5.1.4. 109511028 黃威凱

We have $K_e(G) \geq K_v(G)$, if G has exactly 1 bridge then $K_e(G) = 1 \longrightarrow K_v(G) \leq 1$. So G cannot be 3-connected.

5.1.20.

We have $\sum_{v \in V} \deg(v) = 2|E| = 14$. and $\delta_{min}(G) \geq K_e(G) \geq K_v(G) = 3$.

There must be at least 5 vertices to have 7 edges $\left(\binom{5}{2} = 10, \binom{4}{2} = 6\right)$

Therefore $\sum_{v \in V} \deg(v) \geq \delta_{min}(G) \times 5 = 15 > 14$, which implies there is no 3-connected graph with exactly 7 edges.

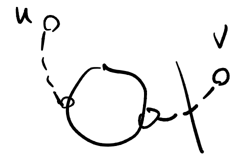
5.1.22

Let u, v be vertices on a unicyclic graph, then:

① u, v both on the cycle: We have to remove 2 cycle edge to make u, v disjoint.

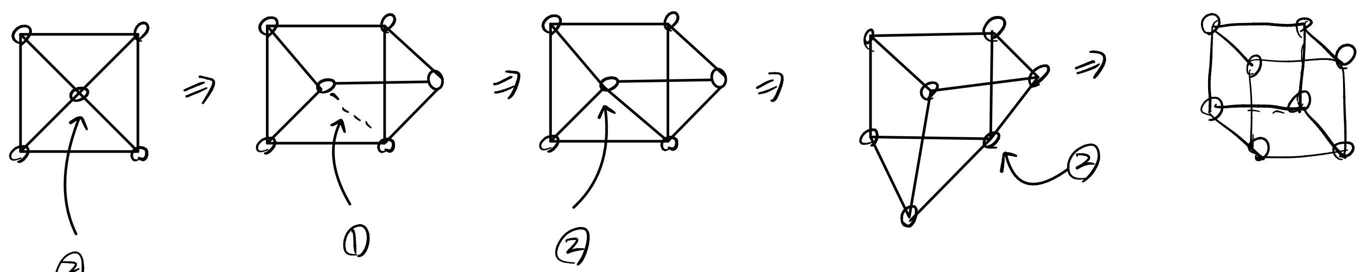
② u is on the cycle; v is not: Since the graph is unicyclic, v can only be on a path "leaving" the cycle, and hence it requires to remove 1 edge to disconnect u, v.

③ Neither of them on the cycle: Similar to ②, both u and v are on a path and hence it requires to remove 1 edge to disconnect u, v.

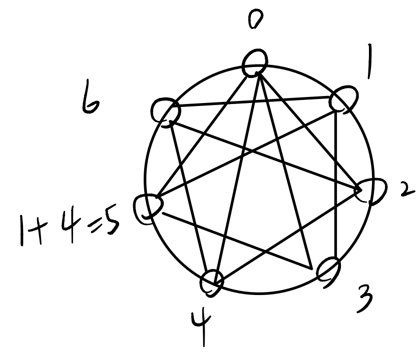


From the above, we know $K_e(G) \leq 2$. #

5.2.6. [Tutte's Synthesis Thm] 3-connected $\longleftrightarrow$ wheel or obtained from wheel by ① add edge to non-adjacent vertices, or ② 把 $\deg \geq 4$ 的 vtx 換成 2個相鄰 vtx，並把原本 neighbor 平分。



5.2.10 $H_{k,n}$: k-connected, n-vertex



① $k = 2r+1$, $r = 2$ ⇒ $H_{2r,n}$

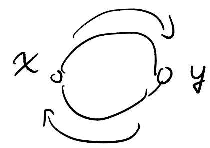
② add $\frac{n+1}{2}$ edges if n odd, from $0 - \frac{n-1}{2}$, $0 - \frac{n+1}{2}$, $i - (i + \frac{n+1}{2})$

add $\frac{n}{2}$ edges : even, 連对面的.

$n = 7$

5.2.15. Case 3. k, n are odd for $H_{k,n}$

Prove any vertices x, y are connected after removing $2r$ vertices (excluding x, y) from $H_{2r+1,n}$.



⇒ Suppose D is a set of $2r$ vertices removed from $H_{2r+1,n}$

⇒ x and y divides the perimeter of the original n-cycle into 2 sectors.

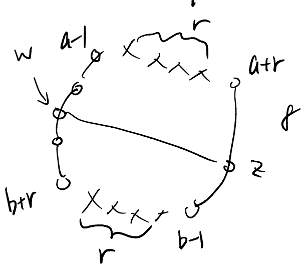
⇒ Both sectors must have r consecutively deleted vertices, otherwise $H-D$ will still be connected by the construction of $H_{2r+1,n}$. ($H_{2r,n}$: $e \in E$ if $|i-j| \le r$)

Let $A = \{a, a+1, \ldots, a+r-1\}$ and $B = \{b, b+1, \ldots, b+r-1\}$ be the deleted vertices

⇒ let $l = |(b+r) - (a-1)|_n$ and let $W = b+r + \lfloor \frac{l}{2} \rfloor$, then w is halfway between $(b+r)$ and $(a-1)$, moving clockwise from $(b+r)$ to $(a-1)$.

⇒ Let $z = w + \frac{(n+1)}{2}$, then z is halfway between $(b+r)$ and $(a-1)$, moving counterclockwise from $(b+r)$ to $(a-1)$ and z is in the other subset of remaining vertices

⇒ Since $|z-w|_n = \frac{n+1}{2}$, there is an edge joining W and Z. for case 3.



5.3.4. $|P_{uv}| = |S_{uv}| = 3$

$S_{uv} = \{a, w, c\}$; $P_{uv} = \{utcv, uwxv, usabv\}$

5.3.17

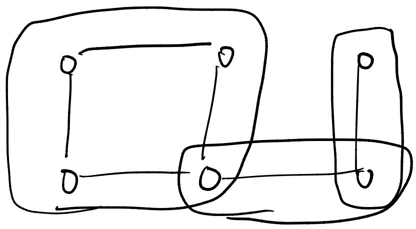
Construct G^x from G by adding a new vertex x into G and add edges (x, w_i) for $i = 1 \sim k$.

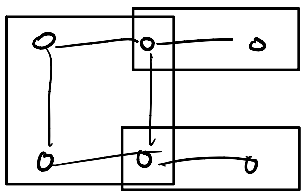
Since G is k-connected, G^x is also k-connected.

By Whitney's k-connected characterization, (G is k-connected $\Leftrightarrow \exists\, k\ P_{uv}$)

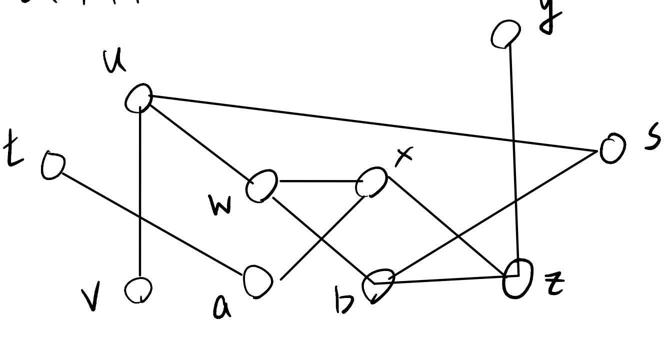
there are k internally disjoint v-x path in G^x. The v-w_i portion of these paths are k internally disjoint path in G.

5.4.6 Blocks of a loopless graph is a maximal connected subgraph s.t. no vtx is a cut-vertex in the block.

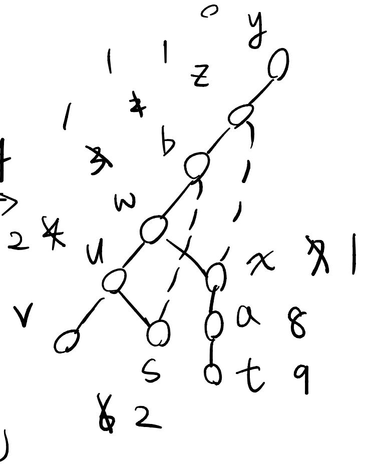




5.4.12



4.4.3 from y $\longrightarrow$



$\Leftarrow$ low =
① df(自己)
② df(non Tree edge 鄰居)
③ low(child)

child

If low(w) $\geq$ df(v) $\longrightarrow$ Cut = $\{z, u, x\}$

① visit x (df=7):

child a: T_a = ⓐ — ⓣ

$\boxed{B_1 = \{x, a, t\}}$

$T' = T - \{a, t\}$

② visit u (df=4)

child v, $T_v = \{v\}$, $\boxed{B_2 = \{u, v\}}$, $T'' = T' - \{v\}$

③ visit z (df=1), child b: $T_b = \{b, w, u, s, x\}$

$\boxed{B_3 = \{z, b, w, u, s, x\}}$, $T''' = T'' - T_b = \{y, z\}$

④ $\boxed{B_4 = \{y, z\}}$

5.4.23 (Prop 5.4.6) G (connected) has at least one cut-vertex
$\Rightarrow$ G has at least 2 leaf blocks.

① Leaf block has only one cut-vertex
② x is a cut-vertex iff x is in 2 different blocks.
③ Two blocks can have at most 1 vertex in common.
$(|B_1 \cap B_2| \leq 1)$
If G has no leaf block, then G has only one block and no cut-vertex.
If G has only one leaf block B_1, by ①, ②, there is another block B_2 that shares this cut-vertex. If B_2 is also a leaf block, we are done.
Assume B_2 is not a leaf block, then it has more than 1 cut vertices (in G), it must share this cut-vertex with another block B_3 by ③.
B_3 cannot share any more cut-vertex with B_1 or B_2 as it will make $B_1 \cup B_2 \cup B_3$ a larger block contradict the maximality of block.
We repeat this process until there are no blocks that share cut vertex with previous block. The last block B_n in the process has only one cut-vertex, so B_n is a leaf block.
Contradict to the assumption that G has only one leaf block.
$\Rightarrow$ If G has < 2 leaf block, either G has no cut-vertex or no such graph exist (1 leaf block), so