1.1.14 Draw a simple graph with the given degree sequence.
a. $\langle 6,4,4,3,3,2,1,1\rangle$ b. $\langle 5,5,5,3,3,3,3,3\rangle$

For each of the number sequences in Exercises 1.1.15 through 1.1.18, either draw a simple graph that realizes it, or explain, without resorting to Corollary 1.1.7 or Algorithm 1.1.1, why no such graph can exist.

1.1.15[S] a. $\langle 2,2,1,0,0\rangle$ b. $\langle 4,3,2,1,0\rangle$
1.1.16 a. $\langle 4,2,2,1,1\rangle$ b. $\langle 2,2,2,2\rangle$
1.1.17 a. $\langle 4,3,2,2,1\rangle$ b. $\langle 4,3,3,3,1\rangle$
1.1.18 a. $\langle 4,4,4,4,3,3,3,3\rangle$ b. $\langle 3,2,2,1,0\rangle$

1.1.19 Apply Algorithm 1.1.1 to each of the following sequences to determine whether it is graphic. If the sequence is graphic, then draw a simple graph that realizes it.
a. $\langle 7,6,6,5,4,3,2,1\rangle$ b. $\langle 5,5,5,4,2,1,1,1\rangle$
c. $\langle 7,7,6,5,4,4,3,2\rangle$ d. $\langle 5,5,4,4,2,2,1,1\rangle$

? **1.1.20** Use Theorem 1.1.6 to prove Corollary 1.1.7.

1.1.21 Write an iterative version of Algorithm 1.1.1 that applies Corollary 1.1.7 repeatedly until a sequence of all zeros or a sequence with a negative term results.

1.1.22[S] Given a group of nine people, is it possible for each person to shake hands with exactly three other people?

1.1.23 Draw a graph whose degree sequence has no duplicate terms.

1.1.24[S] What special property of a function must the *endpts* function have for a graph to have no multi-edges?

1.1.25 Draw a digraph for each of the following indegree and outdegree sequences, such that the indegree and outdegree of each vertex occupy the same position in both sequences.
a. in: $\langle 1,1,1\rangle$ out: $\langle 1,1,1\rangle$
b. in: $\langle 2,1\rangle$ out: $\langle 3,0\rangle$

DEFINITION: A pair of sequences $\langle a_1,a_2,\ldots,a_n\rangle$ and $\langle b_1,b_2,\ldots,b_n\rangle$ is called **digraphic** if there exists a simple digraph with vertex-set $\{v_1,v_2,\ldots,v_n\}$ such that $outdegree(v_i)=a_i$ and $indegree(v_i)=b_i$ for $i=1,2,\ldots,n$.

1.1.26 Determine whether the pair of sequences $\langle 3,1,1,0\rangle$ and $\langle 1,1,1,2\rangle$ is digraphic.

1.1.27 Establish a result like Corollary 1.1.7 for a pair sequences to be digraphic.

1.1.28 How many different degree sequences can be realized for a graph having three vertices and three edges?

1.1.29 Given a list of three vertices and a list of seven edges, show that 3^7 different formal specifications for simple graphs are possible.

1.1.30 Given a list of four vertices and a list of seven edges, show that $\binom{7}{2}5^6 2^{10}$ different formal specifications are possible if there are exactly two self-loops.

1.1.31 Given a list of three vertices and a list of seven edges, how many different formal specifications are possible if exactly three of the edges are directed?

1.1.32[S] Does there exist a simple graph with five vertices, such that every vertex is incident with at least one edge, but no two edges are adjacent?

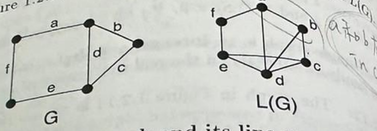
Line Graphs

Line graphs are a special case of ...

DEFINITION: The ***line graph*** $L(G)$ of a graph G has a vertex for each edge ..., two vertices in $L(G)$ are adjacent if and only if the corresponding edges ... vertex in common.

Thus, the line graph $L(G)$ is the intersection graph corresponding to the ... of the edges of G.

Example 1.2.13: Figure 1.2.16 shows a graph G and its line graph $L(G)$.



a和b相 In Com

Figure 1.2.16 A graph and its line graph.

EXERCISES for Section 1.2

1.2.1[S] Find the number of edges for each of the following graphs.
a. K_n b. $K_{m,n}$

1.2.2 What is the maximum possible number of edges in a simple bipartite graph on m vertices?

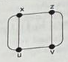
1.2.3 Draw the smallest possible non-bipartite graph.

1.2.4[S] Determine the values of n for which the given graph is bipartite.
a. K_n b. C_n c. P_n

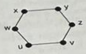
1.2.5 Draw a 3-regular bipartite graph that is not $K_{3,3}$.

In Exercises 1.2.6 and 1.2.7, determine whether the given graph is bipartite. In each case, give a vertex bipartition or explain why the graph is not bipartite.

1.2.6[S]



1.2.7



1.2.8 Label the vertices of the cube graph in Figure 1.2.5 with 3-digit binary strings so that the labels on adjacent vertices differ in exactly one digit.

1.2.9 For each of the platonic graphs, is it possible to trace a *tour* of all the vertices by starting at one vertex, traveling only along edges, never revisiting a vertex, and never lifting the pen off the paper? Is it possible to make the tour return to the starting vertex?

1.2.10[S] Prove or disprove: There does not exist a 5-regular graph on 11 vertices.

DEFINITION: A ***tournament*** is a digraph whose underlying graph is a complete graph. every pair of vertice is joined by an edge

1.2.11 a. Draw all the 3-vertex tournaments whose vertices are u, v, x.
b. Determine the number of 4-vertex tournaments whose vertices are u, v, x, y.

1.2.12 Prove that every tournament has at most one vertex of indegree 0 and at most one vertex of outdegree 0.

1.2.13[S] Suppose that n vertices $v_1, v_2, \ldots, v_n$ are drawn in the plane. How many different n-vertex tournaments can be drawn on those vertices?

1.2.14 Chartrand and Lesniak [ChLe04] define a pair of sequences of nonnegative integers $\langle a_1, a_2, \ldots, a_r\rangle$ and $\langle b_1, b_2, \ldots, b_t\rangle$ to be **bigraphical** if there exists a bipartite graph G with bipartition subsets $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_t\}$ such that $deg(u_i) = a_i$, $i = 1, 2, \ldots, r$, and $deg(w_i) = b_i$, $i = 1, 2, \ldots, t$. Prove that a pair of non-increasing sequences of nonnegative integers $\langle a_1, a_2, \ldots, a_r\rangle$ and $\langle b_1, b_2, \ldots, b_t\rangle$ with $r \geq 2$, $0 < a_1 \leq t$, and $0 < a_1 \leq r$ is bigraphical if and only if the pair $\langle a_2, \ldots, a_r\rangle$ and $\langle b_1 - 1, b_2 - 1, \ldots, b_{a_1} - 1, b_{a_1+1}, b_{a_1+2}, \ldots, b_t\rangle$ is bigraphical.

1.2.15 Find all the 4-vertex circulant graphs.

1.2.16 Show that each of the following graphs is a circulant graph.

a.

b.

c.

√ **1.2.17** State a necessary and sufficient condition on the positive integers n and k for $circ(n : k)$ to be the cycle graph C_n.

1.2.18 Find necessary and sufficient conditions on the positive integers n and k for $circ(n : k)$ to be the graph consisting of $n/2$ mutually non-adjacent edges.

1.2.19 Determine the size of a smallest dominating set (defined in §1.1 exercises) in the graph indicated.

a. K_n b. $K_{m,n}$ c. C_n d. P_n e. CL_n

1.2.20 Determine the size of a smallest vertex cover (defined in §1.1 exercises) in the graph indicated.

a. K_n b. $K_{m,n}$ c. C_n d. P_n e. CL_n

1.4.13 Is there an even-length walk between two antipodal vertices (i.e., endpoints of a long diagonal) of a cube?

1.4.14 Draw an 8-vertex connected graph with as few edges as possible.

1.4.15[S] Draw a 7-vertex connected graph such that the removal of any one edge results in a non-connected graph.

1.4.16 Draw an 8-vertex connected graph with no closed walks, except those that retrace at least one edge.

1.4.17 Draw a 5-vertex connected graph that remains connected after the removal of any two of its vertices.

In Exercises 1.4.18 through 1.4.28, determine the diameter, radius, and central vertices of the graph indicated.

1.4.18[S] The graph in Exercise 1.4.1.

1.4.19 The graph in Exercise 1.4.2.

1.4.20 Path graph P_n, $n \geq 3$ (from §1.2).

1.4.21 Cycle graph C_n, $n \geq 4$ (from §1.2).

1.4.22 Complete graph K_n, $n \geq 3$.

1.4.23 Complete bipartite graph $K_{m,n}$, $m \geq n \geq 3$ (from §1.2).

1.4.24 The Petersen graph (from §1.2).

1.4.25 Hypercube graph Q_3; can you generalize to Q_n? (from §1.2).

1.4.26 Circular ladder graph CL_n, $n \geq 4$ (from §1.2).

1.4.27 Dodecahedral graph (from §1.2).

1.4.28 Icosahedral graph (from §1.2).

1.4.29 Determine the radius and diameter of each of the following circulant graphs (from §1.2).

a. $circ(5 : 1, 2)$; b. $circ(6 : 1, 2)$; c. $circ(8 : 1, 2)$.

1.4.30 Describe the diameter and radius of the circulant graph $circ(n : m)$ in terms of integer n and connection m.

1.4.31 Describe the diameter and radius of the circulant graph $circ(n : a, b)$ in terms of integer n and the connections a and b.

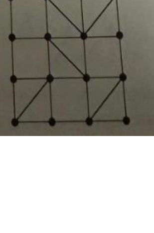
1.4.32[S] Let G be a connected graph. Prove that

$$rad(G) \leq diam(G) \leq 2 \cdot rad(G)$$

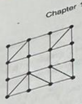
1.4.33 Prove that a digraph must have a closed directed walk if each of its vertices has nonzero outdegree.

In Exercises 1.4.34 through 1.4.36, use a bipartite graph model to determine whether the given rectangular framework is rigid.

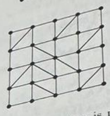
1.4.34[S]



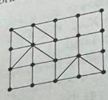
1.4.35



1.4.36



1.4.37 *Show that the framework shown below is not rigid. Can moving one brace make the framework rigid?*



1.4.38[S] Let u and v be any two vertices of a connected graph G. Prove that there exists a u-v walk containing all the vertices of G.

1.4.39 Prove that a shortest walk between two vertices cannot repeat a vertex or an edge.

1.4.40 Let D be a digraph. Prove that mutual reachability is an equivalence relation on V_D.

1.4.41[S] Let x and y be two different vertices in the complete graph K_4. Find the number of x-y walks of length 2 and of length 3.

1.4.42 Let x and y be two different adjacent vertices in the complete bipartite graph $K_{3,3}$. Find the number of x-y walks of length 2 and of length 3.

1.4.43 Let x and y be two different non-adjacent vertices in the complete bipartite graph $K_{3,3}$. Find the number of x-y walks of length 2 and of length 3.

1.4.44 Prove that the distance function d on a graph G satisfies the *triangle inequality*. That is,

$$\text{For all } x, y, z \in V_G, \quad d(x,z) \le d(x,y) + d(y,z)$$

1.4.45 Let G be a connected graph. Prove that if $d(x,y) \ge 2$, then there exists a vertex w such that $d(x,y) = d(x,w) + d(w,y)$.

1.4.46[S] Let G be an n-vertex simple graph such that $deg(v) \ge \frac{(n-1)}{2}$ for every vertex $v \in V_G$. Prove that graph G is connected.

1.5 PATHS, CYCLES, AN

By stripping away its extraneo
to one that has no repetition of vert
simplifies discussion in many situat

Trails and Paths

DEFINITION: A **trail** is a walk
DEFINITION: A **path** is a trail
final vertices).

DEFINITION: A walk, trail,

TERMINOLOGY NOTE: Un
for walks, trails, and pat
walk and *simple path* ins
best to check the termin

Example 1.5.1: Fo
the edge sequence of
$\langle v, a, b, c, d, e, u\rangle$ is a

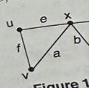


Figure 1

Remark: Dire
analogous to t
directed walk
the context t
will be omit

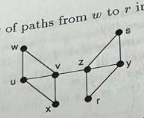
Deleting C

DEFINITIO
closed su
walk W
起终同

Thus,
v_k. O

Exa
wal

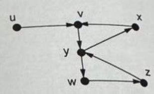
1.5.5[S] Determine the number of paths from w to r in the following graph.



1.5.6 Draw a copy of the Petersen graph (from §1.2) with vertex labels. Then find

a. a trail of length 5.
b. a path length 9.
c. cycles of lengths 5, 6, 8, and 9.

1.5.7 Consider the digraph shown at the right. Find



a. an open directed walk that is not a directed trail.
b. an open directed trail that is not a directed path.
c. a closed directed walk that is not a directed cycle.

1.5.8[S] Give an example of a nontrivial, closed walk that does not contain a cycle.

1.5.9 Let x and y be two different vertices in the complete graph K_5. Find the number of x-y paths of length 2 and of length 3.

1.5.10 Let x and y be two different vertices in the complete graph K_n, $n \ge 5$. Find the number of x-y paths of length 4.

1.5.11[S] Let x and y be two adjacent vertices in the complete bipartite graph $K_{3,3}$. Find the number of x-y paths of length 2, of length 3, and of length 4.

1.5.12 Let x and y be two different non-adjacent vertices in the complete bipartite graph $K_{3,3}$. Find the number of x-y paths of length 2, of length 3, and of length 4.

1.5.13 Let x and y be two adjacent vertices in the complete bipartite graph $K_{n,n}$, $n \ge 3$. Find the number of x-y paths of length 2, of length 3, and of length 4.

1.5.14 Let x and y be two different non-adjacent vertices in the complete bipartite graph $K_{n,n}$, $n \ge 3$. Find the number of x-y paths of length 2, of length 3, and of length 4.

In Exercises 1.5.15 through 1.5.22, determine the girth of the graph indicated.

1.5.15[S] Complete bipartite graph $K_{3,7}$.

1.5.16 Complete bipartite graph $K_{m,n}$, $m \ge n \ge 3$.

1.5.17 Complete graph K_n.

1.5.18 Hypercube graph Q_5. Can you generalize to Q_n?

1.5.19 Circular ladd

1.5.25 Describe the g
and the connections a a

1.5.26 Find necessar
to be hamiltonian.

1.5.27 Determine w

1.5.28 Determine

1.5.29 Determine

1.5.30[S] Prove tha

1.5.31 Prove or

1.5.32[S] Prove th
y is a directed pa

1.5.33 Suppos
a. Prove th
b. Prove t

1.5.34 State

1.5.35 Stat

1.5.36 Sta

1.5.37 Fi
trail in Exa
one of the
trail).

1.5.38
adherenc

1.5.3

1.5.25 Describe the girth of the circulant graph $circ(n : a, b)$ in terms of integer n and the connections a and b.

1.5.26 Find necessary and sufficient conditions for the circulant graph $circ(n : a, b)$ to be hamiltonian.

1.5.27 Determine whether the hypercube graph Q_3 is hamiltonian.

1.5.28 Determine whether the Petersen graph is hamiltonian.

1.5.29 Determine whether the circular ladder graph CL_n, $n \geq 3$, is hamiltonian.

1.5.30[S] Prove that if v is a vertex on a nontrivial, closed trail, then v lies on a cycle.

1.5.31 Prove or disprove: Every closed walk of odd length contains a cycle.

1.5.32[S] Prove that in a digraph, a shortest directed walk from a vertex x to a vertex y is a directed path from x to y.

1.5.33 Suppose G is a simple graph whose vertices all have degree at least k.
a. Prove that G contains a path of length k.
b. Prove that if $k \geq 2$, then G contains a cycle of length at least k.

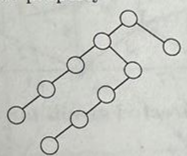
1.5.34 State and prove the digraph version of Theorem 1.5.2.

1.5.35 State and prove the digraph version of Proposition 1.5.5.

1.5.36 State and prove the digraph version of Theorem 1.5.6.

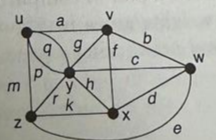
1.5.37 Find a collection of cycles whose edge-sets partition the edge-set of the closed trail in Example 1.5.4 but that is not a decomposition of the closed trail (i.e., at least one of the cycles in the collection is neither a subwalk nor a reduced walk of the closed trail).

1.5.38 a. In the following binary tree, store the ten 3-digit keys of Figure 1.5.11, in adherence to the binary-search-tree property.



b. Is this tree easier or harder to search than the one in Figure 1.5.11?

1.5.39 Find an eulerian tour in the following graph.



1.5.40[S] Describe how to use a rooted tree to represent all possible moves of a game of tic-tac-toe, where each node in the tree corresponds to a different board configuration.

1.5.41 Which of the platonic graphs are bipartite?

1.5.42 Prove that if G is a digraph such that every vertex has positive indegree, then G contains a directed cycle.

1.5.43[S] Prove that if G is a digraph such that every vertex has positive outdegree, then G contains a directed cycle.