Chap 7. Planarity And Kuratowski's Theorem

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The sources of most figure images are from the course slides (Graph Theory) of Prof. Gross

Outline

- Planar Drawings and Some Basic Surfaces
- Subdivision and Homeomorphism
- Extending Planar Drawings
- Kuratowski's Theorem
- Algebraic Tests for Planarity

7.1 Planar Drawings and Some Basic Surfaces

Planar Drawings

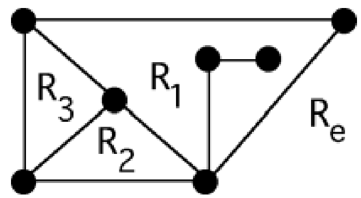
- **DEFINITION**: A ***planar drawing*** of a graph is a drawing of the graph in the plane without edge-crossings.
- **DEFINITION**: A graph is said to be ***planar*** if there exists a planar drawing of it.
- **Example 7.1.1:**
  - Minimum crossing drawing

Nonplanar drawing and planar drawing of K_4.

- **Example 7.1.2:** ***Utilities Problem***. Three houses connect to three utilities, such as electricity, gas, and water, without any crossings of the utility lines. (ex. Three houses to three lines)
- **Remark:** A graph G and a given drawing of G are categorically different objects. That is, a graph is combinatorial and a drawing is topological. In particular, the *vertices* and *edges* in a drawing of a graph are actually *images* of the vertices and edges in that graph.
- **TERMINOLOGY**: Intuitively, we see that in a planar drawing of a graph, there is exactly one ***exterior*** (or ***infinite***) region whose area is infinite.

Planar Drawings

- **Example 7.1.3:**



The four regions of a planar drawing of a graph.

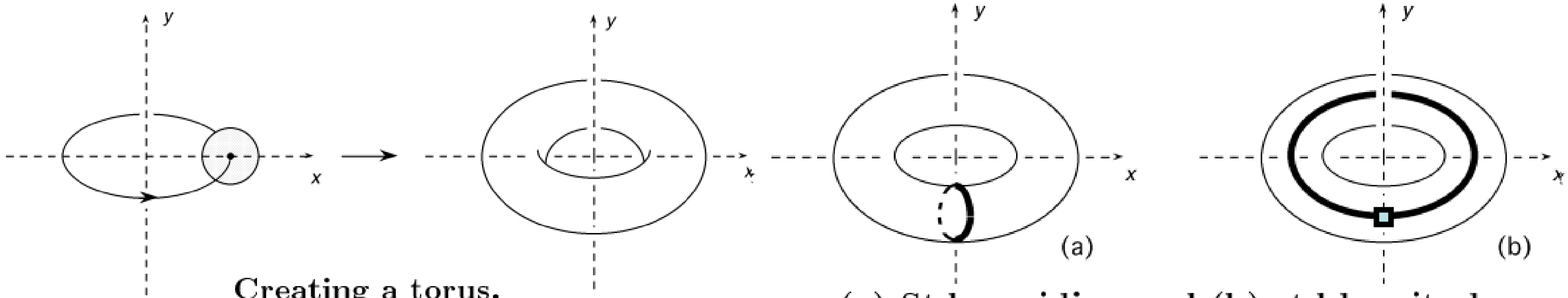
- **Remark:** If we consider a planar drawing of a graph on a piece of paper, then the intuitive notion of *region* corresponds to the pieces of paper that result from cutting the paper along the length of every edge in the drawing. If one adds new edges to a graph without crossing an existing edge, then a region may be subdivided into more regions.
- **TERMINOLOGY NOTE**: We restrict the usage of the word "regions" to the case of crossing-free drawings, since many assertions that are true in that case may be untrue when there are edge-crossings.

Three Basic Surfaces

- All of the surfaces under consideration in this chapter are subsets of Euclidean 3-space.
- **DEFINITION**: A ***plane*** in Euclidean 3-space $\mathbb{R}^3$ is a set of points (x, y, z) such that there are numbers a, b, c, and d with $ax + by + cz = d$.

Three Basic Surfaces

- **DEFINITION**: A ***sphere*** is a set of points in $\mathbb{R}^3$ equidistant from a fixed point.
- **DEFINITION**: The ***standard torus*** is the surface of revolution obtained by revolving a circle of radius 1 centered at (2,0) in the *xy*-plane disk around the *y*-axis in 3 space, as depicted in Figure 7.1.3. The solid inside is called the ***standard donut***.



Creating a torus.

(a) Std meridian and (b) std longitude.

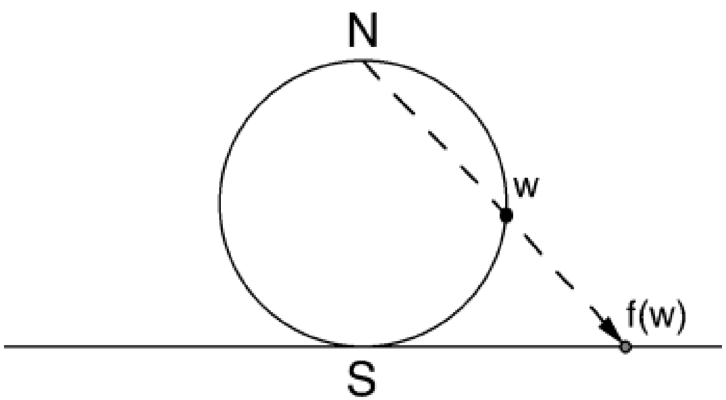
- **DEFINITION**: The circle of intersection of the standard torus with the half-plane $\{(x, y, z,) \mid x = 0, z \leq 0\}$ is called the ***standard meridian* (longitude)**. We observe that the standard meridian bounds a disk inside the standard donut.
- **DEFINITION**: The circle of tangent intersection of the standard torus with the plane $y = 1$ is called the ***standard latitude***. Figure 7.1.4(b) illustrates the standard longitude. We observe that the standard longitude bounds a disk in the plane $y = 1$ that lies outside the standard donut.

Three Basic Surfaces

- **TERMINOLOGY**: Any closed curve that circles the torus once in the meridian direction (the "short" direction) without circling in the latitude direction (the "long" direction) is called a ***meridian***. Any closed curve that circles the torus once in the latitude direction without circling in the meridian direction is called a ***latitude***.
- **Remark**: The three surfaces described above are all relatively uncomplicated. In §8.2, the *Möbius band* and the *Klein bottle* are defined, and it is described how the surfaces generally fall into two infinite sequences.

Riemann Stereographic Projection

- **DEFINITION**: The ***Riemann stereographic projection*** is the function ρ that maps each point w of the unit-diameter sphere (tangent at the region (0,0,0) to the xz-plane in Euclidean 3-space) to the point $\rho(x)$ where the ray from the north pole (0,1,0) through point w intersects the xz-plane.



The Riemann stereographic projection.

Riemann Stereographic Projection

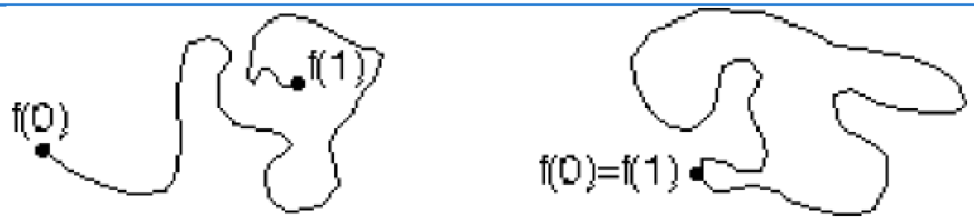
- **Remark**: Whereas a planar drawing of a graph has one exterior region (containing the "point as infinity"), a crossing-free graph drawn on the sphere, has one region that contains the north pole. Given a graph drawn on the sphere, the Riemann stereographic projection enables us to move the drawing to the plane so that the point at infinity is deleted from whatever region we choose. That is, we simply rotate the sphere so that a point in the designated region is at the north pole.
- **Proposition 7.1.1.** *A graph is planar if and only if it can be drawn without edge-crossing on the sphere.*
  - ✓ This is an immediate consequence of the Riemann stereographic projection.

<u>Jordan Separation Property</u>

- **DEFINITION**: By a ***Euclidean set***, we mean a subset of any Euclidean space $\mathbb{R}^n$.
- **DEFINITION**: An ***open path from*** *s* ***to*** *t* in a Euclidean set *X* is the image of a continuous bijection *f* from the unit interval [0,1] to a subset of *X* such that $f(0) = s$ and $f(1) = t$. (One may visualize a path as the trace of a particle traveling through space for a fixed length of time.)
- **DEFINITION**: An ***closed path*** *or* ***closed curve*** in a Euclidean set is the image of a continuous function *f* from the unit interval [0,1] to a subset of that space such that $f(0) = f(1)$, but which is otherwise a bijection. (For instance, this would include a "knotted circle" in space.)

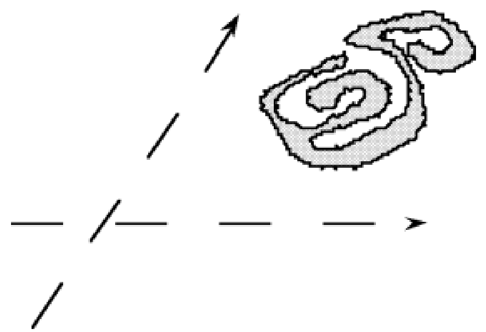
Jordan Separation Property

- **Example 7.1.4:**
- **Example 7.1.5:**



Open path and closed curve (= closed path).

- **DEFINITION**: A Euclidean set X is ***connected*** if for every pair of points $s, t \in X$, there exists a path within X from s to t.
- **DEFINITION**: A Euclidean set X ***separates*** the connected Euclidean set Y if there exists a pair of points s and t in $Y - X$, such that every path in Y from s to t intersects the set X.
- **Example 7.1.6:**

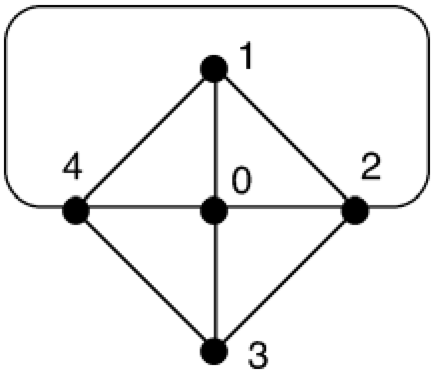


A closed curve separating the plane.

- Jordan separation property makes the plane and the sphere the simplest surfaces for drawing graphs.
- **DEFINITION**: A Euclidean set X has the ***Jordan Separation Property*** if every closed curve in X separates X.

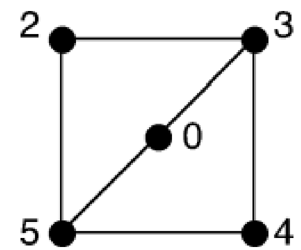
Applying The Jordan Curve Theorem to the Nonplanarity of K_5 and $K_{3,3}$

- **Theorem 7.1.2 (Jordan curve Theorem).** *Every closed curve in the sphere (plane) has the Jordan separation property, that is, it separates the sphere (plane) into two regions, one of which contains the north pole (contains "infinity").*
- **Corollary 7.1.3.** *A path from one point on the boundary of a disk through the interior to another point on the boundary separates the disk.*
- Neither a meridian nor a parallel separates the surface of a torus into two parts.
- **Theorem 7.1.4.** *Every drawing of the complete graph K_5 in the sphere (or plane) contains at least one edge-crossing.*
  - ✓ Label the vertices 0, …, 4. By the Jordan Curve Theorem, any drawing of the cycle <1, 2, 3, 4, 1> separates the sphere into two regions.
  - ✓ Consider the region with vertex 0 in its interior as the "inside" of the cycle.
  - ✓ By the Jordan Curve Theorem, the edges joining vertex 0 to each of the vertex 1, 2, 3, and 4 must also lie entirely inside the cycle,
  - ✓ Edge 24 must lie to the exterior of the cycle <1, 2, 3, 4, 1>.
  - ✓ Cycle formed by edges 24, 40, and 02 separates vertices 1 and 3, which implies it's impossible to draw edge 13 without crossing.



Applying The Jordan Curve Theorem to the Nonplanarity of K_5 and $K_{3,3}$

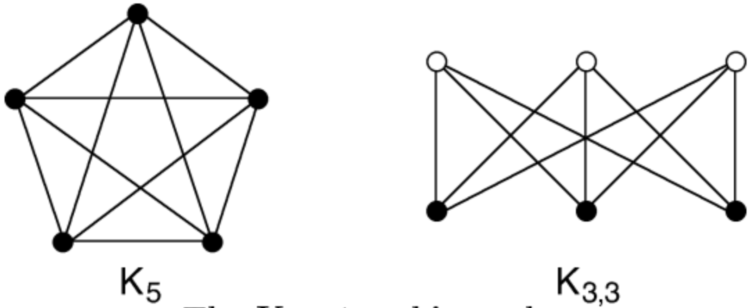
- **Theorem 7.1.5.** *Every drawing of the complete bipartite graph $K_{3,3}$ in the sphere (or plane) contains at least one edge-crossing.*
  - ✓ Label the vertices of one partite set 0, 2, 4, and of the other 1, 3, 5.
  - ✓ By the Jordan Curve Theorem, cycle <2, 3, 4, 5, 2> separates the sphere into two regions, and we regard the region containing vertex 0 as the "inside" of the cycle.
  - ✓ By the Jordan Curve Theorem, the edges joining vertex 0 to each of the vertex 3 and 5 lie entirely inside that cycle, and each of the cycles <0, 3, 2, 5, 0> and <0, 3, 4, 5, 0> separates the sphere.
  - ✓ There are three regions, and at least one crossing is needed no matter which region contains vertex 1 since each region contains only two vertices to be connected on its boundary.



Drawing most of $K_{3,3}$ in the sphere.

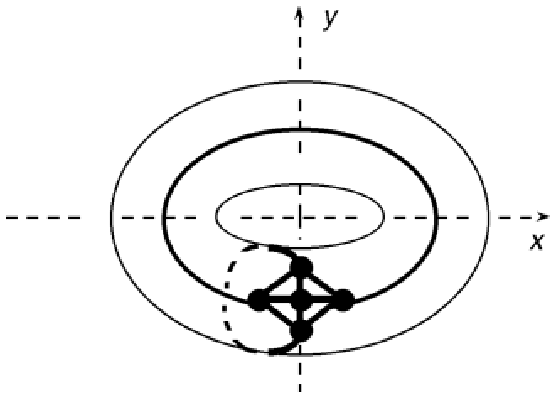
- **Corollary 7.1.6.** *If either K_5 or $K_{3,3}$ is a subgraph of a graph G then every drawing of G in the sphere (or plane) contains at least one edge-crossing.*
- **DEFINITION**: The complete graph K_5 and the complete bipartite graph $K_{3,3}$ are called the ***Kuratowski graphs***.

Applying The Jordan Curve Theorem to the Nonplanarity of K_5 and $K_{3,3}$



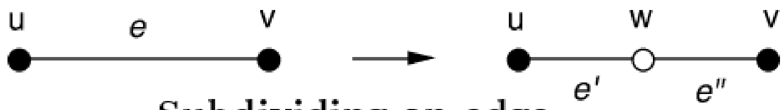
The Kuratowski graphs.

- **Example 7.1.7:**

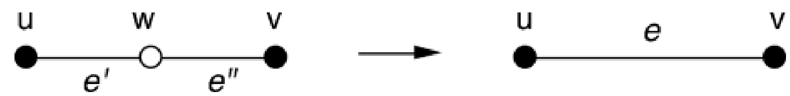


A crossing-free drawing of K_5 on the torus.

7.2 Subdivision and Homeomorphism



Subdividing an edge.

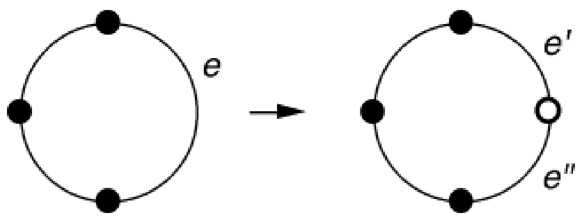


Smoothing away a vertex.

- **DEFINITION**: Let e be an edge with endpoints $\{u, v\}$ in a graph G. ***Subdividing the edge*** e means that a new vertex w is added to V_G, and that edge e is replaced in E_G by an edge e' with endpoints $\{u, w\}$ and an edge e'' with endpoints $\{w, v\}$.

- **DEFINITION**: Let w be a 2-valent vertex in a graph G, such that two proper edge e' and e'' meet at w. ***Smoothing away*** or (***smoothing out***) vertex w means replacing edge e' and e'' by a new edge e that joins the other endpoints of e' and e''.

- **Example 7.2.1:**



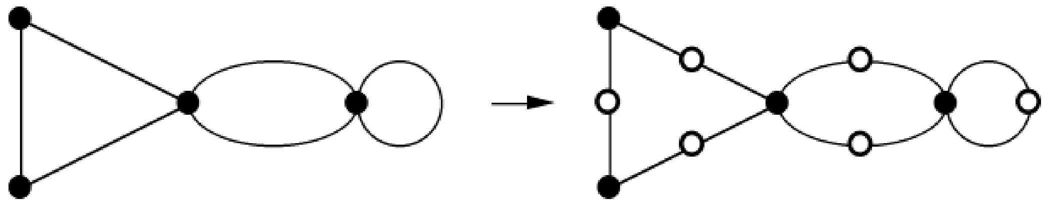
Subdividing an edge of C_3 yields C_4.

- **DEFINITION**: ***Subdividing a graph*** G means performing a sequence of edge-subdivision operations. The resulting graph is called a ***subdivision of the graph*** G.

- **Proposition 7.2.1.** A subdivision of a graph can be drawn without edge-crossings on a surface if and only if the graph itself can be drawn without edge-crossings on that surface.
  - ✓ When the operations of subdivision and smoothing are performed on a copy of the graph already drawn on the surface, they neither introduce nor remove edge-crossings.

Barycentric Subdivision

- **DEFINITION**: The (***first***) ***barycentric subdivision*** of a graph is the subdivision in which one new vertex is inserted in the interior of each edge.
- **Example 7.2.3:**

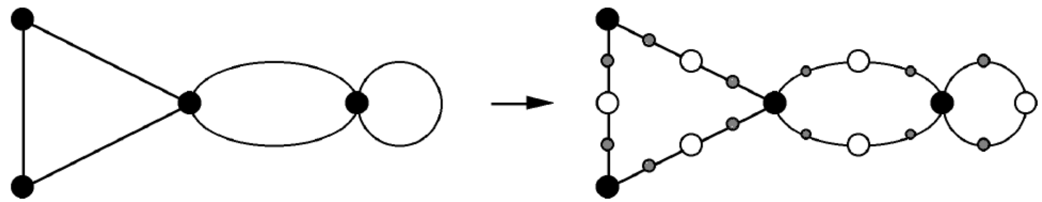


A graph and its barycentric subdivision.

- **Proposition 7.2.2.** The barycentric subdivision of any graph is a bipartite graph.
  - ✓ Let G' denote the barycentric division of graph G. One endpoint of each edge in G' is an "old" vertex (i.e., from V_G), and the other endpoint is "new" (i.e., from subdividing).
- **Proposition 7.2.3.** Barycentric subdivision of any graph yields a loopless graph.
  - ✓ By Proposition 7.2.2, a barycentric subdivision of a graph is bipartite, and a bipartite graph has no self-loops.
- **Proposition 7.2.4.** Barycentric subdivision of any loopless graph yields a simple graph.
  - ✓ Clearly, barycentric subdivision of a loopless graph cannot create loops.
  - ✓ Each "new" vertex connects to two distinct old vertices since there is no self loop and each new vertex must connects to two endpoints of original old edge.

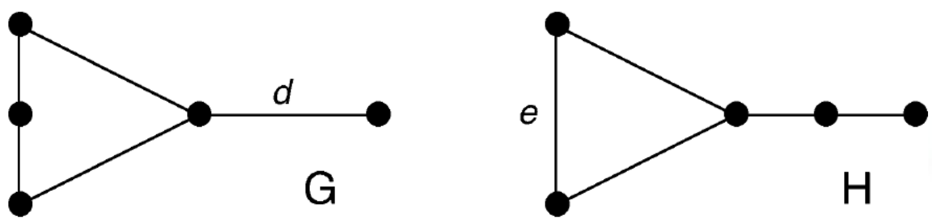
Barycentric Subdivision

- **DEFINITION**: The n^{th} ***barycentric subdivision*** of a graph is the first barycentric subdivision of the $(n-1)^{st}$ barycentric subdivision .
- **Proposition 7.2.5.** The second barycentric subdivision of any graph is a simple graph.
  - ✓ By Proposition 7.2.3, the first barycentric subdivision is loopless,
  - ✓ And thus, by Proposition 7.2.4, the second barycentric subdivision is simple.



A graph and its 2nd barycentric subdivision.

- **DEFINITION**: The graph *G* and *H* are ***homeomorphic graphs*** if there is an isomorphism from a subdivision of G to a subdivision of *H*.
- **Example 7.2.4:**



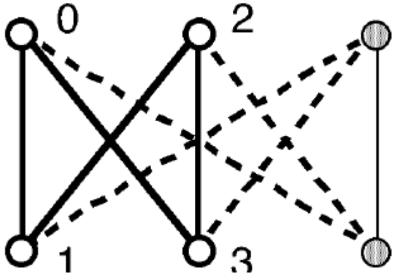
Two non-isom graphs that are homeomc.

Graph Homeomorphism

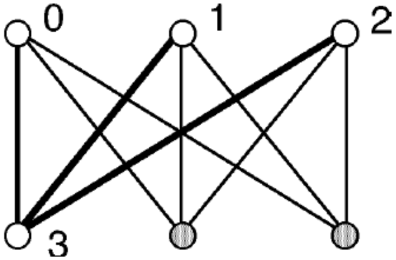
- **Remark**: Notice in Example 7.2.4 that no subdivision of graph G is isomorphic to graph H, and that no subdivision of graph H is isomorphic to graph G.
  - ✓ Isomorphism implies homermorphism.
- **Proposition 7.2.6.** Let G and H be homeomorphic graphs. Then G can be drawn without edge-crossings on a surface S if and only if H can be drawn on S without edge-crossings.
  - ✓ This follows from iterated application of Proposition 7.2.1.
- **Proposition 7.2.7.** Every graph is homeomorphic to a bipartite graph.
  - ✓ By Proposition 7.2.2, the barycentric subdivision of a graph is bipartite. Of course, a graph is homeomorphic to a subdivision of itself.

Subgraph Homeomorphism Problem

- **Example 7.2.5:**

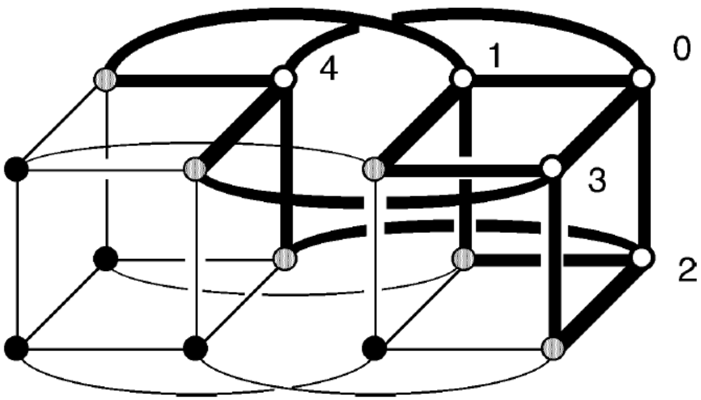


A homeomorphic copy of K_4 in $K_{3,3}$.



An impossible way to place K_4 into $K_{3,3}$.

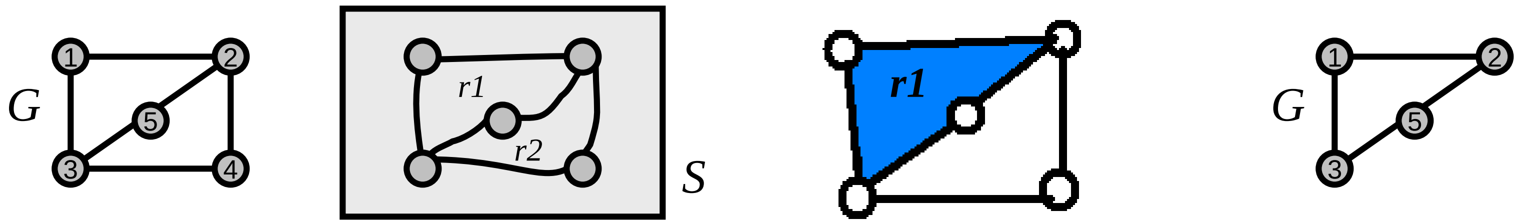
- **Example 7.2.6:**



A homeomorphic copy of K_5 in Q_4.

7.3 Extending Planar Drawings

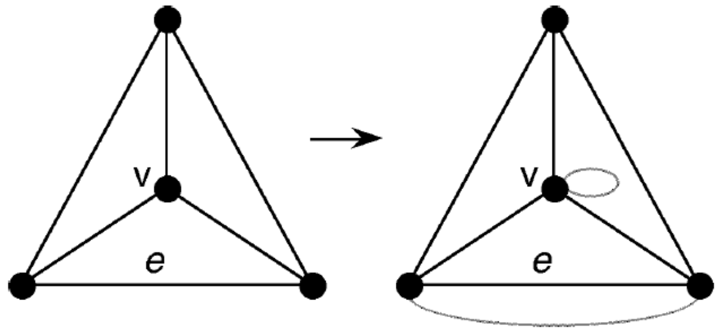
- **DEFINITION**: An ***imbedding of a graph*** G on a surface S is a drawing without any edge-crossings. We may denote the imbedding $\iota : G \to S$.
- **DEFINITION**: A ***region*** of a graph imbedding $G \to S$ is a component of the Euclidean set that results from deleting the image of G from the surface S.
- **DEFINITION**: The ***boundary of a region*** r of a graph imbedding $\iota : G \to S$ is the subgraph of G that comprises all vertices that abut the region and all edges whose interiors abut the region.



- **Remark:** When a 2-connected graph is imbedding on the sphere, the boundary of every region is a cycle on the perimeter of the region. However, it is a dangerous misconception to imagine that this is the general case.
- **DEFINITION**: A ***face*** of a graph imbedding $\iota : G \to S$ is the union of a region and its boundary .
- **Remark:** A drawing does *not* have faces unless it is an imbedding.

Planar Extensions of a Planar Subgraph

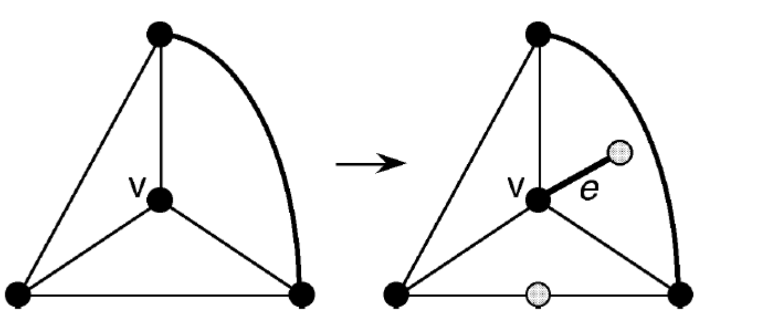
- A standard way to construct a planar drawing of a graph is to draw a subgraph in the plane, and then to extend the drawing by adding the remaining parts of the graph.
- **Proposition 7.3.1.** *A planar graph G remains planar if a multiple edge or self-loop is added to it.*
  - ✓ Multiple edges can be added by drawing them very close to original edges.
  - ✓ Self-loops can be added by drawing them with very small regions.



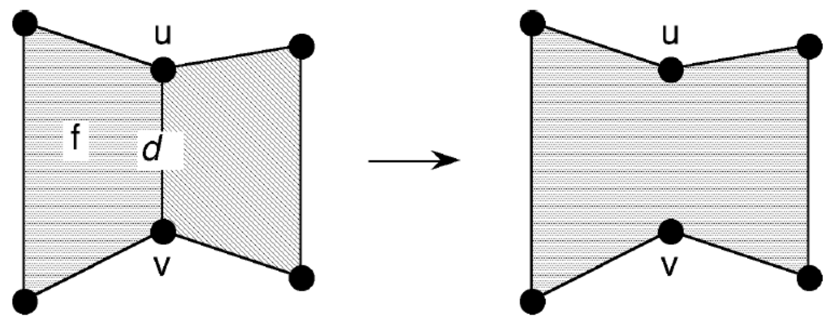
+ self-loop or multi-edge preserves planarity.

- **Corollary 7.3.2.** *A nonplanar graph remains nonplanar if a self-loop is deleted or if one edge of a multi-edge is deleted.*
- **Proposition 7.3.3.** *A planar graph G remains planar if any edge is subdivided or a new edge e is attached to a vertex $v \in V_G$ (with the other endpoint of e added as a new vertex).*
  - ✓ Clearly, placing a dot in the middle of an edge in a planar drawing of *G* to indicate subdivision does not create edge-crossings. Moreover, it is easy enough to insert edge *e* into any region that is incident on vertex *v*.

Planar Extensions of a Planar Subgraph

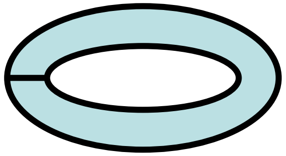


Subdividing an edge; adding a spike.



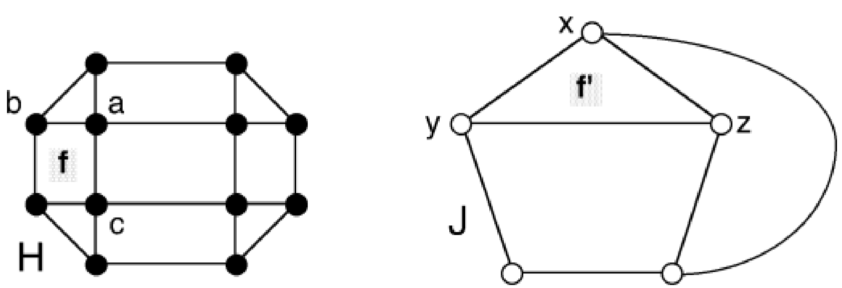
Merging two regions by deleting an edge.

- **Proposition 7.3.4.** *Let $\iota : G \to S$ be a graph imbedding on a sphere or on any other surface. Let d be an edge of G with endpoints u and v. Then the imbedding of the graph G – d obtained by deleting edge d from the imbedding $\iota : G \to S$ has a face whose boundary contains both of the vertices u and v.*
  - ✓ In the *imbedding $\iota : G \to S$*, let *f* be a face whose boundary contains edge *d*. When edge *d* is deleted, face *f* is merged with whatever face lies on the other side of edge *d*.
  - ✓ On surface *S*, the boundary of the merged face is the union of the boundaries of the faces containing edge *d*, minus the interior of edge *d*. (This is true even when the same face *f* lies on both sides of edge *d*.) Vertices *u* and *v* both lie on the boundary of that merged face, exactly as illustrated.

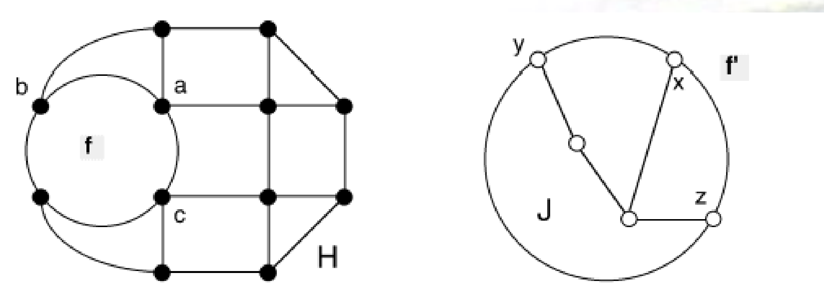


Amalgamating Planar Graphs

- **Proposition 7.3.5.** *Let f be a face of a planar drawing of a connected graph G. Then there is a planar drawing of G in which the boundary walk of face f bounds a disk in the plane that contains the entire graph G. That is, in the new drawing, face f is the ”outer” face.*
  - ✓ Copy the planar drawing of *G* onto the sphere so that the north pole lies in the interior of face *f*. Then apply the Riemann stereographic projection.
- **Proposition 7.3.6.** *Let f be a face of a planar drawing of a graph H, and let* $u_1,\ldots,u_n$ *be a* <u>*subsequence of vertices in the boundary walk of f*</u>*. Let f’ be a face of a planar drawing of a graph J, and let* $w_1,\ldots,w_n$ *be* <u>*a subsequence of vertices in the boundary walk of f’*</u>*. Then the amalgamated graph* $(H \cup J)/\{u_1=w_1,\ldots, u_n=w_n\}$ is planar.
  - ✓ Planar drawings of graph *H* and *J* with *n*=3 are illustrated in the following. First redraw the plane imbedding of *H* so that the unit disk lies wholly inside face *f*. Next redraw graph *J* so that the boundary walk of face *f’* surrounds the rest of graph *J*, which is possible according to Proposition 7.3.5.



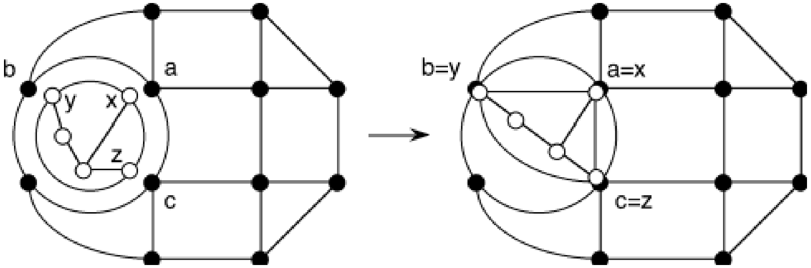
Two planar graph drawings.



Redrawings of the planar imbeddings of *H* & *J*.

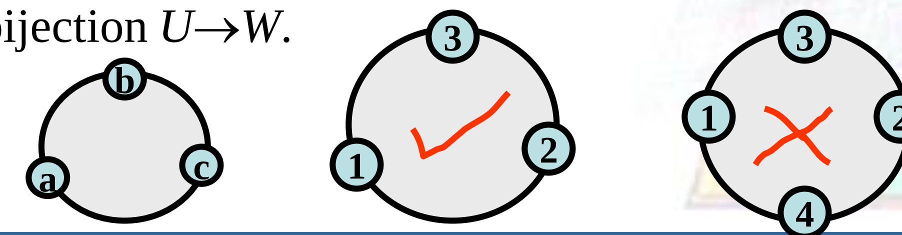
Amalgamating Planar Graphs

- ✓ We may assume that the cyclic orderings of the vertex sequences $\{u_1,\ldots,u_n\}$ and $\{w_1,\ldots,w_n\}$ are consistent with each other, since the drawing of graph J can be reflected, if necessary, to obtain cyclic consistency.
- ✓ Now shrink the drawing of graph J so that it fits inside the unit disk in face f. Then stretch the small copy of J outward, thereby obtaining a crossing-free drawing of the amalgamation $(H \cup J)/\{a=x, b=y, c=z\}$.



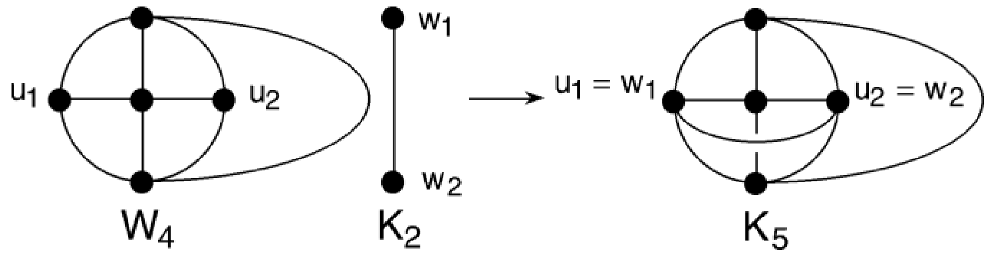
Position H & J and then amalg by stretching.

- **Corollary 7.3.7.** *Let H and J be planar graphs. Let U be a set of one, two, or three vertices* <u>*in the boundary of a face f*</u> *of the drawing of H, and let W be a set of the same number of vertices in the boundary of a face f' of the drawing of J. Then the amalgamated graph* $(H \cup J)/\{U=W\}$ is planar. **(There is no explicit vertex sequence here)**
  - ✓ Whenever there are at most three vertices in the vertex subsequences to be amalgamated, there are only two possible cyclic orderings. Since reflection of either of the drawings is possible, the vertices of sets U and W in the boundaries of face f and f', respectively, can be aligned to correspond to any bijection $U \rightarrow W$.



Amalgamating Planar Graphs

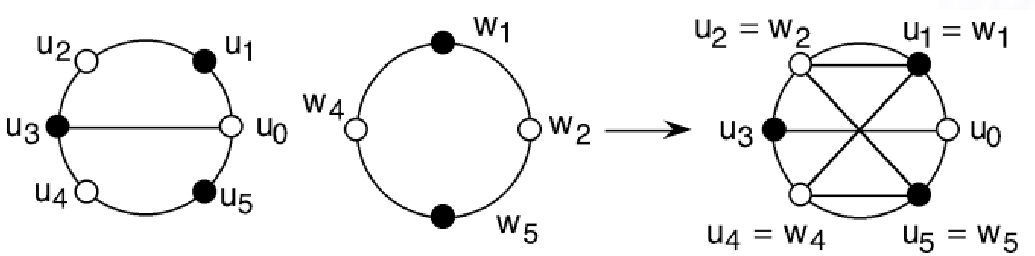
- **Remark:** The requirement that vertices of amalgamation be selected in their respective graphs from the boundary of a single face cannot be relaxed. Amalgamating two planar graphs across two arbitrarily selected vertices may yield a nonplanar graph.
  - ✓ For instance, the nonplanar K_5 can be derived as the amalgamation of the planar graphs W_4 and K_2. This does not contradict Corollary 7.3.7, because the two vertices of amalgamation in W_4 do not lie on the same face of any planar drawing of W_4.



A 2-vertex amalg of two planar graphs into K_5.

- **Remark:** Amalgamating two planar graphs across sets of four or more vertices per face may yield a nonplanar graph. Four vertices are not selected in a subsequence of a walk in a face.

If three vertices are selected in this case, still planar?



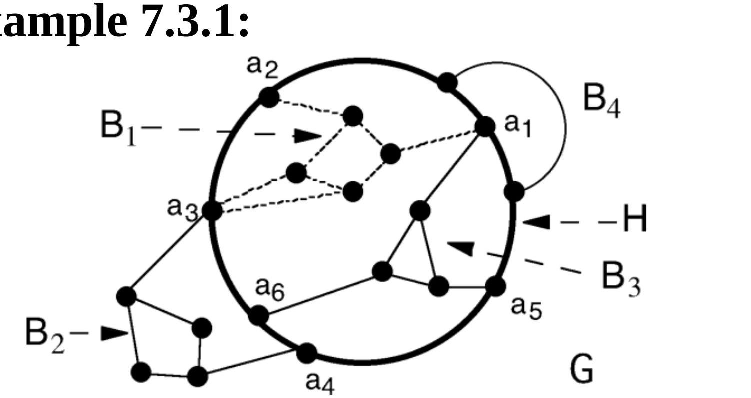
A 4-vertex amalg of two planar graphs into $K_{3,3}$.

Appendages to a Subgraph

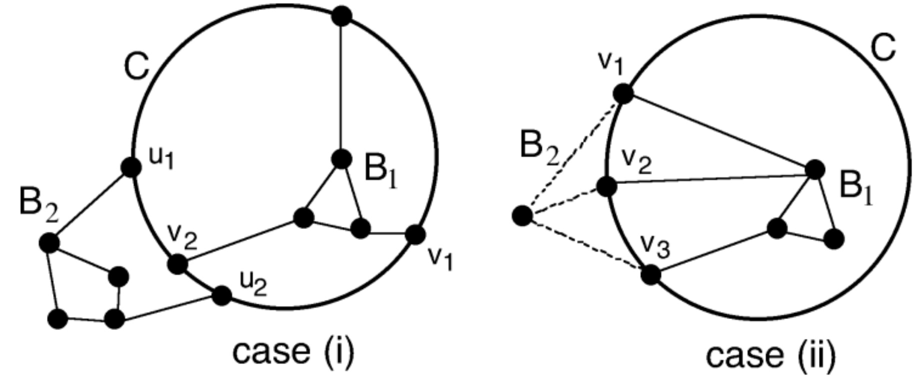
- The property that a subgraph H of a graph G separates two edge explains that it is impossible to get from one edge to the other without going through H. This is helpful for constructing a planar graph starting from a selected large cycle.
- **DEFINITION**: Let H be a subgraph of a connected graph G. Two edges e_1 and e_2 of E_G-E_H are ***unseparated by subgraph*** H if there exists a walk in G that contains both e_1 and e_2, but whose internal vertices are not in H.
- **Remark:** The relation *unseparated by subgraph H* is an equivalence relation on E_G-E_H; that is, it is reflexive, symmetric, and transitive.
- **DEFINITION**: Let H be a subgraph of a graph G. Then an ***appendage to subgraph*** H is the induced subgraph on an equivalence class of edges of E_G-E_H under the relation *unseparated by H*.
- **DEFINITION**: Let H be a subgraph of a graph. An appendage to H is called a ***chord*** if it contains only one edge. Thus, a chord joins two vertices of H, but does not lie in the subgraph H itself.
- **DEFINITION**: Let H be a subgraph of a graph, and let B be an appendage to H. Then a ***contact point*** of B is a vertex of $B \cap H$.

Appendages to a Subgraph & Overlapping Appendages

- **Example 7.3.1:**



Subgraph H and appendages B_1, B_2, B_3, and B_4.



The two ways that appendages can overlap.

- Also notice that each non-chord appendage contains a single component of the deletion subgaph $G - V_H$. In addition to a component of $G - V_H$, a non-chord appendage also contains every edge extending from that component to a contact point, and the contact point as well.

- **Remark:** Every subgraph of a graph (not just cycles) has appendages. Even a subgraph comprising a set of vertices and no edges would have appendages.

- **DEFINITION**: Let C be a cycle in a graph. The appendages B_1 and B_2 of C ***overlap*** if either of these conditions holds:
  i. Two contact points of B_1 alternate with two contact points of B_2 on cycle C.
  ii. B_1 and B_2 have three contact points in common.

- **Example 7.3.2**:

Overlapping Appendages

- **Proposition 7.3.8.** *Let C be a cycle in a planar drawing of a graph, and let B_1 and B_2 be overlapping appendages of C. Then one appendage lies inside cycle C and the other outside.*
  - ✓ Overlapping appendages on the same side of cycle *C* would cross, by Corollary 7.1.3 to the Jordan Curve Theorem.
- **TERMINOLOGY**: Let *C* be a cycle of a connected graph, and suppose that *C* has been drawn in the plane. Relative to that drawing, an appendage of *C* is said to be *inner* or *outer*, according to whether that appendage is drawn inside or outside of *C*.

7.4 Kuratowski's Theorem

- TERMINOLOGY: Any graph homeomorphic either to K_5 or to $K_{3,3}$ is called a ***Kuratowski subgraph***.

- **Theorem 7.4.1.** ***Kuratowski's Theorem*** *[1930] A graph is planar if and only if it contains no subgraph homeomorphic to K_5 or to $K_{3,3}$.*
  - ✓ $\rightarrow$ Theorem 7.1.4 and 7.1.5 have established that K_5 and $K_{3,3}$ are both non-planar. Proposition 7.2.6 implies that a planar graph cannot contain a homeomorphic copy of either. Thus, containing no Kuratowski subgraph is necessary for planarity.
  - ✓ $\leftarrow$ If the absence of Kuratowski subgraphs were not sufficient, then there would exist nonplanar graphs with no Kuratowski subgraphs. If there were any such counterexamples, then <u>some counterexample graph would have the minimum number of edges among all counterexamples.</u>
  - ✓ The strategy is to derive some properties that this minimum counterexample would have to have, which ultimately establish that could not exist. The main steps within this strategy are proofs of three statements:

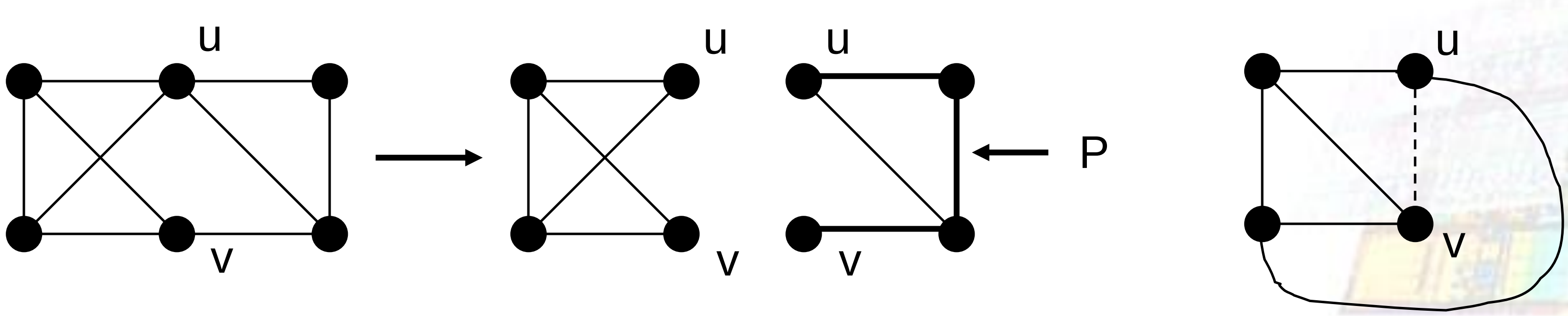
Step 1

- *Step1*: The minimum counterexample would be simple and 3-connected.
- *Step2*: The minimum counterexample would contain a cycle with three mutually overlapping appendages.
- *Step3*: Any configuration comprising a cycle and three mutually overlapping appendages must contain a Kuratowski subgraph.

- ***Step 1: The minimum counterexample would be simple and 3-connected.***
- **Assertion 1.1.** *Let G be a nonplanar connected graph with no Kuratowski subgraph and with the minimum number of edges for any such graph. Then G is a simple graph.*
  - ✓ Suppose that *G* is not simple. By Corollary 7.3.2, deleting a self-loop or one edge of a multi-edge would result in a smaller nonplanar graph, still with no Kuratowski subgraph, contradicting the minimality of *G*.
- **Assertion 1.2.** *Let G be a non-planar connected graph with no Kuratowski subgraph and with the minimum number of edges for any such graph. Then graph G has no cut-vertex.*
  - ✓ If graph *G* had a cut-vertex *v*, then removing *v* can yield smaller non-planar connected graph. By minimality of *G*, contradiction.

Step 1

- **Assertion 1.3.** *Let G be a nonplanar connected graph containing no Kuratowski subgraph with the minimum number of edges for any such graph. Let* $\{u,v\}$ *be a vertex-cut in G, and let L be a non-chord appendage of* $\{u,v\}$*. Then there is a planar drawing of L with a face whose boundary contains both vertices u and v.*
  - ✓ Since $\{u,v\}$ is a vertex-cut of graph G, the graph $G-\{u,v\}$ has at least two components. Thus, the vertex set $\{u,v\}$ must have at least one more non-chord appendage besides L.
  - ✓ Since, by Assertion 1.2, $\{u,v\}$ is a minimal cut, it follows that both u and v must be contact points of this other non-chord appendage. Since this other non-chord appendage is connected and has no edge joining u and v, it must contain a u-v path P of length at least 2, as illustrated in Figure 7.4.1.
  - ✓ The subgraph $H=(V_L \cup V_P, E_L \cup E_P)$ (obtained by adding path P to subgraph L) has no Kuratowski subgraph, because it is contained in graph G, which by premise contains no Kuratowski subgraphs.



Step 1

- ✓ Let d be the edge obtained from path P by smoothing away all the internal vertices. Then the graph $L+d$ contains no Kuratowski subgraphs (because $L+d$ is homeomorphic to subgraph H). Moreover, the graph $L+d$ has fewer edges than the minimal counter-example G (because P has at least one internal vertex). Thus, the graph $L+d$ is planar.
- ✓ In every planar drawing of the graph $L+d$, vertices u and v both lie on the boundary of each face containing edge d. Discarding edge d from any such planar drawing of $L+d$ yields a planar drawing of appendage L such that vertices u and v lie on the same face (by Proposition 7.3.4).

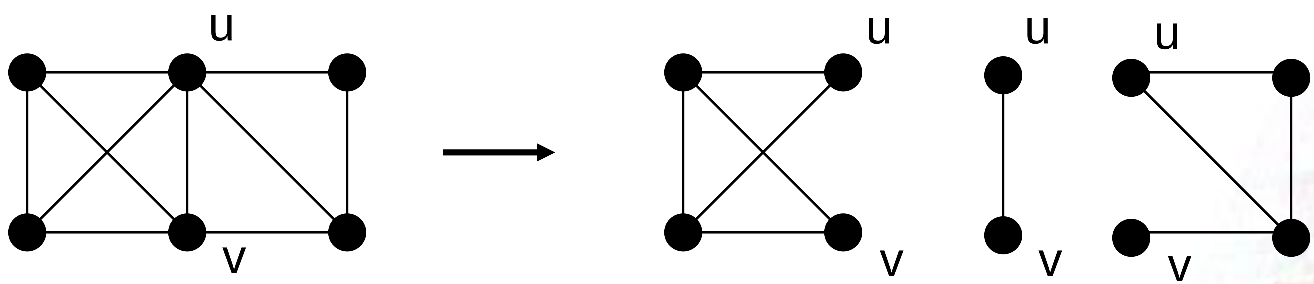
- ***Assertion 1.4.*** *Let G be a nonplanar connected graph containing no Kuratowski subgraph, with the minimum number of edges for any such graph. Then graph G has no vertex-cut with exactly two vertices.*
  - ✓ Suppose that graph G has a minimal vertex-cut $\{u,v\}$. Clearly, a chord appendage of $\{u,v\}$ would have a planar drawing with only one face, whose boundary contains both the vertices u and v.

Step 1

- ✓ By Assertion 1.3, every non-chord appendage of $\{u,v\}$ would also have a planar drawing with vertices u and v on the same face boundary. Figure 7.4.2 illustrates the possible decomposition of graph G into appendages.
- ✓ By Proposition 7.3.6, when graph G is reassembled by iteratively amalgamating these planar appendages at vertices u and v, the result is a planar graph.

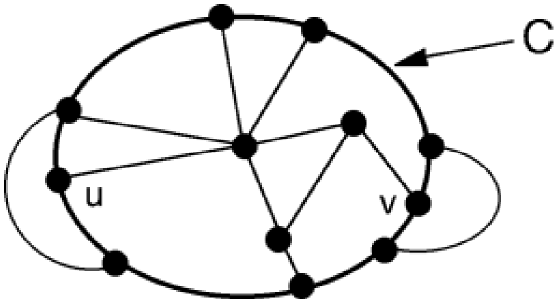
Completion of Step 1. *Let G be a nonplanar connected graph containing no Kuratowski subgraph and with the minimum number of edges for any such graph. Then graph G is simple and 3-connected.*

- ✓ This summarizes the four preceding assertions. The connectedness premise serves only to avoid isolated vertices.



Step 2

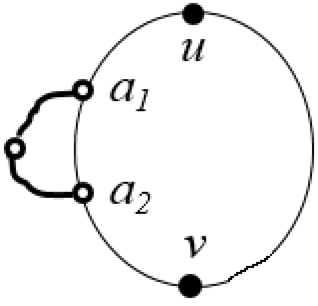
- ***Step 2: Finding a Cycle with Three Mutually Overlapping Appendages***
  - ✓ Let G be a nonplanar graph containing no Kuratowski subgraph and with the minimum number of edges for any such graph. Let e be any edge of graph G, say with endpoints u and v, and consider a planar drawing of $G-e$.
  - ✓ Since graph G is 3-connected (by Step 1), it follows (see §5.1) that $G-e$ is 2-connected. This implies (see §5.1) that there is a cycle in $G-e$ through vertices u and v.
  - ✓ Among all such cycles, choose cycle C, as illustrated in Figure 7.4.3, so that the number of edges "inside" C is as large as possible. The next few assertions establish that cycle C must have two overlapping appendages in $G-e$ that both overlap edge e.
- **Assertion 2.1.** *Cycle C has at least one outer appendage.*
  - ✓ Otherwise, edge e could be drawn in the outer region, thereby completing a planar drawing of G.

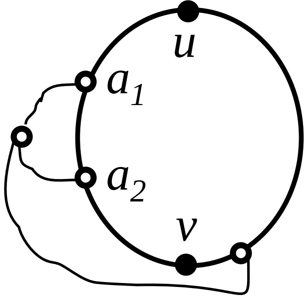


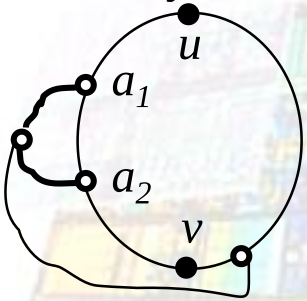
Cycle C thru u, v has max # of edges inside it.

Step 2

- **Assertion 2.2.** *Let B be an appendage of C that has only two contact points, neither of which is u or v. Then appendage B is a chord.*
  - ✓ If *B* were a non-chord appendage, then those two contact points of *B* would separate vertices *u* and *v* in *G* from the other vertices of *B*, which would contradict the 3-connectivity of *G*.
- **Assertion 2.3.** *Let d be an outer appendage of cycle C. Then d is a chord, and its endpoints alternate on C with u and v, so that edges e and d are overlapping chords of cycle C.*
  - ✓ Suppose that two contact points a_1 and a_2 of appendage *d* do not alternate with vertices *u* and *v* on cycle *C*. Then appendage *d* would contain a path *P* between vertices a_1 and a_2 with no contact point in the interior of *P*.
  - ✓ Under such a circumstance, cycle *C* could be "enlarged" by replacing its arc between a_1 and a_2 by path *P*. The enlarged cycle would still pass through vertices *u* and *v* and would have more edges inside it than cycle *C*, thereby contradicting the choice of cycle *C*.







Step 2

- ✓ Moreover, if outer appendage *d* had three or more contact points, then there would be at least one pair of them that does not alternate with *u* and *v*. Thus, appendage *d* has only two contact points (is a chord by Assertion 2.2), and they alternate with *u* and *v*.

- **Assertion 2.4.** *There is an inner appendage of cycle C that overlaps edge e and also overlaps some outer chord.*

  - ✓ One or more inner appendages must overlap chord *e*, because otherwise edge *e* could be drawn inside cycle *C* without crossing any inner appendages → a planar drawing of graph *G*.

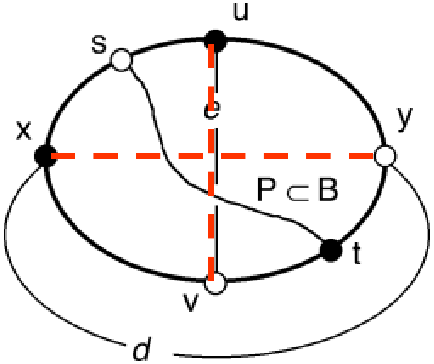
  - ✓ Moreover, at least one inner appendage that overlaps *e* also overlaps some outer appendage *d*. Otherwise, every such inner appendage could be redrawn outside cycle *C*, which would permit edge *e* to be drawn inside without edge crossing. By Assertion 2.3, outer appendage *d* must be a chord.

- **Completion of Step 2.** *Let G be a nonplanar connected graph containing no Kuratowski subgraph and with the minimum number of edges for any such graph. Then graph G contains a cycle that has three mutually overlapping appendages, two of which are chords.*

  - ✓ The number of edges "inside" *C* is as large as possible. A second appendage *B* that overlaps chord *e* and some outer chord *d* must exist (Assertion 2.4). Chord *d* also overlaps chord *e* (Assertion 2.3). Thus, appendages *e*, *B*, and *d* are mutually overlapping.

Step 3

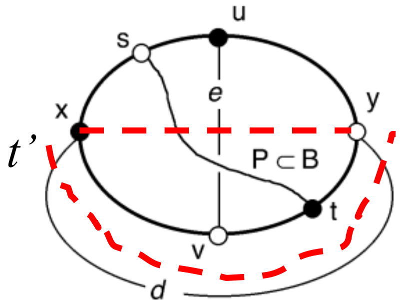
- **Step 3: Analyzing the Cycle-and-Appendages Configuration**
  we want to show that the cycle-and-appendages configuration whose existence is guaranteed by Step 2 must contain a *Kuratowski subgraph.*
- **Step 3:** *Let C be a cycle in a connected graph G. Let edge e be an inner chord, edge d an outer chord, and B an inner appendage, such that e, d, and B are mutually overlapping. Then graph G has a Kuratowski subgraph.*
  - ✓ Let *u* and *v* be the contact points of inner chord *e*, and let *x* and *y* be the contact points of outer chord *d*. These pairs of contact points alternate on cycle *C*. Observe that the union of cycle *C*, chord *e*, and chord *d* forms the complete graph K_4. There are two cases to consider, according to the location of the contact points of appendage *B*.
  - ✓ **Case 1.** *Suppose that appendage B has at least one contact point s that differs from u, v, x, and y.*
    By symmetry of the *C-e-d* configuration, it suffices to assume that contact point *s* lies between vertices *u* and *x* on cycle *C*. In order to overlap chord *e*, appendage *B* must have a contact point *t* on the other side of *u* and *v* from vertex *s*.



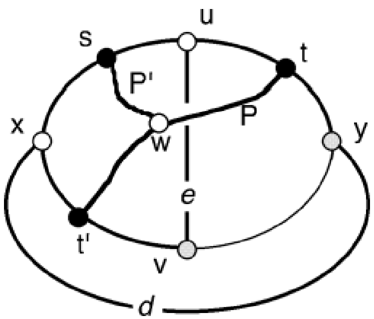
Subcase 1a.

Step 3

- In order to overlap chord *d*, appendage *B* must have a contact point *t'* on the other side of *x* and *y* from vertex *s*.
- **Subcase 1a.** Contact point *t* lies in the interior of the arc between *v* and *y* on cycle *C*, in which case it may be considered that $t' = t$. In this subcase, let *P* be a path in appendage *B* between contact points *s* and *t*. Then the union of cycle *C*, path *P*, and chord *e* and *d* forms a homeomorphic copy of $K_{3,3}$.
- **Subcase 1b.** Contact point *t* lies on arc *uy*, with $t \neq u$ (so that appendage *B* overlaps chord *e*), and contact point *t'* lies on arc *xv*, with $t' \neq x$ (*v* in textbook is a typo)(so that *B* overlaps chord *d*). In this subcase, let *P* be a path in appendage *B* between contact points t' and t. Let *w* be an internal vertex on path *P* such that there is a *w*-*s* path *P*' in *B* with no internal vertices in *P*. (Such a path *P*' exists, because appendage *B* is connected.) Then this configuration contains a homeomorph of $K_{3,3}$ in which each of vertices *w*, *x*, and *u* is joined to each of the vertices *s*, *t*, and *t*'. It does not matter if $t = y$ or if $t' = v$.



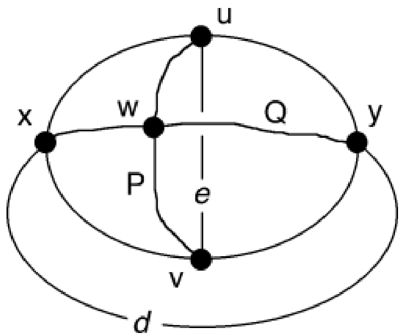
Subcase 1a.



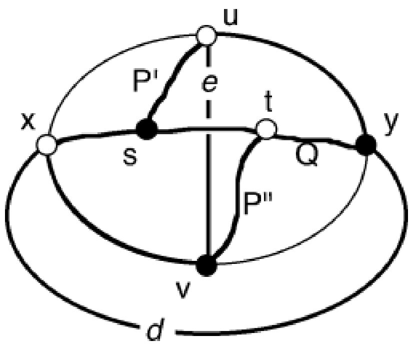
Subcase 1b.

Step 3

- **Case 2.** *Appendage B has no contact points other than u, v, x, and y.* Vertices x and y must be contact points of B so that it overlaps chord e. Vertices u and v must be contact points of B so that it overlaps chord d. Thus, all four vertices u, v, x, and y must be contact points of appendage B. Appendage B has a u-v path P and an x-y path Q whose internal vertices are not contact points of B.
- **Subcase 2a.** Paths P and Q intersect in a single vertex w. Then the union of the C-e-d configuration with paths P and Q yields a configuration homeomorphic to K_5.
- **Subcase 2b.** Paths P and Q intersect in more than one vertex. Let s be the intersection nearest on path P to contact point u, and t the intersection nearest on path P to contact point v. Also, let P' be the us-subpath of P and P'' the tv-subpath of P. Then the union of chords e and d with the paths Q, P', and P'' and with arcs vx and uy on cycle C form a subgraph homeomorphic to $K_{3,3}$, with bipartition $(\{u, x, t\}, \{v, y, s\})$.



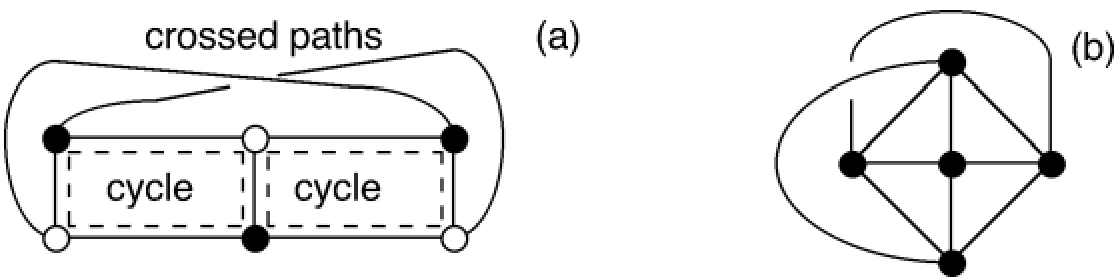
Subcase 2a.



Subcase 2b.

Finding $K_{3,3}$ or K_5 in Small Nonplanar Graphs

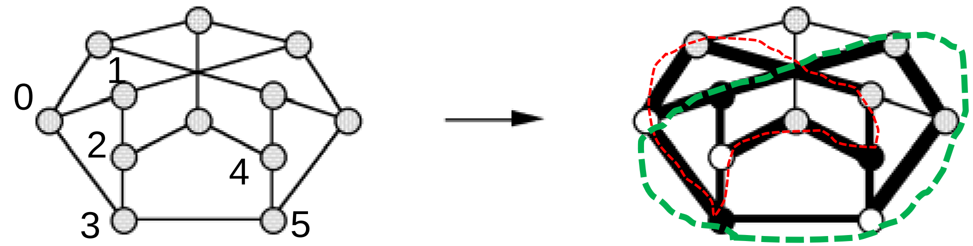
- **Corollary 7.4.2.** *Let G be a nonplanar graph formed by the amalgamation of subgraphs H and J at vertices u and v, such that J is planar. Then the graph obtained from graph H by joining vertices u and v is nonplanar.*
  - ✓ By Theorem 7.4.1, graph G must contain a Kuratowski subgraph K. All the vertices of the underlying Kuratowski graph would have to lie on the same side of the cut $\{u,v\}$, because K is 3-connected. At most, a subdivided edge of the Kuratowski subgraph crosses through u and back through v. The conclusion follows.
- A simple way to find a homeomorphic copy of $K_{3,3}$ in a small graph is to identify two cycles C and C' (necessarily of length at least 4) that meet on a path P, such that there is a pair of paths between $C-P$ and $C'-P$ that "cross", thereby forming a subdivided Möbius ladder ML_3, as illustrated in Figure 7.4.10(a).
- To find a homeomorphic copy of K_5, look for a 4-wheel with a pair of disjoint paths joining two pair of vertices that alternate on the rim, as illustrated in Figure 7.4.10(b).



Finding $K_{3,3}$ (a) or K_5 (b) by visual inspection.

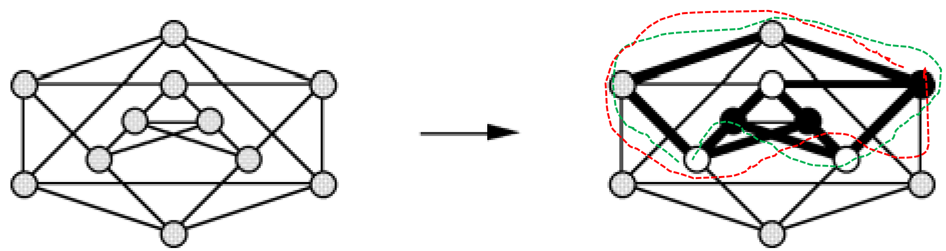
Finding $K_{3,3}$ or K_5 in Small Nonplanar Graphs

- **Example 7.4.1.**
- **Example 7.4.2.**
- **Example 7.4.3.**

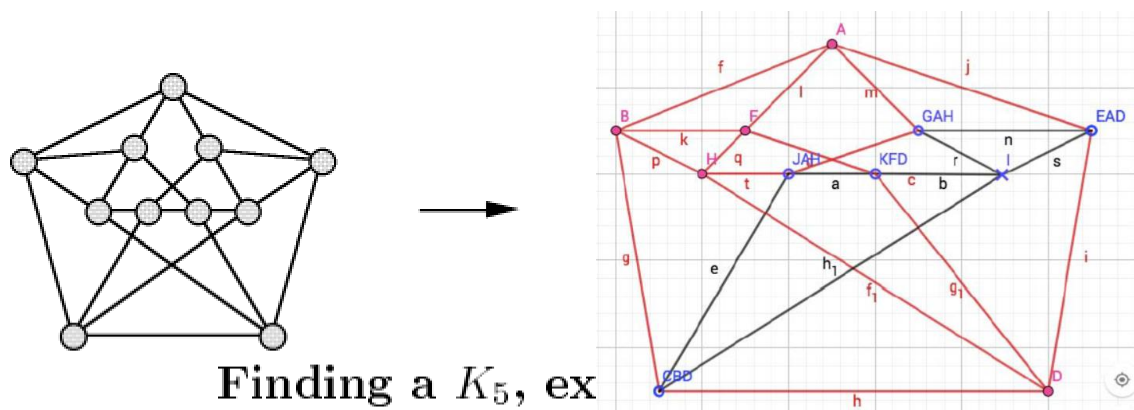


C = {0, 1, 2, 3, 0}
C' = {1, 2, 4, 5, 1}
P = {1, 2}; C-P = {0, 3}; C'-P = {4, 5}

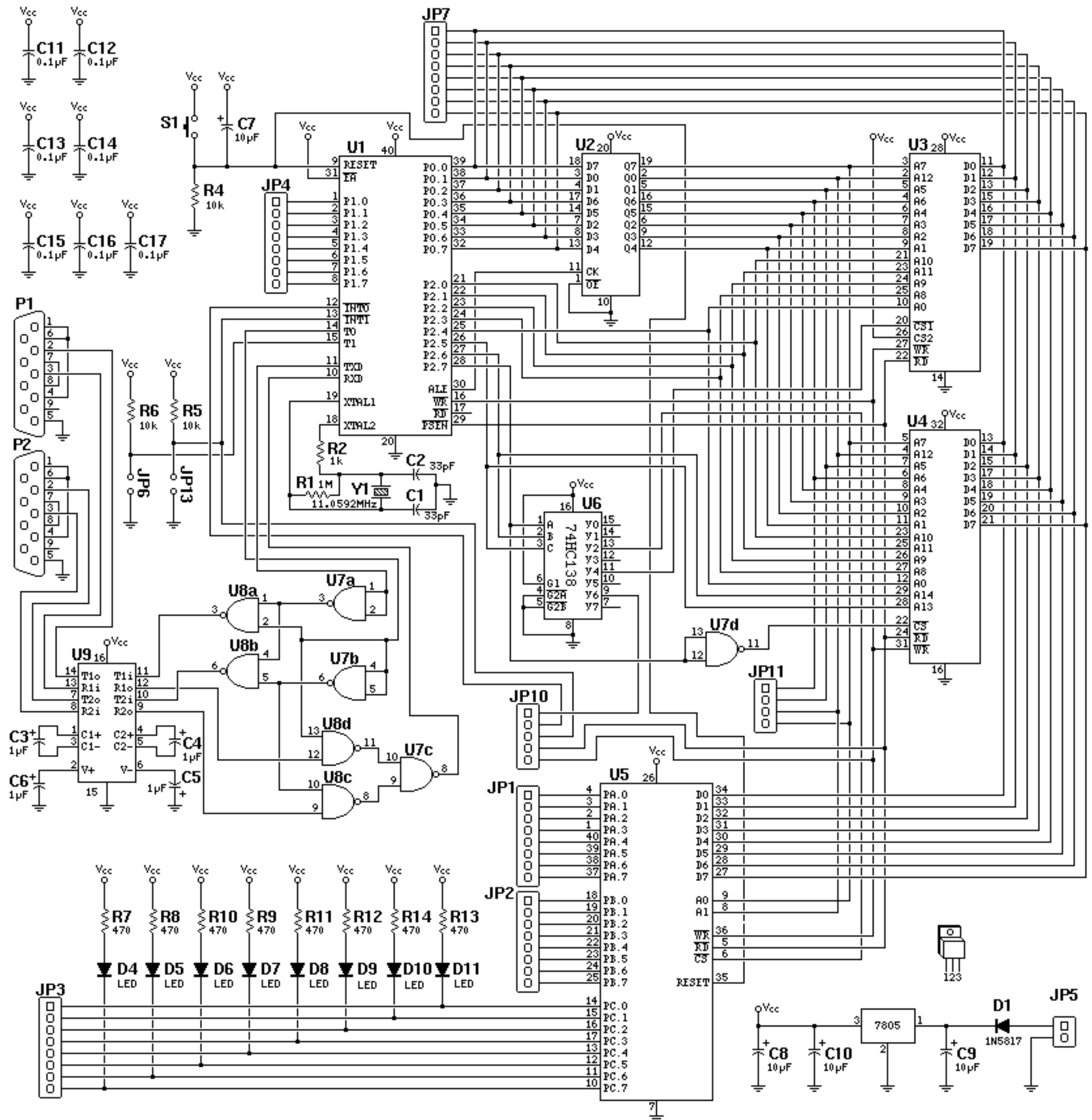
Finding a $K_{3,3}$, example.



Finding a $K_{3,3}$, example.



Finding a K_5, ex



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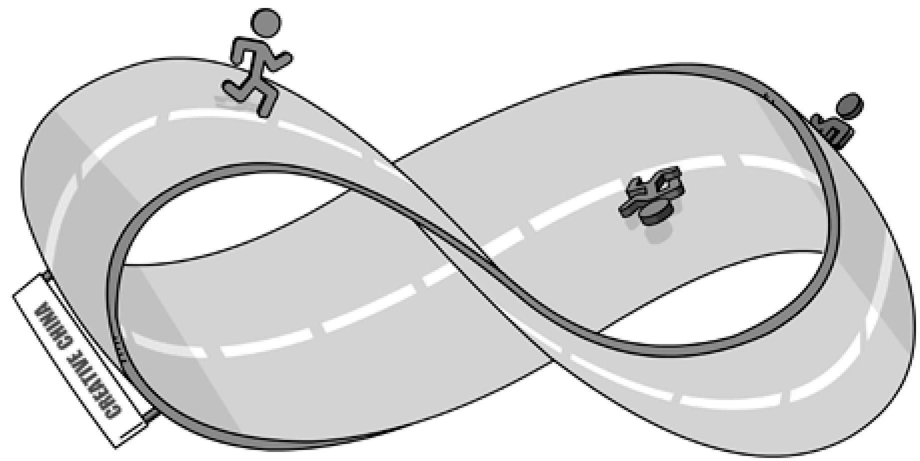
Möbius band



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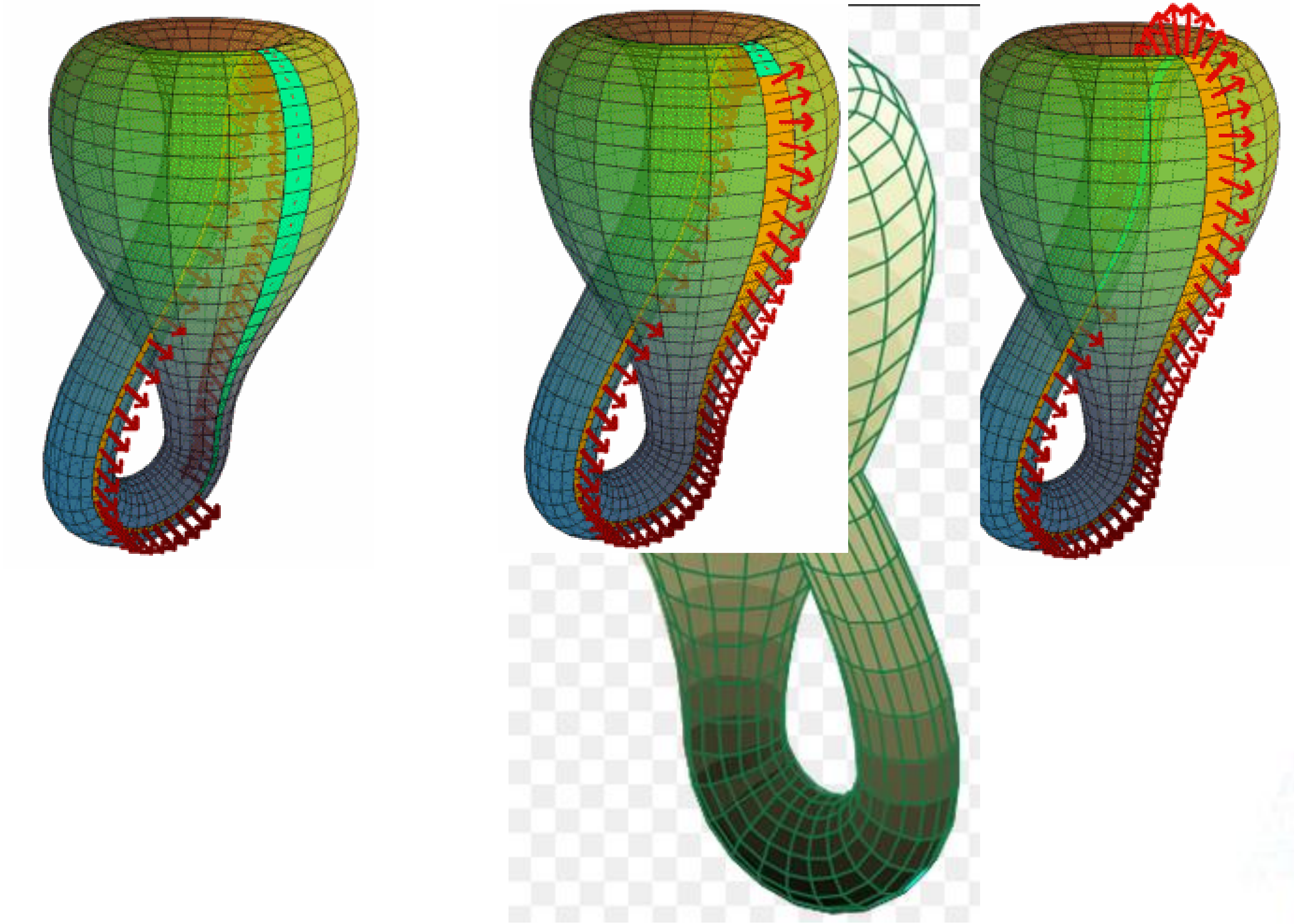
Klein Bottle



http://plus.maths.org/content/os/issue26/features/mathart/kleinBottle_anim

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