For integers $k \ge 0$ and $j \ge 1$, we define the function $A_k(j)$ as

$$A_k(j) = \begin{cases} j+1, & \text{if } k=0 \\ A_{k-1}^{(j+1)}(j), & \text{if } k \ge 1 \end{cases}, \quad \text{where } \underline{A_{k-1}^{(0)}(j) = j} \text{ and } A_{k-1}^{(i)}(j) = A_{k-1}\left(A_{k-1}^{(i-1)}(j)\right) \text{ for } i \ge 1.$$

Lemma 21.2

For any integer $j \ge 1$, we have $A_1(j) = 2j+1$.

→ First use induction to prove $A_0^{(i)}(j) = j+i$. For $i=0$, $A_0^{(0)}(j) = j = j+0$.

→ Assume that for some $i>0$, we have $A_0^{(i-1)}(j) = j+(i-1)$

→ Then, $A_0^{i}(j) = A_0\left(A_0^{(i-1)}(j)\right) = \underline{A_0(j+(i-1)) = (j+(i-1))+1 = j+i}$ ①

→ Finally, $\underbrace{A_1(j) = A_0^{(j+1)}}_{\text{by def.}}(j) = j+(j+1) = 2j+1$ #

Lemma 21.3

For any integer $j \ge 1$, we have $A_2(j) = 2^{j+1}(j+1)-1$.

→ First use induction to prove $A_1^{(i)}(j) = 2^{i}(j+1)-1$. For $i=0$, $A_1^{0}(j) = j = 2^{0}(j+1)-1$.

→ Assume that for some $i>0$, we have $A_1^{(i-1)}(j) = 2^{i-1}(j+1)-1$.

→ $A_1^{(i)}(j) = A_1\left(A_1^{(i-1)}(j)\right) = A_1\left(2^{i-1}(j+1)-1\right) = 2 \times \left(2^{i-1}(j+1)-1\right)+1 = 2^{i}(j+1)-1$, by Lemma 21.2

→ Finally, $\underbrace{A_2(j) = A_1^{(j+1)}(j)}_{\text{by def.}} = 2^{j+1}(j+1)-1$ #

$$A_4(1) = A_3^{(2)}(1)$$

$$= A_3(A_3(1))$$

$$= A_3(A_2^{(2)}(1))$$

$$= A_3(A_2(A_2(1)))$$

$$= A_3(A_2(A_1^{(2)}(1)))$$

$$= A_3(A_2(A_1(A_1(1))))$$

$$= A_3(A_2(A_1(2\times1+1)))$$

$$= A_3(A_2(2\times3+1))$$

$$= A_3(2^8\times8-1)$$

$$= A_2^{(2^8\times8-1+1)}(2^{11}-1)$$

$$\gg A_2(2047)$$

$$= 2^{2048}\cdot(2048)-1$$

$$> 2^{2048}$$

#

$$\log_{10} 2^{2048}$$

$$= 2048 \log_{10} 2$$

$$\approx 2048 \times 0.3$$

$$= 614.4 \Rightarrow 615 \text{ digits in } 2^{2048}$$

$$\Rightarrow 2^{2048} \gg 10^{80}$$

#