1. Define the amortized cost $\hat{c}_i$ of the i-th operation as $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$, where c_i is the actual cost of the i-th operation, D_i is the data structure after i-th operation.

Define $\underline{size(D)}$ as the size of table and $\underline{num(D)}$ as the number of items in the table.

Define the potential function as in textbook: $\Phi(D_i) = \begin{cases} 2\,num(D_i) - size(D_i) & \text{if } \alpha(D_i) \geq \frac{1}{2} \\ size(D_i)/2 - num(D_i) & \text{if } \alpha(D_i) < \frac{1}{2} \end{cases}$

Ⓐ If the i-th operation is insertion:

Ⓐ-1: if $\alpha_{i-1} \geq \frac{1}{2}$:

- If it triggers expansion: $\hat{c}_i = c_i + \big(2num(D_i) - size(D_i)\big) - \big(2num(D_{i-1}) - size(D_{i-1})\big)$

$= num(D_i) + 2num(D_i) - (2num(D_i) - 2) - (2num(D_i) - 2 - (num(D_i) - 1))$

$= num(D_i) + 2 - num(D_i) + 1 = 3$ #

- If no expansion: $\hat{c}_i = c_i + \big(2num(D_i) - size(D_i)\big) - \big(2num(D_{i-1}) - size(D_{i-1})\big)$

$= 1 + \big(2num(D_i) - size(D_i)\big) - \big(2num(D_i) - 2 - size(D_i)\big) = 3$ #

Ⓐ-2: if $\alpha_{i-1} < \frac{1}{2}$, but $\alpha_i \geq \frac{1}{2}$: $\hat{c}_i = c_i + \big(2num(D_i) - size(D_i)\big) - \big(size(D_{i-1})/2 - num(D_{i-1})\big)$

(no expansion)

$= 1 + (2num(D_{i-1}) + 2 - size(D_{i-1})) - (size(D_{i-1})/2 - num(D_{i-1}))$

$\alpha_{i-1} = \dfrac{num(D_{i-1})}{size(D_{i-1})}$

$= 3num(D_{i-1}) - \frac{3}{2}size(D_{i-1}) + 3$

$= 3\alpha_{i-1}\, size(D_{i-1}) - \frac{3}{2}size(D_{i-1}) + 3 < \frac{3}{2}size(D_{i-1}) - \frac{3}{2}size(D_{i-1}) + 3 = 3$ #

Ⓐ-3: if $\alpha_{i-1} < \alpha_i < \frac{1}{2}$: $\hat{c}_i = c_i + \big(size(D_i)/2 - num(D_i)\big) - \big(size(D_{i-1})/2 - num(D_{i-1})\big)$

(no expansion)

$= 1 + (size(D_i)/2 - num(D_i)) - (size(D_i)/2 - (num(D_i) - 1))$

$= 1 - 1 = 0$ #

(B) If the i-th operation is deletion: ($num(D_i) = num(D_{i-1}) - 1$)

(B)-1 : if $\alpha_{i-1} < \frac{1}{2}$:

- if it triggers contraction : $\hat{c}_i = c_i + \left(\frac{size(D_i)}{2} - num(D_i)\right) - \left(\frac{size(D_{i-1})}{2} - num(D_{i-1})\right)$

($num(D_{i-1}) = \frac{size(D_{i-1})}{4} = \frac{size(D_i)}{2}$)

$$= (num(D_i)+1) + \left(\frac{size(D_i)}{2} - num(D_i)\right) - \left(size(D_i) - \frac{size(D_i)}{2}\right)$$

$$= 1 \ \#$$

- If no contraction : $\hat{c}_i = c_i + \left(\frac{size(D_i)}{2} - num(D_i)\right) - \left(\frac{size(D_{i-1})}{2} - num(D_{i-1})\right)$

($size(D_{i-1}) = size(D_i)$)

$$= 1 + \left(\frac{size(D_i)}{2} - num(D_i)\right) - \left(\frac{size(D_{i-1})}{2} - (num(D_i)+1)\right) = 2 \ \#$$

(B)-2 : if $\alpha_{i-1} \geq \frac{1}{2}$, $\alpha_i < \frac{1}{2}$: $\hat{c}_i = c_i + \left(\frac{size(D_i)}{2} - num(D_i)\right) - \left(2num(D_{i-1}) - size(D_{i-1})\right)$

(no contraction)

$$= 1 + \frac{3}{2} size(D_{i-1}) - 3\, num(D_{i-1}) + 1$$

$$= 2 + \frac{3}{2} size(D_{i-1}) - 3\alpha_{i-1} size(D_{i-1}) \leq 2 + \frac{3}{2} size(D_{i-1}) - \frac{3}{2} size(D_{i-1}) = 2 \ \#$$

(B)-3 : if $\frac{1}{2} \leq \alpha_i < \alpha_{i-1}$: $\hat{c}_i = c_i + (2num(D_i) - size(D_i)) - (2num(D_{i-1}) - size(D_{i-1}))$

(no contraction)

$$= 1 + (2\, num(D_{i-1}) - 2 - size(D_{i-1})) - (2num(D_{i-1}) - size(D_{i-1}))$$

$$= -1 \ \#$$