Study of lifting of 1D adinkras to 2D

Kevin Iga and Yan X Zhang

February 25, 2015

1 Definitions

1.1 1-d Adinkras

Adinkras in [references] will be referred to as 1-d Adinkras in this paper, since they relate to supersymmetry in 1 dimension. We will review a definition of 1-d Adinkras now. Note that while it is conventional to specify a natural number N to denote the number of supersymmetries, we break with convention and instead specify C, a set of N different colors.

Definition 1.1. A 1-d Adinkra with color set C is (V, E, χ, Δ, g) where

- V is a finite set of vertices
- $E \subset V \times V$ is a set of edges
- $\chi : E \to C$ is a map called the coloring
- $\Delta : E \to \{1, -1\}$ is a map called the dashing
- $g : V \to \mathbf{Z}$ is a map called the grading

These are required to satisfy the following:

- If $(v, w) \in E$, then $(w, v) \in E$. Furthermore, $\chi(v, w) = \chi(w, v)$ and $\Delta(v, w) = \Delta(w, v)$.
- For every $v \in V$ and $c \in C$, there exist exactly one $w \in V$ so that $(v, w) \in E$ and $\chi(v, w) = c$.
- If c_1, $c_2 \in C$ with $c_1 \neq c_2$, and $v \in V$, then there exist w, x, and $y \in V$ so that (v, w), (w, x), (x, y), and $(y, v) \in E$, and $\chi(v, w) = \chi(x, y) = c_1$ and $\chi(w, x) = \chi(y, v) = c_2$ and $\Delta(v, w)\Delta(w, x)\Delta(x, y)\Delta(y, v) = -1$.
- If $(v, w) \in E$, then $|g(v) - g(w)| = 1$.

Note that in [Reference], there is also a bigrading of the vertices. This is rendered obsolete by the grading, since $g(v)$ is even if and only if v is a boson.

Let $V(A,r) = g^{-1}(\{r\})$ be the vertices of A with rank r. We will assume 0 to be the lowest rank in the range of g.

Recall that a 1-d adinkra has a code L, which is necessarily doubly-even. The code has parameters (n,k), where $n = |C|$ is the number of colors and k is the dimensions of the code. In this situation, the vertices of A are labeled by cosets $\{0,1\}^n + L$, so $|V| = 2^{n-k}$.

1.2 2-d Adinkras

The notion of 2-d Adinkras is described in Ref...[fill in references to various things]. We use a definition here that is equivalent to the one found in [some reference]: the proof is found in Appendix...[maybe?]

A 2-d Adinkra is similar to a 1-d Adinkra except that some colors are called "left-moving" and the other colors called "right-moving". Edges are called "left-moving" if they are colored by left-moving edges, and right-moving otherwise. Furthermore, there are two gradings, one that is affected by the left-moving edges and the other for the right-moving edges.

More formally,

Definition 1.2. A 2-d Adinkra with disjoint color sets C_L and C_R is $(V, E, \chi, \Delta, g_L, g_R)$ where

- V is a finite set of vertices
- $E \subset V \times V$ is a set of edges
- $\chi : E \to C_L \cup C_R$ is a map called the coloring
- $\Delta : E \to \{1, -1\}$ is a map called the dashing
- $g_L : V \to \mathbf{Z}$ and $g_R : V \to \mathbf{Z}$ are maps called the left grading and right grading.

These are required to satisfy:

- The first requirement for a 1-d Adinkra still holds.
- The second and third requirements for a 1-d Adinkra still hold with $C = C_L \cup C_R$.
- The fourth requirement is replaced by: if $(v,w) \in E$ and $\chi(v,w) \in C_L$, then $|g_L(v) - g_L(w)| = 1$ and $g_R(v) = g_R(w)$. If $(v,w) \in E$ and $\chi(v,w) \in C_R$, then $|g_R(v) - g_R(w)| = 1$ and $g_L(v) = g_R(w)$.

2 Product of Adinkras

One way to get 2-d Adinkras is to take a product of two 1-d Adinkras, where the first Adinkra uses only left-moving colors and the second Adinkra uses only right-moving colors.

Definition 2.1. Let C_L and C_R be disjoint color sets. Let $A_1 = (V_1, E_1, \chi_1, \Delta_1, g_1)$ be a 1-d Adinkra with color set C_L; and let $A_2 = (V_2, E_2, \chi_2, \Delta_2, g_2)$ be a 1-d Adinkra with color set C_R. We can define the product of these Adinkras as the following 2-Adinkra with color sets (C_L, C_R).

$$A_1 \times A_2 = (V, E, \chi, \Delta, g_L, g_R)$$

where

$$\begin{aligned}
V &= V_1 \times V_2 \\
E &= E_1 \cup E_2 \text{ where} \\
E_1 &= \{((v_1, w), (v_2, w)) \,|\, (v_1, v_2) \in E_1, \text{ and } w \in V_2\} \\
E_2 &= \{((v, w_1), (v, w_2)) \,|\, v \in V, \text{ and } (w_1, w_2) \in E_2\} \\
\chi((v_1, w), (v_2, w)) &= c_1(v_1, v_2) \text{ for all } ((v_1, w), (v_2, w)) \in E_1 \\
\chi((v, w_1), (v, w_2)) &= c_2(w_1, w_2) \text{ for all } (v, w_1), (v, w_2) \in E_2 \\
g_L(v, w) &= g_1(v) \\
g_R(v, w) &= g_2(w) \\
\Delta((v_1, w), (v_2, w)) &= \Delta_1(v_1, v_2) \\
\Delta((v, w_1), (v, w_2)) &= (-1)^{g_1(v)} \Delta_2(w_1, w_2)
\end{aligned}$$

3 Structure Theorems

3.1 Bigrading

A really important consequence of having a bigrading is that we get to "complete the square" with the following Corollary:

Corollary 3.1. *In a 2-d adinkra, suppose we have a path* $(x, y \pm_1 1) \to (x, y) \to (x \pm_2 1, y)$*, where each* $\pm_i$ *corresponds to a choice of sign, the first and the last vertices are connected to* $(x \pm_2 1, y \pm_1 1)$ *via the corresponding colors in a square.*

Proof. Because we have an (1-d) adinkra, the two edges in this path correspond to two different colors (WLOG 1 and 2 in order) respectively, and if we use the colors 2 and 1 in order we must also reach $(x \pm_2 1, y)$ from $(x, y \pm_1 1)$. Because left-moving colors only correspond to y-axis moves in the $\mathbf{Z}^2$ bigrading, and right-moving colors only correspond to x-axis moves, the first move must have displacement $(\pm_2 1, 0)$ and the second move must have displacement $(0, \pm_1)$. This is exactly equivalent to the statement. $\square$

3.2 Rectangle

Let the *support* of a 2-d adinkra (and/or its bigrading function g) be defined as the range of g, its bigrading function. Now, we show that the support of 2-d adinkra must form a rectangle in $\mathbf{Z}^2$. In this and the following sections, it helps to have some standard assumptions:

- Recall that any 1-d adinkra has vertices labeled by equivalence classes of $\{0,1\}$ by some (n,k) doubly-even code L. We will refer to vertices by these equivalence classes (or particular elements of the equivalence classes).
- Recall that we have $n = p + q$ colors. WLOG, let the first p colors be left-moving and the last q colors be right-moving. We will also refer to these colors as elements of $\{1, 2, \ldots, n\}$, thinking of them as the indices of the vertices labeled as elements of $\{0,1\}^n$. So an edge of color i moves a vertex $(v_1 v_2 \cdots v_n)$ by changing the i-th bit to $(1 - v_i)$ and leaving the other vertices intact.
- We will always define $\overline{0}$ to be the vertex corresponding to the equivalence class of $(00\cdots0)$. We will also assume that $\overline{0}$ has the coordinate $(0,0)$. (to be precise, this means $g(\overline{0}) = (0,0)$).

For every vertex pair (v,w) in our 2-d adinkra, there exist (many) paths from v to w. Ignoring the dashings for now, let the sequence of colors on any path be called a *color sequence* for the path. So, for example, in a chromotopology corresponding to the unique trivial $(4,0)$ code, the path with color sequence $(3,2,1,1)$ carries $\overline{0} = 0000$ to 0010, 0110, 1110, and finally 0110.

Now, define a map s that takes a color sequence and returns an element of $V = \{0,1\}^n$ where the i-th element is the number of times (modulo 2) that color i appears in the sequence. For example, $s(3,2,1,1) = 0110$. Note that $s(d)$ will return (a member of the equivalence class of) the bitstring obtained by applying the sequence d to $\overline{0}$. In particular that we may permute a sequence from $d = (d_n)$ to $d' = (d'_n)$ without changing the result of s.

The previous work on 1-d adinkras states that if we start at any vertex, following two paths with color sequences d and d' must end up at the same resulting vertex if and only if $s(d) = s(d') \pmod{L}$. Let $l(d)$ be a rearrangement of d that moves all the left-moving colors to the beginning, and let $r(d)$ be a rearrangement of d that moves all the right-moving colors to the beginning. Thus, suppose $p = q = 2$, we have $l((3,2,1,1)) = (2,1,1,3)$ and $r((3,2,1,1)) = (3,2,1,1)$. We always have $s(l(d)) = s(r(d))$ in general, since $l(d)$ and $r(d)$ are just permutations of each other.

Proposition 3.2. *Suppose we have any path from (x,y) to (v,w) in a 2-d adinkra A; then the vertices (x,w) and (v,y) are in A's support.*

Proof. Let this path take color sequence d. $l(d)$ and $r(d)$ must both end up at (v,w), but $l(d)$ only changes along the y-axis in the first part of moves that only use left-moving colors, so it must end up at coordinate (x, w) to be able to end up at (v, w) (since it can only use right-moving / x-axis colors afterwards). Using the same argument for $r(d)$ shows we must end up in (v, y) at some point in the path. $\square$

Corollary 3.3. *The support of a 2-d adinkra is exactly some $k \times l$ rectangle. In other words, the set of points in $\mathbf{Z}^2$ in the range of the bigrading map can be taken to the set of points (x, y), $0 \leq x < l$, $0 \leq y < k$.*

Proof. If the support is not a rectangle, then there must be some coordinates (x, y) and (v, w) in the support such that one of the other two diagonal coordinates are missing. This violates Proposition 2.2. $\square$

3.3 Factored by the boundary

Now, we show the main classification theorem, which is that a 2-d adinkra is completely controlled by its boundary. Let the *left boundary* A_L and *right boundary* A_R of the rectangle be the subgraphs of A induced by vertices of A that occur at the sets $\{(i, 0) | 0 \leq i < l\}$ and $\{(0, j) | 0 \leq j < k\}$ respectively. It helps to picture the normal Cartesian plane rotated 45 degrees ounterclockwise, because then adjectives refer to the direction of "movement": this way, the left boundary literally corresponds to the left-most vertices in the support rectangle and vice-versa for the right.

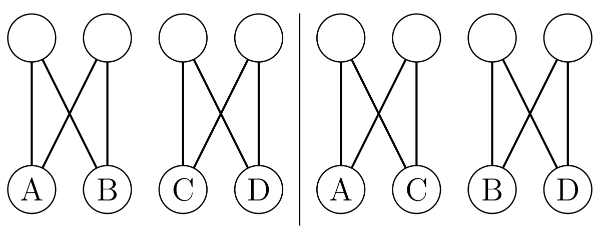

Figure 1: Tensoring the two adinkras here with the following identification gives a non-disconnected adinkra with 16 vertices.

Note that we can have both A_L and A_R be disconnected graphs. An example that we will often return to is displayed in Figure 1. However, we show that the disconnectedness is not really an issue:

Lemma 3.4. *All disconnected components in A_L (and respectively A_R) are isomorphic as graded posets.*

[$\star\star\star$ TODO $\star\star\star$]

Theorem 3.5. *A 2-d adinkra A of type (n, k) is uniquely determined by A_L, A_R, and an identification of $V(A_L, 0)$ and $V(A_R, 0)$.*

Here is another way to factor the adinkra:

Theorem 3.6. *Pick any vertex $v \in V(A,(0,0))$. Let the connected component of the right boundary that v belongs to be labeled A_R^v; similarly, let the connected component in the left boundary that v belongs to be labeled A_L^v. The adinkra A is uniquely determined by the vertices and edges of A_R^v and A_L^v.*

Proof. Consider the color sequence d of any path from v to a vertex u in A_R^v. Pick any left-moving color c. Examine the sequence cd. Corollary 2.1 will force the cd_1 to end with the same x-axis displacement as d_1, cd_1d_2 to end with the same x-axis displacement as d_1d_2, etc. so inductively any color sequence cd will end with the same x-axis displacement as d. Using induction, we get that for every one of the elements u of A_L^0, the right-moving (x-axis) colors starting at u forms a copy A_R^v whose x-axis positions are forced by the right boundary.

The key observation is that every element u of the adinkra can be reached starting at v via moving to an element of the left boundary and then using only right-moving moves (i.e. by taking $l(d)$ for any color sequence d that goes from v to u). Because of the discussion in our previous paragraph, A_R^v and A_L^v uniquely tell us how to obtain the coordinates of u. $\square$

We will worry less about exact rankings right now with the following construction: for an 1-d adinkra A let $\mathrm{Val}(A)$ be the valise form of A. For any 2-d adinkra A with boundary pair (A_L, A_R), the adinkra defined by the boundary pair $(\mathrm{Val}(A_L), \mathrm{Val}(A_R))$ is again a 2-d adinkra. We will abuse notation and call this new adinkra $\mathrm{Val}(A)$, and call such an adinkra (i.e. where the rectangle is a 2×2 square) a *valise* (2-d) adinkra. We seek the following theorem, which is very similar to Theorem 2.5

Corollary 3.7. *A valise 2-d adinkra A of type (n,k) is uniquely determined by A_L, A_R, and an identification of $V(A_L,0)$ and $V(A_R,0)$. Furthermore, $|A_L| = |A_R| = 2^{n-k-1}$.*

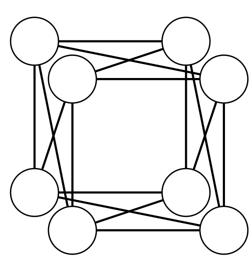

Figure 2: This is a tight valise which cannot be lifted.

Problem 1. What are all the 2-d adinkras A with the same valise adinkra $\mathrm{Val}(A)$? Not all lifts are possible. For example, the adinkra in Figure 2 cannot be lifted to any non-valise form!

4 Quotienting

To understand quotients, we first define the homomorphism from one 2-d Adinkra to another. This will be similar to the definition of homomorphism of graphs [give some standard reference to this terminology].

Definition 4.1. Let $A_1 = (V_1, E_1, \chi_1, \Delta_1, g_{L1}, g_{R1})$ and $A_2 = (V_2, E_2, \chi_2, \Delta_2, g_{L2}, g_{R2})$ be 2-Adinkras with the same color set C. A homomorphism from A_1 to A_2 is a map

$$\phi : V_1 \to V_2$$

satisfying the following:

- If $(v, w) \in E_1$, then $\phi(v, w) \in E_2$ and $\chi_1(v, w) = \chi_2(\phi(v, w))$.
- If $v \in V_1$ then $g_{1L}(v) = g_{2L}(\phi(v))$.
- If $v \in V_1$ then $g_{1R}(v) = g_{2R}(\phi(v))$.

Note that there is no condition on the dashings Δ_1 and Δ_2.

Define a *even-split doubly-even (ESDE) code* to be a doubly-even code isomorphic to a direct sum $C_L \oplus C_R$ of even codes. It was proved [**?**] that 1-d adinkras are ismorphic to quotienting the hamming cube H^n by a doubly-even code, so any adinkra A has a well-defined associated code $C(A)$ that is uniquely determined by just the graph structure of the adinkra.

Our goal is to show

Theorem 4.2. *The image of C of the set of* 2*-d adinkras are exactly the ESDE codes.*

This answers a conjecture of Hübsch [].

To do this we need to show two things:

Proposition 4.3. *Any valise* 2*-d adinkra A has $C(A)$ equal to a ESDE code.*

Proof. Note that the automorphism group of the adinkra is defined by the basepoint.

TODO □

Proposition 4.4. *Any ESDE code defines a valise* 2*-d adinkra A.*

Proof. TODO □

Here are some other problems:

Problem 2. Given two valise 1-d adinkras A_L and A_R of equal size, what identifications of $V(A_L, 0)$ and $V(A_R, 0)$ are possible?

Proof. Data: if we have $\{0, 1\} \cup \{2, 3\}$ on one side, the other side must be $\{0, 2\} \cup \{1, 3\}$. □

[$\star\star\star$ There are two kinds of quotienting that we can think of: one quotient is directly quotienting the mega hypercube adinkra by a ESDE code; one quotient is given the valise adinkra with associated A_L, A_R each with 2^{d+1} vertices, the necessary d-dimensional quotienting that occurs when we naively tensor the two parts (which gives 2^{2d} vertices in each corner, for 2^{2d+2} total vertices, when in the end we just want 2^{d+2} vertices. $\star\star\star$]

5 ESDE Codes

In this section, we classify all ESDE codes.

[$\star\star\star$ TODO! $\star\star\star$]

A Equivalence with other notions of 2-d Adinkras

If I read Tristan's stuff right, we can completely translate the combinatorial rules to: a *2-d adinkra* (of dimension n) is a finite simple connected graph A such that:

- It is an 1-d adinkra (with the associated ranking, dashing, etc.).
- It has $p+q=n$ colors, where the first p-colors are called "left-moving" and the second q-colors are called "right-moving."
- A coherence condition: for any cycle, we imagine the following sum: going up (here "up" comes from the grading we have from the engineering dimension in our ranking for the 1-d adinkra) a left-handed edge adds -1, and going up a right-handed edge adds 1; going down the edges give contributions with opposite signs. The sum of this around any cycle must be 0. (in particular, this rules out things like ambidextrous bow-ties)

Assuming I interpreted these rules correctly, now I can do combinatorics without needing any physics.

The first structural fact we can impose is a bi-grading that is compatible with the grading we already have from the 1-d adinkra structure, in the sense that the 1-d grading is simply one of the coordinates of our bi-grading.

Proposition A.1. *A* 1*-d adinkra can be extended to a* 2*-d adinkra if and only if the* 1*-d adinkra has a* bigrading *to* $\mathbf{Z}^2$*. This is a map* $g : V \to \mathbf{Z}^2$*, such that all left-moving edges correspond to displacements of* $(0,1)$ *and right-moving edges correspond to displacements of* $(1,0)$*.*

Proof. Proof delayed until talking more with Kevin and Tristan about the easiest way to write things up to avoid reinventing wheels. □