1

UNIT

Matrices

PART-1

Types of Matrices : Symmetric, Skew-Symmetric and Orthogonal Matrices, Complex Matrices.

CONCEPT OUTLINE : PART-1

Definition of Matrix : An arrangement of mn numbers in the form of m horizontal and n vertical lines is called a matrix of type m by n or order $m \times n$.

For Example : $\begin{bmatrix} 2 & 2 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ is a 2×3 matrix and

$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is an $m \times n$ matrix.

Transpose of a Matrix : Given a matrix A, then the matrix obtained by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A' or A^T.

Types of Matrices :

Symmetric Matrix : A square matrix is said to be symmetric if $A' = A$.

For Example : $A = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 2 & 7 \\ 5 & 7 & 3 \end{bmatrix}$

Skew Symmetric Matrix : A square matrix is called skew symmetric if $A' = -A$.

For Example : $A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

Complex Conjugate of a Matrix : A matrix in which complex elements are replaced by its corresponding conjugate complex numbers is called complex conjugate of that matrix. It is denoted as $\overline{A}$.

For Example : If $A = \begin{bmatrix} 1+i & 2+3i \\ 2 & 3i \end{bmatrix}$

Then $\overline{A} = \begin{bmatrix} 1-i & 2-3i \\ 2 & -3i \end{bmatrix}$

Hermitian Matrix : A square matrix A is said to be Hermitian if $(\overline{A})' = A$ or $A^{\theta} = A$.

Skew Hermitian Matrix : A square matrix A is said to be skew Hermitian if $(\bar{A})' = -A$ or $A^{\theta} = -A$.

Orthogonal Matrix : A square matrix A is called orthogonal if $AA' = A'A = I$.

Unitary Matrix : A square matrix A is said to be unitary if $AA^{\theta} = A^{\theta}A = I$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Express the following matrix as the sum of a symmetric and a skew-symmetric matrix,**

$$\begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$$

Answer

Given matrix is $A = \begin{bmatrix} -1 & 7 & 1 \\ 2 & 3 & 4 \\ 5 & 0 & 5 \end{bmatrix}$ $\therefore A' = \begin{bmatrix} -1 & 2 & 5 \\ 7 & 3 & 0 \\ 1 & 4 & 5 \end{bmatrix}$

If B is symmetric and C is a skew-symmetric matrix

Then,
$$B = \frac{1}{2}(A + A') = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix}$$

$$C = \frac{1}{2}(A - A') = \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$\therefore \quad A = B + C = \begin{bmatrix} -1 & 9/2 & 3 \\ 9/2 & 3 & 2 \\ 3 & 2 & 5 \end{bmatrix} + \begin{bmatrix} 0 & 5/2 & -2 \\ -5/2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

Que 1.2. **Express $\begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix}$ as the sum of a lower triangular matrix and an upper triangular matrix with zero leading diagonal.**

Answer

Let $L = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ be a lower triangular matrix

and $U = \begin{bmatrix} 0 & b' & a' \\ 0 & 0 & c' \\ 0 & 0 & 0 \end{bmatrix}$ be an upper triangular matrix with zero leading diagonal element.

Then
$$\begin{bmatrix} 2 & 5 & -7 \\ -9 & 12 & 4 \\ 15 & -13 & 6 \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix} + \begin{bmatrix} 0 & b' & a' \\ 0 & 0 & c' \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b' & a' \\ b & c & c' \\ d & e & f \end{bmatrix}$$

Equating the corresponding elements on both sides, we get
$a = 2, b' = 5, a' = -7, b = -9, c = 12, c' = 4, d = 15, e = -13, f = 6$

$$\therefore \quad L = \begin{bmatrix} 2 & 0 & 0 \\ -9 & 12 & 0 \\ 15 & -13 & 6 \end{bmatrix} \text{ and } U = \begin{bmatrix} 0 & 5 & -7 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Que 1.3. **If $A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$, show that $A^2 = 0$.**

Answer

$$A^2 = A.A = \begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \\ -1 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2-3 & 1+2-3 & 3+6-9 \\ 2+4-6 & 2+4-6 & 6+12-18 \\ -1-2+3 & -1-2+3 & -3-6+9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O, \text{ where } O \text{ is a null matrix.}$$

Que 1.4. **Show that the matrix $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is orthogonal.**

Answer

A is orthogonal if $AA' = A'A = I$

$$A' = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$AA' = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & -\cos\alpha\sin\alpha + \sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha + \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A'A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \cos\alpha\sin\alpha - \cos\alpha\sin\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Since $AA' = A'A = I$, hence A is orthogonal.

Que 1.5. **Determine the value of α, β, γ when $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is orthogonal.**

Answer

Let, $$A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$$

A is orthogonal if, $AA' = A'A = I$

$$A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

$$AA' = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$$

$$= \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix}$$

By definition of orthogonal, $AA' = I$

$$\therefore \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equating the corresponding elements, we get

$$\left.\begin{array}{l} 4\beta^2 + \gamma^2 = 1 \\ 2\beta^2 - \gamma^2 = 0 \end{array}\right\} \Rightarrow \beta = \pm\frac{1}{\sqrt{6}},\ \gamma = \pm\frac{1}{\sqrt{3}}$$

and $\alpha^2 + \beta^2 + \gamma^2 = 1$

As $\beta = \pm\dfrac{1}{\sqrt{6}},\ \gamma = \pm\dfrac{1}{\sqrt{3}}$

Then $\alpha = \pm\dfrac{1}{\sqrt{2}}$

Que 1.6. **If $A = \begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$ then prove that $\overline{A}$ is Hermitian.**

Answer

$$\overline{A} \text{ (Conjugate of } A) = \begin{bmatrix} 3 & 2+3i & 3-5i \\ 2-3i & 5 & -i \\ 3+5i & i & 7 \end{bmatrix} = B \text{ (Say)}$$

Then $$B' = \begin{bmatrix} 3 & 2-3i & 3+5i \\ 2+3i & 5 & i \\ 3-5i & -i & 7 \end{bmatrix}$$

Now $$(\overline{B})' = \begin{bmatrix} 3 & 2+3i & 3-5i \\ 2-3i & 5 & -i \\ 3+5i & i & 7 \end{bmatrix} = B$$

$\therefore$ B is Hermitian *i.e.*, $\overline{A}$ is Hermitian.

Que 1.7. **Express the Hermitian matrix :**

$$A = \begin{bmatrix} 1 & -i & 1+i \\ i & 0 & 2-3i \\ 1-i & 2+3i & 2 \end{bmatrix}$$

as $P + iQ$ where P is a real symmetric and Q is a real skew symmetric matrix.

Answer

The given matrix can be written as

$$A = \begin{bmatrix} 1+0i & 0-i & 1+i \\ 0+i & 0+0i & 2-3i \\ 1-i & 2+3i & 2+0i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix} + i \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$$

$$A = P + iQ$$

Where $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 2 \end{bmatrix}$ is a real symmetric matrix and $Q = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$ is a real skew symmetric matrix.

Que 1.8. **Show that $A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$ is a unitary matrix, where ω is complex cube root of unity.**

AKTU 2016-17, Marks 3.5

Answer

We know that, $\omega^2 + \omega + 1 = 0$

Here by quadratic formula,

$$\omega = \frac{-1 \pm \sqrt{1 - 4 \times 1}}{2 \times 1} = \frac{-1 \pm \sqrt{-3}}{2} \Rightarrow \omega = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$$

and
$$\omega^2 = \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^2$$

$$= \left(-\frac{1}{2}\right)^2 + \left(\frac{i\sqrt{3}}{2}\right)^2 + 2 \times \left(-\frac{1}{2}\right) \times \frac{i\sqrt{3}}{2}$$

(Take positive sign)

$$\omega^2 = -\frac{1}{2} - \frac{i\sqrt{3}}{2} \qquad (\because i^2 = -1)$$

So,
$$A = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix}$$

$$\bar{A} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix}$$

$$A^\theta = (\bar{A})' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix}$$

So,

$$AA^\theta = \frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{bmatrix} \times \begin{bmatrix} 1 & 1 & 1 \\ 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 + \frac{1}{4} + \frac{3}{4} + \frac{1}{4} + \frac{3}{4} & 1 + \frac{1}{4} - \frac{3}{4} - \frac{i\sqrt{3}}{2} + \frac{1}{4} - \frac{3}{4} + \frac{i\sqrt{3}}{2} \\ 0 & 1 + \frac{1}{4} - \frac{3}{4} - \frac{i\sqrt{3}}{2} + \frac{1}{4} - \frac{3}{4} + \frac{i\sqrt{3}}{2} & 1 + \frac{1}{4} + \frac{3}{4} + \frac{1}{4} + \frac{3}{4} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & \frac{12}{4} & 1 - \frac{1}{2} - \frac{1}{2} \\ 0 & 1 - \frac{1}{2} - \frac{1}{2} & \frac{12}{4} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Since $AA^\theta = I$, hence A is a unitary matrix.

Que 1.9. **Show that the matrix $\begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$.**

AKTU 2017-18, Marks 3.5

Answer

$$A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$$

$$\bar{A} = \begin{bmatrix} \alpha - i\gamma & -\beta - i\delta \\ \beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

$$A^\theta = (\bar{A})' = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$$

For unitary matrix, $AA^\theta = I$

Now, $AA^\theta = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$

$$= \begin{bmatrix} (\alpha+iy)(\alpha-iy)+(-\beta+i\delta)(-\beta-i\delta) & (\alpha+iy)(\beta-i\delta)+(-\beta+i\delta)(\alpha+iy) \\ (\beta+i\delta)(\alpha-iy)+(\alpha-iy)(-\beta-i\delta) & (\beta+i\delta)(\beta-i\delta)+(\alpha-iy)(\alpha+iy) \end{bmatrix}$$

$$= \begin{bmatrix} \alpha^2+y^2+\beta^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+y^2+\delta^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is a unitary matrix.

PART-2

Inverse and Rank of Matrix using Elementary Transformation, Rank-Nullity Theorem.

CONCEPT OUTLINE : PART-2

Inverse of a Matrix : If A and B be two square matrices of same order such that $AB = BA = I$, then B is called inverse of A, *i.e.*, $B = A^{-1}$.

Properties of Inverse :

i. If A is invertible then A^{-1} is also invertible.
ii. $(AB)^{-1} = B^{-1}A^{-1}$
iii. $(A^{1})^{-1} = (A^{-1})^{1}$

Elementary Operations :

The following operations on a matrix are called elementary operations :

i. Interchange of two rows or two columns, e.g., $R_1 \leftrightarrow R_2$, $C_1 \leftrightarrow C_2$
ii. Multiplication of a row or column by a non zero number K, e.g., $R_1 \to KR_1$ or $C_1 \to KC_1$
iii. Addition of K times the elements of a row (or column) to the corresponding elements of another row (or column), $K \neq 0$, e.g., $R_1 \to R_1 + KR_3$, $R_2 \to R_2 - 3R_3$, $C_1 \to C_1 + 2C_3$

Note : If we write $R_1 \to R_1 + KR_3$ that means R_1 is replaced by $R_1 + K$ times R_3.

Rank of a Matrix : A number r is said to be the rank of a matrix A if it possesses the following properties :

i. There is at least one square sub-matrix of A of order r whose determinant is not equal to zero.
ii. If the matrix A contains any square sub-matrix of order $r + 1$, then the determinant of every square sub-matrix of A of order $r + 1$ should be zero.

Therefore the rank of a matrix is the order of any highest order non zero minor of the matrix.

Rank of a Matrix by Echelon Form : A Matrix A is said to be in Echelon Form if :

i. Every row of A which has all its entries 0 occurs below every row which has a non-zero entry.

ii. The first non zero entry in each non zero row is equal to 1.
iii. The number of zeros before the first non zero element in a row is less than the number of such zeros in the next row.

Note : The rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix.

Rank by Normal Form (Canonical Form) : By elementary operations any non zero matrix can be reduced to a form that is called normal form, e.g.,

$$I_r,\ [I_r, 0],\ \begin{bmatrix} I_r \\ 0 \end{bmatrix},\ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

Then the number r denotes the unit matrix of order r and this r is called the rank of the matrix.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.10. **Compute the inverse of the matrix $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$ by employing elementary row transformation.**

AKTU 2017-18, Marks 3.5

Answer

We know that, $A = IA$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \qquad R_1 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -8 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -3 & 1 \end{bmatrix} A \qquad R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & 3 & 1 \\ \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix} A \qquad R_1 \to \frac{R_1}{2}$$

But $I = A^{-1}A$

Hence
$$A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 4 & 3 & 1 \\ \frac{5}{2} & \frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Que 1.11. **Find the inverse of the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, by employing elementary transformation.**

AKTU 2013-14, Marks 05

Answer

Same as Q. 1.10, Page 1–10C, Unit-1.

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Que 1.12. **If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ then evaluate the value of the expression $A + 5I + 2A^{-1}$.**

AKTU 2016-17, Marks 3.5

Answer

$$A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} -3 & 2 \\ -1 & 0 \end{vmatrix} \text{ or } |A| = 2$$

Cofactor matrix of $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, adj $A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$

$$A^{-1} = \frac{\text{adj } A}{|A|} \Rightarrow A^{-1} = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}$$

Substituting the value of A^{-1}, A and I in given equation,

$$A + 5I + 2A^{-1} = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2}{2}\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I$$

Que 1.13. **Prove that the point (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear if the rank of the matrix $\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$ is less than 3.**

Answer

If the rank of the given matrix is less than 3, then the determinant of order 3 of this matrix must be zero.

$$i.e., \quad \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \qquad ...(1.13.1)$$

Now the area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3)

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \qquad \text{(From eq. (1.13.1))}$$

Since the area of this triangle is zero, so its vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) are collinear.

Que 1.14. **Reduce A to Echelon form and then to its row canonical form where $A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}$. Hence find the rank of A.**

AKTU 2014-15, Marks 10

Answer

$$A = \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & -11 & 5 & -3 \\ 0 & -11 & 5 & -3 \end{pmatrix} \quad R_3 \to R_3 - 2R_1, R_4 \to R_4 - 4R_1$$

$$= \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_3 \to R_3 + R_2, R_4 \to R_4 + R_2$$

$$= \begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 1 & -5/11 & 3/11 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad R_2 \to R_2/11$$

Using Echelon form,

Rank of A = Number of non-zero rows

$\rho(A) = 2.$

Que 1.15. **Find the rank of the matrix by reducing to normal form** $\begin{pmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{pmatrix}$

AKTU 2015-16, Marks 05

Answer

Given matrix $= \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 7 & 4 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 2 & -1 \\ 4 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - (R_1 + R_2)$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad C_1 \to C_1 - C_2$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to R_2/2$$

$$= \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - R_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} C_3 \to C_3 + C_2 \\ C_2 \to -C_2 \end{matrix}$$

$$= \begin{bmatrix} I_2 \\ 0 \end{bmatrix}$$

Rank of matrix = 2.

Que 1.16. **Using elementary transformations, find the rank of the following matrix :**

$$A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

AKTU 2017-18, Marks 3.5

Answer

$$A = \begin{bmatrix} -2 & -1 & 3 & -1 \\ 1 & 2 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ -2 & -1 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad R_1 \leftrightarrow R_2$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & -2 & 4 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 + 2R_1, \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & -2 & 4 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_2/3$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} \quad R_3 \to R_3 + 2R_2/3$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 3 & -3 & -3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

$$= \begin{bmatrix} 1 & 2 & -3 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2/3 \\ R_3 \to R_3/2 \end{matrix}$$

Number of non-zero rows = 3

Rank of given matrix = 3

Que 1.17. **State Rank-Nullity theorem with example.**

Answer

A. Rank-Nullity Theorem :

The rank-nullity theorem states that the rank and the nullity (the dimension of the kernel) sum to the number of columns in a given matrix M. If there is a matrix M with x rows and y columns over a field, then

$$\text{rank}(M) + \text{nullity}(M) = y$$

The rank-nullity theorem is useful in calculating either one by calculating the other instead, which is useful as it is often much easier to find the rank than the nullity (or vice versa).

B. Example : Consider the matrix :

$$A = \begin{pmatrix} 3 & 1 \\ -6 & -2 \end{pmatrix}$$

Here, the rank is 1, since the basis $\left\{\begin{pmatrix} 3 \\ -6 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$ can be reduced to $\left\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right\}$. The kernel of A is vectors such that $Av = 0$, which is a vector space spanned by $\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\}$ and has dimension 1. Hence the rank and nullity are both 1, and sum to 2, the number of columns in A.

PART-3

System of Linear Equations, Characteristic Equation, Cayley-Hamilton Theorem and its Application, Eigen Values and Eigen Vectors, Diagonalisation of a Matrix.

CONCEPT OUTLINE : PART-3

Condition for Consistency : The system of equations $AX = B$ is consistent *i.e.*, possesses a solution if the coefficient matrix A and the augmented matrix $[AB]$ are of same rank.

Procedure to find the Solution of $AX = B$:

1. Let the coefficient matrix be of type $m \times n$.
2. Write the augmented matrix $[AB]$ and reduce it to Echelon form by applying elementary row operation.
3. From the Echelon form we can find the rank of coefficient matrix A and the rank of augmented matrix $[AB]$.

Now the following cases arise :

Case I : If rank $[A]$ < rank $[AB]$, then the system of equations is inconsistent, *i.e.*, they have no solution.

Case II : If rank $[A]$ = rank $[AB] = r$ (say), then the system of equations $AX = B$ is consistent, *i.e.*, they possesses a solution.

1. If $r = n$, then the system of equations has unique solution.
2. If $r < n$, then there will be infinite solution. Only $n - r + 1$ solutions will be linearly independent and rest of the solutions will be linear combination of them.

Linear Dependence and Linear Independence : A set of n-vectors $x_1, x_2, \ldots\ldots x_n$ are said to be linearly dependent if there exists n scalars, $a_1, a_2, \ldots\ldots a_n$ not all zero such that $a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n = 0$ and if all $a_1, a_2, \ldots\ldots a_n$ are zero, vectors are said to be linearly independent.

Characteristic Matrix : Let A be any square matrix of order n and λ an indeterminate. The matrix $[A - \lambda I]$ is called the characteristic matrix of A where I is an identity matrix of order n.

Also the determinant $|A - \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & : & a_{1n} \\ a_{21} & a_{22}-\lambda & : & a_{2n} \\ : & : & : & : \\ a_{n1} & a_{n2} & : & a_{nn}-\lambda \end{vmatrix}$

is called the characteristic polynomial of A and the roots of this equation are called the characteristic roots, latent roots or eigen values of A.

If λ is a characteristic root of an $n \times n$ matrix A, then a non-zero vector x such that $AX = \lambda X$ is called a characteristic vector or eigen vector of A corresponding to the eigen value λ.

Cayley Hamilton Theorem : Every square matrix satisfies its own characteristic equation *i.e.*, if for a square matrix A of order n,

$$|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots\ldots + a_n]$$

then the matrix equation

$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots\ldots + a_n I = 0$ is satisfied by $X = A$.

$$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots\ldots + a_n I = 0$$

i.e., $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots + a_n I = 0$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.18. Solve by calculating the inverse by elementary row operations :

$x_1 + x_2 + x_3 + x_4 = 0$, $x_1 + x_2 + x_3 - x_4 = 4$, $x_1 + x_2 - x_3 + x_4 = -4$, $x_1 - x_2 + x_3 + x_4 = 2$.

AKTU 2014-15, Marks 10

Answer

$$AX = B \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

$$X = A^{-1} B$$

We have, $A = IA$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & -2 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A \qquad \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{-1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{-1}{2} \end{bmatrix} A \qquad \begin{matrix} R_2 \to R_2/(-2) \\ R_3 \to R_3/(-2) \\ R_4 \to R_4/(-2) \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & \frac{-1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{-1}{2} \end{bmatrix} A \qquad R_1 \to R_1 - (R_2 + R_3 + R_4)$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 \end{bmatrix} A \qquad R_2 \leftrightarrow R_4$$

$$I = A^{-1} A$$

Thus
$$X = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{-1}{2} \\ \frac{1}{2} & 0 & \frac{-1}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ -1 \\ 2 \\ -2 \end{bmatrix}$$

Thus, $x_1 = 1, x_2 = -1, x_3 = 2, x_4 = -2$

Que 1.19. Test the consistency and hence, solve the following set of equations :

$$\begin{aligned} 10y + 3z &= 0, \\ 3x + 3y + 2z &= 1, \\ 2x - 3y - z &= 5, \\ x + 2y &= 4. \end{aligned}$$

Answer

$$[A : B] = \left[\begin{array}{ccc:c} 0 & 10 & 3 & 0 \\ 3 & 3 & 2 & 1 \\ 2 & -3 & -1 & 5 \\ 1 & 2 & 0 & 4 \end{array}\right]$$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 2 & -3 & -1 & 5 \\ 0 & 10 & 3 & 0 \end{array}\right] \quad R_1 \leftrightarrow R_4$$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & -7 & -1 & -3 \\ 0 & 10 & 3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & -17/3 & 68/3 \\ 0 & 0 & 29/3 & -110/3 \end{array}\right] \quad \begin{array}{l} R_3 \to R_3 - \dfrac{7}{3} R_2 \\ R_4 \to R_4 + \dfrac{10}{3} R_2 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & -17/3 & 68/3 \\ 0 & 0 & 0 & 2 \end{array}\right] \quad R_4 \to R_4 + \frac{29}{17} R_3$$

$\rho(A) \neq \rho(A : B)$, equations are inconsistent.

Que 1.20. **Investigate for what values of λ and μ, the system of equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$, has**

i. No solution.

ii. Unique solution.

iii. Infinite number of solutions.

AKTU 2013-14, Marks 05

AKTU 2017-18, Marks 3.5

OR

Determine the values 'a' and 'b' for which the following system of equation has.

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + az &= b \end{aligned}$$

i. No solution.

ii. Unique solution.

iii. Infinite number of solutions.

AKTU 2015-16, Marks 10

Answer

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 3z &= 10 \\ x + 2y + \lambda z &= \mu \end{aligned}$$

In matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix}$$

$$AX = B$$

$$C = [A : B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda - 1 & \mu - 6 \end{array}\right] \quad R_3 \to R_3 - R_2$$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda - 3 & \mu - 10 \end{array}\right]$$

i. For no solution :

$$R(A) \neq R(C)$$

i.e., $\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 \neq 0$ or $\mu \neq 10$.

ii. A unique solution :

$$R(A) = R(C) = 3$$

i.e., $\lambda - 3 \neq 0$ or $\lambda \neq 3$ and μ may have any value.

iii. Infinite solutions :

$$R(A) = R(C) = 2$$

i.e., $\lambda - 3 = 0$ or $\lambda = 3$ and $\mu - 10 = 0$ or $\mu = 10$.

Que 1.21. **State and prove Cayley-Hamilton theorem.**

Answer

A. Cayley-Hamilton Theorem : Every square matrix satisfies its own characteristic equation.

If $|A - \lambda I| = (-1)^n [\lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \ldots\ldots + a_n]$ be the characteristics polynomial of $(n \times n)$ matrix $A = a_{ij}$, then the matrix equation,

$X^n + a_1 X^{n-1} + a_2 X^{n-2} + \ldots\ldots + a_n I = 0$ is satisfied by $X = A$

i.e., $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots\ldots + a_n I = 0$

B. Proof : Since the elements of $A - \lambda I$ are at most of the first degree in λ, the elements of adj$(A - \lambda I)$ are at most of degree $(n-1)$ in λ. Thus, adj$(A - \lambda I)$ in matrix form can be written as

$\text{Adj}(A - \lambda I) = B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-2}\lambda + B_{n-1}$

where $B_0, B_1, \ldots, B_{n-1}$ are the matrices of type $n \times n$.

Now, $(A - \lambda I)\,\text{adj}(A - \lambda I) = |A - \lambda I|\, I$

$(A - \lambda I)(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1}) = (-1)^n [\lambda^n + a_1\lambda^{n-1} + \ldots + a_n] I$

Comparing coefficients of like powers of λ on both sides,

$$-IB_0 = (-1)^n I$$
$$AB_0 - IB_1 = (-1)^n a_1 I$$
$$AB_1 - IB_2 = (-1)^n a_2 I$$
$$\vdots \qquad \vdots$$
$$AB_{n-1} = (-1)^n a_n I$$

On multiplying these equations by $A^n, A^{n-1}, \ldots, I$ respectively and adding, we get

$$0 = (-1)^n [A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I]$$

Thus, $A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$

Que 1.22. **Verify Cayley-Hamilton theorem for $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$. Hence find A^{-1}.**

AKTU 2014-15, Marks 10

Answer

Characteristic equation, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 5 \\ 3 & 5 & 6-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(4-\lambda)(6-\lambda) - 25] - 2[2(6-\lambda) - 15] + 3[10 - 3(4-\lambda)] = 0$

$(1-\lambda)[24 - 10\lambda + \lambda^2 - 25] - 2[12 - 2\lambda - 15] + 3[-2 + 3\lambda] = 0$

$(1-\lambda)(\lambda^2 - 10\lambda - 1) + 2(3 + 2\lambda) + 3(3\lambda - 2) = 0$

$\lambda^2 - 10\lambda - 1 - \lambda^3 + 10\lambda^2 + \lambda + 6 + 4\lambda + 9\lambda - 6 = 0$

$-\lambda^3 + 11\lambda^2 + 4\lambda - 1 = 0$

$\lambda^3 - 11\lambda^2 - 4\lambda + 1 = 0$

For verification of Cayley-Hamilton theorem we need to verify,

$A^3 - 11A^2 - 4A + I = 0$ ...(1.22.1)

$$4A = 4\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 20 \\ 12 & 20 & 24 \end{bmatrix}$$

$$A^2 = AA = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}$$

$$11A^2 = 11\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} = \begin{bmatrix} 154 & 275 & 341 \\ 275 & 495 & 616 \\ 341 & 616 & 770 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix}$$

From eq. (1.22.1), we have

$$A^3 - 11A^2 - 4A + I = \begin{bmatrix} 157 & 283 & 353 \\ 283 & 510 & 636 \\ 353 & 636 & 793 \end{bmatrix} - \begin{bmatrix} 154 & 275 & 341 \\ 275 & 495 & 616 \\ 341 & 616 & 770 \end{bmatrix}$$
$$- \begin{bmatrix} 4 & 8 & 12 \\ 8 & 16 & 20 \\ 12 & 20 & 24 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley-Hamilton theorem is verified.

To find A^{-1} multiplying eq. (1.22.1) both sides by A^{-1}, we get

$A^2 - 11A - 4I + A^{-1} = 0$

$$A^{-1} = -(A^2 - 11A - 4I)$$
$$= -\begin{bmatrix} 14 & 25 & 31 \\ 25 & 45 & 56 \\ 31 & 56 & 70 \end{bmatrix} + \begin{bmatrix} 11 & 22 & 33 \\ 22 & 44 & 55 \\ 33 & 55 & 66 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Que 1.23. **Verify Cayley-Hamilton theorem for $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$**

AKTU 2013-14, Marks 05

Answer Same as Q. 1.22, Page 1–21C, Unit-1.

Que 1.24. **Verify Cayley-Hamilton theorem for the matrix**

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

AKTU 2017-18, Marks 3.5

OR

Find the characteristic equation of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$ **and verify Cayley-Hamilton theorem.**

Also evaluate $A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$.

AKTU 2015-16, Marks 10

Answer

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)\{(2-\lambda)^2 - 1\} + 1\{-(2-\lambda) + 1\} + 1\{1 - (2-\lambda)\} = 0$$

$$(2-\lambda)(3-\lambda)(1-\lambda) - 2(1-\lambda) = 0$$

$$(1-\lambda)[6 - 5\lambda + \lambda^2 - 2] = 0$$

$$(1-\lambda)(\lambda^2 - 5\lambda + 4) = 0$$

$$\lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$$

To verify Cayley-Hamilton theorem we need to prove,

$$A^3 - 6A^2 + 9A - 4I = 0$$

$$A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

$$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix}$$

$$+ \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Hence Cayley-Hamilton theorem is verified.

Now computing the given equation,

$= A^6 - 6A^5 + 9A^4 - 2A^3 - 12A^2 + 23A - 9I$

$= A^6 - 6A^5 + 9A^4 - 4A^3 + 4A^3 - 2A^3 - 12A^2 + 23A - 9I$

$= A^3(A^3 - 6A^2 + 9A - 4I) + 2A^3 - 12A^2 + 23A - 9I$

$= A^3(0) + 2(A^3 - 6A^2 + 9A - 4I) + 5A - I$

$= A^3(0) + 2(0) + 5A - I$

$$= 5\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & -5 & 5 \\ -5 & 10 & -5 \\ 5 & -5 & 10 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & -5 & 5 \\ -5 & 9 & -5 \\ 5 & -5 & 9 \end{bmatrix}$$

Que 1.25. **Show that row vectors of the matrix** $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ **are linearly independent.**

Answer

$$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = 1(3) - 2(-1) - 2(2) \neq 0$$

Hence given matrix is non-singular.

Now, $$A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} \quad R_2 \to R_2 + R_1$$

$$= \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & 0 & 1/5 \end{bmatrix} \quad R_3 \to R_3 + \frac{2}{5} R_2$$

Number of non-zero rows = 3

Rank $(A) = 3$ = Order of matrix

Hence the given row vectors are linearly independent.

Que 1.26. **Find the eigen values of the matrix**

$$A = \begin{bmatrix} 8 & 6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Answer

Characteristic equation of matrix A is,

$$|A - \lambda I| = \begin{vmatrix} 8-\lambda & 6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$(8-\lambda)\,[(7-\lambda)(3-\lambda) - 16] - 6\,[-6(3-\lambda) + 8] + 2\,[24 - 2(7-\lambda)] = 0$

$(8-\lambda)\,[21 - 7\lambda - 3\lambda + \lambda^2 - 16] + 6[18 - 6\lambda - 8] + 2[24 - 14 + 2\lambda] = 0$

$(8-\lambda)\,[\lambda^2 - 10\lambda + 5] + 108 - 36\lambda - 48 + 48 - 28 + 4\lambda = 0$

$8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda - 32\lambda + 80 = 0$

$-\lambda^3 + 18\lambda^2 - 117\lambda + 120 = 0$

$\lambda^3 - 18\lambda^2 + 117\lambda - 120 = 0$

$\lambda = 1.248$

($\because$ other eigen values are in complex form)

Que 1.27. **Show that the matrix** $A = \begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$, **has less than three linearly independent eigen vectors. Is it possible to obtain a similarity transformation that will diagonalize this matrix ?**

AKTU 2013-14, Marks 10

Answer

Characteristic equation of given matrix is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 10 & 5 \\ -2 & -3-\lambda & -4 \\ 3 & 5 & 7-\lambda \end{vmatrix} = 0$$

$(3-\lambda)\,[(-3-\lambda)(7-\lambda) + 20] - 10\,[-14 + 2\lambda + 12] + 5\,[-10 + 9 + 3\lambda] = 0$

$(3-\lambda)\,[-21 + 3\lambda - 7\lambda + \lambda^2 + 20] - 10\,(2\lambda - 2) + 5\,(3\lambda - 1) = 0$

$(3-\lambda)\,(\lambda^2 - 4\lambda - 1) - 20\lambda + 20 + 15\lambda - 5 = 0$

$3\lambda^2 - 12\lambda - 3 - \lambda^3 + 4\lambda^2 + \lambda - 5\lambda + 15 = 0$

$-\lambda^3 + 7\lambda^2 - 16\lambda + 12 = 0$

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$

By solving above equation, we get

$\lambda = 2, 2, 3$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 1 & 10 & 5 \\ -2 & -5 & -4 \\ 3 & 5 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x + 10y + 5z = 0$ ...(1.27.1)

$-2x - 5y - 4z = 0 \quad \Rightarrow \quad 2x + 5y + 4z = 0$ ...(1.27.2)

$3x + 5y + 5z = 0$ ...(1.27.3)

Using cross-Multiplication in eq. (1.27.1) and eq. (1.27.2), we get

$$\frac{x}{40-25} = \frac{y}{10-4} = \frac{z}{5-20}$$

$$\frac{x}{15} = \frac{y}{6} = -\frac{z}{15} \Rightarrow \frac{x}{5} = \frac{y}{2} = \frac{z}{-5} = k$$

$$\text{Eigen vector} = \begin{bmatrix} 5 \\ 2 \\ -5 \end{bmatrix}$$

Eigen vector for $\lambda = 3$

$$\begin{bmatrix} 3-3 & 10 & 5 \\ -2 & -3-3 & -4 \\ 3 & 5 & 7-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 10 & 5 \\ -2 & -6 & -4 \\ 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$0x + 10y + 5x = 0$ ...(1.27.4)

$-2x - 6y - 4z = 0$ ...(1.27.5)

$3x + 5y + 4z = 0$ ...(1.27.6)

Using cross multiplication in eq. (1.27.4) and eq. (1.27.5), we get

$$\frac{x}{-40+30} = \frac{y}{-10-0} = \frac{z}{0+20}$$

$$\frac{x}{-10} = \frac{y}{-10} = \frac{z}{20} \Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{-2} = k$$

$\therefore$ Eigen vector = $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

As there are two repeating roots so their eigen vectors are similar. So that, the problem has less than three linearly independent eigen vectors. Since the eigen vectors λ_1 and λ_2 are not linearly independent hence similarity transformation is not possible.

Que 1.28. **Find the eigen value of the matrix $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ corresponding to the eigen vector $\begin{bmatrix} 101 \\ 101 \end{bmatrix}$.**

AKTU 2016-17, Marks 3.5

Answer

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}, \text{ Eigen vector matrix } (X) = \begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

We know that, $AX = \lambda X$

Where, λ is the eigen value matrix.

$$\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} 101 \\ 101 \end{bmatrix}$$

$$\begin{bmatrix} 101 \times 4 + 101 \times 2 \\ 101 \times 2 + 101 \times 4 \end{bmatrix} = \begin{bmatrix} 101\,\lambda_1 \\ 101\,\lambda_2 \end{bmatrix}$$

$$101\,\lambda_1 = 101 \times 6 \Rightarrow \lambda_1 = 6$$

$$101\,\lambda_2 = 101 \times 6 \Rightarrow \lambda_2 = 6$$

Que 1.29. **Reduce the matrix $P = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$ to diagonal form.**

AKTU 2016-17, Marks 04

Answer

$$P = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Characteristic equation of matrix P is $|P - \lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix} = 0$$

$$(1-\lambda)\,[-\lambda(2-\lambda) + 1] - 2(-\lambda + 1) - 2(-1 + 2 - \lambda) = 0$$

$$(1-\lambda)\,(\lambda^2 - 2\lambda - 3) = 0$$

$$(1-\lambda)\,(\lambda + 1)\,(\lambda - 3) = 0$$

$$\lambda = 1, -1, 3$$

We know that diagonal matrix is the identity matrix with eigen values at diagonal, so

$$\text{Diagonal matrix } P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Que 1.30. **Find a matrix P which diagonalizes the matrix**

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}, \text{ verify } P^{-1} AP = D$$

where D is the diagonal matrix.

Answer

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation of the matrix A is

$$\begin{vmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\,(3-\lambda) - 2 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$(\lambda - 2)\,(\lambda - 5) = 0$$

$$\lambda = 2, 5$$

Eigen vector for $\lambda = 2$

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2x_1 + x_2 = 0$$

$$2x_1 + x_2 = 0$$

Let, $x_1 = k_1$

$x_2 = -2k_1$

Eigen vector is $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Eigen vector for $\lambda = 5$

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 = 0$$
$$x_1 - x_2 = 0$$
$$2x_1 - 2x_2 = 0$$

Let, $$x_2 = k_2$$
$$x_1 = k_2$$

Eigen vector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Thus, $$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$$

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 2 & -2 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 6 & 0 \\ 0 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D \text{ (Diagonal matrix)}$$

☺☺☺

2
UNIT

Differential Calculus-I

PART-1

Introduction to Limits, Continuity and Differentiability, Rolle's Theorem, Lagrange's Mean Value Theorem and Cauchy's Mean Value Theorem

CONCEPT OUTLINE : PART-1

Limit : The function $f(x, y)$ tends to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$. Then

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l$$

The function $f(x, y)$ in region R tends to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number $\in (a, b)$, there exists another positive number δ such that

$$|f(x, y) - l| < \in \text{ for } 0 < (x - a)^2 + (y - b)^2 < \delta^2$$

for every point (x, y) in R.

Continuity : A function $f(x, y)$ is said to be continuous at the point (a, b) if

$\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b)$ irrespective of the path along with $x \to a, y \to b$.

Rolle's Theorem : If $f(x)$ is

i. Continuous in $[a, b]$

ii. Derivable in (a, b) and

iii. $f(a) = f(b)$.

Then there exists at least one value $c \in (a, b)$ such that $f'(c) = 0$.

Lagrange's Mean Value Theorem : Let f be

i. Continuous in $[a, b]$ and

ii. Derivable in (a, b).

Then there exists at least one value $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Cauchy's Mean Value Theorem : Let $f(x)$ and $g(x)$ be two functions which are both derivable in $[a, b]$ and $g'(x) \neq 0$ for any value of x in $[a, b]$. Then there exists at least one value c in between a and b such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. Evaluate $\lim\limits_{\substack{x \to 1 \\ y \to 2}} \dfrac{3x^2 y}{x^2 + y^2 + 5}$.

Answer

$$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{3x^2 y}{x^2 + y^2 + 5} = \lim_{x \to 1}\left[\lim_{y \to 2} \frac{3x^2 y}{x^2 + y^2 + 5}\right] = \lim_{x \to 1} \frac{3x^2 (2)}{x^2 + (2)^2 + 5}$$

$$= \lim_{x \to 1} \frac{6x^2}{x^2 + 9} = \frac{6}{1 + 9} = \frac{3}{5}$$

Que 2.2. Evaluate $\lim\limits_{\substack{x \to \infty \\ y \to 2}} \dfrac{xy + 4}{x^2 + 2y^2}$.

Answer

$$\lim_{\substack{x \to \infty \\ y \to 2}} \frac{xy + 4}{x^2 + 2y^2} = \lim_{x \to \infty}\left[\lim_{y \to 2} \frac{xy + 4}{x^2 + 2y^2}\right] = \lim_{x \to \infty} \frac{2x + 4}{x^2 + 8}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{4}{x}}{x + \frac{8}{x}} = \frac{2 + 0}{\infty + 0} = 0$$

Que 2.3. Show that the function $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.

Answer

Let $(a, b) \in R^2$ then $f(a, b) = a - b$

$$\therefore \quad |f(x, y) - f(a, b)| = |(x - y) - (a - b)|$$
$$= |(x - a) + (b - y)|$$
$$\leq |x - a| + |y - b| \qquad [\because |x| = |-x|] \quad ...(2.3.1)$$

Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{2}$ then for $|x - a| < \delta$ and $|y - b| < \delta$, we have from eq. (2.3.1)

$$|f(x, y) - f(a, b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence the function $f(x, y) = x - y$ is continuous for all $(a, b) \in R^2$. But (a, b) is an arbitrary element of R^2 so $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.

Que 2.4. **State Rolle's theorem and also give its proof.**

Answer

A. Rolle's theorem : If $f(x)$ is
i. Continuous in $[a, b]$
ii. Derivable in (a, b) and
iii. $f(a) = f(b)$.

Then there exists at least one value $c \in (a, b)$ such that $f'(c) = 0$.

B. Proof :

1. If $f(x) = 0$ for all x, then $f'(x) = 0$ for all x.
2. Since f is continuous in [a, b], it is bounded and attains its maximum M and minimum m say at two number c and d lying in between a and b such that

$$f(c) = M \text{ and } f(d) = m$$

Case a : If $M = m$, then f is a constant function so that f' is zero for all x in (a, b).

Case b : If $M \neq m$, then $M = f(c) \geq f(c + h)$ for values of h both positive and negative. Then

$$\frac{f(c+h) - f(c)}{h} \leq 0 \text{ for } h > 0 \quad \text{...(2.4.1)}$$

and
$$\frac{f(c+h) - f(c)}{h} \geq 0 \text{ for } h < 0 \quad \text{...(2.4.2)}$$

Since f is differentiable in (a, b) from eq. (2.4.1) and eq. (2.4.2) as $h \to 0$, we have

$$f'(c) \leq 0 \text{ and } f'(c) \geq 0$$

Hence $f'(c) = 0$ for some value c in (a, b). Similarly if $m = f(d) \leq f(d + h)$ for values of h both positive and negative it follow that

$$\frac{f(d+h) - f(d)}{h} \geq 0 \text{ for } h > 0 \quad \text{...(2.4.3)}$$

and
$$\frac{f(d+h) - f(d)}{h} \leq 0 \text{ for } h < 0$$

As $h \to 0$, $f'(d) \geq$ and $f'(d) \leq 0$. Hence $f'(d) = 0$.

Que 2.5. **Verify Rolle's theorem for $f(x) = x(x-2)e^{3x/4}$ in (0, 2).**

Answer

$f(0) = 0, f(2) = 0$, f is continuous and differentiable, so by Rolle's theorem, $f'(c) = 0$.

Here
$$f'(x) = [(x-2) + x + \frac{3}{4}(x)(x-2)]e^{\frac{3}{4}x}$$
$$= 0 \quad \text{or} \quad 3x^2 + 2x - 8 = 0$$

$\therefore \quad c = -2$ or $\frac{8}{6}$ but $c = -2$ does not lie in (0, 2) thus $c = \frac{8}{6} \in (0,2)$.

Que 2.6. **State and prove Lagrange's mean value theorem.**

Answer

A. Lagrange's Mean Value Theorem : Let f be
i. Continuous in $[a, b]$ and
ii. Derivable in (a, b).

Then there exists at least one value $c \in (a, b)$ such that,

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

B. Proof :

1. Choose $\phi(x) = f(x) + x A$, $x \in [a, b]$.
2. Since f and x are continuous in $[a, b]$ and derivable in (a, b), therefore ϕ is continuous in $[a, b]$ and derivable in (a, b).
3. Now choose the unknown constant A such that

$$f(b) + b A = \phi(b) = \phi(a) = f(a) + a A$$

or
$$A = \frac{f(b) - f(a)}{a - b}$$

Thus ϕ satisfies all the three conditions of the Rolle's theorem.

4. Therefore by Rolle's theorem there exists a number $c \in (a, b)$ such that

$$0 = \phi'(c) = f'(c) + A$$

Thus
$$f'(c) = -A = \frac{f(b) - f(a)}{b - a}$$

Que 2.7. **Verify Lagrange's mean value theorem for $f(x) = x^2$ in (1, 5).**

Answer

f is continuous and differentiable in (1, 5), so by Lagrange's mean value theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c), \text{ for some } c \text{ in } (a, b).$$

Here $f(5) = 25, f(1) = 1, f'(x) = 2x$. Therefore

$$\frac{25 - 1}{5 - 1} = 2c$$

$$\therefore \quad c = 3 \in (1, 5).$$

Que 2.8. **State and prove Cauchy's mean value theorem.**

Answer

A. Cauchy's Mean Value Theorem :

Let $f(x)$ and $g(x)$ be two functions which are both derivable in $[a, b]$ and $g'(x) \neq 0$ for any value of x in $[a, b]$. Then there exists at least one value c in between a and b such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

B. Proof :

1. Let $\phi(x) = f(x) + Ag(x)$ where A is an unknown constant.
2. ϕ is derivable in $[a, b]$ because f and g are derivable in $[a, b]$ by hypothesis.
3. Choose A such that $\phi(b) = \phi(a)$ i.e.,

$$f(b) + Ag(b) = f(a) + Ag(a)$$

or $$A = \frac{f(b) - f(a)}{g(a) - g(b)}$$

with $g(a) - g(b) \neq 0$. If $g(a) - g(b) = 0$ then $g(a) = g(b)$ and g satisfies conditions of Rolle's theorem and then $g'(c) = 0$ for some c. This contradicts the hypothesis that $g'(x) \neq 0$ for any x. Thus $g(a) - g(b) \neq 0$.

4. Now the new function ϕ satisfies the conditions of Rolle's theorem. Therefore there exists at least one $c \in (a, b)$ such that

$$0 = \phi'(c) = f'(c) + Ag'(c)$$

Thus $$\frac{f'(c)}{g'(c)} = -A = \frac{f(b) - f(a)}{g(b) - g(a)}$$

with $g(b) - g(a) \neq 0$ as $g'(c) \neq 0$ for any c.

Que 2.9. **Verify Cauchy's mean value theorem for the functions**

i. $f(x) = x^4$, $g(x) = x^2$ in the interval $[a, b]$.

ii. $f(x) = \ln x$, $g(x) = \dfrac{1}{x}$ in $[1, e]$.

Answer

i. $f(x) = x^4$, $g(x) = x^2$, $[a, b]$

$f'(x) = 4x^3$, $g'(x) = 2x$

By Cauchy's mean value theorem,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\frac{b^4 - a^4}{b^2 - a^2} = \frac{4c^3}{2c}$$

$$\therefore \quad c^2 = \frac{1}{2}(a^2 + b^2)$$

$$c = \frac{1}{\sqrt{2}}\sqrt{(a^2 + b^2)} \in (a, b)$$

ii. $f = \ln x$, $\quad g(x) = \dfrac{1}{x}$, $[1, e]$

$$f'(x) = \frac{1}{x}, g'(x) = -\frac{1}{x^2}$$

By Cauchy's mean value theorem,

$$\frac{\ln e - \ln 1}{\frac{1}{e} - 1} = \frac{-c^2}{c}$$

$$\therefore \quad c = \frac{e}{e - 1} \in (1, e)$$

PART-2

Successive Differentiation (n^{th} Order Derivatives), Leibnitz Theorem and its Application

CONCEPT OUTLINE : PART-2

Successive Differentiation :

If $y = f(x)$ is a differentiable function of x, then dy/dx is called the first differential coefficient of y w.r.t. x.

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ is called the second differential coefficient and in a similar way $\dfrac{d^ny}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)$ is the n^{th} differential coefficient of y w.r.t. x. This type of differentiation is called successive differentiation. n^{th} differential coefficient can be denoted by various ways as :

$\dfrac{d^ny}{dx^n}$, D^ny, y_n, $f^n(x)$ etc.

Some Standard Results for n^{th} Derivative are :

1. $D^n (ax + b)^m = m(m-1)(m-2)......(m-n+1)\, a^n (ax+b)^{m-n}$
2. $D^n (ax + b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-n-1}$
3. $D^n e^{ax+b} = a^n e^{ax+b}$
4. $D^n a^x = a^x (\log a)^n$
5. $D^n \log(ax + b) = \dfrac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

6. $D^n \log x = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$

7. $D^n \sin(ax+b) = a^n \sin\left(n\dfrac{\pi}{2} + ax + b\right)$

8. $D^n \cos(ax+b) = a^n \cos\left(n\dfrac{\pi}{2} + ax + b\right)$

9. $D^n e^{ax} \sin(bx+c) = r^n e^{ax} \sin(bx + c + n\phi)$

where
$$r = \sqrt{a^2+b^2}$$
$$\phi = \tan^{-1}(b/a)$$

10. $D^n e^{ax} \cos(bx+c) = r^n e^{ax} \cos(bx + c + n\phi)$

Leibnitz Theorem : If u and v are any two functions of x such that all their desired differential coefficients exist, then the n^{th} differential coefficient of their product is given by

$$D^n(uv) = {}^nC_0 D^n u.v + {}^nC_1 D^{n-1} u\, Dv + {}^nC_2 D^{n-2} u\, D^2 v + \ldots\ldots + {}^nC_r D^{n-r} u\, D^r v + \ldots\ldots + u.D^n v$$

Determination of the Value of the n^{th} Derivative of a Function for $x = 0$:

Step I : Suppose the given function equals to y.

Step II : Find $\quad y_1 = \dfrac{dy}{dx}$

i. Simplify the expression by taking LCM if possible.
ii. Square both sides to avoid square root.
iii. Convert y_1 in terms of y (if possible).

Step III : Find $\quad y_2 = \dfrac{d^2y}{dx^2}$

Step IV : Differentiate n times by Leibnitz theorem.

Step V : Put $x = 0$ in Steps (I), (II), (III) and (IV).

Step VI : Put $n = 1, 2, 3, 4$ in last equation of step V.

Step VII : Discuss the cases for n i.e., even or odd.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.10. **Find the n^{th} differential coefficient of**

ii. **$\sin ax \cos bx$**
iii. **$\sin x \cos 3x$**

Answer

i. Let
$$y = \cos^4 x = \left[\frac{1}{2}(1+\cos 2x)\right]^2 = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$
$$= \frac{1}{4}\left[1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)\right]$$
$$y = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$
$$\because \quad D^n \cos(ax+b) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$
$$\therefore \quad y_n = 0 + \frac{1}{2}\times 2^n \cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8}\times 4^n \cos\left(4x + \frac{n\pi}{2}\right)$$
$$y_n = 2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right) + 2^{2n-3}\cos\left(4x + \frac{n\pi}{2}\right)$$

ii.
$$y = \cos bx \sin ax$$
$$y = \frac{1}{2}(2\sin ax \cos bx)$$
$$= \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$$
$$\because \quad D^n \sin(ax+b) = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$
$$\therefore \quad y_n = \frac{1}{2}\left[(a+b)^n \sin\left\{(a+b)x + \frac{n\pi}{2}\right\} + (a-b)^n \sin\left\{(a-b)x + \frac{n\pi}{2}\right\}\right]$$

iii.
$$y = \sin x \cos 3x$$
or
$$y = \frac{1}{2}(\sin 4x - \sin 2x)$$
$$y_n = \frac{1}{2}\left[4^n \sin\left(4x + \frac{n\pi}{2}\right) - 2^n \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

Que 2.11. **Find the n^{th} derivative of $\tan^{-1}\left\{\dfrac{2x}{1-x^2}\right\}$.**

Answer

$$y_1 = \frac{1}{1+\left(\dfrac{2x}{1-x^2}\right)^2}\frac{d}{dx}\left(\frac{2x}{1-x^2}\right)$$
$$= \frac{(1-x^2)^2}{(1+x^4-2x^2+4x^2)}\cdot\frac{(1-x^2)\,2 - 2x(-2x)}{(1-x^2)^2}$$

$$= \frac{2(1+x^2)}{(1+x^2)^2} = \frac{2}{1+x^2}$$

We know that,

$$y_n = \frac{d^n}{dx^n}\left\{\frac{1}{x^2+a^2}\right\} = \frac{(-1)^n\, n!}{a^{n+2}} \sin (n+1)\theta \sin^{n+1}\theta$$

Where

$$\theta = \cot^{-1}\left(\frac{x}{a}\right) = \tan^{-1}\left(\frac{a}{x}\right)$$

Now differentiating y_1, $(n-1)$ times

$$y_n = \frac{d^{n-1}}{dx^{n-1}} y_1 = \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{2}{1+x^2}\right\}$$

$$= 2(-1)^{n-1}(n-1)!\, \sin n\theta \sin^n\theta$$

Where

$$\theta = \tan^{-1}\frac{1}{x} = \cot^{-1} x$$

Que 2.12. **If $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$, show that $I_n = nI_{n-1} + (n-1)!$**

AKTU 2016-17, Marks 3.5

Answer

$$I_n = \frac{d^n}{dx^n}[x^n \log x] = \frac{d^{n-1}}{dx^{n-1}}\left\{\frac{d}{dx}(x^n \log x)\right\}$$

$$I_n = \frac{d^{n-1}}{dx^{n-1}}\left[nx^{n-1}\log x + \frac{x^n}{x}\right]$$

$$= \frac{d^{n-1}}{dx^{n-1}}(nx^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1})$$

$$= n\frac{d^{n-1}}{dx^{n-1}}[x^{n-1}\log x] + (n-1)! = nI_{n-1} + (n-1)!$$

$$I_n = nI_{n-1} + (n-1)!$$

Que 2.13. **If $u = \sin nx + \cos nx$, then prove that**

$$u_r = n^r[1 + (-1)^r \sin 2nx]^{1/2},$$

where u_r is the r^{th} differential coefficient of u w.r.t. x.

AKTU 2017-18, Marks 3.5

Answer

$$u = \sin nx + \cos nx$$

The r^{th} differential coefficient of u is given as

$$u_r = n^r \sin\left(nx + \frac{r\pi}{2}\right) + n^r \cos\left(nx + \frac{r\pi}{2}\right)$$

On squaring and taking the square root, we get

$$u_r = n^r\left[\left\{\sin\left(nx+\frac{r\pi}{2}\right) + \cos\left(nx+\frac{r\pi}{2}\right)\right\}^2\right]^{1/2}$$

$$= n^r\left[1 + 2\sin\left(nx+\frac{r\pi}{2}\right)\cos\left(nx+\frac{r\pi}{2}\right)\right]^{1/2}$$

$$= n^r[1 + \sin(2nx + r\pi)]^{1/2}$$

$$= n^r[1 + \sin 2nx \cos r\pi + \cos 2nx \sin r\pi]^{1/2}$$

$$u_r = n^r[1 + (-1)^r \sin 2nx]^{1/2}$$

$$(\because \cos r\pi = (-1)^r \text{ and } \sin r\pi = 0)$$

Que 2.14. **If $y = e^{m\cos^{-1}x}$, then show that $(1-x^2)\,y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ and hence calculate y_n, when $x = 0$.**

AKTU 2013-14, Marks 10

OR

If $\cos^{-1} x = \log(y)^{1/m}$, then show $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ and hence calculate y_n when $x = 0$.

AKTU 2015-16, Marks 10

Answer

$$y = e^{m\cos^{-1}x} \quad \text{...(2.14.1)}$$

$$y_1 = e^{m\cos^{-1}x}\frac{-m}{\sqrt{1-x^2}} = -\frac{me^{m\cos^{-1}x}}{\sqrt{1-x^2}} \quad \text{...(2.14.2)}$$

$$\sqrt{1-x^2}\; y_1 = -my$$

Squaring on both sides

$$(1-x^2)\,y_1^2 = m^2y^2 \quad \text{...(2.14.3)}$$

Differentiating eq. (2.14.3), we get

$$(1-x^2)\,2y_1y_2 - 2x\,y_1^2 = m^2\,2y\,y_1$$

or $\quad (1-x^2)\,y_2 - xy_1 - m^2y = 0 \quad$ ...(2.14.4)

Differentiating eq. (2.14.4) n times by Leibnitz theorem, we get

$$(1-x^2)\,y_{n+2} + n(-2x)\,y_{n+1} + \frac{n(n-1)}{2}(-2)\,y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

or $(1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+m^2)\,y_n = 0 \quad$ ...(2.14.5)

From eq. (2.14.1), $y(0) = e^{m\cos^{-1}0} = e^{\frac{m\pi}{2}}$

From eq. (2.14.2), $y_1(0) = -\dfrac{me^{m\cos^{-1}0}}{\sqrt{1-0}} = -me^{\frac{m\pi}{2}}$

From eq. (2.14.4), $y_2(0) = m^2 y(0) = m^2 e^{\frac{m\pi}{2}}$

Put $n = 1, 2, 3, 4, \ldots\ldots$ in eq. (2.14.5), we get

$$y_3(0) = (1^2 + m^2)\, y_1(0) = (1^2 + m^2)\left(-me^{\frac{m\pi}{2}}\right) = -m(1^2 + m^2)\, e^{\frac{m\pi}{2}}$$

$$y_4(0) = (2^2 + m^2)\, y_2(0) = (2^2 + m^2)\left(m^2 e^{\frac{m\pi}{2}}\right) = m^2 (2^2 + m^2)\, e^{\frac{m\pi}{2}}$$

$$y_5(0) = (3^2 + m^2)\, y_3(0) = (3^2 + m^2)\left(-m(1^2 + m^2) e^{\frac{m\pi}{2}}\right)$$

$$y_5(0) = -m\,(1^2 + m^2)(3^2 + m^2)\, e^{\frac{m\pi}{2}}$$

$$y_6(0) = (4^2 + m^2)\, y_4(0) = m^2 (2^2 + m^2)(4^2 + m^2)\, e^{\frac{m\pi}{2}}$$

$$\therefore y_n(0) = \begin{cases} -m(1^2 + m^2)(3^2 + m^2) \ldots\ldots [(n-2)^2 + m^2]\, e^{\frac{m\pi}{2}}, & \text{if } n \text{ is odd} \\ m^2(2^2 + m^2)(4^2 + m^2) \ldots\ldots [(n-2)^2 + m^2]\, e^{\frac{m\pi}{2}}, & \text{if } n \text{ is even} \end{cases}$$

$$\text{and } y_{n+2} = \begin{cases} -m(1^2 + m^2)(3^2 + m^2) \ldots\ldots [n^2 + m^2]\, e^{\frac{m\pi}{2}}, & \text{if } n \text{ is odd} \\ m^2(2^2 + m^2)(4^2 + m^2) \ldots\ldots [n^2 + m^2]\, e^{\frac{m\pi}{2}}, & \text{if } n \text{ is even} \end{cases}$$

Que 2.15. **If $y^{1/m} + y^{-1/m} = 2x$, prove that**

$(x^2 - 1)\, y_{n+2} + (2n + 1)\, y_{n+1} x + (n^2 - m^2)\, y_n = 0$.

Answer

Given : $y^{1/m} + y^{-1/m} = 2x$

Let $z = y^{1/m}$

So, $z + z^{-1} = 2x$

$z^2 + 1 - 2xz = 0$

$$z = [x \pm \sqrt{x^2 - 1}]$$

Taking positive sign,

$$y = [x + \sqrt{x^2 - 1}]^m \qquad \{\because z = y^{1/m}\}$$

$$y_1 = m[x + \sqrt{x^2 - 1}]^{m-1}\left[1 + \frac{2x}{2\sqrt{x^2 - 1}}\right]$$

$$= m\,[x + \sqrt{x^2 - 1}]^m \frac{1}{\sqrt{x^2 - 1}}$$

$$\sqrt{x^2 - 1}\; y_1 = my$$

On squaring both sides

$$(x^2 - 1)\, y_1^{\,2} = m^2 y^2 \qquad \ldots(2.15.1)$$

Differentiating eq. (2.15.1),

$$(x^2 - 1)\, 2y_1 y_2 + 2xy_1^{\,2} = 2m^2\, yy_1$$

$$\text{or } (x^2 - 1)\, y_2 + xy_1 - m^2 y = 0 \qquad \ldots(2.15.2)$$

Differentiating eq. (2.15.2) n times by Leibnitz theorem,

$$(x^2 - 1)\, y_{n+2} + n2x\, y_{n+1} + \frac{n(n-1)}{2!} 2y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$

$$(x^2 - 1)\, y_{n+2} + (2n + 1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

Que 2.16. **If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$ prove that**

$(x^2 - 1)\, y_{n-2} + (2n + 1)xy_{n-1} + (n^2 - m^2)y_n = 0$.

AKTU 2014-15, Marks 05

Answer

Same as Q. 2.15, Page 2–12C, Unit-2.

Use $n - 2$ in place of $n + 2$ and $n - 1$ in place of $n + 1$.

Que 2.17. **If $y = (x + \sqrt{1 + x^2})^m$, then find the n^{th} derivative of y at $x = 0$.**

Answer

$$\text{Given : } \quad y = (x + \sqrt{1 + x^2})^m \qquad \ldots(2.17.1)$$

Differentiating eq. (2.17.1),

$$y_1 = m(x + \sqrt{1 + x^2})^{m-1}\left(1 + \frac{x}{\sqrt{1 + x^2}}\right)$$

$$\sqrt{1 + x^2}\; y_1 = my \qquad \ldots(2.17.2)$$

On squaring both sides,

$$(1 + x^2)y_1^{\,2} = m^2 y^2$$

Differentiating above equation again,

$$(1 + x^2)2y_1 y_2 + 2xy_1^{\,2} = 2m^2 yy_1$$

$$\text{or} \quad (1 + x^2)y_2 + xy_1 - m^2 y = 0 \qquad \ldots(2.17.3)$$

Differentiating eq. (2.17.3) n times using Leibnitz theorem,

$$(1 + x^2)y_{n+2} + n(2x)y_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$

$$\text{or} \quad (1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0 \qquad \ldots(2.17.4)$$

Putting $x = 0$ in eq. (2.17.1), eq. (2.17.2), eq. (2.17.3) and eq. (2.17.4) respectively, we get

$$y(0) = 1$$

$$y_1(0) = my(0)$$
$$y_1(0) = m$$
$$y_2(0) = m^2$$
$$y_{n+2}(0) = (m^2 - n^2)y_n(0) \quad ...(2.17.5)$$

Put $n = 1, 2, 3, 4, 5, ...$ in eq. (2.17.5),

$$y_3(0) = (m^2 - 1^2)\, y_1(0) = (m^2 - 1^2)m$$
$$y_4(0) = (m^2 - 2^2)\, y_2(0) = (m^2 - 2^2)m^2$$
$$y_5(0) = (m^2 - 3^2)\, y_3(0) = (m^2 - 3^2)\,(m^2 - 1^2)m$$
$$y_6(0) = (m^2 - 4^2)\,(m^2 - 2^2)m^2$$

and so on.

Thus using above results, we have

$$y_n(0) = \begin{cases} m(m^2 - 1^2)(m^2 - 3^2) ...... \{m^2 - (n-2)^2\}, \text{ if } n \text{ is odd.} \\ m^2(m^2 - 2^2)(m^2 - 4^2) ...... \{m^2 - (n-2)^2\}, \text{ if } n \text{ is even.} \end{cases}$$

Que 2.18. **If $y = e^{\tan^{-1}x}$, then prove that $(1 + x^2)y_2 + (2x - 1)y_1 = 0$ and $(1 + x^2)\, y_{n+2} + [2(n + 1)x - 1]\, y_{n+1} + n(n + 1)y_n = 0.$**

AKTU 2017-18, Marks 3.5

Answer

Given : $\quad y = e^{\tan^{-1}x} \quad$ ...(2.18.1)

Differentiating eq. (2.18.1) w.r.t x, we get

$$y_1 = e^{\tan^{-1}x}\frac{1}{1+x^2}$$
$$(1 + x^2)\, y_1 = y \quad ....(2.18.2)$$

Differentiating eq. (2.18.2) w.r.t x, we get

$$(1 + x^2)y_2 + 2xy_1 = y_1$$
$$(1 + x^2)y_2 + 2xy_1 - y_1 = 0$$
$$(1 + x^2)y_2 + (2x - 1)y_1 = 0 \quad ....(2.18.3)$$

Now, differentiating eq. (2.18.2) n times by Leibniz theorem, we get

$$y_{n+1}(1+x^2) + {}^nC_1 y_n(2x) + {}^nC_2 y_{n-1}(2) = y_n$$

$$(1+x^2)y_{n+1} + (2nx - 1)y_n + n(n-1)y_{n-1} = 0 \quad ...(2.18.4)$$

Differentiating eq. (2.18.4) again w.r.t x, we get

$$y_{n+2}(1 + x^2) + 2xy_{n+1} + (2nx - 1)y_{n+1} + 2ny_n + n(n - 1)y_n = 0$$
$$(1 + x^2)y_{n+2} + [2(n + 1)x - 1]y_{n+1} + n(n + 1)y_n = 0$$

PART-3

Envelope, Involutes and Evolutes, Curve Tracing : Cartesian and Polar Coordinates.

CONCEPT OUTLINE : PART-3

Envelope : Envelope E of a given family of curves c is a curve which touches every member of the family of curves c and at each point of the envelope E is touched by same member of the family of curves c.

Curve Tracing : It is the method of finding approximate shape of curves from their cartesian, polar or parametric equation without plotting a large number of data points.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.19. **Write the procedure for obtaining envelope.**

Answer

Step I : Envelope is obtained generally by eliminating the parameter α between the equation of the given family curves

$$f(x, y, \alpha) = 0 \quad ...(2.19.1)$$

and $\quad \dfrac{\partial f}{\partial \alpha} = f_\alpha(x, y, \alpha) = 0 \quad$ ...(2.19.2)

Where $\dfrac{\partial f}{\partial \alpha}$ is the partial derivative of f w.r.t. α.

Step II : In case α cannot be eliminated between eq. (2.19.1) and eq. (2.19.2), then solve eq. (2.19.1) and eq. (2.19.2) for x and y in terms of α. Then the envelope is given in the parametric form by the equations

$$x = x(\alpha) \text{ and } y = y(\alpha)$$

Step III : If the eq. (2.19.1) is a quadratic in the parameter α or quadratic in some parameter λ which is a function of α, then the envelope is given by discriminant equated to zero.

Suppose $f(x, y, a) = 0$ is rewritten as a quadratic equation

$$A\lambda^2 + B\lambda + C = 0 \quad ...(2.19.3)$$

Where A, B, C are functions of x, y while λ is either α or function of α.

Differentiating eq. (2.19.3) w.r.t λ.

$$2A\lambda + B = 0 \quad \text{or} \quad \lambda = -\frac{B}{2A} \qquad \text{...(2.19.4)}$$

Eliminating λ from eq. (2.19.3) by using eq. (2.19.4) we get the equation of the required envelope as

$$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C = 0$$

i.e., $\quad B^2 - 4AC = \text{Discriminant} = 0$

Step IV : Envelope of the family of normals to a given curve C is the evolute of the curve C.

Step V : For a given two parameter family of curves

$$f(x, y, \alpha, \beta) = 0 \qquad \text{...(2.19.5)}$$

with a given relation $g(\alpha, \beta) = 0$ between the parameters α, β, eq. (2.19.5) can be reduced to a one parameter family by elimination of one of the parameters say β in terms of α by using the given relation $g(\alpha, \beta)$. Then proceed as in Step I.

Que 2.20. **Find the envelope of the one parameter family of curves $y = mx + am^p$ where m is the parameter and a, p are constants.**

Answer

$$y = mx + am^p \qquad \text{...(2.20.1)}$$

Differentiating the given curves w.r.t. the parameter 'm', we get

$$0 = x + apm^{p-1}$$

or $\quad m = \left(-\frac{x}{pa}\right)^{\frac{1}{(p-1)}} \qquad \text{...(2.20.2)}$

Using eq. (2.20.2) eliminate m from eq. (2.20.1)

$$y = \left(\frac{-x}{pa}\right)^{\frac{1}{(p-1)}} x + a\left(\frac{-x}{pa}\right)^{\frac{p}{(p-1)}}$$

$$y^{(p-1)} = \left(\frac{-x}{pa}\right) x^{(p-1)} + a^{(p-1)}\left(\frac{-x}{pa}\right)^p$$

or $\quad ap^p y^{p-1} = -x^p p^{p-1} + (-x)^p \qquad \text{...(2.20.3)}$

Eq. (2.20.3) is the equation of required envelope of given family of curve.

Que 2.21. **Determine the envelope of the two parameter family of parabolas.**

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1$$

Where the two parameter a and b are connected by the relation $a + b = c$ where c is a given constant.

Answer

Using the given relation

$$a + b = c$$

Eliminate $b = c - a$ from the given family

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{c-a}} = 1 \qquad \text{...(2.21.1)}$$

Which is now a one-parameter family of parabolas with a as the parameter.

Differentiating eq. (2.21.1) w.r.t 'a',

$$\sqrt{x}\left(-\frac{1}{2}\right)\frac{1}{a^{3/2}} + \sqrt{y}\left(-\frac{1}{2}\right) \times \frac{1}{(c-a)^{\frac{3}{2}}}(-1) = 0$$

$$\left(\frac{c-a}{a}\right)^{\frac{3}{2}} = \left(\frac{y}{x}\right)^{\frac{1}{2}}$$

$$\frac{c}{a} = \frac{x^{1/3} + y^{1/3}}{x^{1/3}}$$

or $\quad a = \frac{cx^{1/3}}{x^{1/3} + y^{1/3}} \qquad \text{...(2.21.2)}$

Substituting eq. (2.21.2) in eq. (2.21.1), we get the required envelope as

$$\left[x\frac{x^{1/3} + y^{1/3}}{cx^{1/3}}\right]^{\frac{1}{2}} + \left[y\frac{1}{\left\{c - \frac{cx^{1/3}}{x^{1/3} + y^{1/3}}\right\}}\right]^{\frac{1}{2}} = 1$$

or $\quad [x^{2/3}(x^{1/3} + y^{1/3})]^{\frac{1}{2}} + [y^{2/3}(x^{1/3} + y^{1/3})]^{\frac{1}{2}} = c^{1/2}$

or $\quad (x^{1/3} + y^{1/3})^{\frac{1}{2}}\,[(x^{2/3})^{\frac{1}{2}} + (y^{2/3})^{\frac{1}{2}}] = c^{1/2}$

$$(x^{1/3} + y^{1/3})^{\frac{3}{2}} = c^{1/2}$$

Thus the envelope is the astroid given by

$$x^{1/3} + y^{1/3} = c^{1/3}.$$

Que 2.22. **Define involute and evolute. How will you find an evolute ?**

Answer

A. Evolute and Involute :

1. As a point P moves along a given curve c_1, the center of curvature corresponding to P describes another curve c_2.

2. The curve c_2 is known as the evolute of the given curve c_1 and c_1 is known as the involute of c_2.

B. Determination of Evolute :

a. Cartesian Form : Let $y = f(x)$ be the equation of the given curve c in cartesian form. Then the coordinates of the center of curvature given by

$$X = x - y_1(1 + y_1^2)/y_2 \quad \text{...(2.22.1)}$$

$$Y = y(x) + (1 + y_1^2)/y_2 \quad \text{...(2.22.2)}$$

form the parametric equations of the evolute of C expressed in terms of the parameter x.

In many cases, the parameter x can be eliminated between eq. (2.22.1) and eq. (2.22.2). This results in a relation between X and Y of the form $f(X, Y) = 0$ which is the equation of the required evolute.

b. Parametric Form : Let the equation of the curve be in parametric form $x = x(t)$, $y = y(t)$ where t is the parameter. Then the parametric equations of the evolute are

$$X(t) = x(t) - \frac{y'(x'^2 + y'^2)}{x' y'' - x'' y'}$$

$$Y(t) = y(t) + \frac{x'(x' + y'^2)}{x' y'' - x'' y'}$$

Que 2.23. **Determine the parametric equations for the evolute of the curve $x = \dfrac{t^4}{4}$, $y = \dfrac{t^5}{5}$.**

Answer

$$x' = \frac{dx}{dt} = t^3, \; x'' = 3t^2, \; y' = \frac{dy}{dt} = t^4, \; y'' = 4t^3$$

The coordinates of the center of curvature in parametric form are

$$X = x(t) - \frac{y'(x'^2 + y'^2)}{x' y'' - x'' y'}$$

$$= \frac{t^4}{4} - \frac{t^4(t^6 + t^8)}{4t^6 - 3t^6} = -\frac{3}{4}t^4 - t^6 \quad \text{...(2.23.1)}$$

$$Y = y(t) + \frac{x'(x'^2 + y'^2)}{x' y'' - x'' y'}$$

$$= \frac{t^5}{5} + \frac{t^3(t^6 + t^8)}{4t^6 - 3t^6} = \frac{6}{5}t^5 + t^3 \quad \text{...(2.23.2)}$$

Note that the parameter t cannot be eliminated between eq. (2.23.1) and eq. (2.23.2). Therefore the equation of the required evolute in the parametric equations $X = x(t)$ and $Y = y(t)$ are given by eq. (2.23.1) and eq. (2.23.2).

Que 2.24. **Find the evolute of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Deduce the evolute of a rectangular hyperbola.**

Answer

Equation of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{...(2.24.1)}$$

Differentiating w.r.t. x, we get

$$\frac{2x}{a^2} - \frac{2y\, y_1}{b_2} = 0$$

or

$$y_1 = \frac{dy}{dx} = \frac{b^2}{a^2}\frac{x}{y} \quad \text{...(2.24.2)}$$

so,

$$y_2 = y'' = \frac{d^2y}{dx^2} = \frac{b^2}{a^4y^3}(a^2y^2 - b^2x^2)$$

$$= -\frac{b^4}{a^2}\frac{1}{y^3} \quad \text{...(2.24.3)}$$

From eq. (2.24.1),

$$\frac{y^2}{b^2} = \frac{x^2 - a^2}{a^2} \text{ or } a^2y^2 - b^2x^2 = -b^2a^2$$

Now the center of curvature is

$$X = x - \left(\frac{b^2}{a^2}\frac{x}{y}\right)\left(1 + \frac{b^4x^2}{a^4y^2}\right)\left(\frac{a^2y^3}{-b^4}\right)$$

$$X = x\left[\frac{b^2a^4 + a^2b^2x^2 - a^4b^2 + b^4x^2}{b^2a^4}\right]$$

$$= \frac{x^3(b^2 + a^2)}{a^4} \quad \text{...(2.24.4)}$$

Similarly,

$$Y = y + \left(1 + \frac{b^4x^2}{a^4y^2}\right)\left(\frac{-a^2y^3}{b^4}\right)$$

$$Y = y\left[\frac{-b^4a^2 + a^4y^2 + b^4x^2}{-b^4a^2}\right] = \frac{-y^3(a^2 + b^2)}{b^4} \quad \text{...(2.24.5)}$$

From eq. (2.24.4) : $x^2 = \left(\dfrac{a^4X}{a^2 + b^2}\right)^{\frac{2}{3}}$ ...(2.24.6)

From eq. (2.24.5), $y^2 = \left(\dfrac{b^4 Y}{a^2 + b^2}\right)^{2/3}$ ...(2.24.7)

Then, $$1 = \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{a^2}\left(\frac{a^4 X}{a^2 + b^2}\right)^{2/3} - \frac{1}{b^2}\left(\frac{b^4 Y}{a^2 + b^2}\right)^{2/3}$$

Thus the required envelope is,

$$(aX)^{2/3} - (bY)^{2/3} = (a^2 + b^2)^{2/3}$$

For a rectangular hyperbola with $a = b$ the envelope reduces to,

$$X^{2/3} - Y^{2/3} = (2a)^{2/3}$$

Que 2.25. **Give the procedure for tracing the curves in cartesian coordinates and polar coordinates.**

Answer

A. Procedure for Tracing Cartesian Curves :

a. Symmetry : See if the curve is symmetrical about any line.

1. A curve is symmetrical about the X-axis, if only even powers of y occur in its equation. (e.g., $y^2 = 4ax$ is symmetrical about X-axis) on if the equation remains same by replacing y by $-y$.
2. A curve is symmetrical about the Y-axis, if only even powers of x occur in its equation. (e.g., $x^2 = 4ay$ is symmetrical about Y-axis).
3. A curve is symmetrical about the line $y = x$, if on interchanging x and y, its equation remains unchanged. (e.g., $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$).

b. Origin :

1. See if the curve passes through the origin. (A curve passes through the origin if there is no constant term in its equation).
2. If it does, find the equation of the tangents at that place by equating to zero the lowest degree terms.
3. If the origin is a double point, find whether the origin is a node, cusp or conjugate point.

c. Asymptotes :

1. See if the curve has any asymptote parallel to the axes.
2. Then find the inclined asymptotes, if need.

d. Points :

1. Find the points where the curve crosses the axes and the asymptotes.
2. Find the points where the tangent is parallel or perpendicular to the X-axis (*i.e.*, the points where $dy/dx = 0$ or ∞).

3. Find the region (or regions) in which no portion of the curve exists.

B. Procedure for Tracing Polar Curves :

a. Symmetry : See if the curve is symmetrical about any line.

1. A curve is symmetrical about the initial line OX, if only $\cos\theta$ (or $\sec\theta$) occur in its equation (*i.e.*, it remains unchanged when θ is changed to $-\theta$) e.g., $r = a(1 + \cos\theta)$ is symmetrical about the initial line.
2. A curve is symmetrical about the line through the pole perpendicular to the initial line (*i.e.*, OY), if only $\sin\theta$ (or cosec θ) occur in its equation (*i.e.*, it remains unchanged when θ is changed to $\pi - \theta$) e.g., $r = a \sin 3\theta$ is symmetrical about OY.
3. A curve is symmetrical about the pole, if only even powers of r occur in the equation (*i.e.*, it remains unchanged when r is changed to $-r$) e.g., $r^2 = a^2 \cos 2\theta$ is symmetrical about the pole.

b. Limits : See if r and θ are confined between certain limits.

1. Determine the numerically greatest value of r, so as to notice whether the curve lies within a circle or not e.g., $r = a \sin 3\theta$ lies wholly within the circle $r = a$.
2. Determine the region in which no portion of the curve lies by finding those values of θ for which r is imaginary. e.g., $r^2 = a^2 \cos 2\theta$ does not lie between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

c. Asymptotes : If the curve possesses an infinite branch, find the asymptotes.

d. Points :

1. Giving successive values to θ, find the corresponding values of r.
2. Determine the points where the tangent coincides with the radius vector or is perpendicular to it (*i.e.*, the points where $\tan\phi = r\, d\theta/dr = 0$ or ∞).

Que 2.26. **Trace the curve $y^2(a - x) = x^3$.**

Answer

Given curve : $y^2(a - x) = x^3$

1. **Symmetry :** All powers of y are even, the curve is symmetrical about X-axis.
2. **Origin :** Equation does not contain any constant term, therefore, it passes through origin.
3. **Asymptote :** Parallel to Y-axis by equating the coefficient of highest degree of y to zero

$$a - x = 0$$
$$x = a$$

We can discuss the position of the curve w.r.t. asymptote $x = a$.

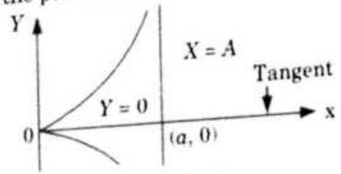

Fig. 2.26.1.

4. **Points of Intersection with Axes :** The curve meet X-axis and Y-axis at the origin only
5. **Region :** Equation of curve

$$y = \sqrt{\frac{x^3}{a - x}}$$

y^2 is negative for $x < 0$ and $x > a$.
Thus curve does not exist for $a < x < 0$.
Required curve is shown in the given Fig. 2.26.1.

Que 2.27. **Trace the curve $y^2(2a - x) = x^3$.**

AKTU 2014-15, 2015-16; Marks 05

Answer

Same as Q. 2.26, Page 2-21C, Unit-2.
Replace a by $2a$

Que 2.28. **Trace the curve : $r^2 = a^2 \cos 2\theta$.**

AKTU 2014-15, Marks 05

Answer

The equation of the curve is $r^2 = a^2 \cos 2\theta$ ...(2.28.1)

1. **Symmetry :** The curve is symmetrical about the initial line, the line $\theta = \dfrac{\pi}{2}$ and pole.
2. **Origin or Pole :** Put $r = 0$ in eq. (2.28.1), we get

$$\cos 2\theta = 0 = \cos\left(\pm\frac{\pi}{2}\right)$$

$$\theta = \pm\frac{\pi}{4}, \text{ which are real.}$$

Hence, the curve passes through the pole and the tangents at the pole are $\theta = \pm\dfrac{\pi}{4}$.

Also when, $\theta = 0, r = \pm a$

Hence, the curve meets the initial line at $(\pm a, 0)$.

3. **Asymptotes :** The curve has no asymptote.
4. **Value of ϕ :** $\phi = \dfrac{\pi}{2} + 2\theta$

When $\theta = 0, \phi = \dfrac{\pi}{2}$

For $\theta = 0, r = \pm a$

Hence, at the points $(a, 0)$ and $(-a, 0)$, the tangents are perpendicular to the initial line.

5. **Special Points and Region :**

From eq. (2.28.1), $r = a\sqrt{\cos 2\theta}$ (Taking +ve root as the curve is symmetrical about the pole)

$$\Rightarrow \quad \frac{dr}{d\theta} = \frac{-a \sin 2\theta}{\sqrt{\cos 2\theta}}$$

For $0 < \theta < \dfrac{\pi}{4}$, $\dfrac{dr}{d\theta}$ is –ve $\Rightarrow r$ decreases in this range

For $\dfrac{3\pi}{4} < \theta < \pi$, $\dfrac{dr}{d\theta}$ is +ve $\Rightarrow r$ increases in this range

When $\theta = 0, r = a$

As θ increases from 0 to $\dfrac{\pi}{4}$, r decreases from a to 0.

For $\dfrac{\pi}{4} < \theta < \dfrac{3\pi}{4}$, r is imaginary. Hence, no portion of the curve lies between the lines $\theta = \dfrac{\pi}{4}$ and $\theta = \dfrac{3\pi}{4}$.

Again, as θ increases from $\dfrac{3\pi}{4}$ to π, r is +ve and increases from 0 to a.

Thus, we can trace the part of the curve about the initial line. The part of the curve below the initial line can be traced by symmetry. Hence, the shape of the curve is as shown in the Fig. 2.28.1.

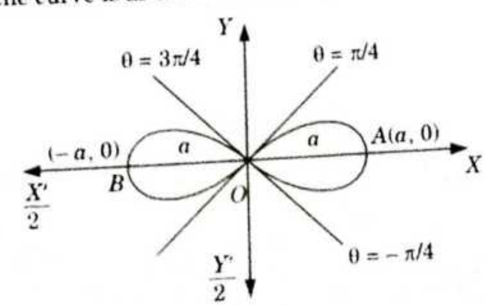

Fig. 2.28.1.

Que 2.29. **Trace the curve : $4ay^2 = x(x - 2a)^2$.**

AKTU 2013-14, Marks 05

Answer

The given curve is, $4ay^2 = x(x - 2a)^2$ ...(2.29.1)

1. **Origin :** The equation of the curve does not contain any constant term. Therefore it passes through origin.
2. **Symmetry :** Equation contains only even powers of y, therefore, it is symmetrical about X-axis.
3. **Point of Intersection with X-axis :** On putting $y = 0$, in eq. (2.29.1), we get

$$x(x - 2a)^2 = 0$$
$$x = 0, 2a$$

On putting $x = 0$, in eq. (2.29.1), we get

$$y = 0$$

So, the point of intersection is $(2a, 0)$.

4. **Region :** $4ay^2 = x(x - 2a)^2$

$$y = \frac{\sqrt{x}}{\sqrt{4a}}(x - 2a)$$

y becomes imaginary if $x < 0$.

Curve does not lie on left of Y-axis.

5. **Asymptotes :** The curve has no asymptote.
6. **Tangent :** Equating the lowest degree term to zero, we get $x = 0$. Therefore $x = 0$ is tangent at origin.

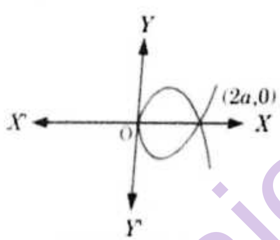

Fig. 2.29.1.

Que 2.30. **Trace the curve $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.**

AKTU 2016-17, Marks 04

Answer

The parametric equations are

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta) \quad ...(2.30.1)$$

Let us trace the curve firstly for values of θ in the interval $[0, 2\pi]$

1. **Symmetry :** On changing θ to $-\theta$ in eq. (2.30.1), changes to $-x$ and y remains unchanged. Therefore curve is symmetrical about Y-axis.
2. **Origin :** Putting $y = 0$ in eq. (2.30.1), we get,

$$a(1 - \cos\theta) = 0$$
$$1 - \cos\theta = 0 \Rightarrow \cos\theta = 1 \text{ or } \theta = 0$$

For $\theta = 0$, $x = a(0 - \sin 0) = 0$. Therefore, the curve passes through the origin.

3. **Asymptotes :** Since for any finite value of θ, x and y does not tends to infinite, therefore the curve has no asymptote.
4. **Points of Intersection :**
   i. **Intersection with X-axis :** Putting $y = 0$ in eq. (2.30.1), we get $\theta = 0$ which give $X = 0$. Therefore, intersection with X-axis is $(0, 0)$.
   ii. **Intersection with Y-axis :** Putting $x = 0$ in eq. (2.30.1), we have $\theta - \sin\theta = 0$, which is satisfied by only $\theta = 0$, which gives $y = 0$.

   Therefore, intersection with Y-axis is $(0, 0)$.
5. **Region :** We know that $-1 \le \cos\theta \le 1$

Multiplying by -1, we get

$$1 \ge -\cos\theta \ge -1 \quad \text{or} \quad -1 \le -\cos\theta \le 1$$
$$1 - 1 \le 1 - \cos\theta \le 1 + 1 \quad \text{or} \quad 0 \le 1 - \cos\theta \le 2$$
$$0 \le a(1 - \cos\theta) \le 2a \quad \text{or} \quad 0 \le y \le 2a$$

Curve lies between the lines $y = 0$ and $y = 2a$.

6. **Special Points :** From eq. (2.30.1),

$$\frac{dx}{d\theta} = a(1 - \cos\theta) \text{ and } \frac{dy}{d\theta} = a\sin\theta$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1 - \cos\theta)} = \cot\frac{\theta}{2} = \frac{1}{\tan\frac{\theta}{2}} \quad ...(2.30.2)$$

Corresponding values of x, y and dy/dx for different values of θ are given below :

θ	0	$\pi/2$	π	$3\pi/2$	2π
x	0	$a\left(\frac{\pi}{2} - 1\right)$	$a\pi$	$a\left(\frac{3\pi}{2} - 1\right)$	$2a\pi$
y	0	a	$2a$	a	0
$\frac{dy}{dx}$	∞	1	0	-1	∞

Tangent at the point $(0, 0)$ and $(2a\pi, 0)$ are parallel to Y-axis.
$(\because\ dy/dx = \infty)$

Tangent at the point $(a\pi, 2a)$ is parallel to X-axis. $(\because \; dy/dx = 0)$

From eq. (2.30.2),

$$\frac{dy}{dx} = \cot\frac{\theta}{2}$$

Differentiate w.r.t. x,

$$\frac{d^2y}{dx^2} = -\operatorname{cosec}^2\frac{\theta}{2}\,\frac{1}{2}\,\frac{d\theta}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{2}\operatorname{cosec}^2\frac{\theta}{2}\times\frac{1}{2a}\operatorname{cosec}^2\frac{\theta}{2} \qquad \left[\because \frac{d\theta}{dx} = \frac{1}{2a}\operatorname{cosec}^2\frac{\theta}{2}\right]$$

or $\dfrac{d^2y}{dx^2} = -\dfrac{1}{4a}\operatorname{cosec}^4\dfrac{\theta}{2}$. Hence, $\dfrac{d^2y}{dx^2}$ is negative for all values of θ

$\therefore$ The curve is concave downwards.

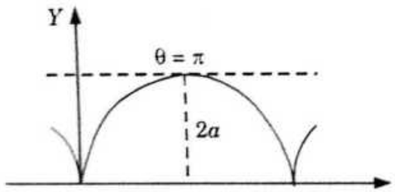

Fig. 2.30.1.

Que 2.31. **Trace the curve : $y^2(a + x) = x^2(3a - x)$.**

AKTU 2017-18, Marks 3.5

Answer

The equation of the curve is

$$y^2(a + x) = x^2(3a - x) \qquad ...(2.31.1)$$

1. **Symmetry :** Since the power of y are even, therefore the curve is symmetrical about X-axis.
2. **Origin :** Equation does not contain any constant term. Therefore, it passes through origin (0, 0). The tangents at the origin are given by
$$y^2 = 3x^2$$
$$y = \pm\sqrt{3}x$$
Which represent two non-coincident straight lines. Hence we may expect a node at the origin.
3. **Asymptotes :** Parallel to Y-axis, by equating the coefficient of highest degree of y to zero.
$$a + x = 0$$
$$x = -a$$
4. **Points of Intersection with Axes :** The curve meets X-axis at (0, 0) and $(3a, 0)$ while it meets Y-axis only at (0, 0).
5. **Region :** Eq. (2.31.1) can be rewritten as

$$y = x\sqrt{\frac{3a - x}{a + x}}$$

If $x = 0$, then $y = 0$

When x is positive and small, y is real

Also $\dfrac{3a - x}{a + x} < \dfrac{3a}{a}$ *i.e.*, < 3

$\therefore$ $y < \sqrt{3}x$

Hence curve lies below tangent $y = \sqrt{3}x$.

At $x = 3a$, $y = 0$

When $x > 3a$, y is imaginary

When $x = a$, $y = a$

When $x = 2a$, $y = \dfrac{2a}{\sqrt{3}} = 1.2\,a$ (nearly)

If we transfer origin at $(3a, 0)$, then by eq. (2.31.1), $x = 0$ will be tangent.

Taking all above points in consideration, the approximate shape of the curve is shown in Fig. 2.31.1.

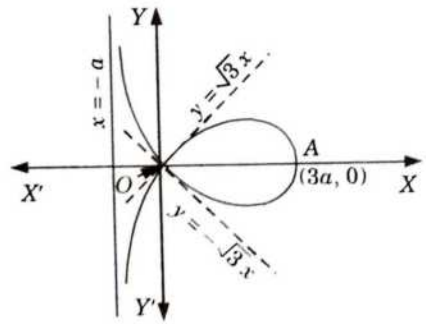

Fig. 2.31.1.

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3

UNIT

Differential Calculus-II

PART-1

Partial Derivatives.

CONCEPT OUTLINE : PART-1

Partial Differentiation : If a derivative of function of several independent variables to be found with respect to any one of them, keeping the others as constants, it is said to be a partial derivative. The operation of finding the partial derivatives of a function of more than one independent variable is called partial differentiation.

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ etc., are the symbols used for partial derivatives. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ can also be denoted as u_x, u_y. Second order partial derivatives are denoted by $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$ or u_{xx}, u_{xy}, u_{yy}.

Note : If $u = f(x, y)$ and its partial derivatives are continuous, then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **If, $z = x^y + y^x$ verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.**

Answer

$$z = x^y + y^x \quad \text{...(3.1.1)}$$

Differentiating w.r.t. x, we have

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

Again differentiating w.r.t. y, we get

$$\frac{\partial^2 z}{\partial y \partial x} = x^{y-1} + yx^{y-1} \log x + xy^{x-1} \log y + y^x \frac{1}{y}$$

$$= x^{y-1}(1 + y \log x) + y^{x-1}(1 + x \log y)$$

Now, differentiating eq. (3.1.1) w.r.t. y, we get

$$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$

Differentiating w.r.t. x, we get

$$\frac{\partial^2 z}{\partial x \partial y} = yx^{y-1}\log x + x^y \frac{1}{x} + y^{x-1} + xy^{x-1}\log y$$

$$= x^{y-1}(1 + y\log x) + y^{x-1}(1 + x\log y)$$

Thus, $$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Que 3.2. **If $z = f(x, y)$ where $x = e^u \cos v$, $y = e^u \sin v$, prove that**

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$$

Answer

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}(e^u \cos v) + \frac{\partial f}{\partial y}(e^u \sin v) \quad ...(3.2.1)$$

Similarly, $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}(-e^u \sin v) + \frac{\partial f}{\partial y}(e^u \cos v) \quad ...(3.2.2)$$

Squaring and adding eq. (3.2.1) and eq. (3.2.2), we get

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = e^{2u}\left(\frac{\partial f}{\partial x}\right)^2 + e^{2u}\left(\frac{\partial f}{\partial y}\right)^2$$

$$e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right] = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Que 3.3. **For what value of n, $u = r^n (3\cos^2\theta - 1)$ satisfies the equation $\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = 0$?**

AKTU 2013-14, Marks 05

Answer

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = 0 \quad ...(3.3.1)$$

$$u = r^n (3\cos^2\theta - 1)$$

Differentiating it w.r.t. r, we get

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}[r^n(3\cos^2\theta - 1)] = nr^{n-1}(3\cos^2\theta - 1)$$

$$r^2 \frac{\partial u}{\partial r} = nr^{n+1}(3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}[nr^{n+1}(3\cos^2\theta - 1)]$$

$$= n(n+1)r^n(3\cos^2\theta - 1) \quad ...(3.3.2)$$

Now, $$u = r^n(3\cos^2\theta - 1) = r^n\left(\frac{3}{2}(1 + \cos 2\theta) - 1\right)$$

Differentiating w.r.t. θ, we get

$$\frac{\partial u}{\partial\theta} = r^n\left(\frac{3}{2}(-2\sin 2\theta)\right) = -3r^n \sin 2\theta$$

$$\sin\theta \frac{\partial u}{\partial\theta} = -6r^n \sin^2\theta\cos\theta \quad (\because \sin 2\theta = 2\sin\theta\cos\theta)$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6r^n(2\sin\theta\cos^2\theta + \sin^2\theta(-\sin\theta))$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6r^n(2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right)\right) = -6r^n(2\cos^2\theta - \sin^2\theta)$$

$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right)\right) = -6r^n(3\cos^2\theta - 1) \quad ...(3.3.3)$$

Now put all these values in eq. (3.3.1), we get

$$n(n+1)r^n(3\cos^2\theta - 1) - 6r^n(3\cos^2\theta - 1) = 0$$

$$r^n(3\cos^2\theta - 1)[n^2 + n - 6] = 0$$

Since, $r^n(3\cos^2\theta - 1) \neq 0$

$$\therefore \quad n^2 + n - 6 = 0$$

$$(n-2)(n+3) = 0$$

$$n = 2, -3$$

Que 3.4. **If $u = f(r)$ where $r^2 = x^2 + y^2$, show that**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r)$$

AKTU 2015-16, Marks 05

Answer

$$r^2 = x^2 + y^2 \quad ...(3.4.1)$$

Differentiating partially w.r.t. x, we get

$$2r\frac{\partial r}{\partial x} = 2x \quad \text{or} \quad \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $$\frac{\partial r}{\partial y} = \frac{y}{r}$$

Now, $$u = f(r)$$

$$\therefore \quad \frac{\partial u}{\partial x} = f'(r)\frac{\partial r}{\partial x} = \frac{x}{r}f'(r)$$

Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r} f'(r) + x\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right) f'(r) + \frac{x}{r} f''(r)\frac{\partial r}{\partial x}$$

$$\left[\because \frac{\partial}{\partial x}(uvw) = vw\frac{\partial}{\partial x}(u) + uw\frac{\partial}{\partial x}(v) + uv\frac{\partial}{\partial x}(w)\right]$$

$$= \frac{1}{r} f'(r) - \frac{x}{r^2}\frac{x}{r} f'(r) + \frac{x}{r} f''(r)\frac{x}{r} = \frac{1}{r} f'(r) - \frac{x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r)$$

$$= \frac{r^2 - x^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r) = \frac{y^2}{r^3} f'(r) + \frac{x^2}{r^2} f''(r)$$

Similarly, $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{x^2}{r^3} f'(r) + \dfrac{y^2}{r^2} f''(r)$ {Using eq. (3.4.1)}

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2}{r^3} f'(r) + \frac{x^2 + y^2}{r^2} f''(r)$$

$$= \frac{r^2}{r^3} f'(r) + \frac{r^2}{r^2} f''(r) = f''(r) + \frac{1}{r} f'(r)$$

Que 3.5. **If $e^{-z/(x^2-y^2)} = x - y$ then show that $y\dfrac{\partial z}{\partial x} + x\dfrac{\partial z}{\partial y} = x^2 - y^2$.**

AKTU 2016-17, Marks 3.5

Answer

Given : $e^{\frac{-z}{x^2-y^2}} = x - y$

Taking log on both sides, we get

$$\frac{-z}{x^2 - y^2} = \log (x - y)$$

$$z = -(x^2 - y^2) \log (x - y)$$

Now, $\dfrac{\partial z}{\partial x} = -\left[(x^2 - y^2)\dfrac{1}{x - y} + \log (x - y)\, 2x\right]$

$$y\frac{\partial z}{\partial x} = -y\,[(x + y) + \log (x - y)\, 2x] \quad ...(3.5.1)$$

and, $\dfrac{\partial z}{\partial y} = -\left[(x^2 - y^2)\dfrac{(-1)}{(x - y)} + \log (x - y)\, 2y\,(-1)\right]$

$$\frac{\partial z}{\partial y} = (x + y) + \log (x - y)\, 2y$$

$$x\frac{\partial z}{\partial y} = x\,(x + y) + 2xy \log (x - y) \quad ...(3.5.2)$$

Now adding eq. (3.5.1) and eq. (3.5.2), we get

$$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = -y(x + y) - 2xy \log (x - y) + x(x + y) + 2xy \log (x - y)$$

$$= x(x + y) - y(x + y) = x^2 - y^2$$

Que 3.6. **If $w = \sqrt{x^2 + y^2 + z^2}$ and $x = u \cos v, y = u \sin v, z = uv$, then prove that $\left[u\dfrac{\partial w}{\partial u} - v\dfrac{\partial w}{\partial v}\right] = \dfrac{u}{\sqrt{1 + v^2}}$.**

AKTU 2016-17, Marks 3.5

Answer

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$= \frac{x}{\sqrt{x^2 + y^2 + z^2}}(\cos v) + \frac{y}{\sqrt{x^2 + y^2 + z^2}}(\sin v) + \frac{zv}{\sqrt{x^2 + y^2 + z^2}}$$

$$u\frac{\partial w}{\partial u} = \frac{xu \cos v + yu \sin v + zuv}{\sqrt{x^2 + y^2 + z^2}}$$

Putting the values of x, y and z

$$u\frac{\partial w}{\partial u} = \frac{u^2 \cos^2 v + u^2 \sin^2 v + u^2 v^2}{u\sqrt{1 + v^2}} = \frac{u^2(1 + v^2)}{u\sqrt{1 + v^2}} = \frac{u(1 + v^2)}{\sqrt{1 + v^2}}$$

Similarly,

$$\frac{\partial w}{\partial v} = \frac{x(-u \sin v) + y(u \cos v) + zu}{\sqrt{x^2 + y^2 + z^2}}$$

$$v\frac{\partial w}{\partial v} = \frac{-xuv \sin v + yuv \cos v + zuv}{\sqrt{x^2 + y^2 + z^2}}$$

$$= \frac{-u^2 v \cos v \sin v + u^2 v \sin v \cos v + u^2 v^2}{u\sqrt{1 + v^2}}$$

$$= \frac{u^2 v^2}{u\sqrt{1 + v^2}} = \frac{uv^2}{\sqrt{1 + v^2}}$$

Now, $u\dfrac{\partial w}{\partial u} - v\dfrac{\partial w}{\partial v} = \dfrac{u(1 + v^2) - uv^2}{\sqrt{1 + v^2}} = \dfrac{u}{\sqrt{1 + v^2}}$

Que 3.7. **If $u = f(r, s, t)$, where $r = \dfrac{x}{y}, s = \dfrac{y}{z}, t = \dfrac{z}{x}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + z\dfrac{\partial u}{\partial z} = 0.$**

AKTU 2017-18, Marks 3.5

Answer

$$u = f(r, s, t), \quad r = \frac{x}{y}, \quad s = \frac{y}{z}, \quad t = \frac{z}{x}$$

So, $\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} + \dfrac{\partial u}{\partial t}\dfrac{\partial t}{\partial x}$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{1}{y}\right) + \frac{\partial u}{\partial s}(0) + \frac{\partial u}{\partial t}\left(\frac{-z}{x^2}\right) = \frac{1}{y}\frac{\partial u}{\partial r} - \frac{z}{x^2}\frac{\partial u}{\partial t}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{y}\frac{\partial u}{\partial r} - \frac{z}{x}\frac{\partial u}{\partial t} \quad ...(3.7.1)$$

and,
$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{-x}{y^2}\right) + \frac{\partial u}{\partial s}\left(\frac{1}{z}\right) + \frac{\partial u}{\partial t}(0) = \frac{-x}{y^2}\frac{\partial u}{\partial r} + \frac{1}{z}\frac{\partial u}{\partial s}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{y}\frac{\partial u}{\partial r} + \frac{y}{z}\frac{\partial u}{\partial s} \quad ...(3.7.2)$$

and,
$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial r}(0) + \frac{\partial u}{\partial s}\left(\frac{-y}{z^2}\right) + \frac{\partial u}{\partial t}\left(\frac{1}{x}\right) = \frac{-y}{z^2}\frac{\partial u}{\partial s} + \frac{1}{x}\frac{\partial u}{\partial t}$$

$$z\frac{\partial u}{\partial z} = \frac{-y}{z}\frac{\partial u}{\partial s} + \frac{z}{x}\frac{\partial u}{\partial t} \quad ...(3.7.3)$$

On adding eq. (3.7.1), eq. (3.7.2) and eq. (3.7.3), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$$

Que 3.8. **If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$; $xy \neq 0$ prove that $\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.**

AKTU 2017-18, Marks 3.5

Answer

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$$

Differentiating w.r.t x, we get

$$\frac{\partial u}{\partial x} = \left[x^2 \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{-y}{x^2}\right) + 2x\tan^{-1}\left(\frac{y}{x}\right) - y^2\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right)\right]$$

$$= \left[\frac{-yx^2}{x^2+y^2} + 2x\tan^{-1}\left(\frac{y}{x}\right) - \frac{y^3}{x^2+y^2}\right]$$

$$= 2x\tan^{-1}\left(\frac{y}{x}\right) - y\frac{(x^2+y^2)}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = 2x\tan^{-1}\left(\frac{y}{x}\right) - y$$

Now, differentiating w.r.t y, we get

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left[2x\tan^{-1}\left(\frac{y}{x}\right) - y\right]$$

$$= 2x\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - 1$$

$$= \frac{2x^2}{x^2+y^2} - 1$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}$$

Que 3.9. **If $V = f(2x - 3y, 3y - 4z, 4z - 2x)$ prove that $6V_x + 4V_y + 3V_z = 0$.**

AKTU 2014-15, Marks 05

Answer

Let, $V = f(P, Q, R)$

Where, $P = 2x - 3y$

$Q = 3y - 4z$

$R = 4z - 2x$

So,
$$V_x = \frac{\partial V}{\partial P}\frac{\partial P}{\partial x} + \frac{\partial V}{\partial Q}\frac{\partial Q}{\partial x} + \frac{\partial V}{\partial R}\frac{\partial R}{\partial x}$$

$$V_x = 2\frac{\partial V}{\partial P} + 0 - 2\frac{\partial V}{\partial R}$$

$$6V_x = 12\frac{\partial V}{\partial P} - 12\frac{\partial V}{\partial R} \quad ...(3.9.1)$$

Similarly,
$$V_y = \frac{\partial V}{\partial P}(-3) + \frac{\partial V}{\partial Q}(3) + \frac{\partial V}{\partial R}(0)$$

$$4V_y = -12\frac{\partial V}{\partial P} + 12\frac{\partial V}{\partial Q} \quad ...(3.9.2)$$

and,
$$V_z = \frac{\partial V}{\partial P}(0) + \frac{\partial V}{\partial Q}(-4) + \frac{\partial V}{\partial R}(4)$$

$$3V_z = -12\frac{\partial V}{\partial Q} + 12\frac{\partial V}{\partial R} \quad ...(3.9.3)$$

On adding eq. (3.9.1), eq. (3.9.2) and eq. (3.9.3), we get

$$6V_x + 4V_y + 3V_z = 0$$

PART-2

Total Derivatives, Euler's Theorem for Homogeneous Functions.

CONCEPT OUTLINE : PART-2

Homogeneous Function : An expression in which every term is of the same degree is called a homogeneous function. An expression of the type $x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$ is a homogeneous function of $\left(\frac{y}{x}\right)$ or $\left(\frac{x}{y}\right)$.

Euler's Theorem : If u is a homogeneous function of x and y of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Prop. 1 : If u is a homogeneous function of degree n in x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Prop. 2 : If $F(u) = V(x, y, z)$, where V is a homogeneous function in x, y, z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)}$$

Total Derivatives :

If $u = f(x, y)$, where $x = f_1(t)$, $y = f_2(t)$, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

du/dt is called the total differential coefficient of u w.r.t. 't'.

Prop. 1 : If u is a function of x and y and y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx}$$

Prop. 2 : If $u = f(x, y)$, $x = f_1(t_1, t_2)$, $y = f_2(t_1, t_2)$, then

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t_1}$$

and, $$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t_2}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.10. **Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$.**

AKTU 2014-15, Marks 05

Answer

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \quad ...(3.10.1)$$

$$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial u}{\partial y} = 2y, \ \frac{\partial u}{\partial z} = 2z$$

$$\frac{dx}{dt} = 2e^{2t}, \ \frac{dy}{dt} = 2e^{2t}\cos 3t - 3e^{2t}\sin 3t$$

$$\frac{dz}{dt} = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t$$

Putting all values in eq. (3.10.1), we get

$$\frac{du}{dt} = 2x(2e^{2t}) + 2y[2e^{2t}\cos 3t - 3e^{2t}\sin 3t] + 2z[2e^{2t}\sin 3t + 3e^{2t}\cos 3t]$$

$$= 4xe^{2t} + 4ye^{2t}\cos 3t - 6ye^{2t}\sin 3t + 4ze^{2t}\sin 3t + 6ze^{2t}\cos 3t$$

$$= 4e^{2t}[x + y\cos 3t + z\sin 3t] - 6e^{2t}[y\sin 3t - z\cos 3t]$$

$$= 4e^{2t}x + 4e^{2t}[e^{2t}\cos^2 3t + e^{2t}\sin^2 3t] - 6e^{2t}[e^{2t}\cos 3t\sin 3t - e^{2t}\cos 3t\sin 3t]$$

$$[\because y = e^{2t}\cos 3t \text{ and } z = e^{2t}\sin 3t]$$

$$= 4e^{2t}x + 4e^{2t}[e^{2t}] = 4e^{2t}e^{2t} + 4e^{2t+2t}$$

$$\frac{du}{dt} = 8e^{4t}$$

Now from direct substitution

$$u = (e^{2t})^2 + (e^{2t}\cos 3t)^2 + (e^{2t}\sin 3t)^2$$

$$u = e^{4t} + e^{4t}[\cos^2 3t + \sin^2 3t]$$

$$u = e^{4t} + e^{4t}$$

$$u = 2e^{4t}$$

Now differentiating, w.r.t. 't'

$$\frac{du}{dt} = 8e^{4t}$$

So, the result is verified.

Que 3.11. State and prove Euler's theorem for homogeneous functions.

Answer

A. Euler's Theorem : If u is a homogeneous function of x and y of degree n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

B. Proof : Let, $u=x^n f\left(\frac{y}{x}\right)$

$$\frac{\partial u}{\partial x}=nx^{n-1}f\left(\frac{y}{x}\right)-\left[x^n f'\left(\frac{y}{x}\right)\right]\frac{y}{x^2}$$

$$x\frac{\partial u}{\partial x}=nx^n f\left(\frac{y}{x}\right)-x^{n-1}y f'\left(\frac{y}{x}\right) \quad ...(3.11.1)$$

Similarly, $$\frac{\partial u}{\partial y}=x^n\frac{1}{x}f'\left(\frac{y}{x}\right)$$

$$y\frac{\partial u}{\partial y}=yx^{n-1}f'\left(\frac{y}{x}\right) \quad ...(3.11.2)$$

Adding eq. (3.11.1) and eq. (3.11.2), we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Que 3.12. Prove that $xu_x+yu_y=\frac{5}{2}\tan u$ if $u=\sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$.

AKTU 2014-15, Marks 05

Answer

$$u=\sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$$

It is not a homogenous function, so first we convert it into a homogenous function.

$$\sin u=x^{5/2}\left[\frac{1+\left(\frac{y}{x}\right)^3}{1+\left(\frac{y}{x}\right)^{1/2}}\right]$$

Now, $\sin u$ is a homogeneous function of degree 5/2.
By using Euler's theorem, we get

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=n(\sin u)$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=\frac{5}{2}\sin u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{2}\tan u$$

Que 3.13. Verify Euler's theorem for $z=\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}$.

AKTU 2013-14, 2015-16; Marks 05

Answer

$$z=\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}=f(x,y)$$

$$z=\frac{x^{1/3}\left[1+\left(\frac{y}{x}\right)^{1/3}\right]}{x^{1/2}\left[1+\left(\frac{y}{x}\right)^{1/2}\right]}$$

$$z=x^{-1/6}f(y/x)$$

So, z is a homogeneous function of degree (–1/6). Thus using Euler's theorem

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{1}{6}z$$

Now, $$\frac{\partial z}{\partial x}=\frac{(x^{1/2}+y^{1/2})\frac{1}{3}x^{-2/3}-(x^{1/3}+y^{1/3})\frac{1}{2}x^{-1/2}}{(x^{1/2}+y^{1/2})^2}$$

$$x\frac{\partial z}{\partial x}=\frac{\frac{1}{3}x^{1/3}(x^{1/2}+y^{1/2})-\frac{1}{2}x^{1/2}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2}$$

Similarly, $$y\frac{\partial z}{\partial y}=\frac{\frac{1}{3}y^{1/3}(x^{1/2}+y^{1/2})-\frac{1}{2}y^{1/2}(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})^2}$$

Now,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=\frac{1}{(x^{1/2}+y^{1/2})^2}\left[\frac{1}{3}(x^{1/3}+y^{1/3})(x^{1/2}+y^{1/2})-\frac{1}{2}(x^{1/2}+y^{1/2})(x^{1/3}+y^{1/3})\right]$$

$$=-\frac{1}{6}\frac{(x^{1/3}+y^{1/3})}{(x^{1/2}+y^{1/2})}=-\frac{1}{6}z$$

Thus, $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y} = -\dfrac{1}{6}z$

Que 3.14. If $u = \sin^{-1}\left(\dfrac{x^{1/4}+y^{1/4}}{x^{1/6}+y^{1/6}}\right)$ then evaluate the value of $\left(x^2\dfrac{\partial^2 u}{\partial x^2}+2xy\dfrac{\partial^2 u}{\partial x\partial y}+y^2\dfrac{\partial^2 u}{\partial y^2}\right)$

AKTU 2016-17, Marks 03

Answer

$$u = \sin^{-1}\left(\frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{6}}+y^{\frac{1}{6}}}\right)$$

It is not a homogeneous function so we convert it into a homogeneous function.

$$\sin u = \frac{x^{\frac{1}{4}}+y^{\frac{1}{4}}}{x^{\frac{1}{6}}+y^{\frac{1}{6}}} = \frac{x^{\frac{1}{4}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}{x^{\frac{1}{6}}\left[1+\left(\frac{y}{x}\right)^{\frac{1}{6}}\right]} \quad \text{or} \quad \sin u = x^{\frac{1}{12}}\frac{\left[1+\left(\frac{y}{x}\right)^{\frac{1}{4}}\right]}{\left[1+\left(\frac{y}{x}\right)^{\frac{1}{6}}\right]}$$

Now, it a homogeneous function of degree $\dfrac{1}{12}$.

By Euler's formula, $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y} = n\dfrac{f(u)}{f'(u)}$

Where, n = Degree, and $f(u) = \sin u$

$$= \frac{1}{12}\frac{\sin u}{\cos u} = \frac{1}{12}\tan u$$

Also, by Euler's formula,

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u)-1] \quad \left[\text{Where, } g(u) = \frac{nf(u)}{f'(u)}\right]$$

$$= \frac{1}{12}\tan u\left[\frac{\sec^2 u}{12}-1\right] = \frac{1}{12}\tan u\left[\frac{\sec^2 u-12}{12}\right]$$

$$= \frac{1}{12\times 12}\tan u\,[\tan^2 u+1-12]$$

$$= \frac{1}{144}\tan u\,[\tan^2 u-11]$$

Que 3.15. If $u = \sin^{-1}\left(\dfrac{x^3+y^3+z^3}{ax+by+cz}\right)$, prove that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+z\dfrac{\partial u}{\partial z} = 2\tan u$.

AKTU 2017-18, Marks 3.5

Answer

$$u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$$

$$\sin u = \frac{x^3+y^3+z^3}{ax+by+cz} = v \text{ (say)}$$

So, v is a homogeneous function of degree 2.

Using Euler's theorem,

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z} = nv$$

$$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)+z\frac{\partial}{\partial z}(\sin u) = 2\sin u$$

$$x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}+z\cos u\frac{\partial u}{\partial z} = 2\sin u$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z} = 2\tan u$$

PART-3

Taylor and Maclaurin's Theorems for a Function of One and Two Variables.

CONCEPT OUTLINE : PART-3

Expansion of Function of One Variable :

Taylor's Series : Let $f(x)$ possesses continuous derivatives of all orders in the interval $[a, a+h]$, then for every positive integer value of n,

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\ldots\ldots+\frac{h^n f^n(a)}{n!} \quad \ldots(1)$$

The series (1) is the Taylor's series for the expansion of $f(a+h)$ in powers of h.

Note :

i. Put $a+h = b$ or $h = b-a$, in eq. (1)

$$f(b) = f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\ldots\ldots+\frac{(b-a)^n}{n!}f^n(a)$$

ii. Put $h = x-a$ in eq. (1), we get

$$f(x) = f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f''(a)+\ldots\ldots+\frac{(x-a)^n}{n!}f^n(a)$$

iii. Put $a = 0$ and $h = x$ in eq. (1), we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0)$$

This is the Maclaurin series.

Maclaurin Series : Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[0, x]$. Then for every positive integral value of n,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^n}{n!}f^n(0) \quad \ldots(2)$$

The series (2) is known as Maclaurin infinite series for the expansion of $f(x)$ in powers of x.

Expansion of a Function of Two Variables : Taylor's theorem for a function of two variables,

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots\ldots$$

Corollary 1 : Put $x = a$, $y = b$

$$f(a+h, b+k) = f(a,b) + [h f_x(a,b) + k f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \ldots\ldots$$

Corollary 2 : In above expression put $a + h = x$, $b + k = y$

$$f(x, y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \ldots\ldots$$

Corollary 3 : Put $a = 0$, $b = 0$ in above expression

$$f(x, y) = f(0, 0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \ldots$$

This is Maclaurin series for two variables.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.16. **Find the Taylor's series expansion of $f(x, y) = x^3 + xy^2$ about point (2, 1).**

Answer

$f(x, y) = x^3 + xy^2$, $\quad f(2, 1) = 10$

$f_x(x, y) = 3x^2 + y^2$, $\quad f_x(2, 1) = 13$

$f_{xx}(x, y) = 6x$, $\quad f_{xx}(2, 1) = 12$

$f_y(x, y) = 2xy$, $\quad f_y(2, 1) = 4$

$f_{xy}(x, y) = 2x$, $\quad f_{xy}(2, 1) = 4$

$f_{yy}(x, y) = 2y$, $\quad f_{yy}(2, 1) = 2$

Using Taylor's theorem,

$$f(x, y) = f(2, 1) + [(x-2)f_x(2, 1) + (y-1)f_y(2, 1)] + \frac{1}{2!}[(x-2)^2 f_{xx}(2, 1) + 2(x-2)(y-1)f_{xy}(2, 1) + (y-1)^2 f_{yy}(2, 1)] + \ldots$$

$$= 10 + 13(x-2) + 4(y-1) + \frac{1}{2!}[12(x-2)^2 + 4(x-2)(y-1) + 4(y-1)^2] + \ldots$$

Que 3.17. **Expand $\log x$ in powers of $(x - 1)$ by Taylor's theorem and hence find $\log(1.1)$.**

Answer

$f(x) = \log x = \log(1 + x - 1)$

$= \log(a + h)$, $a = 1$, $h = x - 1$

$f(x) = \log x$, $\quad f(1) = 0$

$f'(x) = \frac{1}{x}$, $\quad f'(1) = 1$

$f''(x) = -\frac{1}{x^2}$, $\quad f''(1) = -1$

$f'''(x) = \frac{2}{x^3}$, $\quad f'''(1) = 2$

$f''''(x) = \frac{-6}{x^4}$, $\quad f''''(1) = -6$

According to Taylor's series,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \ldots\ldots$$

Putting values in above equation, we get

$$f(x) = \log x = 0 + (x-1)1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \ldots\ldots$$

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots\ldots$$

Put $x = 1.1$, we get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots\dots + \frac{x^n}{n!}f^n(0)$$

This is the Maclaurin series.

Maclaurin Series : Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[0, x]$. Then for every positive integral value of n,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \dots\dots + \frac{x^n}{n!}f^n(0) \quad \dots(2)$$

The series (2) is known as Maclaurin infinite series for the expansion of $f(x)$ in powers of x.

Expansion of a Function of Two Variables : Taylor's theorem for a function of two variables,

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$$
$$+ \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \dots\dots$$

Corollary 1 : Put $x = a$, $y = b$

$$f(a+h, b+k) = f(a,b) + [h\, f_x(a,b) + k\, f_y(a,b)] +$$
$$\frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \dots\dots$$

Corollary 2 : In above expression put $a + h = x$, $b + k = y$

$$f(x, y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] +$$
$$\frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \dots$$

Corollary 3 : Put $a = 0$, $b = 0$ in above expression

$$f(x, y) = f(0, 0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0)$$
$$+ y^2 f_{yy}(0, 0)] + \dots$$

This is Maclaurin series for two variables.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.16. **Find the Taylor's series expansion of $f(x, y) = x^3 + xy^2$ about point (2, 1).**

Answer

$f(x, y) = x^3 + xy^2$, $\quad f(2, 1) = 10$

$f_x(x, y) = 3x^2 + y^2$, $\quad f_x(2, 1) = 13$

$f_{xx}(x, y) = 6x$, $\quad f_{xx}(2, 1) = 12$

$f_y(x, y) = 2xy$, $\quad f_y(2, 1) = 4$

$f_{yy}(x, y) = 2x$, $\quad f_{yy}(2, 1) = 4$

$f_{xy}(x, y) = 2y$, $\quad f_{xy}(2, 1) = 2$

Using Taylor's theorem,

$$f(x, y) = f(2, 1) + [(x-2)f_x(2, 1) + (y-1)f_y(2, 1)]$$
$$+ \frac{1}{2!}[(x-2)^2 f_{xx}(2, 1) + 2(x-2)(y-1)f_{xy}(2, 1) + (y-1)^2 f_{yy}(2, 1)] + \dots$$
$$= 10 + 13(x-2) + 4(y-1) + \frac{1}{2!}[12(x-2)^2 + 4(x-2)(y-1) + 4(y-1)^2] + \dots$$

Que 3.17. **Expand $\log x$ in powers of $(x - 1)$ by Taylor's theorem and hence find log (1.1).**

Answer

$$f(x) = \log x = \log(1 + x - 1)$$
$$= \log(a + h),\ a = 1,\ h = x - 1$$

$f(x) = \log x$, $\quad f(1) = 0$

$f'(x) = \dfrac{1}{x}$, $\quad f'(1) = 1$

$f''(x) = -\dfrac{1}{x^2}$, $\quad f''(1) = -1$

$f'''(x) = \dfrac{2}{x^3}$, $\quad f'''(1) = 2$

$f''''(x) = \dfrac{-6}{x^4}$, $\quad f''''(1) = -6$

According to Taylor's series,

$$f(x) = f(1) + (x-1)f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \dots\dots$$

Putting values in above equation, we get

$$f(x) = \log x = 0 + (x-1)1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \dots\dots$$
$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \dots\dots$$

Put $x = 1.1$, we get

$$\log(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4 + \ldots$$

$$\log(1.1) = 0.095305 \text{ (Approximate)}$$

Que 3.18. **Expand $f(x, y) = e^x \tan^{-1} y$ in powers of $(x-1)$ and $(y-1)$ upto two terms of degree 2.**

Answer

$f(x, y) = e^x \tan^{-1} y$, $\qquad f(1, 1) = e^1 \tan^{-1} 1 = \pi e/4$

$f_x(x, y) = e^x \tan^{-1} y$, $\qquad f_x(1, 1) = \pi e/4$

$f_{xx}(x, y) = e^x \tan^{-1} y$, $\qquad f_{xx}(1, 1) = e^1 \tan^{-1} 1 = \pi e/4$

$f_y(x, y) = e^x/(1+y^2)$, $\qquad f_y(1, 1) = e/2$

$f_{yy}(x, y) = -2ye^x/(1+y^2)^2$, $\qquad f_{yy}(1, 1) = -2e/4 = -e/2$

$f_{xy}(x, y) = \dfrac{e^x}{1+y^2}$, $\qquad f_{xy}(1, 1) = e/2$

Using Taylor's expansion,

$$f(x, y) = f(1, 1) + [(x-1) f_x(1, 1) + (y-1) f_y(1, 1)]$$
$$+ \frac{1}{2!}[(x-1)^2 f_{xx}(1, 1) + 2(x-1)(y-1) f_{xy}(1, 1)$$
$$+ (y-1)^2 f_{yy}(1, 1)] + \ldots$$

$$= \frac{\pi e}{4} + (x-1)\frac{\pi e}{4} + (y-1)\frac{e}{2} + \frac{1}{2!}\left[(x-1)^2 \frac{\pi e}{4} + 2(x-1)(y-1)\frac{e}{2}\right.$$
$$\left. + (y-1)^2\left(-\frac{e}{2}\right)\right] + \ldots$$

$$f(x, y) = \frac{\pi e}{4} + (x-1)\frac{\pi e}{4} + (y-1)\frac{e}{2}$$
$$+ \frac{1}{2!}\left[(x-1)^2 \frac{\pi e}{4} + e(x-1)(y-1) - \frac{e}{2}(y-1)^2\right] + \ldots$$

Que 3.19. **Obtain Taylor's expansion of $\tan^{-1}\left(\dfrac{y}{x}\right)$, about (1, 1) up to and including the second degree terms.**

AKTU 2013-14, Marks 05

Answer

Let $\qquad f(x, y) = \tan^{-1}\left(\dfrac{y}{x}\right)$, $\quad f(1, 1) = \dfrac{\pi}{4}$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}, \; f_x(1, 1) = -\frac{1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}, \; f_y(1, 1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}, \quad f_{xx}(1, 1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}, \quad f_{yy}(1, 1) = -\frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, \; f_{xy}(1, 1) = 0$$

By Taylor's theorem,

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right] +$$
$$\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2} + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 0 + (y-1)^2\left(-\frac{1}{2}\right)\right]$$
$$= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

Que 3.20. **Expand $e^x \log(1+y)$ is powers of x and y upto terms of third degree.**

AKTU 2014-15, Marks 10

Answer

$f(x, y) = e^x \log(1+y)$

$f_x = e^x \log(1+y)$, $\qquad f_x(0, 0) = e^0 \log 1 = 0$

$f_{xx} = e^x \log(1+y)$, $\qquad f_{xx}(0, 0) = 0$

$f_{xxx} = e^x \log(1+y)$, $\qquad f_{xxx}(0, 0) = 0$

$f_{yxx} = \dfrac{e^x}{1+y}$, $\qquad f_{yxx}(0, 0) = \dfrac{e^0}{1+0} = 1$

$f_y = \dfrac{e^x}{1+y}$, $\qquad f_y(0, 0) = 1$

$f_{yy} = \dfrac{-e^x}{(1+y)^2}$, $\qquad f_{yy}(0,0) = -1$

$f_{xyy} = \dfrac{-e^x}{(1+y)^2}$, $\qquad f_{xyy}(0, 0) = -1$

$f_{yyy} = \dfrac{2e^x}{(1+y)^3}$, $\qquad f_{yyy}(0, 0) = 2$

$$f_{xy} = \frac{e^x}{1+y}, \qquad f_{xy}(0, 0) = 1$$

According to Maclaurin's series,

$$f_{xy} = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \ldots$$

Put all these values in Maclaurin's series,

$$e^x \log(1+y) = y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2 y - \frac{1}{2}xy^2 + \frac{1}{3}y^3$$

Que 3.21. **Express the function $f(x, y) = x^2 + 3y^2 - 9x - 9y + 26$ as Taylor's series expansion about the point (1, 2).**

AKTU 2016-17, 2017-18 ; Marks 3.5

Answer

$$f(x, y) = x^2 + 3y^2 - 9x - 9y + 26$$

Here, $a + h = x$ or $1 + h = x$ or $h = x - 1$

$b + k = y$ or $2 + k = y$ or $k = y - 2$

		$x = 1, y = 2$
$f(x, y)$	$x^2 + 3y^2 - 9x - 9y + 26$	12
$f_x(x, y)$	$2x - 9$	–7
$f_y(x, y)$	$6y - 9$	3
$f_{xx}(x, y)$	2	2
$f_{xy}(x, y)$	0	0
$f_{yy}(x, y)$	6	6

Now, Taylor's series is

$$f(a + h, b + k) = f(a, b) + \left[h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right]_{(a, b)} + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right]_{(a, b)}$$

$$= 12 + [(x - 1)(-7) + (y - 2) \times 3] + \frac{1}{2}[(x - 1)^2 \times 2 + 2(x - 1)(y - 2) \times 0 + (y - 2)^2 \times 6]$$

$$x^2 + 3y^2 - 9x - 9y + 26 = 12 - 7(x - 1) + 3(y - 2) + (x - 1)^2 + 3(y - 2)^2$$

PART-4

Maxima and Minima of Function of Several Variables, Lagrange's Method of Multipliers.

CONCEPT OUTLINE : PART-4

Maxima and Minima of Functions of Single Independent Variable :

A function $f(x)$ is said to be maximum at $x = a$, if there exists a positive number δ such that,

$f(a + h) < f(a)$ for all values of h, other than zero in the interval $(-\delta, \delta)$.

A function $f(x)$ is said to be minimum at $x = a$, if there exists a positive number δ such that,

$f(a + h) > f(a)$ for all values of h, other than zero in the interval $(-\delta, \delta)$.

Working Rule for Maxima and Minima of $f(x)$:

i. Find $f'(x)$ and equate it to zero.

ii. Solve the resulting equation for x. Let its roots be a_1, a_2.......then $f(x)$ is stationary at $x = a_1, a_2$,..... Thus $x = a_1, a_2$, ... are the only points at which $f(x)$ can be maximum or minimum.

iii. Find $f''(x)$ and substitute in it by turns $x = a_1, a_2$, ...

iv. If $f''(a_1)$ is negative we have a maximum at $x = a_1$. If $f''(a_1)$ is positive, we have a minimum at $x = a_1$.

v. If $f''(a_1) = 0$, find $f'''(x)$ and put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f''''(x)$ and put $x = a_1$ in it. If f'''' is negative, we have a maximum at $x = a_1$, if it is positive, there is minimum at $x = a_1$. If $f''''(a_1)$ is zero, we must find $f'''''(x)$, and so on.

Repeat the above process for each root of the equation $f'(x) = 0$.

Maxima and Minima of Functions of Two Independent Variables :

Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively. Then $f(a, b)$ is said to be a maximum or a minimum values of $f(x, y)$ according to $f(a + h, b + k)$ is less or greater than $f(a, b)$ for all sufficiently small independent values of h and k, positive or negative, provided both of them are not equal to zero.

Working Rule for Maxima and Minima of $f(x, y)$: Suppose $f(x, y)$ is a given function of x and y. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and solve the simultaneous equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. In order to solve these equations we may either eliminate one of the variables, to factorise the equations. In the latter case each factor of the first equation must be solved in conjunction with each factor of the second equation. Suppose on solving these equations we get the pairs of values of x and y as (a_1, b_1), (a_2, b_2) etc., then all these pairs of roots will give stationary values of $f(x, y)$.

To discuss the maximum or minimum at $x = a_1$, $y = b_1$ we should find

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a_1\\y=b_1}}, \quad s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a_1\\y=b_1}}, \quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a_1\\y=b_1}}$$

Then calculate $rt - s^2$.

i. If $rt - s^2 > 0$ and r is negative, $f(x, y)$ is maximum at $x = a_1$, $y = b_1$.

ii. If $rt - s^2 > 0$ and r is positive, $f(x, y)$ is minimum at $x = a_1$, $y = b_1$.

iii. If $rt - s^2 < 0$, $f(x, y)$ is neither maximum nor minimum at $x = a_1$, $y = b_1$.

iv. If $rt - s^2 = 0$, the case is doubtful and further investigation will be required to decide it.

Lagrange's Method of Undetermined Multipliers :

Let $f(x, y, z)$ be a function of three variables x, y and z and the variables are connected by the relation,

$$f(x, y, z) = 0. \qquad ...(1)$$

Then the necessary condition for $f(x, y, z)$ to be maximum or minimum is

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

Thus, $$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0 \qquad ...(2)$$

Total differentiation of eq. (1) is

$$\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz = 0 \qquad ...(3)$$

Multiply eq. (3) by λ and adding to eq. (2), we get the required condition

$$\left.\begin{aligned}\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0\\ \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0\\ \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0\end{aligned}\right\}$$

Solving the above three equations, the values of x, y, z and λ are find out for which $f(x, y, z)$ is maximum or minimum.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.22. **Find the extreme values of : $f(x, y) = x^3 + y^3 - 3axy$.**

Answer

Here $\quad f(x, y) = x^3 + y^3 - 3axy$

$f_x = 3x^2 - 3ay$, $f_y = 3y^2 - 3ax$, $r = f_{xx} = 6x$, $s = f_{xy} = -3a$, $t = f_{yy} = 6y$

Now, $\quad f_x = 0$ and $f_y = 0$

$$x^2 - ay = 0 \qquad ...(3.22.1)$$

and $$y^2 - ax = 0 \qquad ...(3.22.2)$$

From eq. (3.22.1), $y = \frac{x^2}{a}$

$\therefore$ From eq. (3.22.2),

$$\frac{x^4}{a^2} - ax = 0 \quad \text{or} \quad x(x^3 - a^3) = 0 \quad \text{or} \quad x = 0, a$$

When $x = 0$, $y = 0$; when $x = a$, $y = a$

$\therefore$ There are two stationary points $(0, 0)$ and (a, a)

Now, $\quad rt - s^2 = 36xy - 9a^2$

At $(0, 0)$: $\quad rt - s^2 = -9a^2 < 0$

There is no extreme value at $(0, 0)$

At (a, a) : $\quad rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

$f(x, y)$ has extreme value at (a, a)

Now, $\quad r = 6a$

If $a > 0$, $r > 0$ so that $f(x, y)$ has a minimum value at (a, a)

Minimum value $= a^3 + a^3 - 3a^3 = -a^3$

If $a < 0$, $r < 0$ so that $f(x, y)$ has a maximum value at (a, a)

Maximum value $= -a^3 - a^3 + 3a^3 = a^3$

Que 3.23. Find the maximum and minimum distance of the point (1, 2, – 1) from the sphere $x^2 + y^2 + z^2 = 24$.

OR

Using Lagrange's method of maxima and minima, find the shortest distance from the point (1, 2, –1) to sphere $x^2 + y^2 + z^2 = 24$.

AKTU 2017-18, Marks 3.5

Answer

Let the coordinates of the point on the sphere be (x, y, z) whose distance from (1, 2, – 1) is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

Let,
$$f(x, y, z) = D^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$$
$$\phi(x, y, z) = x^2 + y^2 + z^2 - 24$$

Let us form a Lagrange's function,

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$$

For maximum or minimum distances, $dF = 0$

i.e., $2(x-1) + 2\lambda x = 0$

or $$x = \frac{1}{1+\lambda}$$

Similarly, $2(y-2) + 2\lambda y = 0$

$$y = \frac{2}{1+\lambda}$$

and, $2(z+1) + 2\lambda z = 0$

$$z = \frac{-1}{1+\lambda}$$

Putting the values of x, y and z in $x^2 + y^2 + z^2 = 24$

$$\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 + \left(\frac{-1}{1+\lambda}\right)^2 = 24$$

On solving, $$\lambda = -\frac{1}{2}, -\frac{3}{2}$$

Thus, when $\lambda = -\frac{1}{2}$

$$x = 2, y = 4, z = -2$$

and, when $\lambda = -\frac{3}{2}$

$$x = -2, y = -4, z = 2$$

Thus, the required points are (2, 4, – 2) and (– 2, – 4, 2).

Thus, the distance for point (2, 4, – 2) is

$$D = \sqrt{(2-1)^2 + (4-2)^2 + (-2+1)^2} = \sqrt{6}$$

and, for (– 2, – 4, 2) is

$$D = \sqrt{(-2-1)^2 + (-4-2)^2 + (2+1)^2} = \sqrt{54}$$

Thus, shortest distance = $\sqrt{6}$ and longest distance = $\sqrt{54}$

Que 3.24. Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Answer

Consider (x, y, z) be a vertex of parallelopiped, then it lies on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Dimensions are $2x$, $2y$ and $2z$, then the volume V is given by

$$V = 2x\, 2y\, 2z = 8xyz$$

$$V^2 = 64\, x^2 y^2 z^2 = 64\, x^2 y^2 c^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

[By converting the problem of three variables into two variables]

$$= 64c^2\left(x^2y^2 - \frac{x^4y^2}{a^2} - \frac{x^2y^4}{b^2}\right) = f(x, y)$$

$$\frac{\partial f}{\partial x} = 64c^2\left(2xy^2 - \frac{4x^3y^2}{a^2} - \frac{2xy^4}{b^2}\right)$$

$$\frac{\partial f}{\partial y} = 64c^2\left(2x^2y - \frac{2x^4y}{a^2} - \frac{4x^2y^3}{b^2}\right)$$

$\therefore$
$$r = \frac{\partial^2 f}{\partial x^2} = 64c^2\left(2y^2 - \frac{12x^2y^2}{a^2} - \frac{2y^4}{b^2}\right)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 64c^2\left(4xy - \frac{8x^3y}{a^2} - \frac{8xy^3}{b^2}\right)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 64c^2\left(2x^2 - \frac{2x^4}{a^2} - \frac{12x^2y^2}{b^2}\right)$$

Now $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ for maximum or minimum of V.

$$128\, c^2 xy^2\left(1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2}\right) = 0 \text{ and } 128\, c^2 x^2 y\left(1 - \frac{x^2}{a^2} - \frac{2y^2}{b^2}\right) = 0$$

$$1 - \frac{2x^2}{a^2} - \frac{y^2}{b^2} = 0 \quad \text{...(3.24.1)}$$

$$1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}=0 \quad ...(3.24.2)$$

Subtracting eq. (3.24.2) from eq. (3.24.1), we get

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2}=0 \text{ or } y=\frac{bx}{a}$$

From eq. (3.24.1), $1-\frac{2x^2}{a^2}-\frac{x^2}{a^2}=0$

$$x=\frac{a}{\sqrt{3}}$$

$$\therefore \quad y=\frac{b}{a}\frac{a}{\sqrt{3}}=\frac{b}{\sqrt{3}}$$

and, $$z^2=c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=c^2\left(1-\frac{1}{3}-\frac{1}{3}\right)=\frac{c^2}{3}$$

Thus, $x=\frac{a}{\sqrt{3}}, y=\frac{b}{\sqrt{3}}$ is a stationary point. At this point,

$$r=64\,c^2\left\{\frac{2b^2}{3}-\frac{12}{a^2}\frac{a^2}{3}\frac{b^2}{3}-\frac{2}{b^2}\frac{b^4}{9}\right\}$$

$$r=\frac{-512}{9}b^2c^2<0$$

$$s=64\,c^2\left\{4\frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}-\frac{8}{a^2}\frac{a^3}{3\sqrt{3}}\frac{b}{\sqrt{3}}-\frac{8}{b^2}\frac{a}{\sqrt{3}}\frac{b^3}{3\sqrt{3}}\right\}$$

$$=-\frac{256}{9}abc^2$$

$$t=64c^2\left\{2\frac{a^2}{3}-\frac{2}{a^2}\frac{a^4}{9}-\frac{12}{b^2}\frac{a^2}{3}\frac{b^2}{3}\right\}$$

$$=-\frac{512}{9}a^2c^2$$

$$rt-s^2=\left(\frac{512}{9}\right)^2a^2b^2c^4-\left(\frac{256}{9}\right)^2a^2b^2c^4$$

$$=\left(\frac{256}{9}\right)^2a^2b^2c^4(4-1)>0$$

Also $r<0$

$\therefore$ V^2 is maximum, hence V is maximum when $x=\frac{a}{\sqrt{3}}, y=\frac{b}{\sqrt{3}}, z=\frac{c}{\sqrt{3}}$

and, its maximum value is

$$V=8\frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}\frac{c}{\sqrt{3}}=\frac{8\,abc}{3\sqrt{3}}$$

Que 3.25. **A rectangle box open at the top is to have 32 cubic ft. Find the dimensions of the box requiring least material for its construction.**

AKTU 2014-15, Marks 10

Answer

Let x, y and z are the dimensions of the box.

Volume, $V=32$

$$xyz=32$$

$$y=\frac{32}{xz} \quad ...(3.25.1)$$

Surface area, $S=2(x+y)z+xy$

Putting the value of y, we get

$$S=2\left(x+\frac{32}{xz}\right)z+\frac{32}{z} \quad ...(3.25.2)$$

$$S=2xz+\frac{64}{x}+\frac{32}{z}$$

$$\frac{\partial S}{\partial x}=2z-\frac{64}{x^2}$$

$$\frac{\partial S}{\partial z}=2x-\frac{32}{z^2}$$

For least material,

$$\frac{\partial S}{\partial x}=0 \quad i.e., \quad z=\frac{32}{x^2} \quad ...(3.25.3)$$

$$\frac{\partial S}{\partial z}=0 \quad i.e., \quad x=\frac{16}{z^2} \quad ...(3.25.4)$$

From eq. (3.25.3), eq. (3.25.4) and eq. (3.25.1),

$$x=4, y=4, z=2$$

$$r=\frac{\partial^2 S}{\partial x^2}=\frac{128}{x^3}=2, s=\frac{\partial^2 S}{\partial x\partial z}=2, t=\frac{\partial^2 S}{\partial z^2}=\frac{64}{z^3}=8$$

$$rs-t^2=2\times 8-(2)^2=16-4=12$$

$$\frac{\partial^2 S}{\partial x^2}=+2, \text{ so } S \text{ is minimum for } x=4, y=4, z=2.$$

Que 3.26. **Find the dimension of rectangular box of maximum capacity whose surface area is given, when box is open at the top.**

AKTU 2013-14, Marks 05

OR

Using the Lagrange's method find the dimension of rectangular box of maximum capacity whose surface area is given (a) box is open at the top (b) box is closed.

AKTU 2015-16, Marks 10

Answer

Let x, y, z be the length, breadth and height of the rectangular box.

Surface area S is given by

$$S = nxy + 2yz + 2zx \quad ...(3.26.1)$$

$$V = xyz \quad ...(3.26.2)$$

According to Lagrange's method of undetermined multiplier,

$$F = V + \lambda(S)$$

$$F = xyz + \lambda(nxy + 2yz + 2zx)$$

$$dF = 0$$

i.e., $$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

$$yz + \lambda(ny + 2z) = 0 \quad ...(3.26.3)$$

$$xz + \lambda(nx + 2z) = 0 \quad ...(3.26.4)$$

$$xy + \lambda(2y + 2x) = 0 \quad ...(3.26.5)$$

Multiply eq. (3.26.3) by x, eq. (3.26.4) by y and eq. (3.26.5) by z, we have

$$xyz + \lambda(nxy + 2zx) = 0 \quad ...(3.26.6)$$

$$xyz + \lambda(nxy + 2zy) = 0 \quad ...(3.26.7)$$

$$xyz + \lambda(2yz + 2xz) = 0 \quad ...(3.26.8)$$

On adding above three equations, we get

$$3xyz + 2\lambda(nxy + 2zx + 2yz) = 0$$

$$3V + 2\lambda S = 0$$

$$\lambda = \frac{-3V}{2S}$$

Substituting λ in eq. (3.26.3), eq. (3.26.4) and eq. (3.26.5), we have

$$yz - \frac{3V}{2S}(ny + 2z) = 0$$

$$yz - \frac{3xyz}{2S}(ny + 2z) = 0$$

$$nxy + 2xz = \frac{2S}{3} \quad ...(3.26.9)$$

Similarly, $$nxy + 2yz = \frac{2S}{3} \quad ...(3.26.10)$$

$$2yz + 2zx = \frac{2S}{3} \quad ...(3.26.11)$$

Subtracting eq. (3.26.10) from eq. (3.26.9), we get

$$x = y$$

Subtracting eq. (3.26.11) from eq. (3.26.10), we get

$$ny = 2z$$

Thus from eq. (3.26.1),

$$S = n\,x\,x + 4\,x\,\frac{nx}{2}$$

$$S = 3nx^2$$

$$x^2 = \frac{S}{3n}$$

a. When box is open at top, $n = 1$

$$x = y = \sqrt{\frac{S}{3}}$$

$$z = \frac{1}{2}\sqrt{\frac{S}{3}}$$

b. When box is closed, $n = 2$

$$x = y = z = \sqrt{\frac{S}{6}}$$

Que 3.27. **A tent of a given volume has a square base of side $2a$, has its four sides, vertical of length b and is surmounted by a regular pyramid of height h. Find the values of a and b in terms of h such that the canvas required for its construction is minimum.**

Answer

Let V is the volume and S is the surface area of the tent.

$$V = 4a^2b + \frac{1}{3}(4a^2)\,h$$

[$\because$ Volume of pyramid = 1/3 (Area of base × Height)]

$$S = 8ab + 4a\sqrt{a^2 + h^2}$$

[$\because$ Surface area of pyramid = 1/2 (Perimeter × Slant height)]

For minimum canvas requirement,

$$\frac{\partial S}{\partial a} + \lambda\frac{\partial V}{\partial a} = 0$$

$$8b + 4\sqrt{a^2 + h^2} + \frac{4a^2}{\sqrt{a^2 + h^2}} + \lambda\left[8ab + \frac{8ah}{3}\right] = 0 \quad ...(3.27.1)$$

$$\frac{\partial S}{\partial b} + \lambda\frac{\partial V}{\partial b} = 0$$

$$8a + 4\lambda a^2 = 0 \quad ...(3.27.2)$$

$$\frac{\partial S}{\partial h} + \lambda\frac{\partial V}{\partial h} = 0$$

$$\frac{4ah}{\sqrt{a^2 + h^2}} + \frac{4}{3}\lambda a^2 = 0 \quad ...(3.27.3)$$

From eq. (3.27.2), $\lambda a + 2 = 0$ ...(3.27.4)

From eq. (3.27.3),

$$3h + \lambda a\sqrt{a^2 + h^2} = 0 \quad ...(3.27.5)$$

From eq. (3.27.4) and eq. (3.27.5),

$$a = \frac{\sqrt{5}}{2}h$$

From eq. (3.27.2),

$$\lambda a = -2$$

Now putting $\lambda a = -2$ and $a = \frac{\sqrt{5}}{2}h$ in eq. (3.27.1), we get

$$8b + 4\sqrt{\frac{5h^2}{4} + h^2} + \frac{5h^2}{\sqrt{\frac{5h^2}{4} + h^2}} - 2\left[8b + \frac{8h}{3}\right] = 0$$

$$8b - 6h + \frac{10h}{3} - 16b - \frac{16h}{3} = 0$$

$$b = \frac{h}{2}$$

Thus, $a = \frac{\sqrt{5}}{2}h$ and $b = \frac{h}{2}$

Que 3.28. **Divide a number into three parts such that the product of first, square of the second and cube of third is maximum.**

AKTU 2016-17, Marks 04

Answer

Let 'N' be the number, $N = x + y + z$ and, $F = xy^2z^3$

Consider Lagrange's function $F(x, y, z) = xy^2z^3 + \lambda(x + y + z - N)$

According to Lagrange's method of undetermined multipliers,

$$y^2z^3 + \lambda = 0 \quad ...(3.28.1)$$

$$2xyz^3 + \lambda = 0 \quad ...(3.28.2)$$

$$3xy^2z^2 + \lambda = 0 \quad ...(3.28.3)$$

Multiplying by x, y and z in eq. (3.28.1), eq. (3.28.2) and eq. (3.28.3) respectively and adding, we get

$$6(xy^2z^3) + \lambda(x + y + z) = 0 \quad \text{or} \quad 6F + \lambda N = 0 \quad \text{or} \quad \lambda = -\frac{6F}{N}$$

From eq. (3.28.1), eq. (3.28.2) and eq. (3.28.3)

$$y^2z^3 - \frac{6F}{N} = 0 \quad ...(3.28.4)$$

$$2xyz^3 - \frac{6F}{N} = 0 \quad ...(3.28.5)$$

$$3xy^2z^2 - \frac{6F}{N} = 0 \quad ...(3.28.6)$$

Multiplying by 'x' in eq. (3.28.4), we get

$$xy^2z^3 - \frac{6Fx}{N} = 0 \quad \text{or} \quad F = \frac{6Fx}{N} = 0 \quad \text{or} \quad x = \frac{N}{6}$$

Similarly, multiplying by 'y' in eq. (3.28.5), we get

$$2xy^2z^3 - \frac{6Fy}{N} = 0 \quad \text{or} \quad 2F = \frac{6Fy}{N} \quad \text{or,} \quad y = \frac{N}{3}$$

Similarly, multiplying by 'z' in eq. (3.28.6), we get

$$3xy^2z^3 - \frac{6Fz}{N} = 0 \quad \text{or} \quad 3F = \frac{6Fz}{N} \quad \text{or,} \quad z = \frac{N}{2}$$

Hence, $\left(\frac{N}{6}, \frac{N}{3}, \frac{N}{2}\right)$ is the stationary point.

Now to find whether 'F' is maximum or minimum.

Let 'F' be a function of x and y.

$$F = xy^2(N - x - y)^3$$

$$F = xy^2[N^3 - (x + y)^3 - 3N(x + y)(N - (x + y))]$$

$$F = xy^2[N^3 - (x^3 + y^3 + 3xy(x + y)) - 3N^2(x + y) + 3N(x + y)^2]$$

$$= [N^3xy^2 - x^4y^2 - xy^5 - 3x^3y^3 - 3x^2y^4 - 3N^2x^2y^2 - 3N^2xy^3 + 3Nx^3y^2 + 3Nxy^4 + 6Nx^2y^3] \quad ...(3.28.7)$$

Differentiating eq. (3.28.7) w.r.t. 'x', we get

$$\frac{\partial F}{\partial x} = N^3y^2 - 4x^3y^2 - y^5 - 9x^2y^3 - 6xy^4 - 6N^2xy^2 - 3N^2y^3 + 9Nx^2y^2 + 3Ny^4 + 12Nxy^3$$

Again differentiating w.r.t. 'x', we get

$$\frac{\partial^2 F}{\partial x^2} = -12x^2y^2 - 18xy^3 - 6y^4 - 6N^2y^2 + 18Nxy^2 + 12Ny^3$$

$$\text{or,} \quad \frac{\partial^2 F}{\partial x^2} = \left[-12\left(\frac{N}{6}\right)^2\left(\frac{N}{3}\right)^2 - 18\left(\frac{N}{6}\right)\left(\frac{N}{3}\right)^3 - 6\left(\frac{N}{3}\right)^4 - 6N^2\left(\frac{N}{3}\right)^2 + 18N\left(\frac{N}{6}\right)\left(\frac{N}{3}\right)^2 + 12N\left(\frac{N}{3}\right)^3\right]$$

$$\frac{\partial^2 F}{\partial x^2} = -\frac{N^4}{9} \quad \text{or,} \quad r = -\frac{N^4}{9} \quad \left[\because r = \frac{\partial^2 F}{\partial x^2}\right]$$

Differentiating eq. (3.28.7) w.r.t. 'y', we get

$$\frac{\partial F}{\partial y} = [2N^3xy - 2x^4y - 5xy^4 - 9x^3y^2 - 12x^2y^3 - 6N^2x^2y - 9N^2xy^2 + 6Nx^3y + 12Nxy^3 + 18Nx^2y^2]$$

Differentiating again w.r.t. 'y', we get

$$\frac{\partial^2 F}{\partial y^2} = [2N^2x - 2x^4 - 20xy^3 - 18x^3y - 36x^2y^2 - 6N^2x^2 - 18N^2xy + 6Nx^3 + 36Nxy^2 + 36Nx^2y]$$

$$\frac{\partial^2 F}{\partial y^2} = \left[2N^2\left(\frac{N}{6}\right) - 2\left(\frac{N}{6}\right)^4 - 20\left(\frac{N}{6}\right)\left(\frac{N}{3}\right)^3 - 18\left(\frac{N}{6}\right)^3\left(\frac{N}{3}\right) - 36\left(\frac{N}{6}\right)^2\left(\frac{N}{3}\right)^2 - 6N^2\left(\frac{N}{6}\right)^2 - 18N^2\left(\frac{N}{6}\right)\left(\frac{N}{3}\right) + 6N\left(\frac{N}{6}\right)^3 + 36N\left(\frac{N}{6}\right)\left(\frac{N}{3}\right)^2 + 36N\left(\frac{N}{6}\right)^2\left(\frac{N}{3}\right)\right]$$

$$\frac{\partial^2 F}{\partial y^2} = -\frac{45}{648}N^4 \text{ or, } t = -\frac{5}{72}N^4 \qquad \left[\because t = \frac{\partial^2 F}{\partial y^2}\right]$$

Now,

$$\frac{\partial^2 F}{\partial x \partial y} = [2N^3y - 8x^3y - 5y^4 - 27x^2y^2 - 24xy^3 - 12N^2xy - 9N^2y^2 + 18Nx^2y + 12Ny^3 + 36Nxy^2]$$

or,

$$\frac{\partial^2 F}{\partial x \partial y} = \left[2N^3\left(\frac{N}{3}\right) - 8\left(\frac{N}{6}\right)^3\left(\frac{N}{3}\right) - 5\left(\frac{N}{3}\right)^4 - 27\left(\frac{N}{6}\right)^2\left(\frac{N}{3}\right)^2 - 24\left(\frac{N}{6}\right)\left(\frac{N}{3}\right)^3 - 12N^2\left(\frac{N}{6}\right)\left(\frac{N}{3}\right) - 9N^2\left(\frac{N}{3}\right)^2 + 18N\left(\frac{N}{6}\right)^2\left(\frac{N}{3}\right) + 12N\left(\frac{N}{3}\right)^3 + 36N\left(\frac{N}{6}\right)\left(\frac{N}{3}\right)^2\right]$$

$$\frac{\partial^2 F}{\partial x \partial y} = -\frac{N^4}{36} \quad \text{or,} \quad s = -\frac{N^4}{36} \qquad \left[\because s = \frac{\partial^2 F}{\partial x \partial y}\right]$$

Now checking the value of $(rt - s^2)$,

$$rt - s^2 = \left(-\frac{N^4}{9}\right)\left(-\frac{5}{72}N^4\right) - \left(-\frac{N^4}{36}\right)^2 = \frac{5N^8}{648} - \frac{N^8}{1296} = \frac{9N^8}{2}$$

$(rt - s^2) > 0$ and $r < 0$ which is the case of maxima.

Hence the numbers would be maximum for the values of x, y and z $\left(\frac{N}{6}, \frac{N}{3}, \frac{N}{2}\right)$.

PART-5

Jacobians, Approximation of Errors.

CONCEPT OUTLINE : PART-5

Jacobian : If $u_1, u_2, \ldots\ldots, u_n$ are functions of n independent variables $x_1, x_2, \ldots\ldots, x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots\ldots, u_n$ with respect to $x_1, x_2, \ldots\ldots, x_n$ and is denoted either by

$$\frac{\partial(u_1, u_2, \ldots\ldots, u_n)}{\partial(x_1, x_2, \ldots\ldots, x_n)} \text{ or by } J(u_1, u_2, \ldots\ldots, u_n)$$

Thus if u and v are functions of two independent variables x and y, then

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = J(u, v)$$

Approximation of Errors : Let $f(x, y)$ be a continuous function of x and y. If δx and δy be the increments in x and y respectively, then new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$ *i.e.*,

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y)$$

Expanding using Taylor's series

$$\delta f = f(x, y) + \delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y} + \ldots\ldots - f(x, y)$$

Neglecting higher powers, $\delta f = \delta x\frac{\partial f}{\partial x} + \delta y\frac{\partial f}{\partial y}$

δx, δy and δf are small changes in x, y and f.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.29. **If J is the Jacobian of u, v with respect to x, y and J' is the Jacobian of x, y with respect to u, v, then**

$$JJ' = 1 \text{ or } \frac{\partial(u,v)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(u,v)} = 1$$

Answer

Let, $u = f_1(x, y)$, $v = f_2(x, y)$ ...(3.29.1)

Obviously x and y can also be expressed as functions of u and v. Differentiating eq. (3.29.1) partially with respect to u and v, we get

$$1 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u},\quad 0 = \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \quad ...(3.29.2)$$

$$0 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u},\quad 1 = \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \quad ...(3.29.3)$$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)}\,\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Que 3.30. **If $x + y + z = u$, $y + z = uv$, $z = uvw$ show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} = u^2v$.**

OR

If $x + y + z = u$, $y + z = uv$, $z = uvw$, then find $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

AKTU 2015-16, Marks 05

Answer

The given relation can be written as,

$$F_1 = x + y + z - u = 0$$
$$F_2 = y + z - uv = 0$$
$$F_3 = z - uvw = 0$$

Now, $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = (-1)^3\,\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} \quad ...(3.30.1)$$

We have, $$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & -uw & -uv \end{vmatrix} = -u^2v$$

and $$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\therefore$ From eq. (3.30.1),

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-(-u^2v)}{1} = u^2v$$

Que 3.31. **The roots of the equation in λ, $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ are u, v, w.**

Prove that : $\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$

OR

If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$

AKTU 2015-16, Marks 10

Answer

The given equation in λ can be written as,

$$3\lambda^3 - 3\lambda^2(x + y + z) + 3\lambda(x^2 + y^2 + z^2) - (x^3 + y^3 + z^3) = 0$$

Since u, v, w are the roots of this equation, therefore,

$$u + v + w = x + y + z$$
$$uv + vw + wu = x^2 + y^2 + z^2$$
$$uvw = 1/3(x^3 + y^3 + z^3)$$

The above relation can be written as

$$F_1 = u + v + w - x - y - z = 0$$
$$F_2 = uv + vw + wu - x^2 - y^2 - z^2 = 0$$
$$F_3 = uvw - 1/3(x^3 + y^3 + z^3) = 0$$

Now, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3\,\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} \quad ...(3.31.1)$$

We have,

$$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= -2\,(y-z)\,(z-x)\,(x-y)$$

Also
$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$

$$= -\,(v-w)\,(w-u)\,(u-v)$$

Hence from eq. (3.31.1),

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3\,\frac{-2\,(y-z)(z-x)(x-y)}{-(v-w)(w-u)(u-v)}$$

$$= -2\,\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Que 3.32. Find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ if $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$.

AKTU 2014-15, Marks 10

Answer

$x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

Multiplying and dividing R_1 by $\sqrt{vw}$, R_2 by $\sqrt{uw}$ and R_3 by $\sqrt{vu}$, we get

$$= \frac{1}{8}\,\frac{1}{\sqrt{vw}}\,\frac{1}{\sqrt{uw}}\,\frac{1}{\sqrt{vu}}\begin{vmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{vmatrix}$$

$$= \frac{1}{8}\,\frac{1}{uvw}\,[-w(-vu) + v(wu)]$$

$$= \frac{1}{8} \times \frac{1}{uvw} \times 2uvw = \frac{1}{4}$$

Again, $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\,[\sin\theta\cos\phi\,(0+\sin\theta\cos\phi) - \cos\theta\cos\phi\,(0-\cos\theta\cos\phi) - \sin\phi\,(-\sin^2\theta\sin\phi - \cos^2\theta\sin\phi)]$$

$$= r^2\sin\theta\,[\sin^2\theta\cos^2\phi + \cos^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi]$$

$$= r^2\sin\theta\,[\cos^2\phi + \sin^2\phi]$$

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \times \frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \frac{1}{4}\,r^2\sin\theta$$

Que 3.33. Show that $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$ are not independent. Find the relation between them.

Answer

$$u = y + z$$
$$v = x + 2z^2$$
$$w = x - 4yz - 2y^2$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4z-4y & -4y \end{vmatrix}$$

$$= -1[-4y - 4z] + 1[-4z - 4y] = 0$$

Hence u, v, w are not independent.

$$w = x - 2y^2 - 4yz = v - 2z^2 - 2y^2 - 4yz \qquad [\because\ x + 2z^2 = v]$$
$$= v - 2(z+y)^2$$
$$w = v - 2u^2$$

or $$2u^2 = v - w$$

Que 3.34. Show that the functions :

$$\mathbf{u = x + y + z,}$$
$$\mathbf{v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx,}$$ **and**

$w = x^3 + y^3 + z^3 - 3xyz$

are functionally related. Find the relation between them.

AKTU 2013-14, Marks 10

Answer

Same as Q. 3.33, Page 3–36C, Unit-3.

(**Ans.** : $4w = 3uv + u^3$)

Que 3.35. **Find the relation between u, v, w for the values**

$$u = x + 2y + z;\ v = x - 2y + 3z;$$
$$w = 2xy - zx + 4yz - 2z^2.$$

AKTU 2016-17, Marks 03

Answer

$$u = x + 2y + z$$
$$v = x - 2y + 3z$$
$$w = 2xy - zx + 4yz - 2z^2$$

We have, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix}$$

$$=1[-2(-x + 2y - 3z) - 3(2x + 4z)] - 2[-x + 2y - 3z - 2(2y - z)] + 1[2x + 4z + 2(2y - z)] = 0$$

Since Jacobian is zero, therefore these functions are not independent and hence there exist a relation between them.

We have, $$u^2 - v^2 = (x+2y+z)^2 - (x-2y+3z)^2$$

$$= (x+2y+z+x-2y+3z)(x+2y+z-x+2y-3z)$$

$$= (2x + 4z)(4y - 2z)$$

$$= 4(x + 2z)(2y - z) = 4(2xy - xz + 4yz - 2z^2)$$

Therefore $u^2 - v^2 = 4w$ is the required relation between u, v and w.

Que 3.36. **If $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$ then show that $\frac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = 1$.**

AKTU 2016-17, Marks 3.5

Answer

Given : $x = v^2 + w^2$, $y = w^2 + u^2$, $z = u^2 + v^2$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 0 & 2v & 2w \\ 2u & 0 & 2w \\ 2u & 2v & 0 \end{vmatrix}$$

$$= -2v(-4uw) + 2w(4uv) = 16uvw$$

Now converting (u, v, w) in terms of (x, y, z), we have

$u^2 = y - w^2 = y - (x - v^2) = y - (x - z + u^2) = y - x + z - u^2$

or, $u^2 = \left(\frac{y}{2} - \frac{x}{2} + \frac{z}{2}\right)$, $v^2 = \left(\frac{z}{2} + \frac{x}{2} - \frac{y}{2}\right)$ and $w^2 = \left(\frac{x}{2} + \frac{y}{2} - \frac{z}{2}\right)$

Now, $$\frac{\partial u}{\partial x} = -\frac{1}{4u}, \frac{\partial u}{\partial y} = \frac{1}{4u}, \frac{\partial u}{\partial z} = \frac{1}{4u}$$

$$\frac{\partial v}{\partial x} = \frac{1}{4v}, \frac{\partial v}{\partial y} = -\frac{1}{4v}, \frac{\partial v}{\partial z} = \frac{1}{4v}$$

$$\frac{\partial w}{\partial x} = \frac{1}{4w}, \frac{\partial w}{\partial y} = \frac{1}{4w}, \frac{\partial w}{\partial z} = -\frac{1}{4w}$$

Now, $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -1/4u & 1/4u & 1/4u \\ 1/4v & -1/4v & 1/4v \\ 1/4w & 1/4w & -1/4w \end{vmatrix}$$

$$= \frac{2}{64\,uvw} + \frac{2}{64\,uvw} = \frac{4}{64\,uvw} = \frac{1}{16\,uvw}$$

$$\therefore \quad \frac{\partial(x,y,z)}{\partial(u,v,w)} \frac{\partial(u,v,w)}{\partial(x,y,z)} = 16\,uvw \times \frac{1}{16\,uvw} = 1$$

Que 3.37. **If $u_1 = \frac{x_2 x_3}{x_1}$, $u_2 = \frac{x_3 x_1}{x_2}$ and $u_3 = \frac{x_1 x_2}{x_3}$, find the value of $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}$.**

AKTU 2017-18, Marks 3.5

Answer

$u_1 = \frac{x_2 x_3}{x_1}$, then $\frac{\partial u_1}{\partial x_1} = -\frac{x_2 x_3}{x_1^2}, \frac{\partial u_1}{\partial x_2} = \frac{x_3}{x_1}, \frac{\partial u_1}{\partial x_3} = \frac{x_2}{x_1}$

$u_2 = \frac{x_3 x_1}{x_2}$, then $\frac{\partial u_2}{\partial x_1} = \frac{x_3}{x_2}, \frac{\partial u_2}{\partial x_2} = \frac{-x_1 x_3}{x_2^2}, \frac{\partial u_2}{\partial x_3} = \frac{x_1}{x_2}$

$u_3 = \frac{x_1 x_2}{x_3}$, then $\frac{\partial u_3}{\partial x_1} = \frac{x_2}{x_3}, \frac{\partial u_3}{\partial x_2} = \frac{x_1}{x_3}, \frac{\partial u_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2}$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \frac{\partial u_1}{\partial x_3} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_3}{\partial x_1} & \frac{\partial u_3}{\partial x_2} & \frac{\partial u_3}{\partial x_3} \end{vmatrix} = \begin{vmatrix} -\frac{x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & -\frac{x_1 x_3}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & -\frac{x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_1 x_3 & x_1 x_2 \\ x_2 x_3 & -x_1 x_3 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(1-1) - 1(-1-1) + 1(1+1) = 0 + 2 + 2 = 4$$

Que 3.38. **Find the percentage of error in calculating the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when error of + 1 % is made in measuring the major and minor axes.**

Answer

Area of the ellipse, $A = \pi ab$

Taking log on both sides,

$$\log A = \log \pi + \log a + \log b$$

Differentiating both side,

$$\frac{\delta A}{A} = 0 + \frac{\delta a}{a} + \frac{\delta b}{b}$$

$$\frac{\delta A}{A} \times 100 = \frac{\delta a}{a} \times 100 + \frac{\delta b}{b} \times 100$$

But, $\quad \frac{\delta a}{a} \times 100 = 1, \frac{\delta b}{b} \times 100 = 1$

∴ Percentage error in area is

$$\frac{\delta A}{A} \times 100 = 1 + 1 = 2 \%$$

Que 3.39. **If the base radius and height of a cone are measured as 4 cm and 8 cm with a possible error of 0.04 and 0.08 inches respectively, calculate the percentage (%) error in calculating volume of the cone.**

Answer

Given : $\quad r = 4$ cm, $h = 8$ cm

$\delta r = 0.04$ inch $= 0.1016$ cm

$\delta h = 0.08$ inch $= 0.2032$ cm

Volume cone of, $\quad V = \frac{1}{3} \pi r^2 h \quad$ ...(3.39.1)

Taking log on both sides of eq. (3.39.1),

$$\log V = \log\left(\frac{\pi}{3}\right) + 2\log r + \log h$$

Differentiating both sides,

$$\frac{\delta V}{V} \times 100 = 0 + \frac{2\delta r}{r} \times 100 + \frac{\delta h}{h} \times 100$$

$$\frac{\delta V}{V} \times 100 = 2\left(\frac{0.1016}{4} \times 100\right) + \frac{0.2032}{8} \times 100 = 7.62 \%$$

Que 3.40. **The two sides of a triangle are measured as 50 cm and 70 cm, and the angle between them is 30°. If there are possible errors of 0.5 % in the measurement of the sides and 0.5 degree in that of the angle, find the maximum approximate percentage error in measuring the area of the triangle.**

Answer

Given : $\quad b = 50$ cm, $c = 70$ cm

$A = 30°$, $\frac{\delta b}{b} \times 100 = 0.5$

$\frac{\delta c}{c} \times 100 = 0.5$, $\delta A = 0.5$

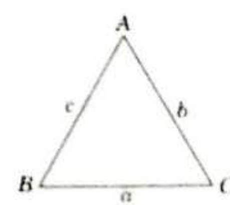

Fig. 3.40.1.

Area of triangle ABC is,

$$\Delta = \frac{1}{2} bc \sin A$$

Taking log on both sides,

$$\log \Delta = \log \frac{1}{2} + \log b + \log c + \log \sin A$$

Differentiating both sides,

$$\frac{\delta\Delta}{\Delta} \times 100 = \frac{\delta b}{b} \times 100 + \frac{\delta c}{c} \times 100 + \frac{\cos A}{\sin A} \delta A \times 100$$

$$\frac{\delta\Delta}{\Delta} \times 100 = 0.5 + 0.5 + \cot 30° (0.5) \times 100 = 1 + 50(\sqrt{3}) = 87.60 \%$$

Que 3.41. **Find the percentage error in measuring the volume of a rectangular box when the error of 1 % is made in measuring the each side.** **AKTU 2016-17, Marks 3.5**

Answer

Volume $(V) = lbh$

Taking log on both sides,

$$\log V = \log l + \log b + \log h$$

Now, differentiating the above,

$$\frac{\delta V}{V} = \frac{\delta l}{l} + \frac{\delta b}{b} + \frac{\delta h}{h}$$

$$= \frac{\delta l}{l} \times 100 + \frac{\delta b}{b} \times 100 + \frac{\delta h}{h} \times 100$$

Error in volume, $\frac{\delta V}{V} \times 100 = 1 + 1 + 1 = 3 \%$

Que 3.42. **A balloon in the form of right circular cylinder of radius 1.5 m and length 4 m is surmounted by hemispherical ends. If the radius is increased by 0.01 m, find the percentage change in the volume of the balloon.** **AKTU 2017-18, Marks 3.5**

Answer

Radius (r) =1.5 m, Length (h) = 4 m, δr = 0.01 m, $\delta h = 0$

Volume, $V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$

On differentiating,

$$\delta V = 2\pi rh\delta r + \pi r^2 \delta h + \frac{4}{3}\pi \times 3r^2 \delta r$$

$$\frac{\delta V}{V} = \frac{2\pi rh\delta r + \pi r^2 \delta h + 4\pi r^2 \delta r}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{\pi r (2h\delta r + r\delta h + 4r\delta r)}{\pi r\left(rh + \frac{4}{3}\pi r^2\right)} = \frac{2h\delta r + r\delta h + 4r\delta r}{rh + \frac{4}{3}r^2}$$

$$= \frac{2 \times 4 \times 0.01 + 1.5 \times 0 + 4 \times 1.5 \times 0.01}{1.5 \times 4 + \frac{4}{3} \times (1.5)^2} \quad (\because \ \delta h = 0)$$

$$= \frac{0.08 + 0.06}{6 + 3}$$

$$\frac{\delta V}{V} \times 100 = \frac{0.14}{9} \times 100 = 1.55 \%$$

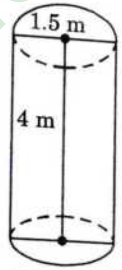

Fig. 3.42.1.

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4 UNIT

Multivariable Calculus-I

PART-1

Double Integral, Triple Integral, Change of Order of Integration, Change of Variables.

CONCEPT OUTLINE : PART-1

Double Integrals Over Rectangles :

Let $f(x, y)$ be defined on a rectangular region R given by

$$R : a \le x \le b,\ c \le y \le d$$

If we divide R into small pieces of area $\Delta A = \Delta x\ \Delta y$ then choosing a point (x_k, y_k) in each piece ΔA_k, then

$$\iint_R f(x, y)\, dA = \lim_{\Delta A \to 0} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

Properties of Double Integrals :

i. $\iint_R K f(x, y)\, dA = K \iint_R f(x, y)\, dA$ (any number K)

ii. $\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$

iii. $\iint_R f(x, y)\, dA \ge 0$ if $f(x, y) \ge 0$ on R.

iv. $\iint_R f(x, y)\, dA \ge \iint_R g(x, y)\, dA$ if $f(x, y) \ge g(x, y)$ on R.

v. $\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

It holds when R is the union of two non-overlapping rectangles R_1 and R_2.

Integrals in Polar Coordinates :

The integral is given as

$$\iint_R f(r, \theta)\, dA = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r, \theta)\, dr\, d\theta$$

Triple Integral : If the function $f(x, y, z)$ is continuous at every point of a region R then triple integral is defined as :

$$\iiint_V f(x, y, z)\, dv \quad \text{or} \quad \int_{x=a}^{x=b} \int_{y=\phi_1(x)}^{y=\phi_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x, y, z)\, dz\, dy\, dx$$

Where the constant limit is taken as outer integral and then the limit of one variable and inner most limit is of two variable.

Change of Order of Integration : With respect to x and y we evaluate a double integral by two ways :

i. First we integrate with respect to x then with respect to y.

ii. First we integrate with respect to y then with respect to x.

In the first case we take horizontal strips or strips parallel to x axis while in the second case we take vertical strips or strips parallel to

y axis. To change the order of integration, first we find out the limit of integration and then change them.

Change of Variables :

Let the double integral be $\iint_R f(x, y)\, dx\, dy$ and the variables x and y be changed to u, v by the relation $x = \phi(u, v)$, $y = \psi(u, v)$ then $dx\, dy$ is given by $|J|\, du\, dv$ where

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

is the Jacobian of transformation from (x, y) to (u, v).

1. To change cartesian coordinates (x, y) to polar coordinates (r, θ).

   Choosing $x = r\cos\theta$, $y = r\sin\theta$ such that $x^2 + y^2 = r^2$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r$$

$$\iint_R f(x, y)\, dx\, dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

2. To change cartesian coordinates (x, y, z) to spherical polar coordinates (r, θ, ϕ).

   Choosing $x = r\sin\theta\cos\phi$
   $y = r\sin\theta\sin\phi$
   $z = r\cos\theta$

   Such that $x^2 + y^2 + z^2 = r^2$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} & \dfrac{\partial x}{\partial \phi} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} & \dfrac{\partial y}{\partial \phi} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \theta} & \dfrac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin^2\theta$$

$$\iiint_V f(x, y, z)\, dx\, dy\, dz$$

$$= \iiint_V f(r\sin\theta\cos\phi, r\sin\theta\sin\phi, r\cos\theta)\, r^2\sin^2\theta\, dr\, d\theta\, d\phi$$

3. To change cartesian coordinates (x, y, z) to cylindrical coordinates (r, z, ϕ).

   Choosing $x = r\cos\phi$

$y = r\sin\phi$
$z = z$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \phi} & \dfrac{\partial x}{\partial z} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \phi} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial z}{\partial r} & \dfrac{\partial z}{\partial \phi} & \dfrac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\phi & -r\sin\phi & 0 \\ \sin\phi & r\cos\phi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r(\cos^2\phi + \sin^2\phi) = r$$

$$\iiint_V f(x, y, z)\, dx\, dy\, dz = \iiint_V f(r\cos\phi, r\sin\phi, z)\, r\, dr\, d\phi\, dz$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. Prove that $\int_1^2\int_3^4 (xy + e^y)\, dy\, dx = \int_3^4\int_1^2 (xy + e^y)\, dx\, dy$.

Answer

Taking LHS :

$$\int_1^2\int_3^4 (xy + e^y)\, dy\, dx = \int_1^2\left[\int_3^4 (xy + e^y)\, dy\right] dx$$

$$= \int_1^2\left[\frac{xy^2}{2} + e^y\right]_3^4 dx = \int_1^2\left(8x + e^4 - \frac{9}{2}x - e^3\right) dx$$

$$= \int_1^2\left(\frac{7}{2}x + e^4 - e^3\right) dx = \left[\frac{7x^2}{4} + (e^4 - e^3)x\right]_1^2$$

$$= 7 + 2(e^4 - e^3) - \frac{7}{4} - (e^4 - e^3) = \frac{21}{4} + e^4 - e^3$$

Taking RHS :

$$\int_3^4\int_1^2 (xy + e^y)\, dx\, dy = \int_3^4\left[\int_1^2 (xy + e^y)\, dx\right] dy = \int_3^4\left[\frac{yx^2}{2} + xe^y\right]_1^2 dy$$

$$= \int_3^4\left(2y + 2e^y - \frac{y}{2} - e^y\right) dy = \int_3^4\left(\frac{3y}{2} + e^y\right) dy$$

$$= \left[\frac{3y^2}{4} + e^y\right]_3^4 = 12 + e^4 - \frac{27}{4} - e^3 = \frac{21}{4} + e^4 - e^3$$

LHS = RHS

Que 4.2. **Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.**

Answer

The region R is shown in the Fig. 4.2.1. Solving $y^2 = 4x$ and $x^2 = 4y$, we get

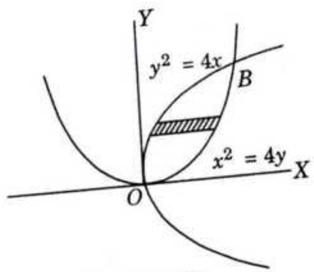


Fig. 4.2.1.

$$\frac{y^4}{16} = 4y \text{ or } y^3 = 4^3 \Rightarrow y = 4 \text{ and } x = 4$$

So, point of intersection of two curves is (4,4).

Taking horizontal strips, the limit of integration of x is $\frac{y^2}{4}$ to $2\sqrt{y}$ and the limit of y is 0 to 4.

Therefore the integral is :

$$\iint_R y\,dx\,dy = \int_0^4 y \int_{y^2/4}^{2\sqrt{y}} dx\,dy$$

$$\int_0^4 y\,[x]_{y^2/4}^{2\sqrt{y}}\,dy = \int_0^4 y\left(2\sqrt{y} - \frac{y^2}{4}\right)dy = \int_0^4 \left(2y^{3/2} - \frac{y^3}{4}\right)dy$$

$$= \left[\frac{2y^{5/2}}{5/2} - \frac{y^4}{16}\right]_0^4$$

$$= \frac{2\times(4)^{5/2}}{5/2} - \frac{4^4}{16} = \frac{2\times(2)^5\times 2}{5} - 4^2$$

$$= \frac{4^3\times 2}{5} - 4^2 = \frac{48}{5}$$

Que 4.3. **Evaluate $\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy$.**

Answer

$$\int_0^1\int_{y^2}^1\int_0^{1-x} x\,dz\,dx\,dy = \int_0^1\int_{y^2}^1 x\,[z]_0^{1-x}\,dx\,dy = \int_0^1\int_{y^2}^1 x(1-x)\,dx\,dy$$

$$= \int_0^1\int_{y^2}^1 (x - x^2)\,dx\,dy$$

$$= \int_0^1\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{y^2}^1 dy$$

$$= \int_0^1\left[\left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{y^4}{2} - \frac{y^6}{3}\right)\right]dy$$

$$= \int_0^1\left[\frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\right]dy$$

$$= \frac{1}{6}[y]_0^1 - \left[\frac{y^5}{10}\right]_0^1 + \left[\frac{y^7}{21}\right]_0^1 = \frac{1}{6} - \frac{1}{10} + \frac{1}{21}$$

$$= \frac{35 - 21 + 10}{210} = \frac{24}{210} = \frac{4}{35}$$

Que 4.4. **Evaluate $\iiint_R (x + y + z)\,dx\,dy\,dz$ where $R : 0 \le x \le 1;\ 1 \le y \le 2;\ 2 \le z \le 3$.**

AKTU 2015-16, Marks 05

AKTU 2017-18, Marks 3.5

Answer

$$\int_0^1 dx \int_1^2 dy \int_2^3 (x + y + z)\,dz = \int_0^1 dx \int_1^2 dy \left[\frac{(x + y + z)^2}{2}\right]_2^3$$

$$= \frac{1}{2}\int_0^1 dx \int_1^2 dy\,[(x + y + 3)^2 - (x + y + 2)^2] = \frac{1}{2}\int_0^1 dx \int_1^2 (2x + 2y + 5)\,dy$$

$$= \frac{1}{2}\int_0^1 dx \left[\frac{(2x + 2y + 5)^2}{4}\right]_1^2 = \frac{1}{8}\int_0^1 dx\,[(2x + 4 + 5)^2 - (2x + 2 + 5)^2]$$

$$= \frac{1}{8}\int_0^1 (4x + 16)\,2\,dx = \int_0^1 (x + 4)\,dx = \left[\frac{x^2}{2} + 4x\right]_0^1 = \frac{1}{2} + 4 = \frac{9}{2}$$

Que 4.5. Evaluate the triple integral

$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (xyz)\, dx\, dy\, dz.$

AKTU 2016-17, Marks 04

Answer

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dx\, dy\, dz$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dx\, dy \int_0^{\sqrt{1-x^2-y^2}} z\, dz = \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dx\, dy \left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dx\, dy \left[\frac{1-x^2-y^2}{2}\right] = \frac{1}{2}\int_0^1 x\, dx \int_0^{\sqrt{1-x^2}} (y - x^2 y - y^3)\, dy$$

$$= \frac{1}{2}\int_0^1 x\, dx \left[\frac{y^2}{2} - \frac{x^2 y^2}{2} - \frac{y^4}{4}\right]_0^{\sqrt{1-x^2}}$$

$$= \frac{1}{4}\int_0^1 x\, dx \left[1 - x^2 - x^2(1-x^2) - \frac{(1-x^2)^2}{2}\right]$$

$$= \frac{1}{8}\int_0^1 x\, dx\, (2 - 2x^2 - 2x^2 + 2x^4 - 1 - x^4 + 2x^2)$$

$$= \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\, dx = \frac{1}{8}\left(\frac{x^2}{2} - \frac{x^4}{2} + \frac{x^6}{6}\right)_0^1 = \frac{1}{8}\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{1}{48}$$

Que 4.6. Evaluate $\int_{z=0}^{z=4} \int_{x=0}^{x=2\sqrt{z}} \int_{y=0}^{y=\sqrt{4z-x^2}} dz\, dx\, dy$.

Answer

The integral is : $\int_{z=0}^{z=4} \int_{x=0}^{x=2\sqrt{z}} \int_{y=0}^{y=\sqrt{4z-x^2}} dy\, dx\, dz$

$$= \int_{z=0}^{z=4} \int_{x=0}^{x=2\sqrt{z}} [y]_0^{\sqrt{4z-x^2}}\, dx\, dz = \int_{z=0}^{z=4} \int_{x=0}^{x=2\sqrt{z}} \sqrt{4z-x^2}\, dx\, dz$$

$$= \int_{z=0}^{z=4} \left[\frac{1}{2}x\sqrt{4z-x^2} + \frac{1}{2}4z \sin^{-1}\left(\frac{x}{\sqrt{4z}}\right)\right]_0^{2\sqrt{z}} dz = \int_{z=0}^{z=4} 2z \sin^{-1}(1) dz$$

$$= \pi \int_0^4 z\, dz = \pi\left[\frac{z^2}{2}\right]_0^4 = 8\pi$$

Que 4.7. Change the order of integration and hence evaluate

$\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$.

AKTU 2014-15, Marks 10

AKTU 2015-16, Marks 05

OR

Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$ and hence evaluate the same.

AKTU 2013-14, Marks 05

AKTU 2016-17, Marks 03

OR

Change the order of integration in $I = \int_0^1 \int_{x^2}^{2-x} f(x, y)\, dy\, dx$.

AKTU 2017-18, Marks 3.5

Answer

As per given integral we have vertical strips but after changing the order of integration we consider horizontal strips.

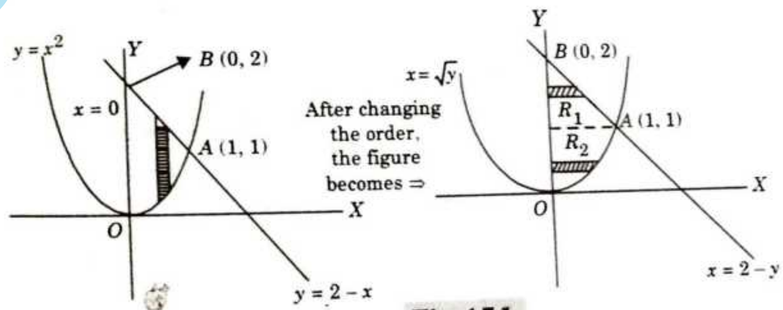


Fig. 4.7.1.

Initially limits are :

$x = 0$ to 1

$y = x^2$ to $2 - x$

After changing the order, we get

$y = 0$ to 1

$x = \sqrt{y}$ to $2 - y$

Now there are two regions R_1 and R_2. In region R_1 the limits of y are 0 to 1 and for x are 0 to $\sqrt{y}$ while in R_2 the limits of x are 0 to $2-y$ and for y are 1 to 2. Therefore

$$\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$= \int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} dy + \int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y} dy$$

$$= \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y\,(2-y)^2\,dy$$

$$= \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[2y^2 - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2 = \frac{3}{8}$$

Que 4.8. **Evaluate by changing the order of integration**

$$\int_0^1\int_{2y}^{2} e^{x^2}\,dx\,dy.$$

Answer

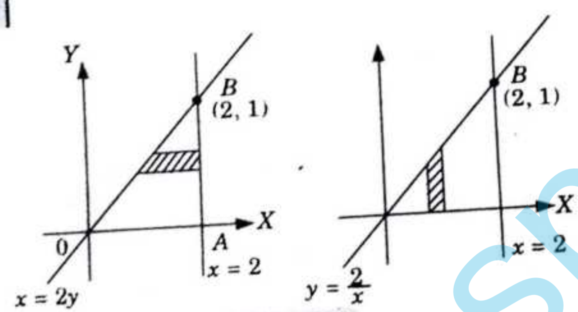


Fig. 4.8.1.

Initially limits are $x = 2y$ to 2 and $y = 0$ to 1.
After changing the order, limits become

$$y = 0 \text{ to } \frac{x}{2} \text{ and } x = 0 \text{ to } 2.$$

Therefore the integral is $\displaystyle\int_0^1\int_{2y}^{2} e^{x^2}\,dx\,dy = \int_0^2\int_0^{x/2} e^{x^2}\,dy\,dx$

$$= \int_0^2 e^{x^2}\,[y]_0^{x/2}\,dx = \frac{1}{2}\int_0^2 x\,e^{x^2}\,dx = \frac{1}{4}\left[e^{x^2}\right]_0^2$$

$$= \frac{e^4-1}{4}$$

Que 4.9. **Change the order of integration and hence evaluate**

$$\int_0^\infty\int_0^y ye^{-y^2/x}\,dxdy.$$

Answer

$$I = \int_0^\infty\int_0^y ye^{-y^2/x}\,dx\,dy$$

By changing its order of integration,

$$I = \int_0^\infty\int_x^\infty ye^{-y^2/x}\,dy\,dx$$

Let, $\dfrac{y^2}{x} = t$

$$\frac{2y}{x}\,dy = dt \Rightarrow \frac{1}{2}\int_0^\infty\int_x^\infty xe^{-t}\,dt\,dx = -\frac{1}{2}\int_0^\infty x[e^{-\infty} - e^{-x}]\,dx$$

$$= \frac{1}{2}\int_0^\infty xe^{-x}\,dx = \frac{1}{2}[-xe^{-x} - e^{-x}]_0^\infty$$

$$= \frac{1}{2}[0 - 0 + e^0] = \frac{1}{2}$$

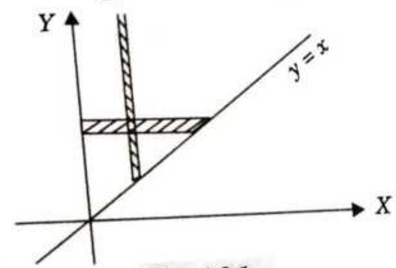


Fig. 4.9.1.

Que 4.10. **Changing the order of integration in the double integral $I = \int_0^8\int_{x/4}^2 f(x, y)\,dydx$ leads to the value $I = \int_r^s\int_p^q f(x, y)dxdy$ what is the value of q?**

AKTU 2016-17, Marks 3.5

Answer

Given : $$I = \int_0^8\int_{x/4}^2 f(x, y)\,dydx$$

Hence the shaded region is bounded as,

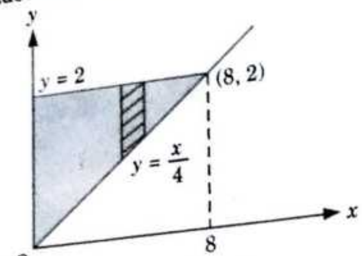


Fig. 4.10.1.

Now on changing the order of limits,

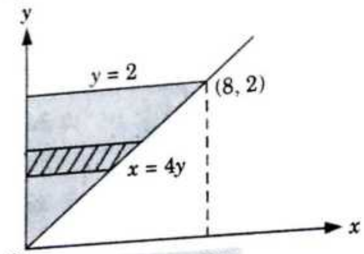


Fig. 4.10.2.

Hence integration becomes, $I = \int_0^2 \int_0^{4y} f(x, y)\, dxdy$ ...(4.10.1)

The above equation is equivalent to this equation,

$$I = \int_r^s \int_p^q f(x, y)\, dxdy \qquad ...(4.10.2)$$

On comparing eq. (4.10.1) and eq. (4.10.2), we get $q = 4y$

Que 4.11. **Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$ by changing into polar coordinates.**

Answer

Putting $x = r\cos\theta$, $y = r\sin\theta$, we get

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & +r\cos\theta \end{vmatrix} = r$$

Hence $dx\, dy$ is given by

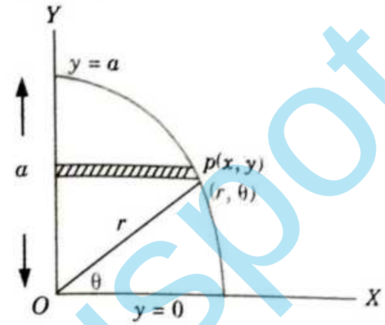


Fig. 4.11.1.

$$J d\theta\, dr = r dr d\theta$$

Also $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

By the limit of x, we find the upper limit

$$x = \sqrt{a^2 - y^2}$$

$$\Rightarrow \quad x^2 + y^2 = a^2$$

Here y varies from 0 to a and x varies from 0 to any point on the circle $x^2 + y^2 = a^2$.

Hence in polar form the circle is $r^2 = a^2$ or $r = a$. Here r varies from 0 to a and θ varies from 0 to $\frac{\pi}{2}$.

$$\Rightarrow \int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dx\, dy$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{4} r^4\right]_0^a d\theta = \frac{1}{4} a^4 [\theta]_0^{\pi/2}$$

$$= \frac{\pi}{8} a^4$$

Que 4.12. **Prove that $\iiint \frac{dx\, dy\, dz}{\sqrt{(1 - x^2 - y^2 - z^2)}} = \frac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real.**

AKTU 2013-14, 2015-16; Marks 10

Answer

$$I = \iiint \frac{dx\,dy\,dz}{\sqrt{1 - x^2 - y^2 - z^2}}$$

$\sqrt{1-x^2-y^2-z^2}$ is real only when $x^2 + y^2 + z^2 < 1$. Hence the given integral is extended for all positive values of the variables, *i.e.*, $x > 0$, $y > 0$, $z > 0$ such that $0 < x^2 + y^2 + z^2 < 1$.

Here the region of integration is bounded by

$z = 0$, $\quad z = \sqrt{1-x^2-y^2}$ $\quad$ (*i.e.*, $x^2 + y^2 + z^2 = 1$)

$y = 0$, $\quad y = \sqrt{1-x^2}$ $\quad$ (*i.e.*, $x^2 + y^2 = 1$)

$x = 0$, $\quad x = 1$

which is the volume of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant.

Changing to spherical polar co-ordinates by putting $x = r \sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ so that $x^2 + y^2 + z^2 = r^2$.

For the volume of sphere $x^2 + y^2 + z^2 = 1$ in the positive octant, r varies from 0 to 1, θ varies from 0 to $\frac{\pi}{2}$ and ϕ varies from 0 to $\frac{\pi}{2}$.

Replacing $dz\,dy\,dx$ by $r^2 \sin\theta\, dr\, d\theta\, d\phi$, we have

$$I = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{\sqrt{1-r^2}}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \frac{1-(1-r^2)}{\sqrt{1-r^2}}\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \left(\frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2}\right)\sin\theta\,dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\sin^{-1} r - \left(\frac{r\sqrt{1-r^2}}{2} + \frac{1}{2}\sin^{-1} r\right)\right]_0^1 d\theta\,d\phi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\left[\frac{\pi}{2} - \frac{1}{2}\cdot\frac{\pi}{2}\right] d\theta\,d\phi$$

$$= \int_0^{\pi/2} \frac{\pi}{4}[-\cos\theta]_0^{\pi/2}\,d\phi = \int_0^{\pi/2}\frac{\pi}{4}\,d\phi$$

$$= \frac{\pi}{4}\cdot\frac{\pi}{2} = \frac{\pi^2}{8}$$

PART-2

Areas and Volumes, Center of Mass and Center of Gravity (Constant and Variable Densities)

CONCEPT OUTLINE : PART-2

Volume by Double Integration :

i. In cartesian coordinates, $V = \iint_R z\,dx\,dy$

ii. In cylindrical coordinates, $V = \iint_P zr\,dr\,d\phi$

Volume by Triple Integration :

$$V = \iiint_V dx\,dy\,dz \quad \text{(Cartesian coordinates)}$$

$$V = \iiint_V r\,dr\,d\phi\,dz \quad \text{(Cylindrical coordinates)}$$

$$V = \iiint_V r^2\sin\theta\,dr\,d\theta\,d\phi \quad \text{(Spherical polar coordinates)}$$

Area by Double Integration :

i. **In Cartesian Coordinates :** The area A of the region bounded by the curves $y = f_1(x)$ and $y = f_2(x)$ and the lines $x = a$, $x = b$ is given by

$$A = \int_a^b \int_{f_1(x)}^{f_2(x)} dy\,dx$$

and if the region is bounded by $x = f_1(y)$, $x = f_2(y)$ and the lines $y = c$ and $y = d$, then

$$A = \int_c^d \int_{f_1(y)}^{f_2(y)} dx\,dy$$

i. **In Polar Coordinates :** Area of region bounded by the curves $r = f_1(\theta)$ and $r = f_2(\theta)$ and $\theta = \alpha$, $\theta = \beta$ is

$$A = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} r\,dr\,d\theta$$

Mass Contained in a Plane Region :

Let $f(x, y) > 0$ be the surface density (mass/unit area) of a given plane region D. Then the amount (quantity) of mass M contained in the plane region D is given by

$$M = \iint_D f(x, y)\,dx\,dy$$

Center of Gravity (Centroid) of a Plane Region D :

The coordinates (x_c, y_c) of the center of gravity (centriod) of a plane region D with surface density $f(x, y)$ and containing mass M are

$$x_c = \frac{\iint_D x f(x, y)\,dx\,dy}{M} \quad \text{and} \quad y_c = \frac{\iint_D y f(x, y)\,dx\,dy}{M}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.13. Find the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Answer

$$\text{Required volume} = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\,dy\,dx$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} 2\sqrt{a^2-x^2}\,dy\,dx = 4\int_{-a}^{a}(a^2-x^2)\,dx$$

$$= 8\int_0^a (a^2-x^2)\,dx = 8\left[a^2x - \frac{x^3}{3}\right]_0^a = \frac{16a^3}{3}.$$

Que 4.14. Find the volume of the region bounded by the surfaces $y = x^2$ and $x = y^2$ and the planes $z = 0$, $z = 3$.

Answer

$$V = \int_0^1\int_{x^2}^{\sqrt{x}}\int_0^3 dz\,dy\,dx$$

$$V = \int_0^1\int_{x^2}^{\sqrt{x}} 3\,dy\,dx = \int_0^1 3\left[y\right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 3(\sqrt{x}-x^2)\,dx = 1$$

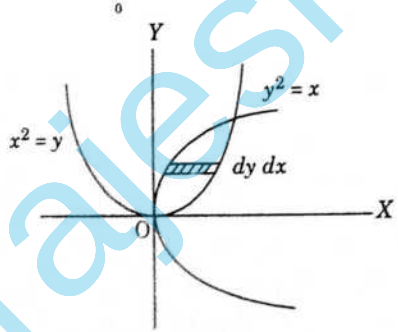


Fig. 4.14.1.

Que 4.15. Find the volume of the cylindrical column standing on the area common to the parabolas $x = y^2$, $y = x^2$ as base and cut off by the surface $z = 12 + y - x^2$.

AKTU 2013-14, Marks 05

Answer

$$V = \int_0^1 dx \int_{x^2}^{\sqrt{x}} dy \int_0^{12+y-x^2} dz = \int_0^1 dx \int_{x^2}^{\sqrt{x}} (12+y-x^2)\,dy$$

$$= \int_0^1 dx\left[12y + \frac{y^2}{2} - x^2y\right]_{x^2}^{\sqrt{x}}$$

$$= \int_0^1\left(12\sqrt{x} + \frac{x}{2} - x^{5/2} - 12x^2 - \frac{x^4}{2} + x^4\right)dx$$

$$= \left[8x^{3/2} + \frac{x^2}{4} - \frac{2}{7}x^{7/2} - 4x^3 - \frac{x^5}{10} + \frac{x^5}{5}\right]_0^1 = \frac{569}{140}$$

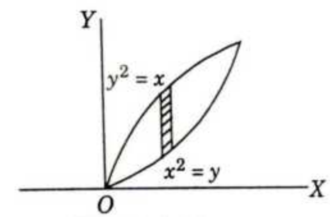


Fig. 4.15.1.

Que 4.16. Determine the area bounded by the curves $xy = 2$, $4y = x^2$ and $y = 4$.

AKTU 2014-15, Marks 10

Answer

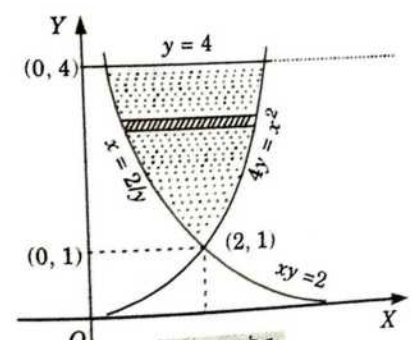


Fig. 4.16.1.

Required area of shaded region

$$= \int_{y=1}^{4}\int_{x=2/y}^{2\sqrt{y}} dx\,dy = \int_1^4\left(2\sqrt{y}-\frac{2}{y}\right)dy$$

$$= 2\left(\frac{2}{3}y^{3/2} - \log y\right)_1^4$$

$$= 2\left[\left(\frac{16}{3} - 2\log 2\right) - \frac{2}{3}\right]$$

$$= \frac{28}{3} - 4\log 2$$

Que 4.17. **Find the area of the region occupied by the curves $y^2 = x$ and $y^2 = 4 - x$.**

Answer

Given curves are $y^2 = x$, $y^2 = 4 - x$

Required area is the shaded portion,

$$A = \iint_{OABCO} dxdy$$

$$A = \int_{-\sqrt{2}}^{\sqrt{2}}\int_{y^2}^{4-y^2} dxdy = \int_{-\sqrt{2}}^{\sqrt{2}} (4 - y^2 - y^2)\,dy$$

$$= 2\int_0^{\sqrt{2}} (4 - 2y^2)\,dy$$

$$= 4\left[2y - \frac{y^3}{3}\right]_0^{\sqrt{2}} = 4\left[2\sqrt{2} - \frac{2\sqrt{2}}{3}\right]$$

$$A = \frac{16\sqrt{2}}{3} \text{ square unit}$$

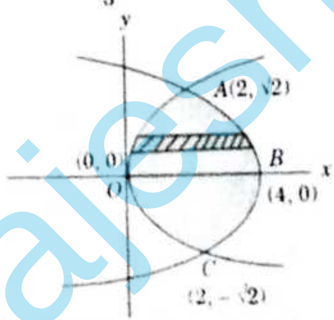


Fig. 4.17.1.

Que 4.18. **Evaluate $\iiint x^2yz\,dx\,dy\,dz$ throughout the volume bounded by planes $x = 0, y = 0, z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.**

AKTU 2016-17, Marks 3.5

Answer

The given integral can be evaluated as follows :

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x^2yz\,dxdy\,dz$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} x^2y\,dx\,dy \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} z\,dz$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} x^2y\,dx\,dy\left[c^2\frac{\left(1-\frac{x}{a}-\frac{y}{b}\right)^2}{2}\right]$$

$$= \frac{c^2}{2}\int_0^a x^2dx \int_0^{b\left(1-\frac{x}{a}\right)} \left[\left(1-\frac{x}{a}\right)^2 y + \frac{y^3}{b^2} - 2\left(1-\frac{x}{a}\right)\frac{y^2}{b}\right]dy$$

$$\frac{c^2}{2}\int_0^a x^2dx\left[\left(1-\frac{x}{a}\right)^2\frac{y^2}{2} + \frac{y^4}{4b^2} - 2\left(1-\frac{x}{a}\right)\frac{y^3}{3b}\right]_0^{b\left(1-\frac{x}{a}\right)}$$

$$= \frac{c^2}{2}\int_0^a x^2dx\left[\left(1-\frac{x}{a}\right)^2 \times b^2\frac{\left(1-\frac{x}{a}\right)^2}{2} + b^4\frac{\left(1-\frac{x}{a}\right)^4}{4b^2} - 2\left(1-\frac{x}{a}\right)\times\frac{\left(1-\frac{x}{a}\right)^3\times b^3}{3b}\right]$$

$$= \frac{c^2}{2}\int_0^a x^2dx\left[\left(1-\frac{x}{a}\right)^4\left(\frac{b^2}{2}+\frac{b^2}{4}-\frac{2}{3}b^2\right)\right]$$

$$= \frac{c^2}{2}\times\frac{b^2}{12}\int_0^a x^2\left(1-\frac{x}{a}\right)^2\left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{c^2}{2}\times\frac{b^2}{12}\int_0^a x^2\left[\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)\left(1+\frac{x^2}{a^2}-\frac{2x}{a}\right)dx\right]$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1+\frac{x^2}{a^2}-\frac{2x}{a}+\frac{x^2}{a^2}+\frac{x^4}{a^4}-\frac{2x^3}{a^3}-\frac{2x}{a}-\frac{2x^3}{a^3}+\frac{4x^2}{a^2}\right)dx$$

$$= \frac{b^2c^2}{24}\int_0^a x^2\left(1+\frac{6x^2}{a^2}-\frac{4x}{a}-\frac{4x^3}{a^3}+\frac{x^4}{a^4}\right)dx$$

$$= \frac{b^2c^2}{24}\int_0^a \left(x^2 + \frac{6x^4}{a^2} - \frac{4x^3}{a} - \frac{4x^5}{a^3} + \frac{x^6}{a^4}\right) dx$$

$$= \frac{b^2c^2}{24}\left[\frac{x^3}{3} + \frac{6x^5}{5a^2} - \frac{4x^4}{4a} - \frac{4x^6}{6a^3} + \frac{x^7}{7a^4}\right]_0^a$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} + \frac{6a^5}{5a^2} - \frac{a^4}{a} - \frac{2a^6}{3a^3} + \frac{a^7}{7a^4}\right]$$

$$= \frac{b^2c^2}{24}\left[\frac{a^3}{3} + \frac{6}{5}a^3 - a^3 - \frac{2}{3}a^3 + \frac{a^3}{7}\right]$$

$$= \frac{a^3b^2c^2}{24}\left[\frac{176-175}{105}\right] = \frac{a^3b^2c^2}{24} \times \frac{1}{105} = \frac{a^3b^2c^2}{2520}$$

Que 4.19. **Find the mass of a plate which is formed by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the density is given by $\rho = kxyz$.**

Answer

Same as Q. 4.18, Page 4–18C, Unit-4. $\left(\textbf{Ans. } \text{Mass} = \frac{ka^2b^2c^2}{720}\right)$

Que 4.20. **Find the mass and coordinates of the center of gravity relative to x-axis, y-axis and origin of a rectangle $0 \le x \le 4$, $0 \le y \le 2$ having mass density xy (see Fig. 4.20.1)**

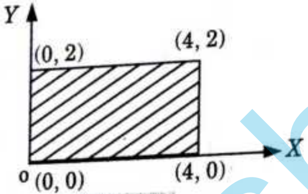


Fig. 4.20.1.

Answer

Here density $f(x, y) = xy$

Mass, $M = \iint_R f(x, y)\, dx\, dy$

$$M = \int_0^4\int_0^2 (xy)\, dy\, dx = \int_0^4 \left[\frac{xy^2}{2}\right]_0^2 dx$$

$$= \int_0^4 2x\, dx = 16$$

Let x_c, y_c be the coordinates of the center of gravity of R, then,

$$x_c = \frac{1}{M}\iint_R x\, f(x, y)\, dx\, dy = \frac{1}{16}\int_0^4\int_0^2 x(xy)\, dy\, dx$$

$$x_c = \frac{1}{16}\int_0^4 x^2\left[\frac{y^2}{2}\right]_0^2 dx = \frac{1}{8}\int_0^4 x^2\, dx = \frac{8}{3}$$

and,

$$y_c = \frac{1}{M}\iint_R y f(x, y)\, dx\, dy = \frac{1}{16}\int_0^4\int_0^2 y(xy)\, dy\, dx$$

$$= \frac{1}{16}\int_0^4 x\left[\frac{y^3}{3}\right]_0^2 dx = \frac{1}{6}\int_0^4 x\, dx = \frac{4}{3}$$

Que 4.21. **Find the mass, centroid of the tetrahedron bounded by the co-ordinates planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.**

Answer

Let ρ be the constant density of the substance (mass/unit volume).

Mass of is the tetrahedron, $M = \iiint_V \rho\, dx\, dy\, dz$

$$M = \rho\int_{x=0}^{a}\int_{y=0}^{b\left(1-\frac{x}{a}\right)}\int_{z=0}^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dz\, dy\, dx$$

$$= \rho\int_0^a\int_0^{b\left(1-\frac{x}{a}\right)} c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dy\, dz$$

$$= \rho c\int_0^a\left[\left(1 - \frac{x}{a}\right)y - \frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= \frac{cb\rho}{2}\int_0^a\left(1 - \frac{x}{a}\right)^2 dx = \frac{\rho bc}{2}\,\frac{a}{3} = \frac{\rho abc}{6}$$

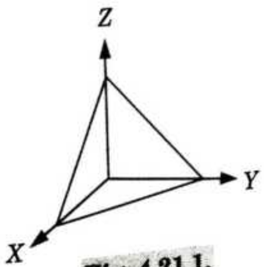


Fig. 4.21.1.

Let (x_c, y_c, z_c) be the coordinates of the centroid.

Then,

$$Mx_c = \rho \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} x \, dz \, dy \, dx$$

$$= \rho \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} cx\left(1-\frac{x}{a}-\frac{y}{b}\right) dy \, dx$$

$$= c\rho \int_0^a \left[x\left(1-\frac{x}{a}\right) y - \frac{xy^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$Mx_c = c\rho b \int_0^a x\left(1-\frac{x}{a}\right)^2 dx = \rho bc \frac{a^2}{12}$$

Therefore, $$x_c = \frac{\rho a^2 bc}{12} \cdot \frac{6}{\rho abc} = \frac{a}{2}$$

Similarly, $$y_c = \frac{b}{2} \text{ and } z_c = \frac{c}{2}$$

☺☺☺

5

UNIT

Vector Calculus

PART-1

Gradient, Curl and Divergence and their Physical Interpretation, Directional Derivatives, Tangent and Normal Planes.

CONCEPT OUTLINE : PART-1

Vectors : A vector in the plane is a directed line segment. Two vectors are equal or the same if they have the same length and direction.

A vector is denoted by a or $\vec{a}$.

Scalars and Scalar Multiples : If C is a non zero real number and v is a vector, the direction of Cv agrees with that of v if C is positive and is opposite to that of v if C is negative. Real numbers are called scalars and multiples like Cv scalar multiples of v.

Components : If $v = a\hat{i} + b\hat{j}$, the vectors $a\hat{i}$ and $b\hat{j}$ are the vector components of v in the direction of $\hat{i}$ and $\hat{j}$. The numbers a and b are the scalar components of v in the direction of $\hat{i}$ and $\hat{j}$.

Equal Vectors : Two vectors are equal if

$$a\hat{i} + b\hat{j} = a'\hat{i} + b'\hat{j} \Leftrightarrow a = a' \text{ and } b = b'$$

Magnitude of a Vector : The magnitude or length of $v = a\hat{i} + b\hat{j}$ is

$|v| = \sqrt{a^2 + b^2}$

Operation on Vectors :

If $v_1 = a_1\hat{i} + b_1\hat{j}$ and $v_2 = a_2\hat{i} + b_2\hat{j}$ then

$v_1 + v_2 = (a_1 + a_2)\hat{i} + (b_1 + b_2)\hat{j}$, and

$v_1 - v_2 = (a_1 - a_2)\hat{i} + (b_1 - b_2)\hat{j}$

Unit Vector : Any vector whose length is 1 is a unit vector. The vectors i and j are unit vectors.

$$|\hat{i}| = |1\hat{i} + 0\hat{j}| = \sqrt{1^2 + 0^2} = 1$$

$$|\hat{j}| = |0\hat{i} + 1\hat{j}| = \sqrt{0^2 + 1^2} = 1$$

Derivatives of a Vector Function : If $\vec{f}(t)$ be a vector function of the scalar variable t and Δt be a small increment in t then

$$\frac{d\vec{f}(t)}{dt} = \lim_{\Delta t \to 0} \frac{\Delta f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} \text{ exists,}$$

The value of this limit is denoted by $\frac{d\vec{f}(t)}{dt}$ and is called the derivative of $\vec{f}(t)$ with respect to t.

Rules for Differentiation : If a, b and c are differentiable vector functions of a scalar t and ϕ is a differentiable scalar function of t, then

1. $\frac{d}{dt}(a \cdot b \times c) = a \cdot b \times \frac{dc}{dt} + a \cdot \frac{db}{dt} \times c + \frac{da}{dt} \cdot b \times c$
2. $\frac{d}{dt}[a \times b \times c] = a \times \left(b \times \frac{dc}{dt}\right) + a \times \left(\frac{db}{dt} \times c\right) + \frac{da}{dt} \times (b \times c)$

Scalars and Vectors Field :

A variable quantity whose value at a point in a space depends upon the position of the point is called a point function.

There are two types of point function :

i. **Scalar Point Function :** If the position of the point is given by a scalar quantity *e.g.*, $\phi(x, y, z)$ then ϕ is called scalar point function.

ii. **Vector Point Function :** If the position of the point is given by a vector *e.g.*, $\vec{v}(x, y, z)$ then $\vec{v}$ is called vector point function.

Gradient of a Scalar Field : If ϕ be a scalar field then gradient is given as :

$$\text{grad } \phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

or $\quad \text{grad } \phi = \nabla\phi$

Properties of Gradient :

i. If ϕ is a constant, then $\nabla\phi = 0$.

ii. If ϕ_1 and ϕ_2 are two scalar point functions, then $\nabla(\phi_1 \pm \phi_2) = \nabla\phi_1 \pm \nabla\phi_2$.

iii. $\nabla(C_1\phi_1 + C_2\phi_2) = C_1\nabla\phi_1 + C_2\nabla\phi_2$ where, C_1 and C_2 are constants.

iv. $\nabla(\phi_1\phi_2) = \phi_1\nabla\phi_2 + \phi_2\nabla\phi_1$

v. $\nabla\left(\frac{\phi_1}{\phi_2}\right) = \frac{\phi_2\nabla\phi_1 - \phi_1\nabla\phi_2}{\phi_2^2}$, $\phi_2 \neq 0$

Divergence of a Vector Point Function : If $\vec{v}$ be a vector point function, then

$$\text{div}\,\vec{v} = \nabla.\ \vec{v} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).\vec{v}$$

$$= \hat{i}\,.\frac{\partial \vec{v}}{\partial x}+\hat{j}\,.\frac{\partial \vec{v}}{\partial y}+\hat{k}\,.\frac{\partial \vec{v}}{\partial z}$$

Curl of a Vector Point Function : The curl of a vector point function $\vec{v}$ is given as :

$$\text{curl}\ \vec{v} = \nabla\times\vec{v} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\times\vec{v}$$

$$= \hat{i}\times\frac{\partial \vec{v}}{\partial x}+\hat{j}\times\frac{\partial \vec{v}}{\partial y}+\hat{k}\times\frac{\partial \vec{v}}{\partial z}$$

or if $\vec{v} = v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ then

$$\text{curl}\ \vec{v} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ v_1 & v_2 & v_3\end{vmatrix}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. **If $f = x^2z\,\hat{i}-2y^3z^2\,\hat{j}+xy^2z\,\hat{k}$ find div f and curl f at (1, – 1, 1).**

Answer

i.
$$\text{div}\, f = \nabla.f$$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right).(x^2z\,\hat{i}-2y^3z^2\,\hat{j}+xy^2z\,\hat{k})$$

$$= \frac{\partial}{\partial x}(x^2z)-\frac{\partial}{\partial y}(2y^3z^2)+\frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2$$

div f at (1, –1, 1) = 2 – 6 + 1 = – 3

ii.
$$\text{curl}\, f = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}$$

$$\text{curl}\, f = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2z & -2y^3z^2 & xy^2z\end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(xy^2z)-\frac{\partial}{\partial z}(-2y^3z^2)\right\}$$

$$-j\left\{\frac{\partial}{\partial x}(xy^2z)-\frac{\partial}{\partial z}(x^2z)\right\}+\hat{k}\left\{\frac{\partial}{\partial x}(-2y^3z^2)-\frac{\partial}{\partial y}(x^2z)\right\}$$

$$= \hat{i}\,(2xyz+4y^3z)-\hat{j}\,(y^2z-x^2)+\hat{k}\,(0-0)$$

$$= \hat{i}\left(2xyz+4y^3z\right)-\hat{j}\left(y^2z-x^2\right)$$

At (1, –1, 1), $\quad \text{curl}\, f = \hat{i}\,(-2-4)-\hat{j}\,(1-1) = -6\,\hat{i}$

Que 5.2. **If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$. Prove that grad u, grad v and grad w are coplanar.**

AKTU 2014-15, Marks 10

Answer

$$\text{grad}\, u = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z)$$

$$= \hat{i}+\hat{j}+\hat{k}$$

$$\text{grad}\, v = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= 2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}$$

$$\text{grad}\, w = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(xy+yz+zx)$$

$$= (y+z)\,\hat{i}+(z+x)\,\hat{j}+(x+y)\,\hat{k}$$

For vectors to be coplanar, their scalar triple product is zero.

grad u. (grad v × grad w)

$$= \begin{vmatrix}1 & 1 & 1\\ 2x & 2y & 2z\\ z+y & z+x & y+x\end{vmatrix}$$

$$= 2\begin{vmatrix}1 & 1 & 1\\ x & y & z\\ z+y & z+x & y+x\end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ z+y & z+x & y+x \end{vmatrix} R_2 \to R_2 + R_3$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ (z+y) & (z+x) & (y+x) \end{vmatrix} \quad [\because R_1 = R_2]$$

grad u. (grad $v \times$ grad w) = 0

Que 5.3. **Prove that div. (curl $\vec{v}$) = $\nabla . (\nabla \times v) = 0$**

OR

Prove that, for every field $\bar{V}$; div curl $\bar{V}$ = 0.

AKTU 2015-16, Marks 05

Answer

Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

then $$\text{curl } v = \nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right\} - \hat{j}\left\{\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right\} + \hat{k}\left\{\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right\}$$

$$\therefore \text{div. (curl } \vec{v}) = \nabla.(\nabla \times \vec{v}) = \frac{\partial}{\partial x}\left\{\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right\} + \frac{\partial}{\partial y}\left\{\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right\} + \frac{\partial}{\partial z}\left\{\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right\}$$

$$= \left(\frac{\partial^2 v_3}{\partial x \partial y} - \frac{\partial^2 v_2}{\partial x \partial z}\right) + \left(\frac{\partial^2 v_1}{\partial y \partial z} - \frac{\partial^2 v_3}{\partial y \partial x}\right) + \left(\frac{\partial^2 v_2}{\partial z \partial x} - \frac{\partial^2 v_1}{\partial z \partial y}\right) = 0$$

Que 5.4. **Prove that div (grad r^n) = $\nabla^2 (r^n) = n(n+1)r^{n-2}$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. Hence show that $\nabla^2\left(\frac{1}{r}\right) = 0$. Hence or otherwise evaluate $\nabla \times \frac{\vec{r}}{r^2}$.**

Answer

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$r^2 = x^2 + y^2 + z^2$$

$$\text{grad } r^n = \hat{i}\frac{\partial}{\partial x}r^n + \hat{j}\frac{\partial}{\partial y}r^n + \hat{k}\frac{\partial}{\partial z}r^n$$

$$= \hat{i}\left(nr^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(nr^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(nr^{n-1}\frac{\partial r}{\partial z}\right) \quad ...(5.4.1)$$

$\because$ $r^2 = x^2 + y^2 + z^2$

So, $2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}$

So, eq. (5.4.1) becomes,

$$\text{grad } r^n = \hat{i}\left(nr^{n-1}\frac{x}{r}\right) + \hat{j}\left(nr^{n-1}\frac{y}{r}\right) + \hat{k}\left(nr^{n-1}\frac{z}{r}\right)$$

$$= nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k}) = nr^{n-2}\vec{r}$$

$$\text{div (grad } r^n) = \nabla.\{nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})\}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).nr^{n-2}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$= \frac{\partial}{\partial x}nr^{n-2}x + \frac{\partial}{\partial y}nr^{n-2}y + \frac{\partial}{\partial z}nr^{n-2}z$$

$$= xn(n-2)r^{n-3}\frac{\partial r}{\partial x} + nr^{n-2} + yn(n-2)r^{n-3}\frac{\partial r}{\partial y} + nr^{n-2} + zn(n-2)r^{n-3}\frac{\partial r}{\partial y} + nr^{n-2}$$

$$= 3nr^{n-2} + xn(n-2)r^{n-3}\frac{x}{r} + yn(n-2)r^{n-3}\frac{y}{r} + zn(n-2)r^{n-3}\frac{z}{r}$$
$$= 3nr^{n-2} + n(n-2)r^{n-4}x^2 + n(n-2)r^{n-4}y^2 + n(n-2)r^{n-4}z^2$$
$$= 3nr^{n-2} + n(n-2)r^{n-4}(x^2 + y^2 + z^2)$$
$$= 3nr^{n-2} + n(n-2)r^{n-2} = (3 + n - 2)nr^{n-2} = n(n+1)r^{n-2}$$

Put $n = -1$, we get $\nabla^2\left(\frac{1}{r}\right) = 0$

We know that curl $(u\vec{a}) = u$ curl $\vec{a}$ + (grad u) $\times \vec{a}$

$$\text{curl}\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}\text{curl } \vec{r} + \left(\text{grad } \frac{1}{r^2}\right) \times \vec{r}$$

$$= 0 - \left(\frac{2}{r^3}\hat{r}\right) \times \vec{r} \quad \{\text{curl } \vec{r} = 0\}$$

$$= -\frac{2}{r^4}(\vec{r} \times \vec{r}) = \vec{0} = 0$$

Que 5.5. Interpret the physical meaning of curl $\vec{F}$ and div $\vec{F}$.

Answer

A. Physical Interpretation of Curl $\vec{F}$:

Consider a rigid body rotating about a fixed axis with uniform angular velocity

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

The vector $\vec{v}$ of any point $P(x, y, z)$ on the body is given by $\vec{v} = \vec{\omega} \times \vec{r}$,

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is the position vector of P.

$$\therefore \quad \vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

$$\text{curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \hat{i}(\omega_1 + \omega_1) - \hat{j}(-\omega_2 - \omega_2) + \hat{k}(\omega_3 + \omega_3)$$

$$\text{curl } \vec{v} = 2\omega$$

$$\vec{\omega} = \frac{1}{2} \text{curl } \vec{v}$$

The angular velocity at any point is equal to half the curl of the linear velocity at that point of the body.

B. Physical Interpretation of Divergence $\vec{F}$:

Let us consider the case of a fluid flow. Consider a small rectangular parallelopiped of dimensions dx, dy, dz parallel to X, Y and Z-axes respectively.

Let $\vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ be the velocity of the fluid at $P(x, y, z)$.

$\therefore$ Mass of fluid flowing in through the face $ABCD$ per unit time $=$ Velocity $\times$ Area of the face

$$= v_x (dy\,dz)$$

Mass of fluid flowing out across the face $PQRS$ per unit time

$$= \left(v_x + \frac{\partial v_x}{\partial x} dx\right)(dy\,dz)$$

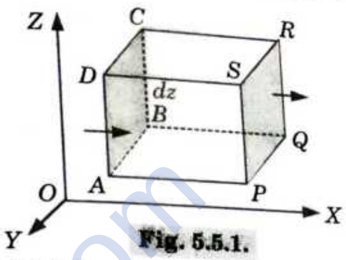


Fig. 5.5.1.

Change in mass of fluid to the flow along X-axis

$$= v_x\,dy\,dz - \left(v_x + \frac{\partial v_x}{\partial x} dx\right) dy\,dz$$

$$= -\frac{\partial v_x}{\partial x} dx\,dy\,dz \quad \text{(Minus sign shows decrease)}$$

Similarly, the decrease in mass of fluid to the flow along Y-axis

$$= \frac{\partial v_y}{\partial y} dx\,dy\,dz$$

and the decrease in mass of fluid to the flow along Z-axis

$$= \frac{\partial v_z}{\partial z} dx\,dy\,dz$$

Total decrease of the amount of fluid per unit time

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) dx\,dy\,dz$$

Thus the rate of loss of fluid per unit volume $= \dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . (\hat{i} v_x + \hat{j} v_y + \hat{k} v_z)$$

$$= \vec{\nabla} . \vec{v} = \text{div } \vec{v}$$

If the fluid is compressible, there can be no gain or loss in the volume element. Hence

$$\text{div } \vec{v} = 0$$

and v is called a solenoidal vector function.

Que 5.6. **Prove that :**

curl $(\vec{F} \times \vec{G}) = \vec{F}$ div $\vec{G}$ − $\vec{G}$ div $\vec{F}$ + $(\vec{G} \cdot \nabla)\vec{F}$ − $(\vec{F} \cdot \nabla)\vec{G}$.

Answer

$$\text{curl } (\vec{F} \times \vec{G}) = \sum i \times \frac{\partial}{\partial x} (\vec{F} \times \vec{G})$$

$$= \sum \hat{i} \times \left(\frac{\partial \vec{F}}{\partial x} \times \vec{G} + \vec{F} \times \frac{\partial \vec{G}}{\partial x} \right)$$

$$= \sum \hat{i} \times \left(\frac{\partial \vec{F}}{\partial x} \times \vec{G} \right) + \sum \hat{i} \times \left(\vec{F} \times \frac{\partial \vec{G}}{\partial x} \right)$$

$$= \sum \left[(\hat{i} \cdot \vec{G}) \frac{\partial \vec{F}}{\partial x} - \left(\hat{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) \vec{G} \right] + \sum \left[\left(\hat{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) \vec{F} - (\hat{i} \cdot \vec{F}) \frac{\partial \vec{G}}{\partial x} \right]$$

$$= \sum (\vec{G} \cdot \hat{i}) \frac{\partial \vec{F}}{\partial x} - \vec{G} \sum \left(\hat{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) + \vec{F} \sum \left(\hat{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) - \sum (\vec{F} \cdot \hat{i}) \frac{\partial \vec{G}}{\partial x}$$

$$= \vec{F} \left(\sum \hat{i} \cdot \frac{\partial \vec{G}}{\partial x} \right) - \vec{G} \sum \left(\hat{i} \cdot \frac{\partial \vec{F}}{\partial x} \right) + \sum (\vec{G} \cdot \hat{i}) \frac{\partial \vec{F}}{\partial x} - \sum (\vec{F} \cdot \hat{i}) \frac{\partial \vec{G}}{\partial x}$$

$$\text{curl}(\vec{F} \times \vec{G}) = \vec{F} \,\text{div}\, \vec{G} - \vec{G} \,\text{div}\, \vec{F} + (\vec{G} \cdot \nabla)\vec{F} - (\vec{F} \cdot \nabla)\vec{G}$$

Hence proved.

Que 5.7. **Define curl of a vector. Prove the following vector identity :**

div $(\vec{u} \times \vec{v}) = \vec{v}$. curl $\vec{u} - \vec{u}$. curl $\vec{v}$

Answer

A. Curl of a Vector :

Let a vector $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Curl of a vector $\vec{F}$ is given by

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$\text{curl } \vec{F} = \hat{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \hat{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \hat{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

B. Proof :

$$\text{div}(\vec{u} \times \vec{v}) = \sum \hat{i} \cdot \frac{\partial}{\partial x}(\vec{u} \times \vec{v}) = \sum \hat{i} \cdot \left(\frac{\partial \vec{u}}{\partial x} \times \vec{v} + \vec{u} \times \frac{\partial \vec{v}}{\partial x} \right)$$

$$= \sum \hat{i} \cdot \left(\frac{\partial \vec{u}}{\partial x} \times \vec{v} \right) - \sum \hat{i} \cdot \left(\frac{\partial \vec{v}}{\partial x} \times \vec{u} \right)$$

$$= \sum \left(\hat{i} \times \frac{\partial \vec{u}}{\partial x} \right) \cdot \vec{v} - \sum \left(\hat{i} \times \frac{\partial \vec{v}}{\partial x} \right) \cdot \vec{u}$$

$$(\because \vec{u} \cdot (\vec{v} \times \vec{c}) = (\vec{u} \times \vec{v}) \cdot \vec{c})$$

$$\text{div}(\vec{u} \times \vec{v}) = (\text{curl } \vec{u}) \cdot \vec{v} - (\text{curl } \vec{v}) \cdot \vec{u} = \vec{v} \cdot (\text{curl } \vec{u}) - \vec{u} \cdot (\text{curl } \vec{v})$$

Que 5.8. **If all second order derivatives of ϕ and $\vec{V}$ are continuous, then show that (i) curl (grad ϕ) = 0 (ii) div(curl $\vec{V}$) = 0.**

AKTU 2017-18, Marks 3.5

Answer

i.
$$\text{grad } \phi = \nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}$$

$$\text{curl (grad } \phi) = \nabla \times (\nabla\phi)$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \times \left(\hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} \right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial^2 \phi}{\partial y\, \partial z} - \frac{\partial^2 \phi}{\partial z\, \partial y} \right) - \hat{j}\left(\frac{\partial^2 \phi}{\partial x\, \partial z} - \frac{\partial^2 \phi}{\partial z\, \partial x} \right) + \hat{k}\left(\frac{\partial^2 \phi}{\partial x\, \partial y} - \frac{\partial^2 \phi}{\partial y\, \partial x} \right) = \vec{0}.$$

ii. Let
$$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$

$$\text{curl } \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right)$$

$$\text{div (curl } \vec{V}) = \nabla \cdot (\nabla \times \vec{V})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) \cdot \left[\hat{i}\left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) + \hat{j}\left(\frac{\partial V_1}{\partial z} - \frac{\partial V_3}{\partial x} \right) + \hat{k}\left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \right]$$

$$= \frac{\partial^2 V_3}{\partial x \partial y} - \frac{\partial^2 V_2}{\partial x \partial z} + \frac{\partial^2 V_1}{\partial y \partial z} - \frac{\partial^2 V_3}{\partial y \partial x} + \frac{\partial^2 V_2}{\partial z \partial x} - \frac{\partial^2 V_1}{\partial z \partial y}$$

$\nabla \cdot (\nabla \times \vec{V}) = 0$ Hence Proved.

Que 5.9. **Find the directional derivative of $\phi_1(x, y, z) = xy^2 + yz^3$ at the point (2, –1, 1) in the direction of the normal to the surface $x \log z - y^2 + 4 = 0$ at (2, –1, 1).**

Answer

$$\phi_1(x, y, z) = xy^2 + yz^3$$

$$\nabla\phi_1 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (xy^2 + yz^3)$$

$$= y^2\hat{i} + (2xy + z^3)\hat{j} + 3yz^2\hat{k}$$

$$(\nabla\phi_1)_{(2,-1,1)} = \hat{i} + (-3)\hat{j} - 3\hat{k}$$

$$\nabla\phi_1 = \hat{i} - 3\hat{j} - 3\hat{k}$$

Surface $\phi_2 = x \log z - y^2 + 4$

Normal to surface,

$$\nabla\phi_2 = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot (x \log z - y^2 + 4)$$

$$= \hat{i}\log z - 2y\hat{j} + \frac{x}{z}\hat{k}$$

$$(\nabla\phi_2)_{(2,-1,1)} = 2\hat{j} + 2\hat{k}$$

Unit vector normal to surface,

$$\hat{a} = \frac{\nabla\phi_2}{|\nabla\phi_2|}$$

$$\hat{a} = \frac{2\hat{j} + 2\hat{k}}{\sqrt{4+4}} = \frac{\hat{j} + \hat{k}}{\sqrt{2}}$$

Directional derivative,

$$= \nabla\phi_1 \cdot \hat{a} = (\hat{i} - 3\hat{j} - 3\hat{k}) \cdot \left(\frac{\hat{j} + \hat{k}}{\sqrt{2}}\right)$$

$$= \frac{-3-3}{\sqrt{2}} = -3\sqrt{2}$$

Que 5.10. **Find the directional derivative of $\left(\frac{1}{r^2}\right)$ in the direction of $\vec{r}$ where $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$.**

AKTU 2016-17, Marks 04

Answer

$$\nabla\left(\frac{1}{r^2}\right) = \frac{2}{r^3}\hat{r} = -\frac{2}{r^4}\cdot\vec{r} \qquad \left(\because \hat{r} = \frac{\vec{r}}{|\vec{r}|}\right)$$

Let $\hat{a}$ be the unit vector in the direction of $\vec{r}$ then $\hat{a} = \hat{r} = \frac{\vec{r}}{r}$

$\therefore$ Directional derivative $= \nabla\left(\frac{1}{r^2}\right)\cdot\hat{a} = -\frac{2}{r^4}\vec{r}\cdot\frac{\vec{r}}{r} = \frac{-2}{r^5}(r^2) = -\frac{2}{r^3}$

Que 5.11. **Find the directional derivative of $f(x, y, z) = x^2 + y^2 + z^2$ at the point (1, 2, 3) in the direction of $3\hat{i} + 4\hat{j} + 10\hat{k}$.**

Answer

$$f = x^2 + y^2 + z^2$$

Directional derivative in the given direction is $\hat{a}$. grad f.

$$\text{grad } f = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(x^2 + y^2 + z^2) + \hat{j}\frac{\partial}{\partial y}(x^2 + y^2 + z^2) + \hat{k}\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$$

The unit vector in the given direction $3\hat{i} + 4\hat{j} + 10\hat{k}$ is

$$\hat{a} = \frac{3\hat{i} + 4\hat{j} + 10\hat{k}}{\sqrt{9 + 16 + 100}} = \frac{3\hat{i} + 4\hat{j} + 10\hat{k}}{5\sqrt{5}}$$

$$\therefore \quad \hat{a} \cdot \text{grad } f = \frac{1}{5\sqrt{5}}(3\hat{i} + 4\hat{j} + 10\hat{k}) \cdot (2x\hat{i} + 2y\hat{j} + 2z\hat{k})$$

At point (1, 2, 3),

$$\hat{a} \cdot \text{grad } f = \frac{2}{5\sqrt{5}}(3 \times 1 + 4 \times 2 + 10 \times 3) = \frac{2 \times 41}{5\sqrt{5}} = \frac{82\sqrt{5}}{25}$$

Que 5.12. **If a vector field is given by $\vec{F} = (x^2 - y^2 + x)\hat{i} - (2xy + y)\hat{j}$. Is this field irrotational ? If so, find its scalar potential.**

AKTU 2013-14, Marks 05

Answer

$$\vec{F} = (x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}$$

For the irrotational vector field, curl $\vec{F} = 0$

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (x^2 - y^2 + x) & (-2xy - y) & 0 \end{vmatrix}$$

$$= \hat{i}\left[0 - \frac{\partial}{\partial z}(-2xy - y)\right] - \hat{j}\left[0 - \frac{\partial}{\partial z}(x^2 - y^2 + x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(-2xy - y) - \frac{\partial}{\partial y}(x^2 - y^2 + x)\right]$$

$$= 0 - 0 + \hat{k}\,(-2y + 2y)$$

$$V = \nabla \times \vec{F} = \vec{0}$$

Hence the field is irrotational.

Scalar Potential : We know that $\vec{F} = \text{grad}\ \phi$

$$\vec{F}\cdot d\vec{r} = \nabla\phi \cdot d\vec{r}$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$$

$$d\phi = \vec{F}\cdot d\vec{r}$$

$$d\phi = [(x^2 - y^2 + x)\,\hat{i} - (2xy + y)\,\hat{j}]\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= (x^2 - y^2 + x)\,dx - (2xy + y)dy$$

$$= (x^2 + x)\,dx - (y^2\,dx + 2xy\,dy) - y\,dy$$

$$= (x^2 + x)dx - d(x\,y^2) - y\,dy$$

or

$$\phi = \frac{x^3}{3} + \frac{x^2}{2} - xy^2 - \frac{y^2}{2} + c$$

Que 5.13. **A fluid motion is given by**

$\vec{v} = (y + z)\hat{i} + (z + x)\,\hat{j} + (x + y)\,\hat{k}$. Show that the motion is irrotational and hence find the velocity potential.

AKTU 2015-16, Marks 05

Answer

$$\text{curl } \vec{v} = \nabla \times \vec{v}$$

$$= \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y + z & z + x & x + y \end{vmatrix}$$

$$= (1 - 1)\,\hat{i} - (1 - 1)\,\hat{j} + (1 - 1)\,\hat{k} = \vec{0}$$

Hence, $\vec{v}$ is irrotational.

To find the corresponding velocity potential ϕ, consider the following relation.

$$\vec{v} = \nabla\phi$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$

$$= \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= \left(\frac{\hat{i}\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi.d\vec{r} = \nabla\phi.d\vec{r} = \vec{v}.d\vec{r}$$

$$= [(y + z)\,\hat{i} + (z + x)\,\hat{j} + (x + y)\,\hat{k}].(\hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz)$$

$$= (y + z)dx + (z + x)dy + (x + y)dz$$

$$= y\,dx + z\,dx + z\,dy + x\,dy + x\,dz + y\,dz$$

$$\phi = \int(y\,dx + x\,dy) + \int(z\,dy + y\,dz) + \int(z\,dx + x\,dz)$$

$$\phi = xy + yz + zx + c$$

Velocity potential $= xy + yz + zx + c$

Que 5.14. **Determine the value of constants a, b, c if, $\vec{F} = (x + 2y + az)\,\hat{i} + (bx - 3y - z)\,\hat{j} + (4x + cy + 2z)\,\hat{k}$ is irrotational.**

AKTU 2017-18, Marks 3.5

Answer

$$\vec{F} = (x + 2y + az)\,\hat{i} + (bx - 3y - z)\,\hat{j} + (4x + cy + 2z)\,\hat{k}$$

It is given that $\vec{F}$ is irrotational *i.e.*, curl $\vec{F} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} = 0$$

$$\left[\frac{\partial}{\partial y}(4x+cy+2z)-\frac{\partial}{\partial z}(bx-3y-z)\right]\hat{i}$$
$$-\left[\frac{\partial}{\partial x}(4x+cy+2z)-\frac{\partial}{\partial z}(x+2y+az)\right]\hat{j}$$
$$+\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]\hat{k}=0$$

$$(c+1)\hat{i}-(4-a)\hat{j}+(b-2)\hat{k}=0$$

Now, $c+1=0 \Rightarrow c=-1$

$4-a=0 \Rightarrow a=4$

$b-2=0 \Rightarrow b=2$

Que 5.15. **Find the unit tangent vector at any point on the curve $x=t^2+2, y=4t-5, z=2t^2-6t$, where t is any variable. Also determine the unit tangent vector at the point $t=2$.**

Answer

If $\vec{r}$ is a position vector for point (x, y, z) on the given curve, then

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}=(t^2+2)\hat{i}+(4t-5)\hat{j}+(2t^2-6t)\hat{k}$$

The vector $\frac{d\vec{r}}{dt}$ is along the tangent at (x, y, z) to the given curve.

$$\frac{d\vec{r}}{dt}=2t\,\hat{i}+4\,\hat{j}+(4t-6)\,\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right|=\sqrt{(2t)^2+4^2+(4t-6)^2}$$

$$=\sqrt{20t^2-48t+52}=2\sqrt{5t^2-12t+13}$$

$\therefore$ The unit tangent vector,

$$\hat{r}=\frac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|}=\frac{t\,\hat{i}+2\hat{j}+(2t-3)\,\hat{k}}{\sqrt{5t^2-12t+13}}$$

Unit tangent vector at $t=2$ is,

$$\frac{2\,\hat{i}+2\hat{j}+(2\times2-3)\,\hat{k}}{\sqrt{5\times4-12\times2+13}}=\frac{1}{3}(2\hat{i}+2\hat{j}+\hat{k})$$

Que 5.16. **A particle moves on the curve $x=2t^2, y=t^2-4t, z=3t-5$, where t is the time. Find the components of velocity and acceleration at time $t=1$ in the direction $\hat{i}-3\hat{j}+2\hat{k}$.**

Answer

If $\vec{r}$ is the position vector for a point (x, y, z) on the curve, then

$$\vec{r}=x\,\hat{i}+y\,\hat{j}+z\hat{k}=2t^2\,\hat{i}+(t^2-4t)\,\hat{j}+(3t-5)\,\hat{k}$$

Velocity, $$\frac{d\vec{r}}{dt}=4t\,\hat{i}+(2t-4)\,\hat{j}+3\hat{k}$$

$$\left(\frac{d\vec{r}}{dt}\right)_{t=1}=4\,\hat{i}-2\,\hat{j}+3\hat{k}$$

Acceleration, $$\frac{d^2\vec{r}}{dt^2}=4\,\hat{i}+2\,\hat{j}$$

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{t=1}=4\,\hat{i}+2\,\hat{j}$$

The unit vector in the given direction $\hat{i}-3\,\hat{j}+2\hat{k}$

$$=\frac{\hat{i}-3\,\hat{j}+2\hat{k}}{\sqrt{1^2+3^2+2^2}}=\frac{\hat{i}-3\hat{j}+2\hat{k}}{\sqrt{14}}=\hat{n}$$

The component of velocity in the given direction

$$\frac{d\vec{r}}{dt}\cdot\hat{n}=(4\hat{i}-2\,\hat{j}+3\hat{k})\cdot\frac{\hat{i}-3\,\hat{j}+2\hat{k}}{\sqrt{14}}$$

$$=\frac{(4+6+6)}{\sqrt{14}}=\frac{16\sqrt{14}}{14}=\frac{8\sqrt{14}}{7}$$

The component of acceleration in the given direction is

$$\frac{d^2\vec{r}}{dt^2}\cdot\hat{n}=(4\hat{i}+2\,\hat{j})\cdot\frac{\hat{i}-3\,\hat{j}+2\hat{k}}{\sqrt{14}}=\frac{4-6}{\sqrt{14}}=\frac{-2}{\sqrt{14}}=\frac{-\sqrt{14}}{7}$$

Que 5.17. **If $\vec{A}=(xz^2\hat{i}+2y\hat{j}-3xz\hat{k})$ and $\vec{B}=(3xz\hat{i}+2yz\hat{j}-z^2\hat{k})$. Find the value of $[\vec{A}\times(\nabla\times\vec{B})]$ and $[(\vec{A}\times\nabla)\times\vec{B}]$.**

AKTU 2016-17, Marks 3.5

Answer

i. $[\vec{A}\times(\nabla\times\vec{B})]$

$$\nabla \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 3xz & 2yz & -z^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial(-z^2)}{\partial y} - \frac{\partial(2yz)}{\partial z}\right] - \hat{j}\left[\frac{\partial(-z^2)}{\partial x} - \frac{\partial(3xz)}{\partial z}\right] + \hat{k}\left[\frac{\partial(2yz)}{\partial x} - \frac{\partial(3xz)}{\partial y}\right]$$

$$= \hat{i}[0-2y] - \hat{j}[0-3x] + \hat{k}[0-0]$$

$$= -2y\,\hat{i} + 3x\,\hat{j} + 0\,\hat{k}$$

Now $$[\vec{A} \times (\nabla \times \vec{B})] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ -2y & 3x & 0 \end{vmatrix}$$

$$= \hat{i}[0+9x^2z] - \hat{j}[0-6xyz] + \hat{k}[3x^2z^2+4y^2]$$

$$= 9x^2z\,\hat{i} + 6xyz\,\hat{j} + (3x^2z^2+4y^2)\hat{k}$$

ii. $[(\vec{A} \times \nabla) \times \vec{B}]$

$$\vec{A} \times \nabla = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ xz^2 & 2y & -3xz \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial(2y)}{\partial z} - \frac{\partial(-3xz)}{\partial y}\right] - \hat{j}\left[\frac{\partial(xz^2)}{\partial z} - \frac{\partial(-3xz)}{\partial x}\right] + \hat{k}\left[\frac{\partial(xz^2)}{\partial y} - \frac{\partial(2y)}{\partial x}\right]$$

$$= \hat{i}[0+0] - \hat{j}[2xz+3z] + \hat{k}[0-0]$$

$$= 0\hat{i} - (2xz+3z)\hat{j} + 0\hat{k}$$

$$[(\vec{A} \times \nabla) \times \vec{B}] = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -(2xz+3z) & 0 \\ 3xz & 2yz & -z^2 \end{vmatrix}$$

$$= [z^2(2xz+3z) - 0]\hat{i} - [0-0]\hat{j} + [0+3xz(2xz+3z)]\hat{k}$$

$$= (2xz^3+3z^3)\hat{i} + (6x^2z^2+9xz^2)\hat{k}$$

PART-2

Line Integral, Surface Integral, Volume Integral, Gauss's Divergence Theorem, Green's Theorem, Stoke's Theorem (without proof) and their Applications.

CONCEPT OUTLINE : PART-2

Line Integral : Any integral which is evaluated along a curve is called a line integral. Let $\vec{F}(t)$ be a continuous vector point function defined at every point of a curve C in space. Let us divide the curve C into n parts by the points.

$$A = t_0, t_1, t_2, \ldots\ldots\ldots\ldots t_{n-1}, t_n = B$$

and let $R_0, R_1, R_2, \ldots\ldots\ldots\ldots, R_{n-1}, R_n$ be position vectors of these points. Let P_i be any point on the arc $t_{i-1}\, t_i$. Then limit of the sum

$$\sum_{i=1}^{n} \vec{F}(P_i) \cdot \delta \vec{R}_i \text{, where } \delta \vec{R}_i = \vec{R}_i - \vec{R}_{i-1}$$

As $n \to \infty$ and every $|\delta \vec{R}_i| \to 0$, if it exists, is called a line integral of $\vec{F}$ along C and is denoted by

$$\int_C \vec{F} . d\vec{R} \quad \text{or} \quad \int_C \vec{F} . \frac{d\vec{R}}{dt}\, dt$$

Line integral is a scalar quantity.

Work Done by a Force :

The work done by force $\vec{F}$ during displacement from A to B is $\int_A^B \vec{F} . d\vec{R}$.

Surface Integral : Any integral which is to be evaluated over a surface is called a surface integral. Surface integral of a vector function $\vec{F}$ over the surface S is defined as the integral of the components of $\vec{F}$ along the normal to the surface.

Component of $\vec{F}$ along the normal $= \vec{F} . \hat{n}$

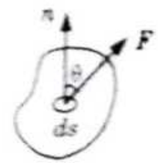
where $\hat{n}$ is the unit normal vector to an element ds and

$$\hat{n} = \frac{\text{grad } f}{|\text{grad } f|}$$

$$ds = \frac{dx\,dy}{(\hat{n}\,.\,\hat{k})}$$

Surface integral of F over

$$S = \sum \vec{F}\,.\,\hat{n} = \iint_S \vec{F}\,.\,\hat{n}\,ds$$



Note : If $\iint_S (\vec{F}\,.\,\hat{n})\,ds = 0$, then $\vec{F}$ is said to be a solenoidal vector point function.

Green's Theorem :

If $F_1(x, y), F_2(x, y), \frac{\partial F_1}{\partial y}, \frac{\partial F_2}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in x-y plane, then

$$\oint_C (F_1 dx + F_2 dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$$

Stoke's Theorem : Surface integral of the component of curl $\vec{F}$ along the normal to the surface S, taken over the surface S bounded by curve C is equal to the line integral of the vector point function $\vec{F}$ taken along the closed curve C.

$$\oint_C \vec{F}\,.\,d\vec{r} = \iint_S \text{curl}\,\vec{F}\,.\,\hat{n}\,ds$$

Where $\hat{n}$ is a unit external normal to any surface ds.

$$\hat{n} = \cos\alpha\,\hat{i} + \cos\beta\,\hat{j} + \cos\gamma\,\hat{k}$$

Gauss's Divergence Theorem :

Relation between Surface and Volume Integrals :

The surface integral of the normal component of a vector function F taken around a closed surface S is equal to the integral of the divergence of F taken over the volume V enclosed by the surface S.

Mathematically, $\iint_S F.\hat{n}\,ds = \iiint_V \text{div}\,\vec{F}\,dv$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.18. **A vector field is given by $\vec{F} = \sin y\,\hat{i} + x(1 + \cos y)\,\hat{j}$ evaluate the line integral over the circular path given by $x^2 + y^2 = a^2$, $z = 0$.**

Answer

The parametric equations of the circular path are $x = a\cos t$, $y = a\sin t$, $z = 0$ and t varies from 0 to 2π.

Particle moves in xy plane ($z = 0$), we take

$$\vec{r} = x\,\hat{i} + y\,\hat{j}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$$

$$\oint_C \vec{F}\,.\,d\vec{r} = \oint_C [\sin y\,\hat{i} + x(1 + \cos y)\,\hat{j}]\,.\,(dx\,\hat{i} + dy\,\hat{j})$$

$$= \oint_C \sin y\,dx + x(1 + \cos y)\,dy$$

$$= \oint_C (\sin y\,dx + x\cos y\,dy) + x\,dy$$

$$= \oint_C d(x\sin y) + \oint_C x\,dy$$

$$= \int_0^{2\pi} d(a\cos t\sin(a\sin t)) + \int_0^{2\pi} a\cos t\;a\cos t\,dt$$

$$= [a\cos t\sin(a\sin t)]_0^{2\pi} + a^2\int_0^{2\pi}\cos^2 t\,dt$$

$$= a^2\int_0^{2\pi}\left(\frac{1 + \cos 2t}{2}\right)dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= \frac{a^2}{2}(2\pi) = \pi a^2$$

Que 5.19. **Evaluate $\int_C \vec{F}\,.\,d\vec{r}$, where $\vec{F} = (x^2 + y^2)\,\hat{i} - 2xy\,\hat{j}$ and C is the rectangle in the xy plane bounded by $y = 0$, $x = a$ and $y = b$, $x = 0$.**

AKTU 2013-14, Marks 05

Answer

$$\int_C \vec{F}\,.\,d\vec{r} = \int_{OA} \vec{F}\,.\,d\vec{r} + \int_{AB} \vec{F}\,.\,d\vec{r} + \int_{BC} \vec{F}\,.\,d\vec{r} + \int_{CO} \vec{F}\,.\,d\vec{r}$$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy$$

Along OA, $x = 0$, $dx = 0$, y from 0 to b
Along AB, $y = b$, $dy = 0$, x from 0 to a
Along BC, $x = a$, $dx = 0$, y from b to 0
Along CO, $y = 0$, $dy = 0$, x from a to 0

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^b 0 + \int_0^a (x^2+b^2)\,dx + \int_b^0 -2ay\,dy + \int_a^0 x^2 dx$$

$$= \left[\frac{x^3}{3} + b^2x\right]_0^a - a\left[y^2\right]_b^0 + \frac{1}{3}\left[x^3\right]_a^0$$

$$= \frac{a^3}{3} + ab^2 + ab^2 - \frac{a^3}{3} = 2\,ab^2$$

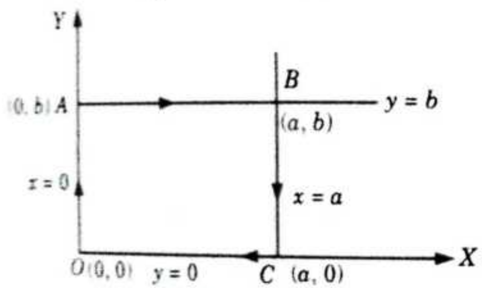


Fig. 5.19.1.

Que 5.20. Evaluate $\iint_S (yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k})\cdot d\vec{s}$, where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ in the first octant.

AKTU 2014-15, Marks 10

Answer

$$\phi = x^2+y^2+z^2-a^2$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\hat{i}+2y\,\hat{j}+2z\,\hat{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{x\,\hat{i}+y\,\hat{j}+z\,\hat{k}}{a}$$

$$A = yz\,\hat{i} + zx\,\hat{j} + xy\,\hat{k}$$

$$A\cdot\hat{n} = \frac{xyz+xyz+xyz}{a} = \frac{3xyz}{a}$$

$$\hat{n}\cdot\hat{k} = \frac{x\,\hat{i}+y\,\hat{j}+z\,\hat{k}}{a}\cdot\hat{k} = \frac{z}{a}$$

$$\iint A\cdot\hat{n}\,d\vec{s} = \iint A\cdot\hat{n}\,\frac{dx\,dy}{\ldots}$$

$$= \iint \frac{3xyz}{a}\,\frac{dx\,dy}{(z/a)} = 3\iint xy\,dx\,dy$$

$$= 3\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\,dy\,dx = \frac{3}{2}\int_0^a x\left[y^2\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{3}{2}\int_0^a x\left(\sqrt{a^2-x^2}\right)^2 dx = \frac{3}{2}\int_0^a x(a^2-x^2)\,dx$$

$$= \frac{3}{2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a = \frac{3}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right] = \frac{3}{8}a^4$$

Que 5.21. If $\vec{F} = (2x^2-3z)\,\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}$, evaluate $\iiint_V \nabla\cdot\vec{F}\,dv$, v is the region $x = 0, y = 0, z = 0, 2x+2y+z = 4$.

Answer

$$\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left((2x^2-3z)\,\hat{i} - 2xy\,\hat{j} - 4x\,\hat{k}\right) = 2x$$

$$dv = dx\,dy\,dz$$

$$\iiint_V \nabla\cdot\vec{F}\,dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x\,dx\,dy\,dz$$

$$= \int_0^2\int_0^{2-x} 2x\,dx\,dy\left[z\right]_0^{4-2x-2y}$$

$$= 2\int_0^2\int_0^{2-x} (4x-2x^2-2xy)\,dy\,dx$$

$$= 2\int_0^2 \left[4xy - 2x^2y - xy^2\right]_0^{2-x} dx$$

$$= 2\int_0^2 \left[4x(2-x) - 2x^2(2-x) - x(2-x)^2\right] dx$$

$$= 2\int_0^2 (x^3 - 4x^2 + 4x)\,dx$$

$$= 2\left[\frac{x^4}{4} - \frac{4}{3}x^3 + 2x^2\right]_0^2 = \frac{8}{3}$$

Que 5.22. If $\vec{A} = (x-y)\,\hat{i} + (x+y)\,\hat{j}$, evaluate $\oint_C \vec{A}\cdot d\vec{r}$ around the curve C consisting of $y = x^2$ and $y^2 = x$.

AKTU 2017-18, Marks 3.5

Answer

$$\vec{A} = (x-y)\hat{i} + (x+y)\hat{j}$$

$$\oint_C \vec{A}\cdot d\vec{r} = \oint_C (x-y)dx + (x+y)dy$$

C consisting of $y = x^2$ and $y^2 = x$

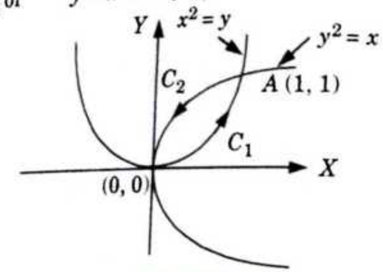


Fig. 5.22.1.

$$\oint_C \vec{A}\cdot d\vec{r} = \oint_{C_1} \vec{A}\cdot d\vec{r} + \oint_{C_2} \vec{A}\cdot d\vec{r}$$

Along C_1,

$$y = x^2$$
$$dy = 2x\,dx$$

$$\oint_{C_1} \vec{A}\cdot d\vec{r} = \int_0^1 (x-x^2)\,dx + (x+x^2)\,2x\,dx$$

$$= \int_0^1 xdx - x^2dx + 2x^2dx + 2x^3dx$$

$$= \int_0^1 (x+x^2+2x^3)\,dx$$

$$= \left[\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{2}\right]_0^1 = \frac{1}{2}+\frac{1}{3}+\frac{1}{2}$$

$$\oint_{C_1} \vec{A}\cdot d\vec{r} = \frac{4}{3}$$

Now Along C_2,

$$x = y^2,\ dx = 2y\,dy$$

$$\oint_{C_2} \vec{A}\cdot d\vec{r} = \int_1^0 (y^2-y)\,2y\,dy + (y^2+y)\,dy$$

$$\oint_{C_2} \vec{A}\cdot d\vec{r} = \int_1^0 2y^3dy - y^2dy + y\,dy = \int_1^0 (2y^3 - y^2 + y)\,dy$$

$$= \left[\frac{y^4}{2}-\frac{y^3}{3}+\frac{y^2}{2}\right]_1^0 = -\left[\frac{1}{2}-\frac{1}{3}+\frac{1}{2}\right] = -\frac{2}{3}$$

Thus, $$\oint_C \vec{A}\cdot d\vec{r} = \frac{4}{3}+\left(-\frac{2}{3}\right) = \frac{2}{3}$$

Que 5.23. **Verify Gauss's divergence theorem for the function $\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$ over unit cube.**

Answer

$$\vec{F} = x^2\hat{i} + z\hat{j} + yz\hat{k}$$

$$\nabla\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x^2\hat{i} + z\hat{j} + yz\hat{k})$$

$$\nabla\cdot\vec{F} = 2x + y$$

$$\text{Volume integral} = \iiint_V \nabla\cdot\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (2x+y)\,dxdydz$$

$$= \int_0^1\int_0^1 (2x+y)\,[z]_0^1\,dxdy$$

$$= \int_0^1 \left[2xy + \frac{y^2}{2}\right]_0^1 dx = \int_0^1 \left(2x+\frac{1}{2}\right)dx = \left(x^2+\frac{x}{2}\right)_0^1 = \frac{3}{2}$$

Now to evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$, where S is the surface bounded by the six surfaces of a unit cube.

Over the face $OABC$, $z = 0$, $dz = 0$, $\hat{n} = -\hat{k}$, $ds = dxdy$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (x^2\hat{i} + z\hat{j} + yz\hat{k})\cdot(-\hat{k})\,dxdy = \int_0^1\int_0^1 -yzdxdy$$

$$[\because z = 0]$$

$$= 0$$

Over the face $BCDE$,

$$y = 1,\ dy = 0,\ \hat{n} = \hat{j},\ ds = dxdz$$

$$\therefore \quad \iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (x^2\hat{i} + z\hat{j} + yz\hat{k})\cdot(\hat{j})\,dxdz = \int_0^1\int_0^1 zdxdz$$

$$= \int_0^1 dx\left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

Over the face $DEFG$, $z = 1$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dxdy$

$$\iint_S \vec F\cdot\hat n\, ds = \int_0^1\int_0^1 (x^2\hat i + z\hat j + yz\hat k)\cdot(\hat k)\,dxdy = \int_0^1\int_0^1 yz\,dxdy = 0$$

$$= \int_0^1 dx\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}\int_0^1 dx = \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

Over the face $OCDG$, $x = 0$, $dx = 0$, $\hat n = -\hat i$, $ds = dydz$

$$\therefore \iint_S \vec F\cdot\hat n\, ds = \int_0^1\int_0^1 (x^2\hat i + z\hat j + yz\hat k)\cdot(-\hat i)\,dydz = \int_0^1\int_0^1 -x^2dydz = 0 \qquad [\because\ x = 0]$$

Over the face $AOGF$, $y = 0$, $dy = 0$, $\hat n = -\hat j$, $ds = dxdz$

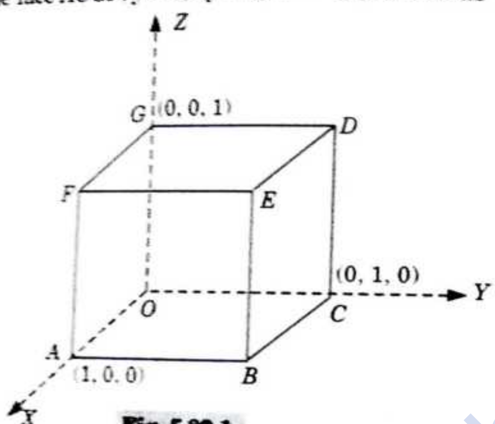


Fig. 5.23.1.

$$\therefore \iint_S \vec F\cdot\hat n\, ds = \int_0^1\int_0^1 (x^2\hat i + z\hat j + yz\hat k)\cdot(-\hat j)\,dxdz = \int_0^1\int_0^1 -z\,dxdz$$

$$= -\int_0^1 dx\left[\frac{z^2}{2}\right]_0^1 = -\frac{1}{2}\int_0^1 dx = -\frac{1}{2}[x]_0^1 = -\frac{1}{2}$$

Over the face $ABEF$, $x = 1$, $dx = 0$, $\hat n = \hat i$, $ds = dydz$

$$\therefore \iint_S \vec F\cdot\hat n\, ds = \int_0^1\int_0^1 (x^2\hat i + z\hat j + yz\hat k)\cdot(\hat i)\,dydz$$

$$= \int_0^1\int_0^1 x^2dydz = \int_0^1 dy\,[z]_0^1 = \int_0^1 dy = [y]_0^1 = 1 \qquad [\because\ x = 1]$$

Thus required surface integral is

$$\iint_S \vec F\cdot\hat n\, ds = 0 + \frac{1}{2} + \frac{1}{2} + 0 - \frac{1}{2} + 1 = \frac{3}{2}$$

Thus, $\iiint_V (\nabla\cdot\vec F)\,dv = \iint_S \vec F\cdot\hat n\, ds$

Hence Gauss's divergence theorem is verified.

Que 5.24. **Verify Gauss Divergence theorem for**

$$\int_c [(x^3yz)\hat i - 2x^2y\hat j + 2\hat k]\hat n\ ds$$ **where s denotes the surface of cube bounded by the planes $x = 0, x = a; y = 0, y = a; z = 0, z = a$.**

AKTU 2016-17, Marks 3.5

Answer

We know $\iiint_V \nabla\vec F\,dv = \iiint_S \vec F.\hat n\,ds$

Taking LHS

$$= \iiint_V \nabla\vec F\,dv$$

$$= \int_0^a\int_0^a\int_0^a\left(\hat i\frac{\partial}{\partial x} + \hat j\frac{\partial}{\partial y} + \hat k\frac{\partial}{\partial z}\right)[(x^3 - yz)\hat i - 2x^2y\,\hat j + 2\hat k]dx\,dy\,dz$$

$$= \int_0^a\int_0^a\int_0^a (3x^2 - 2x^2)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a\int_0^a x^2\,dx\,dy\,dz = \int_0^a\int_0^a x^2\,[z]_0^a dy\,dz = a\int_0^a\int_0^a x^2dy\,dx$$

$$= a\int_0^a x^2[y]_0^a\,dx = a^2\int_0^a x^2dx = a^2\left[\frac{x^3}{3}\right]_0^a = \frac{a^5}{3} \qquad ...(5.24.1)$$

Taking RHS

$$= \iint_s \vec F.\hat n\,ds$$

$$= \iint_{OABC}\vec F.\hat n\,ds + \iint_{BCDE}\vec F.\hat n\,ds + \iint_{DEFG}\vec F.\hat n\,ds + \iint_{OCDG}\vec F.\hat n\,ds$$
$$+ \iint_{AOGF}\vec F.\hat n\,ds + \iint_{ABEF}\vec F.\hat n\,ds \qquad ...(5.24.2)$$

Now, over the face $OABC$, $z = 0$, $dz = 0$, $\hat n = -\hat k$, $ds = dx\,dy$

$$\iint_{OABC}\vec F.\hat n\,ds = \int_0^a\int_0^a [x^3\,\hat i - 2x^2y\,\hat j + 2\,\hat k]\cdot[-\hat k]\,dx\,dy = \int_0^a\int_0^a -2dx\,dy$$

$$= -2\int_0^a [y]_0^a\,dx = -2a\int_0^a dx = -2a[x]_0^a = -2\,a^2$$

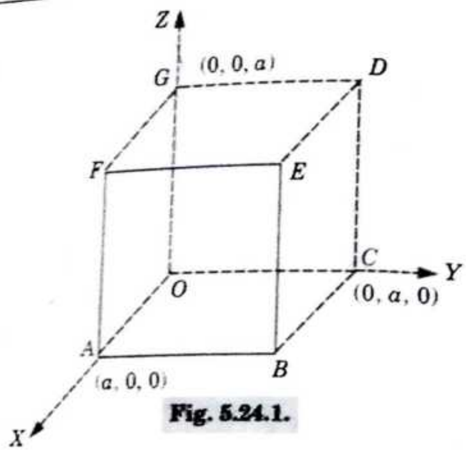


Fig. 5.24.1.

Over the face $BCDE$, $y = a$, $dy = 0$, $\hat{n} = \hat{j}$, $ds = dx\,dz$

$$\iint_{BCDE} \bar{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(x^2 - az)\hat{i} - 2x^2 a\,\hat{j} + 2\hat{k}]\cdot\hat{j}\,dx\,dz$$

$$= \int_0^a\int_0^a -2x^2 a\,dx\,dz$$

$$= -2a\int_0^a x^2[z]_0^a\,dx = -2a \times a\int_0^a x^2 dx = -2a^2\left[\frac{x^3}{3}\right]_0^a$$

$$= -2a^2 \times \frac{a^3}{3} = -\frac{2}{3}a^5$$

Over the face $DEFG$, $z = a$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dx\,dy$

$$\iint_{DEFG} \bar{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(x^2 - ay)\hat{i} - 2x^2 y\,\hat{j} + 2\hat{k}]\cdot\hat{k}\,dx\,dy$$

$$= \int_0^a\int_0^a 2dx\,dy = 2\int_0^a [y]_0^a dx = 2a\int_0^a dx = -2a^2[x]_0^a$$

$$= 2a^2$$

Over the face $OGDC$, $x = 0$, $dx = 0$, $\hat{n} = -\hat{i}$, $ds = dy\,dz$

$$\iint_{OGDC} \bar{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(-yz\,\hat{i} + 2\hat{k})\cdot(-\hat{i})\,dy\,dz = \int_0^a\int_0^a yz\,dy\,dz$$

$$= \int_0^a\left[\frac{z^2}{2}\right]_0^a dy = \frac{a^2}{2}\int_0^a y\,dy = -\frac{a^2}{2}\left[\frac{y^2}{2}\right]_0^a = \frac{a^4}{4}$$

Over the face $AOGF$, $y = 0$, $dy = 0$, $\hat{n} = -\hat{j}$, $ds = dx\,dz$

$$\iint_{AOGF} \bar{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [x^3\,\hat{i} + 2\hat{k}]\cdot(-\hat{j})\,dx\,dz = 0$$

Over the face $ABEF$, $x = a$, $dx = 0$, $\hat{n} = \hat{i}$, $ds = dy\,dz$

$$\iint_{ABEF} \bar{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(a^3 - yz)\,\hat{i} + 2a^2 y\,\hat{j} + 2\hat{k}]\,\hat{i}\,dy\,dz$$

$$= \int_0^a\int_0^a (a^3 - yz)\,dy\,dz$$

$$= \int_0^a\left[a^3 z - y\frac{z^2}{2}\right]_0^a dy = \int_0^a\left[a^4 - y\frac{a^2}{2}\right]dy$$

$$= \left[a^4 y - y^2\frac{a^2}{2\times 2}\right]_0^a = \left[a^4 a - a^2\frac{a^2}{2\times 2} - 0 - 0\right]$$

$$= a^5 - \frac{a^4}{2\times 2} = a^5 - \frac{a^4}{4}$$

Now putting all these values in eq. (5.24.2), we get

$$\iint_S \bar{F}\cdot\hat{n}\,ds = -2a^2 - \frac{2}{3}a^5 + 2a^2 + \frac{a^4}{4} + 0 + a^5 - \frac{a^4}{4}$$

$$= a^5 - \frac{2}{3}a^5 = \frac{3a^5 - 2a^5}{3}$$

$$= \frac{a^5}{3} \qquad ...(5.24.3)$$

From eq. (5.24.1) and eq. (5.24.3), we have

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

Hence verified.

Que 5.25. **Use divergence theorem in cartesian form to evaluate $\iint x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, where the surface s is the sphere $x^2 + y^2 + z^2 = a^2$.**

Answer

$$F_1 = x,\ F_2 = y,\ F_3 = z$$

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

$$= \iint_S (x\,\hat{i} + y\,\hat{j} + z\,\hat{k})(\hat{i}\,dy\,dz + \hat{j}\,dz\,dx + \hat{k}\,dx\,dy)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz \text{ (Using Gauss Divergence theorem)}$$

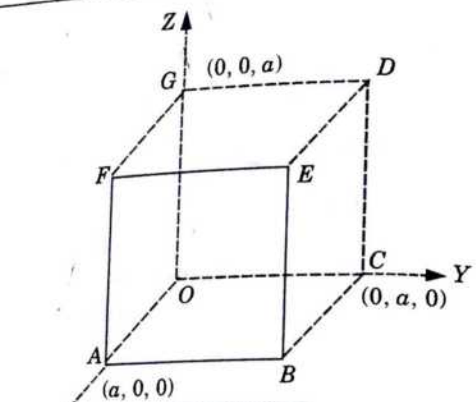


Fig. 5.24.1.

Over the face $BCDE$, $y = a$, $dy = 0$, $\hat{n} = \hat{j}$, $ds = dx\,dz$

$$\iint_{BCDE} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(x^2 - az)\hat{i} - 2x^2 a\,\hat{j} + 2\hat{k}]\cdot\hat{j}\,dx\,dz$$

$$= \int_0^a\int_0^a -2x^2 a\,dx\,dz$$

$$= -2a\int_0^a x^2[z]_0^a dx = -2a\times a\int_0^a x^2 dx = -2a^2\left[\frac{x^3}{3}\right]_0^a$$

$$= -2a^2 \times \frac{a^3}{3} = -\frac{2}{3}a^5$$

Over the face $DEFG$, $z = a$, $dz = 0$, $\hat{n} = \hat{k}$, $ds = dx\,dy$

$$\iint_{DEFG} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(x^2 - ay)\hat{i} - 2x^2 y\,\hat{j} + 2\hat{k}]\cdot\hat{k}\,dx\,dy$$

$$= \int_0^a\int_0^a 2dx\,dy = 2\int_0^a [y]_0^a dx = 2a\int_0^a dx = -2a^2[x]_0^a$$

$$= 2a^2$$

Over the face $OGDC$, $x = 0$, $dx = 0$, $\hat{n} = -\hat{i}$, $ds = dy\,dz$

$$\iint_{OGDC} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(-yz\,\hat{i} + 2\hat{k})\cdot(-\hat{i})\,dy\,dz = \int_0^a\int_0^a yz\,dy\,dz$$

$$= \int_0^a\left[\frac{z^2}{2}\right]_0^a dy = \frac{a^2}{2}\int_0^a y\,dy = -\frac{a^2}{2}\left[\frac{y^2}{2}\right]_0^a = \frac{a^4}{4}$$

Over the face $AOGF$, $y = 0$, $dy = 0$, $\hat{n} = -\hat{j}$, $ds = dx\,dz$

$$\iint_{AOGF} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [x^3\,\hat{i} + 2\hat{k}]\cdot(-\hat{j})\,dx\,dz = 0$$

Over the face $ABEF$, $x = a$, $dx = 0$, $\hat{n} = \hat{i}$, $ds = dy\,dz$

$$\iint_{ABEF} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a [(a^3 - yz)\,\hat{i} + 2a^2 y\,\hat{j} + 2\hat{k}]\,\hat{i}\,dy\,dz$$

$$= \int_0^a\int_0^a (a^3 - yz)\,dy\,dz$$

$$= \int_0^a\left[a^3 z - y\frac{z^2}{2}\right]_0^a dy = \int_0^a\left[a^4 - y\frac{a^2}{2}\right]dy$$

$$= \left[a^4 y - y^2\frac{a^2}{2\times 2}\right]_0^a = \left[a^4 a - a^2\frac{a^2}{2\times 2} - 0 - 0\right]$$

$$= a^5 - \frac{a^4}{2\times 2} = a^5 - \frac{a^4}{4}$$

Now putting all these values in eq. (5.24.2), we get

$$\iint_S \vec{F}\cdot\hat{n}\,ds = -2a^2 - \frac{2}{3}a^5 + 2a^2 + \frac{a^4}{4} + 0 + a^5 - \frac{a^4}{4}$$

$$= a^5 - \frac{2}{3}a^5 = \frac{3a^5 - 2a^5}{3}$$

$$= \frac{a^5}{3} \qquad ...(5.24.3)$$

From eq. (5.24.1) and eq. (5.24.3), we have

$$\iiint_V \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot\hat{n}\,ds$$

Hence verified.

Que 5.25. **Use divergence theorem in cartesian form to evaluate $\iint x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$, where the surface s is the sphere $x^2 + y^2 + z^2 = a^2$.**

Answer

$$F_1 = x,\ F_2 = y,\ F_3 = z$$

$$\iint_S x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$$

$$= \iint_S (x\,\hat{i} + y\,\hat{j} + z\,\hat{k})(\hat{i}\,dy\,dz + \hat{j}\,dz\,dx + \hat{k}\,dx\,dy)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dx\,dy\,dz \quad \text{(Using Gauss Divergence theorem)}$$

$$= \iiint_V (1+1+1)\, dx\, dy\, dz = 3\iiint dx\, dy\, dz = 3 \text{ (Volume of sphere)}$$

$$= 3\left(\frac{4}{3}\right)\pi a^3 = 4\pi a^3$$

Que 5.26. **Verify the Green's theorem to find the line integral $\int_C (2y^2\, dx + 3x\, dy)$, where C is the boundary of the closed region bounded by $y = x$ and $y = x^2$.**

AKTU 2015-16, Marks 10

Answer

Using Green's theorem

$$\int_C 2y^2\, dx + 3x\, dy = \iint \left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(2y^2)\right] dx\, dy$$

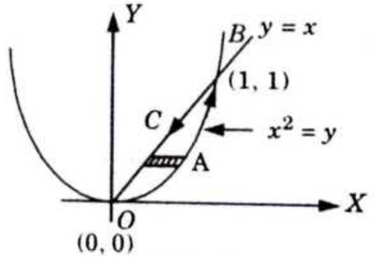


Fig. 5.26.1.

$$= \iint (3-4y)\, dx\, dy = \int_0^1 \int_{x=y}^{\sqrt{y}} (3-4y)\, dx\, dy$$

$$= \int_0^1 [3x - 4xy]_y^{\sqrt{y}}\, dy$$

$$= \int_0^1 (3\sqrt{y} - 4y^{3/2} - 3y + 4y^2)\, dy$$

$$= \left[3 \times \frac{2}{3} y^{3/2} - 4 \times \frac{2}{5} y^{5/2} - \frac{3y^2}{2} + \frac{4y^3}{3}\right]_0^1$$

$$= \frac{6}{3} - \frac{8}{5} - \frac{3}{2} + \frac{4}{3} = \frac{7}{30}$$

Verification of the Green's Theorem :

$$\int_C 2y^2\, dx + 3x\, dy = \int_{OAB} (2y^2\, dx + 3x\, dy) + \int_{BCO} (2y^2\, dx + 3x\, dy)$$

Along OAB, $y = x^2$, $dy = 2x\, dx$

Along BCO, $y = x$, $dy = dx$

$$= \int_0^1 [2x^4 + 3x(2x)]\, dx + \int_1^0 (2x^2 + 3x)\, dx$$

$$= \left[\frac{2x^5}{5} + 2x^3\right]_0^1 + \left[\frac{2x^3}{3} + \frac{3x^2}{2}\right]_1^0$$

$$= \left(\frac{2}{5} + 2\right) - \left(\frac{2}{3} + \frac{3}{2}\right) = \frac{12}{5} - \frac{13}{6} = \frac{72-65}{30} = \frac{7}{30}$$

Que 5.27. **Verify Green's theorem, evaluate $\int_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is square formed by lines $x = \pm 1$, $y = \pm 1$.**

AKTU 2017-18, Marks 3.5

Answer

Here the closed curve C consists of straight lines AB, BC, CD and DA where coordinates of A, B, C and D are $(-1, -1)$, $(1, -1)$, $(1, 1)$ and $(-1, 1)$ respectively.

By Green's theorem, we have

$$\int_C (x^2 + xy)dx + (x^2 + y^2)dy = \iint_R \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(x^2 + xy)\right] dx\, dy \quad \text{...(5.27.1)}$$

$$= \iint_R (2x - x)dx\, dy$$

$$= \int_{x=-1}^{1}\int_{y=-1}^{1} x\, dx\, dy = \int_{-1}^{1} x(y)_{-1}^{1}\, dx$$

$$= \int_{-1}^{1} 2x\, dx = 0 \quad \text{...(5.27.2)}$$

Now we evaluate LHS of eq. (5.27.1) along AB, BC, CD and DA.

$$\int_C (x^2 + xy)\, dx + (x^2 + y^2)\, dy = \int_{AB} (x^2 + xy)\, dx + (x^2 + y^2)\, dy + \int_{BC} (x^2 + xy)\, dx + (x^2 + y^2)\, dy$$
$$+ \int_{CD} (x^2 + xy)\, dx + (x^2 + y^2)\, dy + \int_{DA} (x^2 + xy)\, dx + (x^2 + y^2)\, dy \quad \text{...(5.27.3)}$$

Now along AB, $y = -1$ and $dy = 0$

$$\therefore \int_{AB} (x^2 + xy)\, dx + (x^2 + y^2)\, dy = \int_{-1}^{1} (x^2 - x)\, dx = \left[\frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left[\frac{1}{3} - \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\right] = \frac{2}{3}$$

Along BC, $x = 1$ and $dx = 0$

$$\therefore \int_{BC} (x^2 + xy)\, dx + (x^2 + y^2)\, dy = \int_{-1}^{1} (1 + y^2)\, dy = \left[y + \frac{y^3}{3}\right]_{-1}^{1}$$

$$= \left[1+\frac{1}{3}+1+\frac{1}{3}\right] = \frac{8}{3}$$

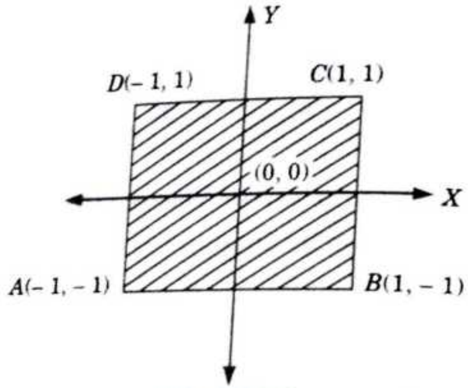


Fig. 5.27.1.

Along CD, $y = 1$ and $dy = 0$

$$\therefore \int_{CD}(x^2+xy)\,dx+(x^2+y^2)\,dy = \int_{1}^{-1}(x^2+x)\,dx = \left[\frac{x^3}{3}+\frac{x^2}{2}\right]_1^{-1}$$

$$= \left[-\frac{1}{3}+\frac{1}{2}-\frac{1}{3}-\frac{1}{2}\right] = -\frac{2}{3}$$

Along DA, $x = -1$ and $dx = 0$

$$\therefore \int_{DA}(x^2+xy)\,dx+(x^2+y^2)\,dy = \int_{1}^{-1}(1+y^2)\,dy = \left[y+\frac{y^3}{3}\right]_1^{-1}$$

$$= \left[-1-\frac{1}{3}-1-\frac{1}{3}\right] = -\frac{8}{3}$$

Now from eq. (5.27.3)

$$\int_C (x^2+xy)\,dx+(x^2+y^2)\,dy = \frac{2}{3}+\frac{8}{3}-\frac{2}{3}-\frac{8}{3} = 0 \quad \text{...(5.27.4)}$$

Equality of eq. (5.27.2) and eq. (5.27.4) verifies the Green's theorem.

Que 5.28. **Verify Stoke's theorem for**

$$\vec{F} = (y-z+2)\,\hat{i} + (yz+4)\,\hat{j} - xz\,\hat{k}$$

over the surface of a cube $x = 0, y = 0, z = 0, x = 2, y = 2, z = 2$ above the XOY plane (open at the bottom).

AKTU 2013-14, Marks 10

Answer

$$\oint_C \vec{F}\cdot d\vec{r} = \oint [(y-z+2)\hat{i}+(yz+4)\hat{j}-(xz)\hat{k}].(\hat{i}\,dx+\hat{j}\,dy+\hat{k}\,dz)$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (y-z+2)\,dx + (yz+4)\,dy - xz\,dz$$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{OA}\vec{F}\cdot d\vec{r} + \int_{AB}\vec{F}\cdot d\vec{r} + \int_{BC}\vec{F}\cdot d\vec{r} + \int_{CO}\vec{F}\cdot d\vec{r}$$

Along OA, $y = 0$, $dy = 0$, $z = 0$, $dz = 0$ and x varies from 0 to 2.
Along AB, $x = 2$, $dx = 0$, $z = 0$, $dz = 0$ and y varies from 0 to 2.
Along BC, $y = 2$, $dy = 0$, $z = 0$, $dz = 0$ and x varies from 2 to 0.
Along CO, $x = 0$, $dx = 0$, $z = 0$, $dz = 0$ and y varies from 2 to 0.

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^2 2dx + \int_0^2 4dy + \int_2^0 4dx + \int_2^0 4dy = 4+8-8-8 = -4$$

$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-z+2 & yz+4 & -xz \end{vmatrix} = -y\,\hat{i} + (z-1)\,\hat{j} - \hat{k}$$

Integration will be about five edges ABDE, OCGE, BCGD, OAEF, DEFG.

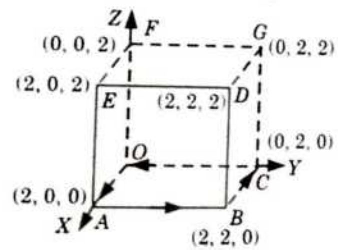


Fig. 5.28.1.

On ABDE $\quad \hat{n} = \hat{i}$, $ds = dy\,dz$, $(x = 2)$

$$\iint(\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint -y\,dy\,dz = -\int_0^2\int_0^2 y\,dy\,dz = -4$$

On OCGF, $\quad \hat{n} = -\hat{i}$, $ds = dy\,dz$, $(x = 0)$

$$\iint(\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint y\,dy\,dz = \int_0^2\int_0^2 y\,dy\,dz = 4$$

On BCGD, $\quad \hat{n} = \hat{j}$, $ds = dx\,dz$, $(y = 2)$

$$\iint(\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint\left[-y\,\hat{i}+(z-1)\,\hat{j}-\hat{k}\right]\hat{j}\,dx\,dz$$

$$= \int_0^2\int_0^2 (z-1)\,dx\,dz = 0$$

On OAEF, $\quad \hat{n} = -\hat{j}$, $ds = dx\,dz$, $(y = 0)$

$$\iint\left(\nabla\times\vec{F}\right).\hat{n}\,ds = -\int_0^2\int_0^2 (z-1)\;dx\,dz = 0$$

On DEFG, $\hat{n} = \hat{k}$, $ds = dx\,dy$, $(z = 2)$

$$\iint\left(\nabla\times\vec{F}\right).\hat{n}\,ds = \iint[-y\,\hat{i}+(z-1)\,\hat{j}-\hat{k}].\hat{k}\;dx\,dy$$

$$= -\iint dx\,dy = \int_0^2\int_0^2 dx\,dy = -4$$

Thus, $\iint_S \text{curl}\,\vec{F}.\hat{n}\,ds = \oint_C \vec{F}.d\vec{r} = -4+4+0+0-4 = -4$

Hence Stoke's theorem is verified.

Que 5.29. **Verify Stoke's theorem $\vec{F} = (2y+z, x-z, y-x)$ taken over the triangle ABC cut from the plane $x + y + z = 1$ by the coordinate planes.**

AKTU 2016-17, Marks 3.5

Answer

$$\oint_C \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\hat{n}ds$$

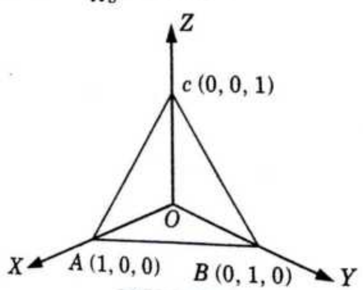


Fig. 5.29.1.

Taking LHS,

$$\oint_{ABC} \vec{F}.d\vec{r} = \int_{AB}\vec{F}.d\vec{r} + \int_{BC}\vec{F}.d\vec{r} + \int_{CA}\vec{F}.d\vec{r} \quad ...(5.28.1)$$

Along AB, $z = 0$, $x + y = 1$, $y = 1 - x$, $dy = -dx$ and $\vec{r} = x\hat{i} + y\hat{j}$

$$\int_{AB}\vec{F}.d\vec{r} = \int_{AB}[2y\hat{i}+x\hat{j}+(y-x)\hat{k}](\hat{i}\,dx+\hat{j}\,dy)$$

$$= \int_{AB} 2y\,dx + x\,dy = \int_{AB} 2(1-x)dx + x(-dx)$$

$(\because\ y = 1 - x \text{ and } dy = -dx)$

$$= \int_{AB} 2dx - 2x\,dx - x\,dx$$

[For line AB, Lower limit = 1, Upper limit = 0]

$$= \int_1^0 (2-3x)dx = \left[2x-\frac{3x^2}{2}\right]_1^0 = -2+\frac{3}{2} = \frac{-1}{2}$$

Along BC, $x = 0$, $y + z = 1$, $z = 1 - y$, $dz = -dy$ and $\hat{r} = y\,\hat{j} + z\,\hat{k}$

$$\int_{BC}\vec{F}.d\vec{r} = \int_{BC}[(2y+z)\hat{i}-z\hat{j}+y\hat{k}](\hat{j}dy+\hat{k}dz)$$

$$= \int_{BC} -z\,dy + y\,dz$$

$$= \int_{BC} -(1-y)dy + y(-dy) = \int_{BC} -dy + y\,dy - y\,dy$$

[For line BC, Lower limit = 1, Upper limit = 0]

$$\int_1^0 -dy = [-y]_1^0 = 0-(-1) = 1$$

Along CA, $y = 0$, $x + z = 1 \Rightarrow x = 1 - z$, $dx = -dz$ and $\hat{r} = x\,\hat{i} + z\,\hat{k}$

$$\int_{CA}\vec{F}.d\vec{r} = \int_{CA}[z\hat{i}+(x-z)j-x\hat{k}](\hat{i}dx+\hat{k}dz) = \int_{CA} z\,dx - x\,dz$$

$$= \int_{CA} z(-dz)-(1-z)dz = \int_{AC} -z\,dz - dz + z\,dz = \int_{AC} -dz$$

[For line CA, lower limit = 1, upper limit = 0]

$$= \int_1^0 -dz = -[z]_1^0 = -[0-1] = 1$$

Putting the value of $\int_{AB}\vec{F}.d\vec{r}$, $\int_{BC}\vec{F}.d\vec{r}$ and $\int_{CA}\vec{F}.d\vec{r}$ in eq. (5.29.1), we have

$$\int_{ABC}\vec{F}.d\vec{r} = \frac{-1}{2}+1+1 = \frac{3}{2} \quad ...(5.29.2)$$

Now, curl $\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2y+z & x-z & y-x \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial(y-x)}{\partial y}-\frac{\partial(x-z)}{\partial z}\right]-\hat{j}\left[\frac{\partial(y-x)}{\partial x}-\frac{\partial(2y+z)}{\partial z}\right]+\hat{k}\left[\frac{\partial(x-z)}{\partial x}-\frac{\partial(2y+z)}{\partial y}\right]$$

$$= \hat{i}\,[1-(-1)]-\hat{j}\,[-1-1]+\hat{k}\,[1-2] = 2\hat{i}+2\hat{j}-\hat{k}$$

Equation of the plane ABC is $x + y + z = 1$

Normal to the plane ABC is

$$\nabla\phi = \left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right](x + y + z - 1) = \hat{i} + \hat{j} + \hat{k}$$

Normal unit vector, $\hat{n} = \dfrac{\hat{i} + \hat{j} + \hat{k}}{\sqrt{3}}$

Now taking RHS

$$\iint_z \text{curl}\,\vec{F}\cdot\hat{n}\,ds = \iint_S (2\hat{i} + 2\hat{j} - \hat{k})\left(\frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\;\frac{dxdy}{\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\cdot\hat{k}}\right)$$

$$= \iint_z \frac{(2+2-1)}{\sqrt{3}}\,\frac{dx\,dy}{\frac{1}{\sqrt{3}}} = 3\iint dx\,dy$$

$$= 3 \times \text{area of } \Delta AOB$$

$$= 3 \times \frac{1}{2} \times 1 \times 1 = \frac{3}{2} \qquad \text{...(2.29.3)}$$

From eq. (5.29.2) and eq. (5.29.3), we have

$$\int_{ABC} \vec{F}.d\vec{r} = \iint_S \text{curl}\,\vec{F}.\hat{n}\,ds$$

Hence Stoke's theorem is verified.

Que 5.30. **Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$ taken around the rectangle bounded by the lines $x = \pm a$, $y = 0$, $y = b$.**

AKTU 2017-18, Marks 3.5

AKTU 2014-15, Marks 10

OR

If $\vec{F} = (x^2 + y^2)\hat{i} - 2xy\hat{j}$, then evaluate the value of $\iint \vec{F}.d\vec{r}$

AKTU 2016-17, Marks 03

Answer

Let C denote the boundary of the rectangle $ABED$, then

$$\oint_C \vec{F}.d\vec{r} = \oint_C [(x^2+y^2)\hat{i} - 2xy\,\hat{j}](\hat{i}\,dx + \hat{j}\,dy)$$

$$= \oint_C [(x^2+y^2)dx - 2xy\,dy]$$

The curve C consists of four lines AB, BE, ED and DA.

Along AB, $x = a$, $dx = 0$ and y varies from 0 to b.

$$\therefore \int_{AB}[(x^2+y^2)dx - 2xy\,dy] = \int_0^b -2ay\,dy = -a[y^2]_0^b = -ab^2 \qquad \text{...(5.30.1)}$$

Along BE, $y = b$, $dy = 0$ and x varies from a to $-a$.

$$\therefore \int_{BE}[(x^2+y^2)dx - 2xy\,dy] = \int_a^{-a}(x^2+b^2)dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= -\frac{2a^3}{3} - 2ab^2 \qquad \text{...(5.30.2)}$$

Along ED, $x = -a$, $dx = 0$ and y varies from b to 0.

$$\therefore \int_{ED}[(x^2+y^2)dx - 2xy\,dy] = \int_b^0 2ay\,dy = a[y^2]_b^0 = -ab^2 \qquad \text{...(5.30.3)}$$

Along DA, $y = 0$, $dy = 0$ and x varies from $-a$ to a.

$$\therefore \int_{DA}[(x^2+y^2)dx - 2dx\,dy] = \int_{-a}^a x^2\,dx = \frac{2a^3}{3} \qquad \text{...(5.30.4)}$$

Adding eq. (5.30.1), eq. (5.30.2), eq. (5.30.3) and eq. (5.30.4), we get

$$\oint_C \vec{F}.d\vec{r} = -ab^2 - \frac{2a^3}{3} - 2ab^2 - ab^2 + \frac{2a^3}{3} = -4ab^2 \qquad \text{...(5.30.5)}$$

Now,
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix} = (-2y - 2y)\hat{k} = -4y\hat{k}$$

For the surface S, $\hat{n} = \hat{k}$

$$\text{curl}\,\vec{F}.\hat{n} = -4y\hat{k}.\hat{k} = -4y$$

$$\therefore \iint_S \text{curl}\,\vec{F}.\hat{n}\,ds = \int_0^b\int_{-a}^a -4y\,dxdy = \int_0^b -4y\,[x]_{-a}^a\,dy$$

$$= -8a\int_0^b y\,dy = -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \qquad \text{...(5.30.6)}$$

The equality of eq. (5.30.5) and eq. (5.30.6) verifies Stoke's theorem.

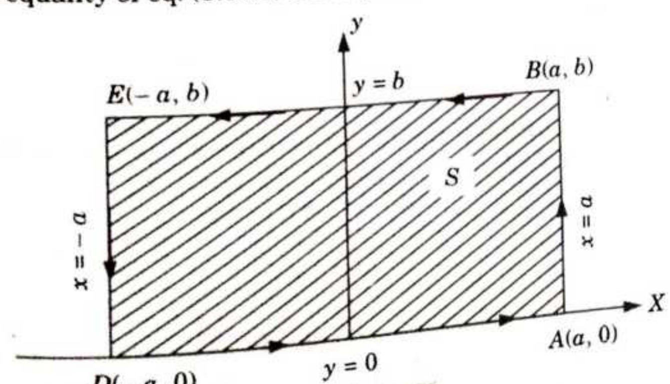


Fig. 5.30.1.

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