ML HW4

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Tags	

Part 1:

Q1:

- k=10

```
kfold_data = cross_validation(x_train, y_train, k=10)
for i, (train_index, val_index) in enumerate(kfold_data):
    print("Split: %s, Training index: %s, Validation index: %s" % (i+
assert len(kfold_data) == 10 # should contain 10 fold of data
assert len(kfold_data[0]) == 2 # each element should contain train fo
assert kfold_data[0][1].shape[0] == 700 # The number of data in each
```

```
0.8s
```

```
(7000,)
Split: 1, Training index: 6300, Validation index: 700
Split: 2, Training index: 6300, Validation index: 700
Split: 3, Training index: 6300, Validation index: 700
Split: 4, Training index: 6300, Validation index: 700
Split: 5, Training index: 6300, Validation index: 700
Split: 6, Training index: 6300, Validation index: 700
Split: 7, Training index: 6300, Validation index: 700
Split: 8, Training index: 6300, Validation index: 700
Split: 9, Training index: 6300, Validation index: 700
Split: 10, Training index: 6300, Validation index: 700
```

- k=13

```
kfold_data = cross_validation(x_train, y_train, k=13)
for i, (train_index, val_index) in enumerate(kfold_data):
    print("Split: %s, Training index: %s, Validation index: %s" % (i+1,
assert len(kfold_data) == 10 # should contain 10 fold of data
assert len(kfold_data[0]) == 2 # each element should contain train fold
assert kfold_data[0][1].shape[0] == 700 # The number of data in each va
```

[22] 0.1s

```
(7000,)
Split: 1, Training index: 6461, Validation index: 539
Split: 2, Training index: 6461, Validation index: 539
Split: 3, Training index: 6461, Validation index: 539
Split: 4, Training index: 6461, Validation index: 539
Split: 5, Training index: 6461, Validation index: 539
Split: 6, Training index: 6461, Validation index: 539
Split: 7, Training index: 6462, Validation index: 538
Split: 8, Training index: 6462, Validation index: 538
Split: 9, Training index: 6462, Validation index: 538
Split: 10, Training index: 6462, Validation index: 538
Split: 11, Training index: 6462, Validation index: 538
Split: 12, Training index: 6462, Validation index: 538
Split: 13, Training index: 6462, Validation index: 538
```

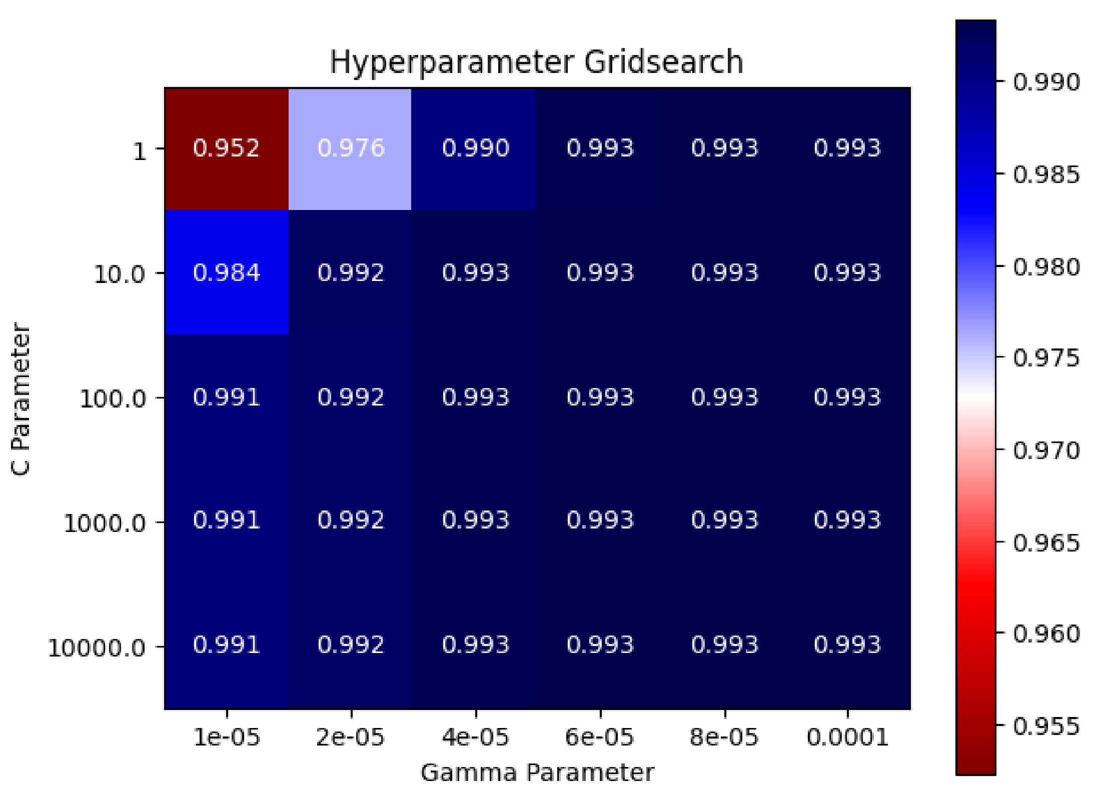
Q2(K=5):

```
print(len(kfold_data))
print(best_parameters)
# print(stat_table)
```

9] ✓ 0.7s

```
5
{'C': 10000.0, 'Gamma': 0.0001, 'kernel': 'rbf'}
```

Q3:


Hyperparameter Gridsearch
C Parameter
1
10.0
100.0
1000.0
10000.0
Gamma Parameter
1e-05
2e-05
4e-05
6e-05
8e-05
0.0001
0.952
0.976
0.990
0.993
0.993
0.993
0.984
0.992
0.993
0.993
0.993
0.993
0.991
0.992
0.993
0.993
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0.993
0.993
0.991
0.992
0.993
0.993
0.993
0.993
0.990
0.985
0.980
0.975
0.970
0.965
0.960
0.955

Part 2:

Q1 :

$$\text{The kernel function of } \mathbf{x}_n \text{ and } \mathbf{x}_m,$$
$$k(\mathbf{x}_n,\ \mathbf{x}_m) = \phi(\mathbf{x}_n)^T\phi(\mathbf{x}_m),\ \text{is symmetric.}$$
$$\text{so we have } k(\mathbf{x}_n,\ \mathbf{x}_m) = k(\mathbf{x}_m,\ \mathbf{x}_n)$$
$$\text{Therefore the kernel matrix K is symmetric.}$$
$$\text{We can write K in the form } K = V\Lambda V^T,$$
$$\text{where V is an orthogonal matrix } \mathbf{v}_t \text{ and } \Lambda \text{ contains the eigenvalues of K}$$
$$\text{if K is positive semidefinite, all eigenvalues are nonnegative}$$
$$\text{Consider the mapping } \phi : \mathbf{x}_i \mapsto (\sqrt{\lambda_t}\mathbf{v}_{ti})_{t=1}^n$$
$$\text{We have } k(\mathbf{x}_i,\ \mathbf{x}_j) = \sum_{t=1}^{n}\lambda_t\mathbf{v}_{ti}\mathbf{v}_{tj} = (\mathbf{V\Lambda V^T})_{ij} = K_{ij} = \phi(\mathbf{x}_i)^T\phi(\mathbf{x}_j)...(1)$$
$$\text{Since } \lambda_t \text{ is in the square root, it has to be nonnegative for the mapping to be valid..(2)}$$
$$\text{Therefore K should be positive semidefinite}$$

Q2 :

$$\text{By Taylor Series, we have } e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$\text{We have } k(\mathbf{x},\ \mathbf{x'}) \times k(\mathbf{x},\ \mathbf{x'}),$$
$$k(\mathbf{x},\ \mathbf{x'}) + k(\mathbf{x},\ \mathbf{x'}),$$
$$\text{and } ck(\mathbf{x},\ \mathbf{x'}) \text{ are also valid kernels when } k(\mathbf{x},\ \mathbf{x'}) \text{ is valid,}$$
$$\text{so } e^{k(\mathbf{x},\ \mathbf{x'})} \text{ is also valid.}$$

Q3 :

$\because k(x, x')$ is valid $\therefore K = \begin{array}{c c} & N \\ N & \begin{bmatrix} k(x_1,x_1) & \cdots & k(x_N,x_1) \\ \vdots & \ddots & \vdots \\ k(x_N,x_1) & \cdots & k(x_N,x_N) \end{bmatrix} \end{array}$ is Positive Semidefinite (P.S.D.)

a.) if $k_1(\underline{x}, \underline{x}')$ use the feature map $\phi_1(\underline{x})$

we can use another mapping $\phi : x \mapsto [\phi_1(\underline{x}), 1]^T$

$$\Rightarrow \langle \phi(\underline{x}), \phi(\underline{x}') \rangle = \langle \phi_1(\underline{x}), \phi_1(\underline{x}') \rangle + 1 = k_1(\underline{x}, \underline{x}') + 1 = k(\underline{x}, \underline{x}')$$

$\Rightarrow \underline{k(\underline{x}, \underline{x}') \text{ is valid}}$

c.) $K_2(\underline{x}, \underline{x}') = \exp(\|\underline{x}\|^2) \exp(\|\underline{x}'\|^2) = \exp(\|\underline{x}\|^2 + \|\underline{x}'\|^2)$

$$\Rightarrow G_2 = \begin{bmatrix} \exp(2\|\underline{x}_1\|^2) & \exp(\|\underline{x}_1\|^2+\|\underline{x}_2\|^2) & \exp(\|x_1\|^2+\|\underline{x}_3\|^2) & \cdots \\ \vdots & \exp(2\|\underline{x}_2\|^2) & \exp(\|x_2\|^2+\|\underline{x}_3\|^2) & \cdots \\ & & \exp(2\|\underline{x}_3\|^2) & \vdots \\ \exp(\|\underline{x}_1\|^2+\|\underline{x}_N\|^2) & \cdots & & \ddots \end{bmatrix}$$

$\exp(2\|\underline{x}_1\|^2) > 0$ (first principal minor)

$\exp(2\|\underline{x}_1\|^2 + 2\|\underline{x}_2\|^2) - \exp(\|\underline{x}_1\|^2+\|\underline{x}_2\|^2)\exp(\|\underline{x}_1\|^2+\|\underline{x}_2\|^2) = 0$ (2nd principal minor)

$$\exp(2\|\underline{x}_1\|^2+2\|\underline{x}_2\|^2+2\|\underline{x}_3\|^2) + \exp(\|\underline{x}_1\|^2+\|\underline{x}_2\|^2+\|x_3\|^2+\|\underline{x}_2\|^2+\|\underline{x}_1\|^2+\|\underline{x}_3\|^2)$$
$$+\exp(\|\underline{x}_1\|^2+\|x_3\|^2+\|\underline{x}_1\|^2+\|\underline{x}_2\|^2+\|x_3\|^2+\|\underline{x}_2\|^2) - \exp(2\|\underline{x}_1\|^2+2\|\underline{x}_2\|^2+2\|\underline{x}_3\|^2)$$
$$-\exp(\|\underline{x}_2\|^2+\|x_3\|^2+\|\underline{x}_3\|^2+\|\underline{x}_2\|^2+2\|\underline{x}_1\|^2) - \exp(2\|x_3\|^2+\|x_2\|^2+\|x_1\|^2+\|x_1\|^2+\|x_2\|^2) = 0$$

$\because$ 對稱的 $\therefore$ 後面都會剛好消掉

(用 cofactor 算的話)

$$\begin{array}{l} \cancel{2x_1 \quad 12 \quad 13 \quad 14} \to 0 \\ \cancel{21 \quad 2x_2 \quad 23 \quad 24} \to 0 \\ \cancel{31 \quad 32 \quad 2x_3} \quad 34 \to 2\cancel{x_1} + 2\cancel{x_2} + \cancel{x_4} + \cancel{x_3} + \cancel{x_2} + \cancel{x_1} + \cancel{x_4} + \cancel{x_2} + \cancel{x_1} + \cancel{x_3} + \cancel{41} + \cancel{12} + \cancel{23} \\ \cancel{41 \quad 42 \quad 43} \quad 2x_4 \to 0 \qquad - \cancel{13} - \cancel{2x_2} - \cancel{14} - \cancel{23} - \cancel{42} - \cancel{2x_1} - \cancel{43} - \cancel{21} - \cancel{12} = 0 \end{array}$$

$\Rightarrow$ by Sylvester's criterion, G_2 is P.S.D.

$\Rightarrow k_2(\underline{x}, \underline{x}')$ and $k_1(\underline{x}, \underline{x}')^2$ are valid

$\Rightarrow \underline{k(\underline{x}, \underline{x}') \text{ is valid.}}$

b.) if $k_1(\underline{x}, \underline{x}') = \begin{cases} 1 & , \underline{x} = \underline{x}' \\ 0 & , \sim \end{cases} \Rightarrow G_1 = \underline{I}_{N \times N} \Rightarrow$ P.S.D. , valid.

$\Rightarrow k(\underline{x}, \underline{x}') = k_1 - 1 = \begin{cases} 0 & , \underline{x} = \underline{x}' \\ -1 & , \sim \end{cases} \Rightarrow G = \begin{bmatrix} 0 & -1 & \cdots & -1 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \\ -1 & \vdots & \cdots & 0 \end{bmatrix}$ is not P.S.D.

e.g. $G = \begin{bmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$,

< 0

$\det(\underline{G} - \lambda \underline{I}) = -\lambda^3 - 2 = 0 \Rightarrow \lambda = -\sqrt[3]{2}, \frac{\sqrt[3]{2} \pm \sqrt{2^{\frac{2}{3}} - 4 \cdot 2^{\frac{2}{3}}}}{2}$

$(\lambda + \sqrt[3]{2})(\lambda^2 - \sqrt[3]{2}\lambda + 2^{\frac{2}{3}})$ $\Rightarrow$ not valid.

d.) $k_1(x, x')$ is valid , $\underline{K}_1$ has eigenvalues ≥ 0

$\Rightarrow$ Sylvester's law of Inertia $\Rightarrow$ $\underline{K}_1$ has pivots ≥ 0

$\Rightarrow \exp(\underbrace{k_1(x, x')}_{\geq 0}) \geq 1 \Rightarrow \exp(k_1(\underline{x}, \underline{x}')) - 1 \geq 0$

$k_1(\underline{x}, \underline{x}')^2 \geq 0$ (only pivots are ensured)

$\Rightarrow k(\underline{x}, \underline{x}') = k_1(\underline{x}, \underline{x}')^2 + \exp(k_1(\underline{x}, \underline{x}')) - 1 \geq 0$

$\Rightarrow \underline{K}$ has all pivots ≥ 0 and $\underline{K}$ is symmetric

$\Rightarrow$ again, $\underline{K}$ has all eigenvalues ≥ 0.

Q4 :

To minimize $f(x)$ subject to an inequality constraint $\underline{g(x) \ge 0}$

Lagrangian function: $L(x, \lambda) = f(x) - \lambda g(x)$

$f(x) = (x-2)^2$, $\underline{g(x) = 3 - (x+3)(x-1)}$

with constraints : $\frac{\partial L(x,\lambda)}{\partial x} = 0 = 2(x-2) - \lambda(-2x-2)$

$$\Rightarrow 2x - 4 + 2\lambda x + 2\lambda = 0$$

$$\Rightarrow x = \frac{4-2\lambda}{2+2\lambda} \quad \cdots ①$$

$$\lambda \ge 0 \quad \cdots ②$$

dual representation :

$$L(\lambda) = \left(\frac{-\lambda+2}{\lambda+1} - 2\right)^2 - \lambda\left[3 - \left(\frac{-\lambda+2}{\lambda+1}\right)^2 - 2\left(\frac{-\lambda+2}{\lambda+1}\right) + 3\right]$$

subject to : $\lambda \ge 0$

$$= t^2 - 4t + 4 - 6\lambda + \lambda t^2 + 2\lambda t$$

$$= (\lambda+1)t^2 + (2\lambda-4)t + 4 - 6\lambda$$

$$= \frac{(-\lambda+2)^2}{\lambda+1} + \frac{(2\lambda-4)(-\lambda+2)}{\lambda+1} + \frac{(4-6\lambda)(\lambda+1)}{\lambda+1}$$

$$= \frac{(1-4+4)+(-2+8-8)+(-6-2+4)}{\lambda+1}$$

$$= \frac{-7\lambda^2+2\lambda}{\lambda+1} = 0$$

$$\frac{(-14\lambda+2)(\lambda+1)+(-7\lambda^2+2\lambda)}{\cancel{(\lambda+1)^2}}$$

$$x = \frac{4-2\lambda}{2+2\lambda} \approx 1.6\cdots \quad ✓$$

$$= -14\lambda^2 - 12\lambda + 2$$

$$\Rightarrow -21\lambda^2 - 10\lambda + 2 = 0$$

$$\frac{-10 \pm \sqrt{100 + \overset{268}{168}}}{42} \approx 0.15$$

- (橫線下面是驗算)，x和KKT算出來($\sqrt{7}-1$)差不多