Chap 9. Graph Colorings

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The sources of most figure images are from the course slides (Graph Theory) of Prof. Gross

Outline

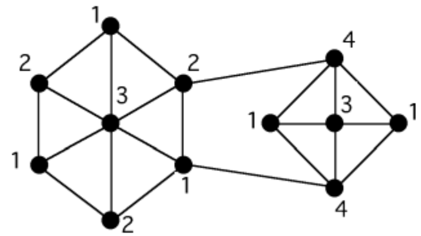
- Vertex-Coloring
- Map-Colorings
- Edge-Colorings

9.1 Vertex Colorings

- **NOTATION:** The maximum degree of a vertex in a graph G is denoted $\delta_{max}(G)$, or simply by δ_{max} when the graph of reference is evident from context.

The Minimization Problem for Vertex-Colorings

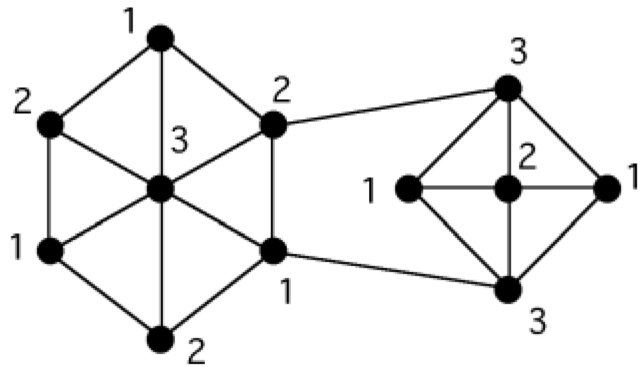
- **DEFINITION**: A ***vertex k-coloring*** is an assignment $f : V_G \to C$ from its vertex-set onto the set $C = \{ 1,\ldots,k \}$, or onto another set of cardinality k, whose elements are called *colors*. For any k, such an assignment is called a ***vertex-coloring***.
- **TERMINOLOGY** : Since vertex-colorings arise more frequently than edge-colorings or map-colorings, one often says *coloring*, instead of *vertex-coloring*, when the context is clear.
- **DEFINITION**: A ***color class*** in a vertex-coloring of a graph G is a subset of V_G containing all the vertices of a given color.
- **DEFINITION**: A ***proper vertex-coloring*** of a graph is a vertex-coloring such that the endpoints of each edge are assigned two different colors.
- **DEFINITION**: A graph is said to be ***vertex k-colorable*** if it has a proper vertex k-coloring.
- **Example 9.1.1:**



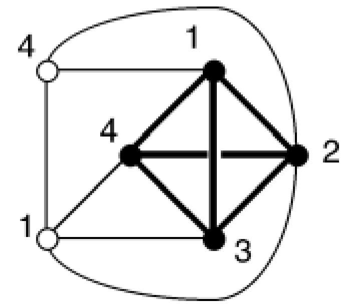
A proper vertex 4-coloring of a graph.

The Minimization Problem for Vertex-Colorings

- **DEFINITION**: The ***(vertex) chromatic number*** of a graph G, denoted $\chi(G)$, is the minimum number of different colors required for a proper vertex-coloring of G. A graph G is **(*vertex*)** ***k-chromatic*** if $\chi(G) = k$.
  - ✓ $\chi(G) = k$, if graph G is k-colorable but not $(k – 1)$-colorable.
- **Example 9.1.1, continued:**
- **Example 9.1.2:**



Proper 3-coloring of graph from Example 9.1.1.



Graph G is 4-colorable.

- **Remark:** The study of vertex-colorings of graphs is customarily restricted to simple graphs. A graph with self-loops is regarded as uncolorable, and a multiple adjacency has no more effect on the colors of its endpoints than a single adjacency.

Sequential Vertex-Coloring Algorithm

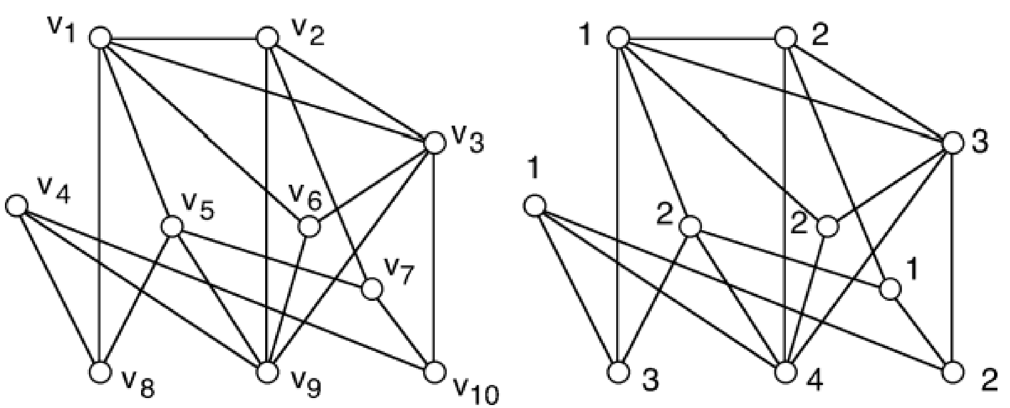
- The problem of calculating the chromatic number of a graph is known to be NP-hard.
- Deciding whether a graph has a 3-coloring is an NP-complete problem.

Algo 9.1.1: Color the Vertices Sequentially

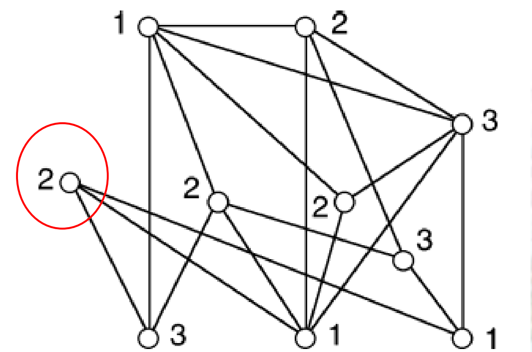
Input: a graph G with vertex list $v_1, v_2, \ldots, v_p$.
Output: a proper vertex-coloring $f : V_G \to \{1, 2, \ldots\}$.

For $i = 1, \ldots, p$
  Let $f(v_i) :=$ the smallest color number not used on any of the smaller-subscripted neighbors of v_i.
Return vertex-coloring f.

- **Example 9.1.3:**



A graph and a sequential 4-coloring.



A nonsequential 3-coloring for the same graph.

Basic Principles for Calculating Chromatic Numbers

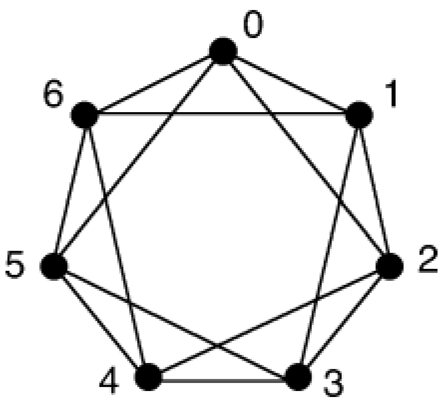
- A few basic principles recur in many chromatic-number calculations.
  - *Upper Bound*: Show $\chi(G) \le k$, most often by exhibiting a proper k-coloring of G.
  - *Lower Bound*: Show $\chi(G) \ge k$, most especially, by finding a subgraph that requires k colors.
- **Proposition 9.1.1.** *Let G be a simple graph. Then* $\chi(G) \le \delta_{max}(G)+1$.
  - ✓ The sequential coloring algorithm never uses more than $\delta_{max}(G)+1$ colors, no matter how the vertices are ordered, since a vertex cannot have more than $\delta_{max}(G)$ neighbors.
- **Proposition 9.1.2.** *Let G be a graph that has k mutually adjacent vertices. Then* $\chi(G) \ge k$.
  - ✓ Using fewer than k colors on graph G would result in a pair from the mutually adjacent set of k vertices being assigned the same color.

- REVIEW FROM §2.3: A ***clique*** in a graph G is a maximal subset of V_G whose vertices are mutually adjacent. The ***clique number*** $\omega(G)$ of a graph G is the number of vertices in a largest clique in G.

Basic Principles for Calculating Chromatic Numbers

- **Proposition 9.1.3.** *Let G be a graph. Then* $\chi(G) \geq \omega(G)$.
- REVIEW FROM §2.3: The ***independence number*** $\alpha(G)$ of a graph G is the number of vertices in an independent set in G of maximum cardinality.
- **Proposition 9.1.4.** *Let G be a graph. Then* $\chi(G) \geq \left\lceil \frac{|V_G|}{\alpha(G)} \right\rceil$.
  - ✓ Since each color class contains at most $\alpha(G)$ vertices, the number of different color classes must be at least $\left\lceil \frac{|V_G|}{\alpha(G)} \right\rceil$.
- **Example 9.1.4:** $\alpha(G) = 2$. $\chi(G) \geq \left\lceil \frac{|V_G|}{\alpha(G)} \right\rceil = \left\lceil \frac{7}{2} \right\rceil = 4$

{0,3}, {1,4}, {2,5}, {6}.



$Circ(7:1,2)$.

- **Proposition 9.1.5.** *Let H be a subgraph of graph G. Then* $\chi(G) \geq \chi(H)$.
  - ✓ Whatever colors are used on the vertices of subgraph H in a minimum coloring of graph G can also be used in a coloring of H by itself.

Basic Principles for Calculating Chromatic Numbers

- REVIEW FROM §2.7: The ***join*** $G+H$ of the graphs G and H is obtained from the graph union $G \cup H$ by adding an edge between each vertex of G and each vertex of H.
- **Proposition 9.1.6.** *The join of graphs G and H has chromatic number* $\chi(G+H) = \chi(G) + \chi(H)$.
  - ✓ *Lower Bound*. In the join $G+H$, no color used on the subgraph G can be the same as a color used on the subgraph H, since every vertex of G is adjacent to every vertex of H. Since $\chi(G)$ colors are required for subgraph G and $\chi(H)$ colors are required for subgraph H, it follows that $\chi(G+H) \geq \chi(G) + \chi(H)$.
  - ✓ *Upper Bound*. Just use any $\chi(G)$ colors to properly color the subgraph G of $G+H$, and use $\chi(H)$ different colors to color the subgraph H.

Chromatic Numbers for Common Graph Families

Table 9.1.1 Chromatic #s for common graph families

Graph G	$\chi(G)$
empty graph	1
bipartite graph	2
nontrivial path graph P_n	2
nontrivial tree T	2
cube graph Q_n	2
even cycle graph C_{2n}	2
odd cycle graph C_{2n+1}	3
even wheel W_{2n}	3
odd wheel W_{2n+1}	4
complete graph K_n	n

- **Proposition 9.1.7.** *A graph G has* $\chi(G) = 1$ *if and only if G has no edges.*
  - ✓ The endpoints of an edge must be colored differently.
- **Proposition 9.1.8.** *A bipartite graph G has* $\chi(G) = 2$, *unless G is edgeless.*
  - ✓ A 2-coloring is obtained by assigning one color to every vertex in one of the bipartition parts and another color to every vertex in the other part. If G is not edgeless, then $\chi(G) \geq 2$, by Proposition 9.1.2.

Chromatic Numbers for Common Graph Families

- **Proposition 9.1.9.** *Path Graphs:* $\chi(P_n) = 2$, *for* $n \geq 2$.
  - ✓ P_n is bipartite. Apply Proposition 9.1.8.
- **DEFINITION**: A ***k-partite graph*** is a loopless graph whose vertices can be partitioned into k independent sets, which are sometimes called the ***partite sets*** of the partition.
- **Proposition 9.1.10.** *Trees:* $\chi(T) = 2$, *for any nontrivial tree T.*
  - ✓ Tree are bipartite.
- **Proposition 9.1.11.** *Cube Graphs:* $\chi(Q_n) = 2$.
  - ✓ The cube graph Q_n is bipartite.
- **Proposition 9.1.12.** *Even Cycles:* $\chi(C_{2n}) = 2$.
  - ✓ An even cycle is bipartite.
- **Proposition 9.1.13.** *Odd Cycles:* $\chi(C_{2n+1}) = 3$.
  - ✓ Clearly, $\alpha(C_{2n+1}) = n$. Thus, by Proposition 9.1.4,

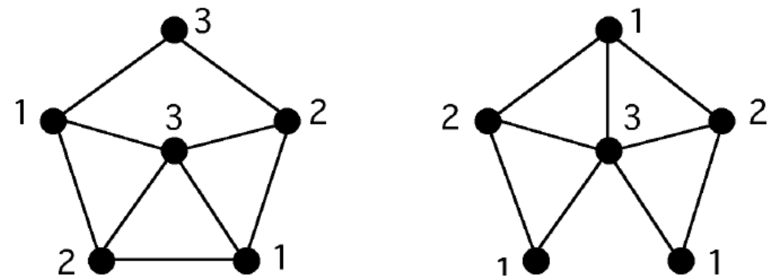
$$\chi(C_{2n+1}) \geq \left\lceil \frac{|V(C_{2n+1})|}{\alpha(V(C_{2n+1}))} \right\rceil = \left\lceil \frac{2n+1}{n} \right\rceil = 3$$

Chromatic Numbers for Common Graph Families

- REVIEW FROM §2.4: The wheel graph $W_n = K_1 + C_n$ is called an ***odd wheel*** if n is odd, and an ***even wheel*** if n is even.
- **Proposition 9.1.14.** *Even Wheels:* $\chi(W_{2m}) = 3$.
  - ✓ Using the fact that $W_{2m} = C_{2m} + K_1$, Proposition 9.1.6 and Corollary 9.1.12 imply that $\chi(W_{2m}) = \chi(C_{2m}) + \chi(K_1) = 2 + 1 = 3$.
- **Proposition 9.1.15.** *Odd Wheels:* $\chi(W_{2m+1}) = 4$, for all $m \geq 1$.
  - ✓ Using the fact that the wheel graph W_{2m+1} is the join $C_{2m+1} + K_1$, Proposition 9.1.6 and 9.1.13 imply that $\chi(W_{2m+1}) = \chi(C_{2m+1}) + \chi(K_1) = 3 + 1 = 4$, for all $m \geq 1$.
- **Proposition 9.1.16.** *Complete Graphs:* $\chi(K_n) = n$.
  - ✓ By Proposition 9.1.2, $\chi(K_n) \geq n$. Moreover, any assignment of n colors to the n vertices of K_n is a proper n-coloring.

Chromatically Critical Subgraphs

- **DEFINITION**: A connected graph G is ***(chromatically) k-critical*** if $\chi(G) = k$ and the edge-deletion subgraph $G - e$ is $(k - 1)$-colorable, for every edge $e \in E_G$ (i.e., if G is an edge-minimal k-chromatic graph).
- **Example 9.1.5:** An odd cycle is 3-critical, since deleting any edge yields a path, thereby reducing the chromatic number from 3 to 2.
- **Example 9.1.6:** The odd wheel W_5 is 4-chromatic. The following figure illustrates how removing either a spoke-edge or a rim-edge reduces the chromatic number to 3. Thus W_5 is 4-critical.



The graph $W_5 - e$ is 3-colorable, for any edge e.

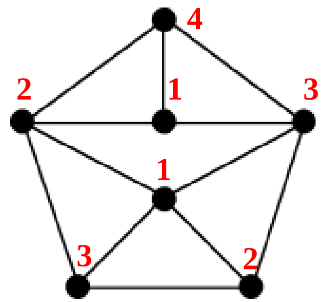
- **Proposition 9.1.17.** *Let G be a chromatically k-critical graph, and let v be any vertex of G. Then the vertex-deletion subgraph $G - v$ is $(k - 1)$-colorable.*

Chromatically Critical Subgraphs

- **Theorem 9.1.18.** *Let G be a chromatically k-critical graph. Then no vertex of G has degree less than* $k-1$.
  - ✓ By way of contradiction, suppose that v were a vertex of degree less than $k-1$. Consider a $(k-1)$-coloring of the vertex-deletion subgraph $G-v$.
  - ✓ The colors assigned to the neighbors of v would not include all the colors of the $(k-1)$ coloring, because vertex v has fewer than $k-1$ neighbors.
  - ✓ Thus, if v were restored to the graph, it could be colored with any one of the $k-1$ colors that was not used on any of its neighbors. This would achieve a $(k-1)$-coloring of G, which is a contradiction.
- **DEFINITION**: An ***obstruction to*** k***-chromaticity*** (or k***-obstruction***) is a subgraph that forces every graph that contains it to have chromatic number greater than k.
- **Example 9.1.7:** The complete graph K_{k+1} is an obstruction to k-chromaticity.
- **Proposition 9.1.19.** *Every* $(k+1)$*-critical graph is an edge-minimal obstruction to k-chromaticity.*
  - ✓ If any edge is deleted from a $(k+1)$-critical graph, then, by definition, the resulting graph is not an obstruction to k-chromaticity.

Chromatically Critical Subgraphs

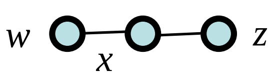
- **DEFINITION**: A set $\{G_j\}$of chromatically $(k+1)$-critical graphs is a ***complete set of obstructions*** if every $(k+1)$-chromatic graph contains at least one member of $\{G_j\}$ as a subgraph.
- **Example 9.1.8:** The singleton set $\{K_2\}$ is a complete set of obstructions to 1-chromaticity.
- REVIEW FROM §1.5: [Characterization of bipartite graphs] A graph is bipartite if and only if it contains no cycle of odd length.
- **Example 9.1.9:** The bipartite-graph characterization above implies that the family $\{C_{2j+1} | j = 1,2, \ldots\}$ of all odd cycles is a complete set of 2-obstructions.
- **Example 9.1.10:** Although the odd wheel graphs W_{2m+1} with $m \geq 1$ are 4-critical, they do not form a complete set of 3-obstructions, since there are 4-chromatic graphs that contain no such wheel.



A 4-chromatic graph that contains no odd wheel.

Brooks's Theorem

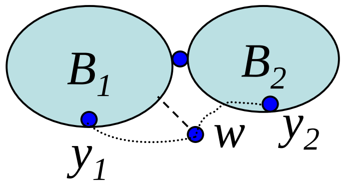
- REVIEW FROM §5.4:
  - ✓ A ***block*** of a loopless graph is a maximal connected subgraph H such that no vertex of H is a cut-vertex of H.
  - ✓ A ***leaf block*** of a graph G is a block that contains exactly one cut-vertex of G.
  - ✓ [1] A graph G with at least one cut-vertex has at least two leaf blocks.
  - ✓ [2] Let B_1 and B_2 be distinct blocks of a connected graph G. Let y_1 and y_2 be vertices in B_1 and B_2, respectively, such that neither is a cut-vertex of G. Then vertex y_1 is not adjacent to vertex y_2.
- **Lemma 9.1.20.** *Let G be a non-complete, k-regular 2-connected graph with $k \geq 3$. Then G has a vertex x with two non-adjacent neighbors y and z such that $G - \{y, z\}$ is a connected graph.*
  - ✓ Let w be any vertex in graph G. First suppose that subgraph $G - w$ is 2-connected. Then let z be any vertex at distance 2 from vertex w (which exists because the graph G is regular and non-complete).

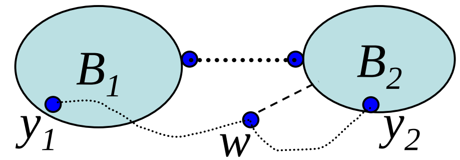


  - ✓ If x is the vertex between w and z, and if vertex y is taken to be w, then the conditions of the assertion are satisfied.
  - ✓ Alternatively, suppose that $\kappa_v(G - w) = 1$. Then let B_1 and B_2 be two leaf blocks of $G - w$ (which exist by review-result [1] above).

Brooks's Theorem

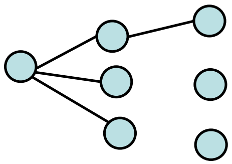
- ✓ Since graph G has no cut-vertices, it follows that vertex w is adjacent to some vertex $y_1 \in B_1$ that is not a cut-vertex of $G - w$ and, likewise, adjacent to some vertex $y_2 \in B_2$ that is not a cut-vertex of $G - w$.
- ✓ By review-result [2], vertex y_1 is not adjacent to vertex y_2. Hence, letting x be w, y be y_1, and z be y_2 completes the proof.





Theorem 9.1.21 [Brooks, 1941]. *Let G be a non-complete, simple connected graph with maximum vertex degree* $\delta_{max}(G) \geq 3$. *Then* $\chi(G) \leq \delta_{max}(G)$.

- ✓ In this proof of [Lo75], the sequential coloring algorithm enables us to make some simplifying reductions. Suppose that $|V_G| = n$.



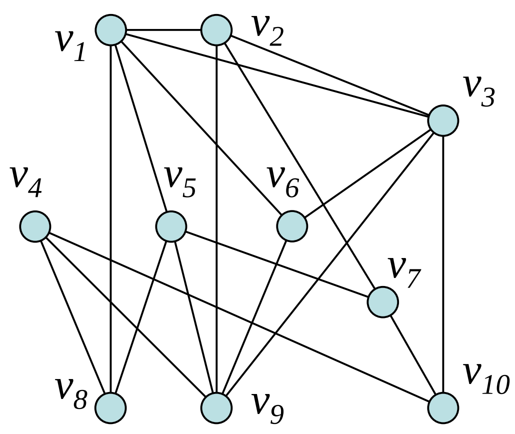
- ✓ *Case 1*. G is not regular.
  - First choose vertex v_n to be any vertex of degree less than $\delta_{max}(G)$.
  - Next grow a spanning tree v_n, assigning indices in decreasing order. In this ordering of V_G, every vertex except v_n has a higher-indexed neighbor along the tree path to v_n.
  - Hence each vertex has at most $\delta_{max}(G) - 1$ lower-indexed neighbors. It follows that the sequential coloring algorithm uses at most $\delta_{max}(G)$ colors.

Brooks's Theorem

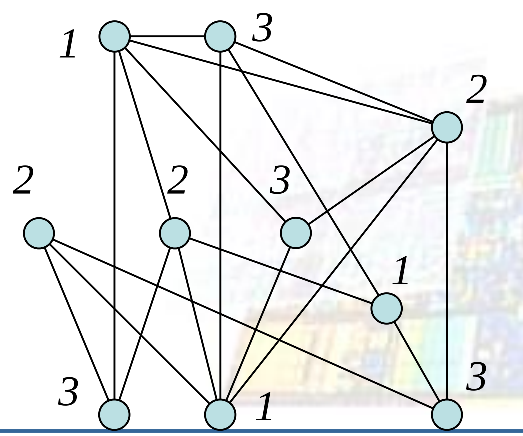
- ✓ *Case 2*. G is regular, and G has a cut-vertex x.
  - Let $C_1,\ldots,C_m$ be the components of the vertex-deletion subgraph $G - x$, and let G_i be the subgraph of G induced on the vertex-set $V_{C_i} \cup x$, $i = 1,\ldots,m$. Then the degree of vertex x in each subgraph G_i is clearly less than $\delta_{max}(G)$.
  - By Case 1, each subgraph G_i has a proper vertex coloring with $\delta_{max}(G)$ colors. <u>By permuting the names of the colors in each such subgraph so that vertex x is always assigned color c_1</u>, one can construct a proper coloring of G with $\delta_{max}(G)$ colors.
- ✓ *Case 3*. G is regular, and 2-connected.
  - By Lemma 9.1.20, graph G has a vertex, which we call w, with two non-adjacent neighbors, which we call x and y, such that $G - \{x, y\}$ is connected.
  - Grow a spanning tree from w in $G - \{x, y\}$, assigning indices in decreasing order such that $w = v_n$. The least-indexed vertex of spanning tree is v_3. Let x and y be indexed as v_2 and v_1. Since v_2 and v_1 are not adjacent, we can assign color 1 to them. As in Case 1, each vertex except v_n has at most $\delta_{max}(G) - 1$ lower-indexed neighbors. It follows that the sequential coloring algorithm uses at most $\delta_{max}(G)$ colors on all vertices except v_n.

Heuristic for Vertex-Coloring

- Many vertex-coloring heuristics for graphs are based on the intuition –
  - ✓ A vertex of larger degree will be more difficult to color later than one of smaller degree.
  - ✓ The one with the denser *neighborhood subgraph* will be harder to color later.
- *Neighborhood subgraph of radius r of v*: the subgraph composed of those vertices whose distance to v is not larger than r.



Algorithm 9.1.2: Vertex-Coloring: Largest Degree First
Input: an n-vertex graph G.
Output: a vertex-coloring f of graph G.
While there are uncolored vertices of G
Among the uncolored vertices with maximum degree, choose vertex v with maximum colored degree.
Assign smallest possible color k to vertex v: $f(v) := k$.
Return graph G with vertex-coloring f.

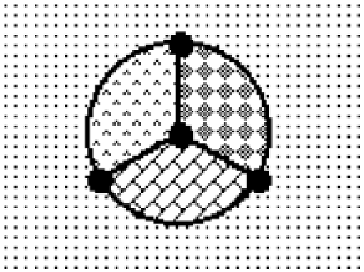


- **Example 9.1.11:** Descending degree order :

$v_1, v_3, v_9, v_2, v_5, v_4, v_6, v_7, v_8, v_{10}$

9.2 Map-Colorings: Dualizing Map-Colorings into Vertex-Colorings

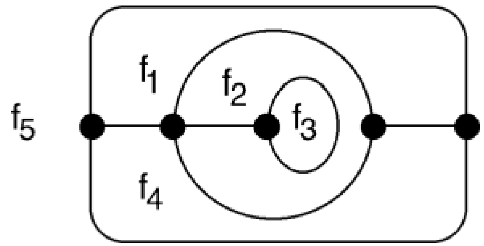
- **REVIEW FROM 8.4:** A ***map*** on a surface is an imbedding of a graph on that surface.
- For every closed surface S, there is a minimum number $chr(S)$ of colors sufficient so that every map on S can be colored *properly* with $chr(S)$ colors, which means that no color meets itself across an edge.
- **DEFINITION**: A ***map k-coloring*** for an imbedding $\iota : G \rightarrow S$ of a graph on a surface is an assignment $f : F \rightarrow C$ from the face-set F onto the set $C = \{1,\ldots,k\}$, or onto another set of cardinality k, whose elements are called colors. For any k, such an assignment is called a ***map-coloring***.
- **DEFINITION**: A map-coloring is ***proper*** if for each edge $e \in E_G$, the regions that meet on edge e are colored differently.
- **DEFINITION**: The ***chromatic number of a map*** $\iota : G \rightarrow S$ is the minimum number $chr(\iota)$ of colors needed for a proper coloring.
- **Example 9.2.1:**



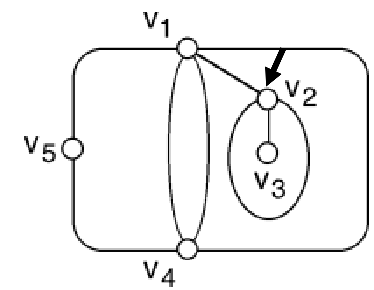
A 4-colored planar map.

Dualizing Map-Colorings into Vertex-Colorings

- **TERMINOLOGY**: A region is said to ***meet itself on edge*** *e* if edge *e* occurs twice in the region's boundary walk. It is said to ***meet itself on vertex*** *v* if vertex *v* occurs more than once in the region's boundary walk.
- **Remark**: A map cannot be properly colored if a region meets itself
- **Example 9.2.2:**



A map with no proper coloring.



The dual graph for the map of Figure 9.2.2.

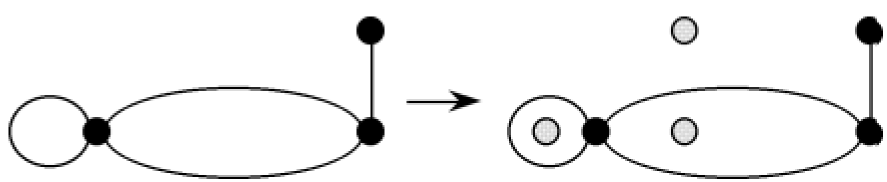
- **(7.5) DEFINITION**: Whatever graph imbedding $\iota : G \to S$ is to be supplied as input to the duality process (whose definition follows below) is called the ***primal graph imbedding***. Moreover, the graph *G* is called the ***primal graph***, the vertices of *G* are called ***primal vertices***, the edges of *G* are called ***primal edges***, and the faces of the imbedding of *G* are called ***primal faces***.
- **(7.5) DEFINITION**: Given a primal graph imbedding $\iota : G \to S$, the **(*Poincaré*) *duality construction*** of a new graph imbedding is a two-step process, one for the dual vertices and one for the dual edge.

Dualizing Map-Colorings into Vertex-Colorings

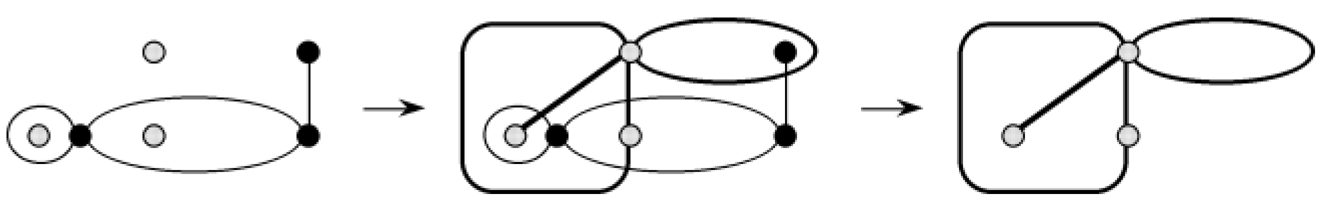
dual vertices: Into the interior of each primal face f, insert a new vertex f^*, to be regarded as dual to face f. The set $\{f^* \mid f \in F_\iota\}$ is denoted V^*.

dual edges: Through each primal edge e, draw a new edge e^*, joining the dual vertex in the primal face on one side of that edge to the dual vertex in the primal face f contains both sides of primal edge e, then dual edge e^* is a self-loop through primal edge e from the dual vertex f^* to itself. The set $\{e^* \mid e \in E_G\}$ is denoted E^*.

- ***Dual graph, dual imbedding, and dual faces***



Inserting dual vertices into primal imbedding.

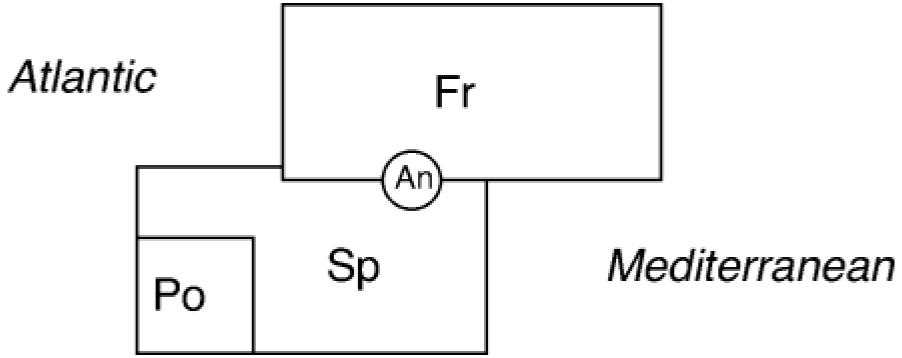


Inserting dual edges completes dual imbedding.

Geographic Maps

- **Proposition 9.2.1.** *The chromatic number of a map equals the chromatic number of its dual graph.*

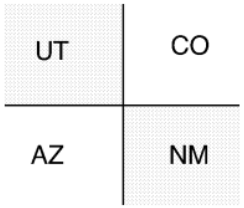
- **Application 9.2.1** *Political Cartography: In the cartography of political maps, various interesting configurations arise. For instance, France and Spain meet two distinct borders, one from Andorra to the Atlantic Ocean, the other from Andorra to the Mediterranean Sea, as represented in Figure 9.2.4. Similar configurations occur where Switzerland meets Austria twice around Liechtenstein. Moreover, India and China have a triple adjacency around Nepal and Bhutan. Multiple adjacency of regions does not affect the rules for colorings a map.*



Double adjacency of France and Spain around Andorra.

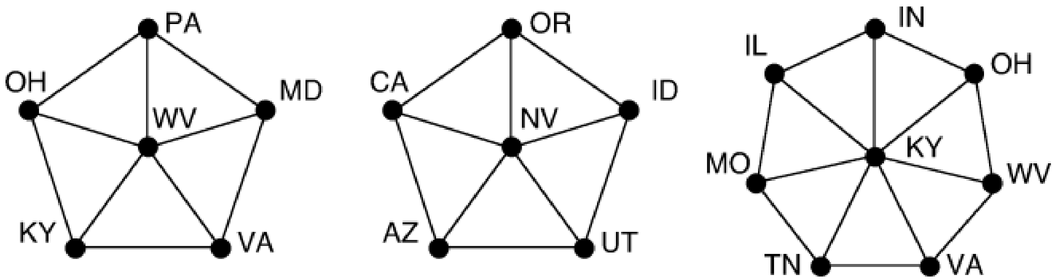
Geographic Maps

- **Remark**: Two faces that meet at a vertex but not along an edge may have the same color in a proper map coloring. thus, a checkerboard configuration such as the Four Corners, USA, represention in Figure 9.2.5, may be properly colored with only two colors.
- **Remark**: Utah meets five other states across an edge in the map, but these five do not quite encircle Utah, since Arizona and Colorado do not meet an edge, even though they do meet at Four Corners. Thus, <u>the Utah configuration is the join $P_5 + K_1$, and not the wheel W_5</u>.



A proper 2-coloring at Four Corners, USA.

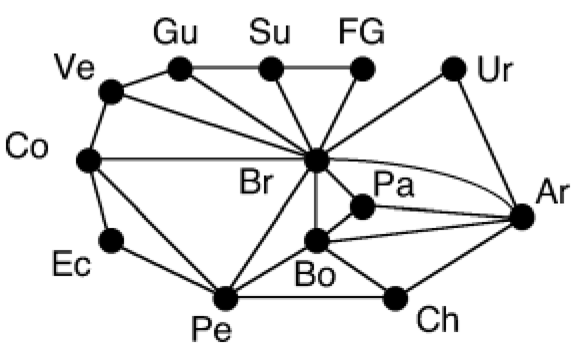
- **Example 9.2.4:**



The three odd wheels in the map of the USA.

Geographic Maps

- **Example 9.2.3:**

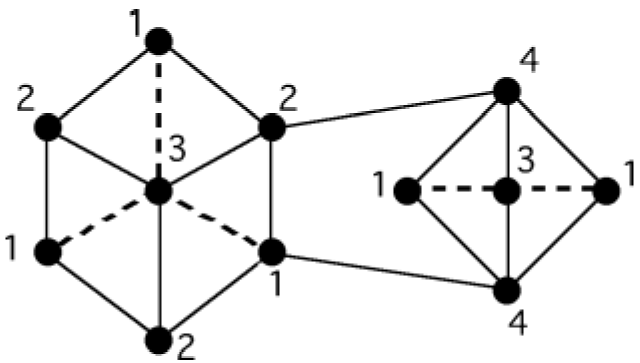


Ar:	Argentina	Gu:	Guyana 蓋亞那
Bo:	Bolivia 玻利維亞	Pa:	Paraguay
Br:	Brazil	Pe:	Peru 祕魯
Ch:	Chile	Su:	Surinam 蘇利南
Co:	Columbia	Ur:	Uruguay
Ec:	Ecuador 厄瓜多爾	Ve:	Venezuela 委內瑞拉
FG:	French Guyana		

The dual graph of the map of South America.

Five-Color Theorem for Planar Graphs and Maps

- An early investigation of the Four Color Problem by A. B. Kempe [1879] introduced a concept that enabled Heawood [1890] to prove without much difficulty that five colors are sufficient.
- **DEFINITION**: The $\{i, j\}$-***subgraph*** of a graph G with a vertex-coloring that has i and j in its color set is the subgraph of G induced on the subset of all vertices that are colored either i or j.
- **DEFINITION**: A ***Kempe*** i-j ***chain*** for a vertex-coloring of a graph is a component of the $\{i, j\}$-subgraph.
- **Example 9.2.5:**

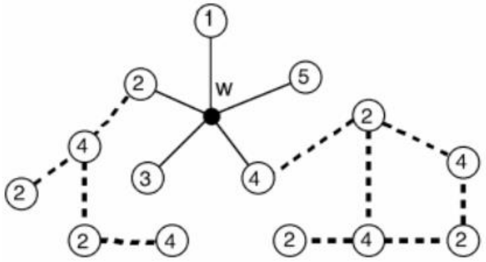


A graph coloring with two Kempe 1-3 chains.

Five-Color Theorem for Planar Graphs and Maps

- **Theorem 9.2.2 [Heawood, 1890].** *The chromatic number of a planar simple graph is at most 5.*
  - ✓ Starting with an arbitrary planar graph, edges are removed until a chromatically critical graph G is obtained. It suffices to prove that G is 5-colorable.
  - ✓ Since $\delta_{avg}(G) < 6$ (by Theorem 8.4.2), there is a vertex $w \in V_G$ of degree at most 5. Theorem 9.1.18 implies that $\chi(G) \le 6$. Thus, the vertex-deletion subgraph $G - w$ is 5-colorable, by Proposition 9.1.17.
  - ✓ Next, consider any 5-coloring of subgraph $G - w$. If not all five colors were used on the neighbors of vertex w, then the 5-coloring of $G - w$ could be extended to graph G by assigning to w a color not used on the neighbors of w.
  - ✓ Thus, we can assume that all five colors are assigned to the neighbors of vertex w. Moreover, there is no loss of generality in assuming that these colors are consecutive in counterclockwise order. Consider the {2, 4}-subgraph, and let K be the Kempe 2-4 chain that contains the 2-neighbor of vertex w (i.e., the neighbor that was assigned color 2).

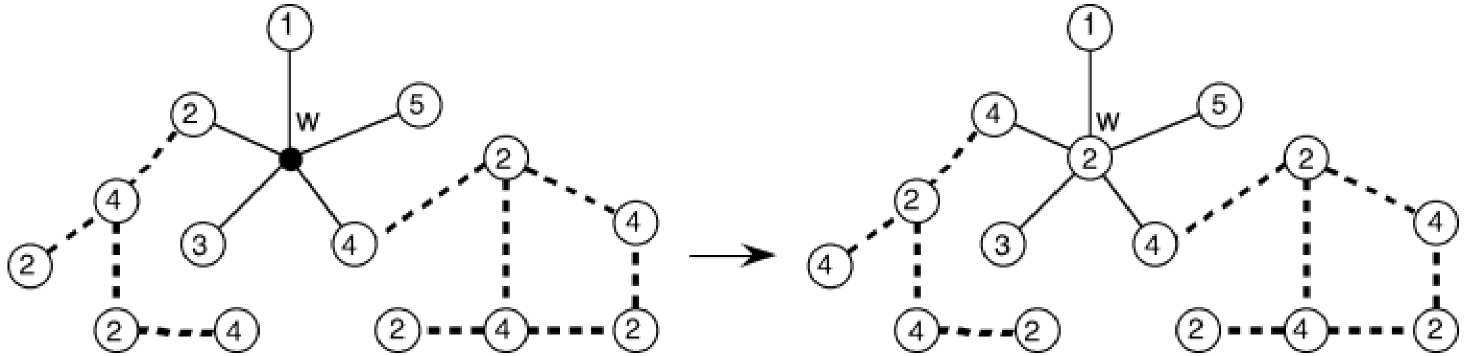
Theorem 9.1.18. *Let G be a chromatically k-critical graph. Then no vertex of G has degree less than k – 1.*
Proposition 9.1.17. *Let G be a chromatically k-critical graph, and let v be any vertex of G. Then the vertex-deletion subgraph G – v is (k –1)-colorable.*



Theorem 8.4.2. Let $\iota : G \to S_0$ be an imbedding of a simple graph on the sphere. Then $\delta_{avg}(G) < 6$

Five-Color Theorem for Planar Graphs and Maps

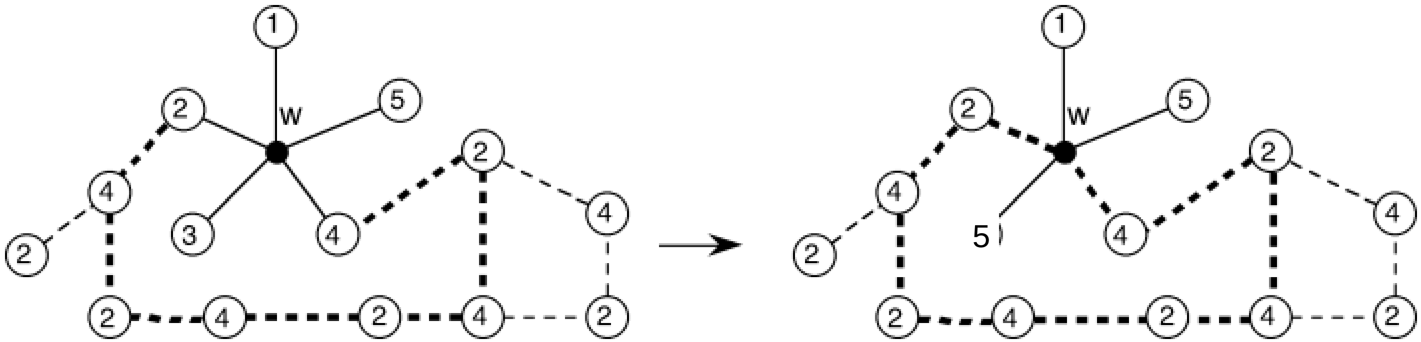
- Case 1. Suppose that Kempe chain *K* does not also contain the 4-neighbor of vertex *w*. Then colors 2 and 4 can be swapped in Kempe chain *K*. The result is a 5-coloring of *G* – *w* that does not use color 2 on any neighbor of *w*. This 5-coloring extends to a 5-coloring of graph *G* when color 2 is assigned to vertex *w*.



Swapping colors in a Kempe 2-4 chain.

- Case 2. Suppose that Kempe chain *K* contains both the 4-neighborand the 2-neighbor of vertex *w*. Then there is a path in Kempe chain *K* from the 2-neighbor to the 4-neighbor. Appending the edge between vertex *w* and both these neighbors extends that path to a cycle. By the Jordan Curve Theorem (§7.1), this cycle separates the plane.

Five-Color Theorem for Planar Graphs and Maps



Extending a path in a Kempe 2-4 chain to a cycle.

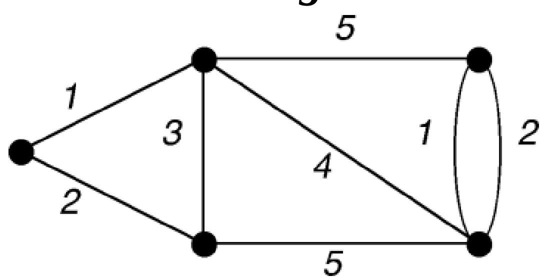
- ✓ Since the 3-neighbor and 5-neighbor of *w* are on different sides of the separation, it follows that the Kempe 3-5 chain *L* containing the 5-neighbor cannot also contain the 3-neighbor. Thus, it is possible to swap colors in Kempe chain *L* and assign color 5 to vertex *w*, thereby completing a 5-coloring of *G*.

- **Theorem 9.2.3 [Appel and Haken. 1976].** *Every planar graph is 4-colorable.*

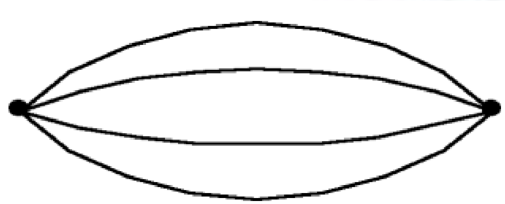
- **Remark**: The proof by Appel and Haken [ApHa76] of the Four Color Theorem is highly specialized, intricate, and long. Following an approach initiated by Heesch, Appel and Haken first reduced the seemingly infinite problem of considering every planar graph to checking a finite, unavoidable set of (over 1900) reducible configurations. Over 1200 hours of computer time were used. Eventually, a more concise proof was derived by Robertson, Sanders, Seymour, and Thomas [RoSaSeTh97].

9.3 Edge-Colorings

- **REVIEW FROM** §1.1: Two different edges are ***adjacent edges*** if they have at least one endpoint in common.
- **DEFINITION**: An ***edge k-coloring*** is an assignment $f: E_G \to C$ from its edge-set onto the set C $= \{1,\ldots,k\}$, or onto another set of cardinality k, whose elements are called *colors*. For any k, such an assignment is called an ***edge-coloring***.
- **DEFINITION**: An ***edge color class*** in an edge-coloring of a graph G is a subset of E_G containing all the edge of some color.
- **DEFINITION**: A ***proper edge-coloring*** of a graph is an edge-coloring such that adjacent edges are assigned different colors.
- **Remark**: Whereas multi-edges have no bearing on the proper vertex-colorings of a graph, they have an obvious effect on the proper edge-colorings and cannot be ignored. Graphs with self-loops are excluded from the present discussion.
- **DEFINITION**: A graph is said to be ***edge k-colorable*** if it has a proper edge k-coloring.
- **Example 9.3.1:**

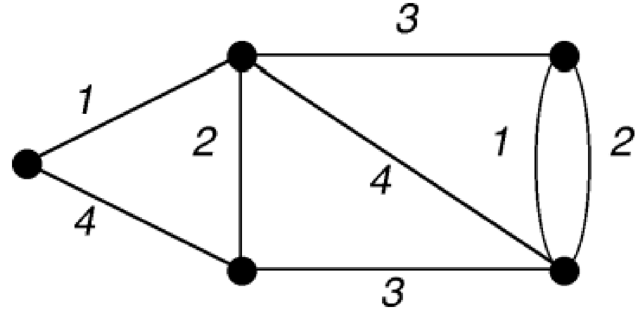


A graph with a proper edge 5-coloring.


A proper edge 4-coloring of the dipole D_4.

The Minimization Problem for Edge-Colorings

- **DEFINITION**: The ***edge-chromatic number*** of a graph G, denoted $\chi'(G)$, is the minimum number of different colors required for a proper edge-coloring of G. A graph G is ***edge** k-**chromatic*** if $\chi'(G)=k$.
  Thus, $\chi'(G)=k$ if graph G is edge k-colorable but not edge $(k-1)$-colorable.
- **Example 9.3.1, continued:**



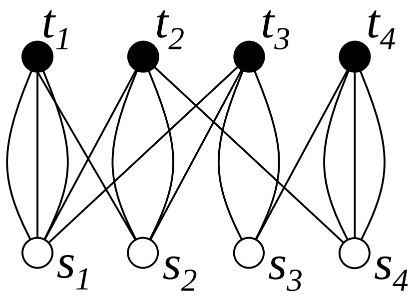
Proper edge 4-coloring of graph in Fig 9.3.2.

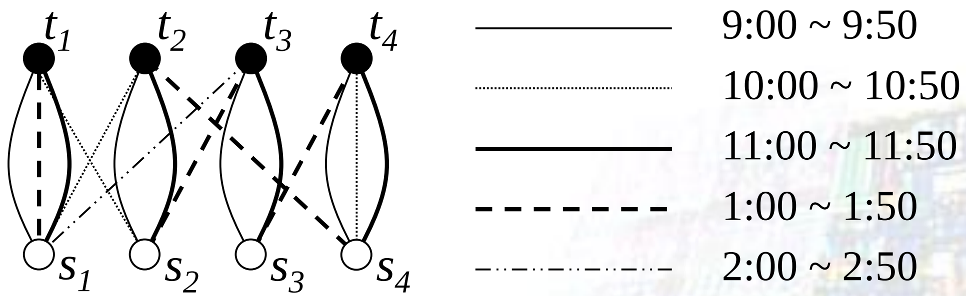
Modeling Applications as Edge-Coloring Problems

- **Application 9.3.1.** *Circuit Boards*: Some electronic devices $x_1, x_2, \ldots x_n$ are on a board. The connecting wires emerging from each device must be colored differently, so that they can be distinguished. The least number if colors required is the edge-chromatic number of the associated network.

Modeling Applications as Edge-Coloring Problems

- **Application 9.3.2.** *Scheduling Class Times*: A high school has teachers $t_1,\ldots,t_m$ to teach courses $s_1,\ldots,s_n$. In particular, teacher t_j must teach $s_{j,k}$ sections of course s_k.
  - ✓ PROBLEM: Calculate the minimum number of time periods required to schedule all the courses so that no two sections of the same course are taught at the same time.
  - ✓ SOLUTION: Form a bipartite graph on the two sets $\{t_1,\ldots,t_m\}$ and $\{s_1,\ldots,s_n\}$ so that there are $s_{j,k}$ edges joining t_j and s_k, for all j and k, as shown in Figure 9.3.3.
    A matching of teachers to courses can be realized in a time period. If each edge-color represents a timeslot in the schedule, then an edge-coloring of the bipartite graph represents a feasible timetable for section of courses. A minimum edge-coloring, as shown in Figure 9.3.4, uses the smallest number of time periods.





Sequential Edge-Coloring Algorithm

- There is a sequential edge-coloring algorithm analogous to the sequential vertex-coloring algorithm of §9.1.
- **DEFINITION**: A ***neighbor of an edge*** *e* is another edge that shares one or both of its endpoints with *e*.

Algo 9.3.1: Sequential Edge-Coloring

Input: a graph G with edge list $e_1, e_2, \ldots, e_p$.
Output: a proper edge-coloring with positive integers

For $i = 1, \ldots, p$
    Let $f(e_i) :=$ the smallest color number not used on any of the smaller-subscripted neighbors of e_i.
Return edge-coloring f.

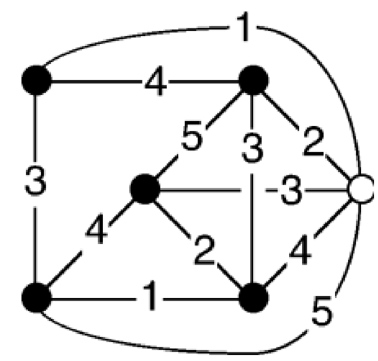
Basic Principles for Calculating Edge-Chromatic Numbers

- Edge-chromatic-number calculations are largely based on a few simple principles, mostly analogous to those used in the two-step vertex-chromatic-number calculations.
  - ✓ *Upper Bound*: Show $\chi'(G) \leq k$ by exhibiting a proper edge k-coloring of G.
  - ✓ *Lower Bound*: Show $\chi'(G) \geq k$ by using properties of graph G.

  The next three results help in establishing a lower bound for the edge-chromatic number. They are immediate consequences of the definitions.
- **Proposition 9.3.1.** *Let G be a graph that has k mutually adjacent edges. Then* $\chi'(G) \geq k$.
- **Corollary 9.3.2.** *For any graph G,* $\chi'(G) \geq \delta_{max}(G)$.
- **Proposition 9.3.3.** *Let H be a subgraph of grpah G. Then* $\chi'(G) \geq \chi'(H)$.
- **Example 9.3.2:** The edge coloring in the right figure indicates:

  $\chi'(G) \leq 5$, and the existence of a 5-valent vertex shows

  $\chi'(G) \geq 5$, by Corollary 9.3.2. $\chi'(G) = 5$



Graph G is edge 5-colorable.

Matchings

- **DEFINITION**: A ***matching*** (or ***independent set of edges***) of a graph G is a subset of edges of G that are mutually non-adjacent.
- **DEFINITION**: A ***maximum matching*** in a graph is a matching with the maximum number of edges.
- **NOTATION**: The cardinality of a maximum matching in a graph G is denoted $\alpha'(G)$, analogous to the independence number $\alpha(G)$ for the vertices.
- **Remark**: It follows immediately from the definition that each color class of a proper edge-coloring of a graph G is a matching of G. The following proposition provides a lower bound on the edge-chromatic number that is based on the size of a maximum matching.
- **Proposition 9.3.4.** *For any graph G*, $\chi'(G) \geq \left\lceil \frac{|E_G|}{\alpha'(G)} \right\rceil$.
- COMPUTATIONAL NOTE: There are low-order polynomial-time algorithms that find maximum matchings, based largely on the work of Jack Edmonds. Bipartite matching is discussed in §13.4, and references to algorithms and applications of matchings in general graphs are given there.

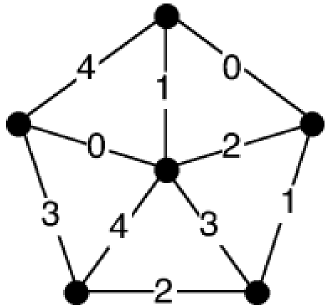
Algo 9.3.2: Edge-Coloring by Maximum Matching

Input: a graph G.
Output: a proper edge k-coloring f.

Initialize color number $k := 0$.
While $E_G \neq \emptyset$
  $k := k + 1$
  Find a maximum matching M of graph G.
  For each edge $e \in M$
    $f(e) := k$
  $G := G - M$ (edge-deletion subgraph)
Return edge-coloring f.

Edge-Chromatic Numbers for Common Graph Families

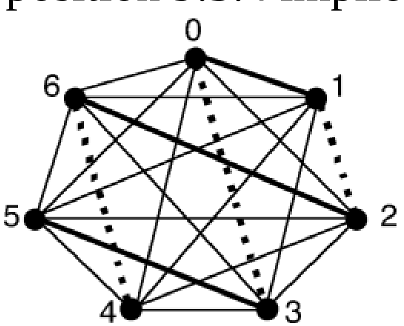
- **Proposition 9.3.5.** *A graph G has* $\chi'(G) = 1$ *if and only if* $\delta_{\max}(G) = 1$.
- **Proposition 9.3.6.** *Path Graphs:* $\chi'(P_n) = 2$, *for* $n \geq 3$.
- **Proposition 9.3.7.** *Even Cycle Graphs:* $\chi'(C_{2n}) = 2$.
- **Proposition 9.3.8.** *Odd Cycle Graphs:* $\chi'(C_{2n+1}) = 3$.
- **Proposition 9.3.9.** *Tree:* $\chi'(T) = \delta_{\max}(T)$, *for any tree T.*
- **Proposition 9.3.10.** *Cube Graphs:* $\chi'(Q_n) = n$.
- **Proposition 9.3.11.** *Wheel Graphs:* $\chi'(W_n) = n$, *for* $n \geq 3$.
- **Example 9.3.3:**



A proper edge 5-coloring of the wheel W_5.

Edge-Chromatic Numbers for Common Graph Families

- **Proposition 9.3.12.** *Odd Complete Graphs:* $\chi'(K_n) = n$*, for all odd* $n \geq 3$.
  - ***Upper bound*:** Draw the complete graph K_n so that its vertices are the vertices of a regular n-gon, labeled $0,1,2,\ldots,n-1$ clockwise around the n-gon.
  - Observe that the edge joining vertices 0 and 1 is parallel to all edges whose endpoints also sum to 1 (mod n). Since no two of these parallel edges are adjacent, they all can be assigned color 1.
  - Similarly, the edges whose endpoints sum to 3 (mod n) form a set of parallel edges that can be assigned the color 3. In all, there are n sets $S_1, S_2, \ldots, S_n$, where S_k is the set of parallel edges whose endpoints sum to k (mod n). Thus, if each of the edges in set S_k is assigned color c_k, then a proper edge n-coloring of K_n is obtained.
  - ***Lower bound*:** The size of a maximum matching in K_n with n odd is $\frac{n-1}{2}$ .Since K_n contains $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, Proposition 9.3.4 implies that $\chi'(K_n) \geq$ n.

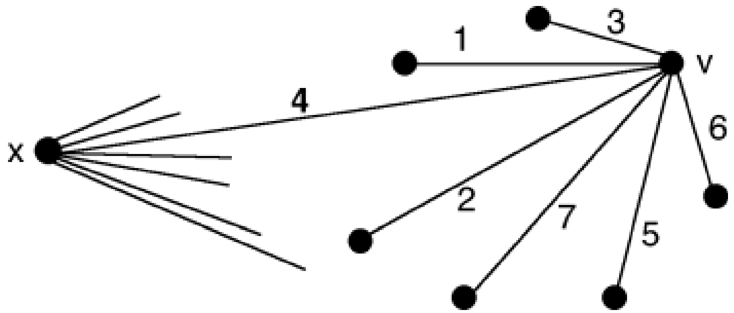


Two sets of parallel edges in K_7.

0+1, …, 0+6 (n-1 sets)
1+6 n sets in total

Edge-Chromatic Numbers for Common Graph Families

- **Corollary 9.3.13.** *Even Complete Graphs:* $\chi'(K_n) = n - 1$, *for all even n.*
  - ✓ The even complete graph K_n is the join of the odd complete graph $K_{n\text{-}1}$ with a single vertex x. The proof of Proposition 9.3.12 constructs a proper edge (n-1)-coloring of $K_{n\text{-}1}$ in which **each edge-color is missing at exactly one vertex**. Thus, the edge-coloring of $K_{n\text{-}1}$ can be extended to an edge-coloring of K_n by assigning the missing color at each vertex $v \in K_{n\text{-}1}$ to the edge joining vertex v to vertex x.



Extending the edge n-coloring from K_{n-1} to K_n.

Table 9.3.1 Edge-chromatic numbers for common graph families

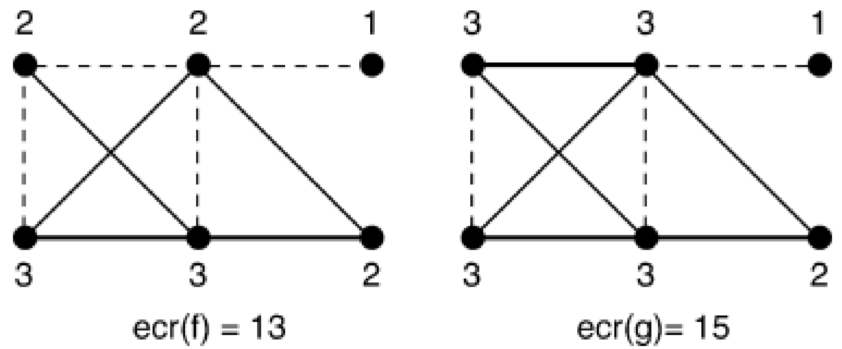
Graph G	$\chi'(G)$
graph G with $\delta_{\max}(G) = 1$	1
path graph P_n, $n \geq 3$	2
even cycle graph C_{2n}	2
odd cycle graph C_{2n+1}	3
bipartite graph G	$\delta_{\max}(G)$
tree T	$\delta_{\max}(T)$
cube graph Q_n	n
wheel W_n, $n \geq 3$	n
even complete graph K_{2n}	$n - 1$
odd complete graph K_{2n+1}	n

Chromatic Incidence

- **DEFINITION**: For a given edge-coloring of a graph, color i is an ***incident edge-color*** on vertex v if some edge incident on v has been assigned color i. Otherwise, color i is an ***absent edge-color*** at vertex v.
- **DEFINITION**: The ***chromatic incidence at*** v of a given edge-coloring f is the number of different edge-colors incident on vertex v. It is denoted $ecr_v(f)$.
- **DEFINITION**: The ***total chromatic incidence*** for an edge-coloring f of a graph G, denoted $ecr(f)$, is the sum of the chromatic incidences of all the vertices. That is,

$$ecr(f) = \sum_{v \in V_G} ecr_v(f)$$

- **Example 9.3.4:**



Total-chromatic-incidence calculations.

Chromatic Incidence

- **Proposition 9.3.14.** *Let f be any edge-coloring of a graph G. Then for every v∈G,*

$$ecr_v(f) \le deg(v)$$

- **Corollary 9.3.15.** *Let f be any edge-coloring of a graph G. Then*

$$\sum_{v \in V_G} ecr_v(f) \le \sum_{v \in V_G} deg(v)$$

- **Proposition 9.3.16.** *An edge-coloring f of a graph G is proper if and only if for every vertex v* $\in V_G$,

$$ecr_v(f) = deg(v)$$

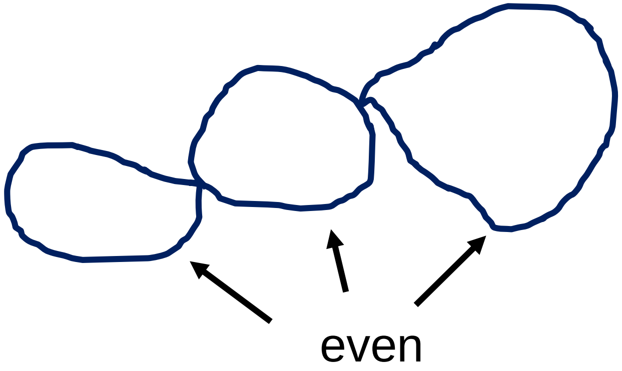
- **Corollary 9.3.17.** *An edge-coloring f of a graph G is proper if and only if*

$$\sum_{v \in V_G} ecr_v(f) = \sum_{v \in V_G} deg(v)$$

- **REVIEW FROM** §1.5: An ***eulerian tour*** in a graph is a closed trail that contains every edge of that graph. An ***eulerian graph*** is a graph that has an eulerian tour.

Edge-Coloring of Bipartite Graphs

- ***Lemma 9.3.18.*** *Let G be a connected graph with at least two edges that is not an odd-cycle graph. Then G has an edge 2-coloring such that both colors are incident on every vertex of degree at least 2.*
  - **Proof:** *Case 1*: **G is an even cycle**. The edge 2-coloring obtained by assigning two edge-colors that alternate around the cycle meets the requirement.
  - *Case 2*: **G is eulerian but not a cycle**. **Consider an eulerian tour that starts (and ends) at a vertex of degree at least 4.** Assign color 1 to the edges that occur as odd terms in the edge sequence of the tour, and assign color 2 to the even-term edges. Then the two colors are incident at least once on each internal vertex of the tour, since each such vertex is an endpoint of both an odd-term edge and an even-term edge. Moreover, since the start vertex has degree at least 4, it also occurs on the tour as an internal vertex. Thus, both colors are incident on every vertex.



Edge-Coloring of Bipartite Graphs

- ✓ *Case 3*: **<u>G is not eulerian</u>**. Construct an auxiliary graph G^* by joining a new vertex w to every odd-degree vertex of G, thereby making each such vertex even-degree in G^*. By a corollary to Euler's degree-sum equation (§1.1), every graph has an even number of odd-degree vertices, so vertex w has even degree. Thus, the auxiliary graph G^* is eulerian, by the eulerian-graph characterization (§4.5). Next, let f be an edge 2-coloring of graph G^*, as specified in Case 2. Then it is easy to verify that the edge-coloring f_G of f restricted to the edges of graph G is an edge-coloring such that both colors are incident on each vertex of G of degree at least 2.

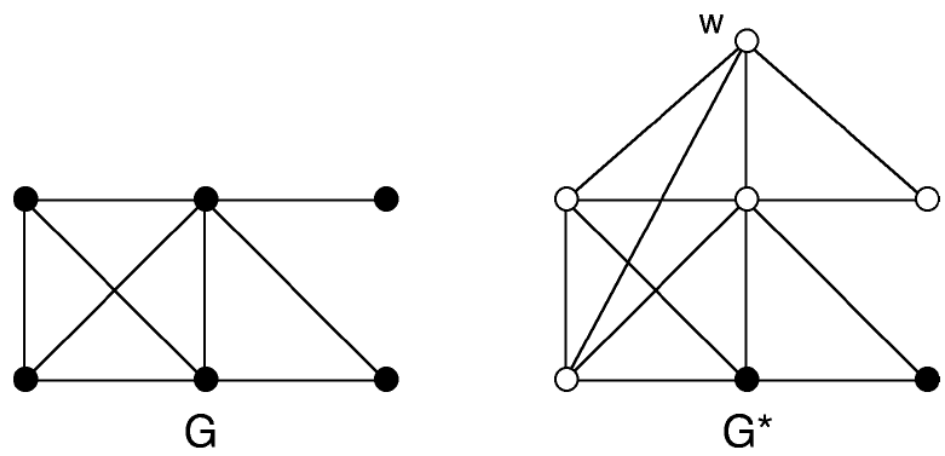
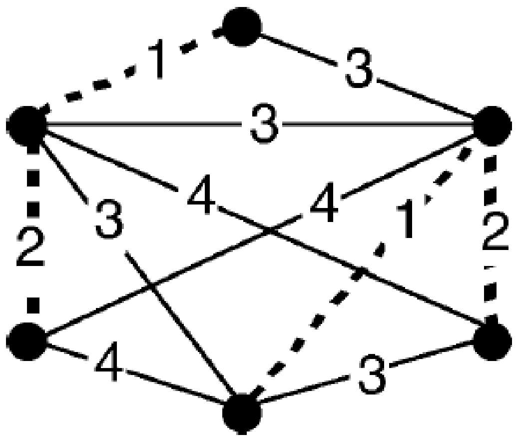
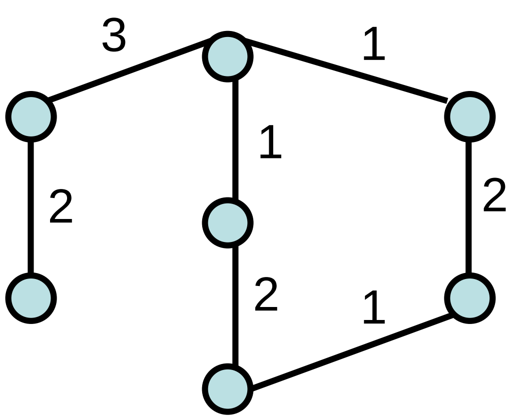


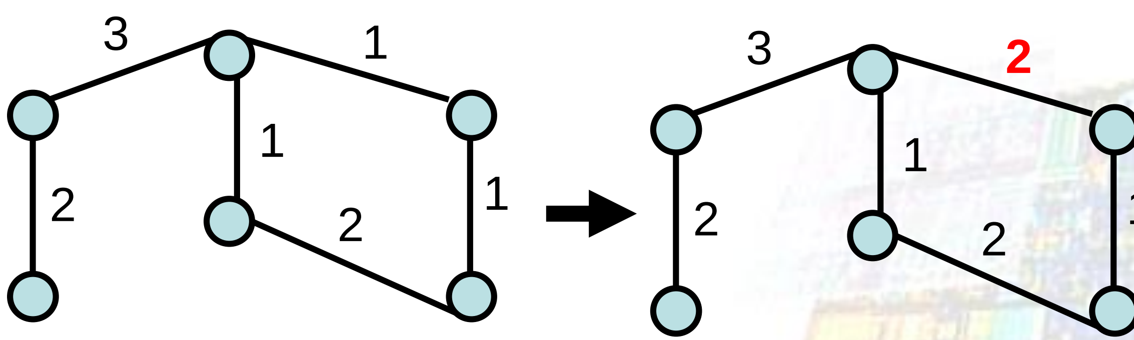
Fig 9.3.10 Constructing the auxiliary graph for Case 3.

Kempe *i-j* Edge-Chain

- **DEFINITION**: In a graph G with a (possibly improper) edge-coloring, a ***Kempe*** i-j ***edge-chain*** is a component of the subgraph of G induced on all the i-colored and j-colored edges.







König Theorem

- **Lemma 9.3.19.** *Let f be an edge k-coloring of a graph G with the largest possible total chromatic incidence. Let v be a vertex on which some color i is incident at least twice and on which some color j is not incident at all. Then the Kempe i-j edge-chain K containing vertex v is an odd cycle.*
  - ✓ **Proof:** By Lemma 9.3.18, if the Kempe *i-j* edge-chain *K* incident on vertex *v* were not an odd cycle, then we could rearrange edge color *i* and *j* within *K* so that the chromatic incidence of the coloring of *K* would be 2 at every vertex. The edge-coloring for *G* thereby obtained would have higher chromatic incidence at vertex *v* and at-least-equal chromatic incidence at every other vertex of *G*. This would contradict the premise that edge-coloring *f* has the maximum possible total chromatic incidence.

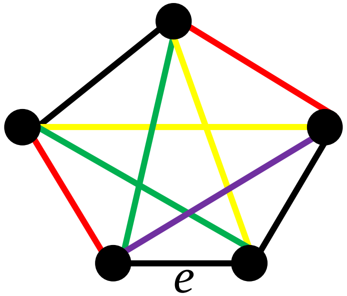
- **Theorem 9.3.20. [König, 1916].** *Let G be a bipartite graph. Then* $\chi'(G) = \delta_{max}(G)$.
  - ✓ **Proof:** Let $\Delta = \delta_{max}(G)$, and, by way of contradiction, suppose that $\chi'(G) \neq \Delta$. Then, by Corollary 9.3.2, $\Delta < \chi'(G)$. Next, let *f* be an edge Δ-coloring of graph *G* for which the total chromatic incidence $ecr_G(f)$ is maximum. Since *f* is not a proper edge-coloring, there is a vertex *v* such that $ecr_G(f) < deg(v)$ (by Proposition 9.3.16). Thus, some color occurs on at least two edges incident on *v*. But there are $\Delta - 1$ other colors and at most $\Delta - 2$ other edges incident on *v*, which means that some other color is not incident on vertex *v*. It follows by Lemma 9.3.19, that graph *G* contains an odd cycle, which contradiction the fact that *G* is bipartite.

Vizing Theorem

- **Vizing's Theorem**
  Complementing the lower bound $\chi'(G) \geq \delta_{max}(G)$ for a simple graph, provided by Corollary 9.3.2, Vizing's theorem provides a sharp upper bound that narrows the range for $\chi'(G)$ to two possible value.

- **DEFINITION**: Let G be a graph, and let f be a proper edge k-coloring of a subset S of the edges of G. Then f is a ***blocked partial edge k-coloring*** if for each unclosed (un-colored) edge e, every color has already been assigned to the edges that are adjacent to e. Thus, f cannot be extended to any edge outside subset S.

- **Lemma 9.3.21.** *Let i and j be two of the colors used in a proper edge-coloring of a graph. Then every Kempe i-j edge-chain K is a path (open or closed).*
  - ✓ **Proof:** Every vertex of Kempe chain K has degree at most 2 (since the edge-coloring is proper), and, by definition, K is a connected subgraph.

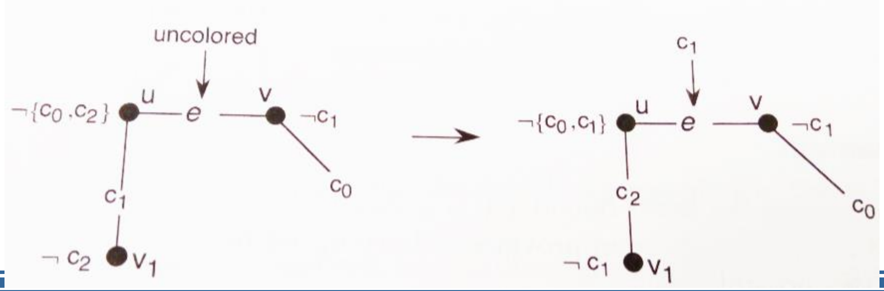


Attempted edge 4-coloring of k_5 that is blocked at edge e

Vizing Theorem

- **Theorem 9.3.22. [Vizing, 1964, 1965] [Gupta, 1966].** *Let G be a simple graph. Then there exists a proper edge-coloring of G that uses at most $\delta_{max}(G) + 1$ colors.*
  - **Proof:**

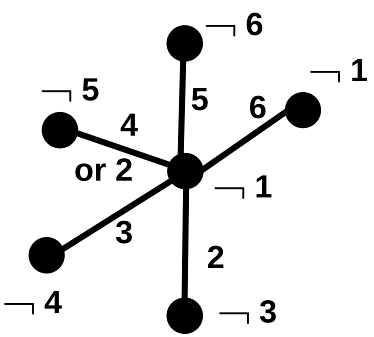
1. To construct such an edge-coloring, start by successively coloring edges, using any method (e.g., Algorithm 9.3.1) until the coloring is blocked or complete.
2. If the set of unclosed edges is empty, then the construction is complete. Or, there is some edge $e = (u, v)$ that remains unclosed. It will be shown that by recoloring some edges, the blocked colored coloring can be transformed into one that can be extended to edge e. The process can then be repeated until all edges have been colored.
3. $\chi'(G) > \delta_{max}(G) \rightarrow \forall v \in V_G$, at least one of the colors is absent. Let c_0 be a color absent at vertex u, and c_1 a color absent at vertex v. Color c_1 (c_0) cannot also be absent at vertex u (v), otherwise we can assign color c_1 (c_0) to edge e.
4. So let e_1 be the c_1-edge incident on vertex u, and let v_1 be its other endpoint. Next, let c_2 be a color absent at v_1. If c_2 is also absent at vertex u, then the color of edge e_1 can be changed from c_1 to c_2, thereby permitting the assignment of color c_1 to edge e. A missing color c at a vertex is indicated by placing $\neg$c alongside that vertex. Several missing colors may be grouped in braces.



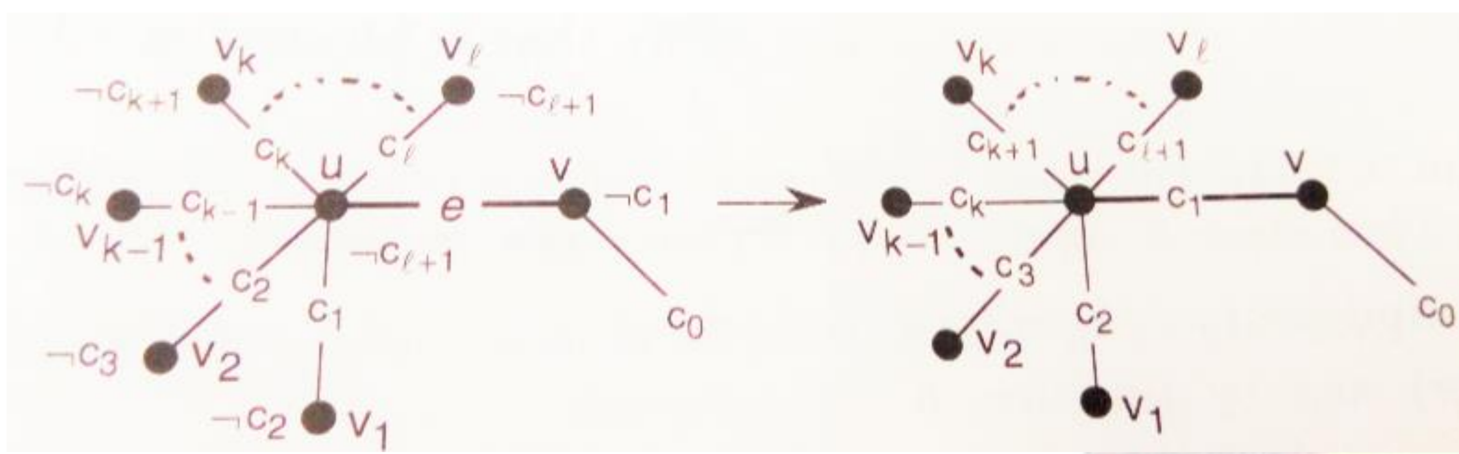
Vizing Theorem

5. If color c_2 does occur at vertex u, then let e_2 be the c_2-edge incident on vertex u, let v_2 be its other endpoint, and let c_3 be a color absent at vertex v_2. Continue iteratively in this way, so that at the jth iteration, e_j is the c_j-edge incident on vertex u, v_j is its other endpoint, and c_{j+1} is the color absent at vertex v_j. Let l be the smallest j such that vertex v_l has a missing color c_{l+1} and that c_{l+1} is also absent at vertex u or is one of the colors in the list $c_1,\ldots,c_l$ (such an l exists, since the set of colors is finite).

✓ *Case 1:* Color c_{l+1} is absent at both vertex v_l and vertex u. *Color Shift*.
Then perform the following *color shift*: for $j = 1,\ldots,l$, change the color of edge e_j from c_j to c_{j+1}. The releases color c_1 from edge e_1, so that it can be reassigned to edge e. Notice that it maintains a proper edge-coloring, because, by the construction, color c_{j+1} was absent at both endpoints of edge e_j before the shift.

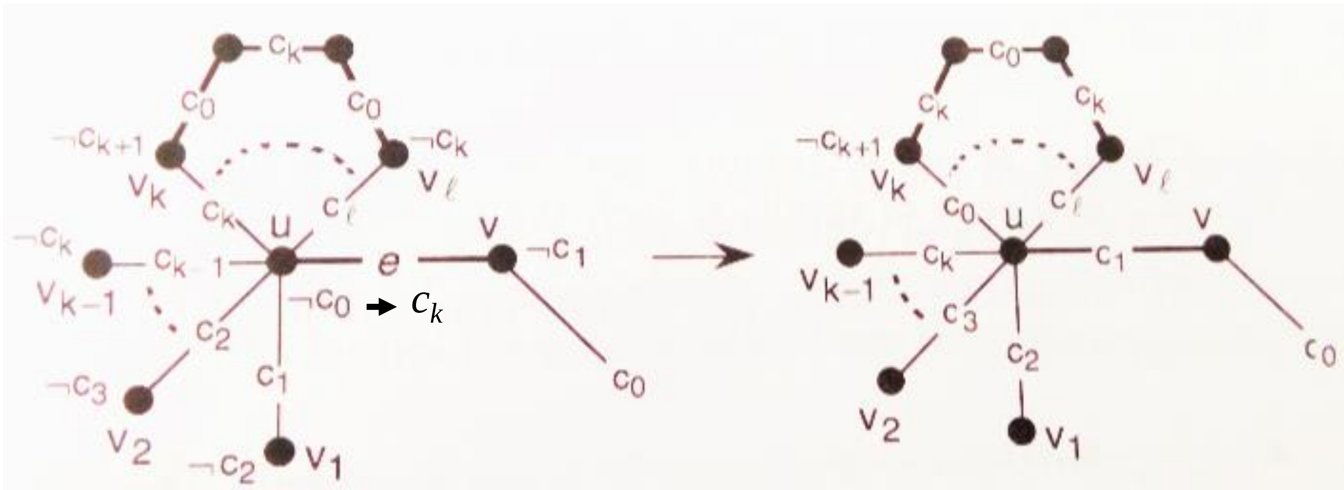


Δ=5



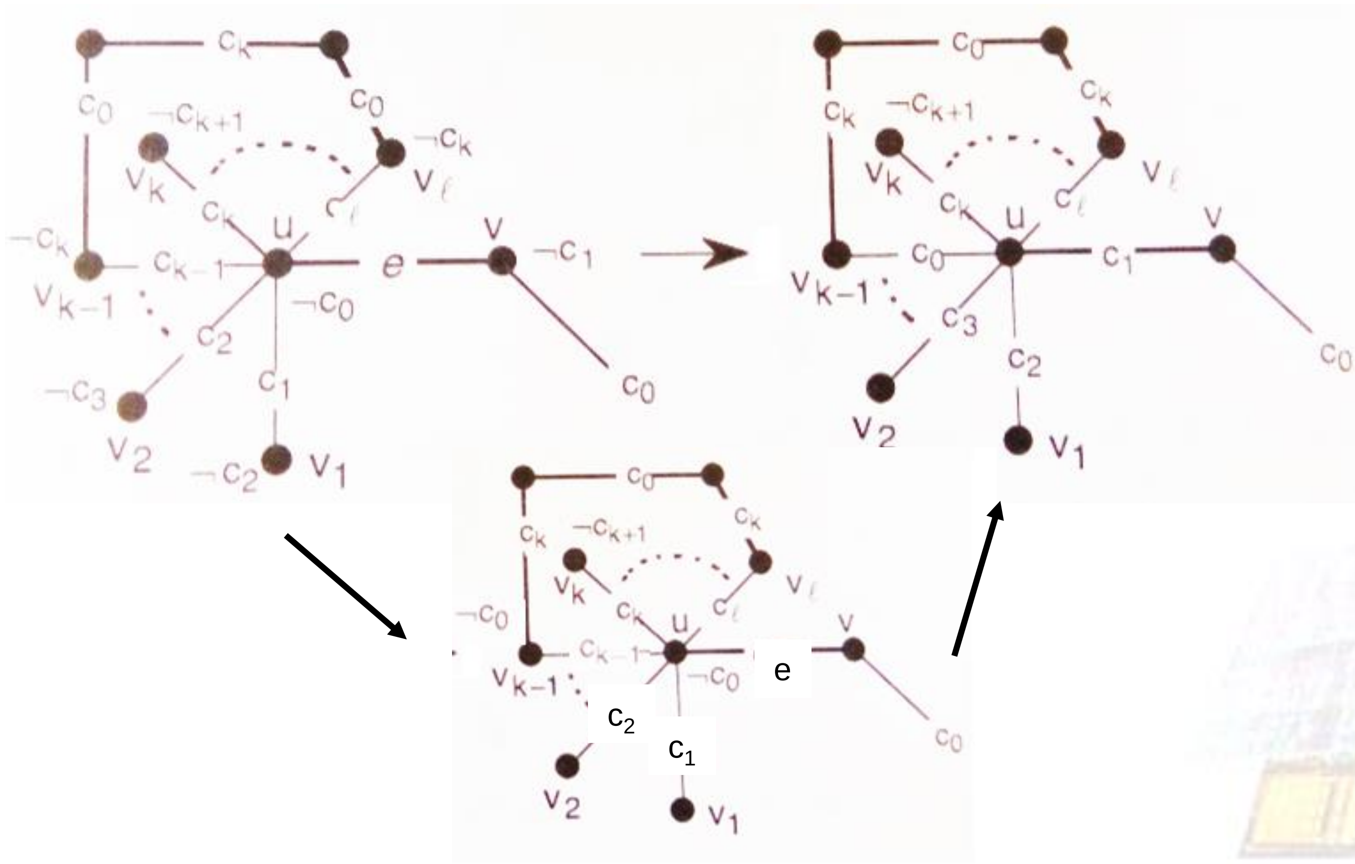
Vizing Theorem

- *Case 2:* Color $c_{l+1} = c_k$, where $1 \leq k \leq l$. *Swap and Shift.*
  Let K be the Kempe c_0-c_k edge-chain incident on vertex v_l. By definition, K includes the c_0-edge incident on v_l, but there is no c_k-edge incident on vertex v_l (by definition of l). By Lemma 9.3.21, Kempe chain K is a path, and one end of this path is vertex v_l. There are three subcases to consider, according to where the other end of the path is. In each of the three subcases, the two colors are swapped so that a Case 1 color shift can then be performed.
  - *Case 2a:* Path K reaches vertex v_k.
    Then swap color c_0 and c_k along path K. As a result of the swap, color c_k no longer occurs at vertex u. This configuration permits a Case 1 color shift that releases color c_1 for edge e.



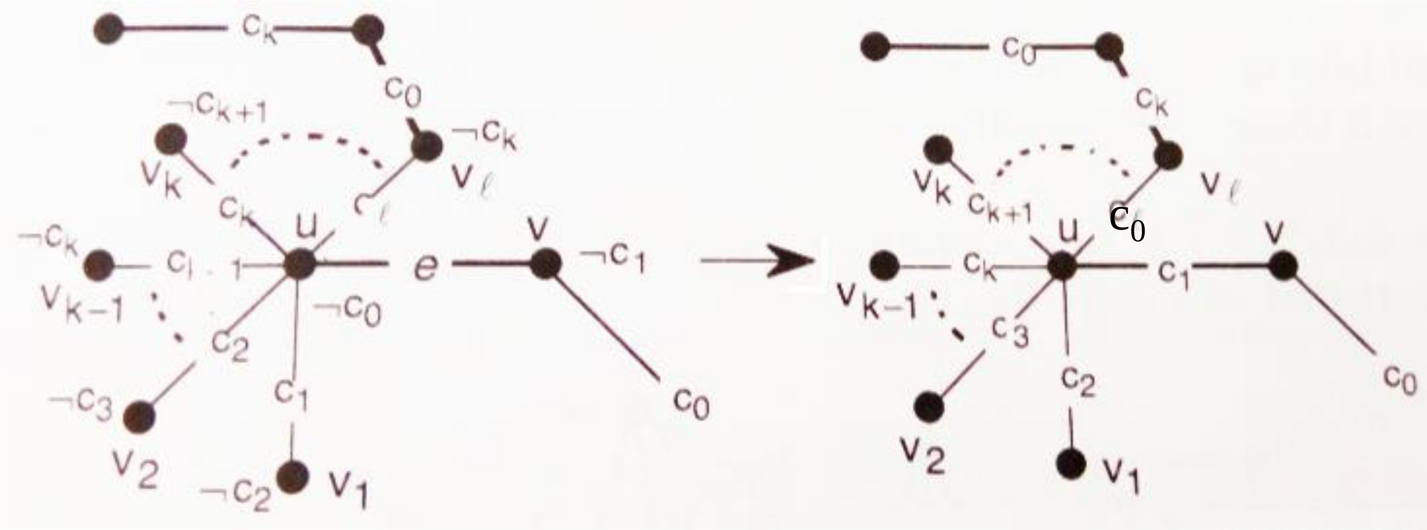
Vizing Theorem

- *Case 2b:* Path K reaches vertex $v_{k\text{-}1}$.
  Then swap color c_0 and c_k along path K. As a result of the swap, color c_0 no longer occurs at vertex $v_{k\text{-}1}$. Thus, edge $e_{k\text{-}1}$ can be recolored c_0. A color shift can now be performed to release color c_1 for edge e.



Vizing Theorem

- *Case 2c:* Path K never reaches vertex $v_{k\text{-}1}$ or vertex v_k.
  Since color c_0 does not occur at vertex u, and since color c_k occurs at only on the edge from v_k, it follows that path K does not reach vertex u. Then swap colors c_0 and c_k along path K, so that c_0 no longer occurs at vertex v_l. Now perform a Case 1 color shift that releases color c_1 for edge e.



9.3 Edge-Colorings

- **Corollary 9.3.23.** *Let G be a simple graph. Then either* $\chi'(G) = \delta_{max}(G)$ or $\chi'(G) = \delta_{max}(G) + 1$.
  - ✓ **Proof:** This follows immediately from Vizing's theorem and Corollary 9.3.2.

- **DEFINITION**: ***Class 1*** is the set of non-empty simple graphs G such that $\chi'(G) = \delta_{max}(G)$. ***Class 2*** is the set of simple graphs G such that $\chi'(G) = \delta_{max}(G) + 1$.
- **COMPUTATIONAL NOTE**: Deciding whether a simple graph is in class 1 is an NP-complete problem [Ho81].
- **DEFINITION**: The ***multiplicity*** is $\mu(G)$ of a graph G is the maximum number of edges joining two vertices of G.
- **Remark**: A more general result of Vizing, beyond simple graphs, which applies to every loopless graph G, is that $\delta_{max}(G) \leq \chi'(G) \leq \delta_{max}(G) + \mu(G)$. The edge-chromatic number achieves the upper bound of Vizing's general formula when all three vertex pairs of a "fat triangle", as illustrated by Figure 9.3.18, are joined by the same multiplicity of edges.

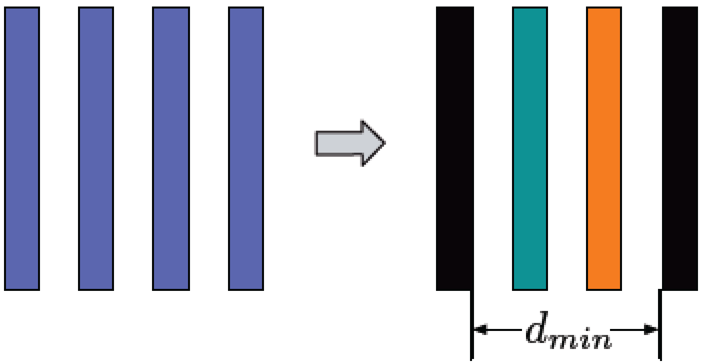
9.3 Edge-Colorings

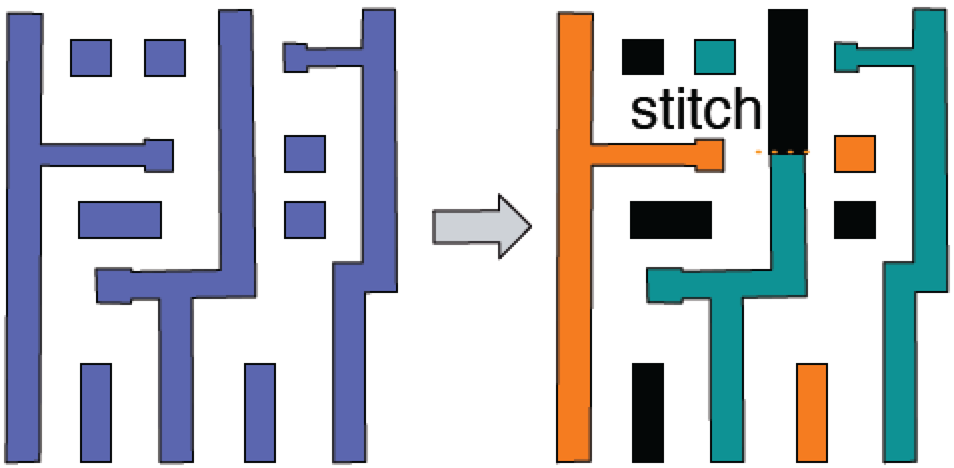
- **REVIEW FROM** §1.2: The ***line graph*** of a graph G is the graph $L(G)$ whose vertices correspond bijectively to the edges of G, and such that two of these vertices are adjacent if and only if their corresponding edges in G have a vertex in common.
- **Proposition 9.3.24.** *The edge-chromatic number of a graph G equals the vertex-chromatic number of its line graph* $L(G)$.
- **Proof:** This follows immediately from the definitions.

- **Remark**: Beineke [Be68] proved that a simple graph G is a line graph of some simple graph if and only if G does not contain any of the graphs in Figure 9.3.20 as an induced subgraph. Since it is an NP-complete problem to decide this subgraph problem, much of the theory of edge-colorings has prospered separately from the theory of vertex-colorings.

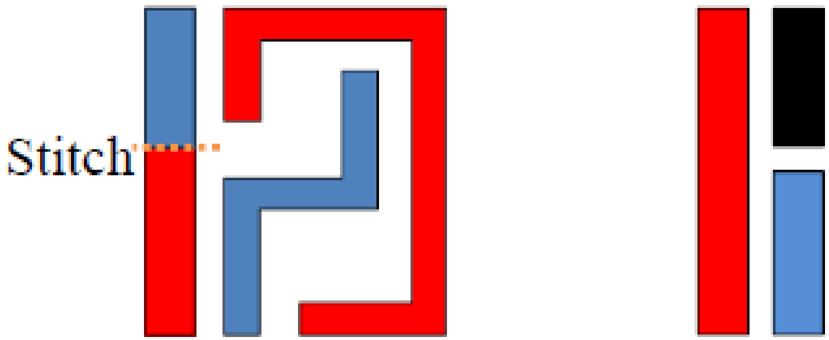
ITRS roadmap

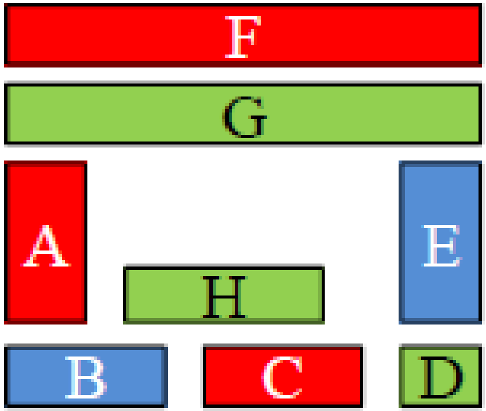
- 28nm single-patterning
- 20nm double-patterning
- 14nm triple-patterning / EUV
- 10nm quadruple-patterning / EUV





return


Stitch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