Problem 1 (20%). How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 = 21$ with

1. $x_i \geq 0$. 2. $x_i > 0$. 3. $0 \leq x_i \leq 12$.

Problem 2 (20%). Prove the following identities **using path-walking argument**.

1. For any $n, r \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq r} \binom{n+k}{k} \;=\; \binom{n+r+1}{r}.$$

2. For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,

$$\sum_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} \;=\; \binom{m+n}{r}.$$

Problem 3 (20%). Let $\mathcal{F}$ be a set family on the ground set X and $d(x)$ be the degree of any $x \in X$, i.e., the number of sets in $\mathcal{F}$ that contains x. Use the double counting principle to prove the following two identities.

$$\sum_{x \in Y} d(x) \;=\; \sum_{A \in \mathcal{F}} |Y \cap A| \;\text{ for any } Y \subseteq X.$$

$$\sum_{x \in X} d(x)^2 \;=\; \sum_{A \in \mathcal{F}} \sum_{x \in A} d(x) \;=\; \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$$

Problem 4 (20%). Let H be a 2α-dense 0-1 matrix. Prove that at least an $\alpha/(1-\alpha)$ fraction of its rows must be α-dense.

Problem 5 (20%). Let $\mathcal{F}$ be a family of subsets defined on an n-element ground set X. Suppose that $\mathcal{F}$ satisfies the following two properties:

1. $A \cap B \neq \emptyset$ for any $A, B \in \mathcal{F}$.
2. For any $A \subsetneq X$, $A \notin \mathcal{F}$, there always exists $B \in \mathcal{F}$ such that $A \cap B = \emptyset$.

Prove that

$$2^{n-1} - 1 \;\leq\; |\mathcal{F}| \;\leq\; 2^{n-1}.$$

Hint: Consider any set $A \subseteq X$ and its complement $\overline{A}$. Apply the conditions given above and prove the two inequalities "$\leq$" and "$\geq$" separately.

Problem 1、

1、 $x_i \geq 0$: $0\cdots0|0\cdots0|0\cdots0|0\cdots0$ ⇒ total 21 "0", 3 "1" ⇒ $\frac{24!}{21!3!} = \binom{24}{3} = 2024$ #

2、 $x_i > 0$: $(x_1'-1)+(x_2'-1)+(x_3'-1)+(x_4'-1) = 17$ ⇒ 17 "0", 3 "1" ⇒ $\frac{20!}{17!\,3!} = \binom{20}{3} = 1140$

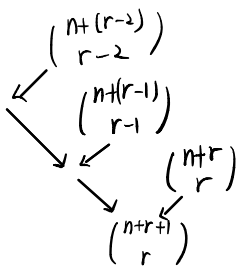
3、 $0 \leq x_i \leq 12$: Let A_i be the event that $x_i > 12$, since there cannot be more than one x greater than 12 (or others will have to be negative), $A_i \cap A_j = \emptyset$, $\forall i \neq j$

⇒ (# solutions with $0 \leq x_i$) − (A_i, $i \in \{1,2,3,4\}$)

⇒ $\binom{24}{3} - \binom{11}{3} \times 4 = 4\times23\times22 - 11\times5\times3\times4 = 1364$ #

Problem 2、

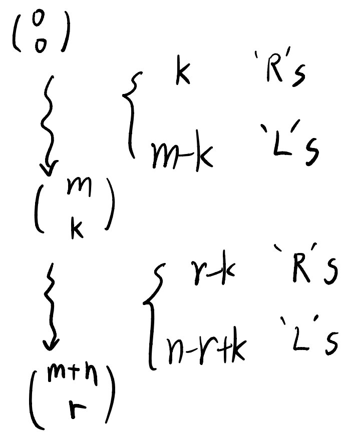
1、 Recognize the term on LHS is # path through $\binom{n+k}{k}$ to $\binom{n+r+1}{r}$ with 1 more "L" and the rest "R" steps.



These paths are disjoint and we can identify these paths by its last "L"

there are $\sum_{0\leq k\leq r} \binom{n+k}{k}$ such paths.

#



2、 Consider the set of all possible downward path to $\binom{m+n}{r}$

- Identify any of such paths by the cell it reaches at m^{th}-row
  - suppose it's $\binom{m}{k}$
  - From the illustration on RHS, there are $\binom{m}{k}\binom{n}{r-k}$ such paths

→ take summation over the cells at n^{th} row

→ there are $\sum_{0\leq k\leq r} \binom{m}{k}\binom{n}{r-k}$ such paths #

⇒ $\binom{m}{k}\binom{n}{r-k}$

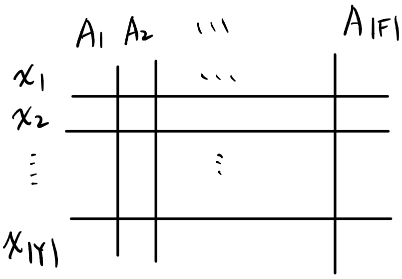
→ By the double-counting principle, they are equal

Problem 3、

1、 $\sum_{x\in Y} d(x) = \sum_{A\in F} |Y \cap A|$ for any $Y \subseteq X$.

- Consider the $|Y| \times |F|$ incidence matrix $M = (m_{x,A})$

  where $m_{x,A} = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{otherwise} \end{cases}$



(縱：所有Y的元素；橫：所有F的元素(X的 subset)

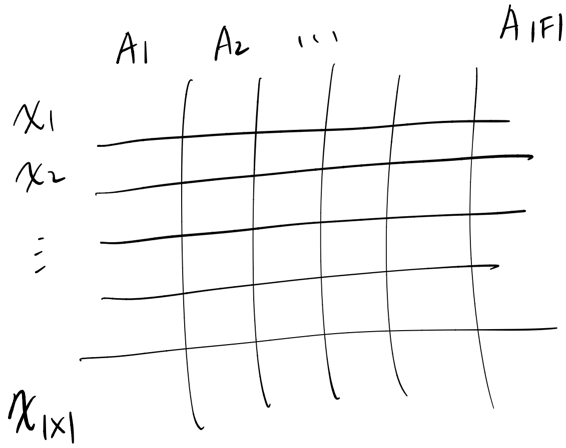
- $\sum_{x\in Y} d(x)$：把每列 1 的個數加起來。
- $\sum_{A\in F} |Y\cap A|$：每行有1對應的 x_i 代表 $A_j \cap Y$ 的元素
  考慮每個 $A\in F$ 就是把每行的1加起來。

→ 2個都是算 matrix 內有多少個 1。

→ By the double-counting principle, they are equal.

Problem 3、

2、 $\sum_{x\in X} d(x)^2 = \sum_{A\in F}\sum_{x\in A} d(x) = \sum_{A\in F}\sum_{B\in F} |A\cap B|$



$\sum_{A\in F}\sum_{x\in A} d(x)$：對每個在F的 A_i，如果 $m_{x_j,A_i}=1$ ($x_j \in A_i$)
就加一次 $d(x_j)$。

如此，每個 x_j 都會加 $d(x_j)$ 次
(因為 $d(x_j)$ 就是 x_j 存在於幾個 A_i 中
既然外層 ($\sum_{A\in F}$) 會走過所有 A，那 x_j 就
必然會走 $d(x_j)$ 次。)

$\Rightarrow d(x_j)\times d(x_j) = d(x_j)^2 \quad \forall j = 1\sim |X|$ … ⓐ

由第一題知：$\sum_{x\in A} d(x) = \sum_{B\in F} |A\cap B|$ (令 Y = A，把原式 A 改成 B) … ⓑ

$$\Rightarrow \sum_{x\in X} d(x)^2 \underset{ⓐ}{=} \sum_{A\in F}\sum_{x\in A} d(x) \underset{ⓑ}{=} \sum_{A\in F}\sum_{B\in F} |A\cap B| .$$

#

Problem 4、

$P \to Q \equiv \neg Q \to \neg P$, P: H is 2α dense
Q: At least $\frac{\alpha}{1-\alpha}$ rows are α-dense.

Let H be a $m\times n$ matrix

If less than $\frac{\alpha}{1-\alpha}$ fraction of its rows are α-dense, ($\neg Q$)

$\Rightarrow$ There are less than $\frac{\alpha}{1-\alpha} m \times n$ 1s in these α-dense rows. (when all 1s in dense rows)

$\Rightarrow$ There are less than $\frac{1-2\alpha}{1-\alpha} m \times \alpha n$ 1s in these non-α-dense rows. (sparse rows with density less than α)

$\Rightarrow$ There will be less than $\alpha mn \times \left(\frac{1-2\alpha}{1-\alpha} + 1\right) = 2\alpha mn$

$\Rightarrow$ H cannot be 2α-dense if $\frac{\alpha}{1-\alpha}$ fraction of rows are not α-dense.

$\Leftrightarrow$ It requires at least $\frac{\alpha}{1-\alpha}$ fraction of rows are α-dense for H to be 2α-dense.

Problem 5.

Consider any $A \subseteq X$ and $\bar{A}$, since $A \cap \bar{A} = \phi$, at most one of them can be in F.

If $A \in F$, then $\bar{A} \notin F$, by ②, $\exists B \in F$ s.t. $\bar{A} \cap B = \phi$.

There are 2^n possible subsets in X, but A and $\bar{A}$ cannot be in F at the same time.

When we choose 1 to be A from 2^n subsets, $\bar{A}$ is also determined, so we can only choose at most $2^n/2 = 2^{n-1}$ subsets from X to be in F $\Rightarrow |F| \leq 2^{n-1}$

For every pair of $A, \bar{A}$, if both A and $\bar{A} \notin F$, then to satisfy property ② subset of A and subset of $\bar{A}$ must be in F simultaneously.

but they are disjoint, which violates property ① that $A \cap B \neq \phi$

$\Rightarrow$ Exactly 1 of A and $\bar{A}$ has to be in F $\Rightarrow |F| \geq 2^{n-1} - 1$ (扣掉 ϕ)

Consider ϕ and X

by ①, we cannot choose ϕ in F, but we can put X in F.

by ②, It only requires $A \subsetneq X$, so we can leave $X \notin F$ and still satisfy $\exists B \in F$ $A \cap B = \phi$.

(X 放外面沒辦法 $A \cap B = \phi$, 但因為 ② 不考慮 X, 所以可以放外面)

$\Rightarrow |F| = 2^{n-1}$, if $X \in F$

$|F| = 2^{n-1} - 1$, if $X \notin F$, 只有這2種可能 $\Rightarrow 2^{n-1} - 1 \leq |F| \leq 2^{n-1}$

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