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組合數學 HW2

Problem 1 (20%). Let X, Y be discrete random variables. The variance of a random variable X is defined as $\mathrm{Var}[X] := \mathrm{E}[(X - \mathrm{E}[X])^2]$. Prove that

1. $\mathrm{E}[aX + bY] = a \cdot \mathrm{E}[X] + b \cdot \mathrm{E}[Y]$ for any constant a, b.
2. If X and Y are independent, then $\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y]$ and $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3. $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. *Hint:* Use the fact that $\mathrm{E}[X \cdot \mathrm{E}[X]] = \mathrm{E}[X]^2$.

1、 $E[X] = \sum_x x \cdot P[X=x]$, $E[Y] = \sum_y y\, P[Y=y]$

$$E[aX+bY] = \sum_x \sum_y (ax+by)\, P[X=x, Y=y]$$
$$= a\sum_x \sum_y x\, P[X=x, Y=y] + b\sum_x \sum_y y\, P[X=x, Y=y]$$
$$= a\sum_x x\, P[X=x] + b\sum_y y\, P[Y=y] = aE[X] + bE[Y] \quad \#$$

2、 $E[X\cdot Y] = \sum_x \sum_y (x\cdot y)\, P[X=x, Y=y]$

(independence) $= \sum_x \sum_y (x\cdot y)\, P[X=x]\, P[Y=y]$

$$= \sum_x x\, P[X=x] \sum_y y\, P[Y=y] = E[X]\, E[Y] \quad \#$$

$\mathrm{Var}[X+Y] = E[(X+Y - E[X+Y])^2] = E[(X - E[X] + Y - E[Y])^2]$ (linearity of expectation)

$$= E[(X-E[X])^2] + E[(Y-E[Y])^2] + \cancel{2E[(X-E[X])(Y-E[Y])]} = 0$$

$$2E[(X-E[X])(Y-E[Y])] = 2E[XY - XE[Y] - YE[X] + E[X]E[Y]]$$
$$= 2E[XY] - 2E[X]E[Y] - \cancel{2E[Y]E[X]} + \cancel{2E[X]E[Y]}$$

$(E[X\cdot Y] = E[X]\, E[Y])$ $= 2(E[X]E[Y] - E[X][Y]) = 0$

$\Rightarrow \mathrm{Var}[X+Y] = E[(X-E[X])^2] + E[(Y-E[Y])^2] = \mathrm{Var}[X] + \mathrm{Var}[Y] \quad \#$

3、 $\mathrm{Var}[X] = E[(X-E[X])^2] = E[X^2 - 2XE[X] + E[X]^2] = E[X^2] - 2E[X]E[X] + E[X]^2$
$$= E[X^2] - (E[X])^2 \quad \#$$

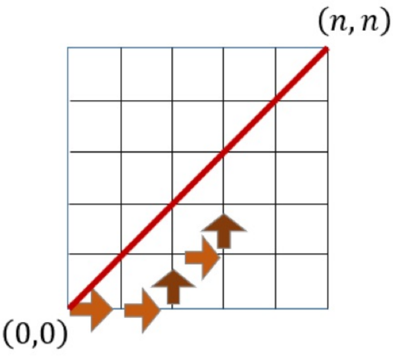
Problem 2 (20%). Consider the slides #2. Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Let $n = 2^m$ and consider the graph K_n. H_i partitions K_n into X and Y,
where $X = \{\vec{u} = (u_1, u_2, \cdots u_m) \mid u \in V(K_n),\ u_i = 0\}$, $Y = \{\vec{u} = (u_1, u_2, \cdots u_m) \mid u \in V(K_n),\ u_i = 1\}$

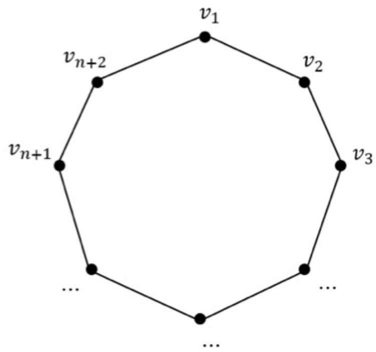
By definition of H_i, $(u,v) \in E(H_i)$ $\forall\, u, v \in V(K_n) \wedge u_i \neq v_i$;
therefore, $\forall\, u \in X, v \in Y$, we have $(u,v) \in E(H_i)$
and $\forall\, u, v \in X$, $(u,v) \notin E(H_i)$ and $\forall\, u, v \in Y$, $(u,v) \notin E(H_i)$;

hence prove that H_i must be a biclique.

Problem 3 (20%). For any integer $n \geq 1$, consider the grid points (r, c) with $1 \leq r, c \leq n$. Let C_n be the number of possible paths from $(0,0)$ to (n,n) that use only $\rightarrow$ and $\uparrow$ and that never cross the diagonal $r = c$. See also the Figure (a) below. For convenience, define $C_0 := 1$.



(a)



(b)

For any integer $n \geq 2$, consider the convex $(n+2)$-gon with vertices labeled with $v_1, v_2, \ldots, v_{n+2}$. Let P_n denote the number of possible ways to triangulate the polygon. It follows that $P_2 = 2$, $P_3 = 5$, etc. For convenience, also define $P_0 := 1$ and $P_1 := 1$.

1. Prove that for any $n \geq 2$, P_n satisfies the recurrence

$$P_n \;=\; \sum_{0 \leq k < n} P_k \cdot P_{n-k-1}.$$

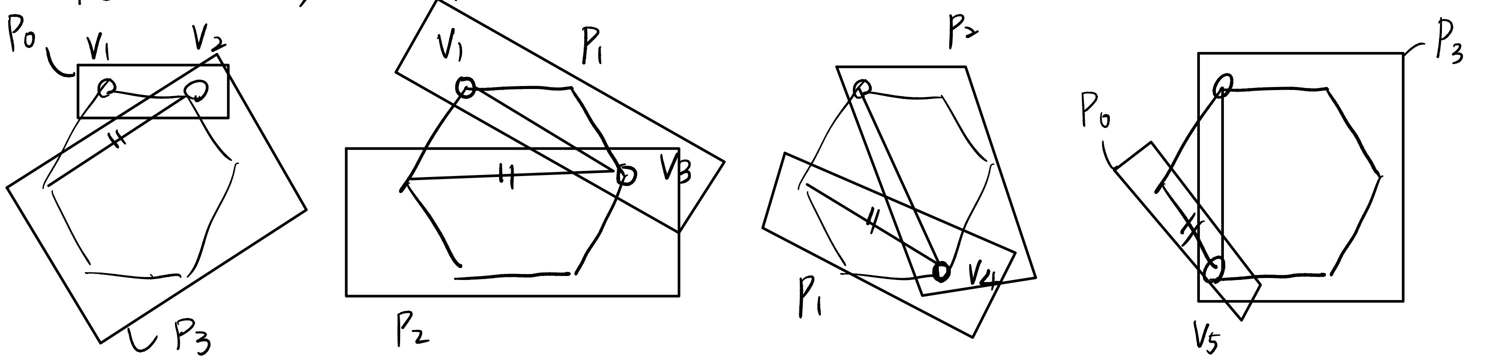
Consider polygon with vertex $V_1, V_2, \ldots, V_{n+2}$, we can split it by adding a diagonal to it. If we connect V_1 to V_k to form a diagonal, then there are $V_2 \sim V_{n+1}$, which means n possible points to choose from.

By forming the diagonal, we try to find triangulation for $(V_1 \sim V_k)$-polygon and $(V_k - V_{n+2})$-polygon; we have $(k-1+1)$ and $(n+2-k+1)$ points respectively, so the number of ways to triangulate the two polygon is $P_{k-2} \times P_{n-k+1}$. By adding all possible values of k, we get $\sum_{2 \leq k < n+2} P_{k-2} \times P_{n-k+1}$, this is the number of ways to triangulate $(n+2)$-polygon.

$$\sum_{2 \leq k < n+2} P_{k-2} \times P_{n-k+1} \underset{k'=k-2}{=} \sum_{0 \leq k' < n} P_{k'} \times P_{n-k'-1} = \sum_{0 \leq k < n} P_k \times P_{n-k-1} \quad \#$$

Example: $n=4 \Rightarrow P_4 = P_0 P_3 + P_1 P_2 + P_2 P_1 + P_3 P_0$

$P_0 P_3$ $(k=2)$ $\quad$ $P_1 P_2$ $(k=3)$ $\quad$ $P_2 P_1$ $(k=4)$ $\quad$ $P_3 P_0$ $(k=5)$

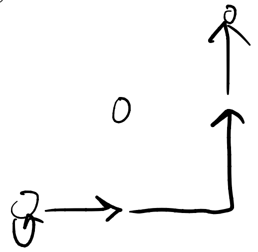


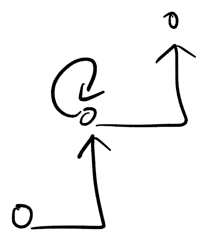
2. Prove that for any $n \geq 2$, C_n satisfies the same recurrence

$$C_n = \sum_{0 \leq k < n} C_k \cdot C_{n-k-1}.$$

$C_0 = C_1 = 1$

$C_2 = C_0 C_1 + C_1 C_0$





$k = 0$ $\qquad$ $k = 1$

$C_0 C_{n-1-k}$ $\qquad$ $C_1 C_0$

Let k be the smallest number that a path crosses (k, k).

For any C_n, we can count all possible paths by considering k, $\forall k \in [0, n)$

The paths can be broken up into two parts:

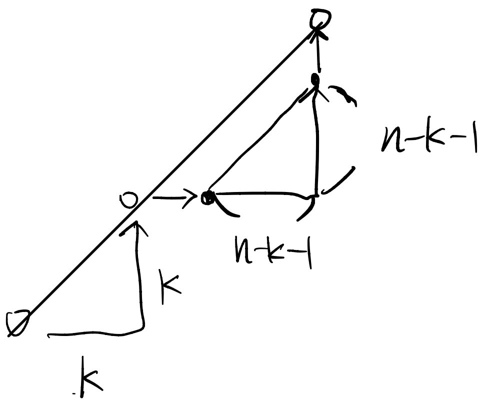
(1) $(0,0) \to (k,k)$

(2) $(k,k) \to (k+1, k) \to (n, n-1) \to (n, n)$

The second part is just the same as "$\to$" $\times$ C_{n-k-1} $\times$ "$\uparrow$"

And there are $C_k \cdot C_{n-k-1}$ for each possible k.

$$\Rightarrow C_n = \sum_{0 \leq k < n} C_k \cdot C_{n-k-1} \quad \#$$



Problem 4 (20%). Let $\mathcal{F}$ be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \qquad \text{and} \qquad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F},\ A \neq B.$$

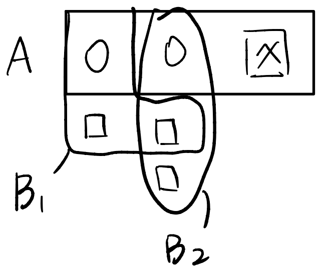
Suppose that $\mathcal{F}$ is not 2-colorable. Let x, y be any elements that appear in $\mathcal{F}$, i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:

1. x belongs to at least two members of $\mathcal{F}$.

2. There exists some $C \in \mathcal{F}$ such that $\{x, y\} \subseteq C$.

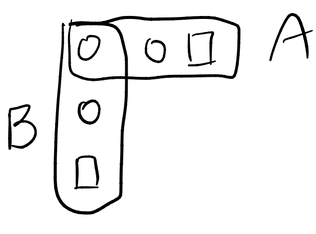
Hint: Construct proper coloring to prove the properties. For (1), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (1). For (2), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (2).

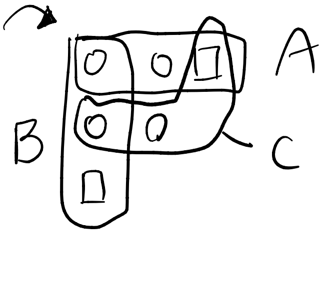
1. Color $A \setminus \{x\}$ red and $\{x\}$ blue.
Since F is not 2-colorable, there must exist some $B \in F$ and $|B \cap A| = 1$, otherwise A itself is 2-colorable and hence a contradiction
If no other $B \in F$ s.t. $A \cap B = \{x\}$, then we can always color $\{x\}$ the remaining color in A, which makes F 2-colorable.
Therefore, there must be another B s.t. $|A \cap B| = \{x\}$, then x must belong to at least 2 members.



(O: red, □: blue)

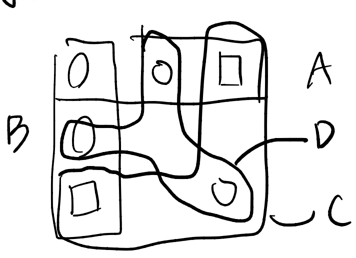
2. Same reason as (1). We color $A \cup B \setminus \{x, y\}$ red, and we can make $\{x, y\}$ blue so that F is still 2 colorable.
If we let $x \in A$ to be in C, then we can choose the other red point $\notin A \cap B$ to be in C and make F 2-colorable.





($\to\leftarrow$)

So there must $\exists\ C \in F$ s.t. $\{x, y\} \in C$.



* D has to be there because of (1).

Problem 5 (20%). Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B. Assume that $|A|d \geq |B|D$.

Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,
2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0, and
3. at least half of the edges leaving A_0 go to B_0.

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0. First prove (3) and then (1).

$\forall a \in A$, $\deg(a) \geq d$; $\forall b \in B$, $\deg(b) \leq D$

$\Rightarrow |A|d \leq |B|D$, and by assumption that $|A|d \geq |B|D$

$\Rightarrow |A|d = |B|D$, and hence it's the actual number of edges.

because $|A|d$ is the minimum number and $|B|D$ is the maximum.

$\Rightarrow \deg(a) = d$; $\deg(b) = D$.

Let B_0 consist of all vertices in B that have at least $\frac{\alpha D}{2}$ neighbors in A_0, and x be the number of edges between A_0 and $(B - B_0)$

The number of edges from A_0 to B is $|A_0|d = |A|\alpha d$.

In (3), we can prove $x \leq \frac{|A|\alpha d}{2}$

$\Rightarrow x \leq (|B| - |B_0|) \times \frac{\alpha D}{2}$ (vertex in $\overline{B_0}$ can have at most $\frac{\alpha D}{2}$)

$$= \frac{\alpha|B|D}{2} - \frac{\alpha|B_0|D}{2} = \underline{\frac{\alpha|A|d}{2} - \frac{\alpha|B_0|D}{2}} < \frac{|A|\alpha d}{2}$$

$\Rightarrow x \leq \frac{|A|\alpha d}{2}$, hence proved (3) #

By (3), we know the number of edges between B_0 and A_0 is at least $\frac{|A|\alpha d}{2}$

$\Rightarrow \frac{|A|\alpha d}{2} = \frac{|B|\alpha D}{2} \leq |B_0|D$, because B_0 can have neighbors in $(A - A_0)$

$\Rightarrow |B_0| \geq \frac{\alpha|B|}{2}$