Computability - Exercise 1

All answers should be proved formally (unless noted otherwise)

Due - May 26

1. What is the language of the automaton below? (remember to prove your answer formally.)

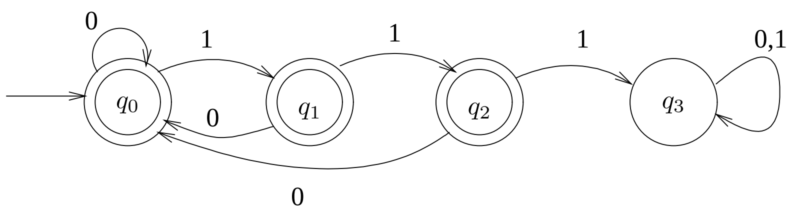

Figure 1: The automaton $\mathcal{A}$

2. Describe a deterministic finite automaton (a.k.a. DFA) for each of the following languages. A drawing can be considered a description, but only if it is *exact* and contains all the information needed to create from it the formal definition of the automaton.

   No need to prove formally the correctness of your construction.
   All languages are over the alphabet $\Sigma = \{0, 1\}$.

   (a) The language that contains all words that end with 0111.

   (b) The language that contains all words that begin with 0111.

   (c) The language that contains all words that contain 0111 as a subword.

   (d) The language that contains all words that do not contain 0111 as a subword.

3. The binary representation of a number is a word of the alphabet $\{0, 1\}$. The value of the word $s = s_1 \dots s_n$, denoted $v(s)$, is defined as follows: For

the empty word, $v(\epsilon) = 0$. For a longer word, the definition is inductive $v(s_1 \dots s_{n-1}s_n) = 2 \cdot v(s_1 \dots s_{n-1}) + s_n$. Note that this means that $v(s_1 \dots s_n) = \sum_{i=1}^{n} 2^{n-i} \cdot s_i$.

Describe a DFA that recognizes binary words that their value mod 3 is 0.

4. Let S be a set. A relation $E \subseteq S \times S$ is a *partition* relation, if there exists some partition of S into a set of subsets $\{S_i\}_{i \in I}$ (where I is a set of indices), such that:

   (a) the set S is covered by the subsets (i.e. $S = \bigcup_{i \in I} S_i$).

   (b) the subsets are disjoint, (i.e., for every $i, j \in I$ for which $i \neq j$ it holds that $S_i \cap S_j = \emptyset$).

   (c) for all $s_1, s_2 \in S$ it holds that $s_1 E s_2$ iff there exists an index $i \in I$ for which $s_1, s_2 \in S_i$. Equivalently, $E = \bigcup_{i \in I} S_i \times S_i$.

   A relation $E \subseteq S \times S$ is an *inverse function* relation, if there exists some set T and a function $f : S \to T$ such that: for all $s_1, s_2 \in S$ it holds that $s_1 E s_2$ iff $f(s_1) = f(s_2)$. Equivalently, $E = \bigcup_{t \in T} (f^{-1}(t) \times f^{-1}(t))$.

   Prove that a relation is an equivalence relation iff it is a partition relation, and that a relation is a partition relation iff it is an inverse function relation.

   Comment: As you shall prove, every equivalence relation is a partition relation. The sets S_i in the partition are called the *equivalence classes* of the equivalence relation.

5. optional ("reshut")
   Prove that there exists a non-regular language over the alphabet $\{0, 1\}$. Hints:

   (a) Prove that the set of all languages over the alphabet $\{0, 1\}$ is not countable.

   (b) Prove that the set of all deterministic finite automata over the alphabet $\{0, 1\}$ is countable.