Problem 1 (20%). Prove that, for any vector $v \in \mathbb{R}^n$,

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組合數學 HW6.

$$\frac{|v|_1}{\sqrt{n}} \le \|v\|_2 \le |v|_1,$$

where $|v|_1 := \sum_i |v_i|$ is the L_1-norm and $\|v\|_2 := \left(\sum_i v_i^2\right)^{1/2}$ is the L_2-norm of v.

Hint: Use the Cauchy-Schwarz inequality, i.e., $|u \cdot v| \le \|u\|_2 \|v\|_2$ for any $u, v \in \mathbb{R}^n$.

① $\frac{|V|_1}{\sqrt{n}} \le \|V\|_2$: Let $\vec{u} = (\operatorname{sgn}(V_1), \operatorname{sgn}(V_2), \dots, \operatorname{sgn}(V_n))$

$\rightarrow |u \cdot v| = |\sum_i u_i \cdot V_i| = |\sum_i \operatorname{sgn}(V_i) \cdot V_i| = \sum_i |V_i|$

$\rightarrow \|u\|_2 \|v\|_2 = (\sum_i u_i^2)^{1/2} \times (\sum_i V_i^2)^{1/2} = \sqrt{n} \cdot \|V\|_2$

By Cauchy-Schwarz : $\sum_i |V_i| = |V|_1 \le \sqrt{n} \|V\|_2$ #

* $\operatorname{sgn}(x) = \begin{cases} +1, & x \ge 0 \\ -1, & x < 0 \end{cases}$

② $\|V\|_2^2 = \sum_i V_i^2 \le \sum_i |V_i|^2 + 2 \sum_{i<j} |V_i| \times |V_j| = (\sum_i |V_i|)^2$

$\rightarrow \|V\|_2 \le |V|_1$ # (take square root).

Problem 2 (20%). Let A be a square symmetric matrix and λ be an eigenvalue of A. Prove that, for any $k \in \mathbb{N}$, λ^k is an eigenvalue of A^k.

Prove by induction on k, for $k=1$, we have that λ is an eigenvalue of A.

Assume for some $k \ge 1$, we have that λ^k is an eigenvalue of A^k.

For $(k+1)$, $A^{k+1} v = A(A^k v) = A \cdot \lambda^k v$, by I.H..

Since λ is a real number, we have $A \lambda^k v = \lambda^k A v = \lambda^k (\lambda v) = \lambda^{k+1} v$

Then we obtain $A^{k+1} v = \lambda^{k+1} v$, which implies that λ^{k+1} is an eigenvalue of A^{k+1}.

By induction, we proved that λ^k is an eigenvalue of A^k $\forall k \in \mathbb{N}$.

Problem 3 (20%). Let G be an n-vertex d-regular bipartite graph and A be the normalized adjacency matrix of G. Prove that, there exists a vector $v \in \mathbb{R}^n$ such that

$$Av = -v.$$

Generalize the construction to non-regular bipartite graphs, i.e., for any bipartite graph G' with column-normalized adjacency matrix A', prove that A' has an eigenvalue -1.

Note: A' is also called the *random-walk* matrix of G'.

① Since G is d-regular, the largest eigenvalue of A is 1. (Stated in slide)

Since G is bipartite, A could be written as $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}$,

where B is a $|X| \times |Y|$ matrix that represents the edges between X and Y.

We can prove that A also has an eigenvalue -1 as follows:

Let $x = \begin{pmatrix} u \\ v \end{pmatrix}$, where $u \in \mathbb{R}^{|X|}$, $v \in \mathbb{R}^{|Y|}$, it follows that $A\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} Bv \\ B^T u \end{pmatrix}$.

If x is an eigenvector corresponding to $\lambda \rightarrow \begin{cases} Bv = \lambda u \\ B^T u = \lambda v \end{cases}$

We then consider another vector $x' = \begin{pmatrix} u \\ -v \end{pmatrix}$

$\rightarrow A\begin{pmatrix} u \\ -v \end{pmatrix} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix}\begin{pmatrix} u \\ -v \end{pmatrix} = \begin{pmatrix} -Bv \\ B^T u \end{pmatrix} = -\lambda \begin{pmatrix} u \\ -v \end{pmatrix}$.

Therefore, if G is a bipartite graph with normalized adjacency matrix A having eigenvalue λ, then $-\lambda$ is also A's eigenvalue.

$\rightarrow -1$ is also A's eigenvalue $\longrightarrow \exists v$ s.t. $Av = -v$ #

② It suffices to prove that 1 is an eigenvalue of A. (non-regular bipartite graph)

Problem 4 (20%). Let $G = (V, E)$ be a d-regular graph and P be a random walk of length t in G. Prove that, for any edge $e \in E$ and any $1 \le i \le t$,

$$\Pr\left[\ e \text{ is the } i^{th}\text{-edge of } P\ \right] = \frac{1}{|E|}.$$

Hint: Prove by induction on i.

Prove by induction on i, for $i = 1$, the probability that $e = (u, v)$ is the 1st edge of P is $\Pr(u \text{ or } v \text{ is starting point}) \times \Pr(\text{edge } e \text{ is chosen})$

$= \frac{2}{|V|} \times \frac{1}{d}$. Since G is d-regular, we have $|V|d = 2|E|$.

$\rightarrow \Pr(e \text{ is the 1st edge of } P) = \frac{2}{|V|} \times \frac{1}{d} = \frac{2}{2|E|} = \frac{1}{|E|}$

Assume for some $i > 1$, we have $\Pr(e \text{ is the } i\text{-th edge of } P) = \frac{1}{|E|}$.

For $\Pr(e \text{ is the } (i+1)\text{-th edge of } P)$, it requires that the i-th edge of P should be edges e' incident to e, therefore :

$$\boxed{\Pr(e \text{ is the } (i+1)\text{-th edge of } P) = \sum_{e' \in E} \Pr(e' \text{ is the } i\text{-th edge of } P \ \underline{\text{AND}} \text{ transition to } e)}$$

$\rightarrow$ The second term is equal to $\sum_{e' \in E} \frac{1}{|E|} \times \frac{1}{d}$, where $\frac{1}{|E|}$ is by I.H. and $\frac{1}{d}$ is by uniform distribution (to select next edge).

$\rightarrow$ When e' is the i-th edge of P, we are already on one of e's end points, $(u \text{ or } v)$ since G is d-regular, there are only d terms in the summation.

不用考慮 u和v 的所有 edge (共 2d-1 個)

$\rightarrow \sum_{e' \in E} \frac{1}{|E|} \times \frac{1}{d} = \frac{1}{|E|} = \Pr(e \text{ is the } (i+1)\text{-th edge of } P)$

By induction, we proved the statement.

Problem 5 (20%). Let $G = (V, E)$ be an (n, d, λ)-expander and $S \subseteq V$ be a vertex subset. Prove that,

$$\Pr_{(u,v)\in E}[\, u, v \in S \,] \le \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right),$$

i.e., for any $(u, v) \in E$, the probability that both u, v are in S is bounded by $\frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right)$.

Hint: Use the fact that $|E(S,S)| = (d|S| - |E(S,T)|)/2$. Apply the crossing lemma.

The crossing Lemma states that: $E(S,T) \ge (1-\lambda)\frac{d|S||T|}{n}$

Some properties: ① $|E| = \frac{nd}{2}$ ② $|T| = n - |S|$

$$\Pr[u,v\in S] = \frac{|E(S,S)|}{|E|} \quad (\text{pick an edge that's entirely in } S)$$

$$= 1 - \frac{1}{|E|}\left(E(T,T) + E(S,T)\right)$$

$$= 1 - \frac{1}{|E|}\left(\frac{d|T| - |E(S,T)|}{2} + |E(S,T)|\right)$$

$$\le 1 - \frac{1}{|E|}\left(\frac{d|T| + (1-\lambda)\frac{d|S||T|}{n}}{2}\right)$$

$$= 1 - \frac{d|T|}{2|E|} - (1-\lambda)\frac{d|S||T|}{2n|E|} = 1 - \frac{|T|}{n} - (1-\lambda)\frac{|S||T|}{n^2}$$

$$= \frac{|S|}{n} - (1-\lambda)\frac{|S|(n-|S|)}{n^2} = \frac{|S|}{n}\left(1 - (1-\lambda)\frac{(n-|S|)}{n}\right)$$

$$= \frac{|S|}{n}\left(\frac{n - n + |S| + n\lambda - \lambda|S|}{n}\right) = \frac{|S|}{n}\left((1-\lambda)\frac{|S|}{n} + \lambda\right)$$

$$\le \frac{|S|}{n}\left(\frac{|S|}{n} + \lambda\right) \quad (\lambda \ge 0\,?)$$