Combinatorial Mathematics

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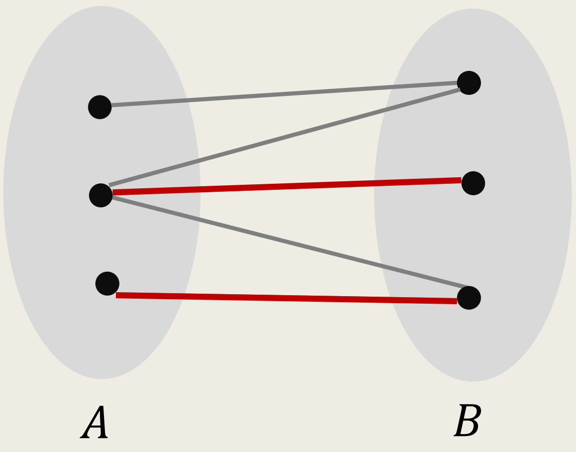
Monday 18:30 – 20:20

Outline

- Hall’s Matching Theorem
- König-Egeváry Theorem
- The Maximum Matching Problem
  - A Generic Algorithm and the Berge’s Theorem
  - The Augmenting Path Problem in Bipartite Graphs
    - A simple DFS-based recursive algorithm

Matching in Bipartite Graphs

- Let $G = (V, E)$ be a bipartite graph with partite sets A and B.
- An edge subset $M \subseteq E$ is called a matching for G,
  if each vertex in V is incident to at most one edge in M.
  - i.e., the endpoints of the edges in M are disjoint.



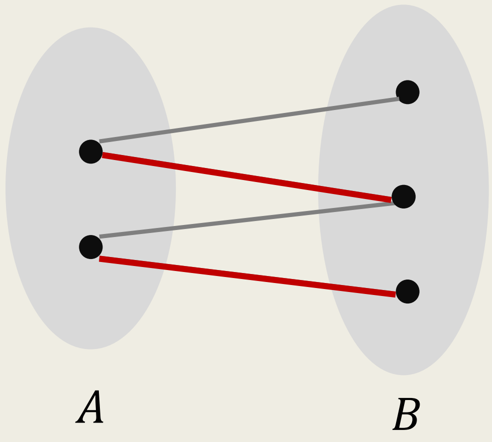
The same definition applies to general graphs, too.

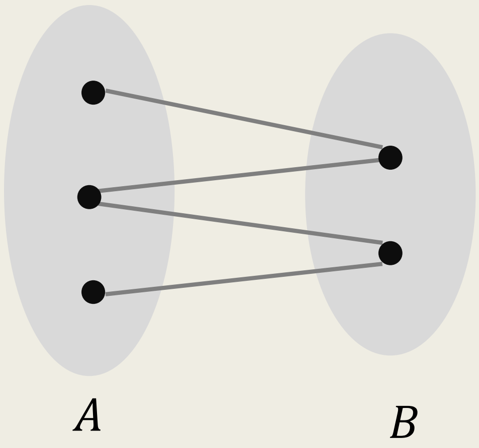
Matching in Bipartite Graphs

- Let $G = (V, E)$ be a bipartite graph with partite sets A and B.
- Let M be a matching for G.
  - For any $u, v \in V$,
    we say that u is matched to v by M (and vice versa),
    if $(u, v) \in M$.
  - For any $U \subseteq A$, we say that M matches U, or,
    M is a matching from U to B, or, M is a matching for U,
    if M matches every vertex in U to some vertex in B.

Matching in Bipartite Graphs

- Let $G = (V, E)$ be a bipartite graph with partite sets A and B.
- Let M be a matching for G.
  - For any $U \subseteq A$, we say that M is a matching for U, if M matches every vertex in U to some vertex in B.





There is no enough candidates to be matched to for A.

Hall’s Matching Condition

The necessary and sufficient condition for a matching in bipartite graphs to exist.

Theorem 5.1 (Hall's Theorem).

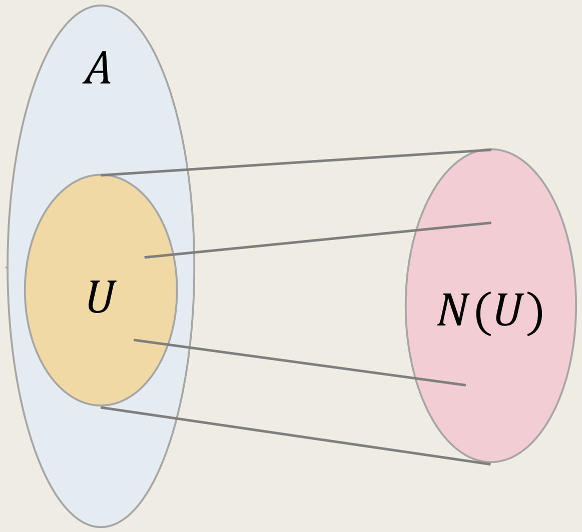
Let $G = (V, E)$ be a bipartite graph with partite sets A and B.

There exists a matching M for A

if and only if

$$|N(U)| \geq |U| \quad \text{for all } U \subseteq A. \qquad (*)$$

i.e., there is always a sufficient number of candidates to be matched to.



$|N(U)| \geq |U|$, for any $U \subseteq A$.

Theorem 5.1 (Hall's Theorem).

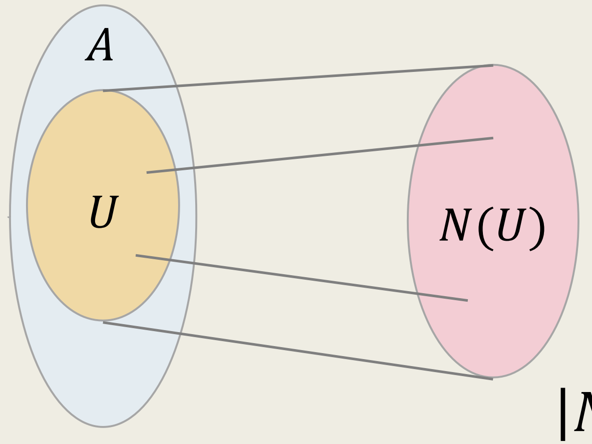
Let $G = (V, E)$ be a bipartite graph with partite sets A and B.

There exists a matching M for A

if and only if

$$|N(U)| \geq |U| \quad \text{for all } U \subseteq A. \qquad (*)$$

- Proof.
  - The direction ($\Longrightarrow$) is clear.
    - M matches each vertex in U to a distinct vertex in B.
    - Hence, $|N(U)| \geq |U|$.



$|N(U)| \geq |U|$

- <u>Proof. (continue)</u>
  - We prove the direction ($\Longleftarrow$) by induction on the size of $|A|$, which we denote by m.
  - The case $m = 1$ holds trivially.
  - Assume that the statement ($\Longleftarrow$) holds for any A with $|A| < m$.

- Proof. (continue)
  - Assume that the statement ($\Longleftarrow$) holds when the number of vertices in the left partite set is $< m$.
  - To prove for $|A| = m$, we distinguish the following two cases.
    1. For any $U \subset A$, we always have $|N(U)| > |U|$.
    2. For some $U \subset A$, $|N(U)| = |U|$.

We *always* have *more candidates* than we need.

The number of candidates for some subset is tight.

We always have more candidates than we need.

- We distinguish following two cases.

1. For any $U \subset A$, we always have $|N(U)| > |U|$.
   - Pick an arbitrary $u \in A$ and any $v \in N(u)$.
     Match u to v and remove v from the graph.

     At most one vertex is removed from $N(U)$.

   - Then, it follows that,
     for any $U \subseteq A - \{u\}$, we still have $|N(U)| \geq |U|$.
     - By the induction hypothesis,
       there exists a matching from $A - \{u\}$ to $B - \{v\}$.
     - Hence, we obtain a matching for A.

- We distinguish following two cases.

The number of candidates for some subset is tight.

2. For some $U \subset A$, $|N(U)| = |U|$.

   - By the induction hypothesis, there exists a matching M_1 from U to $N(U)$.

   Remove $N(U)$ from the graph.

- Then, we claim that, for any $U' \subseteq A - U$, we still have $|N(U')| \geq |U'|$.

- We distinguish following two cases.

The number of candidates for some subset is tight.

2. For some $U \subset A$, $|N(U)| = |U|$.

   - Remove $N(U)$ from the graph.

   - Then, we claim that,
     for any $U' \subseteq A - U$, we always have $|N(U')| \geq |U'|$.

     – If not, then before $N(U)$ is removed, we have

       $$|N(U' \cup U)| \leq |N(U')| + |N(U)| < |U'| + |U|,$$

       which is a contradiction.

- We distinguish following two cases.

The number of candidates for some subset is tight.

2. For some $U \subset A$, $|N(U)| = |U|$.
   - By the induction hypothesis, there exists a matching M_1 from U to $N(U)$.

     Remove $N(U)$ from the graph.
   - Then, we claim that, for any $U' \subseteq A - U$, we always have $|N(U')| \geq |U'|$.
   - By induction hypothesis, there exists a matching M_2 for $A - U$.
   - Together, we obtain a matching for A.

Application -

System of Distinct Representatives

Distinct Representative of Sets in a Family

- Let $F = \{S_1, S_2, \dots, S_m\}$ be a set family.

- The elements $x_1, x_2, \dots, x_m$ is called a set of <u>*distinct representatives*</u> for F, if the following two conditions hold.
  - $x_i \in S_i$ for all $1 \le i \le m$.
  - The elements $x_1, x_2, \dots, x_m$ are distinct, i.e., $x_i \ne x_j$ for all $i \ne j$.

Corollary.

The set family $S_1, S_2, \dots, S_m$ has a set of distinct representatives ***if and only if***

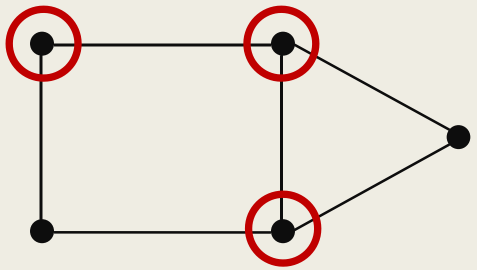
$$\left|\bigcup_{i \in I} S_i\right| \geq |I| \quad \text{for all } I \subseteq \{1,2,\dots,m\}.$$

- Construct a bipartite graph for the set family, and this corollary follows directly from the Hall’s theorem.

Matching v.s. Vertex Cover

Weak-duality between matching and vertex cover.

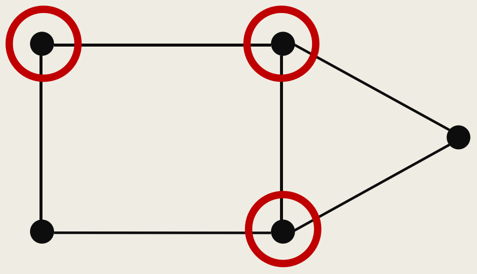
Vertex Cover of a Graph



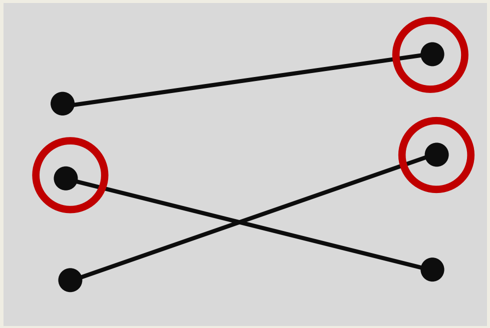
- Let $G = (V, E)$ be a graph.

- A ***vertex cover*** of G is a subset $U \subseteq V$ of vertices such that, every edge $e \in E$ has at least one endpoint in U.
  - Intuitively, we use the vertices in U to cover the edges in E.

Matching v.s. Vertex Cover



- Let $G = (V, E)$ be a graph,
  - $M \subseteq E$ be a matching, and
  - $C \subseteq V$ be a vertex cover for G.



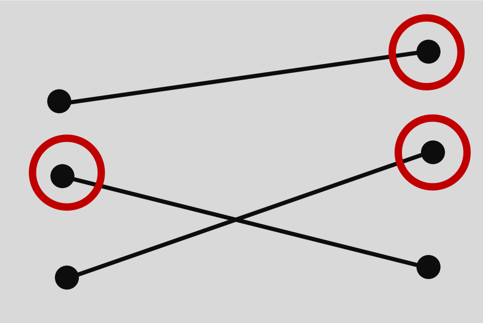
The matching M

- It follows that

$$|\mathrm{M}| \leq |C|.$$

  - The endpoints of the edges in M are distinct.
  - It takes <u>at least one vertex</u> to cover <u>each edge in M</u>, i.e., at least one endpoint of each edge has to be selected in C.

Matching v.s. Vertex Cover



The matching M

- Let $G = (V, E)$ be a graph,
  $M \subseteq E$ be a matching, and $C \subseteq V$ be a vertex cover for G.

- Then, it follows that $|\mathrm{M}| \leq |C|$.

  – This property is called the *weak-duality* between the matching and vertex cover.

  – It implies that, in any graph, the size of maximum matching is at most the size of minimum vertex cover.

The König-Egeváry Theorem

<u>In bipartite graphs</u>, the size of the ***maximum matching*** is equal to the size of the ***minimum vertex cover***.

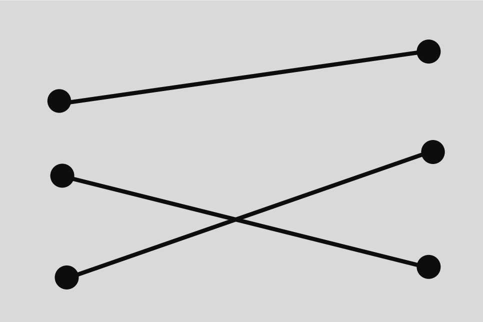
Theorem 5.5 (König-Egeváry 1931).

In a bipartite graph, the size of ***maximum matching*** is equal to the size of ***minimum vertex cover***.

Proof.

- Let G be a bipartite graph with partite sets U and V.
- Let M be a *maximum matching* and

  C be a *minimum vertex cover* for G, respectively.
- It suffices to prove that $|M| \geq |C|$.

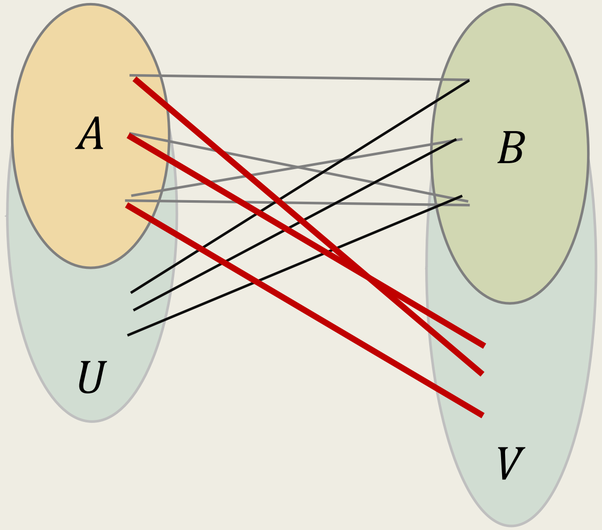


The matching M

Theorem 5.5 (König-Egeváry 1931).

In a bipartite graph, the size of ***maximum matching*** is equal to the size of ***minimum vertex cover***.

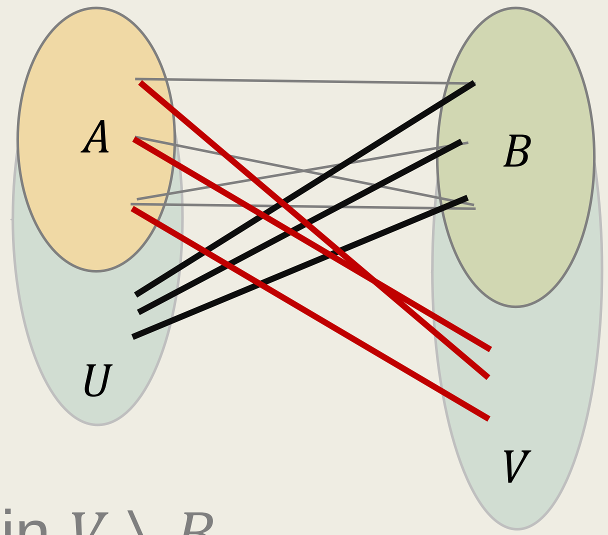
Proof.



- It suffices to prove that $|M| \geq |C|$.
  - Let $A \coloneqq U \cap C$ and $B \coloneqq V \cap C$.
  - We will prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.

Proof.

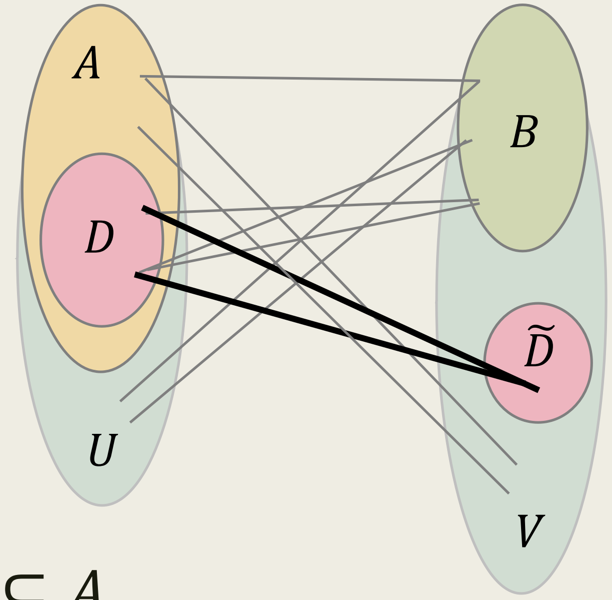
- It suffices to prove that $|M| \geq |C|$.
- Let $A \coloneqq U \cap C$ and $B \coloneqq V \cap C$.
  - We will prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.
  - **If the above is true**, then by a similar argument, there exists a matching M_B for B to $U \setminus A$.
  - The endpoints of the edges in $M_A \cup M_B$ are distinct.
    - So, $M_A \cup M_B$ is a matching of size $|A| + |B| = |C|$.
    - Hence, this will prove that $|M| \geq |A| + |B| = |C|$.



It suffices to prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.

- Suppose that there exists no such matching.
- Then, by Hall's matching theorem, there exists some $D \subseteq A$, such that

$$| \; N(D) \; \cap \; (V \setminus B) \; | \; < \; |D| \, .$$



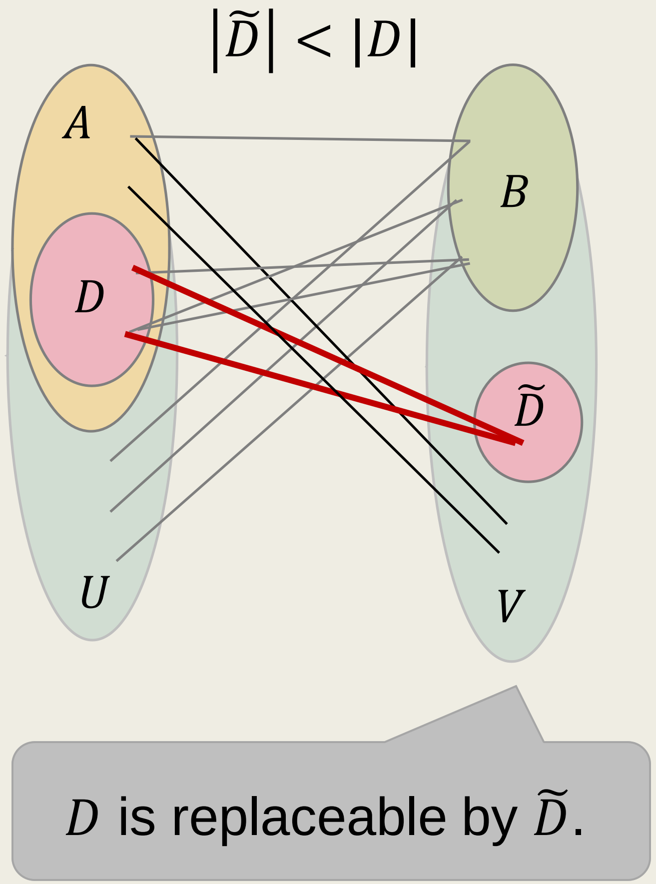
  - Indeed, if $|N(D) \cap (V \setminus B)| \geq |D|$ holds for all $D \subseteq A$, then there exists a matching from A to $V \setminus B$.
  - Since there is no such matching, there must be such a $D \subseteq A$ with $|N(D) \cap (V \setminus B)| < |D|$.

It suffices to prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.

- If not, there exists some $D \subseteq A$, such that

$$| \; N(D) \;\; \cap \;\; (V \setminus B) \; | \;\; < \;\; |D| \, .$$

- Let $\widetilde{D} \coloneqq N(D) \cap (V \setminus B)$, then $|\widetilde{D}| < |D|$.
- We claim that,
  $(A \setminus D) \cup \widetilde{D} \cup B$ is a valid vertex cover for G.
  - If this is true, we obtain a vertex cover with size smaller than $|A| + |B| = |C|$, a contradiction.

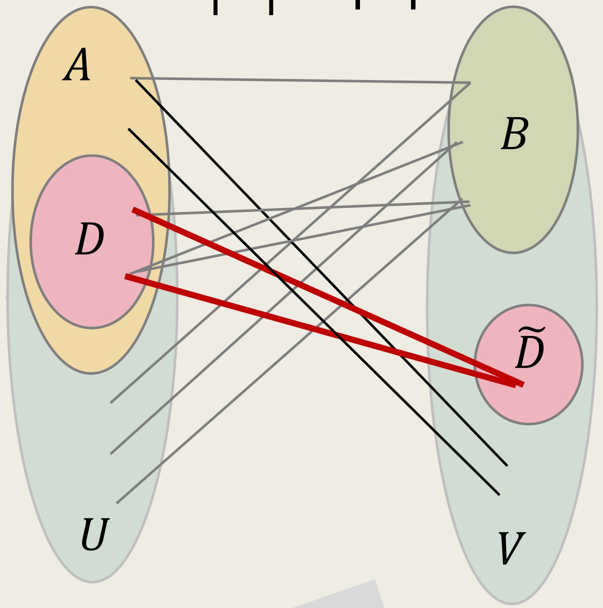


It suffices to verify that,

$(A \setminus D) \cup \widetilde{D} \cup B$ is a valid vertex cover for G.

- Let $\widetilde{D} \coloneqq N(D) \cap (V \setminus B)$.
- There are four categories of edges in G.
  - $E_{A,B}$, $E_{U \setminus A,B}$ --- covered by B.
  - $E_{A \setminus D, V \setminus B}$ --- covered by $A \setminus D$.
  - $E_{D,\widetilde{D}}$ --- covered by $\widetilde{D}$.
- All the edges are covered.

$|\widetilde{D}| < |D|$



Since $C = A \cup B$ is a vertex cover,
there is not edge between $U \setminus A$ and $V \setminus B$.

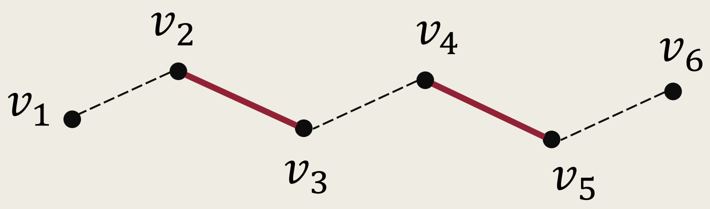
The Maximum Matching Problem

To compute a maximum-size matching for the input graph.

The Maximum Matching Problem

- Input :
  - A graph $G = (V, E)$.
- Output :
  - A matching $M \subseteq E$ that has the maximum size among all possible matchings.

Alternating Path & Augmenting Path

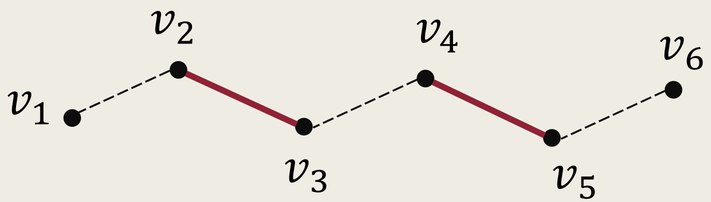


- Let M be a matching for a graph G.
  - An M-alternating path is a path that alternates between edges in M and edges not in M.
  - An ***M*-augmenting path** is an M-alternating path that both starts and ends at unmatched vertices.

Both v_1, v_2, v_3 and v_2, v_3, v_4, v_5 are M-alternating paths.

$v_1, v_2, v_3, v_4, v_5, v_6$ is an M-augmenting paths.

Observation

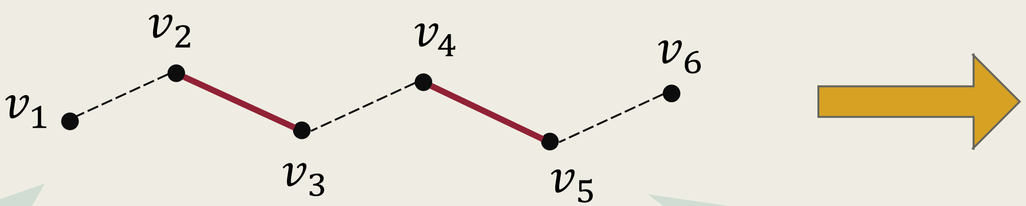


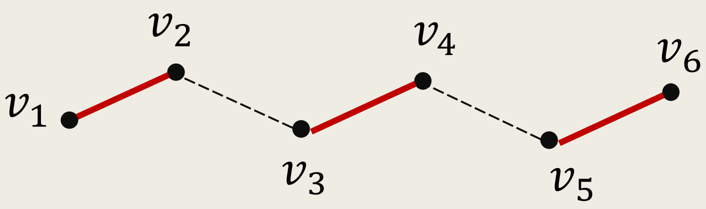
- We can see that,

  each M-augmenting path is a way to enlarge the size of M by 1.

  - This is done by swapping the status of the edges on the path.
    - Matched edges $\Longrightarrow$ *unmatched*
    - Unmatched edges $\Longrightarrow$ *matched*

So, this is still a valid matching with size increased by 1.

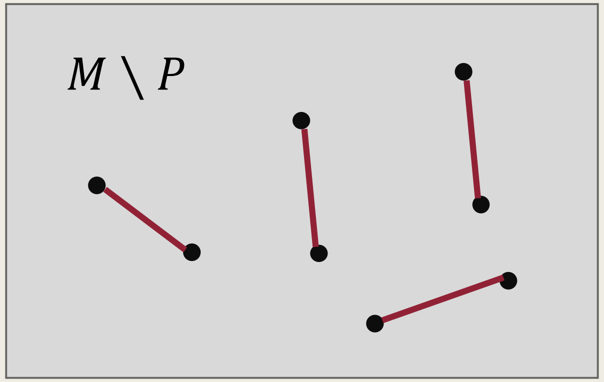


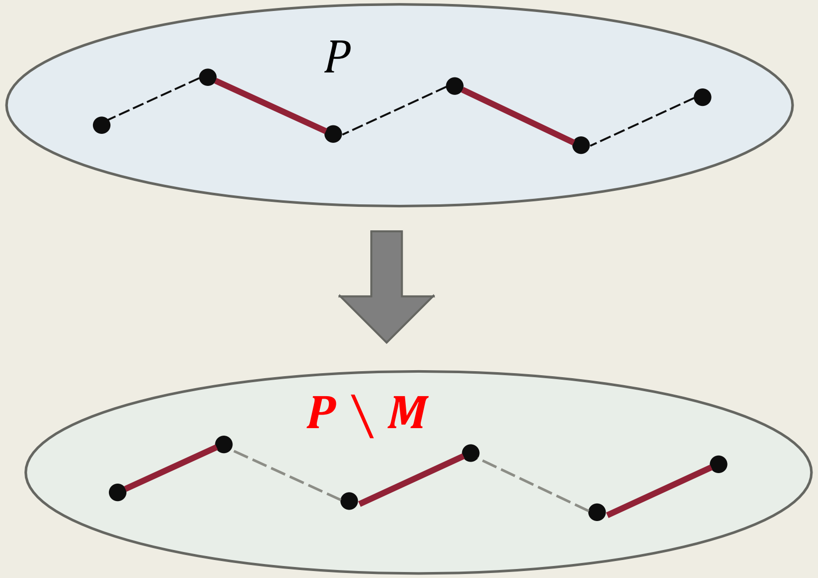


v_1 and v_6 were unmatched.

All internal vertices are matched only by edges on the path.

Observation





- We can see that,

  each M-augmenting path P is a way to enlarge the size of M by 1.

- $M' \coloneqq (M \setminus P) \cup (P \setminus M)$ is a valid matching with $|M'| = |M| + 1$.

$M \Delta P$: the edges that appear exactly once in M and P.

A Simple Greedy Algorithm

- The observation suggests the following *greedy algorithm*.
  - Let $G = (V, E)$ be the input graph.

1. $M \leftarrow \emptyset$.
2. Repeat until there is no M-augmenting path in G.
   a. Compute an M-augmenting path P.
   b. Set $M \leftarrow (M \setminus P) \cup (P \setminus M)$.
3. Output M.

1. $M \longleftarrow \emptyset$.
2. Repeat until there is no M-augmenting path in G.
   a. Find an M-augmenting path P.
   b. Set $M \longleftarrow (M \setminus P) \cup (P \setminus M)$.
3. Output M.

The philosophy behind the algorithm is very simple :

“Make the current matching larger until no augmenting path exists.”

- A direct question is that,

  “Does it always output a maximum matching?”

Theorem 1. (Berge 1957).

A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.

- Theorem 1 assures the correctness of the greedy algorithm.
  - When there is no M-augmenting path, M is guaranteed to be maximum.
- We begin with some definition & helper lemma.

Symmetric Difference

- Let $G = (V, E)$ be a graph, and $A, B \subseteq E$ be two edge sets.
  - The symmetric difference of A and B is defined as

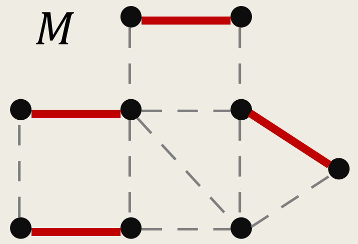
$$A \triangle B \coloneqq (A \setminus B) \cup (B \setminus A)\,.$$

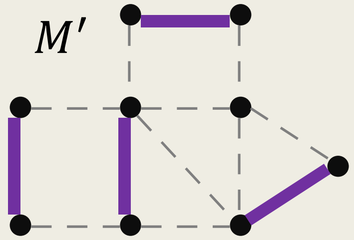
  - i.e., the set of edges that appear exactly once in A and B.

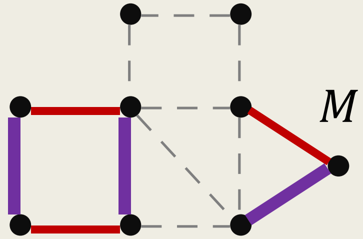
Lemma 2.

Let M, M' be matchings for a graph G. Then, every component of $M \triangle M'$ is a either *path* or a *cycle with an even length*.

- Let $F \coloneqq M \triangle M'$.
  - Each vertex in G is incident to at most two edges in F.
  - Hence, each component in F is either a path or a cycle.
- Consider any cycle in F.
  - The cycle alternates between edges in M and M'.
  - It must have an even length.







Theorem 1. (Berge 1957).

A matching M in a graph G is a maximum matching if and only if G has no M-augmenting path.

- Let us prove Theorem 1.
  - The direction ($\Longrightarrow$) is clear.
  - For the direction ($\Longleftarrow$),
    we prove the contrapositive statement.
    - We show that, if M' is a matching with $|M'| > |M|$, then G must have an M-augmenting path.

It suffices to prove that, if M' is a matching with $|M'| > |M|$, then G must have an M-augmenting path.

- Let $F \coloneqq M \triangle M'$.
  - By Lemma 2, F is a union of paths and even cycles.
- Since $|M'| > |M|$, there must be a component in F that has more edges from M' than M.
  - The component must be a path.
    Furthermore, it must start and ends with edges in M'.
  - The path is then an M-augmenting path.