

# Combinatorial Mathematics

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Monday 18:30 – 20:20

# Outline

- The Double-Counting Principle (P.38)
  - The Handshaking Lemma
  - Average Number of Divisors
  - Turán Number
  - Pascal Triangle & Binomial Identities
  - Catalan Numbers
- The Density of 0-1 Matrices (P.46)

# The Double Counting Principle

If the elements of a set are counted in two different ways,  
the answers are the same.

**Lemma 1.**

In any graph  $G = (V, E)$ ,  
the number of vertices with odd degrees is even.

- For each  $v \in V$ , let  $d(v)$  denote the degree of  $v$ .

Then, we have

$$\sum_{v \in V} d(v) = 2 \cdot |E|.$$

Each edge is counted exactly twice.

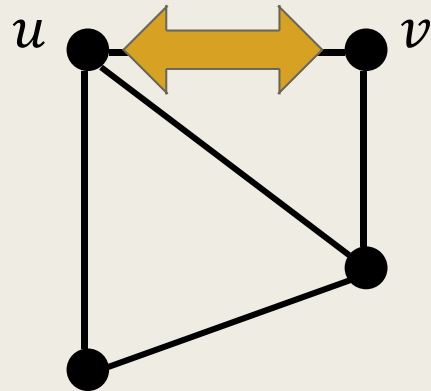
$2 \cdot |E|$  is an even number.

Hence, the number of vertices  
with odd degree must be even.

### Handshaking Lemma.

At a party, the number of guests who shake hands an odd number of times is even.

- Consider the graph  $G = (V, E)$  defined on the guests, where  $(u, v) \in E$  if and only if guest  $u$  and  $v$  have shaken hands.



- Let  $F$  be a **set family** on a ground set  $X$ , i.e.,  $F$  is a collection of subsets of  $X$ .
  - For any  $x \in X$ , let  $d(x)$  be the number of sets in  $F$  that contain the element  $x$ , i.e., the degree of  $x$  in  $F$ .

#### The sets in $F$

$$A_1 = \{e_2, e_4\}$$

$$A_3 = \{e_3, e_2, e_6\}$$

$$A_2 = \{e_1, e_5, e_7\}$$

$$A_4 = \{e_1, e_2, e_6, e_7\}$$

#### The ground set $X$ of elements

●  $e_1$

●  $e_2$

●  $\vdots$

●  $e_n$

- The previous identity,  $\sum_v \deg(v) = 2|E|$ , in Lemma 1 is a special case of the following general identity.

**Proposition 2.**

Let  $F$  be a family of subsets of some ground set  $X$ . Then

$$\sum_{x \in X} d(x) = \sum_{A \in F} |A| .$$

- Consider the  $|X| \times |F|$  incidence matrix  $M = (m_{x,A})$ , where

$$m_{x,A} = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

	$A_1$	$A_2$	$A_3$	$\dots$	$A_{ F }$
$x_1$					
$x_2$					
$\vdots$					
$x_{ X }$					

The matrix  $M$

## Proposition 2.

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- Consider the  $|X| \times |F|$  incidence matrix  $M = (m_{x,A})$ ,

where

$$m_{x,A} = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

- Then,

- The  $d(x)$  is the number of 1s in the  $x$ -th row.
- $|A|$  is the number of 1s in the  $A$ -th column.

	$A_1$	$A_2$	$A_3$	$\cdots$	$A_{ F }$
$x_1$					
$x_2$					
$\vdots$					
$x_{ X }$					

The matrix  $M$

- ***The identity counts the number of 1s in the matrix  $M$ .***



- Note that, the concept of ***set family*** is equivalent to ***hypergraphs***, where
  - The elements in  $X$  are the set vertices, and
  - The subsets in  $F$  are the set of hyperedges.

# Average Number of Divisors

# The Number of Divisors

- How many numbers from  $\{1, 2, \dots, n\}$  is a divisor of  $n$  ?
  - For any  $n \geq 1$ , let  $t(n)$  be the number of divisors of  $n$ .

■ Ex.

$t(p) = 2$  for any prime number  $p$ .

$t(2^m) = m + 1$  for any integer  $m \geq 1$ .

# Average Number of Divisors

- How many numbers from  $\{1, 2, \dots, n\}$  is a divisor of  $n$  ?
  - For any  $n \geq 1$ , let  $t(n)$  be the number of divisors of  $n$ .
  - While  $t(n)$  varies a lot for different choices of  $n$ , we will see that, **the average number of divisors,**

$$T(n) := \frac{1}{n} \cdot \sum_{1 \leq i \leq n} t(i)$$

is quite stable and is roughly  $\ln n$  for all  $n$ .

**Proposition 1.10.**

For any  $n \geq 1$ ,

$$|T(n) - \ln n| \leq 1.$$

- Consider the  $n \times n$  0-1 matrix  $M = (m_{i,j})$ , where  $m_{i,j} = 1$  if and only  $i$  is a divisor of  $j$ .

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		1		1		1		1	
3			1			1			1
4				1				1	

The number of 1s  
in the  $i$ -th column is  
exactly  $t(i)$ .

- Consider the  $n \times n$  0-1 matrix  $M = (m_{i,j})$ , where  
 $m_{i,j} = 1$  if and only  $i$  is a divisor of  $j$ .

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		1		1		1		1	
3			1			1			1
4				1				1	

The number of 1s  
in the  $i$ -th column is  
exactly  $t(i)$ .

The number of 1s  
in the  $i$ -th row is  $\lfloor n/i \rfloor$ .

- Counting the number of 1s in the matrix,

we have

$$\sum_{1 \leq i \leq n} \left\lfloor \frac{n}{i} \right\rfloor = \sum_{1 \leq i \leq n} t(i) = n \cdot T(n).$$

by the definition of  $T(n)$ .

- We have 
$$\sum_{1 \leq i \leq n} \left\lfloor \frac{n}{i} \right\rfloor = \sum_{1 \leq i \leq n} t(i) = n \cdot T(n).$$

- Since  $x - 1 \leq \lfloor x \rfloor \leq x$  holds for every real number  $x$ , we obtain

$$n \cdot \sum_{1 \leq i \leq n} \frac{1}{i} - n \leq n \cdot T(n) \leq n \cdot \sum_{1 \leq i \leq n} \frac{1}{i}.$$

- The  $n^{\text{th}}$ -harmonic number,  $H_n := \sum_{1 \leq i \leq n} \frac{1}{i}$ , satisfies  $H_n = \ln n + \gamma_n$  for some  $0 \leq \gamma_n \leq 1$ .
- Hence, we obtain  $\ln n - 1 \leq T(n) \leq \ln n + 1$ .

# Turán Number



# Turán Number $T(n, k, l)$

- Consider any ground set  $X$  with  $n$  elements.
- For any  $l \leq k \leq n$ ,  
*the Turán number  $T(n, k, l)$  is the smallest number of  $l$ -element subsets of  $X$  such that*

*every  $k$ -element subset of  $X$  contains*

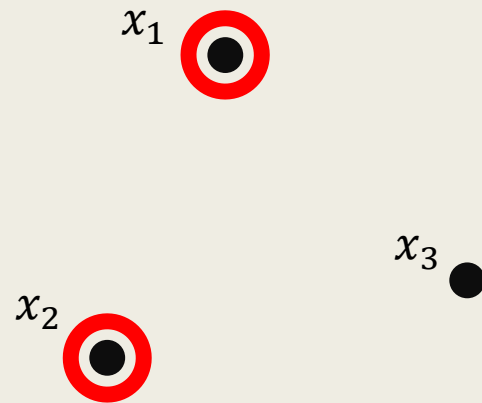
*at least one of these  $l$ -element subsets.*

How many  $l$ -element subsets do we need?

# Turán Number $T(n, k, l)$

- For any  $n = 3, k = 2, l = 1$ , we have

$$T(3, 2, 1) = 2.$$



One way to achieve this

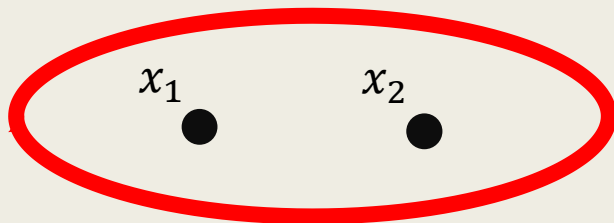
Any 2-element subset  
must contain  $\{x_1\}$  or  $\{x_2\}$ .

It won't suffice,  
if only one 1-element subset was chosen.

# Turán Number $T(n, k, l)$

- For any  $n = 4$ ,  $k = 3$ ,  $l = 2$ , we have

$$T(4, 3, 2) = 2.$$



Any 3-element subset  
must contain  $\{x_1, x_2\}$  or  $\{x_3, x_4\}$ .

One way to achieve this

It won't suffice,  
if only one 2-element subset was chosen.

**Proposition 1.9.**

For all positive integers  $l \leq k \leq n$ ,

$$T(n, k, l) \geq \binom{n}{l} / \binom{k}{l}.$$

- Let  $F$  be a smallest  $l$ -uniform family over  $X$  such that every  $k$ -element subset of  $X$  contains at least one member of  $F$ .
  - In the following,  
we derive a lower-bound on  $|F|$ .

- Consider a 0-1 matrix  $M = (m_{A,B})$  with size  $|F| \times \binom{n}{k}$ , where
  - The rows are indexed by sets  $A$  in  $F$  and
  - The columns are indexed by all possible  $k$ -element subsets of  $X$ ,

and

$$m_{A,B} = \begin{cases} 1, & \text{if } A \subseteq B, \\ 0, & \text{otherwise.} \end{cases}$$

For each  $l$ -element subset  $A$ , the number of  $k$ -element subsets that contains  $A$  is exactly  $\binom{n-l}{k-l}$ .

	$B_1$	$B_2$	$B_3$	$\dots$	$B_{\binom{n}{k}}$
$A_1$					
$A_2$					
$\vdots$					
$A_{ F }$					

The matrix  $M$

Since every  $k$ -element subset of  $X$  contains at least one member of  $F$ , there is at least one 1 in each column.

For each  $l$ -element subset  $A$ ,  
the number of  $k$ -element subsets  
containing the set  $A$  is exactly  $\binom{n-l}{k-l}$ .

	$B_1$	$B_2$	$B_3$	$\dots$	$B_{\binom{n}{k}}$
$A_1$					
$A_2$					
$\vdots$					
$A_{ F }$					

The matrix  $M$

Since every  $k$ -element subset of  $X$   
contains at least one member of  $F$ ,  
there exists at least one 1  
in each column.

- Let  $r_A$  be the number of 1s in row  $A$  and  $c_B$  the number of 1s in column  $B$ .
- Counting the number of 1s in  $M$ , we have

$$|F| \cdot \binom{n-l}{k-l} = \sum_{A \in F} r_A = \sum_B c_B \geq 1 \cdot \binom{n}{k},$$

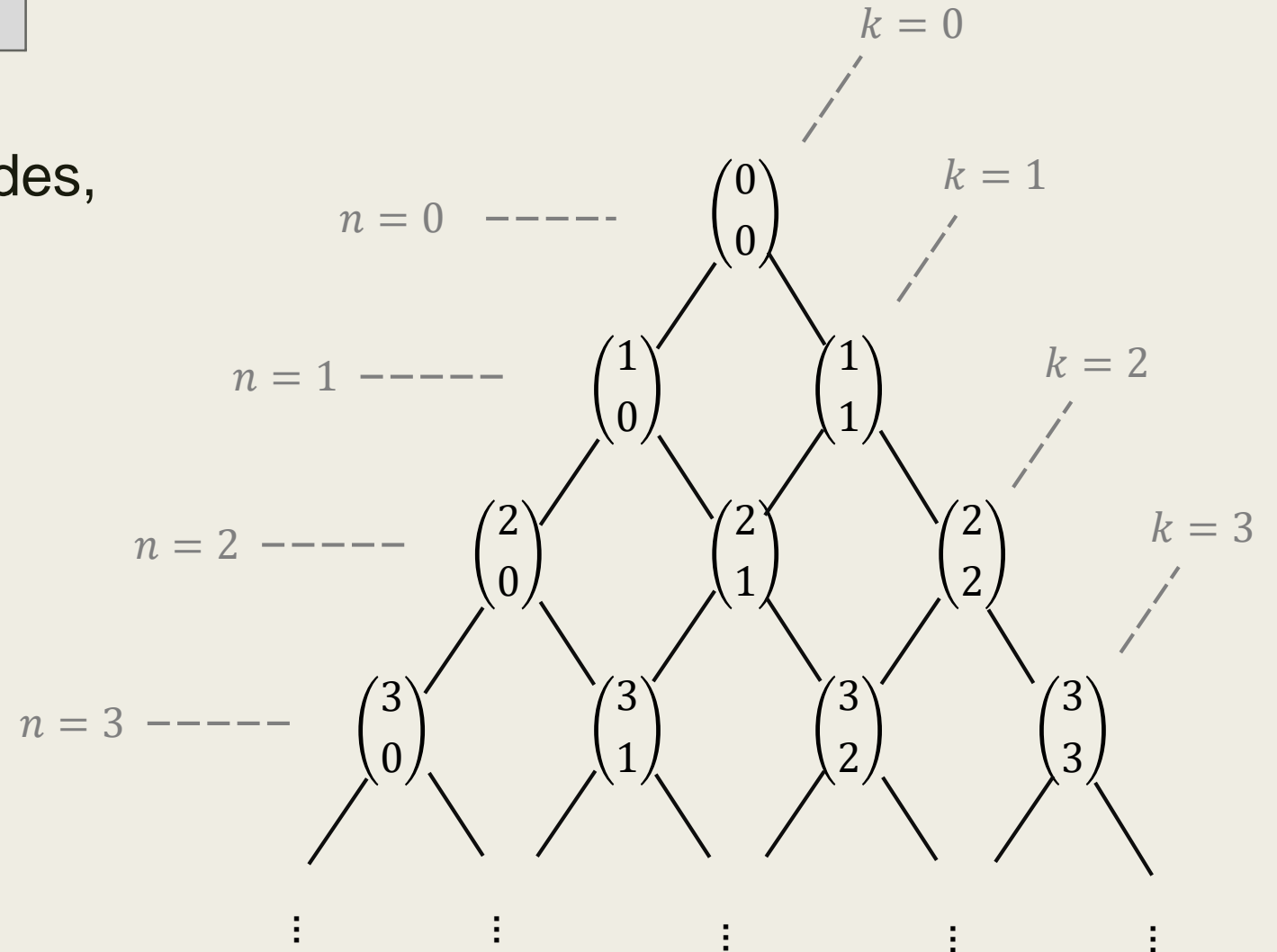
and

$$T(n, k, l) = |F| \geq \binom{n}{k} / \binom{n-l}{k-l} = \binom{n}{l} / \binom{k}{l}.$$

# The Pascal Triangle & Binomial Identities

## The Pascal's Triangle

- Consider the pyramid of nodes, formed by the binomial coefficients.

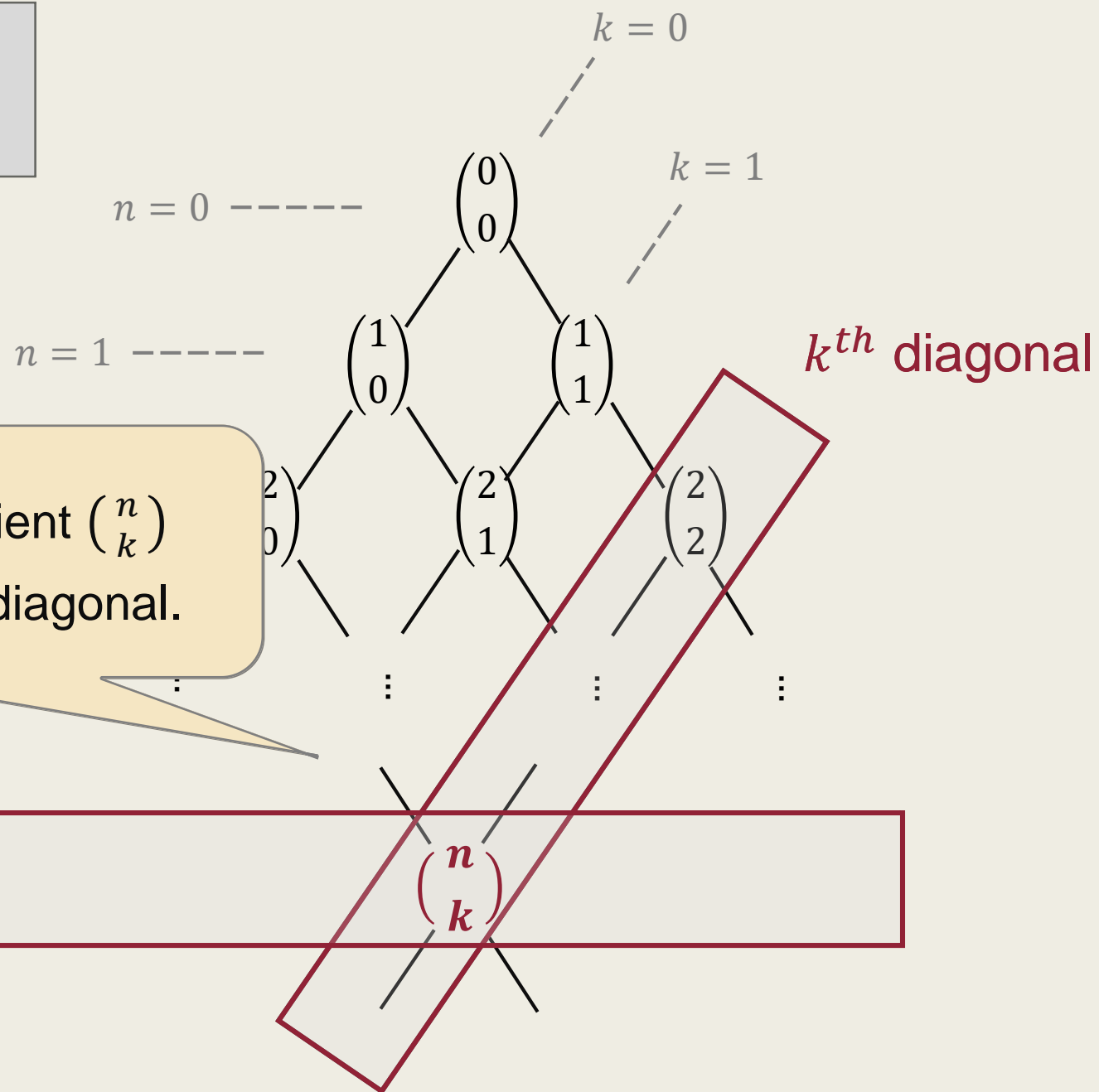




## The Pascal's Triangle

We place the binomial coefficient  $\binom{n}{k}$  in the cell at the  $n^{th}$  row,  $k^{th}$  diagonal.

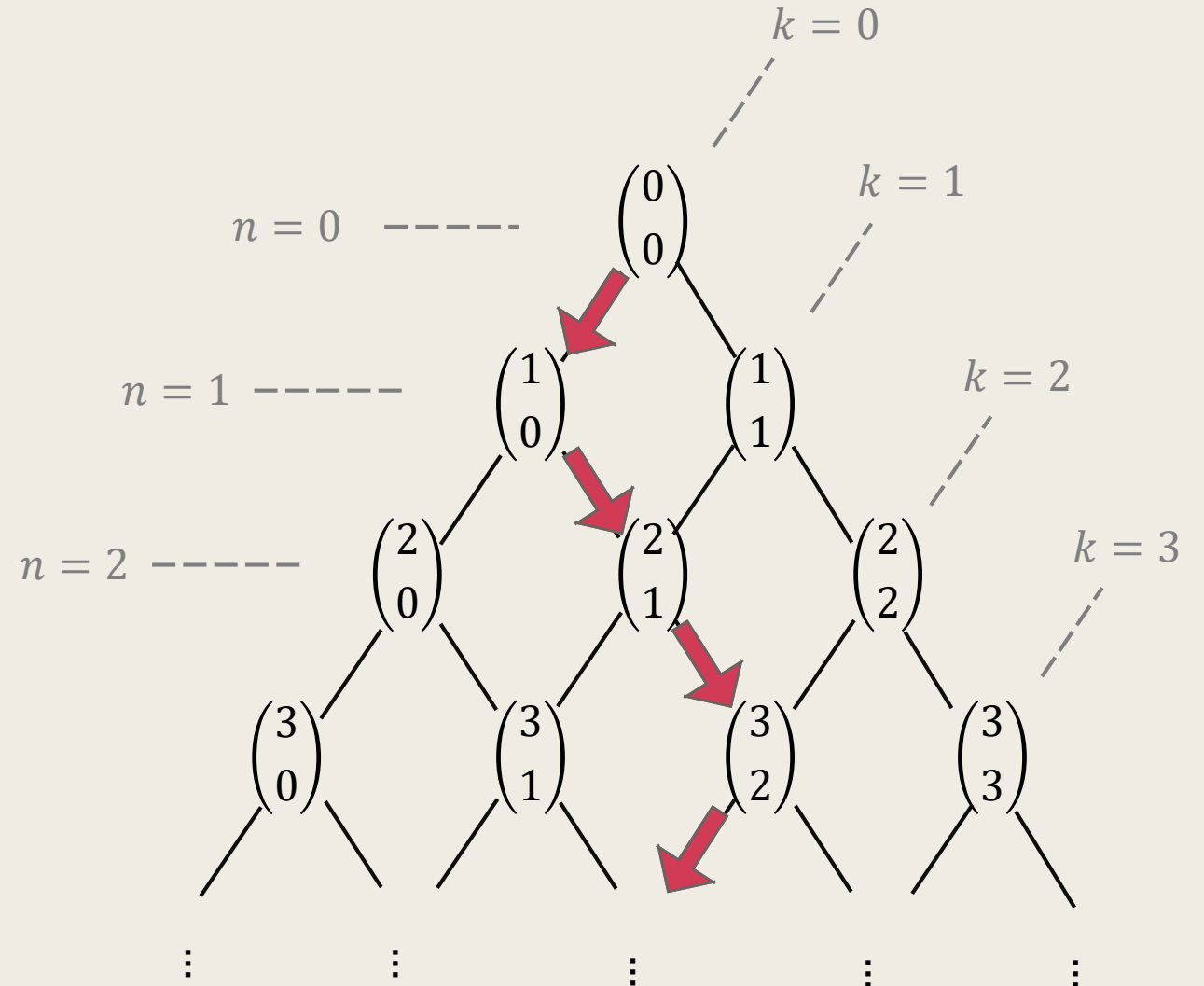
$n^{th}$  row



## The Pascal's Triangle

- Consider any downward path from  $(0,0)$  to  $(n,k)$ .
  - Only 'L' or 'R' is allowed.

- It must use  $\begin{cases} k & \text{'R's} \\ n - k & \text{'L's} \end{cases}$ .
- The number of such paths is exactly  $\binom{n}{k}$ .

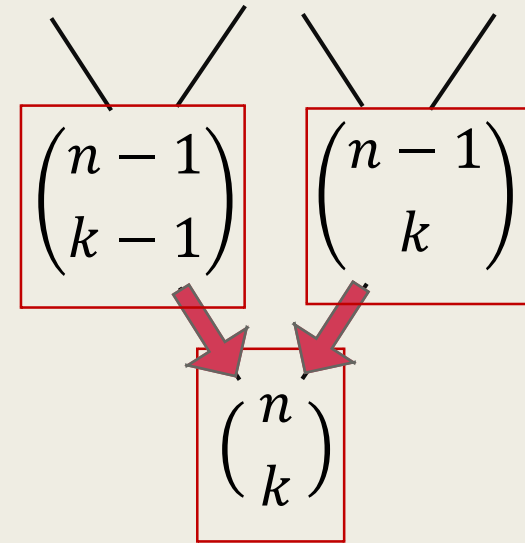


### Lemma 3.

For any  $n, k \in \mathbb{N}$  with  $n > k$ , we have

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

- Any downward path to  $(n, k)$  must pass  $(n-1, k)$  or  $(n-1, k-1)$ .
- The number of downward paths to  $(n, k)$  equals the sum of number of paths to  $(n-1, k)$  and  $(n-1, k-1)$ .



#### Lemma 4.

For any  $n \in \mathbb{Z}^{\geq 0}$ , we have 
$$\sum_{0 \leq k \leq n} \binom{n}{k} = 2^n.$$

- Consider the number of all possible *downward paths to the  $n^{\text{th}}$  row*.
  - It is the sum of the number of possible paths to each cell, which is  $\sum_{0 \leq k \leq n} \binom{n}{k}$ .
  - It is also the number of possible arrangements (permutations) with a total number of  $n$  'L's or 'R's, and is hence  $2^n$ .
  - By the double-counting principle, they are equal.

**Lemma 5.**

For any  $n, r \in \mathbb{Z}^{\geq 0}$ ,  $n \geq r$ ,

$$\sum_{0 \leq k \leq n-r} \binom{r+k}{r} = \binom{n+1}{r+1}.$$

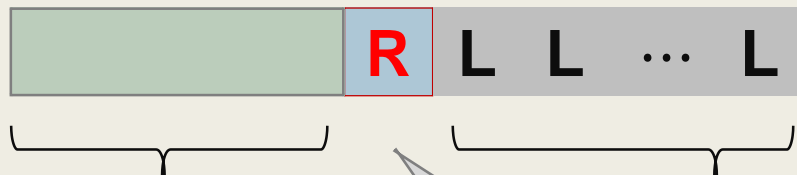
- Consider the set of all possible *downward paths* to  $(n+1, r+1)$ .
  - There are  $\binom{n+1}{r+1}$  such paths.
  - Any of such paths must **go to some cell at the  $r^{\text{th}}$ -diagonal**, then followed by **exactly one ‘R’** and then **some ‘L’s**.

### Lemma 5.

For any  $n, r \in \mathbb{Z}^{\geq 0}$ ,  $n \geq r$ ,

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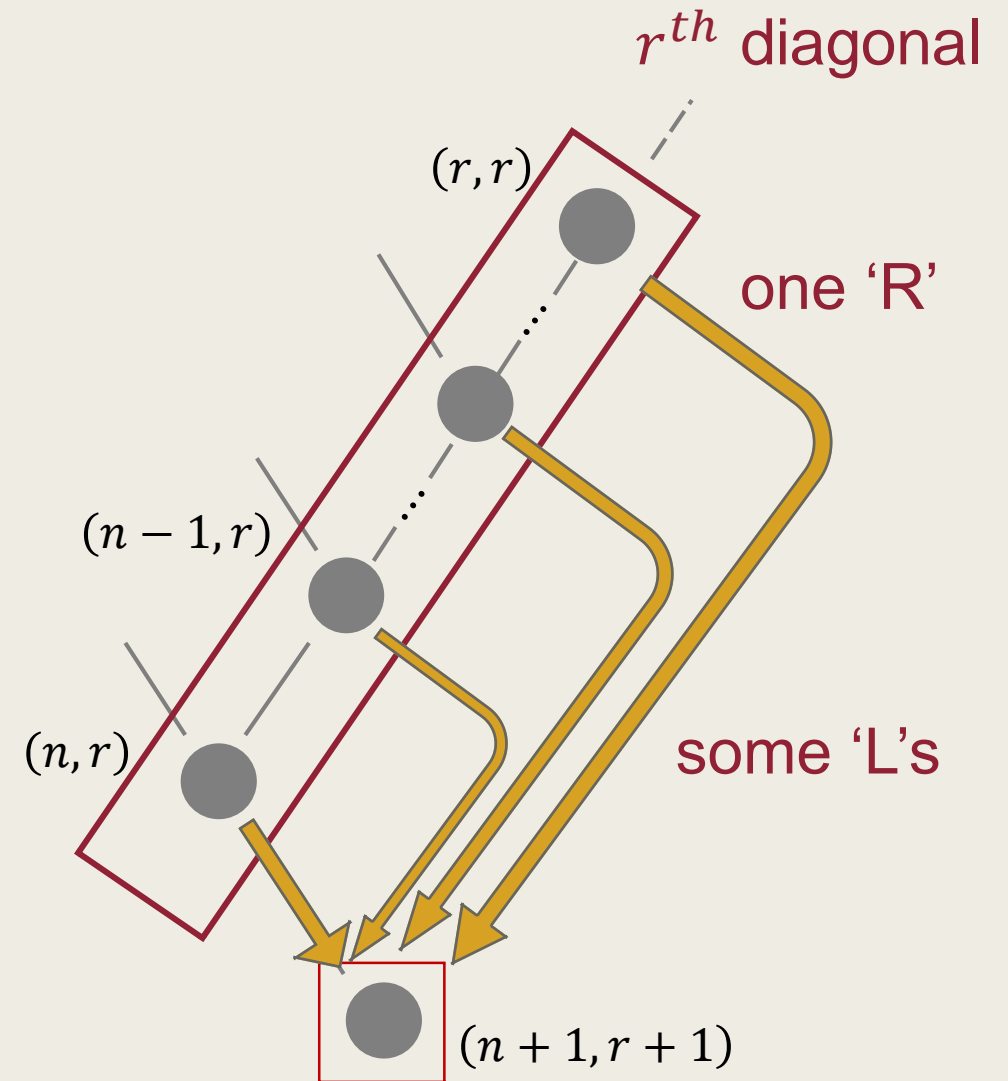
A downward path to  $(n+1, r+1)$



A downward path  
to  $(\ell, r)$   
for some  $r \leq \ell \leq n$

Zero or more 'L's

The last 'R' in the path.



- Consider the set of all possible *downward paths* to  $(n + 1, r + 1)$ .
  - There are  $\binom{n+1}{r+1}$  such paths.
  - Any of such paths must ***go to some cell at the  $r^{\text{th}}$ -diagonal***, then followed by ***exactly one ‘R’*** and then ***some ‘L’s’***.
    - Identify such paths by its last ‘R’.
    - Then, there are  $\sum_{r \leq \ell \leq n} \binom{\ell}{r} = \sum_{0 \leq k \leq n-r} \binom{r+k}{r}$  such paths.
  - By double counting principle, they are equal.

**Lemma 6.**

For any  $n \in \mathbb{Z}^{\geq 0}$ ,

$$\sum_{0 \leq k \leq n} \binom{n}{k}^2 = \binom{2n}{n}.$$

- Consider the set of all possible *downward paths* to  $(2n, n)$ .
  - There are  $\binom{n+1}{r+1}$  such paths.
  - Identify any of such paths by the cell it reaches at the  $n^{th}$ -row.
    - Suppose that it is  $(n, k)$ .
    - Let's count the number of such paths.

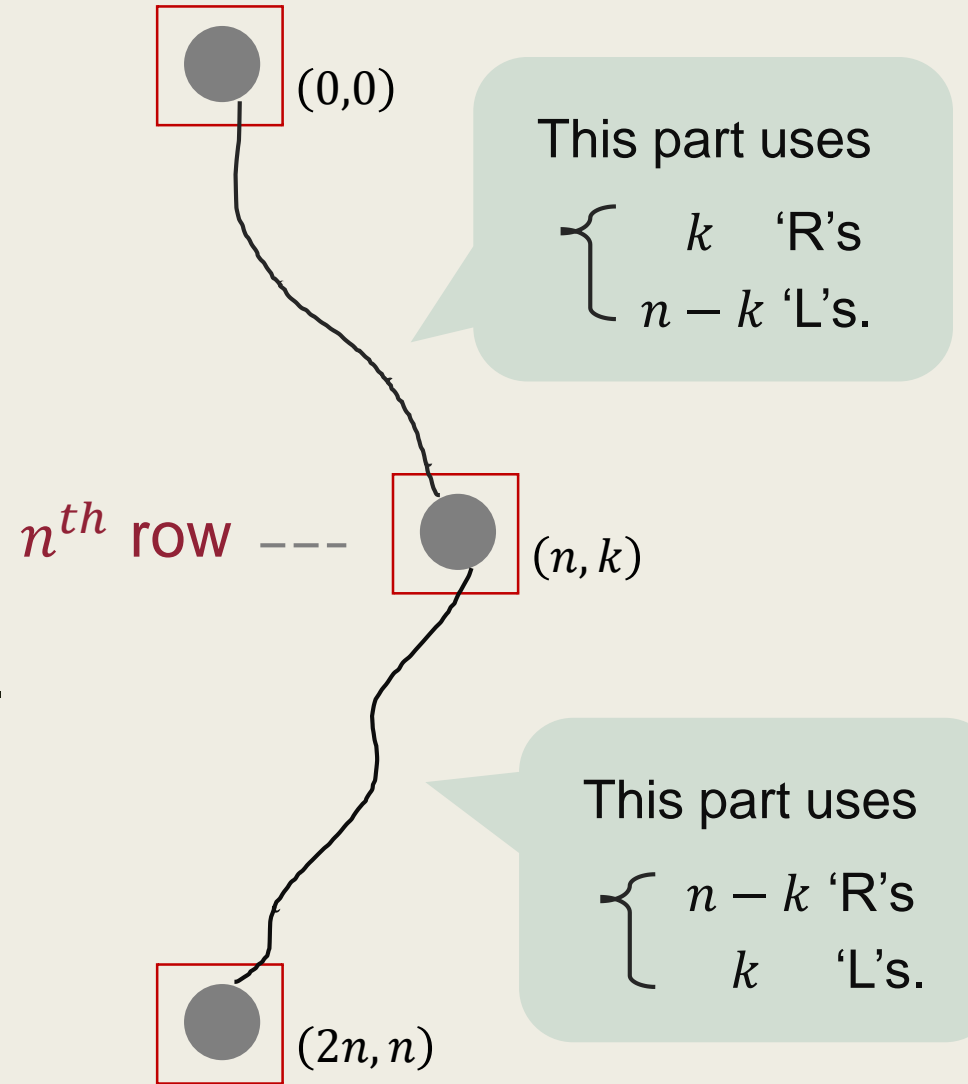


**Lemma 6.**

For any  $n \in \mathbb{Z}^{\geq 0}$ ,

$$\sum_{0 \leq k \leq n} \binom{n}{k}^2 = \binom{2n}{n}.$$

- The upper-part uses  $k$  'R's and  $n - k$  'L's.
  - There are  $\binom{n}{k}$  such paths.
- The lower-part uses  $n - k$  'R's and  $k$  'L's.
  - There are  $\binom{n}{n-k} = \binom{n}{k}$  such paths.

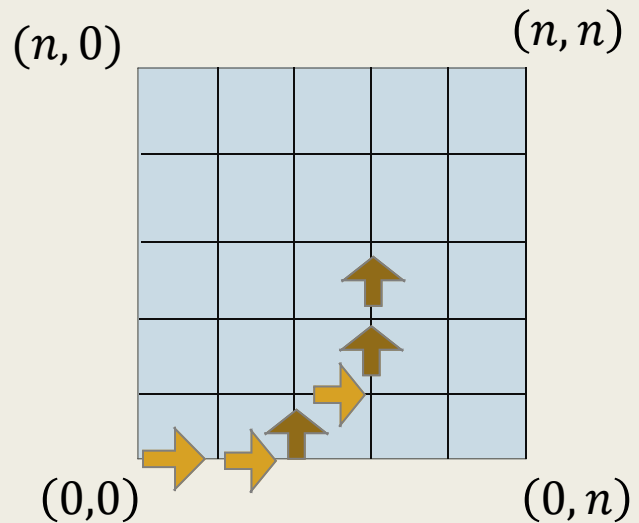


- Consider the set of all possible *downward paths* to  $(2n, n)$ .
  - There are  $\binom{n+1}{r+1}$  such paths.
  - Identify any of such paths by the cell it reaches at the  $n^{th}$ -row.
    - Suppose that it is  $(n, k)$ .
    - By the above argument, there are  $\binom{n}{k}^2$  such paths.
  - Taking summation over the cells at the  $n^{th}$ -row,  
there are  $\sum_{0 \leq k \leq n} \binom{n}{k}^2$  such paths.
- By the double-counting principle, they are equal.

# The Catalan Numbers

# The Catalan Number $C_n$

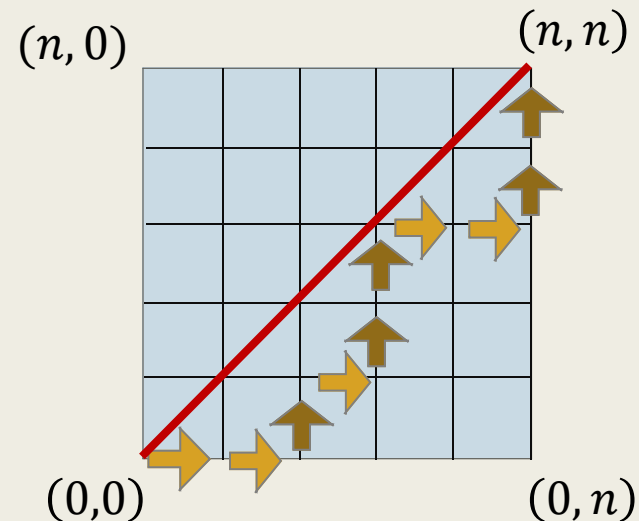
- Consider the  $n \times n$  grid points and any path from  $(0,0)$  to  $(n,n)$  that uses only 'up' and 'right'.



# The Catalan Number $C_n$

- Consider the  $n \times n$  grid points and any path from  $(0,0)$  to  $(n,n)$  that uses only 'up' and 'right'.
- Define the Catalan number  $C_n$  to be the number of possible paths that never cross the diagonal connecting  $(0,0)$  and  $(n,n)$ .
- We will prove that

$$C_n = \binom{2n}{n} - \binom{2n}{n-1}.$$

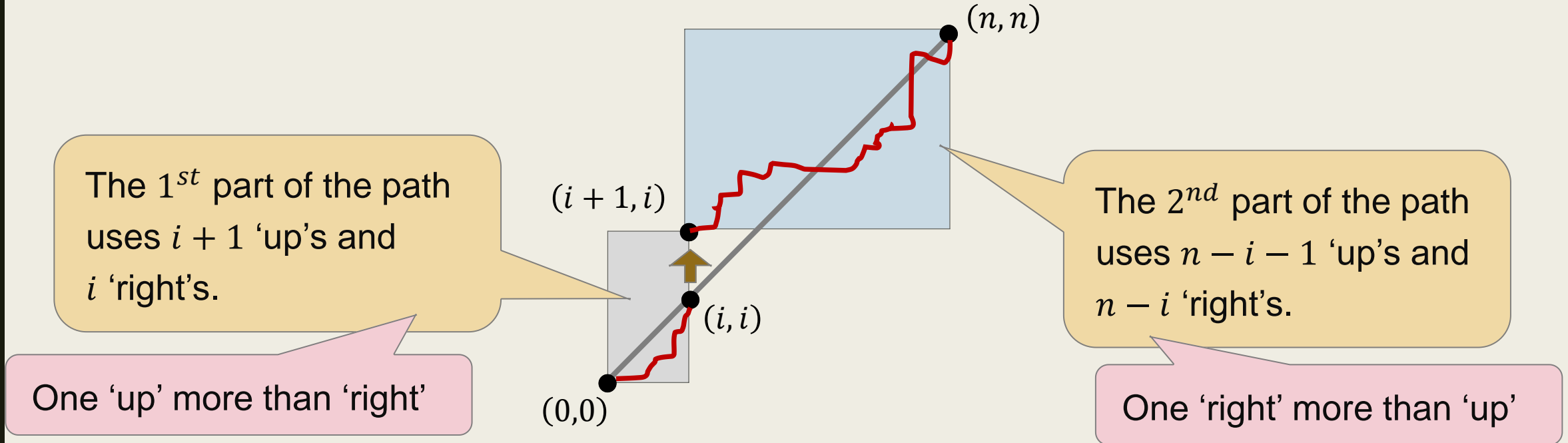


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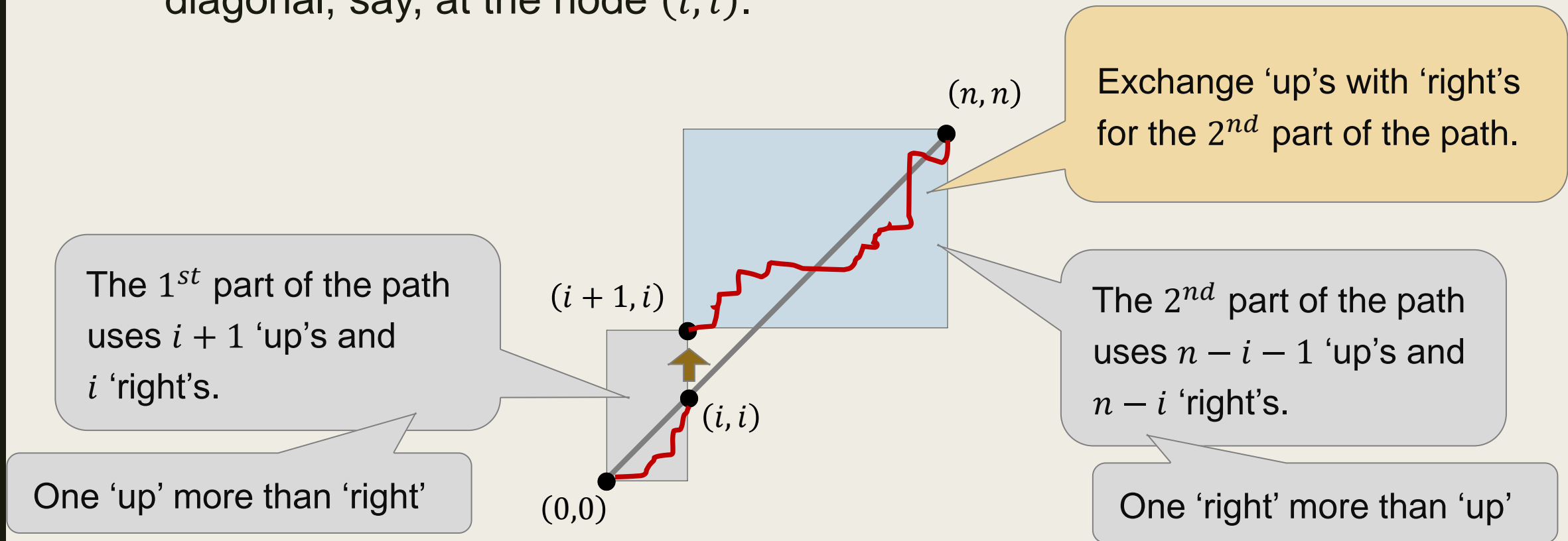
- It is clear that without extra restrictions, the set of all possible paths that go from  $(0,0)$  to  $(n,n)$  is  $\binom{2n}{n}$ .

- It suffices to prove that,  
the number of paths that have crossed the diagonal is exactly  $\binom{2n}{n-1}$ .

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- Consider any of such path and the first time that it crosses the diagonal, say, at the node  $(i, i)$ .



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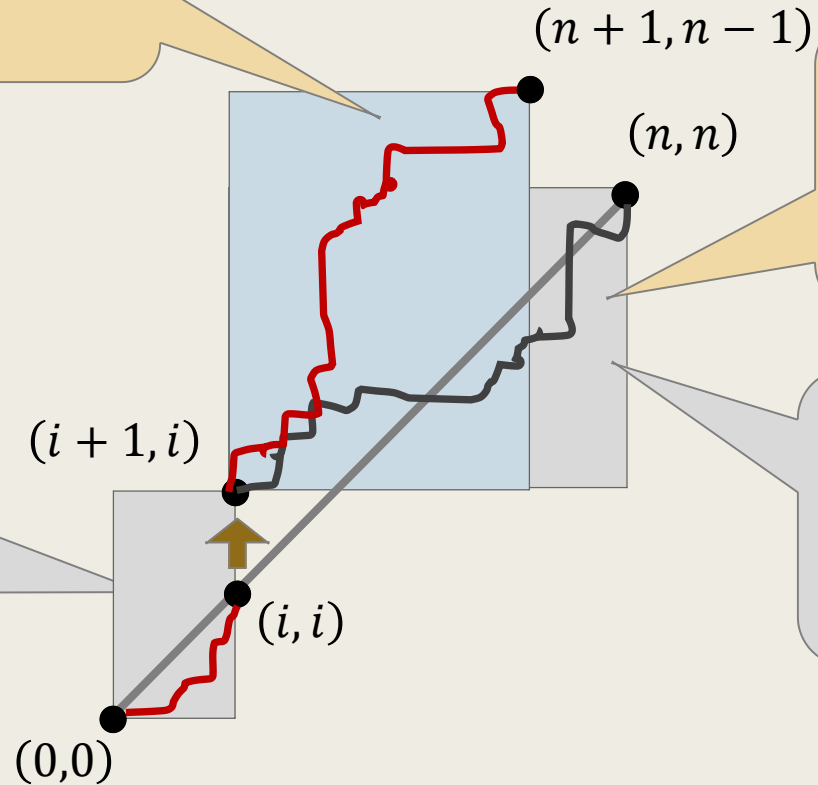
Hence, it will reach  $(n + 1, n - 1)$ .

The new path has  $n - i$  'up's and  $n - i - 1$  'right's.

One 'up' more than 'right'

The 1<sup>st</sup> part of the path uses  $i + 1$  'up's and  $i$  'right's.

One 'up' more than 'right'

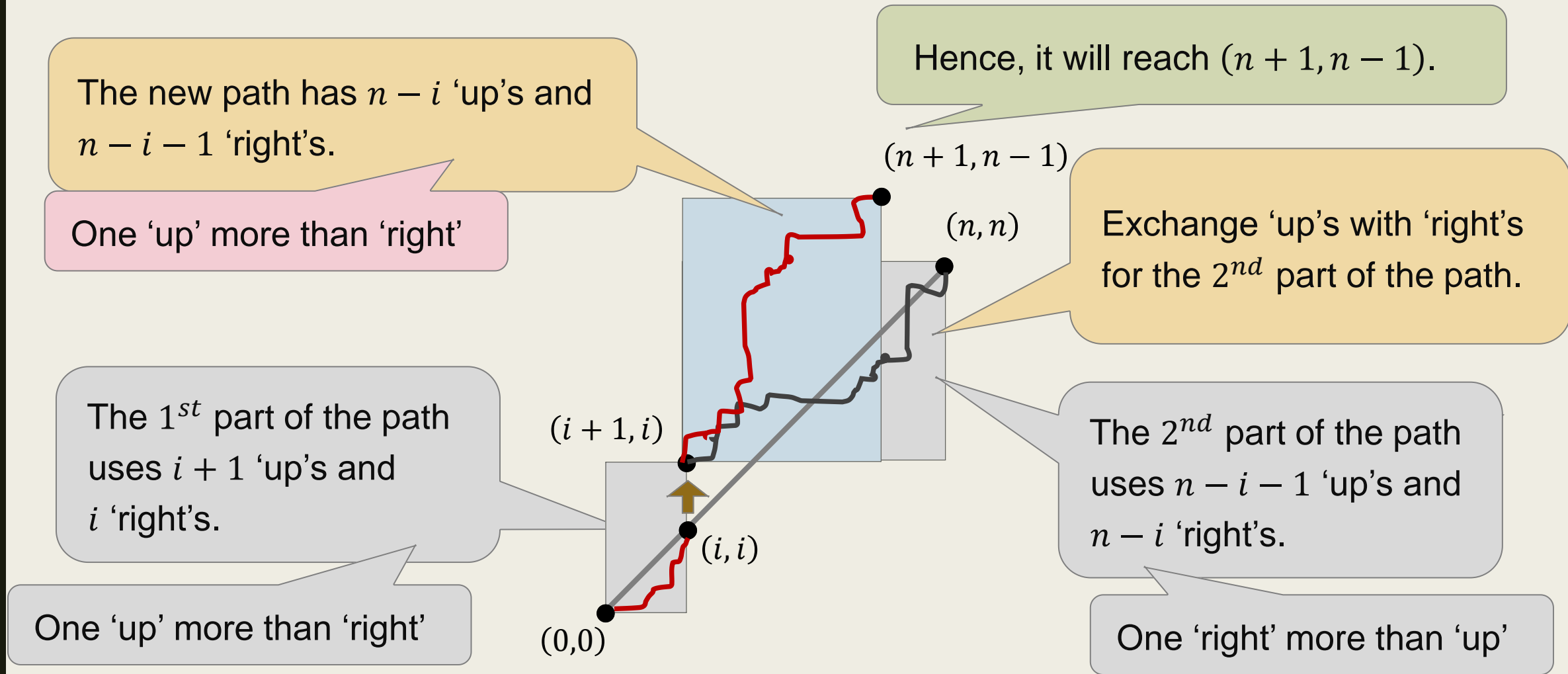


Exchange 'up's with 'right's for the 2<sup>nd</sup> part of the path.

The 2<sup>nd</sup> part of the path uses  $n - i - 1$  'up's and  $n - i$  'right's.

One 'right' more than 'up'

- Observe that, this is a one-to-one correspondence between paths that cross the diagonal and paths that reach  $(n + 1, n - 1)$ .



- It suffices to prove that,  
the number of paths that have crossed the diagonal is  $\binom{2n}{n-1}$ .
- Consider any of such path and the first time that it crosses the diagonal, say, at the node  $(i, i)$ .
  - There is a one-to-one correspondence between paths that cross the diagonal and paths that reach  $(n + 1, n - 1)$ .
  - Hence,  
the number of paths that have crossed the diagonal is  $\binom{2n}{n-1}$ .

# The Density of 0-1 Matrices

- Let  $H$  be an  $m \times n$  0-1 matrix and  $0 \leq \alpha \leq 1$  be a real number.
  - We say that  $H$  is  **$\alpha$ -dense**,  
if at least an  $\alpha$ -fraction of its entries are 1s,  
i.e.,  
$$\# \text{ of 1s in } H \geq \alpha \cdot mn .$$
  - Similarly, a row (column) is  **$\alpha$ -dense**,  
if at least an  $\alpha$ -fraction of its entries are 1s.

**Lemma 2.13 (Grigni and Sipser 1995).**

Note that,  $\sqrt{\alpha} \geq \alpha$   
when  $\alpha \leq 1$ .

If  $H$  is  $2\alpha$ -dense, then either

1. There exists a row which is  $\sqrt{\alpha}$ -dense, or
2. At least  $\sqrt{\alpha} \cdot m$  of the rows are  $\alpha$ -dense.


- If  $H$  is dense, then either
  - There exists a row that is very dense, or
  - A certain fraction of rows must be dense enough.
- Intuitively,  
if the first condition does not hold, then the second must hold.

**Lemma 2.13 (Grigni and Sipser 1995).**


If  $H$  is  $2\alpha$ -dense, then either

1. There exists a row which is  $\sqrt{\alpha}$ -dense, or
2. At least  $\sqrt{\alpha} \cdot m$  of the rows are  $\alpha$ -dense.

■ Suppose that both of the cases do not hold.

- 
- By 2, less than  $\sqrt{\alpha} \cdot m$  rows are  $\alpha$ -dense.
  - By 1, each of the above rows has less than  $\sqrt{\alpha} \cdot n$  1s.
  - Hence, the total number of 1s in ***these  $\alpha$ -dense rows*** is less than  $\sqrt{\alpha} \cdot \sqrt{\alpha} \cdot mn = \alpha \cdot mn$ .

- Suppose that both of the cases do not hold.

- 
- By 2, less than  $\sqrt{\alpha} \cdot m$  rows are  **$\alpha$ -dense**.
  - By 1, each of the above rows has less than  $\sqrt{\alpha} \cdot n$  1s.
  - Hence, the total number of 1s in ***these  $\alpha$ -dense rows*** is less than  $\sqrt{\alpha} \cdot \sqrt{\alpha} \cdot mn = \alpha \cdot mn$ .

- Consider the remaining ***non- $\alpha$ -dense rows***.

- The number of 1s in these rows is less than  $\alpha \cdot mn$ .

- Hence, the total number of 1s in  $H$  is strictly less than  $2\alpha \cdot mn$ , a contradiction.



The Number of Sufficiently Dense Rows  
(Columns) in a Dense Matrix.

Q: How many rows or columns of an  $\alpha$ -dense matrix will be “***dense enough***?”

- In the following,  
let's use a general setting to answer this question!

- Let  $A_1, A_2, \dots, A_k$  be finite sets, and  
consider the Cartesian product of  $A_i$

$$A = A_1 \times A_2 \times \cdots \times A_k .$$

Then, the Cartesian product  
 $A = \{ (i, j) : 1 \leq i \leq m, 1 \leq j \leq n \}$   
is the coordinates of the entries.

To relate the two concepts, for  $m \times n$  0-1 matrix,  
we have  $A_1 = \{1, 2, \dots, m\}$ ,  $A_2 = \{1, 2, \dots, n\}$ .

- Let  $A_1, A_2, \dots, A_k$  be finite sets, and consider the Cartesian product of  $A_i$

To relate the two concepts,  
for an  $m \times n$  0-1 matrix, we have  
 $A_1 = \{1, 2, \dots, m\}$ ,  $A_2 = \{1, 2, \dots, n\}$ .

$$A = A_1 \times A_2 \times \dots \times A_k .$$

The set of entries that are 1.

- Let  $D \subseteq A$  be the set of elements of interests.

- For any  $b \in A_i$ ,  
define the **degree of  $b$**  in  $D$  as

number of 1s in  
the  $b^{th}$ -row (column).

$$d_D(b) := |\{a \in D : a_i = b\}| ,$$

i.e., the number of elements of interests  
whose  $i^{th}$ -coordinate is  $b$ .

- We say that a point  $b \in A_i$  is popular in  $D$ , if

$$d_D(b) \geq \frac{1}{2k} \cdot \frac{|D|}{|A_i|} ,$$

i.e., the degree of  $b$  is at least  $1/2k$  times the average degree of the elements in  $A_i$ .

- Let  $P_i \subseteq A_i$  be the set of all popular elements in  $A_i$ , and let

$$P := P_1 \times P_2 \times \cdots \times P_k .$$

↔ number of 1s (or, density) in that row (column) is at least  $1/4$  of the average.

↔ The entries formed by popular rows and columns.

**Lemma 2.14 (Håstad).**

$$|P| > |D|/2.$$

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$$|P| > |D|/2.$$

Since

$$|P| \geq |P \cap D| = |D| - |D \setminus P|$$

- It suffices to prove that  $|D \setminus P| < |D|/2$ .

- For every non-popular point  $b \in A_i$ , we have

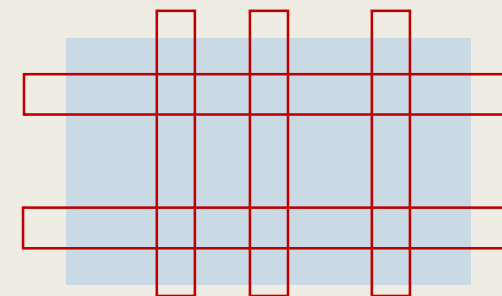
$$d_D(b) = |\{a \in D : a_i = b\}| < \frac{1}{2k} \cdot \frac{|D|}{|A_i|}.$$

- Counting the elements in  $|D \setminus P|$ , we have

$$|D \setminus P| \leq \sum_{1 \leq i \leq k} \sum_{b \in A_i \setminus P_i} d_D(b) < \sum_{1 \leq i \leq k} \sum_{b \in A_i \setminus P_i} \frac{1}{2k} \cdot \frac{|D|}{|A_i|}$$

Any element in  $|D \setminus P|$  is counted at least once in the summation.

$$|A_i \setminus P_i| \leq |A_i| \leq \sum_{1 \leq i \leq k} \frac{1}{2k} \cdot |D| = \frac{1}{2} |D|.$$



Q: How many rows or columns of an  $\alpha$ -dense matrix will be “***dense enough***?”

- Interpret  $D$  as the entries with 1 in the matrix  $M$ .
  - Lemma 2.14 says that,  
the size of  $|P| = |P_1| \cdot |P_2|$  is lower-bounded by  $|D|/2$ .
  - Provided that  $|D|$  is large, at least one of  $|P_1|$ ,  $|P_2|$  must be large.

**Corollary 2.15.**

In any  $2\alpha$ -dense 0-1 matrix  $H$ , either a  $\sqrt{\alpha}$ -fraction of its rows or a  $\sqrt{\alpha}$ -fraction of its columns (or both) are  $\alpha/2$ -dense.

### Corollary 2.15.

In any  $2\alpha$ -dense 0-1 matrix  $H$ , either a  $\sqrt{\alpha}$ -fraction of its rows or a  $\sqrt{\alpha}$ -fraction of its columns (or both) are  $\alpha/2$ -dense.

- Let  $H$  be an  $m \times n$  matrix and  $D$  be the set of entries of 1.
- Let  $P_1$  and  $P_2$  be the set of popular rows and columns, respectively.
  - For any  $i \in P_1$ , we have

$$\deg_D(i) \geq \frac{1}{2k} \cdot \frac{|D|}{|A_1|} \geq \frac{1}{4} \cdot \frac{2\alpha \cdot mn}{m} \geq \frac{\alpha}{2} \cdot n,$$

which implies that  $i$  is  $\alpha/2$ -dense.

- Similarly, any  $j \in P_2$  is also  $\alpha/2$ -dense.

- Let  $H$  be an  $m \times n$  matrix and  $D$  be the set of entries of 1.
- Let  $P_1$  and  $P_2$  be the set of popular rows and columns, respectively.
  - Any row / column  $b \in P_1 \cup P_2$  is  $\alpha/2$ -dense.
- By Lemma 2.14,  $|P_1| \cdot |P_2| \geq |D| / 2 \geq \alpha \cdot mn$ ,  
 which implies that  $\frac{|P_1|}{m} \cdot \frac{|P_2|}{n} \geq \alpha$ .
- Hence, either  $\frac{|P_1|}{m} \geq \sqrt{\alpha}$  or  $\frac{|P_2|}{n} \geq \sqrt{\alpha}$  must hold.
- This proves the statement of this corollary.