# Study of lifting of 1D adinkras to 2D

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# 1 Preliminaries

## 1.1 1-d Adinkras

Adinkras in [references] will be referred to as 1-d Adinkras in this paper, since they relate to supersymmetry in 1 dimension. We will review a definition of 1-d Adinkras now. Note that while it is conventional to specify a natural number $N$ to denote the number of supersymmetries, we break with convention and instead specify $C$, a set of $n$ different colors.

**Definition 1.1.** Let $n$ be a non-negative integer. A 1-d Adinkra with $n$ colors is $(V, E, c, \mu, g)$ where

- $V$ is a finite set of vertices
- $E \subset V \times V$ is a set of edges
- $c : E \to \{1, \ldots, n\}$ is a map called the coloring
- $\mu : E \to \{1, -1\}$ is a map called the dashing
- $h : V \to \mathbf{Z}$ is a map called the grading

These are required to satisfy the following:

- If $(v, w) \in E$, then $(w, v) \in E$. Furthermore, $c(v, w) = c(w, v)$ and $\mu(v, w) = \mu(w, v)$. Intuitively, the edges are undirected.
- For every $v \in V$ and $c \in \{1, \ldots, n\}$, there exist exactly one $w \in V$ so that $(v, w) \in E$ and $c(v, w) = c$.
- If $c_1$, $c_2 \in \{1, \ldots, n\}$ with $c_1 \neq c_2$, and $v \in V$, then there exist $w$, $x$, and $y \in V$ so that $(v, w)$, $(w, x)$, $(x, y)$, and $(y, v) \in E$, and $c(v, w) = c(x, y) = c_1$ and $c(w, x) = c(y, v) = c_2$ and $\mu(v, w)\mu(w, x)\mu(x, y)\mu(y, v) = -1$.
- If $(v, w) \in E$, then $|h(v) - h(w)| = 1$.

Note that in [Reference], there is also a bipartition of the vertices. This is rendered obsolete by the grading, since $h(v)$ is even if and only if $v$ is a boson.

Recall that a 1-d adinkra has a code $L(A)$, which is necessarily doubly-even. The code has parameters $(n, k)$, where $n$ is the number of coordinates and $k$ is the dimensions of the code. In this situation, the vertices of $A$ are labeled by cosets $\{0,1\}^n + L(A)$, so $|V(A)| = 2^{n-k}$.

Let $A$ be a 1-d Adinkra with $n$ colors, with vertex set $V$. For all $1 \leq i \leq n$, define

$$q_i : V \to V$$

so that for all $v \in V$, $q_i(v)$ is the unique vertex joined to $v$ by an edge of color $i$.

If $(v, w)$ is an edge in the Adinkra, $(q_i(v), q_i(w))$ is an edge. This follows from items 2 and 3 in the definition of an Adinkra above. Therefore $q_i$ is a graph homomorphism. From item 2 in the definition, we have that $q_i(q_i(v)) = v$ for all $v \in V$, so $q_i$ is invertible and thus a graph isomorphism. From item 3 in the definition we also have for all $i$ and $j$, $q_i(q_j(v)) = q_j(q_i(v))$. As a result, we can then define an action of $\mathbf{Z}_2^n$ on the graph $(V, E)$ underlying the Adinkra in the following way:

$$(x_1, \ldots, x_N)v = q_1^{x_1} \circ \cdots \circ q_n^{x_n}(v)$$

## 1.2 The code for an Adinkra

Pick a vertex $v \in A$. Define $C(A, v)$ to be the stabilizer of $v$ under this action of $\mathbf{Z}_2^n$. Since it is a subgroup of $\mathbf{Z}_2^n$, $C(A, v)$ is a binary block code of length $n$.

**Proposition 1.2.** *The Adinkra $A$ is connected if and only if the $\mathbf{Z}_2^n$ action is transitive on the vertex set of $A$.*

*Proof.* Let $v$, $w$ be vertices of $A$. If $A$ is connected, then there is a path in $A$ connecting $v$ to $w$. The edges in this path have colors $c_1, \ldots, c_k$. Then $q_{c_1} \cdots q_{c_k}(v) = w$. By the commutativity of $\mathbf{Z}_2^n$, we can rearrange $q_{c_1} \cdots q_{c_k}$ to be in increasing order of $c_i$. This is then of the form $q_1^{x_1} \circ \cdots \circ q_N^{x_n}$ for some $(x_1, \ldots, x_n) \in \{0,1\}^n$. □

**Proposition 1.3.** *If $A$ is connected, then the code $C(A, v)$ does not depend on $v$.*

*Proof.* Let $w \in V$. By the previous proposition, there exists a $\gamma \in \mathbf{Z}_2^n$ so that $\gamma v = w$.

Let $\alpha \in \mathbf{Z}_2^n$. The result follows from the sequence of equivalences:

$$\alpha w = w \Leftrightarrow \alpha\gamma v = \gamma v \Leftrightarrow \gamma\alpha v = \gamma v \Leftrightarrow \alpha v = v$$

□

Thus, from now on we will refer to the code $C(A, v)$ as $C(A)$.

The following Adinkra was defined in [reference] and will be useful now.

**Definition 1.4.** For any non-negative integer $n$, we have an Adinkra $I^n = (V, E, c, \mu, h)$ with

- $V = \mathbf{Z}_2^n = \{0, 1\}^n$,
- $E = \{(v, w) \mid v$ and $w$ differ in precisely one coordinate$\}$,
- $c(v, w)$ is the coordinate where $v$ and $w$ differ,
- $\mu(v, w)$ is the number of 1s in $v$ before the coordinate $c(v, w)$, modulo 2,
- $h(v) = \mathrm{wt}(v)$ is the number of 1s in $v$.

**Definition 1.5.** Given $C$ a linear block code of length $n$, we can define a graph with edge colors $I^n/C$ as the orbit space of the action of $C \subset \mathbf{Z}_2^n$ on $I^n$ as a graph.

**Definition 1.6.** A graph homomorphism from a graph $(V_1, E_1)$ to a graph $(V_2, E_2)$ is a map $\phi : V_1 \to V_2$ so that if $(v, w) \in E_1$ is an edge, then $(\phi(v), \phi(w)) \in E_2$ is an edge. If there is a coloring $c_1 : E_1 \to \{1, \ldots, n\}$ and a coloring $c_2 : E_2 \to \{1, \ldots, n\}$, we say that $\phi$ preserves colors if $c_1(v, w) = c_2(\phi(v), \phi(w))$.

**Theorem 1.7.** *If $A$ is a connected Adinkra, then there is a graph isomorphism from $I^n/C(A)$ to $A$ that preserves colors.*

*Proof.* Choose a vertex $\overline{0}$ in $A$. Let

$$\phi : I^n \to A$$

$$\phi(x_1, \ldots, x_n) = (x_1, \ldots, x_n)\overline{0}$$

where we are using the action of $\mathbf{Z}_2^n$ on $A$ as described above.

To see that this is a graph homomorphism, let $(x_1, \ldots, x_n) \in \mathbf{Z}_2^n$ and let $(y_1, \ldots, y_n)$ be another vertex connected to $(x_1, \ldots, x_n)$ with an edge of color $i$. Then $y_j = x_j$ for all $j \neq i$ and $y_i = 1 - x_i$. Then $(y_1, \ldots, y_n)\overline{0} = q_1^{y_1} \cdots q_n^{y_n}\overline{0} = q_1^{x_1} \cdots q_i^{1-x_i} q_n^{x_n}\overline{0} = q_i(q_1^{x_1} \cdots q_n^{x_n}\overline{0}) = q_i(x_1, \ldots, x_n)$. So $\phi(x_1, \ldots, x_n)$ and $\phi(y_1, \ldots, y_n)$ are connected by an edge of color $i$. Note that this shows that the graph homomorphism preserves colors.

We now prove that $\phi$ is surjective. Since $A$ is connected, $\mathbf{Z}_2^n$ acts transitively on the vertex set of $A$, and so for any vertex $v$ of $A$, there exists an element $\alpha \in \mathbf{Z}_2^n$ so that $\alpha\overline{0} = v$. Then $v = \phi(\alpha)$.

To prove the isomorphism from $I^n/C(A)$ to $A$, we consider the necessary and sufficient conditions for $\phi(x) = \phi(y)$ for $x$, $y \in \mathbf{Z}_2^n$. The condition $\phi(x) = \phi(y)$ is equivalent to $\phi(x - y)\overline{0} = \overline{0}$. This is equivalent to saying $x - y \in C(A)$. Thus, the map $\phi$ descends to $I^n/C(A)$ and gives an isomorphism. ☐

Theorem from other paper: $C(A)$ is doubly even. Furthermore, if a binary block code $C$ of length $n$ is doubly even, then there exists an Adinkra with $C(A) = C$.

## 1.3 2-d Adinkras

The notion of 2-d Adinkras is described in Ref...[fill in references to various things]. We use a definition here that is equivalent to the one found in [some reference]: the proof is found in Appendix...[maybe?]

A 2-d Adinkra is similar to a 1-d Adinkra except that some colors are called "left-moving" and the other colors called "right-moving". Edges are called "left-moving" if they are colored by left-moving edges, and right-moving otherwise. Furthermore, there are two gradings, one that is affected by the left-moving edges and the other for the right-moving edges.

More formally,

**Definition 1.8.** Let $p$ and $q$ be non-negative integers. A 2-d Adinkra with $(p, q)$ colors is a 1-d Adinkra $(V, E, c, \mu, h)$ with $p + q$ colors, and two grading functions $h_L : V \to \mathbf{Z}$ and $h_R : V \to \mathbf{Z}$ so that

- $h(v) = h_L(v) + h_R(v)$
- if $(v, w) \in E$ and $c(v, w) \leq p$, then $|h_L(v) - h_L(w)| = 1$ and $h_R(v) = h_R(w)$. If $(v, w) \in E$ and $c(v, w) > p$, then $|h_R(v) - h_R(w)| = 1$ and $h_L(v) = h_R(w)$.

## 1.4 Product of Adinkras

One way to get 2-d Adinkras is to take a product of two 1-d Adinkras, where the first Adinkra uses only left-moving colors and the second Adinkra uses only right-moving colors.

**Definition 1.9.** Let $p$ and $q$ be non-negative integers. Let $A_1 = (V_1, E_1, c_1, \mu_1, h_1)$ be a 1-d Adinkra with $p$ colors and let $A_2 = (V_2, E_2, c_2, \mu_2, h_2)$ be a 1-d Adinkra $q$ colors. We can define the product of these Adinkras as the following 2-Adinkra with $(p, q)$ colors:

$$A_1 \times A_2 = (V, E, c, \mu, h_L, h_R)$$

where

$$\begin{aligned}
V &= V_1 \times V_2 \\
E &= E_1 \cup E_2 \text{ where} \\
E_1 &= \{((v_1, w), (v_2, w)) \,|\, (v_1, v_2) \in E_1, \text{ and } w \in V_2\} \\
E_2 &= \{((v, w_1), (v, w_2)) \,|\, v \in V, \text{ and } (w_1, w_2) \in E_2\} \\
c((v_1, w), (v_2, w)) &= c_1(v_1, v_2) \text{ for all } ((v_1, w), (v_2, w)) \in E_1 \\
c((v, w_1), (v, w_2)) &= p + c_2(w_1, w_2) \text{ for all } (v, w_1), (v, w_2) \in E_2 \\
h_L(v, w) &= h_1(v) \\
h_R(v, w) &= h_2(w) \\
\mu((v_1, w), (v_2, w)) &= \mu_1(v_1, v_2) \\
\mu((v, w_1), (v, w_2)) &= (-1)^{h_1(v)} \mu_2(w_1, w_2)
\end{aligned}$$

**Definition 1.10.** Let $p$ and $q$ be non-negative integers and let $n = p + q$.

Given a binary block code $C$ of length $p$, we can define a binary block code $\hat{C}$ of length $n$ by appending to the end of every code word in $C$ a string of 0s of length $q$.

Likewise, given a binary block code $C$ of length $q$, we can define a binary block code $\check{C}$ of length $n$ by prepending to the beginning of every code word in $C$ a string of 0s of length $p$.

**Proposition 1.11.** *Let $A_1$ and $A_2$ be as above. Then*

$$C(A_1 \times A_2) = \hat{C}(A_1) \oplus \check{C}(A_2).$$

*Proof.* Let $(v_1, v_2) \in A_1 \times A_2$. Let $g \in \mathbf{Z}_2^N$. We can write $g = g_1 + g_2$ where $g_1$ is zero in the last $q$ bits and $g_2$ is zero in the first $p$ bits. Now

$$g(v_1, v_2) = (g_1 + g_2)(v_1, v_2) = (g_1 v_1, g_2 v_2).$$

This means that $g(v_1, v_2) = (v_1, v_2)$ if and only if $g_1 v_1 = v_1$ and $g_2 v_2 = v_2$.

So $g \in C(A_1 \times A_2)$ if and only if $g_1 \in \hat{C}(A_1)$ and $g_2 \in \check{C}(A_2)$. $\square$

# 2 Structural Theorems

In this section, we show that the coherence conditions of 2-d adinkras force a lot of structure onto them. In particular, we can think of the vertices of 2-d adinkras as arranged in a rectangle, with the stucture of the entire adinkra basically determined by a horizontal and a vertical "slice" of the picture.

## 2.1 A 2-d Adinkra Fits in a Rectangle

Let the *support* of a 2-d adinkra (and/or its bigrading function $g$) be defined as the range of $g$, its bigrading function. Now, we show that the support of 2-d adinkra must form a rectangle in $\mathbf{Z}^2$. In this and the following sections, it helps to have some standard assumptions:

- Recall that any 1-d adinkra has vertices labeled by equivalence classes of $\mathbf{Z}_2^n$ by some $(n, k)$ doubly-even code $L$. We will refer to vertices by these equivalence classes (or their representatives). We use the notation $(v_1, \ldots, v_n)$ and $(v_1 v_2 \cdots v_n)$ interchangeably.

- We will always define $\overline{0}$ to be the vertex corresponding to the equivalence class of $(00 \cdots 0)$. We will also assume that $g(\overline{0}) = (0, 0)$.

For every vertex pair $(v, w)$ in our 2-d adinkra, there exist (many) paths from $v$ to $w$. Ignoring the dashings for now, let the sequence of colors on any path be called a *color sequence* for the path. So, for example, in a chromotopology corresponding to the unique trivial $(4, 0)$ code, the path with color sequence $(3, 2, 1, 1)$ carries $\overline{0} = 0000$

to 0010, 0110, 1110, and finally 0110. Note that a color sequence $(i_1, i_2, \ldots, i_k)$ sends $v$ to $q_{i_k} q_{i_{k-1}} \cdots q_{i_1}(v)$.

Now, define a map $s$ that takes a color sequence and returns an element of $V = \{0,1\}^n$ where the $i$-th element is the number of times (modulo 2) that color $i$ appears in the sequence. For example, $s(3,2,1,1) = 0110$. Note that $s(d)$ will return (a member of the equivalence class of) the bitstring obtained by applying the sequence $d$ to $\overline{0}$. In particular that we may permute a sequence from $d = (d_n)$ to $d' = (d'_n)$ without changing the result of $s$. Thinking about paths is very important for us, because:

**Proposition 2.1.** *Start at any vertex $v$ in an adinkra $A$ with code $L(A)$, following two paths with color sequences $d$ and $d'$ ends up at the same resulting vertex if and only if $s(d) = s(d') \pmod{L}$.*

**Proposition 2.2.** *See [reference]. The main idea is that the $q_i$'s commute, so where we end up after a color sequence $d$ only depends on $s(d)$. By the definition of $L$, we end up at the same vertex only if the difference between the two sets of edges correspond to an element of $L$.*

Let $l(d)$ be a rearrangement of $d$ that moves all the left-moving colors to the beginning, and let $r(d)$ be a rearrangement of $d$ that moves all the right-moving colors to the beginning. Thus, suppose $p = q = 2$, we have $l((3,2,1,1)) = (2,1,1,3)$ and $r((3,2,1,1)) = (3,2,1,1)$. We always have $s(l(d)) = s(r(d))$ in general, since $l(d)$ and $r(d)$ are just permutations of each other.

**Proposition 2.3.** *Suppose we have any path from $(x,y)$ to $(v,w)$ in a 2-d adinkra $A$; then the vertices $(x,w)$ and $(v,y)$ are in $A$'s support.*

*Proof.* Let this path take color sequence $d$. $l(d)$ and $r(d)$ must both end up at $(v,w)$, but $l(d)$ only changes along the $y$-axis in the first part of moves that only use left-moving colors, so it must end up at coordinate $(x,w)$ to be able to end up at $(v,w)$ (since it can only use right-moving / $x$-axis colors afterwards). Using the same argument for $r(d)$ shows we must end up in $(v,y)$ at some point in the path. □

**Corollary 2.4.** *The support of a 2-d adinkra is exactly some $k \times l$ rectangle. In other words, the set of points in $\mathbf{Z}^2$ in the range of $g$, the bigrading map, can be taken to the set of points $(x,y)$, $0 \le x < l$, $0 \le y < k$.*

*Proof.* If the support is not a rectangle, then there must be some coordinates $(x,y)$ and $(v,w)$ in the support such that one of the other two diagonal coordinates are missing. This violates Proposition 2.3. □

While it is neat that the vertices of a 2-d adinkra $A$ line up nicely in a rectangle, we now show that there is even more regularity in its structure. Let $A_L$ (resp. $A_R$) be the subgraphs of $A$ induced by left-moving (resp. right-moving) edges of $A$.

**Lemma 2.5.** *If $X$ is a connected component of $A_L$ and $i$ is a right-moving color, then $q_i(X)$ is the vertex set of a connected component of $A_L$. The analogous statement for $A_R$ also holds.*

*Proof.* Given two vertices $q_i(x)$ and $q_i(x')$ in $q_i(X)$, we know that $x$ and $x'$ are in $X$. Since $X$ is connected in $A_L$, there is a path with left-moving colors from $x$ to $x'$. This means

$$x' = q_{j_1} q_{j_2} \cdots q_{j_k} x,$$

so

$$q_i(x') = q_i q_{j_1} q_{j_2} \cdots q_{j_k}(x) = q_{j_1} q_{j_2} \cdots q_{j_k} q_i(x),$$

meaning $q_i(x')$ and $q_i(x)$ are connected in $A_L$. The converse is equally easy. □

**Proposition 2.6.** *All disconnected components in $A_L$ (and respectively $A_R$) are isomorphic as graded posets.*

*Proof.* Let $X$ and $Y$ be two connected components of $A_L$. Pick vertices $x \in X$ and $y \in Y$. Since $A$ is connected, there is a path from $x$ to $y$ in $A$. Reorder the path so that the right-moving edges occur before the left-moving edges. Since the left-moving edges stay in $Y$, the right-moving edges alone take $x$ to a vertex $y' \in Y$. By Lemma 2.5, each such edge puts us in a new connected component isomorphic to $X$, so $X$ is isomorphic to $Y$. They are all isomorphic as graded posets since translating via right-moving colors does not change the $y$-coordinate of the grading. □

With all this redundancy, what is the minimal amount of information required for us to understand a 2-d adinkra? Proposition 2.6 suggests we just need a single connected component for each direction to give us all the data; this turns out to basically be true, as we see in the next section.

# 3 Quotienting

To understand quotients, we first define the homomorphism from one 2-d Adinkra to another. This will be similar to the definition of homomorphism of graphs [give some standard reference to this terminology].

**Definition 3.1.** Let $A_1 = (V_1, E_1, c_1, \mu_1, h_{L1}, h_{R1})$ and $A_2 = (V_2, E_2, c_2, \mu_2, h_{L2}, h_{R2})$ be 2-Adinkras with $(p, q)$ colors. A homomorphism from $A_1$ to $A_2$ is a map

$$\phi : V_1 \to V_2$$

satisfying the following:

- If $(v, w) \in E_1$, then $\phi(v, w) \in E_2$ and $c_1(v, w) = c_2(\phi(v, w))$.
- If $v \in V_1$ then $h_{1L}(v) = h_{2L}(\phi(v))$.

- If $v \in V_1$ then $h_{1R}(v) = h_{2R}(\phi(v))$.

Note that there is no condition on the dashings $\mu_1$ and $\mu_2$.

**Proposition 3.2.**
$$\hat{C}(A_L^0) \oplus \check{C}(A_R^0) \subseteq C(A).$$

*Proof.* Let $g \in \hat{C}(A_L^0)$ and $h \in \check{C}(A_R^0)$. Then $g\overline{0} = \overline{0}$ because applying $g$ to $\overline{0}$ results in a path that lies completely inside $A_L^0$, and so the fact that $g\overline{0} = \overline{0}$ in $A_L^0$ (since $g \in \hat{C}(A_L^0)$) results in $g\overline{0} = \overline{0}$ in $A$. Likewise $h\overline{0} = \overline{0}$. So $(g+h)\overline{0} = g(h(\overline{0})) = \overline{0}$ and $g + h \in C(A)$. □

**Theorem 3.3.** *There is a binary block code $K$ of length $n$ so that as colored graphs,*

$$A \cong (A_L^0 \times A_R^0)/K$$

*and $K \cap \hat{C}(A_L^0) = 0$ and $K \cap \check{C}(A_R^0) = 0$.*

*Proof.* From the previous proposition and basic linear algebra, there exists a vector subspace $K$ of $\mathbf{Z}_2^N$ that is a vector space complement of $\hat{C}(A_L^0) \oplus \check{C}(A_R^0)$ in $C(A)$. That is,

$$C(A) = \hat{C}(A_L^0) \oplus \check{C}(A_R^0) \oplus K.$$

Then

$$\begin{aligned} A &\cong \mathbf{Z}_2^N/C(A) \\ &= \mathbf{Z}_2^N/(\hat{C}(A_L^0) \oplus \check{C}(A_R^0) \oplus K) \\ &= (\mathbf{Z}_2^N/(\hat{C}(A_L^0) \oplus \check{C}(A_R^0)))/K \\ &= (\mathbf{Z}_2^N/C(A_L^0 \times A_R^0))/K \\ &= (A_L^0 \times A_R^0)/K \end{aligned}$$

□

**Theorem 3.4.** *The bigrading $(h_L, h_R)$ on $(A_L^0 \times A_R^0)$ descends to*

$$(A_L^0 \times A_R^0)/K$$

*and the above isomorphism preserves this bigrading.*

*Proof.* Let $(v, w)$ be a vertex of $A_L^0 \times A_R^0$. Let $\alpha \in K$. Then $\alpha \in C(A)$. We write $\alpha = \alpha_L + \alpha_R$ where $\alpha_L$ has 0s for the last $q$ coordinates and $\alpha_R$ has 0s for the first $p$ coordinates. Then $\alpha_L(v) \in A_L^0$, and $\alpha_R(\alpha_L(v)) = v$. Since $\alpha_R$ preserves $h_L$, we have $h_L(\alpha_L(v)) = h_L(v)$.

Then we have

$$h_L(\alpha(v, w)) = h_L((\alpha_L(v), \alpha_R(w)) = h_L(\alpha_L(v)) = h_L(v)$$

Likewise, $h_R(\alpha(v,w)) = h_R(v)$.

So the bigrading $(h_L, h_R)$ descends to the quotient.

By definition,

$$h_L(\overline{0},\overline{0}) = h_L(\overline{0}) = h_L(\phi(\overline{0},\overline{0})).$$

Now let $(v,w) \in A_L^0 \times A_R^0$ and consider $h_L(v,w)$ and $h_L(\phi(v,w))$. ... I guess we don't have $\phi$ defined explicitly here. [to be completed later]

The proof is similar for $h_R$. □

[Now discuss dashing]

**Theorem 3.5.** *Any 2-d adinkra $A$ is isomorphic to $A_L^0 \times A_R^0/G$ for some group $G$.*

*Proof.* □

[$\star\star\star$ TODO $\star\star\star$]

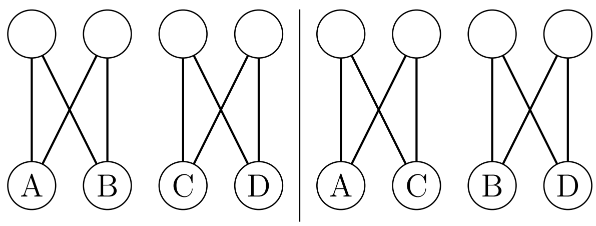


Figure 1: Taking the product of the two adinkras here with the following identification gives a non-disconnected adinkra with 16 vertices.

# 4 ESDE Codes

Define a *even-split doubly-even (ESDE) code* to be a doubly-even code isomorphic to a direct sum $C_L \oplus C_R$ of even codes. Recall that 1-d chromotopologies are in bijection with quotients of the hamming cube $I^n$ by a doubly-even code $L$, so any adinkra $A$ has a well-defined associated code $L(A)$ that is uniquely determined by just the graph structure of the adinkra. Our goal is to show that ESDE codes are exactly the codes that appear for 2-d adinkras. To do this, we introduce a special family of 2-d adinkras.

For an 1-d adinkra $A$ with grading function $g$, let $\mathrm{Val}(A)$, the *valise* of $A$, be defined as a 1-d adinkra $A'$ identical to $A$ except for its grading function $g'$, defined as $g'(v) = 0$ if $g(v) \in 2\mathbf{Z}$ and 1 otherwise. It is easy to see that $\mathrm{Val}(A)$ is also an adinkra. Similarly, for any 2-d adinkra $A$ with bigrading function $g$, there is a unique 2-d adinkra $A'$ with the same chromotopology as $A$ such that $(A')_L^0 = \mathrm{Val}(A_L^0)$ and $(A')_R^0 = \mathrm{Val}(A_R^0)$, defined with a bigrading function $g'(v) = (x', y')$, where $x', y' \in \{0,1\}$ depending on $x$ and $y \pmod{()2}$ respectively given $g(v) = (x,y)$. We similarly denote $A'$ by $\mathrm{Val}(A)$ and call such an adinkra a *valise* (2-d) adinkra.

Equivalently, a valise adinkra is one where the supporting rectangle is a $2 \times 2$ square, and any 2-d adinkra could be put into a valise form.

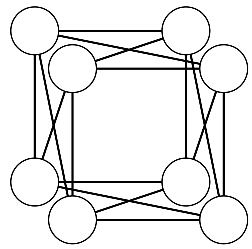

Figure 2: A valise 2-d adinkra that cannot be put into non-valise form.

**Theorem 4.1.** *For a code $L \subset \mathbf{Z}_2^n$, there exists a 2-d adinkra $A$ with $L(A) = L$ if and only if $L$ is a ESDE code.*

*Proof.* Suppose $L(A) = L$ for some 2-d adinkra $A$. We know that $L$ is doubly-even. Consider any codeword $l \in L$. Starting at $\overline{0} \in A$, moving by a path corresponding to $l$ must end up back at $\overline{0}$. In particular, it must use an even number of left-moving (resp. right-moving) edges since each of which changes the $x$-coordinate (resp. $y$-coordinate) by 1 in absolute value. Thus, $L$ must be ESDE.

Now, take an ESDE code $L$. $L$ is doubly-even, so there exists a 1-d adinkra $A$ with code $L(A) = L$. We can assume $A$ is a 1-d valise by taking $A = \mathrm{Val}(A)$ if necessary, which does not change the code. Now, any vertex $v \in A$ corresponds to an equivalence class $c + L \subset \mathbf{Z}_2^n$. consider the function $g' \colon V(A) \to \{0, 1\}$ that sends $v$ to $(x, y)$, where $x$ (resp. $y$) corresponds to the parity of the weight of the left-moving colors of any element in $c + L$. The fact that $L$ is ESDE precisely makes this notion well-defined. Since $g'$ gives $A$ a bigrading, $A$ is realizable as a 2-d valise adinkra $A$ with code $L$, so we are done. $\square$

# A Equivalence with other notions of 2-d Adinkras

If I read Tristan's stuff right, we can completely translate the combinatorial rules to: a *2-d adinkra* (of dimension $n$) is a finite simple connected graph $A$ such that:

- It is an 1-d adinkra (with the associated ranking, dashing, etc.).
- It has $p + q = n$ colors, where the first $p$-colors are called "left-moving" and the second $q$-colors are called "right-moving."
- A coherence condition: for any cycle, we imagine the following sum: going up (here "up" comes from the grading we have from the engineering dimension in our ranking for the 1-d adinkra) a left-handed edge adds $-1$, and going up a right-handed edge adds 1; going down the edges give contributions with

opposite signs. The sum of this around any cycle must be 0. (in particular, this rules out things like ambidextrous bow-ties)

Assuming I interpreted these rules correctly, now I can do combinatorics without needing any physics.

The first structural fact we can impose is a bi-grading that is compatible with the grading we already have from the 1-d adinkra structure, in the sense that the 1-d grading is simply one of the coordinates of our bi-grading.

**Proposition A.1.** *A* 1*-d adinkra can be extended to a* 2*-d adinkra if and only if the* 1*-d adinkra has a* bigrading *to* $\mathbf{Z}^2$*. This is a map* $g : V \to \mathbf{Z}^2$*, such that all left-moving edges correspond to displacements of* $(0,1)$ *and right-moving edges correspond to displacements of* $(1,0)$*.*

*Proof.* Proof delayed until talking more with Kevin and Tristan about the easiest way to write things up to avoid reinventing wheels. □

# B Misc. (unorganized)

**Corollary B.1.** *Consider the vertex* $\overline{0} \in A$*. Let the connected component of* $A_L$ *(resp.* $A_R$*) that* $\overline{0}$ *belongs to be labeled* $A_L^0$ *(resp.* $A_R^0$*). The adinkra* $A$ *is uniquely determined*[1] *by* $A_L^0$ *and* $A_R^0$*.*

*Proof.* Consider the color sequence $d$ of any path from $\overline{0}$ to a vertex $v$. We can permute the sequence so that the left-moving colors all occur before the right-moving colors. Thus, we first make some moves in $A_L^0$, then by Proposition 2.6 we make the remaining moves in a copy of $A_R^0$. □

**Corollary B.2.** *A valise* 2*-d adinkra* $A$ *of type* $(n,k)$ *is uniquely determined by* $B_L(A)$*,* $B_R(A)$*, and an identification of* $V(B_L(A),0)$ *and* $V(B_R(A),0)$*. Furthermore,* $|B_L(A)| = |B_R(A)| = 2^{n-k-1}$*.*

**Problem 1.** What are all the 2-d adinkras $A$ with the same valise adinkra $\mathrm{Val}(A)$? Not all lifts are possible. For example, the adinkra in Figure 2 cannot be lifted to any non-valise form!

Here are some other problems:

**Problem 2.** Given two valise 1-d adinkras $B_L(A)$ and $B_R(A)$ of equal size, what identifications of $V(B_L(A),0)$ and $V(B_R(A),0)$ are possible?

*Proof.* Data: if we have $\{0,1\} \cup \{2,3\}$ on one side, the other side must be $\{0,2\} \cup \{1,3\}$. □

[1] This is fairly nuanced; we need to know not just the shape of $A_L^0$ and $A_R^0$, but also their vertices

[$\star\star\star$ There are two kinds of quotienting that we can think of: one quotient is directly quotienting the mega hypercube adinkra by a ESDE code; one quotient is given the valise adinkra with associated $B_L(A)$, $B_R(A)$ each with $2^{d+1}$ vertices, the necessary $d$-dimensional quotienting that occurs when we naively tensor the two parts (which gives $2^{2d}$ vertices in each corner, for $2^{2d+2}$ total vertices, when in the end we just want $2^{d+2}$ vertices. $\star\star\star$]