Combinatorial Mathematics

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Monday 18:30 – 20:20

Outline

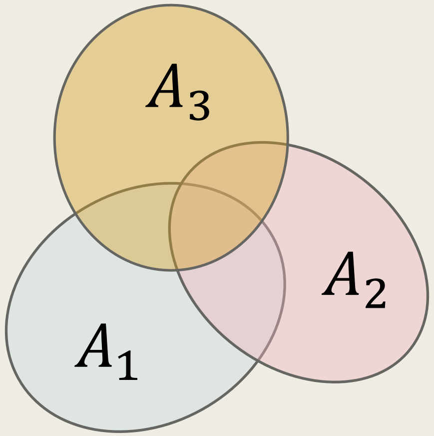
- The Lovász Local Lemma
  - Symmetric & Asymmetric versions
- Ex 1. Disjoint Cycles
- Ex 2. 2-Colorable Families

The Scenario

- To prove that $\Pr\left[\bigcap_i \overline{A_i}\right] > 0$ for a collection of *bad events* A_i, where
  - A_1 : undesirable event #1
  - A_2 : undesirable event #2

    …

  - $\bigcap_i \overline{A_i}$: the event that none of the bad events happen



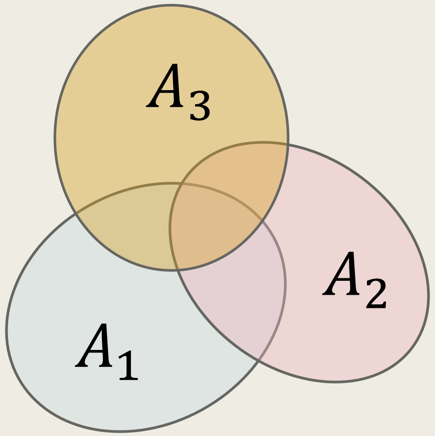
The Scenario

- To prove that $\Pr[\cap_i \overline{A_i}] > 0$ for a collection of bad events A_i

- When A_i are *mutually independent* and $\Pr[A_i] < 1$ for all i, then

$$\Pr\left[\bigcap_i \overline{A_i}\right] = \prod_i \Pr[\overline{A_i}] = \prod_i (1 - \Pr[A_i]) > 0 .$$

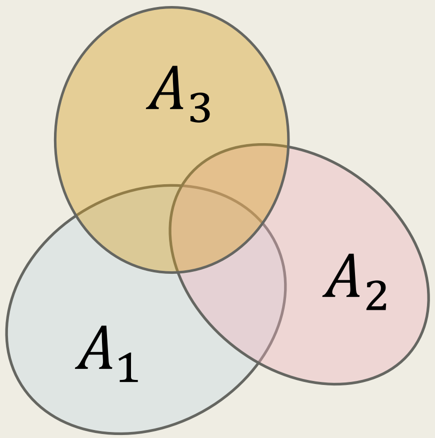
  - This argument works only when A_i satisfy strong independent requirement.



- When A_i are not independent, but $\sum_i \Pr[A_i] < 1$,

  we can apply *union bound* on A_i.

$$\boxed{\Pr\left[\bigcup_i A_i\right] \le \sum_i \Pr[A_i]\,,} \qquad \text{and}$$



$$\Pr\left[\bigcap_i \overline{A_i}\right] = \Pr\left[\overline{\bigcup_i A_i}\right] = 1 - \Pr\left[\bigcup_i A_i\right]$$

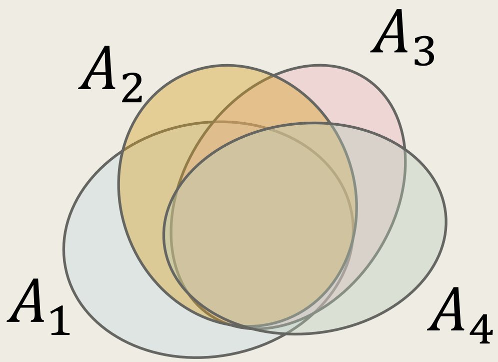
$$\ge 1 - \sum_i \Pr[A_i] > 0\,.$$

However, when $\sum_i \Pr[A_i] \ge 1$, the approach is inconclusive.

The Pros and the Cons

- Method 1 has the exact probability on $\Pr\left[\bigcap_i \overline{A_i}\right]$.
  However, it works only when A_i are independent.

- Method 2 can be used with dependency.
  However, union bound is loose and often it becomes inconclusive.



The shared area is counted a number of times in the union bound.

The Lovász Local Lemma (LLL)

- The Lovász Local Lemma provides a possible solution to the above scenario.
  - Roughly speaking, it says that,

We need to define what they mean.

when the events are "*mostly independent*" and individually "*not too likely to happen*",

then there is a positive probability that ***none of the events will occur***.

A *revised union bound* that takes the dependency of the events into considerations.

Some Definitions

Mutual Independence

- An event A is *mutually independent* of the events $B_1, B_2, \ldots, B_k$,

  if for any *Boolean combination* $C = \{C_1, C_2, \ldots, C_k\}$ of $B_1, B_2, \ldots, B_k$,

  where $C_i \in \{B_i, \overline{B_i}\}$ for all $1 \leq i \leq k$,

  we always have

$$\Pr[A|C] = \Pr[A] .$$

Mutual Independence

- Note that, by the definition,

  if A is *mutually independent* of the events $B_1, B_2, \dots, B_k$,

  then A is *mutually independent* of ***any subsets of*** $B_1, B_2, \dots, B_k$.

Refer to the jamboard for a sketch of the proof.

Why do we need this definition?

- Note that, it is possible that

  An event A is *individually independent* of the events $B_1, B_2, \ldots, B_k$, but *depends on some combination of them*.

  - For example, suppose that a fair coin is flipped twice, and let

    A : both flips are the same.

    B_i : the i^{th}-flip is a head.

    $\Pr[\, A \mid B_i \,] = 1/2 = \Pr[A]$.

  Then A is independent of B_1 and B_2 separately, but

$$\Pr[\, A \mid B_1 B_2 \,] = 1 \neq \Pr[A].$$

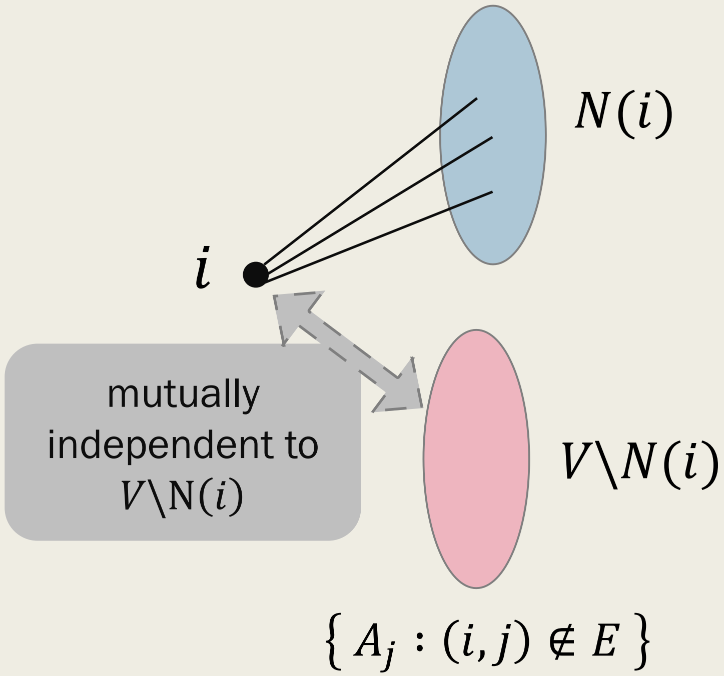
Dependency Graph of the Events

- Let $A_1, A_2, \dots, A_n$ be events.
  - A graph $G = (V, E)$ with $V = \{1,2,\dots,n\}$ is said to be

    a *dependency graph* for $A_1, A_2, \dots, A_n$

    if for all $1 \le i \le n$,

    A_i is *mutually independent* to $\{\, A_j : (i,j) \notin E \,\}$.



Note that, by the definition, dependency graph is not unique.

The Lovász Local Lemma

(Symmetric version)

The Lovász Local Lemma

Theorem 19.1 (Erdös-Lovász 1975).

Let $A_1, A_2, \ldots, A_n$ be events with $\mathbf{Pr}[\boldsymbol{A_i}] \leq \boldsymbol{p}$ for all i, and $\boldsymbol{d}$ ***be the maximum degree*** of a dependency graph for the events.

If $\boldsymbol{ep(d+1) \leq 1}$, then

$$\Pr\left[\overline{A_1}\,\overline{A_2}\cdots\overline{A_n}\right] > 0\,.$$

A slightly weaker version

- The following (weaker) version is sometimes more handy to apply.

Theorem (Erdös-Lovász 1975).

Let $A_1, A_2, \ldots, A_n$ be events with $\Pr[A_i] \le p$ for all i, and let d be the maximum degree of a dependency graph.

If $\boldsymbol{4pd \le 1}$, then $\Pr\left[\overline{A_1}\,\overline{A_2}\cdots\overline{A_n}\right] > 0$.

In fact, this was the original version of LLL when it first appeared.

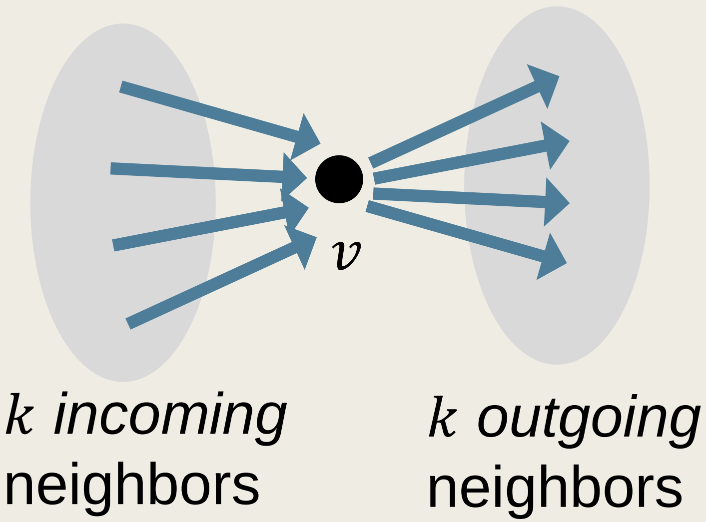
Ex1.

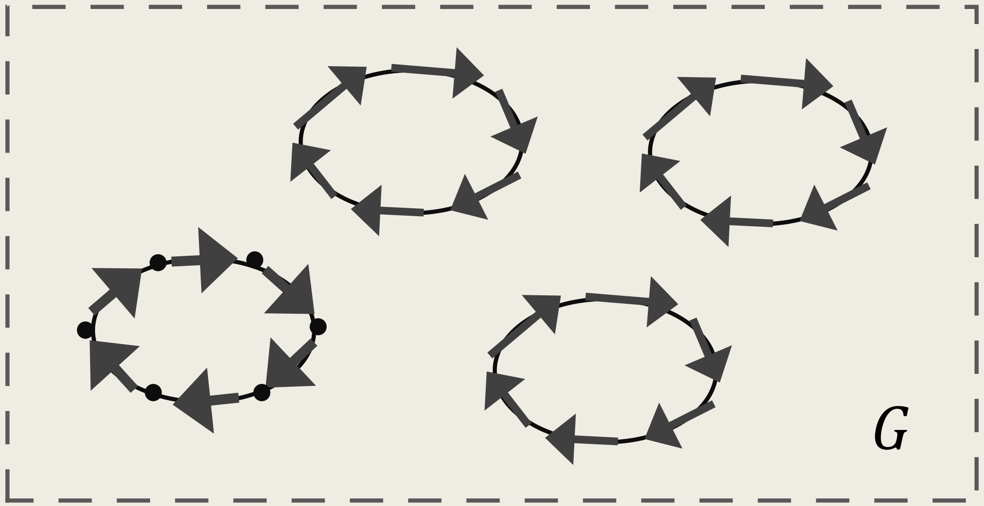
Disjoint Cycles

- A directed graph is said to be ***k*-regular**,
  if the *in-degree* and the *out-degree* of every vertex are both k.

Theorem 19.4 (Erdös 1963a).

Every finite k-regular directed graph has a collection of $\lfloor k/(3 \ln k) \rfloor$ vertex disjoint cycles.





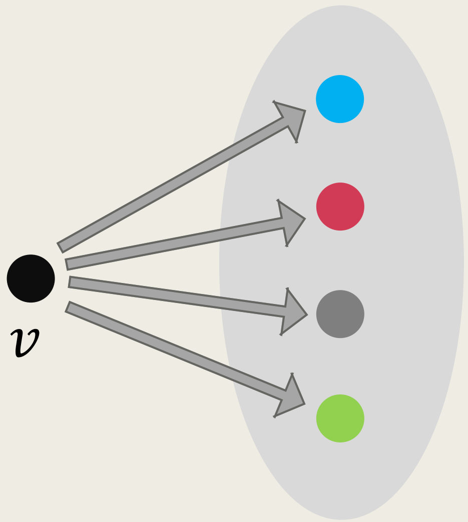
Theorem 19.4 (Erdös 1963a).

Every finite k-regular directed graph has a collection of $\lfloor k/(3\ln k)\rfloor$ vertex disjoint cycles.

- Consider a *uniform random coloring* of the vertices using $r := \lfloor k/(3\ln k)\rfloor$ colors.

  To prove the lemma, we will show that,
  there exists a coloring such that,

  every vertex has all the r colors in its out-neighbors.



Why does this suffice?

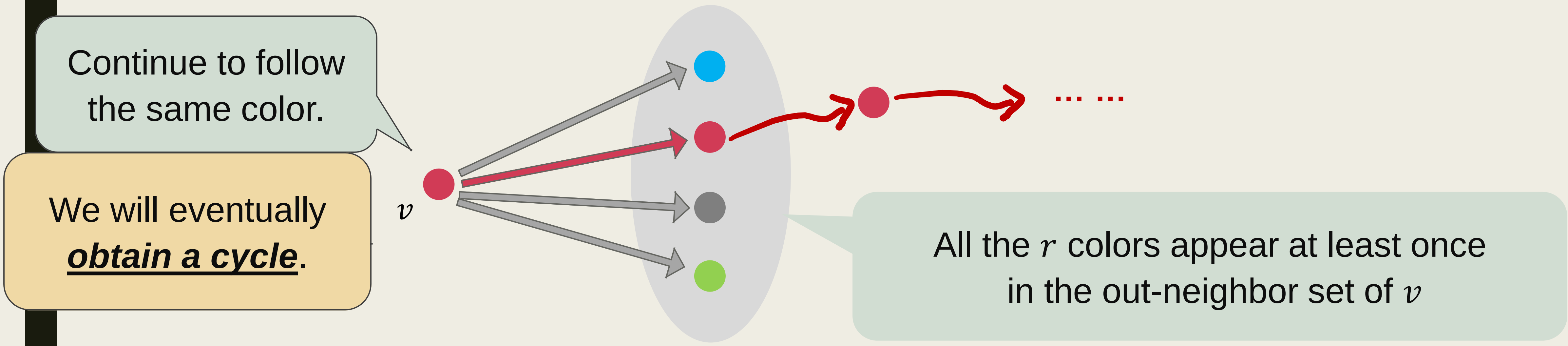
Theorem 19.4 (Erdös 1963a).

Every finite k-regular directed graph has a collection of $\lfloor k/(3\ln k)\rfloor$ vertex disjoint cycles.

- Consider a random coloring of the vertices using $r \coloneqq \lfloor k/(3\ln k)\rfloor$ colors.

  We will show that, there exists a coloring such that,

  every vertex has all the r colors in its out-neighbors.



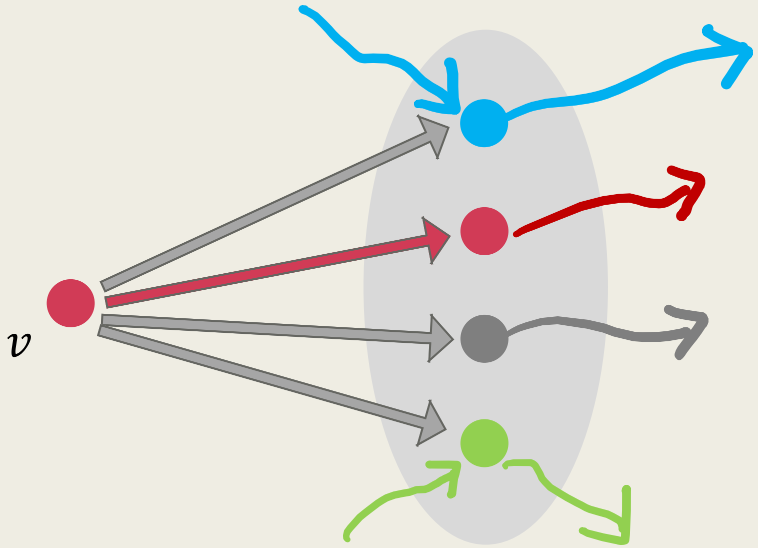
Theorem 19.4 (Erdös 1963a).

Every finite k-regular directed graph has a collection of $\lfloor k/(3\ln k)\rfloor$ vertex disjoint cycles.

- Consider a random coloring of the vertices using $r \coloneqq \lfloor k/(3\ln k)\rfloor$ colors.

  We will show that, there exists a coloring such that,

  every vertex has *all the r colors* in its *out-neighbors*.



Apply the same argument for all colors.

This will imply the conclusion of the theorem.

- Consider a random coloring of the vertices using $r \coloneqq \lfloor k/(3\ln k) \rfloor$ colors.

- For any $v \in V$, let A_v denote the event that

  <u>not every color is used</u> *in the out-neighbors of* v.

  – For any $1 \le i \le r$, let $A_{i,v}$ denote the event that

    the i^{th} *color is not used in the out-neighbors of* v.

  – Then we obtain

$$\Pr[A_v] = \Pr\left[\bigcup_{1\le i\le r} A_{i,v}\right] \le r \cdot \left(1 - \frac{1}{r}\right)^k \le r \cdot e^{-\frac{k}{r}} \le \frac{1}{3k^2 \ln k}.$$

- Consider a random coloring of the vertices using $r \coloneqq \lfloor k/(3\ln k) \rfloor$ colors.

- For any $v \in V$, let A_v denote the event that

  <u>not every color is used</u> *in the out-neighbors of v.*

  – Then we obtain

$$\Pr[A_v] \le r \cdot \left(1 - \frac{1}{r}\right)^k \le r \cdot e^{-\frac{k}{r}} \le \frac{1}{3k^2 \ln k}.$$

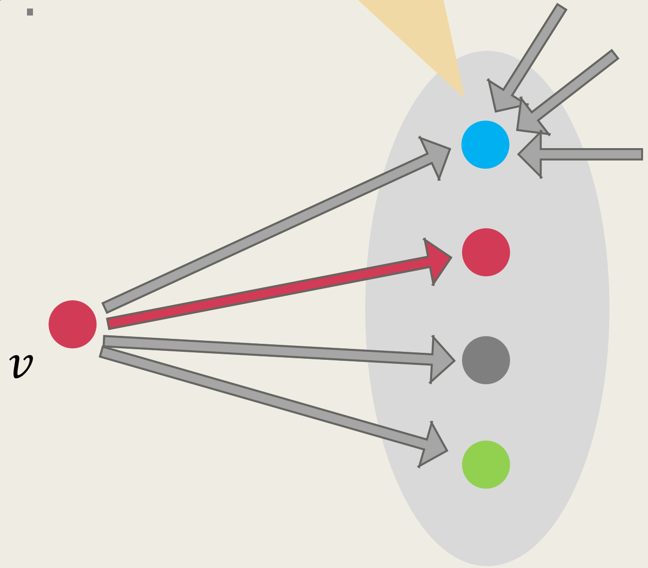
  – Let $N(v)$ denote the out-neighbors of v.

  Then, A_v is mutually independent to

$$\{ A_u : N(u) \cap N(v) = \emptyset \}.$$

  So, A_v shares dependency with at most k^2 events.

The events of the k in-neighbors of this vertex will share dependency with A_v



A_v : not every color is used

- Consider a random coloring of the vertices using $r \coloneqq \lfloor k/(3\ln k) \rfloor$ colors.
- For any $v \in V$, let A_v denote the event that
  not every color is used in the out-neighbors of v.
  - Then $\Pr[A_v] \le r \cdot \left(1 - \frac{1}{r}\right)^k \le r \cdot e^{-\frac{k}{r}} \le \frac{1}{3k^2 \ln k}$.
  - The maximum degree of dependency $d \le k^2$.
  - Since $$e \cdot \frac{1}{3k^2 \ln k} \cdot \left(k^2 + 1\right) \le 1, \qquad \forall\, k \ge 3,$$
    $\Pr\left[\cap\, \overline{A_v}\right] > 0$ by the Lovász Local Lemma.
  - Hence, there must exist such a good coloring for $k \ge 3$.
  - For $k \le 2$, $r = 0$ and the statement of the lemma holds trivially.

Some remarks

- If you have read the textbook,
  the proof of the Theorem 19.4 in the textbook is *incorrect*.
  - The reason is that, it sets the event A_v as
    "the color of v does not appear in the out-neighbors of v."
  - As a result, it didn't consider the event that not all the r colors are used.
  - When this happens, we don't get r disjoint cycles, and

    what it has truly proved is that,

    "*there exists a cycle in the graph*."

The Lovász Local Lemma

--- The Asymmetric Version

When the universal bounds of the events are not good enough.

Theorem 19.2.

Let $G = (V, E)$ be a dependency graph of events $A_1, A_2, \ldots, A_n$.

Suppose that there exists real numbers $x_1, x_2, \ldots, x_n$ with $0 \leq x_i < 1$ such that, for all i,

$$\Pr[A_i] \leq x_i \cdot \prod_{j:(i,j)\in E} (1 - x_j) .$$

Then

$$\Pr\left[\overline{A_1}\, \overline{A_2} \cdots \overline{A_n} \right] \geq \prod_{1\leq i\leq n} (1 - x_i) .$$

In particular, with positive probability, no A_i occurs.

Ex2.

2-Colorable Families

2-Colorable Families

- In Lecture 2, we use simple union bound to show that

  when the size of a k-uniform family is no more than 2^{k-1}, it is 2-colorable.

- We use the Lovász Local Lemma to prove a stronger statement, which takes the structure of the family into consideration.

  It says that, when the dependency of the members is bounded within 2^{k-3}, the family is 2-colorable.

Theorem 19.5 (Erdös-Lovász 1975).

If every member of a k-uniform family intersects at most 2^{k-3} other members, then the family is 2-colorable.

Theorem 19.5 (Erdös-Lovász 1975).

If every member of a k-uniform family $F = \{S_1, S_2, \ldots, S_m\}$ intersects at most 2^{k-3} other members, then the family is 2-colorable.

Proof.

- Let X be the ground set. Consider a random 2-coloring on X and let A_i be the event that S_i is monochromatic.
  - We have $\Pr[A_i] = 2 \cdot 2^{-|S_i|} = 2^{1-k}$.
  - Any A_i is mutually independent to $\{ A_j : S_i \cap S_j = \emptyset \}$. So the maximum degree of dependency $d \leq 2^{k-3}$.
- Since $4pd = 2^0 \leq 1$, the LLL guarantees that $\Pr\left[\overline{A_1}\, \overline{A_2} \cdots \overline{A_n} \right] > 0$.

- For non-uniform families, we have the following theorem.

Theorem 19.6 (Beck 1980).

Let $F = \{S_1, S_2, \dots, S_m\}$ be a family of sets, each of which has at least $k \geq 2$ elements. Suppose that for each element v in the ground set,

$$\sum_{S \in F : v \in S} \left(1 - \frac{1}{k}\right)^{-|S|} \cdot 2^{-|S|+1} \leq \frac{1}{k}.$$

Then F is 2-colorable.

- Consider a random coloring, and let A_i be the event that S_i is monochromatic.

  Include an edge for A_i, A_j in the dependency graph if and only if $S_i \cap S_j \neq \emptyset$,

  and define $x_i \coloneqq \left(1 - \frac{1}{k}\right)^{-|S_i|} \cdot 2^{-|S_i|+1}$ for all i.

Theorem 19.6 (Beck 1980).

Let $F = \{S_1, S_2, \dots, S_m\}$ be a family of sets, each of which has at least $k \geq 2$ elements.

Suppose that for each element v, $\sum_{S \in F : v \in S} \left(1 - \frac{1}{k}\right)^{-|S|} \cdot 2^{-|S|+1} \leq \frac{1}{k}$.

Then F is 2-colorable.

- Consider a random coloring, and let A_i be the event that S_i is monochromatic. Include an edge for A_i, A_j in the dependency graph if and only if $S_i \cap S_j \neq \emptyset$, and define $x_i := \left(1 - \frac{1}{k}\right)^{-|S_i|} \cdot 2^{-|S_i|+1}$ for all i.

- To apply the local lemma, we need to show that

$$x_i \cdot \prod_{j:(i,j) \in E} \left(1 - x_j\right) \geq \Pr[A_i], \qquad \forall 1 \leq i \leq m .$$

Theorem 19.6 (Beck 1980).

Let $F = \{S_1, S_2, \dots, S_m\}$ be a family of sets, each of which has at least $k \geq 2$ points.

Suppose that for each point v, $\sum_{S \in F : v \in S} \left(1 - \frac{1}{k}\right)^{-|S|} \cdot 2^{-|S|+1} \leq \frac{1}{k}$.

Then F is 2-colorable.

Define $x_i \coloneqq \left(1 - \frac{1}{k}\right)^{-|S_i|} \cdot 2^{-|S_i|+1}$ for all i .

- We have

$0 \leq x_i < 1$

$$x_i \cdot \prod_{j:(i,j) \in E} \left(1 - x_j\right) \geq x_i \cdot \prod_{v \in S_i} \prod_{j: v \in S_j} \left(1 - x_j\right)$$

Interprete x_i as probabilities of some other independent events.

Refer to jamboard for details.

$$\geq x_i \cdot \prod_{v \in S_i} \left(1 - \sum_{j: v \in S_j} x_j\right) \geq x_i \cdot \left(1 - \frac{1}{k}\right)^{|S_i|}$$

by the assumption of the theorem

$$= 2^{-|S_i|+1} = \Pr[A_i] \ .$$

by the definition of x_i

Can we actually construct the object ?

Some remark

- The Lovász Local Lemma, and the probabilistic method we introduced, aims to prove the existence of an object satisfying a set of constraints.
  - A natural question is that,

    can we actually *compute* such an object *efficiently*?

  - When the conditions in the Lovász Local Lemma are met, the answer is ***yes*!**

    Such an object can be constructed in expected $O\left(\sum_A \frac{x(A)}{1-x(A)}\right)$ time.

    We will talk about this in lecture #15 (as supplementary content).

Proof of Theorem 19.1

(Symmetric LLL, weaker version)

Proof of the LLL (weaker version)

- We will prove the theorem under a slightly stronger condition, i.e., $4pd \leq 1$.

Theorem (Erdös-Lovász 1975).

Let $A_1, A_2, \dots, A_n$ be events with $\Pr[A_i] \leq p$ for all i, and let d be the maximum degree of their dependence.

If $4pd \leq 1$, then $\Pr\left[\overline{A_1}\,\overline{A_2} \cdots \overline{A_n}\right] > 0$.

- In HW3, you will use asymmetric LLL to prove the stronger version of symmetric LLL with $ep(d+1) \leq 1$.

Tools to Use

- We will use the following two identities for conditional probability.
  - $\Pr[A \mid BC] = \frac{\Pr[AB \mid C]}{\Pr[B \mid C]}$.
  - $\Pr[ABC] = \Pr[A \mid BC] \cdot \Pr[B \mid C] \cdot \Pr[C]$.

They follow directly from the definition of conditional probability.

Tools to Use

- In general,

$$\Pr[\,A \mid B_1B_2\cdots B_m\,] = \frac{\Pr\left[\,AB_1B_2\cdots B_j \,\middle|\, B_{j+1}B_{j+2}\cdots B_m\,\right]}{\Pr\left[\,B_1B_2\cdots B_j \,\middle|\, B_{j+1}B_{j+2}\cdots B_m\,\right]} \quad \forall 1 \le j \le m,$$

$(*)$

and

$$\Pr[\,A_1A_2\cdots A_m\,] = \prod_{1\le j\le m} \Pr\left[\,A_j \,\middle|\, A_{j+1}A_{j+2}\cdots A_m\,\right]$$

$(**)$

Refer to the jamboard for the details.

Theorem (Erdös-Lovász 1975).

Let $A_1, A_2, \ldots, A_n$ be events with $\Pr[A_i] \le p$ for all i, and let d be the maximum degree of their dependence. If $\boldsymbol{4pd} \le 1$, then $\Pr\left[\overline{A_1}\,\overline{A_2} \cdots \overline{A_n}\right] > 0$.

Proof.

- Fix a dependency graph with maximum degree d.
- We will prove that, for ***any subset of events*** of $A_1, A_2, \ldots, A_n$, denoted $B_1, B_2, \ldots, B_m$ for convenience,

we always have

$$\Pr\left[\, B_1 \,\middle|\, \overline{B_2}\,\overline{B_3} \cdots \overline{B_m} \,\right] \le 2p.$$

- We will prove that, for ***any subset of m events*** of $A_1, A_2, \dots, A_n$, denoted $B_1, B_2, \dots, B_m$ for convenience,

  we always have

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] \le 2p.$$

- If this holds, then by $(**)$, we have

$$\Pr[\, \overline{A_1}\, \overline{A_2} \cdots \overline{A_n} \,] = \prod_{1 \le j \le n} \Pr[\, \overline{A_j} \mid \overline{A_{j+1}}\, \overline{A_{j+2}} \cdots \overline{A_n} \,]$$

$$= \prod_{1 \le j \le n} \left(1 - \Pr[\, A_j \mid \overline{A_{j+1}}\, \overline{A_{j+2}} \cdots \overline{A_n} \,] \right) \ge (1 - 2p)^n > 0\,.$$

It suffices to show that, for any subset of m events of $A_1, A_2, \ldots, A_n$, denoted $B_1, B_2, \ldots, B_m$, we always have

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] \le 2p.$$

- Prove by induction on m.
  - The base case $m = 1$ is trivial.
  - For $m \ge 2$, assume without loss of generality that, B_1 and $B_{k+1}, \ldots, B_m$ are mutually independent.
    - i.e., B_1 share dependency only with $B_2, B_3, \ldots, B_k$.

Hence, $k - 1 \le d$.

It suffices to show that, for any subset of m events of $A_1, A_2, \dots, A_n$, denoted $B_1, B_2, \dots, B_m$, we always have

$$\Pr[\, B_1 \mid \overline{B_2}\,\overline{B_3} \cdots \overline{B_m} \,] \le 2p.$$

- For $m \ge 2$, assume without loss of generality that, B_1 and $B_{k+1}, \dots, B_m$ are mutually independent.
  - i.e., B_1 share dependency only with $B_2, B_3, \dots, B_k$.

Hence, $k - 1 \le d$.

- By $(*)$, we have

$$\Pr[\, B_1 \mid \overline{B_2}\,\overline{B_3} \cdots \overline{B_m} \,] = \frac{\Pr[\, B_1 \overline{B_2}\,\overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m} \,]}{\Pr[\, \overline{B_2}\,\overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}$$

Consider the numerator and the denominator separately.

- Assume that B_1 is mutually independent to $B_{k+1}, \dots, B_m$.

  By $(*)$, we have

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m}\,] = \frac{\Pr[\, B_1 \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}\,]}{\Pr[\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]} \; .$$

- For the numerator, we have

$$\Pr[\, B_1 \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}\,] \le \Pr[B_1 \mid \overline{B_{k+1}} \cdots \overline{B_m}]$$

$$= \Pr[B_1] \le p \, .$$

Since B_1 is mutually independent of $B_{k+1}, \dots, B_m$

It suffices to show that, for any subset of m events of $A_1, A_2, \dots, A_n$, denoted $B_1, B_2, \dots, B_m$, we always have

$$\Pr[\, B_1 \,\big|\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] \le 2p.$$

- For the denominator,

$$\Pr[\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \,\big|\, \overline{B_{k+1}} \cdots \overline{B_m}] \;=\; 1 \;-\; \Pr[\, B_2 \cup \cdots \cup B_k \,\big|\, \overline{B_{k+1}} \cdots \overline{B_m}]$$

Union bound

$$\ge\; 1 - \sum_{2 \le i \le k} \Pr[\, B_i \mid \overline{B_{k+1}} \cdots \overline{B_m} \,]$$

Induction hypothesis

$$\ge\; 1 - 2p(k-1) \;\ge\; \frac{1}{2}\;,$$

since $2p(k-1) \;\le\; 2pd \;\le\; 1/2$.

Instead of applying union bound directly,
this lemma applies
when the events are properly conditioned.

It suffices to show that, for any subset of m events of $A_1, A_2, \dots, A_n$, denoted $B_1, B_2, \dots, B_m$, we always have

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] \le 2p.$$

- Then, we obtain

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] = \frac{\Pr[\, B_1 \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m} \,]}{\Pr[\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]} \; .$$

$$\le \frac{p}{1/2} = 2p\,.$$

Proof of the Asymmetric LLL

Theorem 19.2.

Let $G = (V, E)$ be a dependency graph of events $A_1, A_2, \dots, A_n$.

Suppose that there exists real numbers $x_1, x_2, \dots, x_n$ with $0 \le x_i < 1$ such that, for all i,

$$\Pr[A_i] \le x_i \cdot \prod_{j:(i,j)\in E} \left(1 - x_j\right) .$$

Then

$$\Pr\left[\overline{A_1}\,\overline{A_2} \cdots \overline{A_n}\right] \ge \prod_{1\le i\le n} (1 - x_i) .$$

In particular, with positive probability, no event A_i holds.

- The proof is analogous to the symmetric version of the lemma.
- We will use induction to prove that,

for any subset of events of $A_1, A_2, \dots, A_n$, say, $B_1, B_2, \dots, B_m$, for convenience, we always have

$$\boldsymbol{Pr[\, B_1 \,\middle|\, \overline{B_2}\,\overline{B_3} \cdots \overline{B_m}\,] \le x_1} \,.$$

Then by $(**)$ we have

$$\Pr[\,\overline{A_1}\,\overline{A_2} \cdots \overline{A_n}\,] \;=\; \prod_{1\le j\le n} \left(1 - \Pr\left[A_j \mid \overline{A_{j+1}}\,\overline{A_{j+2}} \cdots \overline{A_n}\right]\right) \;\ge\; \prod_{1\le i\le n} (\,1 - x_i\,)\,.$$

- The induction base $m = 1$ follows from the assumption of the lemma.

  For $m \ge 2$, we consider an arbitrary combination of m events.

It suffices to show that,
for any subset of m events of $A_1, A_2, \ldots, A_n$, say, $B_1, B_2, \ldots, B_m$, we always have

$$\Pr[\, B_1 \mid \overline{B_2}\,\overline{B_3} \cdots \overline{B_m}\,] \le x_1.$$

- W.L.O.G., let $B_2, B_3, \ldots, B_k$ be events that share dependency with B_1, while $B_{k+1}, \ldots, B_m$ are mutually independent to B_1.

- By $(*)$, we have $\Pr[\, B_1 \mid \overline{B_2}\,\overline{B_3} \cdots \overline{B_m}\,] = \dfrac{\Pr\left[\, B_1 \overline{B_2}\,\overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}\,\right]}{\Pr\left[\, \overline{B_2}\,\overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}\right]}$.

- For the numerator,

$$\Pr[\, B_1 \overline{B_2}\,\overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}\,] \le \Pr[\, B_1 \mid \overline{B_{k+1}} \cdots \overline{B_m}\,] = \Pr[\, B_1\,]$$

$$\le x_1 \cdot \prod_{j:(i,j)\in E} \left(1 - x_j\right) \le x_1 \cdot \prod_{2\le j\le k} \left(1 - x_j\right).$$

It suffices to show that,
for any subset of m events of $A_1, A_2, \ldots, A_n$, say, $B_1, B_2, \ldots, B_m$, we always have

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m}\,] \le x_1.$$

- W.L.O.G., let $B_2, B_3, \ldots, B_k$ be events that share dependency with B_1, while $B_{k+1}, \ldots, B_m$ are mutually independent to B_1.

- By $(*)$, we have $\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m}\,] = \dfrac{\Pr[\, B_1 \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}\,]}{\Pr[\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]}$.

- For the denominator, apply $(**)$ and the induction hypothesis, we obtain

$$\Pr[\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}] = \prod_{2 \le j \le k} \Pr[\, \overline{B_j} \mid \overline{B_{j+1}} \cdots \overline{B_m}\,] \ge \prod_{2 \le j \le k} (1 - x_j) \; .$$

It suffices to show that,
for any subset of m events of $A_1, A_2, \ldots, A_n$, say, $B_1, B_2, \ldots, B_m$, we always have

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] \le x_1.$$

- W.L.O.G., let $B_2, B_3, \ldots, B_k$ be events that share dependency with B_1, while $B_{k+1}, \ldots, B_m$ are mutually independent to B_1.

- By $(*)$, we have $\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] = \dfrac{\Pr\left[\, B_1 \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m} \,\right]}{\Pr\left[\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}\right]}$.

- Combine the two inequalities. We obtain

$$\Pr[\, B_1 \mid \overline{B_2}\, \overline{B_3} \cdots \overline{B_m} \,] = \frac{\Pr[\, B_1 \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m} \,]}{\Pr[\, \overline{B_2}\, \overline{B_3} \cdots \overline{B_k} \mid \overline{B_{k+1}} \cdots \overline{B_m}]} \le \frac{x_1 \cdot \prod_{2 \le j \le k} (1 - x_j)}{\prod_{2 \le j \le k} (1 - x_j)} = x_1 \,.$$

Some remark

- In HW3, you will prove that Theorem 19.2 leads to Theorem 19.1.
  - This is done as follows.

Set $x_i = \frac{1}{d+1}$ for each event A_i, and apply the inequality that

$$\frac{1}{e} \leq \left(1 - \frac{1}{d+1}\right)^d .$$

This can be obtained from the limit formula $e = \lim_{d\to\infty}\left(1 + \frac{1}{d}\right)^d$ and the fact that it converges from the above.