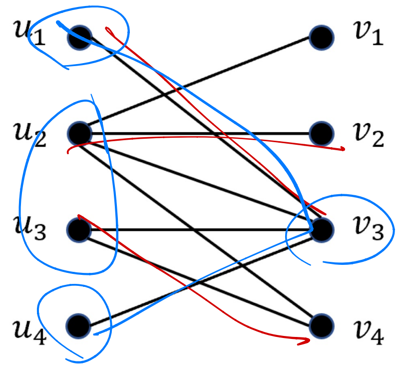
Problem 1 (20%). Consider the following graph. Identify a maximum-size matching and a minimum-size vertex cover for it.

Maximum-Size Matching:

$\{(u_1, v_3), (u_2, v_2), (u_3, v_4)\}$

Minimum-Size Vertex Cover:

$\{u_2, u_3, v_3\}$



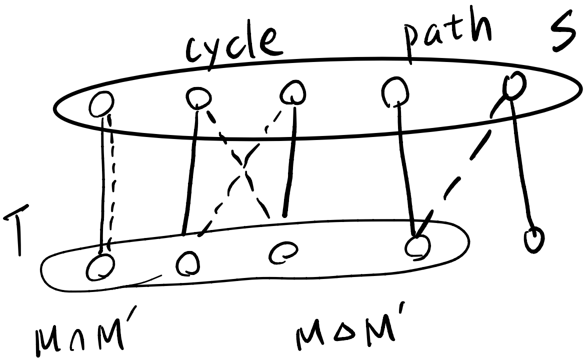
Problem 2 (20%). Let G be a bipartite graph with partite sets A and B, and M, M' be two matchings. Suppose that, M matches the vertices in $S \subseteq A$ and M' matches the vertices in $T \subseteq B$. Prove that there is a matching that matches all the vertices in $S \cup T$.

Hint: Consider $M \cup M'$.

$M \cup M' = (M \triangle M') \cup (M \cap M')$

For $(M \cap M')$, since $S' = S \cap (M \cap M')$, $T' = T \cap (M \cap M')$ only in $M \cap M'$, no other matched edge connects to T' or S', we should add $M \cap M'$ to the final matching.

For $(M \triangle M')$, by theorem, every segment is a cycle or an augmenting path. For a cycle, pick either M or M' to the matching. For an augmenting path, pick the one with more edge on the path.



M: 實線

M': 虛線

$\Rightarrow$ Final matching $= (M \cap M') \cup (M)_{cycle} \cup (\max(M, M'))_{path}$

Problem 3 (20%). Let G be a bipartite graph with partite sets X and Y. Prove that G has a matching of size t if and only if for all $A \subseteq X$,

$$|N(A)| \geq |A| + t - |X| = t - |X - A|.$$

Hint: Add $|X| - t$ new vertices to Y and connect these vertices to every vertex in X.

($\Rightarrow$) G has a mathing of size $t \Rightarrow |N(A)| \geq |A| + t - |X|$

Let G' be the graph constructed from G as hint. Then,

$G' = (X \cup Y', E_{G'})$. $E_{G''} = \{(x_i, y_j) \mid \forall\, 1 \leq i \leq |X|, |Y| < j \leq |Y| + |X| - t.\}$

$E_{G'} = E_G \cup E_{G''}$.

Suppose there is a complete matching for X in G', since at most $|X| - t$ edges connect to $Y' - Y$, there must be a matching of size t for X in G.

By Hall's Theorem, we have $|N_{G'}(A)| \geq |A|\ \forall A \subseteq X$ in G'.

Since $|N_{G'}(A)| = |N_G(A)| + |X| - t$, we have $|N_G(A)| + |X| - t \geq |A|$

$\rightarrow \underline{|N_G(A)| \geq |A| + t - |X|}$. (if G has a matching of size t)

($\Leftarrow$) $|N(A)| \geq |A| + t - |X| \Rightarrow G$ has a mathing of size t

By Hall's Theorem, if we have $|N_{G'}(A)| \geq |A|\ \forall A \subseteq X$, then G' has a matching for X, which implies $|N_G(A)| \geq |A| - |X| + t$ by the equation $|N_{G'}(A)| = |N_G(A)| + |X| - t$. Removing $Y' - Y$ from G' (which is G) eliminate the matching between X and $Y' - Y$, whose size is exactly $|X| - t$ by its construction. Therefore, G has a matching of size t.

Problem 4 (20%). Let G be a bipartite graph with partite sets X and Y. Define

$$\delta(G) := \max_{A \subseteq X} (|A| - |N(A)|),$$

i.e., $\delta(G)$ measures the worst violation of the Hall's matching condition. Note that, $\delta(G) \geq 0$ since $A = \emptyset$ is considered as a subset of X. Use the statement in Problem 3 to prove that, G has a maximum matching of size $|X| - \delta(G)$.

By definition, $|A| - |N(A)| \leq \delta(G)$, $\forall A \subseteq X$ ... ①

By problem 3, if $|N(A)| \geq |A| - (|X| - t)$ $\forall A \subseteq X$, then G has a matching of size t.

By ①, $|N(A)| \geq |A| - \delta(G) = |A| - (|X| - t)$, if $t = |X| - \delta(G)$.

Therefore, G has a matching of size $|X| - \delta(G)$ #

Problem 5 (20%). Let G be a bipartite graph with partite sets X and Y. Assume the same notation $\delta(G)$ as Problem 4. Show that, the largest independent set of G has size $|Y| + \delta(G)$.

Let I be the maximum independent set (MIS), $X' = I \cap X$, $Y' = I \cap Y$.

Since I is the maximum, $Y' = Y / N(X')$, i.e., all $y \in Y$ that is not a neighbor of X' should be in I, otherwise I is not the maximum.

Therefore, $|I| = |X'| + |Y'| = |X'| + |Y / N(X')| = |X'| + |Y| - |N(X')|$.

Since we have $|A| - |N(A)| \leq \delta(G)$ $\forall A \subseteq X$ from problem 4.

$|I| = |X'| - |N(X')| + |Y| \underline{\leq |Y| + \delta(G)}$ #