Problem 1.

For any positive integer n, there are n possible modulo from 0 to n-1. Consider all numbers of the form $a_i = 7...7, 777, 77, 7$, for $i = 1, 2, ..., n+1$, and consider the value a_i modulo n. Since there are $n+1$ different a_i but only n possible modulo, there must be at least two different a_i share a same remainder, say a_i and a_j. Then we take the value: $a_i - a_j = 0$ (mod n), notice that $a_i - a_j$ contains only 7 and 0, so we have proved the statement.

Problem 2.

(1.) A pair of consecutive numbers

The set: $\{1, 2, ..., 2n\}$ can be transformed into the set: $\{(2i, 2i-1) \mid \forall\ 1 \leq i \leq n\}$. There are n elements in the second set, and each element consists of two consecutive numbers. When we choose $n+1$ numbers from the second set, since there are only n elements, there are at least one element we have to choose both numbers from the pair, which guarantees that we will always choose a pair of consecutive number.

(2.) A pair whose sum is $2n+1$

The set: $\{1, 2, ..., 2n\}$ can be transformed into the set: $\{(i, 2n-i+1) \mid \forall\ 1 \leq i \leq n\}$. There are n elements in the second set, and each element consists of a pair of numbers whose sum is $2n+1$. Applying the same argument($n+1$ numbers from n pairs) as in (1.) and we are guaranteed to choose a pair of numbers whose sum is $2n+1$.

Problem 3.

If we color the vertices in some order $v_1, v_2, v_3, ..., v_n$. For each v_k, let N_k be the number of neighbors precedes it in the ordered sequence, we can color v_k with the color C_{N_k+1}. Since each v_k contains at most $\Delta(G)$ neighbors, the color used won't exceed $C_{\Delta(G)+1}$, which means we can color the graph with $\Delta(G)+1$ colors.

Problem 4.

Note that an independent set in graph G is a clique in its complement $\overline{G}$.

By Turan's theorem, if G contains no (k+1)-cliques, where $k \geq 2$, then $|E| \leq (1 - \frac{1}{k})\frac{n^2}{2}$.

For this particular G with $\frac{nk}{2}$ edges, we consider the complement $\overline{G}$:

$$\begin{gathered}
\tfrac{n(n-1)}{2} - \tfrac{nk}{2} \leq (1 - \tfrac{1}{\omega(\overline{G})})\tfrac{n^2}{2} \\
\tfrac{n(n-1)}{2} - \tfrac{nk}{2} \leq (1 - \tfrac{1}{\alpha(G)})\tfrac{n^2}{2} \\
n - (k+1) \leq (1 - \tfrac{1}{\alpha(G)})n \\
\alpha(G) \geq \tfrac{n}{k+1}
\end{gathered}$$

Problem 5.

Prove by contradiction: If less than $(1-\lambda)|Y|$ elements of Y are λ-large, then we have: $\sum_{B_i \text{ is not } \lambda\text{-large}} |B_i| > \lambda|Y|$.
Also from the definition of λ-large, we have: $\sum_{B_i \text{ is not } \lambda\text{-large}} |B_i| < \sum_{B_i \text{ is not } \lambda\text{-large}} \lambda\frac{|Y|}{|X|}|A_i|$
From this relation, we could derive: $\sum_{B_i \text{ is not } \lambda\text{-large}} \lambda\frac{|Y|}{|X|}|A_i| < \sum_i \lambda\frac{|Y|}{|X|}|A_i| = \lambda\frac{|Y|}{|X|}|X| = \lambda|Y|$ As a result, we get two contradictory relation: $\sum_{B_i \text{ is not } \lambda\text{-large}} |B_i| > \lambda|Y|$, and $\sum_{B_i \text{ is not } \lambda\text{-large}} |B_i| < \lambda|Y|$