A proof of the Hübsch conjecture on constructing 2D Adinkras from 1D Adinkras

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1 Preliminaries

1.1 1-d Adinkras

Adinkras in [references] will be referred to as 1-d Adinkras in this paper, since they relate to supersymmetry in 1 dimension. We will review a definition of 1-d Adinkras now.

Definition 1.1. Let n be a non-negative integer. A 1-d Adinkra with n colors is (V, E, c, μ, g) where

- (V, E) is a finite undirected graph (called the underlying graph of the Adinkra) with vertex set V and edge set $E \subset V \times V$,
- $c : E \to \{1, \ldots, n\}$ is a map called the coloring,
- $\mu : E \to \{1, -1\}$ is a map called the dashing,
- $h : V \to \mathbf{Z}$ is a map called the grading.

These are required to satisfy the following:

1. If $(v, w) \in E$, then $(w, v) \in E$. Furthermore, $c(v, w) = c(w, v)$ and $\mu(v, w) = \mu(w, v)$. Intuitively, the edges are undirected.
2. For every $v \in V$ and $c \in \{1, \ldots, n\}$, there exist exactly one $w \in V$ so that $(v, w) \in E$ and $c(v, w) = c$.
3. If c_1, $c_2 \in \{1, \ldots, n\}$ with $c_1 \neq c_2$, and $v \in V$, then there exist w, x, and $y \in V$ so that (v, w), (w, x), (x, y), and $(y, v) \in E$, and $c(v, w) = c(x, y) = c_1$ and $c(w, x) = c(y, v) = c_2$ and $\mu(v, w)\mu(w, x)\mu(x, y)\mu(y, v) = -1$.
4. If $(v, w) \in E$, then $|h(v) - h(w)| = 1$.

Note that in [Reference], there is also a bipartition of the vertices, where some vertices are represented by open circles and called bosons, and other vertices are represented by filled circles and called fermions. This is not necessary to include in our definition, because a vertex v is a boson if and only if $h(v)$ is even.

1.2 The action of $\mathbf{Z}_2^n$ and the code

Let A be a 1-d Adinkra with n colors, with vertex set V. For all $1 \leq i \leq n$, define

$$q_i : V \to V$$

so that for all $v \in V$, $q_i(v)$ is the unique vertex joined to v by an edge of color i.

Proposition 1.2. *The map q_i is a graph isomorphism from the underlying graph of A to itself which preserves colors. It is an involution and for all i, j, we have*

$$q_i \circ q_j = q_j \circ q_i. \tag{1.3}$$

Proof. The statement that q_i is a graph homomorphism means that if (v, w) is an edge in A, then so is $(q_i(v), q_i(w))$. This follows from items 2 and 3 in the definition of an Adinkra above, using $j = c(v, w)$.

The fact that $q_i(q_i(v)) = v$ for all $v \in V$ follows from item 2 in the definition. This means that q_i is an involution and in particular is an isomorphism.

The equation (1.3) follows from item 3 of the definition when $i \neq j$ and is trivial when $i = j$. □

By combining the $q_1, \ldots, q_n$ maps, we can define an action of $\mathbf{Z}_2^n$ on the graph (V, E) underlying the Adinkra in the following way:

Definition 1.4. The action of $\mathbf{Z}_2^n$ on the graph (V, E) underlying the Adinkra is given on vertices by

$$(x_1, \ldots, x_N)v = q_1^{x_1} \circ \cdots \circ q_n^{x_n}(v)$$

and the fact that it acts on edges can be seen by the fact that each q_i is a graph homomorphism. It preserves colors since each q_i does.

Definition 1.5. A path is a finite sequence of edges of the form $((v_1, v_2), (v_2, v_3), \ldots, (v_{k-1}, v_k))$. The *color sequence* of the path is the sequence $(c(v_1, v_2), c(v_2, v_3), \ldots, c(v_{k-1}, v_k))$.

So if there is a path from v to w with color sequence $(i_1, \ldots, i_k)$, we have $w = q_{i_k} \circ \cdots \circ q_{i_1}(v)$.

Proposition 1.6. *Let A be an Adinkra and let v be a vertex of A. Let σ be a color sequence. There exists a unique path in A that starts at v and has color sequence σ.*

Proof. This can be proved by induction on the length of σ, and using the fact that given a vertex v_i of A, and a color c_i, there exists a unique vertex v_{i+1} so that (v_i, v_{i+1}) is an edge of A with color c_i. □

Now, define a map s that takes a color sequence and returns an element of $\mathbf{Z}_2^n = \{0,1\}^n$ where the i-th coordinate is the number of times (modulo 2) that color i appears in the sequence. For example, $s(3,1,2,1) = 0110$. Note that $s(\sigma)$ does not depend on the ordering of the color sequence σ. This relates to paths in Adinkras because of the following:

Proposition 1.7. *Let A be an Adinkra. Let v be a vertex of A and let p be a path that begins at v. Let σ be the color sequence obtained from p. Then the path p ends at the vertex $s(\sigma)v$.*

Proof. If the color sequence is $\sigma = (i_1, \ldots, i_k)$, then the path p ends at $q_{i_k} \circ \cdots \circ q_{i_1}(v)$. By the commutativity of the q_i, we can order them in non-decreasing order of i_j. If any of the q_i appear more than once, we use the fact that q_i^2 is the identity to reduce the number of q_i modulo 2. The result is $s(\sigma)v$. $\square$

Corollary 1.8. *Let A be an Adinkra. Let v be a vertex of A and let p and p' be paths that begin at v. Let σ and σ' be the color sequences obtained from p and p', respectively. If $s(\sigma) = s(\sigma')$, then p and p' both end at the same point.*

Definition 1.9. Pick a vertex $v \in A$. Define $C(A, v)$ to be the stabilizer of v under this action of $\mathbf{Z}_2^n$. Since it is a subgroup of $\mathbf{Z}_2^n$, $C(A, v)$ is a binary block code of length n.

Proposition 1.10. *The Adinkra A is connected if and only if the $\mathbf{Z}_2^n$ action is transitive on the vertex set of A.*

Proof. Let v, w be vertices of A. If A is connected, then there is a path in A connecting v to w. Let σ be the color sequence obtained from this path. Then by Proposition 1.7, $s(\sigma)v = w$.

Conversely, suppose the action is transitive. Let v and w be vertices of A. Then there exists a $\mathbf{x} \in \mathbf{Z}_2^n$ so that $w = \mathbf{x}v$. Write $\mathbf{x} = (x_1, \ldots, x_n)$ and construct a color sequence σ by taking the i for which $x_i = 1$. By Proposition 1.6, there is a path starting at v that has σ as its color sequence. By Proposition 1.7, this path ends at $s(\sigma)v = \mathbf{x}v = w$. $\square$

Proposition 1.11. *If A is connected, then the code $C(A, v)$ does not depend on v.*

Proof. Let $w \in V$. By Proposition 1.10, there exists a $\mathbf{y} \in \mathbf{Z}_2^n$ so that $\mathbf{y}v = w$.

Let $\mathbf{x} \in \mathbf{Z}_2^n$. The result follows from the sequence of equivalences:

$$\mathbf{x}w = w \Leftrightarrow \mathbf{x}\mathbf{y}v = \mathbf{y}v \Leftrightarrow \mathbf{y}\mathbf{x}v = \mathbf{y}v \Leftrightarrow \mathbf{x}v = v$$

$\square$

Definition 1.12. Given a connected Adinkra A, the code for A, called $C(A)$, is defined to be $C(A, v)$, where v is a vertex of A.

Proposition 1.13. *Let A be an Adinkra. Let v be a vertex of A and let p and p' be paths that begin at v. Let σ and σ' be the color sequences obtained from p and p', respectively. The paths p and p' end at the same vertex if and only if*

$$s(\sigma) - s(\sigma') \in C(A).$$

Proof. Suppose p and p' end at the same vertex. Then by Proposition 1.7,

$$s(\sigma)v = s(\sigma')v.$$

Then[1]

$$v = s(\sigma')^{-1}(s(\sigma)v) = (s(\sigma) - s(\sigma'))v.$$

Thus, $s(\sigma) - s(\sigma') \in C(A)$.

Conversely, suppose $s(\sigma) - s(\sigma') \in C(A)$. Then by reversing the above argument,

$$s(\sigma)v = s(\sigma')v$$

and thus, by Proposition 1.7, p and p' end at the same vertex. □

The following 1-d Adinkra was defined in [reference].

Construction 1.14. *For any non-negative integer n, we have an Adinkra $I^n = (V, E, c, \mu, h)$ with*

- $V = \mathbf{Z}_2^n = \{0, 1\}^n$,
- $E = \{(v, w) \mid v$ *and* w *differ in precisely one coordinate*$\}$,
- $c(v, w)$ *is the coordinate where* v *and* w *differ,*
- $\mu(v, w)$ *is the number of* 1*s in* v *before the coordinate* $c(v, w)$*, modulo* 2*,*
- $h(v) = \mathrm{wt}(v)$ *is the number of* 1*s in* v*.*

Definition 1.15. Given C a linear block code of length n, we can define a graph with edge colors I^n/C as the orbit space of the action of $C \subset \mathbf{Z}_2^n$ on I^n as a graph.

Definition 1.16. A graph homomorphism from a graph (V_1, E_1) to a graph (V_2, E_2) is a map $\phi : V_1 \to V_2$ so that if $(v, w) \in E_1$ is an edge, then $(\phi(v), \phi(w)) \in E_2$ is an edge. If there is a coloring $c_1 : E_1 \to \{1, \ldots, n\}$ and a coloring $c_2 : E_2 \to \{1, \ldots, n\}$, we say that ϕ preserves colors if $c_1(v, w) = c_2(\phi(v), \phi(w))$. If ϕ is bijective, we say that it is a graph isomorphism.

Theorem 1.17. *If A is a connected Adinkra, then there is a graph isomorphism from $I^n/C(A)$ to A that preserves colors.*

[1] Note that in this sequence of equations, $\mathbf{Z}_2^n$ is written additively but the group action is written multiplicatively.

Proof. Choose a vertex $\overline{0}$ in A. Let

$$\phi : I^n \to A$$

$$\phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n)\overline{0}$$

where we are using the action of $\mathbf{Z}_2^n$ on A as described above.

To see that this is a graph homomorphism, let $(x_1, \ldots, x_n) \in \mathbf{Z}_2^n$ and let $(y_1, \ldots, y_n)$ be another vertex connected to $(x_1, \ldots, x_n)$ with an edge of color i. Then $y_j = x_j$ for all $j \neq i$ and $y_i = 1 - x_i$. Then $(y_1, \ldots, y_n)\overline{0} = q_1^{y_1} \cdots q_n^{y_n}(\overline{0}) = q_1^{x_1} \cdots q_i^{1-x_i} \cdots q_n^{x_n}(\overline{0}) = q_i(q_1^{x_1} \cdots q_n^{x_n}(\overline{0})) = q_i(x_1, \ldots, x_n)$. So $\phi(x_1, \ldots, x_n)$ and $\phi(y_1, \ldots, y_n)$ are connected by an edge of color i. Note that this also shows that the graph homomorphism preserves colors.

We now prove that ϕ is surjective. Since A is connected, $\mathbf{Z}_2^n$ acts transitively on the vertex set of A, and so for any vertex v of A, there exists an element $\mathbf{x} \in \mathbf{Z}_2^n$ so that $\mathbf{x}\overline{0} = v$. Then $v = \phi(\mathbf{x})$.

To prove the isomorphism from $I^n/C(A)$ to A, we consider the necessary and sufficient conditions for $\phi(\mathbf{x}) = \phi(\mathbf{y})$ for $\mathbf{x}$, $\mathbf{y} \in \mathbf{Z}_2^n$. The condition $\phi(\mathbf{x}) = \phi(\mathbf{y})$ is equivalent to $\phi(\mathbf{x} - \mathbf{y})\overline{0} = \overline{0}$. This is equivalent to saying $\mathbf{x} - \mathbf{y} \in C(A)$. Thus, the map ϕ descends to $I^n/C(A)$ and gives an isomorphism. □

In Reference ..., it was proved that if A is a connected Adinkra, then $C(A)$ is a doubly even code. Furthermore, if C is any doubly even binary block code of length n, then there is an Adinkra with $C(A) = C$.

Lemma 1.18. *Given connected* 1*-d Adinkras A and B and vertices a of A and b of B, there is at most one graph isomorphism from A to B that preserves colors and sends a to b.*

Proof. Suppose $\phi : A \to B$ is a graph isomorphism that preserves colors and sends a to b.

Let v be a vertex of A. Since A is connected, there is a path from a to v. This produces a color sequence σ. Since ϕ is a graph isomorphism that preserves colors, ϕ sends this path to a path in B starting from b with the same color sequence. By Proposition 1.6, there is only one path in B with this property. The end of this path is $\phi(v)$. □

1.3 2-d Adinkras

The notion of 2-d Adinkras is described in Ref...[fill in references to various things]. We use a definition here that is equivalent to the one found in [some reference]: the proof is found in Appendix...[maybe?]

A 2-d Adinkra is similar to a 1-d Adinkra except that some colors are called "left-moving" and the other colors called "right-moving". Edges are called "left-moving" if they are colored by left-moving edges, and right-moving otherwise. Furthermore, there are two gradings, one that is affected by the left-moving edges and the other for the right-moving edges.

More formally,

Definition 1.19. Let p and q be non-negative integers. A 2-d Adinkra with (p, q) colors is a 1-d Adinkra (V, E, c, μ, h) with $p + q$ colors, and two grading functions $h_L : V \to \mathbf{Z}$ and $h_R : V \to \mathbf{Z}$ so that

- $h(v) = h_L(v) + h_R(v)$
- if (v, w) is a left-moving edge, then $|h_L(v) - h_L(w)| = 1$ and $h_R(v) = h_R(w)$. If (v, w) is a right-moving edge, then $|h_R(v) - h_R(w)| = 1$ and $h_L(v) = h_R(w)$.

1.4 Product of Adinkras

One way to get 2-d Adinkras is to take a product of two 1-d Adinkras, where the first Adinkra uses only left-moving colors and the second Adinkra uses only right-moving colors.

Construction 1.20. *Let p and q be non-negative integers. Let $A_1 = (V_1, E_1, c_1, \mu_1, h_1)$ be a 1-d Adinkra with p colors and let $A_2 = (V_2, E_2, c_2, \mu_2, h_2)$ be a 1-d Adinkra q colors. We can define the product of these Adinkras $A_1 \times A_2$ as the following 2-Adinkra with (p, q) colors:*

$$A_1 \times A_2 = (V, E, c, \mu, h_L, h_R)$$

where

$$\begin{aligned}
V &= V_1 \times V_2 \\
E &= E_1 \cup E_2 \textit{ where} \\
E_1 &= \{((v_1, w), (v_2, w)) \,|\, (v_1, v_2) \in E_1, \textit{ and } w \in V_2\} \\
E_2 &= \{((v, w_1), (v, w_2)) \,|\, v \in V, \textit{ and } (w_1, w_2) \in E_2\} \\
c((v_1, w), (v_2, w)) &= c_1(v_1, v_2) \textit{ for all } ((v_1, w), (v_2, w)) \in E_1 \\
c((v, w_1), (v, w_2)) &= p + c_2(w_1, w_2) \textit{ for all } (v, w_1), (v, w_2) \in E_2 \\
h_L(v, w) &= h_1(v) \\
h_R(v, w) &= h_2(w) \\
\mu((v_1, w), (v_2, w)) &= \mu_1(v_1, v_2) \\
\mu((v, w_1), (v, w_2)) &= (-1)^{h_1(v)} \mu_2(w_1, w_2)
\end{aligned}$$

Definition 1.21. Let p and q be non-negative integers and let $n = p + q$.

Given a binary block code C of length p, we can define a binary block code $\hat{C}$ of length n by appending to the end of every code word in C a string of 0s of length q.

Likewise, given a binary block code C of length q, we can define a binary block code $\check{C}$ of length n by prepending to the beginning of every code word in C a string of 0s of length p.

Proposition 1.22. *Let A_1 and A_2 be as above. Then*

$$C(A_1 \times A_2) = \hat{C}(A_1) \oplus \check{C}(A_2).$$

Proof. Let $(v_1, v_2) \in A_1 \times A_2$. Let $\mathbf{x} \in \mathbf{Z}_2^N$. We can write $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1$ is zero in the last q bits and $\mathbf{x}_2$ is zero in the first p bits. Now

$$\mathbf{x}(v_1, v_2) = (\mathbf{x}_1 + \mathbf{x}_2)(v_1, v_2) = (\mathbf{x}_1 v_1, \mathbf{x}_2 v_2).$$

This means that $\mathbf{x}(v_1, v_2) = (v_1, v_2)$ if and only if $\mathbf{x}_1 v_1 = v_1$ and $\mathbf{x}_2 v_2 = v_2$.

So $\mathbf{x} \in C(A_1 \times A_2)$ if and only if $\mathbf{x}_1 \in \hat{C}(A_1)$ and $\mathbf{x}_2 \in \check{C}(A_2)$. $\square$

2 Structural Theorems

In this section, we show that the coherence conditions of 2-d adinkras force a lot of structure onto them. In particular, we can think of the vertices of 2-d adinkras as arranged in a rectangle, with the stucture of the entire adinkra basically determined by a horizontal and a vertical "slice" of the picture.

2.1 A 2-d Adinkra Fits in a Rectangle

Let the *support* of a 2-d adinkra (and/or its bigrading function (h_L, h_R)) be defined as the range of (h_L, h_R), its bigrading function. Now, we show that the support of a connected 2-d adinkra must form a rectangle in $\mathbf{Z}^2$.

Let σ be a color sequence. Let $l(\sigma)$ be a rearrangement of σ that moves all the left-moving colors to the beginning, and let $r(\sigma)$ be a rearrangement of σ that moves all the right-moving colors to the beginning. Thus, suppose $p = q = 2$, we have $l((3,1,2,1)) = (2,1,1,3)$ and $r((3,1,2,1)) = (3,2,1,1)$. We always have $s(l(\sigma)) = s(r(\sigma))$ in general, since $l(\sigma)$ and $r(\sigma)$ are just permutations of each other.

Proposition 2.1. *Let A be a connected Adinkra. Suppose (x_1, y_1) and (x_2, y_2) are in the support of A. Then (x_1, y_2) and (x_2, y_1) are also in the support of A.*

Proof. The statement that (x_1, y_1) and (x_2, y_2) is in the support of A means that there exist vertices v_1 and v_2 of A with $(h_L(v_1), h_R(v_1)) = (x_1, y_1)$ and $(h_L(v_2), h_R(v_2)) = (x_2, y_2)$, respectively. Since A is connected, there exists a path from v_1 to v_2.

Let σ be the color sequence of this path. Now $l(\sigma)$ is a color sequence that can be written as a concatenation $\alpha\beta$ where α is a color sequence only involving left-moving colors, and β is a color sequence only involving right-moving colors. Consider the vertex $u = s(\alpha)v_1$. The color sequence α describes a path from v_1 to u involving only left-moving colors, and so $h_R(u) = h_R(v_1) = y_1$. The color sequence β describes a path from u to v_2 involving only right-moving colors, and so $h_L(u) = h_L(v_2) = x_2$. To summarize, $(h_L(u), h_R(u)) = (x_2, y_1)$.

Repeating this procedure with $r(\sigma)$ likewise provides a vertex w with $(h_L(w), h_R(w)) = (x_1, y_2)$. $\square$

Corollary 2.2. *The support of a connected 2-d adinkra is a rectangle. That is, there exist integers x_0, x_1, y_0, and y_1 so that the support is*

$$\{(x, y) \in \mathbf{Z}^2 \,|\, x_0 \leq x \leq x_1 \text{ and } y_0 \leq y \leq y_1\}$$

Proof. Let x_0 and y_0 be the minima of the x and y coordinates, respectively, of the support of the Adinkra. By Proposition 2.1, (x_0, y_0) is in the support as well.

Likewise, if x_1 and y_1 are the maxima of the x and y coordinates, respectively, of the support of the Adinkra, then (x_1, y_1) is in the support. By Proposition 2.1, (x_1, y_0) and (x_0, y_1) are also in the support.

Since the Adinkra is connected, there must be paths from vertices with bigrading (x_0, y_0) to vertices with bigrading (x_1, y_1). Since h_L and h_R can change by at most 1 along these paths, we see that for all $x_0 \leq x \leq x_1$, there must exist y_x so that (x, y_x) is in the support. Likewise for all $y_0 \leq y \leq y_1$, there must exist x_y so that (x_y, y) is in the support. By application of Proposition 2.1 again, we get that (x, y) is in the support for all $x_0 \leq x \leq x_1$ and $y_0 \leq y \leq y_1$. □

While it is neat that the vertices of a 2-d adinkra A line up nicely in a rectangle, we now show that there is even more regularity in its structure. Let A_L (resp. A_R) be the subgraphs of A induced by left-moving (resp. right-moving) edges of A.

Lemma 2.3. *Let A be a 2-d Adinkra. If X is a connected component of A_L and i is a right-moving color, then there is a graph isomorphism between X and $q_i(X)$ that preserves colors and h_L. The analogous statement for A_R also holds.*

Proof. Propostion 1.2 states that q_i is a graph isomorphism from the underlying graph of A to itself that preserves colors. If we restrict q_i to a connected component X of A_L, the restricted map is an isomorphism from X to $q_i(X)$ that preserves colors.

Since i is a right-moving color, then for all vertices $v \in X$, $h_L(v) = h_L(q_i(v))$. □

Lemma 2.4. *Let A be a 2-d Adinkra. If X is a connected component of A_L and i is a right-moving color, then $q_i(X)$ is the vertex set of a connected component of A_L. The analogous statement for A_R also holds.*

Proof. Because the property of connectedness is preserved under graph isomorphism, we know that $q_i(X)$ is connected. If we let X' be the connected component of A_L that contains $q_i(X)$, then the same argument proves that $q_i(X')$ is connected as well. Since X was assumed to be a connected component of A_L, we have that $q_i(X') \subseteq X$. But since q_i^2 is the identity, $X = q_i^2(X) \subseteq q_i(X') \subseteq X$. This means $q_i(X') = X$. By the fact that q_i^2 is the identity, we also have $q_i(X) = X'$. □

Proposition 2.5. *Let A be a connected 2-d Adinkra. All connected components of A_L (and respectively A_R) are isomorphic as graded posets.*

Proof. Let X and Y be two connected components of A_L. Pick vertices $x \in X$ and $y \in Y$. Since A is connected, there is a path from x to y in A. Reorder the path so that the right-moving edges occur before the left-moving edges. Since the left-moving edges stay in Y, the right-moving edges alone take x to a vertex $y' \in Y$. The sequence of right-moving edges provides a color sequence $i_1, \ldots, i_k$, and thus, a sequence of compositions $q_{i_k} \circ \cdots \circ q_{i_1}$. Now $q_{i_k} \circ \cdots \circ q_{i_1}(x) = y'$. By repeated application of Lemma 2.4, we have that $q_{i_k} \circ \cdots \circ q_{i_1}(X)$ is a connected component of A_L that contains y', which is Y. By repeated application of Lemma 2.3, we have an isomorphism of graphs that preserves colors and the grading h_L. $\square$

With all this redundancy, what is the minimal amount of information required for us to understand a 2-d adinkra? Proposition 2.5 suggests we just need a single connected component for each direction to give us all the data; this turns out to basically be true, as we see in the next section.

3 Quotienting

To understand quotients, we first define the homomorphism from one 2-d Adinkra to another. This will be similar to the definition of homomorphism of graphs [give some standard reference to this terminology].

Definition 3.1. Let $A_1 = (V_1, E_1, c_1, \mu_1, h_{L1}, h_{R1})$ and $A_2 = (V_2, E_2, c_2, \mu_2, h_{L2}, h_{R2})$ be 2-Adinkras with (p, q) colors. A homomorphism from A_1 to A_2 is a graph homomorphism

$$\phi : (V_1, E_1) \to (V_2, E_2)$$

that preserves colors and so that:

- If $v \in V_1$ then $h_{1L}(v) = h_{2L}(\phi(v))$.
- If $v \in V_1$ then $h_{1R}(v) = h_{2R}(\phi(v))$.

Note that there is no condition on the dashings μ_1 and μ_2.

The main result of this section is the following theorem:

Theorem 3.2. *Let A be a connected 2-d Adinkra. Fix a vertex $\overline{0}$ in A and let A_L^0 (resp. A_R^0) be the connected component of A_L (resp. A_R) containing $\overline{0}$.*

There is a binary block code K of length n so that $K \cap \hat{C}(A_L^0) = 0$ and $K \cap \check{C}(A_R^0) = 0$, and there is a graph isomorphism

$$A \cong (A_L^0 \times A_R^0)/K$$

of the underlying graphs that preserves colors and the bigrading.

Before we begin it will help to understand the relationship of the codes.

Lemma 3.3.

$$\hat{C}(A_L^0) \oplus \check{C}(A_R^0) \subseteq C(A).$$

Proof. Let $g \in \hat{C}(A_L^0)$ and $h \in \check{C}(A_R^0)$. Then $g\overline{0} = \overline{0}$ because applying g to $\overline{0}$ results in a path that lies completely inside A_L^0, and so the fact that $g\overline{0} = \overline{0}$ in A_L^0 (since $g \in \hat{C}(A_L^0)$) results in $g\overline{0} = \overline{0}$ in A. Likewise $h\overline{0} = \overline{0}$. So $(g+h)\overline{0} = g(h(\overline{0})) = \overline{0}$ and $g + h \in C(A)$. □

Proof of Theorem 3.2. From Lemma 3.3 and basic linear algebra, there exists a vector subspace K of $\mathbf{Z}_2^N$ that is a vector space complement of $\hat{C}(A_L^0) \oplus \check{C}(A_R^0)$ in $C(A)$. That is,

$$C(A) = \hat{C}(A_L^0) \oplus \check{C}(A_R^0) \oplus K.$$

Then by Theorem 1.17, there is an isomorphism

$$i_1 : A \cong I^n/C(A)$$

that preserves colors. The proof of this began with choosing a special vertex in A, and we specify here that we should use our chosen vertex $\overline{0}$ of A. This isomorphism then sends $\overline{0}$ to the coset containing $(0, \dots, 0)$ in $I^n/C(A)$.

We now note that

$$\begin{aligned} I^n/C(A) &= I^n/(\hat{C}(A_L^0) \oplus \check{C}(A_R^0) \oplus K) \\ &= (I^n/(\hat{C}(A_L^0) \oplus \check{C}(A_R^0)))/K \\ &= (I^n/C(A_L^0 \times A_R^0))/K \end{aligned}$$

and again apply Theorem 1.17 to establish an isomorphism

$$i_2 : (I^n/C(A_L^0 \times A_R^0))/K \cong (A_L^0 \times A_R^0)/K$$

that preserves colors and sends the coset of $(0, \dots, 0)$ to $(\overline{0}, \overline{0})$.

We compose these isomorphisms to define

$$i_3 : A \to (A_L^0 \times A_R^0)/K$$

an isomorphism of graphs that preserves colors.

We now restrict i_3 to A_L^0. It turns out that this sends every vertex v of A_L^0 to $(v, \overline{0})$ in $(A_L^0 \times A_R^0)/K$. This can be seen by noting that both the restriction of i_3 to A_L^0 and the map sending v to $(v, \overline{0})$ are graph isomorphisms that preserve colors and send $\overline{0}$ to $(\overline{0}, \overline{0})$. By Lemma 1.18, such a map is unique. Therefore, $i_3(v) = (v, \overline{0})$ for all vertices v in A_L^0. Likewise, a similar proof shows that if $v \in A_R^0$, then $i_3(v) = (\overline{0}, v)$.

Now let v be any vertex of A. There is a path from $\overline{0}$ to v. This gives a color sequence σ. Consider the reordered color sequence $l(\sigma)$. Write this as $\alpha\beta$ where α only involves left-moving colors and β only uses right-moving colors. Then define $x = s(\alpha)\overline{0}$. Because α only involves left-moving colors, x is in A_L^0.

Now since β consists of right-moving colors, and connects $i_3(x)$ with $i_3(v)$, we have $h_L(i_3(v)) = h_L(i_3(x)) = h_L(x, \overline{0}) = h_L(x)$. This is $h_L(v)$ because β only involves right-moving colors and connects x with v. Thus, $h_L(i_3(v)) = h_L(v)$.

A similar proof shows that $h_R(i_3(v)) = h_R(v)$. □

[Now discuss dashing]

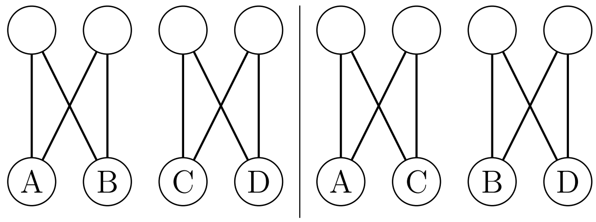

Figure 1: Taking the product of the two adinkras here with the following identification gives a non-disconnected adinkra with 16 vertices.

4 ESDE Codes

Let p and q be non-negative integers so that $p+q=n$, and let C be a linear block code of length n. For every element $\mathbf{x}=(x_1,\ldots,x_n)\in\mathbf{Z}_2^n$, we define $\mathrm{wt}_L(\mathbf{x})$ to be the number of 1s of $(x_1,\ldots,x_p)$ in the first p components, and $\mathrm{wt}_R(\mathbf{x})$ to be the number of 1s of $(x_{p+1},\ldots,x_n)$ in the last q components. Then $\mathrm{wt}(\mathbf{x})=\mathrm{wt}_L(\mathbf{x})+\mathrm{wt}_R(\mathbf{x})$.

Define a *even-split doubly-even (ESDE) code* of length (p,q) to be a doubly-even code of length n so that every word $\mathbf{x}$ in the code has $\mathrm{wt}_L(\mathbf{x})$ and $\mathrm{wt}_R(\mathbf{x})$ even.

Recall that 1-d chromotopologies are in bijection with quotients of the hamming cube I^n by a doubly-even code C, so any adinkra A has a well-defined associated code $C(A)$ that is uniquely determined by just the graph structure of the adinkra. Our goal is to show that ESDE codes are exactly the codes that appear for 2-d adinkras. To do this, we introduce a special family of 2-d adinkras.

In [references], there was a class of 1-d Adinkras that were particularly easy to construct and study, called *Valise* Adinkras, whose height function had values in two adjacent integers (usually 0 and 1). By analogy, given a ESDE code, we construct a 2-d Adinkra called the *Valise 2-d Adinkra* that has support in a 2×2 square.

Construction 4.1. *Let C be an ESDE code. We will describe a construction that provides a 2-d Adinkra with code C, called the* Valise 2-d Adinkra.

C is doubly-even, so there exists a connected 1-d Adinkra A with code $C(A)=C$. [reference: this was a construction somewhere else]

We now wish to define a bigrading (h_L,h_R). To do this we do not use the grading h that comes from the 1-d Adinkra A. Instead, fix a vertex $\overline{0}$ of A. For every vertex v of A, pick a path from $\overline{0}$ to v. This produces a color sequence d. Define $h_L(v)$ to be the number modulo 2 of left-moving colors in d and likewise $h_R(v)$ is the number modulo 2 of right-moving colors in d. These functions are well-defined if every path from $\overline{0}$ to v has the same parity in number of left-moving colors and in number of right-moving colors. This occurs if and only if every loop starting from $\overline{0}$ has an even number of left-moving colors and an odd number of right-moving colors. If C is ESDE, then this is the case.

Theorem 4.2. *This construction gives a 2-d Adinkra with code C.*

Proof. By construction, A is a 1-d Adinkra with code C. The only thing left to check is whether the h_L and h_R that are defined satisfy the correct properties on edges.

Let (v, w) be a left-moving edge in the Adinkra A. Then the color of the edge is left-moving and so $h_L(v) \neq h_L(w)$ and $h_R(v) = h_R(w)$. Since the possible values for h_L and h_R are 0 and 1, it follows that $|h_R(v) - h_R(w)| = 1$.

The proof is similar for right-moving edges. □

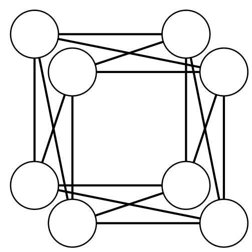

Figure 2: A valise 2-d adinkra that cannot be put into non-valise form.

Theorem 4.3. *For a code $C \subset \mathbf{Z}_2^n$, there exists a 2-d adinkra A with $C(A) = C$ if and only if C is a ESDE code.*

Proof. Suppose $C(A) = C$ for some 2-d Adinkra A. We know that C is doubly-even. Consider any codeword $\alpha \in C$. Starting at $\overline{0} \in A$, moving by a path corresponding to α must end up back at $\overline{0}$. In particular, it must use an even number of left-moving (resp. right-moving) edges since each of such edge changes the h_L (resp. h_R) by 1 in absolute value. Thus, C must be ESDE.

Conversely, given an ESDE code C, we can use Construction 4.1 to get a 2-d Adinkra with code C. □

4.1 Finding K

Given a 2-d Adinkra A, with special basepoint $\overline{0}$ and corresponding A_L^0 and A_R^0, there remains the question of finding the code K. Recall that

$$C(A) = \hat{C}(A_L^0) \oplus \check{C}(A_R^0) \oplus K.$$

There is some choice involved in defining K; it merely has to be a vector space complement to

$$C' = \hat{C}(A_L^0) \oplus \check{C}(A_R^0).$$

K will more naturally be defined as $C(A)/C'$, though computationally we can choose a representative to be K.

Let $V^0 = A_L^0 \cap A_R^0$. Now for every $v \in V^0$, we have $h_L(v) = h_L(\overline{0})$ and $h_R(v) = h_R(\overline{0})$ so in particular, V^0 has no edges; only vertices.

We construct a bijection between V^0 and $C(A)/C'$.

Construction 4.4. *For every $v \in V^0$, there is a path p_L from $\overline{0}$ to v in A_L^0 and a path p_R from v to $\overline{0}$ in A_R^0. These paths give color sequences σ_L and σ_R with $\beta_L = s(\sigma_L)$ and $\mathbf{x}_R = s(\sigma_R)$ so that $\mathbf{x}_L$ is 0 in the last q coordinates and $\mathbf{x}_R$ is 0 in the first p coordinates. Since the paths p_L and p_R form a loop in A, we know that $\mathbf{x}_L + \mathbf{x}_R \in C(A)$.*

The paths p_L and p_R are not canonical, but if p'_L and p'_R are other paths with those properties, then $p_L(p'_L)^{-1}$ is a loop in A_L^0 from $\overline{0}$. Let σ'_L and σ'_R be the corresponding color sequences, then $s(\sigma_L) + s(\sigma'_L) \in C(A_L^0)$. Likewise, $s(\sigma_R) + s(\sigma'_R) \in C(A_R^0)$.

Thus, the choice of $\mathbf{x}_L + \mathbf{x}_R$ is well-defined modulo $C' = \hat{C}(A_L^0) \oplus \check{C}(A_R^0)$. This provides a map from V^0 to $K = C(A)/C'$.

Theorem 4.5. *The above construction provides a bijective map*

$$\Phi : V^0 \to C(A)/C'$$

Proof. We now prove this map is bijective by providing its inverse. Suppose $\mathbf{x} \in C(A)$. Then by Proposition 1.6, there is a loop in A starting at $\overline{0}$ that follows the colors indicated by $\mathbf{x}$. This gives a color sequence σ so that $s(\sigma) = \mathbf{x}$. Write $l(\sigma) = \alpha\beta$ where α only involves left-moving colors and β only involves right-moving colors. Let $v = s(\alpha)\overline{0}$. Since α only involves left-moving colors, we have $v \in A_L^0$. Since $s(\beta)v = \overline{0}$, and β only involves right-moving colors, we have that $v \in A_R^0$. Therefore $v \in V^0$. Note that v does not depend on the order of the left-moving colors in the color sequence, as long as they all come before the right-moving colors. Thus we have a map

$$\Psi : C(A) \to V^0.$$

Now suppose $\mathbf{y} \in \hat{C}(A_L^0)$ and $\mathbf{z} \in \check{C}(A_R^0)$. We consider how the loop in the previous paragraph changes if we replace $\mathbf{x}$ with $\mathbf{x} + \mathbf{y} + \mathbf{z}$. We use Proposition 1.6 to create a loop p_1 in A_L^0 that starts at $\overline{0}$ and follows the colors indicated by $\mathbf{y}$. Then let p_2 be the path from $\overline{0}$ following the colors indicated by $s(\alpha)$ and p_3 the path that continues this using $s(\beta)$. Finally, let p_4 be the path from there using the colors of $\mathbf{z}$. By Proposition 1.6, since p_2 starts from $\overline{0}$, it must still end at the same point v. Therefore the map Ψ descends to a map

$$\tilde{\Psi} : C(A)/C' \to V^0.$$

We now show that this function is the inverse of Construction 4.4. Let $v \in V^0$. The construction gives a path p_L of left-moving edges starting from $\overline{0}$ and ending in v and a path p_R of right-moving edges starting from v to $\overline{0}$. Composing these paths gives the result of the Construction. Then we apply $\tilde{\Psi}$. This takes the code word and finds the loop, which by Proposition 1.6 must be $p_L p_R$. Then Ψ of the code word is the vertex at the end of p_L, which is v.

[show the other composition is the identity]

□

To do:

See if there is a place for a Lemma that says you can reorder a path so left colors come first.

A Equivalence with other notions of 2-d Adinkras

If I read Tristan's stuff right, we can completely translate the combinatorial rules to: a *2-d adinkra* (of dimension n) is a finite simple connected graph A such that:

- It is an 1-d adinkra (with the associated ranking, dashing, etc.).
- It has $p + q = n$ colors, where the first p-colors are called "left-moving" and the second q-colors are called "right-moving."
- A coherence condition: for any cycle, we imagine the following sum: going up (here "up" comes from the grading we have from the engineering dimension in our ranking for the 1-d adinkra) a left-handed edge adds -1, and going up a right-handed edge adds 1; going down the edges give contributions with opposite signs. The sum of this around any cycle must be 0. (in particular, this rules out things like ambidextrous bow-ties)

Assuming I interpreted these rules correctly, now I can do combinatorics without needing any physics.

The first structural fact we can impose is a bi-grading that is compatible with the grading we already have from the 1-d adinkra structure, in the sense that the 1-d grading is simply one of the coordinates of our bi-grading.

Proposition A.1. *A* 1*-d adinkra can be extended to a* 2*-d adinkra if and only if the* 1*-d adinkra has a* bigrading *to* $\mathbf{Z}^2$*. This is a map* $g : V \to \mathbf{Z}^2$*, such that all left-moving edges correspond to displacements of* $(0, 1)$ *and right-moving edges correspond to displacements of* $(1, 0)$.

Proof. Proof delayed until talking more with Kevin and Tristan about the easiest way to write things up to avoid reinventing wheels. □

B Misc. (unorganized)

Corollary B.1. *Consider the vertex* $\overline{0} \in A$*. Let the connected component of* A_L *(resp.* A_R*) that* $\overline{0}$ *belongs to be labeled* A_L^0 *(resp.* A_R^0*). The adinkra* A *is uniquely determined*[2] *by* A_L^0 *and* A_R^0.

Proof. Consider the color sequence d of any path from $\overline{0}$ to a vertex v. We can permute the sequence so that the left-moving colors all occur before the right-moving colors. Thus, we first make some moves in A_L^0, then by Proposition 2.5 we make the remaining moves in a copy of A_R^0. □

Corollary B.2. *A valise* 2*-d adinkra* A *of type* (n, k) *is uniquely determined by* $B_L(A)$*,* $B_R(A)$*, and an identification of* $V(B_L(A), 0)$ *and* $V(B_R(A), 0)$*. Furthermore,* $|B_L(A)| = |B_R(A)| = 2^{n-k-1}$.

[2]This is fairly nuanced; we need to know not just the shape of A_L^0 and A_R^0, but also their vertices

Problem 1. What are all the 2-d adinkras A with the same valise adinkra $\mathrm{Val}(A)$? Not all lifts are possible. For example, the adinkra in Figure 2 cannot be lifted to any non-valise form!

Here are some other problems:

Problem 2. Given two valise 1-d adinkras $B_L(A)$ and $B_R(A)$ of equal size, what identifications of $V(B_L(A), 0)$ and $V(B_R(A), 0)$ are possible?

Proof. Data: if we have $\{0,1\} \cup \{2,3\}$ on one side, the other side must be $\{0,2\} \cup \{1,3\}$. □

[$\star\star\star$ There are two kinds of quotienting that we can think of: one quotient is directly quotienting the mega hypercube adinkra by a ESDE code; one quotient is given the valise adinkra with associated $B_L(A)$, $B_R(A)$ each with 2^{d+1} vertices, the necessary d-dimensional quotienting that occurs when we naively tensor the two parts (which gives 2^{2d} vertices in each corner, for 2^{2d+2} total vertices, when in the end we just want 2^{d+2} vertices. $\star\star\star$]