Study of lifting of 1D adinkras to 2D

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February 28, 2015

1 Preliminaries

1.1 1-d Adinkras

Adinkras in [references] will be referred to as 1-d Adinkras in this paper, since they relate to supersymmetry in 1 dimension. We will review a definition of 1-d Adinkras now. Note that while it is conventional to specify a natural number N to denote the number of supersymmetries, we break with convention and instead specify C, a set of N different colors.

Definition 1.1. A 1-d Adinkra with color set C is (V, E, χ, Δ, g) where

- V is a finite set of vertices
- $E \subset V \times V$ is a set of edges
- $\chi : E \to C$ is a map called the coloring
- $\Delta : E \to \{1, -1\}$ is a map called the dashing
- $g : V \to \mathbf{Z}$ is a map called the grading

These are required to satisfy the following:

- If $(v, w) \in E$, then $(w, v) \in E$. Furthermore, $\chi(v, w) = \chi(w, v)$ and $\Delta(v, w) = \Delta(w, v)$.
- For every $v \in V$ and $c \in C$, there exist exactly one $w \in V$ so that $(v, w) \in E$ and $\chi(v, w) = c$.
- If c_1, $c_2 \in C$ with $c_1 \neq c_2$, and $v \in V$, then there exist w, x, and $y \in V$ so that (v, w), (w, x), (x, y), and $(y, v) \in E$, and $\chi(v, w) = \chi(x, y) = c_1$ and $\chi(w, x) = \chi(y, v) = c_2$ and $\Delta(v, w)\Delta(w, x)\Delta(x, y)\Delta(y, v) = -1$.
- If $(v, w) \in E$, then $|g(v) - g(w)| = 1$.

Note that in [Reference], there is also a bipartition of the vertices. This is rendered obsolete by the grading, since $g(v)$ is even if and only if v is a boson.

Recall that a 1-d adinkra has a code $L(A)$, which is necessarily doubly-even. The code has parameters (n, k), where n is the number of coordinates and k is the dimensions of the code. In this situation, the vertices of A are labeled by cosets $\{0,1\}^n + L(A)$, so $|V(A)| = 2^{n-k}$.

Let A be a 1-d Adinkra with color set C, with vertex set V. Let $N = |C|$. For all $1 \leq i \leq N$, define

$$q_i : V \to V$$

so that for all $v \in V$, $q_i(v)$ is the unique vertex joined to v by an edge of color i.

Note that for all $v \in V$, $q_i(q_i(v)) = v$ and $q_i(q_j(v)) = q_j(q_i(v))$. As a result, we can then define an action of $\mathbf{Z}_2^N$ on V in the following way:

$$(x_1, \ldots, x_N)v = q_1^{x_1} \circ \cdots \circ q_N^{x_N}(v)$$

Pick a vertex $v \in A$. Define $C(A, v)$ to be the stabilizer of v under this action of $\mathbf{Z}_2^N$. Since it is a subgroup of $\mathbf{Z}_2^N$, $C(A, v)$ is a binary block code of length N.

Proposition 1.2. *The Adinkra A is connected if and only if the $\mathbf{Z}_2^N$ action is transitive on V.*

Proof. Let v, w be vertices in V. If A is connected, then there is a path in A connecting v to w. The edges in this path have colors $c_1, \ldots, c_k$. Then $q_{c_1} \cdots q_{c_k}(v) = w$. By the commutativity of $\mathbf{Z}_2^N$, we can rearrange $q_{c_1} \cdots q_{c_k}$ to be in increasing order of c_i. This is then of the form $q_1^{x_1} \circ \cdots \circ q_N^{x_N}$ for some $(x_1, \ldots, x_N) \in \{0,1\}^N$. □

Proposition 1.3. *If A is connected, then the code $C(A, v)$ does not depend on v.*

Proof. Let $w \in V$. By the previous proposition, there exists $h \in \mathbf{Z}_2^N$ so that $hv = w$.

The result follows from the sequence of equivalences:

$$gw = w \leftrightarrow ghv = hv \leftrightarrow hgv = hv \leftrightarrow gv = v$$

□

Thus, from now on we will refer to the code $C(A, v)$ as $C(A)$.

Theorem 1.4. *If A is a connected Adinkra, then $A \cong \mathbf{Z}_2^N / C(A)$.*

Proof. The relationship on vertices follows from standard group action theory.

For edges, note that if (v, w) is an edge in an Adinkra A, then $(q_i(v), q_i(w))$ is also an edge in A. Therefore

...there is a graph homomorphism from $\mathbf{Z}_2$ to A...[do we introduce graph homomorphisms definition here?] □

Theorem from other paper: $C(A)$ is doubly even. Furthermore, if a binary block code C of length N is doubly even, then there exists an Adinkra with $C(A) = C$.

1.2 2-d Adinkras

The notion of 2-d Adinkras is described in Ref...[fill in references to various things]. We use a definition here that is equivalent to the one found in [some reference]: the proof is found in Appendix...[maybe?]

A 2-d Adinkra is similar to a 1-d Adinkra except that some colors are called "left-moving" and the other colors called "right-moving". Edges are called "left-moving" if they are colored by left-moving edges, and right-moving otherwise. Furthermore, there are two gradings, one that is affected by the left-moving edges and the other for the right-moving edges.

More formally,

Definition 1.5. A 2-d Adinkra with disjoint color sets C_L and C_R is $(V, E, \chi, \Delta, g_L, g_R)$ where

- V is a finite set of vertices
- $E \subset V \times V$ is a set of edges
- $\chi : E \to C_L \cup C_R$ is a map called the coloring
- $\Delta : E \to \{1, -1\}$ is a map called the dashing
- $g_L : V \to \mathbf{Z}$ and $g_R : V \to \mathbf{Z}$ are maps called the left grading and right grading.

These are required to satisfy:

- The first requirement for a 1-d Adinkra still holds.
- The second and third requirements for a 1-d Adinkra still hold with $C = C_L \cup C_R$.
- The fourth requirement is replaced by: if $(v, w) \in E$ and $\chi(v, w) \in C_L$, then $|g_L(v) - g_L(w)| = 1$ and $g_R(v) = g_R(w)$. If $(v, w) \in E$ and $\chi(v, w) \in C_R$, then $|g_R(v) - g_R(w)| = 1$ and $g_L(v) = g_R(w)$.

Alternative definition (maybe use instead?)

Definition 1.6. A 2-d Adinkra with disjoint color sets C_L and C_R is a 1-d Adinkra (V, E, χ, Δ, g) with color set $C = C_L \cup C_R$, and two grading functions $g_L : V \to \mathbf{Z}$ and $g_R : V \to \mathbf{Z}$ so that

- $g(v) = g_L(v) + g_R(v)$
- if $(v, w) \in E$ and $\chi(v, w) \in C_L$, then $|g_L(v) - g_L(w)| = 1$ and $g_R(v) = g_R(w)$. If $(v, w) \in E$ and $\chi(v, w) \in C_R$, then $|g_R(v) - g_R(w)| = 1$ and $g_L(v) = g_R(w)$.

1.3 Product of Adinkras

One way to get 2-d Adinkras is to take a product of two 1-d Adinkras, where the first Adinkra uses only left-moving colors and the second Adinkra uses only right-moving colors.

Definition 1.7. Let C_L and C_R be disjoint color sets. Let $A_1 = (V_1, E_1, \chi_1, \Delta_1, g_1)$ be a 1-d Adinkra with color set C_L; and let $A_2 = (V_2, E_2, \chi_2, \Delta_2, g_2)$ be a 1-d Adinkra with color set C_R. We can define the product of these Adinkras as the following 2-Adinkra with color sets (C_L, C_R).

$$A_1 \times A_2 = (V, E, \chi, \Delta, g_L, g_R)$$

where

$$\begin{aligned}
V &= V_1 \times V_2 \\
E &= E_1 \cup E_2 \text{ where} \\
E_1 &= \{((v_1, w), (v_2, w)) \,|\, (v_1, v_2) \in E_1, \text{ and } w \in V_2\} \\
E_2 &= \{((v, w_1), (v, w_2)) \,|\, v \in V, \text{ and } (w_1, w_2) \in E_2\} \\
\chi((v_1, w), (v_2, w)) &= c_1(v_1, v_2) \text{ for all } ((v_1, w), (v_2, w)) \in E_1 \\
\chi((v, w_1), (v, w_2)) &= c_2(w_1, w_2) \text{ for all } (v, w_1), (v, w_2) \in E_2 \\
g_L(v, w) &= g_1(v) \\
g_R(v, w) &= g_2(w) \\
\Delta((v_1, w), (v_2, w)) &= \Delta_1(v_1, v_2) \\
\Delta((v, w_1), (v, w_2)) &= (-1)^{g_1(v)} \Delta_2(w_1, w_2)
\end{aligned}$$

Definition 1.8. Let p and q be non-negative integers and let $N = p + q$.

Given a binary block code C of length p, we can define a binary block code $\hat{C}$ of length N by appending to the end of every code word in C a string of 0s of length q.

Likewise, given a binary block code C of length q, we can define a binary block code $\check{C}$ of length N by prepending to the beginning of every code word in C a string of 0s of length p.

Proposition 1.9. *Let A_1 and A_2 be as above. Then*

$$C(A_1 \times A_2) = \hat{C}(A_1) \oplus \check{C}(A_2).$$

Proof. Let $(v_1, v_2) \in A_1 \times A_2$. Let $g \in \mathbf{Z}_2^N$. We can write $g = g_1 + g_2$ where g_1 is zero in the last q bits and g_2 is zero in the first p bits. Now

$$g(v_1, v_2) = (g_1 + g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2).$$

This means that $g(v_1, v_2) = (v_1, v_2)$ if and only if $g_1 v_1 = v_1$ and $g_2 v_2 = v_2$.

So $g \in C(A_1 \times A_2)$ if and only if $g_1 \in \hat{C}(A_1)$ and $g_2 \in \check{C}(A_2)$. $\square$

2 Structural Theorems

In this section, we show that the coherence conditions of 2-d adinkras force a lot of structure onto them. In particular, we can think of the vertices of 2-d adinkras as arranged in a rectangle, with the stucture of the entire adinkra basically determined by a horizontal and a vertical "slice" of the picture.

2.1 A 2-d Adinkra Fits in a Rectangle

Let the *support* of a 2-d adinkra (and/or its bigrading function g) be defined as the range of g, its bigrading function. Now, we show that the support of 2-d adinkra must form a rectangle in $\mathbf{Z}^2$. In this and the following sections, it helps to have some standard assumptions:

- Recall that any 1-d adinkra has vertices labeled by equivalence classes of $\mathbf{Z}_2^n$ by some (n, k) doubly-even code L. We will refer to vertices by these equivalence classes (or their representatives). We use the notation $(v_1, \ldots, v_n)$ and $(v_1 v_2 \cdots v_n)$ interchangeably.
- Without loss of generality, let our color set $C_L \cup C_R = \{1, 2, \ldots, n\}$, with $C_L = \{1, 2, \ldots, p\}$ and $C_R = \{p+1, p+2, \ldots, p+q = n\}$. We now identify these n elements with the n indices of the vertices thinking of them as the indices of $\mathbf{Z}_2^n$. For all $i \in C_L \cup C_R$, define the map q_i that takes a vertex v and returns the unique vertex joined to v by an edge of color i. So if $v = (v_1, \ldots, v_n)$, $q_i(v) = (v_1, cdots, v_{i-1}, 1 - v_i, v_{i+1}, \cdots, v_n)$. Note that $q_i^2(v) = v$.
- We will always define $\overline{0}$ to be the vertex corresponding to the equivalence class of $(00 \cdots 0)$. We will also assume that $g(\overline{0}) = (0, 0)$.

For every vertex pair (v, w) in our 2-d adinkra, there exist (many) paths from v to w. Ignoring the dashings for now, let the sequence of colors on any path be called a *color sequence* for the path. So, for example, in a chromotopology corresponding to the unique trivial $(4, 0)$ code, the path with color sequence $(3, 2, 1, 1)$ carries $\overline{0} = 0000$ to 0010, 0110, 1110, and finally 0110. Note that a color sequence $(i_1, i_2, \ldots, i_k)$ sends v to $q_{i_k} q_{i_{k-1}} \cdots q_{i_1}(v)$.

Now, define a map s that takes a color sequence and returns an element of $V = \{0, 1\}^n$ where the i-th element is the number of times (modulo 2) that color i appears in the sequence. For example, $s(3, 2, 1, 1) = 0110$. Note that $s(d)$ will return (a member of the equivalence class of) the bitstring obtained by applying the sequence d to $\overline{0}$. In particular that we may permute a sequence from $d = (d_n)$ to $d' = (d'_n)$ without changing the result of s. Thinking about paths is very important for us, because:

Proposition 2.1. *Start at any vertex v in an adinkra A with code $L(A)$, following two paths with color sequences d and d' ends up at the same resulting vertex if and only if $s(d) = s(d') \pmod{L}$.*

Proposition 2.2. *See [reference]. The main idea is that the q_i's commute, so where we end up after a color sequence d only depends on $s(d)$. By the definition of L, we end up at the same vertex only if the difference between the two sets of edges correspond to an element of L.*

Let $l(d)$ be a rearrangement of d that moves all the left-moving colors to the beginning, and let $r(d)$ be a rearrangement of d that moves all the right-moving colors to the beginning. Thus, suppose $p = q = 2$, we have $l((3, 2, 1, 1)) = (2, 1, 1, 3)$ and $r((3, 2, 1, 1)) = (3, 2, 1, 1)$. We always have $s(l(d)) = s(r(d))$ in general, since $l(d)$ and $r(d)$ are just permutations of each other.

Proposition 2.3. *Suppose we have any path from (x, y) to (v, w) in a 2-d adinkra A; then the vertices (x, w) and (v, y) are in A's support.*

Proof. Let this path take color sequence d. $l(d)$ and $r(d)$ must both end up at (v, w), but $l(d)$ only changes along the y-axis in the first part of moves that only use left-moving colors, so it must end up at coordinate (x, w) to be able to end up at (v, w) (since it can only use right-moving / x-axis colors afterwards). Using the same argument for $r(d)$ shows we must end up in (v, y) at some point in the path. □

Corollary 2.4. *The support of a 2-d adinkra is exactly some $k \times l$ rectangle. In other words, the set of points in $\mathbf{Z}^2$ in the range of g, the bigrading map, can be taken to the set of points (x, y), $0 \leq x < l$, $0 \leq y < k$.*

Proof. If the support is not a rectangle, then there must be some coordinates (x, y) and (v, w) in the support such that one of the other two diagonal coordinates are missing. This violates Proposition 2.3. □

While it is neat that the vertices of a 2-d adinkra A line up nicely in a rectangle, we now show that there is even more regularity in its structure. Let A_L (resp. A_R) be the subgraphs of A induced by left-moving (resp. right-moving) edges of A.

Lemma 2.5. *If X is a connected component of A_L and i is a right-moving color, then $q_i(X)$ is the vertex set of a connected component of A_L. The analogous statement for A_R also holds.*

Proof. Given two vertices $q_i(x)$ and $q_i(x')$ in $q_i(X)$, we know that x and x' are in X. Since X is connected in A_L, there is a path with left-moving colors from x to x'. This means

$$x' = q_{j_1} q_{j_2} \cdots q_{j_k} x,$$

so

$$q_i(x') = q_i q_{j_1} q_{j_2} \cdots q_{j_k}(x) = q_{j_1} q_{j_2} \cdots q_{j_k} q_i(x),$$

meaning $q_i(x')$ and $q_i(x)$ are connected in A_L. The converse is equally easy. □

Proposition 2.6. *All disconnected components in A_L (and respectively A_R) are isomorphic as graded posets.*

Proof. Let X and Y be two connected components of A_L. Pick vertices $x \in X$ and $y \in Y$. Since A is connected, there is a path from x to y in A. Reorder the path so that the right-moving edges occur before the left-moving edges. Since the left-moving edges stay in Y, the right-moving edges alone take x to a vertex $y' \in Y$. By Lemma 2.5, each such edge puts us in a new connected component isomorphic to X, so X is isomorphic to Y. They are all isomorphic as graded posets since translating via right-moving colors does not change the y-coordinate of the grading. □

With all this redundancy, what is the minimal amount of information required for us to understand a 2-d adinkra? Proposition 2.6 suggests we just need a single connected component for each direction to give us all the data; this turns out to basically be true, as we see in the next section.

3 Quotienting

To understand quotients, we first define the homomorphism from one 2-d Adinkra to another. This will be similar to the definition of homomorphism of graphs [give some standard reference to this terminology].

Definition 3.1. Let $A_1 = (V_1, E_1, \chi_1, \Delta_1, g_{L1}, g_{R1})$ and $A_2 = (V_2, E_2, \chi_2, \Delta_2, g_{L2}, g_{R2})$ be 2-Adinkras with the same color set C. A homomorphism from A_1 to A_2 is a map

$$\phi : V_1 \to V_2$$

satisfying the following:

- If $(v, w) \in E_1$, then $\phi(v, w) \in E_2$ and $\chi_1(v, w) = \chi_2(\phi(v, w))$.
- If $v \in V_1$ then $g_{1L}(v) = g_{2L}(\phi(v))$.
- If $v \in V_1$ then $g_{1R}(v) = g_{2R}(\phi(v))$.

Note that there is no condition on the dashings Δ_1 and Δ_2.

Proposition 3.2.

$$\hat{C}(A_L^0) \oplus \check{C}(A_R^0) \subseteq C(A).$$

Proof. Let $g \in \hat{C}(A_L^0)$ and $h \in \check{C}(A_R^0)$. Then $g\overline{0} = \overline{0}$ because applying g to $\overline{0}$ results in a path that lies completely inside A_L^0, and so the fact that $g\overline{0} = \overline{0}$ in A_L^0 (since $g \in \hat{C}(A_L^0)$) results in $g\overline{0} = \overline{0}$ in A. Likewise $h\overline{0} = \overline{0}$. So $(g + h)\overline{0} = g(h(\overline{0})) = \overline{0}$ and $g + h \in C(A)$. □

Theorem 3.3. *There is a binary block code K of length n so that*

$$A \cong (A_L^0 \times A_R^0)/K$$

and $K \cap \hat{C}(A_L^0) = 0$ and $K \cap \check{C}(A_R^0) = 0$.

Proof. From the previous proposition and basic linear algebra, there exists a vector subspace K of $\mathbf{Z}_2^N$ that is a vector space complement of $\hat{C}(A_L^0) \oplus \check{C}(A_R^0)$ in $C(A)$. That is,

$$C(A) = \hat{C}(A_L^0) \oplus \check{C}(A_R^0) \oplus K.$$

Then

$$\begin{aligned} A &\cong \mathbf{Z}_2^N / C(A) \\ &= \mathbf{Z}_2^N / (\hat{C}(A_L^0) \oplus \check{C}(A_R^0) \oplus K) \\ &= (\mathbf{Z}_2^N / (\hat{C}(A_L^0) \oplus \check{C}(A_R^0)))/K \\ &= (\mathbf{Z}_2^N / C(A_L^0 \times A_R^0))/K \\ &= (A_L^0 \times A_R^0)/K \end{aligned}$$

□

Theorem 3.4. *Any 2-d adinkra A is isomorphic to $A_L^0 \times A_R^0/G$ for some group G.*

Proof. □

[⋆⋆⋆ TODO ⋆⋆⋆]

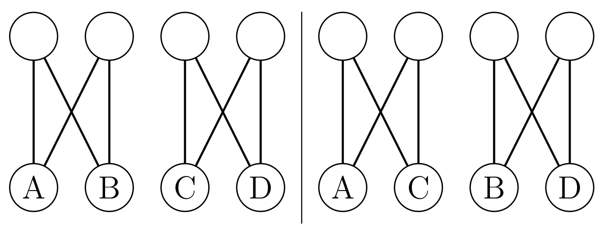

Figure 1: Taking the product of the two adinkras here with the following identification gives a non-disconnected adinkra with 16 vertices.

4 ESDE Codes

Define a *even-split doubly-even (ESDE) code* to be a doubly-even code isomorphic to a direct sum $C_L \oplus C_R$ of even codes. Recall that 1-d chromotopologies are in bijection with quotients of the hamming cube I^n by a doubly-even code L, so any adinkra A has a well-defined associated code $L(A)$ that is uniquely determined by just the graph structure of the adinkra. Our goal is to show that ESDE codes are exactly the codes that appear for 2-d adinkras. To do this, we introduce a special family of 2-d adinkras.

For an 1-d adinkra A with grading function g, let $\mathrm{Val}(A)$, the *valise* of A, be defined as a 1-d adinkra A' identical to A except for its grading function g', defined as $g'(v) = 0$ if $g(v) \in 2\mathbf{Z}$ and 1 otherwise. It is easy to see that $\mathrm{Val}(A)$ is also an adinkra. Similarly, for any 2-d adinkra A with bigrading function g, there is a unique 2-d adinkra A' with the same chromotopology as A such that $(A')^0_L = \mathrm{Val}(A^0_L)$ and $(A')^0_R = \mathrm{Val}(A^0_R)$, defined with a bigrading function $g'(v) = (x', y')$, where $x', y' \in \{0, 1\}$ depending on x and y (mod ()2) respectively given $g(v) = (x, y)$. We similarly denote A' by $\mathrm{Val}(A)$ and call such an adinkra a *valise* (2-d) adinkra. Equivalently, a valise adinkra is one where the supporting rectangle is a 2×2 square, and any 2-d adinkra could be put into a valise form.

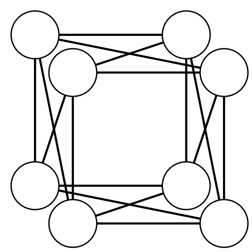

Figure 2: A valise 2-d adinkra that cannot be put into non-valise form.

Theorem 4.1. *For a code $L \subset \mathbf{Z}_2^n$, there exists a 2-d adinkra A with $L(A) = L$ if and only if L is a ESDE code.*

Proof. Suppose $L(A) = L$ for some 2-d adinkra A. We know that L is doubly-even. Consider any codeword $l \in L$. Starting at $\overline{0} \in A$, moving by a path corresponding to l must end up back at $\overline{0}$. In particular, it must use an even number of left-moving (resp. right-moving) edges since each of which changes the x-coordinate (resp. y-coordinate) by 1 in absolute value. Thus, L must be ESDE.

Now, take an ESDE code L. L is doubly-even, so there exists a 1-d adinkra A with code $L(A) = L$. We can assume A is a 1-d valise by taking $A = \mathrm{Val}(A)$ if necessary, which does not change the code. Now, any vertex $v \in A$ corresponds to an equivalence class $c + L \subset \mathbf{Z}_2^n$. consider the function $g' \colon V(A) \to \{0, 1\}$ that sends v to (x, y), where x (resp. y) corresponds to the parity of the weight of the left-moving colors of any element in $c + L$. The fact that L is ESDE precisely makes this notion well-defined. Since g' gives A a bigrading, A is realizable as a 2-d valise adinkra A with code L, so we are done. $\square$

A Equivalence with other notions of 2-d Adinkras

If I read Tristan's stuff right, we can completely translate the combinatorial rules to: a *2-d adinkra* (of dimension n) is a finite simple connected graph A such that:

- It is an 1-d adinkra (with the associated ranking, dashing, etc.).
- It has $p + q = n$ colors, where the first p-colors are called "left-moving" and the second q-colors are called "right-moving."

- A coherence condition: for any cycle, we imagine the following sum: going up (here "up" comes from the grading we have from the engineering dimension in our ranking for the 1-d adinkra) a left-handed edge adds -1, and going up a right-handed edge adds 1; going down the edges give contributions with opposite signs. The sum of this around any cycle must be 0. (in particular, this rules out things like ambidextrous bow-ties)

Assuming I interpreted these rules correctly, now I can do combinatorics without needing any physics.

The first structural fact we can impose is a bi-grading that is compatible with the grading we already have from the 1-d adinkra structure, in the sense that the 1-d grading is simply one of the coordinates of our bi-grading.

Proposition A.1. *A* 1*-d adinkra can be extended to a* 2*-d adinkra if and only if the* 1*-d adinkra has a* bigrading *to* $\mathbf{Z}^2$*. This is a map* $g : V \to \mathbf{Z}^2$*, such that all left-moving edges correspond to displacements of* $(0, 1)$ *and right-moving edges correspond to displacements of* $(1, 0)$*.*

Proof. Proof delayed until talking more with Kevin and Tristan about the easiest way to write things up to avoid reinventing wheels. □

B Misc. (unorganized)

Corollary B.1. *Consider the vertex* $\overline{0} \in A$*. Let the connected component of* A_L *(resp.* A_R*) that* $\overline{0}$ *belongs to be labeled* A_L^0 *(resp.* A_R^0*). The adinkra* A *is uniquely determined*[1] *by* A_L^0 *and* A_R^0*.*

Proof. Consider the color sequence d of any path from $\overline{0}$ to a vertex v. We can permute the sequence so that the left-moving colors all occur before the right-moving colors. Thus, we first make some moves in A_L^0, then by Proposition 2.6 we make the remaining moves in a copy of A_R^0. □

Corollary B.2. *A valise* 2*-d adinkra* A *of type* (n, k) *is uniquely determined by* $B_L(A)$*,* $B_R(A)$*, and an identification of* $V(B_L(A), 0)$ *and* $V(B_R(A), 0)$*. Furthermore,* $|B_L(A)| = |B_R(A)| = 2^{n-k-1}$*.*

Problem 1. What are all the 2-d adinkras A with the same valise adinkra $\mathrm{Val}(A)$? Not all lifts are possible. For example, the adinkra in Figure 2 cannot be lifted to any non-valise form!

Here are some other problems:

Problem 2. Given two valise 1-d adinkras $B_L(A)$ and $B_R(A)$ of equal size, what identifications of $V(B_L(A), 0)$ and $V(B_R(A), 0)$ are possible?

[1] This is fairly nuanced; we need to know not just the shape of A_L^0 and A_R^0, but also their vertices

Proof. Data: if we have $\{0,1\} \cup \{2,3\}$ on one side, the other side must be $\{0,2\} \cup \{1,3\}$. □

[$\star\star\star$ There are two kinds of quotienting that we can think of: one quotient is directly quotienting the mega hypercube adinkra by a ESDE code; one quotient is given the valise adinkra with associated $B_L(A)$, $B_R(A)$ each with 2^{d+1} vertices, the necessary d-dimensional quotienting that occurs when we naively tensor the two parts (which gives 2^{2d} vertices in each corner, for 2^{2d+2} total vertices, when in the end we just want 2^{d+2} vertices. $\star\star\star$]