Chap 5. Connectivity

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The sources of most figure images are from the course slides (Graph Theory) of Prof. Gross

Outline

- Vertex- And Edge-Connectivity
- Constructing Reliable Networks
- MAX-MIN Duality and Menger's Theorems
- Block Decompositions

5.1 Vertex- And Edge-Connectivity

- **TERMINOLOGY:** Let S be a subset of vertices or edges in a connected graph G. The removal of S is said to ***disconnect*** G if the deletion subgraph $G - S$ is not connected.

  REVIEW FROM §2.4:

  A ***vertex-cut*** in a graph G is a vertex-set U such that $G - U$ has more components than G.

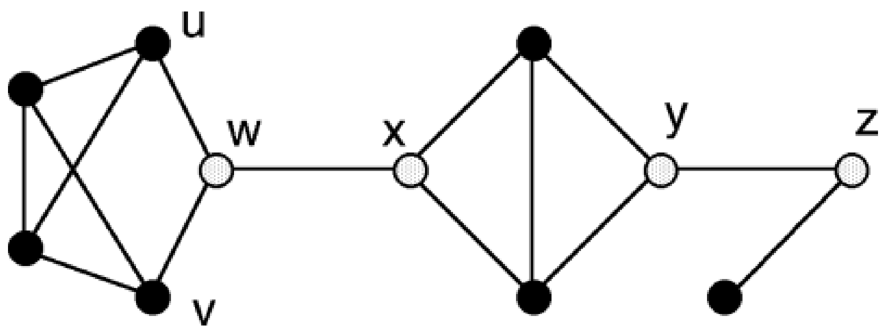
  A ***cut-vertex*** (or ***cutpoint***) is a vertex-cut consisting of a single vertex.

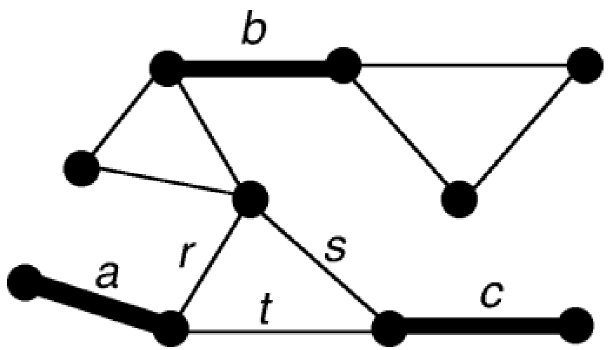
  An ***edge-cut*** in a graph G is a set of edges D such that $G - D$ has more components than G.

  A ***cut-edge*** (or ***bridge***) is an edge-cut consisting of a single edge.

  An edge is a cut-edge if and only if it is not a cycle-edge.

- DEFINITION: The ***vertex-connectivity*** of a connected graph G, denoted $\kappa_v(G)$, is the minimum number of vertices whose removal can either disconnect G or reduce it to a 1-vertex graph.

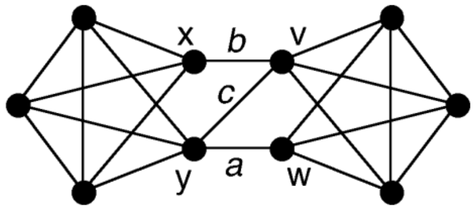




5.1 Vertex- And Edge-Connectivity

- DEFINITION: A graph G is ***k*-connected** if G is connected and $\kappa_v(G) \geq k$. If G has non-adjacent vertices, then G is k-connected if every vertex-cut has at least k vertices.
- DEFINITION: The ***edge-connectivity*** of a connected graph G, denoted $\kappa_e(G)$, is the minimum number of edges whose removal can disconnect G.
- DEFINITION: A graph G is ***k-edge-connected*** if G is connected and every edge-cut has at least k edges (i.e., $\kappa_e(G) \geq k$).
- **Example 5.1.1:** $\kappa_v(G) = 2$, $\{x,y\}$, $\{v,w\}$, $\{y,v\}$.

  $\kappa_e(G) = 3$, $\{a,b,c\}$



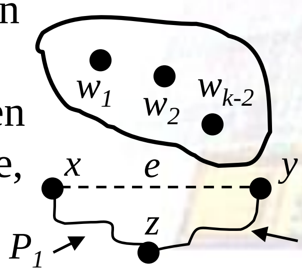
G **has** $\kappa_v(G) = 2$**,** $\kappa_e(G) = 3$**, and** $\delta_{min}(G) = 4$**.**

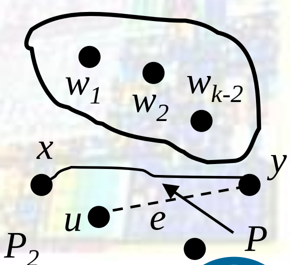
5.1 Vertex- and Edge-Connectivity

- **Application 5.1.1.** *Network Survivability*: The connectivity measure κ_v and κ_e are used in a quantified model of *network survivability*, which is the capacity of a network to retain connections among its nodes after some edges or nodes are removed.
- **Remark:** We will assume that all graphs under consideration throughout this chapter are loopless, unless otherwise specified.
- **Proposition 5.1.1.** Let G be a graph. Then the edge-connectivity $\kappa_e(G)$ is less than or equal to the minimum degree $\delta_{min}(G)$.
  - ✓ Let v be a vertex of graph G, with degree $k = \delta_{min}(G)$. Then the deletion of the k edges that are incident on vertex v separates v from the other vertices of G.
- REVIEW FROM §4.5: A ***partition-cut*** $\langle X_1, X_2 \rangle$ is an edge-cut each of whose edges has one endpoint in each of the vertex bipartition sets X_1 and X_2.
- **Proposition 5.1.2.** A graph G is k-edge-connected if and only if every partition-cut contains at least k edges.
  - ✓ ($\Rightarrow$) Suppose that graph G is k-edge-connected. Then every partition-cut of G has at least k edges, since a partition-cut is an edge cut.
  - ✓ ($\Leftarrow$) Suppose that every partition-cut contains at least k edges. By Proposition 4.5.4, every minimal edge-cut is a partition-cut. Thus, every edge-cut contains at least k edges.

Relationship Between Vertex- and Edge-Connectivity

- **Proposition 5.1.3.** Let e be any edge of a k-connected graph G, for $k \geq 3$. Then the edge-deletion subgraph $G - e$ is $(k - 1)$-connected.
  - ✓ Let $W = \{w_1, w_2, \ldots, w_{k-2}\}$ be any set of $k - 2$ vertices in $G - e$, and let x and y be any two different vertices in $(G - e) - W$. It suffices to show there is an x-y walk in $(G - e) - W$.
  - ✓ ***Condition I***. At least one of the endpoints of edge e is contained in W. Since the vertex-deletion subgraph $G - W$ is connected (2-connected), there is an x-y path in $G - W$. This path cannot contain edge e and, hence, it is an x-y path in the subgraph $(G - e) - W$.
  - ✓ ***Condition II***. Neither endpoint of edge e is in W. Then there are two cases to consider.
    - *case* 1: Vertices x and y are the endpoints of edge e. **Graph G has at least k + 1 vertices** (G is k-connected). So there exists an x-z path P_1 in the vertex-deletion subgraph $G - \{w_1, w_2, \ldots, w_{k-2}, y\}$ and a z-y path P_2 in the subgraph $G - \{w_1, w_2, \ldots, w_{k-2}, x\}$. Neither of these paths contains edge e, and, therefore, their concatenation is an x-y walk in the subgraph G–e–$\{w_1, w_2, \ldots, w_{k-2}\}$.
    - *case* 2: At least one of the vertices x and y, say x, is not an endpoint of edge e. Let u be an endpoint of edge e that is different from vertex x. Since graph G is k-connected, the subgraph $G - W'$ ($W' = \{w_1, w_2, \ldots, w_{k-2}, u\}$) is connected. Hence, there is an x-y path P in $G - W'$. It follows that P is an x-y path in $G - W'$ that does not contain vertex u and, hence, excludes edge e (even if P contains the other endpoint of e, which it could). Therefore, P is an x-y path in $(G - e) - \{w_1, w_2, \ldots, w_{k-2}\}$.





Relationship Between Vertex- and Edge-Connectivity

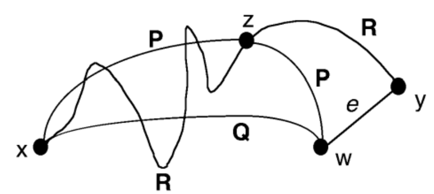
- **Corollary 5.1.4.** Let G be a k-connected graph, and let D be any set of m edges of G, for $m \leq k - 1$. Then the edge-deletion subgraph $G - D$ is $(k - m)$-connected.
  - ✓ The result follows by the iterative application of Proposition 5.1.3.
- **Corollary 5.1.5** Let G be a connected graph. Then $\kappa_e(G) \geq \kappa_v(G)$.
  - ✓ Let $k = \kappa_v(G)$, and let S be any set of $k - 1$ edges in graph G. Since G is k-connected, the graph $G - S$ is 1-connected, by Corollary 5.1.4. Thus, edge subset S is not an edge-cut of graph G, which implies that $\kappa_e(G) \geq k$.
- **Corollary 5.1.6** Let G be a connected graph. Then $\kappa_v(G) \leq \kappa_e(G) \leq \delta_{min}(G)$.
  - ✓ The assertion simply combines Proposition 5.1.1 and Corollary 5.1.5.

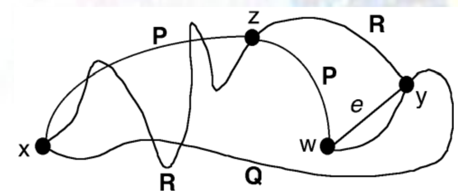
Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

- **TERMINOLOGY:** A vertex of a path P is an ***internal vertex*** of P if it is neither the initial nor the final vertex of that path.
- **DEFINITION:** Let u and v be two vertices in a graph G. A collection of u-v paths in G is said to be ***internally disjoint*** if no two paths in the collection have an internal vertex in common.

Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

- **Theorem 5.1.7 [Whitney's 2-Connected Characterization].** Let G be a connected graph with three or more vertices. Then G is 2-connected if and only if for each pair of vertices in G, there are two internally disjoint paths between them.
  - ✓ ($\Leftarrow$) Arguing by contrapositive, suppose that G is not 2-connected. Then let v be a cut-vertex of G. Since $G - v$ is not connected, there must be two vertices x and y such that there is no x-y path in $G - v$. It follows that v is an internal vertex of every x-y path in G.
  - ✓ ($\Rightarrow$) Let x and y be any two vertices in G. For $d(x , y)=1$ (there is an edge e joining vertices x and y), then the edge-deletion subgraph $G - e$ is connected. Thus, there is an x-y paths P in $G - e$. It follows that path P and edge e are two internally disjoint x-y paths G.
  - ✓ For $d(x,y)=n$, choose w as next-to-last vertex on an x-y path of length n. Let P, Q be two internal-disjoint x-w paths. Since G is 2-connected, there must exist a x-y path R not passing w (for the case of removing w). Let z be the last vertex on path R that precedes vertex y and is also on one of the paths P or Q (z might be vertex x).
    - ➢ ***Case* 1**: y is not in P nor Q. $P_{xz} + R_{zy}$ and $Q+e$ are two internal disjoint x-y paths.
    - ➢ ***Case* 2**: y is in P or Q. $P_{xz} + R_{zy}$ and Q_{xy} are two internal disjoint x-y paths.





Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

- **Corollary 5.1.8.** Let G be a graph with at least three vertices. Then G is 2-connected if and only if any two vertices of G lie on a common cycle.
  - ✓ This follows from Theorem 5.1.7,since two vertices x and y lie on a common cycle if and only if there are two internally disjoint x-y paths.
- **Remark:** Theorem 5.1.7 is a prelude to Whitney's more general result for k-connected graph, which appears in 5.3.

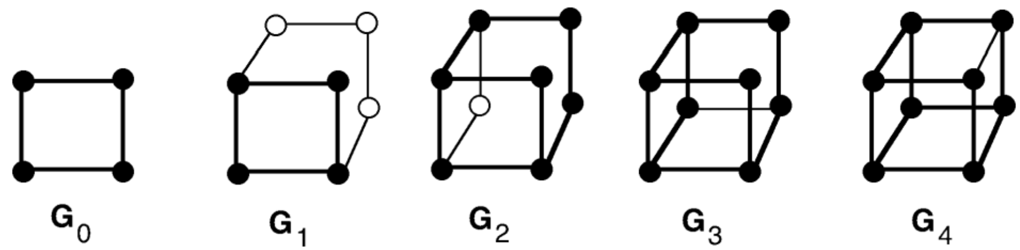
Internally Disjoint Paths and Vertex-Connectivity: Whitney's Theorem

- **Theorem 5.1.9 [Characterization of 2-connected Graph].** Let G be a connected graph with at least three vertices. Then the following statements are equivalent.

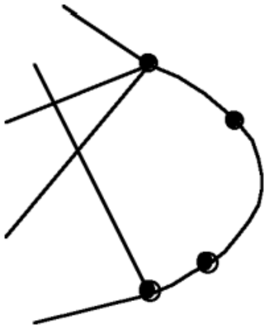
  1. Graph G is 2-connected.
  2. For any two vertices of G, there is a cycle containing both.
  3. For any vertex and any edge of G, there is a cycle containing both.
  4. For any two edges of G, there is a cycle containing both.
  5. For any two vertices and one edge of G, there is a path containing all three.
  6. For any three distinct vertices of G, there is a path containing all three.
  7. For any three distinct vertices of G, there is a path containing any two of them which does not contain the third.

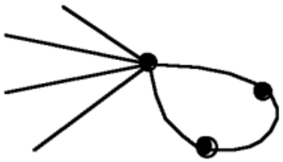
5.2 Constructing Reliable Networks

- **DEFINITION**: A ***path addition*** to a graph G is the addition to G of a path between two existing vertices of G, such that the edges and internal vertices of the path are not in G. A ***cycle addition*** is the addition to G of a cycle that has exactly one vertex in common with G.
- **DEFINITION**: A ***Whitney-Robbins synthesis*** of a graph G from a graph H is a sequence of graphs, $G_0, G_1, \ldots, G_l$, where $G_0 = H$, $G_l = G$, and G_i is the result of a path or cycle addition to G_{i-1}, for $i = 1, \ldots, l$. If each G_i is the result of a path addition only, then the sequence is called a ***Whitney synthesis***.
- **Example 5.2.1:**



A Whitney synthesis of the cube graph Q_3.

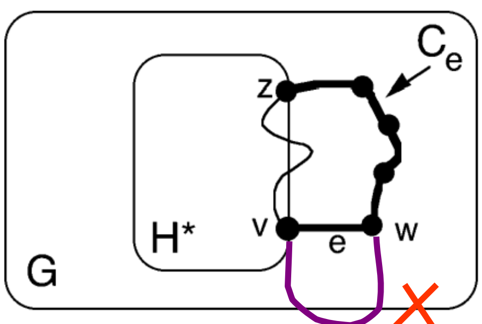


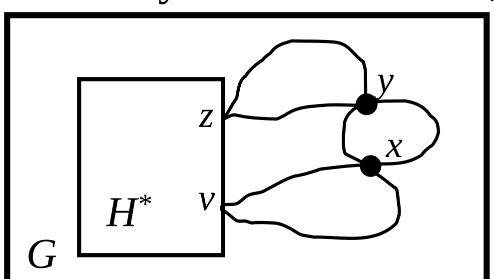


- **Lemma 5.2.1.** *Let H be 2-connected graph. Then the graph G that results from a path addition to H is 2-connected.*
  - ✓ **Proof:** The property that every two vertices lie on a common cycle is preserved under path addition. Thus, by Corollary 5.1.8, graph G is 2-connected.

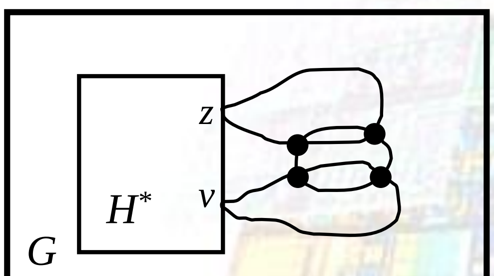
Whitney's Synthesis of 2-Connected Graphs and 2-Edge-Connectted Graphs

- **Theorem 5.2.2 [Whitney Synthesis Theorem].** *A graph G is 2-connected if and only if G is a cycle or a Whitney synthesis from a cycle.*
  - ✓ **Proof:** (⇐) Suppose that $C = G_0, G_1, \ldots, G_l = G$ is a Whitney Synthesis from a cycle C. Since a cycle is 2-connected, iterative application of Lemma 5.2.1 implies that graph G_i is 2-connected for $i = 1, \ldots, l$. In particular, $G = G_l$ is 2-connected.
  (⇒) Suppose that G is a 2-connected graph, and let C be any cycle in G. Consider the collection $\mathcal{H}$ of all subgraphs of G that are Whitney synthesis from cycle C. Since the collection $\mathcal{H}$ is nonempty ($C \in \mathcal{H}$), there exists a subgraph $H^* \in \mathcal{H}$ with the maximum number of edges. Suppose that $H^* \neq G$. Then, the connectedness of G implies that there exists an edge $e = vw \in \boldsymbol{E_G} - \boldsymbol{E_{H^*}}$ whose endpoint v lies in H^*. Since G is 2-connected, every edge is a cycle-edge, from which it follows that there exists a cycle containing edge e. Moreover, since endpoint v is not a cut-vertex, there must be at least one such cycle, say C_e ,that meets subgraph H^* at a vertex other than v. Let z be the first vertex on C_e at which the cycle returns to H^*. Then the portion of C_e from v to z that includes edge e is a path addition to H^*. Thus, H^* is extendible by a path addition, contradicting the maximality of H^*. Therefore, $H^* = G$.





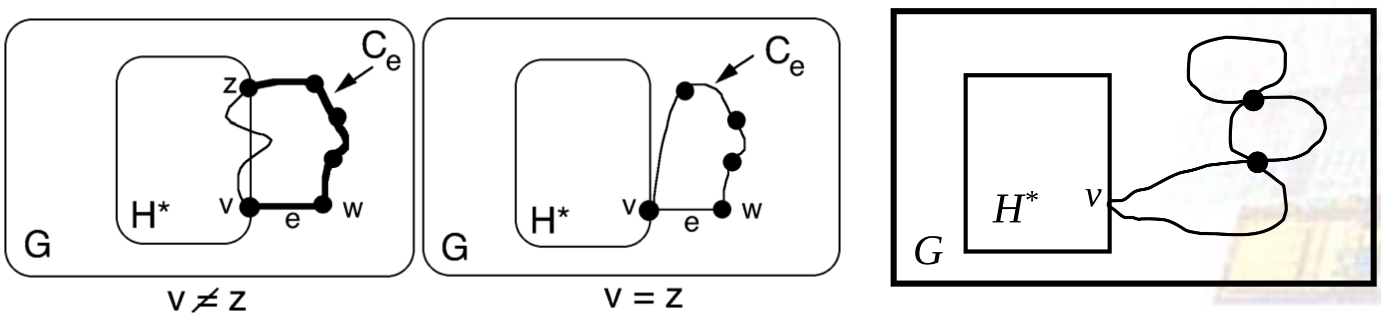
*H** is extendible by four path additions.



*H** is extendible by six path additions.

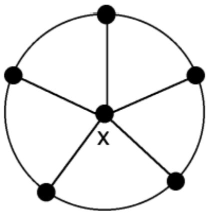
Whitney's Synthesis of 2-Connected Graphs and 2-Edge-Connectted Graphs

- **Lemma 5.2.3.** *Let H be a 2-edge-connected (i.e., bridgeless) graph. Then the graph that results from a path or cycle addition to H is 2-edge-connected.*
- **Theorem 5.2.4 [Whitney-Robbins Synthesis Theorem].** *A graph G is 2-edge-connected if and only if G is a cycle or a Whitney-Robbins synthesis from a cycle.*
  - ✓ **Proof:** ($\Leftarrow$) Suppose that $C = G_0, G_1, \ldots, G_l = G$ is a Whitney-Robbins synthesis from a cycle C. Since a cycle is 2-edge-connected, iterative application of Lemma 5.2.3 implies that G is 2-edge-connected.
    ($\Rightarrow$) Let C be any cycle in G. Among all subgraphs of G that are Whitney-Robbins synthesis from cycle C, let H^* be one with the maximum number of edges. Suppose that $H^* \neq G$. As in the proof of Theorem 5.2.2, there exists an edge $e = vw \in \boldsymbol{E_G} - \boldsymbol{E_{H^*}}$ whose endpoint v lies in H^*. Moreover, edge e must be part of some cycle C_e (because G is 2-edge-connected). Again, let z be the first vertex at which the cycle returns to subgraph H^*. Because v can be a cut-vertex, there are now two possibilities, as shown in Figure5.2.3. Thus, H^* is extendible by a path addition or a cycle addition, contradicting the maximality of H^*. Therefore, $H^* = G$.



Tutte's Synthesis of 3-Connected Graphs

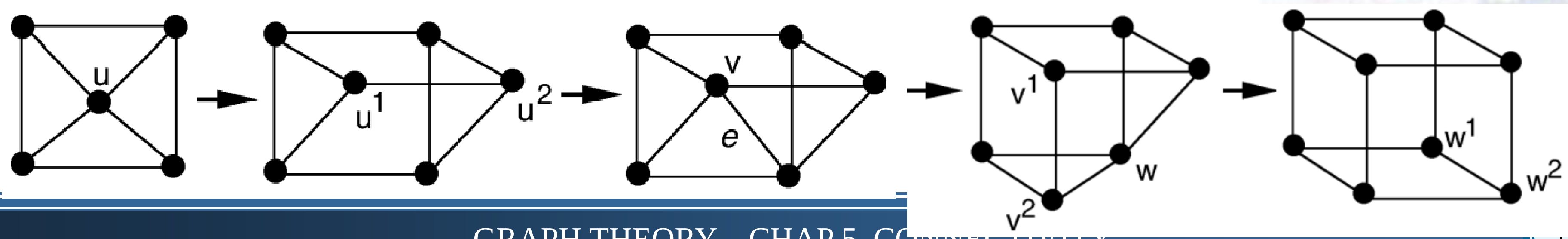
- **TERMINOLOGY:** Path additions and cycle additions are often called, respectively, ***open-ear*** and ***close-ear additions***, since the new paths in drawings like Figure 5.2.3 are imagined to look like human ears. Moreover, the Whitney and Whitney-Robbins synthesis are called ***ear decompositions***.



- **Example 5.2.2:** 5-spoke wheel W_5.
- **Theorem 5.2.5 [Tutte Synthesis Theorem].** *A graph is 3-connected if and only if it is a wheel or can be obtained from a wheel by a sequence of operations of the following two types.*

  1. *Adding an edge between two non-adjacent vertices.*
  2. *Replacing a vertex v with degree at least 4 by two new vertices* v^1 *and* v^2*, joined by a new edge; each vertex that was adjacent to v in G is joined by an edge to exactly one of* v^1 *and* v^2 *so that* $deg(v^1) \geq 3$ and $deg(v^2) \geq 3$.

- **Example 5.2.3:**



Tutte's Synthesis of 3-Connected Graphs

- **Application 5.2.1.** *Construction of a Class of Reliable Networks*

  Given positive integers n and k, with k < n, find a k-connected n-vertex graph having the smallest possible number of edges.

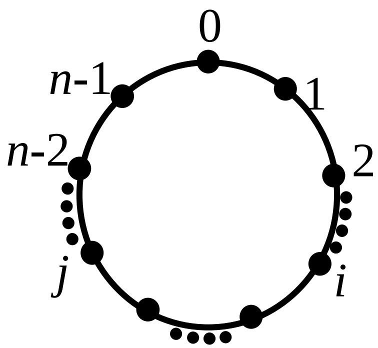
- Let $h_k(n)$ denote the minimum number of edges that a k-connected graph on n vertices must have.
  - ✓ $h_1(n) = n - 1$

- **DEFINITION**: For any real number x, the ***floor*** of x, denoted $\lfloor x \rfloor$, is the greatest integer less than or equal to x, and $\lceil x \rceil$, the ***ceiling*** of x, is the smallest integer greater than or equal to x.

- **Proposition 5.2.6.** *Let G be a k-connected graph on n vertices. Then the number of edges in G is at least* $\left\lceil \frac{kn}{2} \right\rceil$ *. That is ,*$h_k(n) \geq \left\lceil \frac{kn}{2} \right\rceil$.
  - ✓ **Proof:** Euler's Degree-sum Theorem (§1.1) implies that $2|E_G| \geq n\delta_{min}(G)$. The result now follows by Corollary 5.1.6. (Let G be a connected graph. Then $\kappa_v(G) \leq \kappa_e(G) \leq \delta_{min}(G)$.)

Harary's Construction of an Optimal k-Connected Graph

- Frank Harary gave a procedure for constructing a k-connected graph $H_{k.n}$ on n vertices that has exactly $\left\lceil \frac{kn}{2} \right\rceil$ edges for $k \geq 2$.
  - ✓ n-cycle graph with consecutively clockwisely numbered vertices $(0, 1, \ldots, n-1)$.
  - ✓ Adjacency between vertices i and j is determined by the distance between i and j along the perimeter of n-cycle.

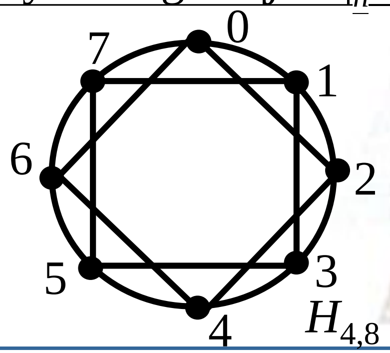


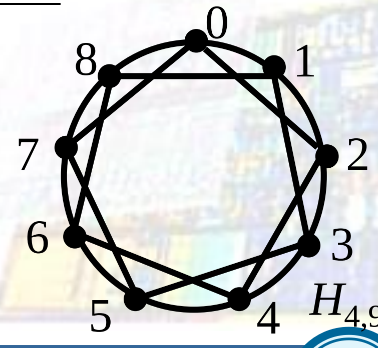
- **DEFINITION:** Let i and j be any two integers from the set $\{0,1,\ldots,n-1\}$. The ***mod n distance*** between i and j, denoted $|j-i|_n$, is the smaller of the two values $|j-i|$ and $n-|j-i|$.

Harary Construction

- The construction of $H_{k,n}$ depends on the parity of k and n and falls into three cases. The construction for the first case (k even) is needed for the last two cases.
- ***Case* 1: *k* even**. Let $k=2r$. Vertices i and j are joined by an edge if $|j-i|_n \leq r$.

It is not hard to show that $H_{2r,n}$ has exactly rn edges (see Exercises). Thus, $H_{2r,n}$ has the desired number of edges, since $rn = \frac{kn}{2} = \left\lceil \frac{kn}{2} \right\rceil$ (kn is even).

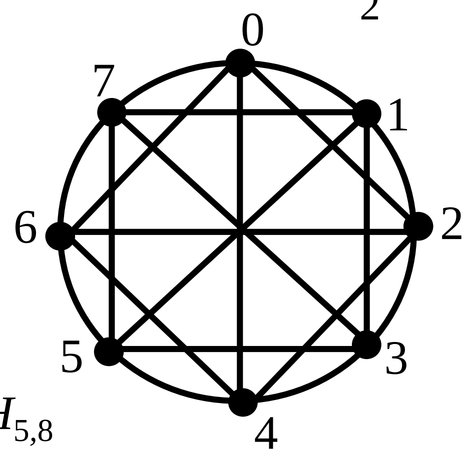




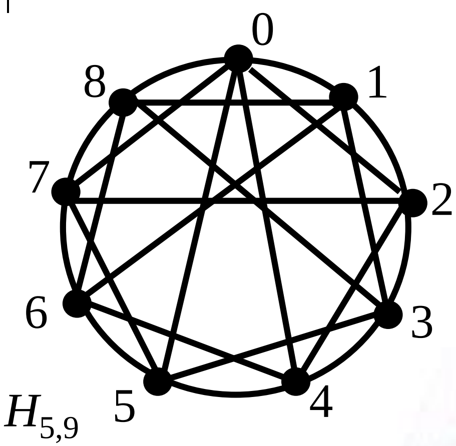
Harary's Construction of an Optimal k-Connected Graph

- ***Case* 2: *k* odd and *n* even:** Let $k = 2r + 1$. Start with graph $H_{2r,n}$, and add the $\frac{n}{2}$ diameters of the original n-cycle. That is, an edge is drawn between vertices i and $i + \frac{n}{2}$, for $i = 0, \ldots, \frac{n}{2} - 1$. Thus, the total number of edges in $H_{2r+1,n}$ equals $rn + \frac{n}{2} = \frac{(2r+1)n}{2} = \frac{kn}{2} = \left\lceil \frac{kn}{2} \right\rceil$.
Graph $H_{5,8}$, shown in Figure 5.2.8 below, is obtained from $H_{4,8}$ by adding the four diameters.

- ***Case* 3: *k* and *n* both odd:** Let $k = 2r + 1$. Start with graph $H_{2r,n}$, and add $\frac{n+1}{2}$ quasi-diameters as follows. First, draw an edge from vertex 0 to vertex $\frac{n-1}{2}$ and from vertex 0 to vertex $\frac{n+1}{2}$. Then draw an edge from vertex i to vertex $(i + \frac{n+1}{2})$, for $i = 1, \ldots, \frac{n-3}{2}$. The total number of edges in $H_{2r+1,n}$ equals $rn + \frac{n+1}{2} = \frac{(2r+1)n+1}{2} = \left\lceil \frac{kn}{2} \right\rceil$ (since kn is odd).



$H_{5,8}$

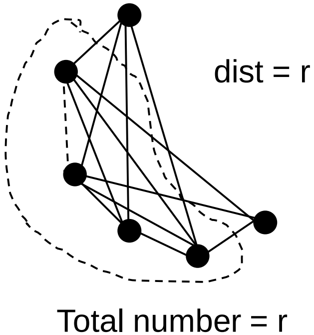


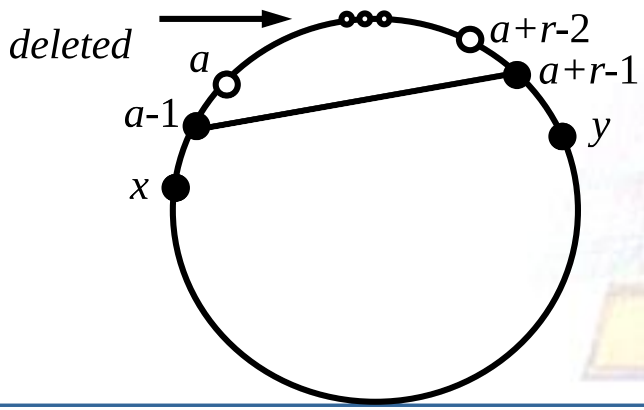
$H_{5,9}$

- **Remark:** An algorithm that realizes the Harary construction appears at the end of this section.

Harary's Construction of an Optimal k-Connected Graph

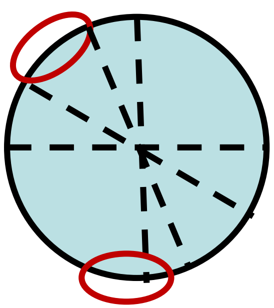
- **Theorem 5.2.7.** *The Harary graph $H_{k,n}$ is k-connected.*
  - ✓ ***Case* 1:** $\underline{k=2r}$. **We want to prove any two vertices x and y are connected after removing $2r-1$ vertices, say $i_1, i_2, \ldots, i_{2r-1}$ (not including x and y), from graph $H_{2r,n}$.**
    - The vertices x and y divide the perimeter of the original n-cycle into two sectors. One of these sectors contains the clockwise sequence of vertices from x to y, and the other sector contains the clockwise sequence of vertices from y to x.
    - **$|x-y| > r$ and at least r vertices between x and y**, or x and y are connected by an edge.
    - We want to prove both sectors are disconnected after removing $2r$-1 vertices.
    - The extreme case is to remove consecutive r and r-1 vertices in two sectors (This extrme case can produce the largest gap between remaining adjacent vertices).
    - But even in this extreme case, the resulting gap is no bigger than r. That is, $|j-i|_n \le r$, for any two consecutive vertices i and j in the sector with r-1 vertices removed.

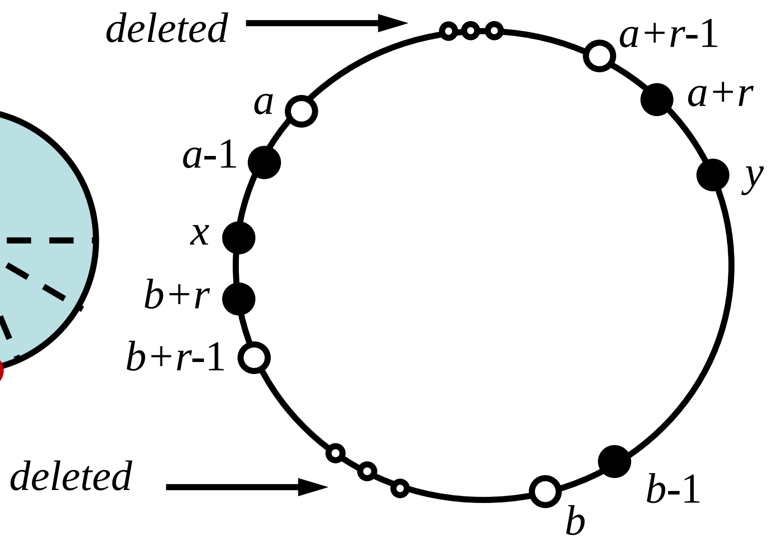


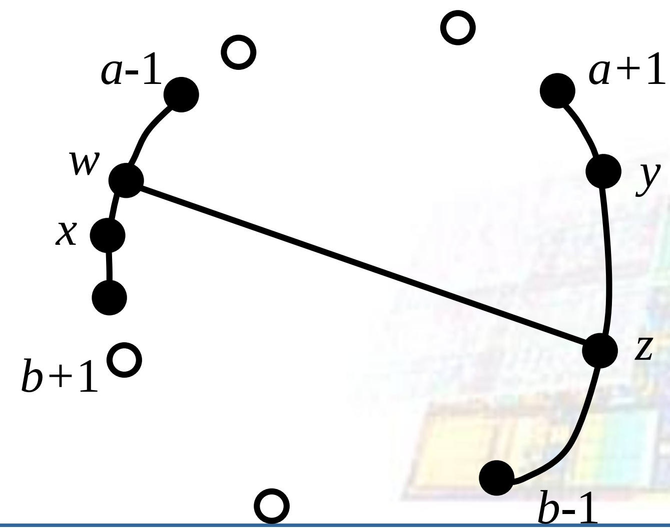


Harary's Construction of an Optimal *k*-Connected Graph

- ✓ ***Case 2***: *$k=2r+1$ and n even*. **We want to prove any two vertices x and y are connected after removing $2r$ vertices in D (not including x and y), from graph $H_{2r+1,n}$.**
  - Suppose D is a set of $2r$ vertices that are deleted from graph $H_{2r+1,n}$.
  - **Both sectors must contain r consecutively deleted vertices,** $A=\{a, a+1,\ldots,a+r-1\}$ for one and $B=\{b, b+1,\ldots,b+r-1\}$ for the other**.**
  - Let $l=|(b+r)-(a-1)|_n$ and let $w=(b+r)+\left\lfloor \frac{l}{2} \right\rfloor$. Then w is "halfway" between vertices $b+r$ and $a-1$, moving clockwise from $b+r$ to $a-1$.
  - Now let $z=w+\frac{n}{2}$. Then z is halfway between $b+r$ and $a-1$, moving counterclockwise from $b+r$ to $a-1$. z is in the other subset of remaining vertices. Moreover, since $|z-w|_n=\frac{n}{2}$, there is an edge joining w and z, by the definition of $H_{2r+1,n}$ for Case 2.







5.2 Constructing Reliable Networks

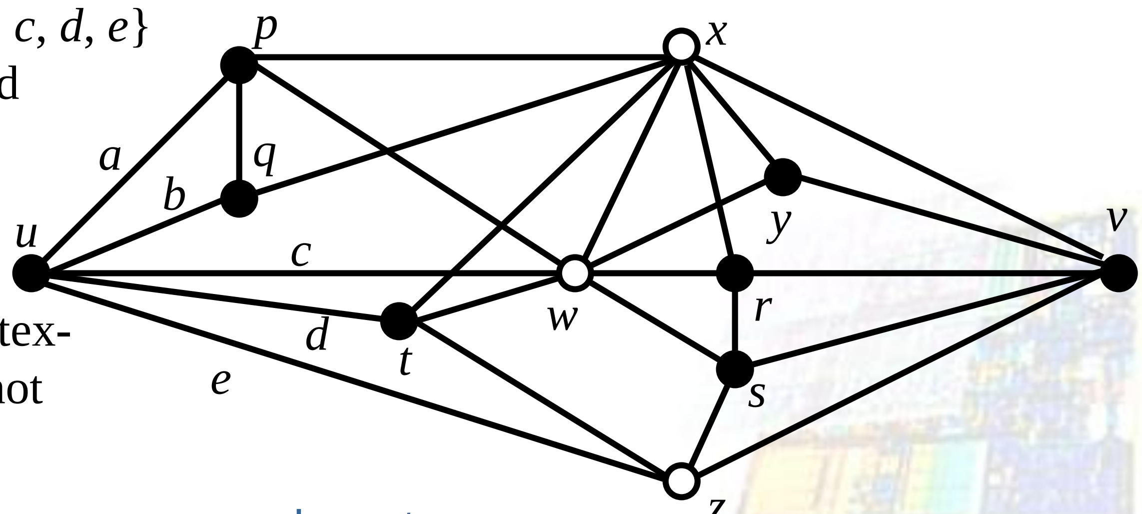
- ✓ ***Case 3*: $k = 2r+1$ and n odd**
  The argument for Case 3 is similar to the one used for Case 2. (Exercises)

- **Corollary 5.2.8** *The Harary graph $H(k,n)$ is a k-edge-connected, n-vertex graph with the fewest possible edges.*

5.3 Max-Min Duality and Menger's Theorems

- ***Primal-dual pair of optimization problems*** – One problem involves minimizing some objective function while the other involves maximizing other objective function.
  - ✓ ***Weak duality*:** A feasible solution of one offers the bound for the optimal value of the other optimization problem.
  - ✓ ***Strong duality*:** The optimal value of one problem is equal to the optimal value of the other.
- **DEFINITION**: Let u and v be distinct vertices in a connected graph G. A vertex subset(or edge subset) S is u-v ***separating***(or ***separate****s* u and v), if the vertices u and v lie in different components of the deletion subgraph G-S.
- **Example 5.3.1:** $\{x, w, z\}$ and $\{a, b, c, d, e\}$ are u-v separating set of vertices and edges of minimum size.

Note: A minimum u-v separating set of edges or vertices need not be a minimum-size edge-cut or vertex-cut. For instance, $\{a, b, c, d, e\}$ is not a minimum-size edge-cut.



<u, p, w, r, x, v> and <u, t, z, s, v>

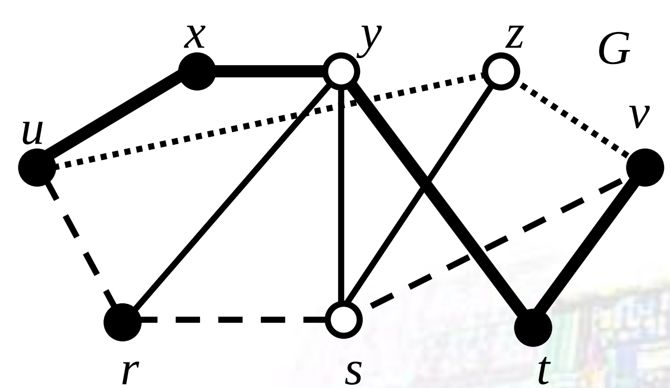
A Primal-Dual Pair of Optimization Problems

- **Connectivity Problems** – we can discuss this problem in terms of the number of internal disjoint paths and the number of vertices needed to separate two vertices, respectively.
  - ✓ The problem of using internal disjoint paths (P_{dis}) is the subpath of a path in P_{dis} also connect two vertices after removing one vertex of the path, decreasing the number of identified internally disjoint paths.
    - ➢ Ex. <*u*, *p*, *w*, *r*, *x*, *v*> and <*u*, *t*, *z*, *s*, *v*> are two internally disjoint paths. When we remove vertex *w*, vertices *u* and *v* are still connected by a smaller path <*u*, *p*, *x*, *v*>.
  - ✓ ***Maximization problem***: Determine the maximum number of internally disjoint *u*-*v* paths.
  - ✓ ***Minimization problem***: Determine the minimum number of vertices needed to separate the vertices *u* and *v*.
- When the number of internally disjoint paths is maximum and the number of vertices to separate two vertices is minimum, these two values are the same, proved by *Menger*.
- **Proposition 5.3.1.** (*Weak Duality*) *Let u and v be any two non-adjacent vertices of a connected graph G. Let P_{uv} be a collection of internally disjoint u-v paths in G, and let S_{uv} be a u-v separating set of vertices in G. Then $|P_{uv}| \leq |S_{uv}|$.*
  - ✓ Since S_{uv} is a *u*-*v* separating set, each *u*-*v* path in P_{uv} must include at least one vertex of S_{uv}. Since the paths in P_{uv} are internally disjoint, no two of them can include the same vertex (exclude the case of larger than S_{uv}). Thus, the number of internally disjoint *u*-*v* paths in *G* is at most $|S_{uv}|$ (by the pigeonhole principle).

A Primal-Dual Pair of Optimization Problems

- **Corollary 5.3.2.** *Let u and v be any two non-adjacent vertices of a connected graph G. Then the maximum number of internally disjoint u-v paths in G is less than or equal to the minimum size of a u-v separating set of vertices in G.*
- **Corollary 5.3.3.[*Certificate of Optimality*]** *Let u and v be any two non-adjacent vertices of a connected graph G. Suppose that P_{uv} is a collection of internally disjoint u-v paths in G, and that S_{uv} is a u-v separating set of vertices in G, such that $|P_{uv}|=|S_{uv}|$. Then P_{uv} is a maximum-size collection of internally disjoint u-v paths, and S_{uv} is a minimum-size u-v separating set.*
- **Example 5.3.2:**

  u-*v* separating set of size 3: {*y*, *r*, *z*}, {*y*, *s*, *z*}, {*s*, *t*, *z*}
  Internally disjoint path set:
  {<*u*, *x*, *y*, *r*, *s*, *z*, *v*>}, {<*u*, *x*, *y*, *s*, *v*>, <*u*, *z*, *v*>},
  {< *u*, *x*, *y*, *t*, *v*>, <*u*, *r*, *s*, *z*, *v*>}, {<*u*, *x*, *y*, *t*, *v*>, <*u*, *z*, *v*>,
  <*u*, *r*, *s*, *v*>}



- **Theorem 5.3.4 [*Menger's Theorem*].** *Let u and v be distinct, non-adjacent vertices in a connected graph G. Then the maximum number of internally disjoint u-v paths in G equals the minimum number of vertices needed to separate u and v.*

Variations and Consequences of Menger's Theorem

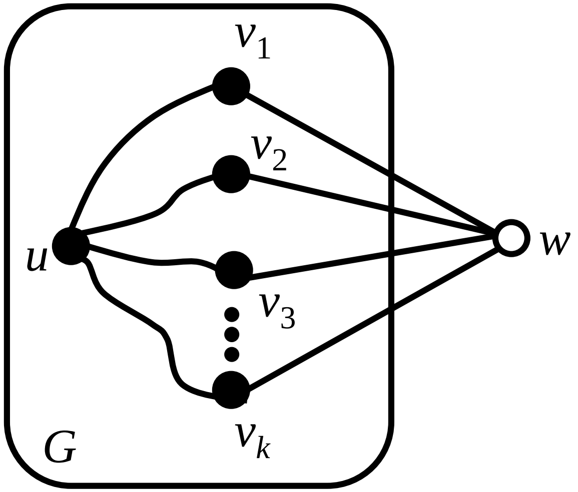
- The vertex connectivity of a graph can be expressed in terms of the *local connectivity* between a given pair of vertices.
- **DEFINITION**: Let s and t be non-adjacent vertices of a connected graph G. Then the ***local vertex-connectivity*** between s and t, denoted $\kappa_v(s,t)$, is the size of a smallest s-t separating vertex set in G.
- **Lemma 5.3.5.** *Let G be a connected graph containing at least one pair of non-adjacent vertices. Then the vertex-connectivity* $\kappa_v(G)$ *is the minimum of the local vertex-connectivity* $\kappa_v(s,t)$*, taken over all pairs of non-adjacent vertices s and t.*
  - ✓ First to prove $\kappa_v(G)\leq\kappa_v(s,t)$, then find a $\kappa_v(s,t)$ such that $\kappa_v(s,t) = \kappa_v(G)$, implying $\kappa_v(G) = \mathrm{MIN}\ (\kappa_v(s,t))$.
    - Since each s-t separating vertex set of the graph G is a vertex-cut, it follows that $\kappa_v(G)\leq\kappa_v(s,t)$, for all pairs of non-adjacent vertices s and t. Thus, $\kappa_v(G)$ is less than or equal to the minimum of $\kappa_v(s,t)$ over all non-adjacent s and t.
    - Since G has at least one pair of non-adjacent vertices, then $\kappa_v(G)$ only happens on disconnecting two non-adjacent vertices (because we cannot disconnect two adjacent vertices and if we want to remove vertices until the remaining vertex is 1, we need to remove (n-1) vertices).

Variations and Consequences of Menger's Theorem

- **Theorem 5.3.6 [*Whitney's k-Connected Characterization*].** *A nontrivial graph G is k-connected if and only if for each pair u, v of vertices, there are at least k internally disjoint u-v paths in G.*
  - ✓ If every two vertices in G are adjacent, then the special case of the vertex-connectivity definition applies, and the theorem assertion is immediately true. So assume that G has at least two non-adjacent vertices.
  - ✓ (⇒)If G is k-connected, then there are at least k vertices in any vertex-cut of G. Thus, there are at least k vertices in any u-v separating set. Theorem 5.3.4 ($\kappa_v(u,v)=max|P_{uv}|$) implies that the maximum number of internally disjoint u-v paths is at least k. Hence, there are at least k internally disjoint u-v paths.
  - ✓ (⇐)Conversely, if for each pair of vertices u and v, there are at least k internally disjoint u-v paths, then Proposition 5.3.1 ($|P_{uv}|\leq|S_{uv}|$) implies that $\kappa_v(u,v)\geq k$, for each pair u, v of non-adjacent vertices. Therefore, $\kappa_v(G)\geq k$, by Lemma 5.3.5.

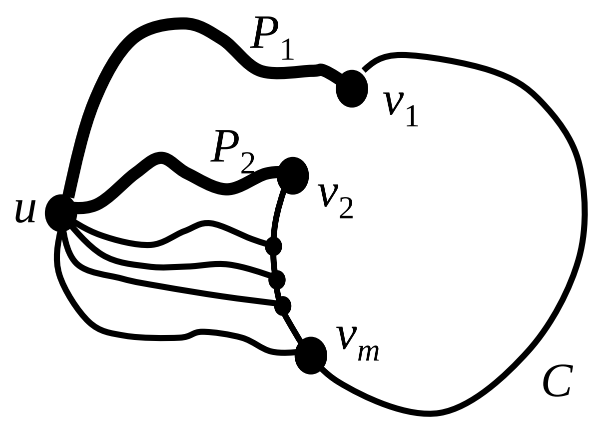
Variations and Consequences of Menger's Theorem

- **Theorem 5.3.7.** *Let G be a k-connected graph and let u, $v_1, v_2, \ldots, v_k$ be any $k+1$ distinct vertices of G. Then there are paths P_i from u to v_i, for $i=1,\ldots,k$, such that the collection $\{P_i\}$ is internally disjoint.*
  - ✓ Construct a new graph G^w from graph G by adding a new vertex w to G together with an edge joining w to v_i, for $i=1,\ldots,k$, as in the following.
  - ✓ Since graph G is k-connected, it follows that graph G^w is also k-connected.
  - ✓ By Theorem 5.3.6, there are k internally disjoint u-w paths in G^w. The u-v_i portions of these paths are k internally disjoint paths in G.



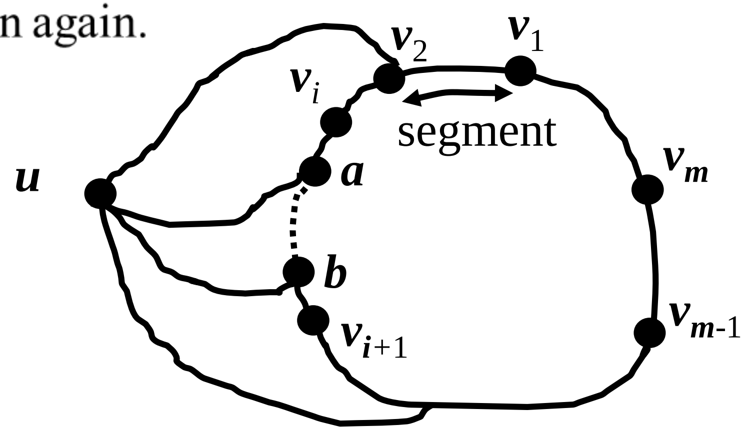
Variations and Consequences of Menger's Theorem

- **Theorem 5.3.8 [*Dirac Cycle Theorem*].** *Let G be a k-connected graph with at least k+1 vertices, for k≥3, and let U be any set of k vertices in G. Then there is a cycle in G containing all the vertices in U.*
  - Let C be a cycle in G that contains the maximum possible number of vertices of set U, and suppose that $U_c = \{v_1,\ldots,v_m\}$ is the subset of vertices of U that lie on C.
  - $k > m$ ($k = m$, proof is done), By Corollary 5.1.8, $m \geq 2$. If there were a vertex $u \in U$ not on cycle C,
  - ***Case* 1:** *cycle C* only contain vertices in U_c. By Corollary 5.3.7, there would exist a set of internally disjoint paths P_i from u to v_i, one for each $i=1,\ldots,m$.
  - But then cycle C could be extended to include vertex u, by replacing the cycle edge between v_1 and v_2 by the paths P_1 and P_2, and this extended cycle would contradict the maximality of cycle C.



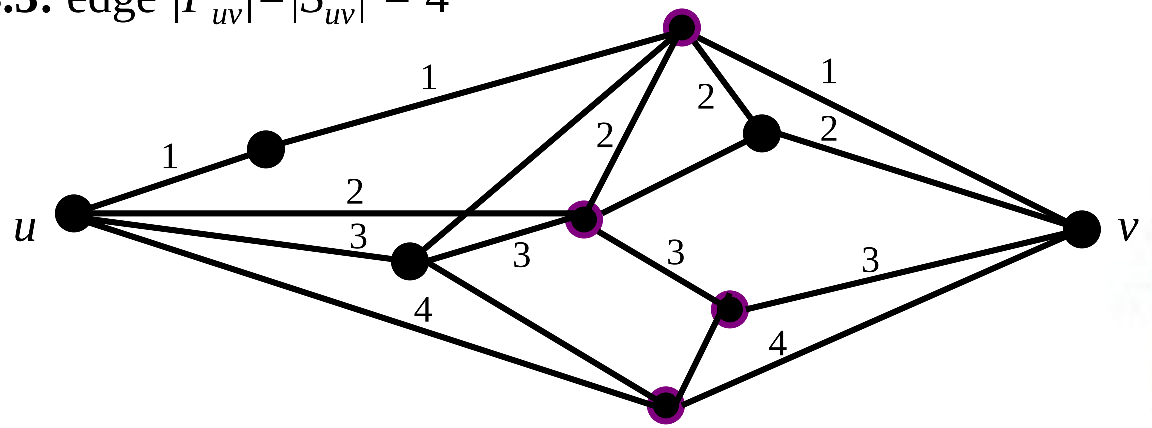
Variations and Consequences of Menger's Theorem

- ✓ ***Case* 2:** Cycle C contains other vertices. In this case, cycle C contains $(m + n)$ vertices and $v'_1, \ldots, v'_n$ are not in U. Cycle C is partitioned into m segments by U_c.
  - ✓ By Corollary 5.3.7, we can obtain $min\{k, m{+}n\}$ internally disjoint paths from u to the cycle (any vertices on the cycle C).
  - ✓ Notice that we can take these internally disjoint paths so that they only touch the cycle once by just cutting the path so it ends at the first vertex of the cycle if they touch it twice or more.
  - ✓ By pigeonhole principle ($min\{k, m{+}n\} > m$), at least two internally disjoint paths connects to the vertices on cycle C within the same segment, say a and b. Now all we have to do is to start at u and go to vertex a, after that go around the cycle touching every vertex in the set U_c $(v_1, \ldots, v_m)$ until you get back to b. After this go back from b to u using the path. Contradiction again.



Analogues of Menger's Theorem

- **Proposition 5.3.9 [*Edge Form of Certificate of Optimality*].** *Let u and v be any two vertices of a connected graph G. Suppose P_{uv} is a collection of edge-disjoint u-v paths in G, and S_{uv} is a u-v separating set of edges in G, such that $|P_{uv}|=|S_{uv}|$. Then P_{uv} is the largest possible collection of edge-disjoint u-v paths, and S_{uv} is the smallest possible u-v separating set. In other words, each is an optimal solution to its respective problem.*
- **Theorem 5.3.10 [*Edge Form of Menger's Theorem*].** *Let u and v be any two distinct vertices in a graph G. Then the minimum number of edges of G needed to separate u and v equals the maximum size of a set of edge-disjoint u-v paths in G.*
- **Theorem 5.3.11 [*Whiteney's k-edge-connected Characterization*].** *A nontrivial graph G is k-edge-connected if and only if for every two distinct vertices u and v of G, there are at least k edge-disjoint u-v paths in G.*
- **Example 5.3.3:** edge $|P_{uv}|=|S_{uv}| = 4$

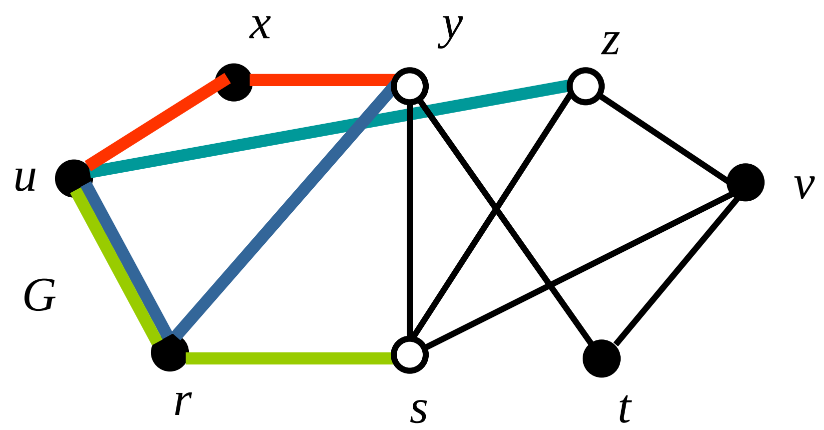


Proof of Menger's Theorem 5.3.4

- **DEFINITION**: Let *W* be a set of vertices in a graph *G* and *x* another vertex not in *W*. A ***strict*** *x*-*W* ***path*** is a path joining vertex *x* to a vertex in *W* and containing no other vertex of *W*. A ***strict*** *W*-*x* ***path*** is the reverse of a strict *x*-*W* path (i.e., its sequence of vertices and edges is in reverse order).

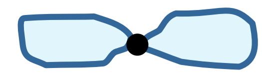
- **Example 5.3.4:**

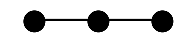
  W={*y*, *s*, *z*}, four strict *u*-*W* paths:
  <*u*, *x*, *y*>, <*u*, *r*, *y*>, <*u*, *r*, *s*>, <*u*, *z*>



Proof of Menger's Theorem 5.3.4

- (1). The proof uses induction on the number of edges. (2). The smallest graph that satisfies the premises of the theorem is the path graph from u to v of length 2 (total three vertices), and the theorem is trivially true for this graph. (3). Assume that the theorem is true for all connected graphs having fewer than m edges, for some $m \geq 3$.





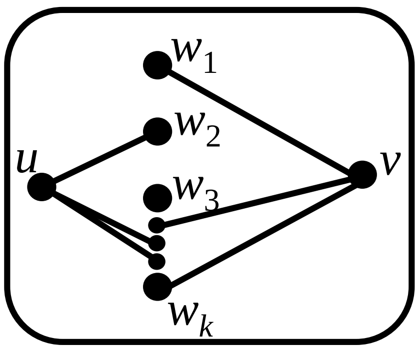
- Now suppose that G is a connected graph with m edges, and let k be the minimum number of vertices needed to separate the vertices u and v. By Corollary 5.3.2, it suffices to show that there exist k internally disjoint u-v paths in G. Since this is clearly true if k=1 (since G is connected), we may assume that $k \geq 2$.

- **Assertion 5.3.4a:** If G contains a u-v path of length 2, then G contains k internally disjoint u-v paths.
  - ✓ Suppose that $P=\langle u, e_1, x, e_2, v\rangle$ is a path in G of length 2. Let W be a smallest u-v separating set for the vertex-deletion subgraph G-x. Since $W \cup \{x\}$ is a u-v separating set for G, the minimality of k implies that $|W| \geq k$-1. By the induction hypothesis, there are at least k-1 internally disjoint u-v paths in G-x. Path P is internally disjoint from any of these, and, hence, there are k internally disjoint u-v paths in G.

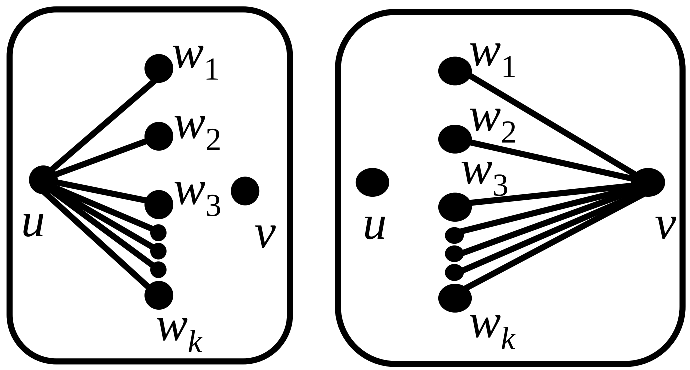
- We just complete the proof for length-2 path. Next are two cases as $dis(u,v) \geq 3$.

Proof of Menger's Theorem 5.3.4

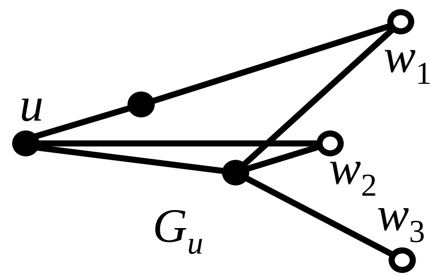
- In Case 1, there exists a minimal (size k) u-v separating set W, where <u>neither u nor v is adjacent to every vertex of W</u>. In Case 2, no such separating set exists. Thus, in every u-v separating set for Case 2, <u>either every vertex is adjacent to u or every vertex is adjacent to v.</u>
  - ✓ *Case 1*: Let G_u be the subgraph induced on the union of the edge-sets of all strict u-W paths in G, and let G_v be the subgraph induced on the union of edge-sets of all strict W-v paths.

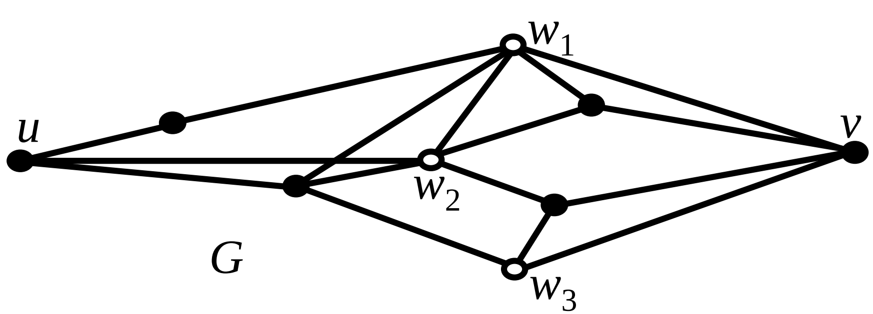


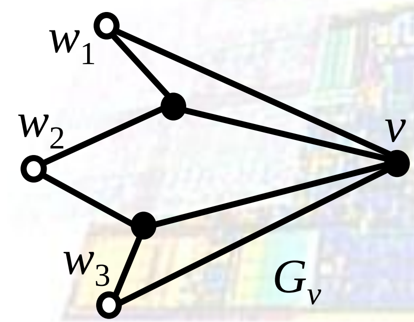
Case 1



Case 2

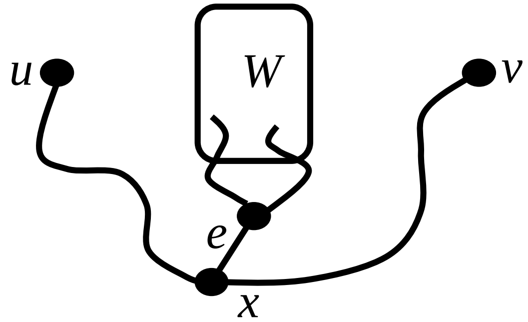


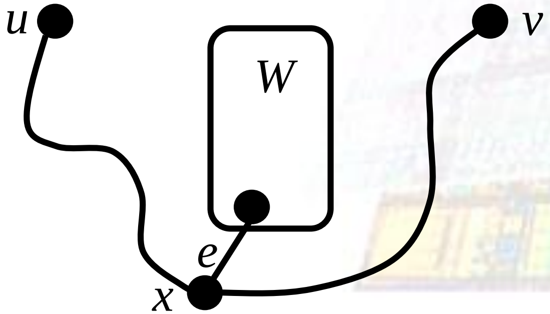




Proof of Menger's Theorem 5.3.4

- **Assertion 5.3.4b:** Both of the subgraph G_u and G_v have more than k edges.
  - ✓ For each $w_i \in W$, there is a u-v path P_{w_i} in G on which w_i is the only vertex of W (or, W-$\{w_i\}$ would still be a u-v separating set, contradicting the minimality of W). The u-w_i subpath of P_{w_i} is a strict u-W path that ends at w_i. Hence, G_u has at least k edges.
  - ✓ The only way G_u could have exactly k edges would be if each of these strict u-W paths consisted of a single edge joining u and w_i, $i = 1,\ldots,k$. But this is ruled out by the condition for Case 1. Therefore, G_u has more than k edges. A similar argument shows that G_v also has more than k edges.
- **Assertion 5.3.4c:** The subgraphs G_u and G_v have no edges in common.
  - ✓ By way of contradiction, suppose that the subgraphs G_u and G_v have an edge e in common. By the definitions of G_u and G_v, edge e is an edge of both a strict u-W path and a strict W-u path. Hence, at least one of its endpoints, say x, is not a vertex in the u-v separating set W,implying the existence of a u-v path in G-W.



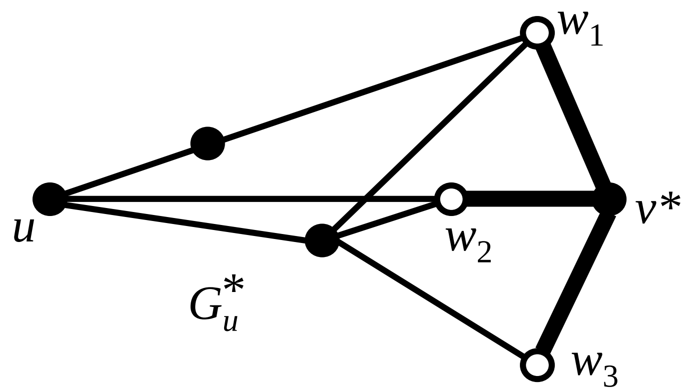


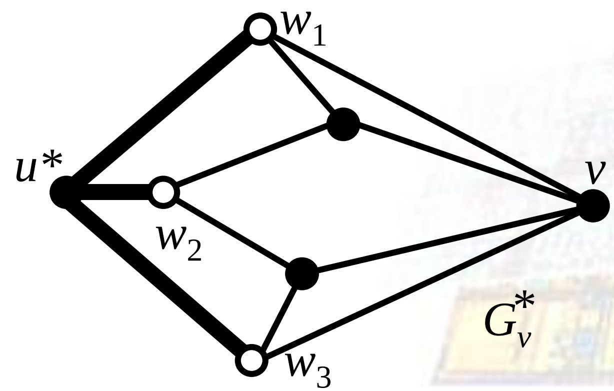
Proof of Menger's Theorem 5.3.4

- We now define two auxiliary graph $G_u{}^*$ and $G_v{}^*$: $G_u{}^*$ is obtained from G by replacing the subgraph G_v with a new vertex v^* and drawing an edge from each vertex in W to v^*.
- **Assertion 5.3.4d:** Both of the auxiliary graphs $G_u{}^*$ and $G_v{}^*$ have fewer edges than G.
  - ✓ The following chain of inequalities shows that graph $G_u{}^*$ has fewer edges than G.

$$
\begin{aligned}
|E_G| &\geq |E_{G_u \cup G_v}| && \text{(since } G_u \cup G_v \text{ is a subgraph of } G\text{)} \\
&= |E_{G_u}| + |E_{G_v}| && \text{(by Assertion 5.3.4c)} \\
&> |E_{G_u}| + k && \text{(by Assertion 5.3.4b)} \\
&= |E_{G_u^*}| && \text{(by the construction of } G_u^*\text{)}
\end{aligned}
$$

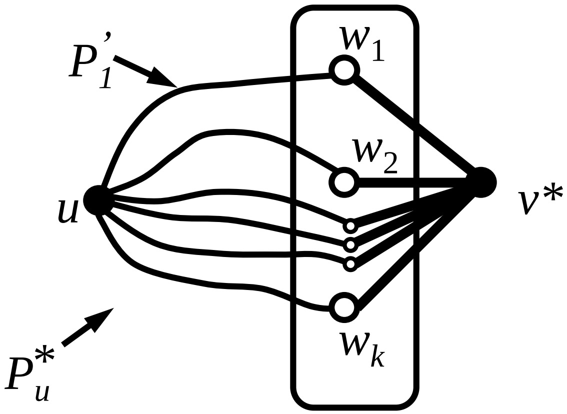
A similar argument shows that $G_v{}^*$ also has fewer edges than G.

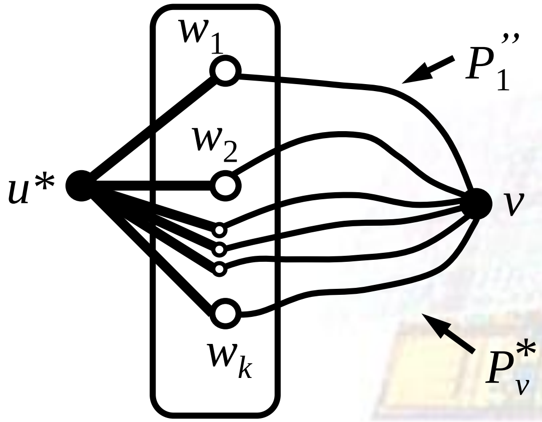




Proof of Menger's Theorem 5.3.4

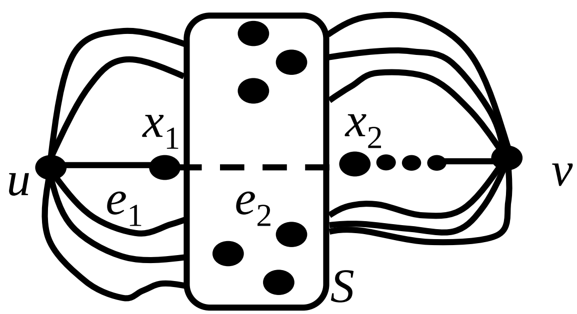
- By the construction of graphs G_u^* and G_v^*, every u-v^* separating set in graph G_u^* and every u^*-v separating set in graph G_v^* is a u-v separating set in graph G. Hence, the set W is a smallest u-v^* separating set in graph G_u^* and a smallest u^*-v separating set in G_v^*. Since G_u^* and G_v^* have fewer edges than G, the induction hypothesis implies the existence of two collections, P_u^* and P_v^*, of k internally disjoint u-v^* paths in G_u^* and k internally disjoint u^*-v paths in G_v^*, respectively .
- Let P_i be the concatenation of paths P_i' and P_i'', for i=1,…,k. Then the set $\{P_i\}$ is a collection of k internally disjoint u-v paths in G.
- *Case 2*: Suppose that for each u-v separating set of size k, one of the vertices u or v is adjacent to all the vertices in that separating set.





Proof of Menger's Theorem 5.3.4

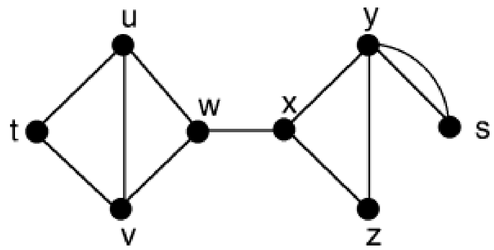
- Let $P=<u, e_1, x_1, e_2, x_2,\ldots,v>$ be a shortest u-v path in G. $|P| \geq 3$ (Cases 1 and 2 are discussing $dis(u,v)\geq 3$) and that vertex x_1 is not adjacent to vertex v. By Proposition 5.1.3, the edge-deletion subgraph G-e_2 is connected. Let S be a smallest u-v separating set in the edge-deletion subgraph G-e_2. Then S is u-v separating set in the vertex-deletion subgraph $G - x_1$ (since G-x_1 is a subgraph of G-e_2).Thus, $S\cup\{x_1\}$ is a u-v separating set in G, which implies that $|S|\geq k$-1, by minimality of k. On the other hand, the minimality of $|S|$ in G-e_2 implies that $|S|\leq k$, since every u-v separating set in G is also a u-v separating set in G-e_2.
- If $|S|=k$, then, by the induction hypothesis, there are k internally disjoint u-v paths in G-e_2 and, hence, in G. If $|S|=k$-1, then x_1 and $x_2 \notin S$ (otherwise S is also a u-v separating set in G too, contradicting the minimality of k). Thus, the sets $S\cup\{x_1\}$ and $S\cup\{x_2\}$ are both of size k and both u-v separating sets of G. The condition for Case 2 and the fact that vertex x_1 is not adjacent to v imply that every vertex in S is adjacent to vertex u. Hence, no vertex in S is adjacent to v (lest there be a u-v path of length 2). But then the condition for Case 2 applied to $S \cup\{x_2\}$ implies that vertex x_2 is adjacent to vertex u. which contradicts the minimality of path P and completes the proof.



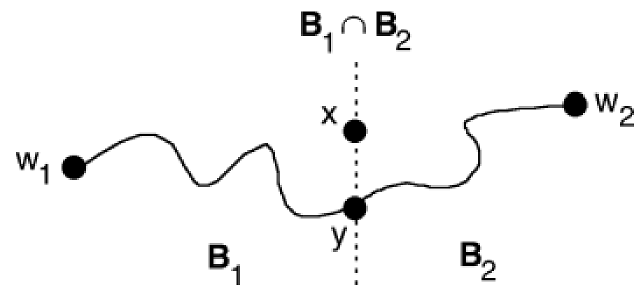
5.4 Block Decompositions

- **DEFINITION**: A ***block*** of a loopless graph is a maximal connected subgraph H such that no vertex of H is a cut-vertex of H.
- **Example 5.4.1:**

  Four blocks: $\{t,u,w,v\}$, $\{w,x\}$, $\{x,y,z\}$, $\{y,s\}$

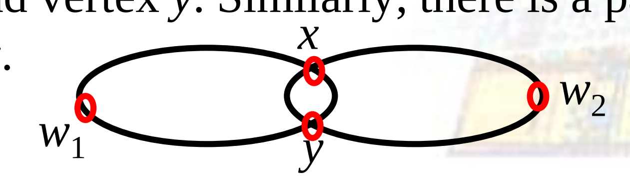


A graph with four blocks.



Two blocks have at most one common vertex.

- **Proposition 5.4.1.** *Every nontrivial connected graph G contains two or more vertices that are not cut-vertices.*
  - ✓ Choose two 1-valent vertices of a spanning tree of graph G.
- **Proposition 5.4.2.** *Two different blocks of a graph can have at most one vertex in common.*
  - ✓ Let B_1 and B_2 be two different blocks of a graph G, and suppose that x and y are vertices in $B_1 \cap B_2$. We want to prove $B_1 \cup B_2$ has no cut vertex $\rightarrow$ contradiction!
  - ✓ Since the vertex-deletion subgraph $B_1 - x$ is a connected subgraph of B_1, there is a path in $B_1 - x$ between any given vertex $w_1 \in B_1 - x$ and vertex y. Similarly, there is a path in $B_2 - x$ from vertex y to any given vertex $w_2 \in B_2 - x$.



5.4 Block Decompositions

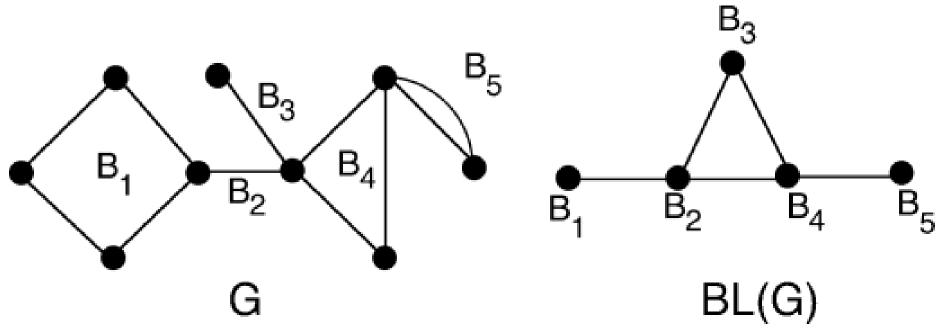
- ✓ The concatenation of these two paths is a $w_1 - w_2$ walk in the vertex-deletion subgraph $(B_1 \cup B_2) - x$, which shows that x is not a cut-vertex of the subgraph $B_1 \cup B_2$. This also holds for other vertex in $B_1 \cap B_2$.
- ✓ Moreover, none of the vertices that are in exactly one of B_1 and B_2 is a cut-vertex of $B_1 \cup B_2$, since such a vertex would be a cut-vertex of that block B_1 or B_2.
- ✓ Thus, the subgraph $B_1 \cup B_2$ has no cut-vertex, which contradicts the maximality of blocks B_1 and B_2.

- **Corollary 5.4.3.** *The edge-sets of the blocks of a graph G partition E_G.*
- **Corollary 5.4.4.** *Let x be a vertex in a graph G. Then x is a cut-vertex of G if and only if x is in two different blocks of G.*
- **Corollary 5.4.5.** *Let B_1 and B_2 be distinct blocks of a connected graph G. Let y_1 and y_2 be vertices in B_1 and B_2, respectively, such that neither is a cut-vertex of G. Then vertex y_1 is not adjacent to vertex y_2.*

- **DEFINITION**: The ***block graph*** of a graph G, denoted $BL(G)$, is the graph whose vertices correspond to the blocks of G, such that two vertices of $BL(G)$ are joined by a single edge whenever the corresponding blocks have a vertex in common.

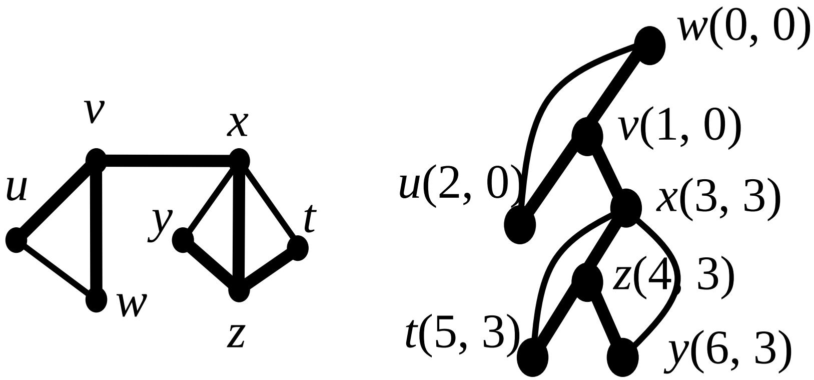
5.4 Block Decompositions

- **Example 5.4.2:**



A graph and its block graph.

- **DEFINITION**: A ***leaf block*** of a graph *G* is a block that contains exactly one cut-vertex of *G*.
- **Proposition 5.4.6.** *Let G be a connected graph with at least one cut-vertex. Then G has at least two leaf blocks.*
- **Algorithm 5.4.1:** Finding blocks.



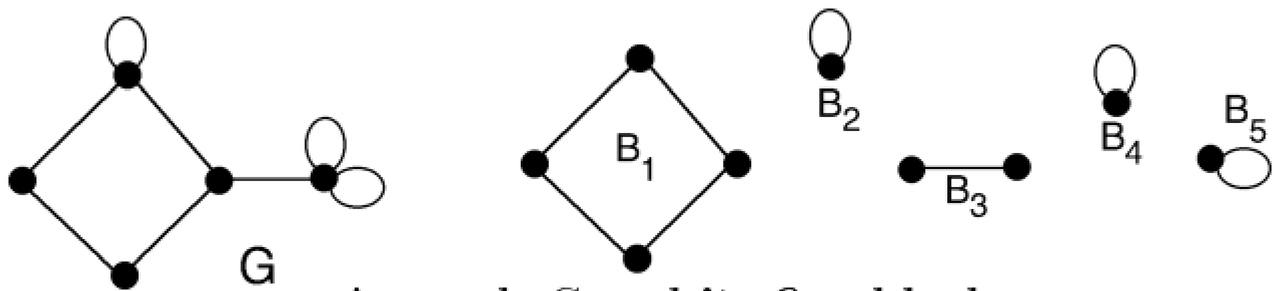
Algo 5.4.1: Block-Finding

Input: a connected graph G.
Output: the vertex-sets $B_1, B_2, \ldots, B_l$ of the blocks of G.

Apply Algo 4.4.3 to find the set K of cutpts of graph G.
Initialize the block counter $i := 0$.
For each cutpt v in set K (in order of decr $dfnumber$)
  For each child w of v in depth-first search tree T
    If $low(w) \geq dfnumber(v)$
      Let T^w be the subtree of T rooted at w.
      $i := i + 1$
      $B_i := V_{T^w} \cup \{v\}$
      $T := T - V_{T^w}$
Return sets $B_1, B_2, \ldots, B_i$.

Block Decomposition of Graphs With Self-Loops

- In a graph with self-loops, each self-loop and its endpoint are regarded as a distinct block, isomorphic to the bouquet B_1.
- **Example 5.4.3:**



A graph G and its five blocks.