{x}_{\text{abs}}), \\ \beta' &= \max\left\{ \frac{1}{i} \left( \sum_{j=1}^i \mathbf{y}_j \right) - \alpha \mid i = 1, \dots, N \right\}, \\ \beta &= \max\{\beta', 0\}. \end{aligned}$$

$$(P_{\{\mathbf{a} \mid \|\mathbf{a}\|_{1,2} \leq \alpha\}}(\mathbf{x}))_{\mathbf{g}_i} = \mathbf{p}_{\mathbf{g}_i}, \quad (10)$$

where

$$\begin{aligned} \mathbf{p}_{\mathbf{g}_i} &= \begin{cases} 0 & \text{if } \|\mathbf{x}_{\mathbf{g}_i}\|_2 = 0, \\ \beta_i \frac{\mathbf{x}_{\mathbf{g}_i}}{\|\mathbf{x}_{\mathbf{g}_i}\|_2} & \text{otherwise,} \end{cases} \\ \beta &= P_{\{\mathbf{a} \mid \|\mathbf{a}\|_1 \leq \alpha\}}([\|\mathbf{x}_{\mathbf{g}_1}\|_2, \dots, \|\mathbf{x}_{\mathbf{g}_N}\|_2]^T). \end{aligned}$$

The proof of equation (7), (9) and (10) can be found in [25, Theorem 3.1 (ii)], [26], [27] accordingly. It is developed a faster algorithm than (9) [28].

### 2.2 Primal-Dual Splitting Algorithm

The Primal-Dual Splitting (PDS) Algorithm [29]–[32] is applied to the following problem:

$$\min_{\mathbf{x} \in \mathbb{R}^N} \{f(\mathbf{x}) + g(\mathbf{x}) + h(\mathbf{L}\mathbf{x})\}, \quad (11)$$

where  $\mathbf{L} \in \mathbb{R}^{M \times N}$ ,  $f$  is differentiable convex function and  $g, h$  are convex functions whose proximity operator can be computed efficiently.

The PDS algorithm solve by iteratively updating the following:

$$\begin{cases} \mathbf{x}^{(k+1)} = \text{prox}_{\gamma_1 g}(\mathbf{x}^{(k)} - \gamma_1 (\nabla f(\mathbf{x}^{(k)}) + \mathbf{L}^T \mathbf{y}^{(k)})), \\ \mathbf{y}^{(k+1)} = \text{prox}_{\gamma_2 h^*}(\mathbf{y}^{(k)} + \gamma_2 \mathbf{L}(2\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)})), \end{cases} \quad (12)$$

where  $\gamma_1, \gamma_2 \in \mathbb{R}$  are step sizes.

### 2.3 Full Waveform Inversion

An objective function of FWI is defined as follows[1]:

$$E(\mathbf{m}) = \frac{1}{2} \|\mathbf{u}_{\text{obs}} - \mathbf{u}_{\text{cal}}(\mathbf{m})\|_2^2, \quad (13)$$

where  $\mathbf{m} \in \mathbb{R}^N$  is velocity model representing subsurface properties,  $\mathbf{u}_{\text{obs}} \in \mathbb{R}^M$  is the observed seismic data, and  $\mathbf{u}_{\text{cal}}(\mathbf{m})$  is the modeled seismic data with the velocity

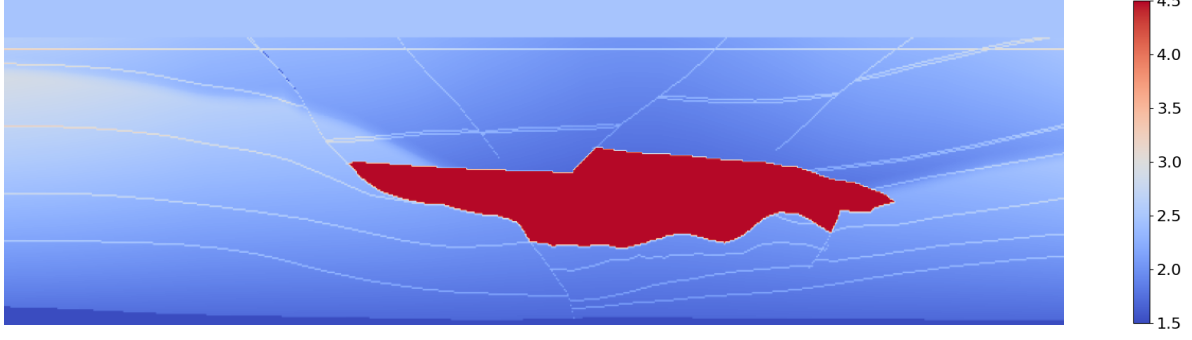


Fig 1: velocity model of salt model [km/s]

model.  $N$  is the number of grid points, and  $M$  is the number of observed signals. In general, velocity model is 2D or 3D grid data, but for simplicity we consider flattened 1D vector.

The standard FWI minimize the objective function and reconstruct the velocity model with following procedures:

$$\mathbf{m}^{(k+1)} = \mathbf{m}^{(k)} - \gamma(\nabla E(\mathbf{m}^{(k)})), \quad (14)$$

where  $\gamma$  is the step size.

The gradient  $\nabla E$  can be computed numerically using the adjoint-state method [33].

### 3 Proposed Method

As shown in fig.1, velocity model of salt model is piecewise smooth. Therefore, we introduce TV constraints to achieve more accurate reconstruction. Also, by introducing a box constraint, we can ensure that the velocity model does not take invalid values, and we show the flexibility of incorporating constraints when using PDS.

We minimize the objective function of the TV and box constrained FWI, which is expressed by the following:

$$\underset{\mathbf{m} \in \mathbb{R}^N}{\operatorname{argmin}} E(\mathbf{m}) \quad \text{s.t.} \quad \|\mathbf{D}\mathbf{m}\|_{1,2} \leq \alpha, \quad \mathbf{m} \in [a, b]^N \quad (15)$$

where  $\alpha \in \mathbb{R}$  is the upper bound of the  $l_{1,2}$  norm, and  $a, b \in \mathbb{R}$  are the lower and upper bound of the velocity model value, respectively.

The constraints can be integrated into the objective function as indicator functions:

$$\underset{\mathbf{m} \in \mathbb{R}^N}{\operatorname{argmin}} E(\mathbf{m}) + \iota_{\|\cdot\|_{1,2} \leq \alpha}(\mathbf{D}\mathbf{m}) + \iota_{[a,b]^N}(\mathbf{m}) \quad (16)$$

As mentioned in section 2.1,  $\iota_{\|\cdot\|_{1,2} \leq \alpha}$  and  $\iota_{[a,b]^N}$  can be computed efficiently(9)(10). Therefore, these functions of  $E$ ,  $\iota_{[a,b]^N}$  and  $\iota_{\|\cdot\|_{1,2} \leq \alpha}$  correspond to  $f$ ,  $g$  and

$h$  in (11), respectively,  $\mathbf{D}$  corresponds to  $\mathbf{L}$ , and the problem (16) can be solved using PDS. The iterative procedures are as follows:

$$\begin{cases} \widetilde{\mathbf{m}}^{(k+1)} = \mathbf{m}^{(k)} - \gamma_1(\nabla E(\mathbf{m}^{(k)}) + \mathbf{D}^T \mathbf{y}^{(k)}) \\ \mathbf{m}^{(k+1)} = P_{[a,b]^N}(\widetilde{\mathbf{m}}^{(k+1)}) \\ \widetilde{\mathbf{y}}^{(k+1)} = \mathbf{y}^{(k)} + \gamma_2 \mathbf{D}(2\mathbf{m}^{(k+1)} - \mathbf{m}^{(k)}) \\ \mathbf{y}^{(k+1)} = \widetilde{\mathbf{y}}^{(k+1)} - \gamma_2 P_{\{\mathbf{a} \mid \|\mathbf{a}\|_{1,2} \leq \alpha\}}\left(\frac{1}{\gamma_2} \widetilde{\mathbf{y}}^{(k+1)}\right) \end{cases}$$

## 4 EXPERIMENTS

### 4.1 Experimental Setup

To demonstrate the effectiveness of the TV and box constrained FWI, we conducted experiments where we compared with the standard FWI with gradient method (14), using the SEG/EAGE Salt and Overthrust Models. The velocity model consists of  $101 \times 51$  grid points. The ground truth velocity model is generated by zooming and cropping fig.1, and the initial velocity model is generated by smoothing the ground truth velocity model with a Gaussian function with a standard deviation of 80. The number of receivers and source shots are 101 and 20, respectively, and are placed on the surface at equal intervals. The source waveform is a Ricker wavelet with a peak wavelet frequency of 10 Hz. The gradient of  $E$  is computed numerically using the Devito framework[34]. In standard FWI, the step size  $\gamma$  is set to  $1.0 \times 10^{-4}$ . In the TV and box constrained FWI, the step size  $\gamma_1$  and  $\gamma_2$  are set to  $1.0 \times 10^{-4}$  and  $1.0 \times 10^2$ , respectively, the upper bound of the  $l_{1,2}$  norm  $\alpha$  is set to 340 and the lower and upper bounds of the velocity model  $a, b$  are set to 1.5[km/s] and 4.5[km/s], respectively. The number of iterations is set to 5000.

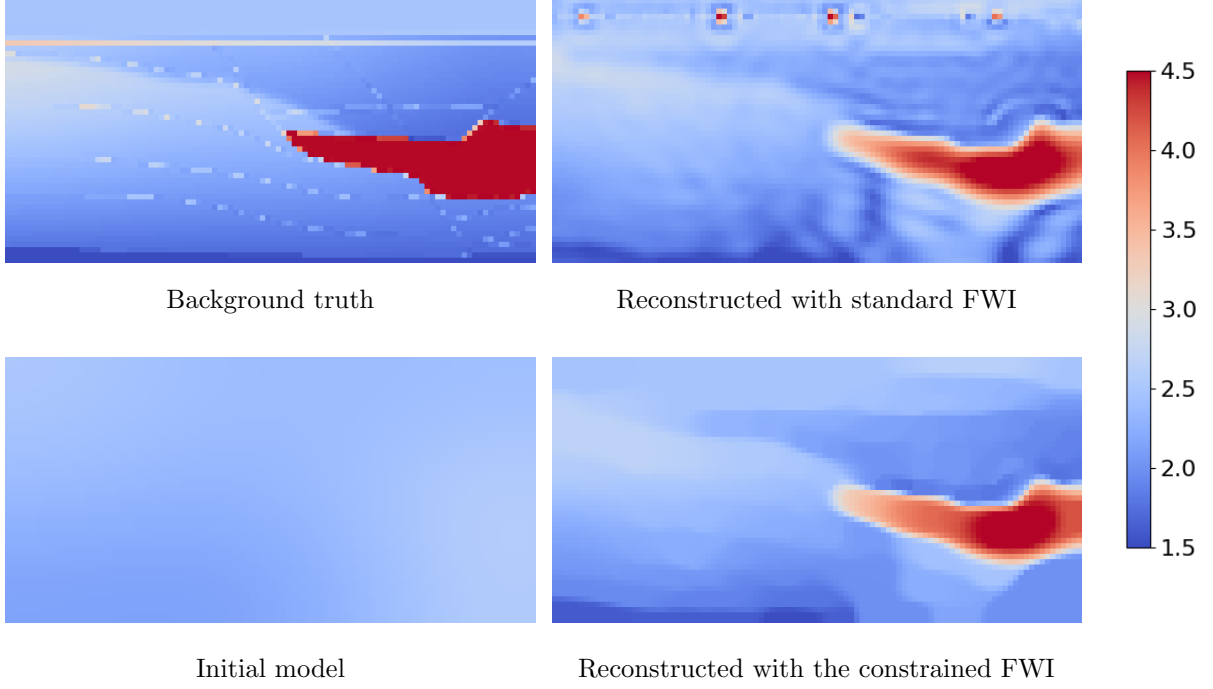


Fig. 2: Velocity models and their corresponding reconstructions.

## 4.2 Results and Discussion

Fig.2 presents the ground truth, the initial model, and the reconstructed velocity models using the standard FWI and the TV and box constrained FWI. It can be observed that the TV and box constrained FWI successfully eliminates wave-like artifacts and noise appearing at the source positions, resulting in a more accurate velocity model reconstruction.

In Fig.3, we plot the Structural Similarity Index Measure (SSIM) against the number of iterations for both methods. The proposed method consistently achieves higher SSIM values than the standard FWI at every iteration, indicating improved reconstruction quality.

Furthermore, since the computational bottleneck lies in the gradient calculation of  $E$ , the introduction of the constraints does not significantly affect the overall computation time. This demonstrates that the proposed method enhances reconstruction accuracy without incurring additional computational costs.

However, it should be noted that parameters such as  $\alpha$ ,  $a$ , and  $b$  were determined by referencing the ground truth data. While this experiment shows that good results can be achieved by appropriately setting these parameters, they need to be determined independently of this framework in practical applications.

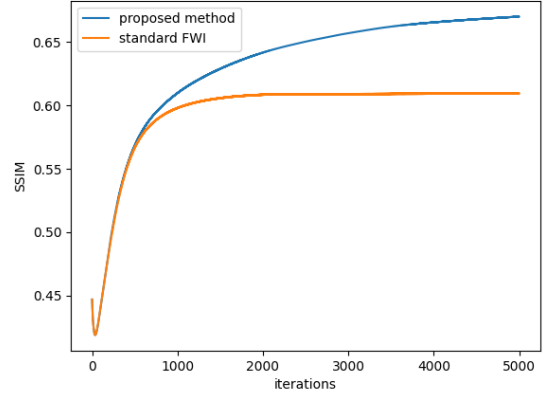


Fig. 3: SSIM against the number of iterations.

## 5 CONCLUSION

In this paper, we developed an efficient algorithm to solve the TV and box constrained FWI problem with neither an inner loop nor approximations based on PDS. We demonstrated the constrained problem can be fully handled within the PDS algorithm. We also demonstrated the piecewise smoothness by the TV constraint is well represented even when the PDS algorithm is used, and that efficient and accurate reconstruction is possible. Furthermore, the PDS framework allows for the incorporation of more complex constraints and regularizations, making it a valuable tool for future research.

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