11 lecture Laplace

niedziela, 8 grudnia 2019 10:16

20.1 -> ~~Exp~~ test
27.1 relative of I, II test.
28.1 results

The Laplace Transform Method

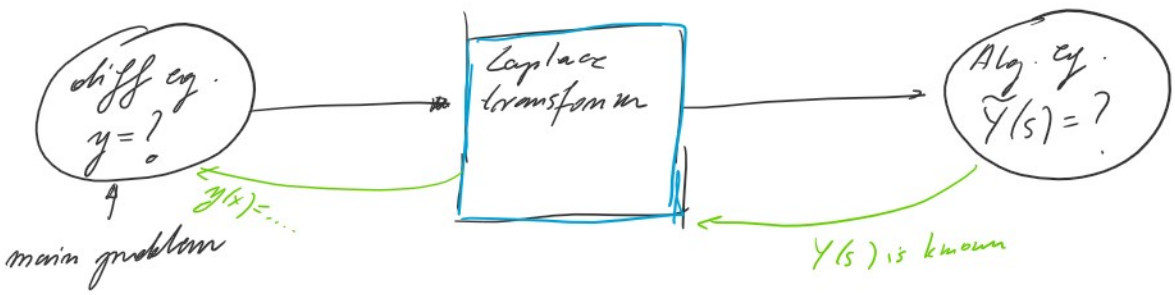
The Laplace Transform is a transformation, meaning that it changes a function into a new function. Actually, it is a linear transformation, because it converts a linear combination of functions into a linear combination of the transformed functions. Even more interesting, the Laplace Transform converts derivatives into multiplications. These two properties make the Laplace Transform very useful to solve linear differential equations with constant coefficients. The Laplace Transform converts such differential equation for an unknown function into an algebraic equation for the transformed function. Usually it is easy to solve the algebraic equation for the transformed function. Then one converts the transformed function back into the original function. This function is the solution of the differential equation.

Solving a differential equation using a Laplace Transform is radically different from all the methods we have used so far. This method, as we will use it here, is relatively new. The Laplace Transform we define here was first used in 1910, but its use grew rapidly after 1920, specially to solve differential equations. Transformations like the Laplace Transform were known much earlier. Pierre Simon de Laplace used a similar transformation in his studies of probability theory, published in 1812, but analogous transformations were used even earlier by Euler around 1737.

transforming functions into functions

$$y(t) \to \tilde{Y}(s) = \int e^{-st}\, y(t)\, dt.$$

$$\mathcal{L}\begin{bmatrix}\text{differential}\\ \text{eq. for } y.\end{bmatrix} \xrightarrow{(1)} \begin{matrix}\text{Algebraic}\\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(2)} \begin{matrix}\text{Solve the}\\ \text{algebraic}\\ \text{eq. for } \mathcal{L}[y].\end{matrix} \xrightarrow{(3)} \begin{matrix}\text{Transform back}\\ \text{to obtain } y.\\ \text{(Use the table.)}\end{matrix}$$



Definition 4.1.1. *The* ***Laplace transform*** *of a function f defined on $D_f = (0, \infty)$ is*

indefine integral - measure

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{4.1.1}$$

defined for all $s \in D_F \subset \mathbb{R}$ where the integral converges.

improper integral

definite integral

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Solution: Following the definition,

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$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,dt = \lim_{N\to\infty} \int_0^N e^{-st}\,dt.$$

Solution:

$$\int e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^N = -\lim_{N\to\infty} -\left(\frac{1}{s} e^{-Ns} + \frac{1}{s}\cdot 1\right) = \frac{1}{s}$$

$$\mathcal{L}[1] = \frac{1}{s}, \qquad s > 0.$$

Example 4.1.2. Compute $\mathcal{L}[e^{at}]$, where $a \in \mathbb{R}$.

Solution: We start with the definition of the Laplace transform,

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st}(e^{at})\,dt = \int_0^\infty e^{-(s-a)t}\,dt.$$

In the case $s = a$ we get

$$\mathcal{L}[e^{at}] = \int_0^\infty 1\,dt = \infty,$$

so the improper integral diverges. In the case $s \neq a$ we get

$$\begin{aligned}
\mathcal{L}[e^{at}] &= \lim_{N\to\infty} \int_0^N e^{-(s-a)t}\,dt, \qquad s \neq a, \\
&= \lim_{N\to\infty} \left[\frac{(-1)}{(s-a)} e^{-(s-a)t} \Big|_0^N \right] \\
&= \lim_{N\to\infty} \left[\frac{(-1)}{(s-a)} (e^{-(s-a)N} - 1) \right].
\end{aligned}$$

Now we have to remaining cases. The first case is:

$$s - a < 0 \quad \Rightarrow \quad -(s-a) = |s-a| > 0 \quad \Rightarrow \quad \lim_{N\to\infty} e^{-(s-a)N} = \infty,$$

so the integral diverges for $s < a$. The other case is:

$$s - a > 0 \quad \Rightarrow \quad -(s-a) = -|s-a| < 0 \quad \Rightarrow \quad \lim_{N\to\infty} e^{-(s-a)N} = 0,$$

so the integral converges only for $s > a$ and the Laplace transform is given by

$$\mathcal{L}[e^{at}] = \frac{1}{(s-a)}, \qquad s > a.$$

Example 4.1.3. Compute $\mathcal{L}[te^{at}]$, where $a \in \mathbb{R}$.

Solution: In this case the calculation is more complicated than above, since we need to integrate by parts. We start with the definition of the Laplace transform,

$$\mathcal{L}[te^{at}] = \int_0^\infty e^{-st} te^{at}\, dt = \lim_{N\to\infty} \int_0^N te^{-(s-a)t}\, dt.$$

This improper integral diverges for $s = a$, so $\mathcal{L}[te^{at}]$ is not defined for $s = a$. From now on we consider only the case $s \neq a$. In this case we can integrate by parts,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N + \frac{1}{s-a} \int_0^N e^{-(s-a)t}\, dt \Big],$$

that is,

$$\mathcal{L}[te^{at}] = \lim_{N\to\infty} \Big[-\frac{1}{(s-a)} te^{-(s-a)t}\Big|_0^N - \frac{1}{(s-a)^2} e^{-(s-a)t}\Big|_0^N \Big]. \tag{4.1.2}$$

In the case that $s < a$ the first term above diverges,

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{-(s-a)N} = \lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{|s-a|N} = \infty,$$

therefore $\mathcal{L}[te^{at}]$ is not defined for $s < a$. In the case $s > a$ the first term on the right hand side in (4.1.2) vanishes, since

$$\lim_{N\to\infty} -\frac{1}{(s-a)} N\, e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)} t\, e^{-(s-a)t}\Big|_{t=0} = 0.$$

Regarding the other term, and recalling that $s > a$,

$$\lim_{N\to\infty} -\frac{1}{(s-a)^2} e^{-(s-a)N} = 0, \qquad \frac{1}{(s-a)^2} e^{-(s-a)t}\Big|_{t=0} = \frac{1}{(s-a)^2}.$$

Therefore, we conclude that

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2}, \qquad s > a.$$

$f(t)$	$F(s) = \mathcal{L}[f(t)]$ or $Y(s)$	D_F
$f(t) = 1$	$F(s) = \frac{1}{s}$	$s > 0$
$f(t) = e^{at}$	$F(s) = \frac{1}{(s-a)}$	$s > a$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$	$s > 0$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2+a^2}$	$s > 0$
$f(t) = \cos(at)$	$F(s) = \frac{s}{s^2+a^2}$	$s > 0$
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2-a^2}$	$s > \lvert a\rvert$
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2-a^2}$	$s > \lvert a\rvert$
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{(n+1)}}$	$s > a$
$f(t) = e^{at}\sin(bt)$	$F(s) = \frac{b}{(s-a)^2+b^2}$	$s > a$
$f(t) = e^{at}\cos(bt)$	$F(s) = \frac{(s-a)}{(s-a)^2+b^2}$	$s > a$

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$f(t) = e^{at}\sinh(bt)$	$F(s) = \dfrac{b}{(s-a)^2-b^2}$	$s - a > \lvert b\rvert$
$f(t) = e^{at}\cosh(bt)$	$F(s) = \dfrac{(s-a)}{(s-a)^2-b^2}$	$s - a > \lvert b\rvert$

Problem

4.1.3. Main Properties. Since we are more or less confident on how to compute a Laplace transform, we can start asking deeper questions. For example, what type of functions have a Laplace transform? It turns out that a large class of functions, those that are piecewise continuous on $[0,\infty)$ and bounded by an exponential. This last property is particularly important and we give it a name.

Definition 4.1.2. *A function f defined on $[0,\infty)$ is of* ***exponential order*** s_0*, where s_0 is any real number, iff there exist positive constants k, T such that*

$$|f(t)| \leqslant k\, e^{s_0 t} \quad \text{for all} \quad t > T. \tag{4.1.3}$$

cannot grow too fast.

Remarks:

(a) When the precise value of the constant s_0 is not important we will say that f is of exponential order.

(b) An example of a function that is not of exponential order is $f(t) = e^{t^2}$.

This definition helps to describe a set of functions having Laplace transform. Piecewise continuous functions on $[0,\infty)$ of exponential order have Laplace transforms.

Theorem 4.1.3 (Convergence of LT)**.** *If a function f defined on $[0,\infty)$ is piecewise continuous and of exponential order s_0, then the $\mathcal{L}[f]$ exists for all $s > s_0$ and there exists a positive constant k such that*

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s-s_0}, \qquad s > s_0.$$

Proof of Theorem 4.1.3: From the definition of the Laplace transform we know that

$$\mathcal{L}[f] = \lim_{N\to\infty} \int_0^N e^{-st}\, f(t)\, dt.$$

The definite integral on the interval $[0, N]$ exists for every $N > 0$ since f is piecewise continuous on that interval, no matter how large N is. We only need to check whether the integral converges as $N \to \infty$. This is the case for functions of exponential order, because

$$\Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant \int_0^N e^{-st}|f(t)|\, dt \leqslant \int_0^N e^{-st} k e^{s_0 t}\, dt = k \int_0^N e^{-(s-s_0)t}\, dt.$$

Therefore, for $s > s_0$ we can take the limit as $N \to \infty$,

$$\big|\mathcal{L}[f]\big| \leqslant \lim_{N\to\infty}\Big|\int_0^N e^{-st} f(t)\, dt\Big| \leqslant k\, \mathcal{L}[e^{s_0 t}] = \frac{k}{(s-s_0)}.$$

Therefore, the comparison test for improper integrals implies that the Laplace transform $\mathcal{L}[f]$ exists at least for $s > s_0$, and it also holds that

$$\big|\mathcal{L}[f]\big| \leqslant \frac{k}{s-s_0}, \qquad s > s_0.$$

This establishes the Theorem. □

Theorem 4.1.4 (Linearity)**.** *If $\mathcal{L}[f]$ and $\mathcal{L}[g]$ exist, then for all $a, b \in \mathbb{R}$ holds*

$$\mathcal{L}[af + bg] = a\,\mathcal{L}[f] + b\,\mathcal{L}[g].$$

Proof of Theorem 4.1.4: Since integration is a linear operation, so is the Laplace transform, as this calculation shows,

$$\begin{aligned}
\mathcal{L}[af + bg] &= \int_0^\infty e^{-st}\big[af(t) + bg(t)\big]\, dt \\
&= a\int_0^\infty e^{-st} f(t)\, dt + b\int_0^\infty e^{-st} g(t)\, dt \\
&= a\,\mathcal{L}[f] + b\,\mathcal{L}[g].
\end{aligned}$$

This establishes the Theorem. □

Example 4.1.5. Compute $\mathcal{L}[3t^2 + 5\cos(4t)]$.

Solution: From the Theorem above and the Laplace transform in Table **??** we know that

$$\begin{aligned}
\mathcal{L}[3t^2 + 5\cos(4t)] &= 3\,\mathcal{L}[t^2] + 5\,\mathcal{L}[\cos(4t)] \\
&= 3\Big(\frac{2}{s^3}\Big) + 5\Big(\frac{s}{s^2+4^2}\Big), \qquad s > 0 \\
&= \frac{6}{s^3} + \frac{5s}{s^2+4^2}.
\end{aligned}$$

$\cos at \to \frac{s}{s^2+a^2}$

$\sin at \to \frac{a}{s^2+a^2}$

Therefore,

Therefore,

$$\mathcal{L}[3t^2 + 5\cos(4t)] = \frac{5s^4 + 6s^2 + 96}{s^3(s^2+16)}, \qquad s > 0.$$

◁

Theorem 4.1.5 (Derivative into Multiplication)**.** *If a function f is continuously differentiable on $[0, \infty)$ and of exponential order s_0, then $\mathcal{L}[f']$ exists for $s > s_0$ and*

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0. \tag{4.1.4}$$

Proof of Theorem 4.1.5: The main calculation in this proof is to compute

$$\mathcal{L}[f'] = \lim_{N\to\infty}\int_0^N e^{-st}\, f'(t)\, dt.$$

We start computing the definite integral above. Since f' is continuous on $[0, \infty)$, that definite integral exists for all positive N, and we can integrate by parts,

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$$\begin{aligned}\int_0^N e^{-st} f'(t)\, dt &= \Big[\Big(e^{-st} f(t)\Big)\Big|_0^N - \int_0^N (-s) e^{-st} f(t)\, dt\Big] \\ &= e^{-sN} f(N) - f(0) + s\int_0^N e^{-st} f(t)\, dt.\end{aligned}$$

We now compute the limit of this expression above as $N \to \infty$. Since f is continuous on $[0,\infty)$ of exponential order s_0, we know that

$$\lim_{N\to\infty} \int_0^N e^{-st} f(t)\, dt = \mathcal{L}[f], \qquad s > s_0.$$

Let us use one more time that f is of exponential order s_0. This means that there exist positive constants k and T such that $|f(t)| \leqslant k\, e^{s_0 t}$, for $t > T$. Therefore,

$$\lim_{N\to\infty} e^{-sN} f(N) \leqslant \lim_{N\to\infty} k\, e^{-sN} e^{s_0 N} = \lim_{N\to\infty} k\, e^{-(s-s_0)N} = 0, \qquad s > s_0.$$

These two results together imply that $\mathcal{L}[f']$ exists and holds

$$\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0.$$

This establishes the Theorem. □

Example 4.1.6. Verify the result in Theorem 4.1.5 for the function $f(t) = \cos(bt)$.

we want to prove : $\mathcal{L}[f'] = s\,\mathcal{L}[f] - f(0), \qquad s > s_0.$

$\mathcal{L}(-b\sin bt) = -b\,\mathcal{L}(\sin bt) = -b\,\frac{b}{s^2+b^2}$

$\mathcal{L}(\cos bt) = s \cdot \frac{s}{s^2+b^2} = \frac{s^2}{s^2+b^2}$ $\qquad s\mathcal{L}[f] - f(0) = \frac{s^2}{s^2+b^2} - 1 = \frac{s^2-s^2-b^2}{s^2+b^2}$ ok.

Theorem 4.1.6 (Higher Derivatives into Multiplication). *If a function f is n-times continuously differentiable on $[0,\infty)$ and of exponential order s_0, then $\mathcal{L}[f''], \cdots, \mathcal{L}[f^{(n)}]$ exist for $s > s_0$ and*

$$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0) \tag{4.1.5}$$

$$\vdots$$

$$\mathcal{L}[f^{(n)}] = s^n\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0). \tag{4.1.6}$$

Proof of Theorem 4.1.6: We need to use Eq. (4.1.4) n times. We start with the Laplace transform of a second derivative,

$$\begin{aligned}\mathcal{L}[f''] &= \mathcal{L}[(f')'] \\ &= s\,\mathcal{L}[f'] - f'(0) \\ &= s\big(s\,\mathcal{L}[f] - f(0)\big) - f'(0) \\ &= s^2\,\mathcal{L}[f] - s\,f(0) - f'(0).\end{aligned}$$

The formula for the Laplace transform of an nth derivative is computed by induction on n.

$$= s^2\,\mathcal{L}[f] - s\,f(0) - f'(0).$$

The formula for the Laplace transform of an nth derivative is computed by induction on n. We assume that the formula is true for $n-1$,

$$\mathcal{L}[f^{(n-1)}] = s^{(n-1)}\,\mathcal{L}[f] - s^{(n-2)}\,f(0) - \cdots - f^{(n-2)}(0).$$

(Mathematical Induction) Th. $A \subset \mathbb{N}$ $A \neq \phi$. (i) $1 \in A$, (ii) $\forall n \in A \Rightarrow n+1 \in A$, then $A = \mathbb{N}$.

Mathematical induction proves that we can climb as high as we like on a ladder, by proving that we can climb onto the bottom rung (the **basis**) and that from each rung we can climb up to the next one (the **step**).

— *Concrete Mathematics*, page 3 margins.

Since $\mathcal{L}[f^{(n)}] = \mathcal{L}[(f')^{(n-1)}]$, the formula above on f' gives

$$\begin{aligned}\mathcal{L}[(f')^{(n-1)}] &= s^{(n-1)}\,\mathcal{L}[f'] - s^{(n-2)}\,f'(0) - \cdots - (f')^{(n-2)}(0)\\ &= s^{(n-1)}\big(s\,\mathcal{L}[f] - f(0)\big) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0)\\ &= s^{(n)}\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - s^{(n-2)}\,f'(0) - \cdots - f^{(n-1)}(0).\end{aligned}$$

This establishes the Theorem. □

Example 4.1.8. Use the Laplace transform to find y solution of

$$y'' + 9\,y = 0, \qquad y(0) = y_0, \quad y'(0) = y_1.$$

diff. eq.

$L(y'') + 9L(y) = 0$

$\mathcal{L}[f''] = s^2\,\mathcal{L}[f] - s\,f(0) - f'(0)$

Laplace transform

$L(y'' + 9y) = L(0) = 0$

$s^2 \cdot L(y) - sy_0 - y_1 + 9L(y) = 0$ ← Algebraic eq.

$L(y)[s^2+9] = sy_0 + y_1 \Rightarrow L(y) = \dfrac{sy_0 + y_1}{s^2+9} = y_0 \dfrac{s}{s^2+9} + y_1 \dfrac{1}{s^2+9}$

$= y_0 L(\cos 3t) + \frac{1}{3} y_1 \underbrace{L(\sin 3t)}_{\frac{3}{s^2+3^2}}$. Therefore

$y(t) = y_0 \cos 3t + \frac{1}{3} y_1 \sin 3t$

Ex $y' - 2y = e^t$ $y(0) = -4$

$L[y'] - 2L[y] = L[e^t]$

$sY(s) + 4 - 2Y(s) = \dfrac{1}{s-1}$

$L(y') - 2L(y) = L[e^t]$

$sY(s) + 4 - 2Y(s) = \dfrac{1}{s-1}$

$Y(s)(s-2) = \dfrac{1}{s-1} - 4 = \dfrac{1-4s+4}{s-1} = \dfrac{5-4s}{s-1}$

$Y(s) = \dfrac{5-4s}{(s-1)(s-2)}$

$\dfrac{5-4s}{(s-1)(s-2)} = \dfrac{A}{s-1} + \dfrac{B}{s-2} = \dfrac{-1}{s-1} + \dfrac{-3}{s-2}$

$y(t) = L^{-1}\left(\dfrac{1}{\ }\right) - e^{2t}$

$\overline{(s-1)(s-2)} = \frac{\quad}{s-1} \quad \frac{\quad}{s-2} \qquad \frac{\quad}{s-1} \quad \frac{\quad}{s-2}$

$L^{-1}\left(\frac{1}{s-1}\right) = e^{t}, \quad L^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$

$y(t) = -e^{t} - 3e^{2t}.$

<u>Ex</u>. $\begin{cases} Y_1' = Y_1 + 2Y_2 + 1 & Y_1(0)=3 \\ Y_2' = 2Y_1 - Y_2 - t & Y_2(0)=0 \end{cases}$

$L(y_1) = Y_1 \quad L(y_2) = Y_2$

$\begin{cases} sY_1 - 3 = Y_1 + 2Y_2 + \frac{1}{s} \\ sY_2 - 0 = 2Y_1 - Y_2 - \frac{1}{s^2} \end{cases}$

$\begin{cases} Y_1(s-1) + Y_2(-2) = \frac{1}{s} + 3 \\ Y_1(-2) + Y_2(s+1) = -\frac{1}{s^2} \end{cases}$ matrix form $W = \begin{vmatrix} s-1 & -2 \\ -2 & s+1 \end{vmatrix} = s^2 - 5$

$W_1 = 3s + 4 + \frac{1}{s} - \frac{2}{s^2}$

$W_2 = \begin{vmatrix} s-1 & 3+\frac{1}{s} \\ -2 & -\frac{1}{s^2} \end{vmatrix} = 6 + \frac{1}{s} + \frac{1}{s^2}$

$Y_1 = \frac{\cdots}{s^2(s-\sqrt{5})(s+\sqrt{5})} = \frac{\quad}{s} + \frac{\quad}{s^2} + \frac{\quad}{s-\sqrt{5}} + \frac{\quad}{s+\sqrt{5}}$

$Y_1 = \ldots \qquad Y_2 = \ldots$

<u>Problem</u> Calculate $L[t^2 e^{3t}]$

$y = t^2 e^{3t} \qquad y' = 2te^{3t} + 3t^2 e^{3t}$

$te^{3t} = y' - 3y$

$y'' = 2e^{3t} + 6te^{3t} + 6te^{3t} + 9t^2 e^{3t}$

$y'' = 2e^{3t} + 6(y' - 3y) + 9y$

$L(y'') = 2L[e^{3t}] + 6L(y' - 3y) + 9L[y]$

$s^2 \cdot Y(s) - s\underset{0}{\underset{\shortparallel}{y(0)}} - \underset{0}{\underset{\shortparallel}{y'(0)}} = \frac{2}{s-3} + 6(sY(s) - 3Y(s)) + 9Y(s)$

$Y(s)(s^2 - 6s + 9) = \frac{2}{s-3}$

<u>Fact:</u> Let $L[y(t)] = Y(s)$. Then

$$\boxed{L[e^{at} \cdot y(t)] = Y(s-a)}$$

<u>Ex</u>

$L(t^2) = \frac{2}{s^3} \qquad L[t^2 e^{3t}] = \frac{2}{(s-3)^3}$

Ex

$$\mathcal{L}(t^2) = \frac{2}{s^3} \qquad \mathcal{L}[t^2 e^{3t}] = \frac{2}{(s-3)^3}$$

Ex

$$\mathcal{L}[e^{-5t} \cdot \cos 2t] = \frac{s+5}{2+(s+5)^2}$$

Proof: $\mathcal{L}[y(t)] = \int_0^\infty e^{-st} y(t)\,dt$

$$\mathcal{L}(s-a) = \int_0^\infty e^{-(s-a)t} y(t)\,dt = \int_0^\infty e^{-st} \underbrace{(e^{at} y(t))}\,dt$$

Fact $\boxed{\mathcal{L}^{-1}[Y(s-a)] = e^{at}\mathcal{L}^{-1}[Y(s)] = e^{at} y(t)}$

Ex $\mathcal{L}^{-1}\left[\frac{4s-7}{(s-2)^2}\right] = \left[\begin{matrix} \bar{s} = s-2 \\ s = \bar{s}+2 \end{matrix}\right] = e^{2t} \cdot \mathcal{L}^{-1}\left(\frac{4s+1}{s^2}\right)$

$$4s-7 = 4(s-2)+1$$

$$= e^{2t}\left(\mathcal{L}^{-1}\left(4 \cdot \frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)\right) = e^{2t}(4+t)$$

Ex $y'' + 6y' + 13y = 0 \qquad y(0) = 1 \quad y'(0) = 3.$

$\mathcal{L}(y) = Y(s)$

$\mathcal{L}(y') = s \cdot Y(s) - 1$

$\mathcal{L}(y'') = s^2 \cdot Y(s) - s1 - 3$

$s^2 \cdot Y - s - 3 + 6sY - 6 + 13Y = 0$

$Y(s^2 + 6s + 13) = 3 + s + 6 = 9 + s$

$Y = \frac{9+s}{s^2+6s+13} \qquad y(t) = \mathcal{L}^{-1}\left(\frac{9+s}{s^2+6s+13}\right)$

$$\frac{9+s}{(s+3)^2+4} = \left[\begin{matrix} s+3 = \bar{s} \\ s = \bar{s}-3 \end{matrix}\right] = \frac{9+\bar{s}-3}{\bar{s}^2+4} =$$

$$= \frac{\bar{s}+6}{\bar{s}^2+4}$$

$$y(t) = e^{-3t} \cdot \mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] + e^{-3t} \cdot \mathcal{L}^{-1}\left[\frac{6}{s^2+4}\right] =$$

$$= e^{-3t}\cos 2t + e^{-3t}\sin 2t \cdot \frac{1}{2}$$

Revision - I Laplace transform

Revision of Laplace transform

diff. eq. $y(t)=?$ $\xrightarrow{(1)}$ Laplace transform $\xrightarrow{(2)}$ Algebraic equation $Y(s)$

Laplace transform $\xrightarrow{(4)}$ $y(t)=\dots$

Algebraic equation $Y(s)$ $\xrightarrow{(3)}$ Laplace transform

For (1), (2) we are using linearity, and the fact that Laplace transform converts derivatives into multiplication.
$\left(\mathcal{L}[f'] = s\,Y(s) - f(0) \quad s > s_0 \right)$

$\mathcal{L}[f^{(n)}] = s^n\,\mathcal{L}[f] - s^{(n-1)}\,f(0) - \cdots - f^{(n-1)}(0).$

Let us think a little bit about step (4).

Solving diff. eq. we are getting
$$\mathcal{L}(y(t)) = Y(s)$$

Now, using Laplace table, we are looking for $h(t)$ s.t. $\mathcal{L}(h(t)) = Y(s)$.

We are getting $\mathcal{L}(y(t)) = \mathcal{L}(h(t))$

Question: $y(t) = h(t)$?

$|f(t)| \leqslant k\,e^{s_0 t}$ for all $t > T$.

Theorem 4.2.1 (One-to-One). *If f, g are continuous on $[0,\infty)$ of exponential order, then*
$$\mathcal{L}[f] = \mathcal{L}[g] \quad \Rightarrow \quad f = g.$$

Before proving the above theorem we need the following tool:

Theorem 4.2.3 (Weierstrass Approximation). *If f is a continuous function on a closed interval $[a,b]$, then for every $\epsilon > 0$ there exists a polynomial q_ϵ such that*
$$\max_{y\in[a,b]} |f(y) - q_\epsilon(y)| < \epsilon.$$

<u>Proof:</u> We want to prove $\mathcal{L}(f) = \mathcal{L}(g) \Rightarrow f = g$.

Let $u = f - g$. $\mathcal{L}(u) = \mathcal{L}[f-g] = \mathcal{L}[f] - \mathcal{L}[g] = 0$

From the definition $\mathcal{L}(u) = \int_0^\infty e^{-st} u(t)\,dt$

f, g are of exponential order $\Rightarrow$ u is of exponential order i.e. $|u(t)| < k\,e^{s_0 t}$, $t > T$. for some k, s_0, T

Let us calculate $\mathcal{L}(u(s))$ for $\boxed{\tilde{s} = s_1 + n + 1}$, where s_1 is arbitrary $s_1 > s_0$, n is positive integer (arbitrary)

usage na pátrání

exp $\longrightarrow$ blow up

e^{at} $\longrightarrow$ $\frac{1}{s-a}$

$$\mathcal{L}(u)\Big|_{\tilde{s}} = \int_0^\infty e^{-(s_1+n+1)t} u(t)\,dt = \int_0^\infty e^{-s_1 t} \cdot e^{-(n+1)t} u(t)\,dt = \int_0^\infty e^{-ts_1} e^{-nt} \cdot e^{-t} u\,dt$$

We now use substitution $\boxed{y = e^{-t} \quad dy = -e^{-t}\,dt}$. $t=0$, $y=1$; $t=\infty$, $y=0$

0 1

We now use substitution $\boxed{y = e \qquad dy = -e\ dt}$. $\quad t = 0 \quad y = 1 \qquad t = \infty \quad y = 0$

$$L(u)\Big|_{\tilde{s}} = \int_1^0 y^{s_1} y^n u(-\ln(y))(-dy) = \int_0^1 y^{s_1} y^n \cdot u(-\ln(y))\,dy$$

Now we put $v(y) = y^{s_1} \cdot u(-\ln(y))$. Function v is well defined since:

$$\lim_{y\to 0^+} |v(y)| = \lim_{t\to\infty} e^{-s_1 t} |u(t)| < \lim_{t\to\infty} e^{-s_1 t} e^{s_0 t} = 0$$

Let us go back to our integral i.e.

$$L(u)\Big|_{\tilde{s}} = \int_0^1 v(y)\, y^n\, dy \qquad n = 1, 2, \ldots$$

We can observe that equation

$\int_0^1 v(y)\, y^n\, dy \underset{(*)}{=} 0$ means that v is perpendicular to every polynomial (since $(*)$ holds for every $n = 1, 2 \ldots$).

$$\int_0^1 v^2(y)\,dy = \int_0^1 (v(y) - p(y))v(y)\,dy + \underbrace{\int_0^1 p(y)v(y)\,dy}_{=0 \text{ because of } (*)}.$$

So

$$\int_0^1 v^2(y)\,dy = \int_0^1 (v(y) - p(y))v(y)\,dy \le \int_0^1 |v(y) - p(y)||v(y)|\,dy \overset{(\square)}{\le} \max_{y\in[0,1]} v(y)\,|v(y) - p(y)|$$

Since, ineq. $(\square)$ holds for every polynomial, from Weierstrass theorem $\exists \varepsilon\ \exists p_\varepsilon \to$ polynomial s.t.

$$\int_0^1 v^2(y)\,dy \overset{(\square)}{\le} \max_{y\in[0,1]} v(y)\,|v(y) - p_\varepsilon(y)| \overset{\text{Th. Weierstrass.}}{\le} \max_{y\in[0,1]} v(y) \cdot \varepsilon$$

v is continuous so $v = 0 \iff u = 0 \iff f = g$. ∎

Only for aesthetic purposes let us write the definition of the inverse.

Definition 4.2.2. *The* ***inverse Laplace transform****, denoted* $\mathcal{L}^{-1}$*, of a function* F *is*

$$\mathcal{L}^{-1}[F(s)] = f(t) \quad \Leftrightarrow \quad F(s) = \mathcal{L}[f(t)].$$

Nonhomogeneous case

Example 4.2.4. Use the Laplace transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 3\,e^t, \qquad y(0) = 0, \qquad y'(0) = 0.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\,e^t] = 3\left(\frac{1}{s-1}\right).$$

The Laplace transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{3}{s-1}.$$

The Laplace transform relates derivatives with multiplication,

Example 4.2.4. Use the Laplace transform to find the solution y to the initial value problem

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The Laplace transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{3}{s-1}.$$

The Laplace transform relates derivatives with multiplication,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{3}{s-1},$$

But the initial conditions are $y(0) = 0$ and $y'(0) = 0$, so

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = \frac{3}{s-1}.$$

Solve the algebraic equation for $\mathcal{L}[y]$,

$$\mathcal{L}[y] = \frac{3}{(s-1)(s^2 - 4s + 4)}.$$

We use *partial fractions* to simplify the right-hand side above. We start finding the roots of the polynomial in the denominator,

$$s^2 - 4s + 4 = 0 \quad \Rightarrow \quad s_\pm = \frac{1}{2}\big[4 \pm \sqrt{16-16}\big] \quad \Rightarrow \quad s_+ = s_- = 2.$$

that is, the polynomial has a single real root, so $\mathcal{L}[y]$ can be written as

$$\mathcal{L}[y] = \frac{3}{(s-1)(s-2)^2}.$$

The partial fraction decomposition of the right-hand side above is

$$\frac{3}{(s-1)(s-2)^2} = \frac{a}{(s-1)} + \frac{b\,s + c}{(s-2)^2} = \frac{a\,(s-2)^2 + (b\,s+c)(s-1)}{(s-1)(s-2)^2}$$

From the far right and left expressions above we get

$$3 = a\,(s-2)^2 + (b\,s+c)(s-1) = a\,(s^2 - 4s + 4) + b\,s^2 - b\,s + c\,s - c$$

Expanding all terms above, and reordering terms, we get

$$(a+b)\,s^2 + (-4a - b + c)\,s + (4a - c - 3) = 0.$$

Since this polynomial in s vanishes for all $s \in \mathbb{R}$, we get that

$$\left.\begin{aligned} a + b &= 0, \\ -4a - b + c &= 0, \\ 4a - c - 3 &= 0. \end{aligned}\right\} \quad \Rightarrow \quad \begin{cases} a = 3 \\ b = -3 \\ c = 9. \end{cases}$$

So we get

$$\mathcal{L}[y] = \frac{3}{(s-1)(s-2)^2} = \frac{3}{s-1} + \frac{-3s+9}{(s-2)^2}$$

One last trick is needed on the last term above,

$$\frac{-3s+9}{(s-2)^2} = \frac{-3(s-2+2)+9}{(s-2)^2} = \frac{-3(s-2)}{(s-2)^2} + \frac{-6+9}{(s-2)^2} = -\frac{3}{(s-2)} + \frac{3}{(s-2)^2}.$$

So we finally get

$$\mathcal{L}[y] = \frac{3}{s-1} - \frac{3}{(s-2)} + \frac{3}{(s-2)^2}.$$

From our Laplace transforms Table we know that

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \quad \Rightarrow \quad \frac{1}{s-2} = \mathcal{L}[e^{2t}],$$

$$\mathcal{L}[te^{at}] = \frac{1}{(s-a)^2} \quad \Rightarrow \quad \frac{1}{(s-2)^2} = \mathcal{L}[te^{2t}].$$

So we arrive at the formula

$$\mathcal{L}[y] = 3\,\mathcal{L}[e^t] - 3\,\mathcal{L}[e^{2t}] + 3\,\mathcal{L}[t\,e^{2t}] = \mathcal{L}\big[3\,(e^t - e^{2t} + t\,e^{2t})\big]$$

So we conclude that $y(t) = 3\,(e^t - e^{2t} + t\,e^{2t})$. $\triangleleft$

Example 4.2.5. Use the Laplace transform to find the solution y to the initial value problem

$$y'' - 4y' + 4y = 3\sin(2t), \qquad y(0) = 1, \qquad y'(0) = 1.$$

Solution: First, compute the Laplace transform of the differential equation,

$$\mathcal{L}[y'' - 4y' + 4y] = \mathcal{L}[3\sin(2t)].$$

The right hand side above can be expressed as follows,

$$\mathcal{L}[3\sin(2t)] = 3\,\mathcal{L}[\sin(2t)] = 3\,\frac{2}{s^2 + 2^2} = \frac{6}{s^2 + 4}.$$

Theorem 4.1.4 says that the Laplace transform is a linear operation,

$$\mathcal{L}[y''] - 4\,\mathcal{L}[y'] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4},$$

and Theorem 4.1.5 relates derivatives with multiplications,

$$\Big[s^2\,\mathcal{L}[y] - s\,y(0) - y'(0)\Big] - 4\Big[s\,\mathcal{L}[y] - y(0)\Big] + 4\,\mathcal{L}[y] = \frac{6}{s^2 + 4}.$$

Reorder terms,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = (s - 4)\,y(0) + y'(0) + \frac{6}{s^2 + 4}.$$

Introduce the initial conditions $y(0) = 1$ and $y'(0) = 1$,

$$(s^2 - 4s + 4)\,\mathcal{L}[y] = s - 3 + \frac{6}{s^2 + 4}.$$

Solve this algebraic equation for $\mathcal{L}[y]$, that is,

$$\mathcal{L}[y] = \frac{(s - 3)}{(s^2 - 4s + 4)} + \frac{6}{(s^2 - 4 + 4)(s^2 + 4)}.$$

From the Example above we know that $s^2 - 4s + 4 = (s - 2)^2$, so we obtain

$$\mathcal{L}[y] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2} + \frac{6}{(s - 2)^2(s^2 + 4)}. \tag{4.2.4}$$

From the previous example we know that

$$\mathcal{L}[e^{2t} - te^{2t}] = \frac{1}{s - 2} - \frac{1}{(s - 2)^2}. \tag{4.2.5}$$

We know use *partial fractions* to simplify the third term on the right hand side of Eq. (4.2.4). The appropriate partial fraction decomposition for this term is the following: Find constants a, b, c, d, such that

$$\frac{6}{(s - 2)^2\,(s^2 + 4)} = \frac{as + b}{s^2 + 4} + \frac{c}{(s - 2)} + \frac{d}{(s - 2)^2}$$

Take common denominator on the right hand side above, and one obtains the system

$$\begin{aligned} a + c &= 0, \\ -4a + b - 2c + d &= 0, \\ 4a - 4b + 4c &= 0, \\ 4b - 8c + 4d &= 6. \end{aligned}$$

The solution for this linear system of equations is the following:

$$a = \frac{3}{8}, \qquad b = 0, \qquad c = -\frac{3}{8}, \qquad d = \frac{3}{4}.$$

Therefore,

$$\frac{6}{(s - 2)^2\,(s^2 + 4)} = \frac{3}{8}\,\frac{s}{s^2 + 4} - \frac{3}{8}\,\frac{1}{(s - 2)} + \frac{3}{4}\,\frac{1}{(s - 2)^2}$$

We can rewrite this expression above in terms of the Laplace transforms given in Table 1, in Sect. **??**, as follows,

$$\frac{6}{(s - 2)^2\,(s^2 + 4)} = \frac{3}{8}\,\mathcal{L}[\cos(2t)] - \frac{3}{8}\,\mathcal{L}[e^{2t}] + \frac{3}{4}\,\mathcal{L}[te^{2t}],$$

and using the linearity of the Laplace transform,

$$\frac{6}{(s - 2)^2\,(s^2 + 4)} = \mathcal{L}\Big[\frac{3}{8}\,\cos(2t) - \frac{3}{8}\,e^{2t} + \frac{3}{4}\,te^{2t}\Big]. \tag{4.2.6}$$

Finally, introducing Eqs. (4.2.5) and (4.2.6) into Eq. (4.2.4) we obtain

$$\mathcal{L}[y(t)] = \mathcal{L}\Big[(1 - t)\,e^{2t} + \frac{3}{8}\,(-1 + 2t)\,e^{2t} + \frac{3}{8}\,\cos(2t)\Big].$$

Since the Laplace transform is an invertible transformation, we conclude that

$$y(t) = (1 - t)\,e^{2t} + \frac{3}{8}\,(2t - 1)\,e^{2t} + \frac{3}{8}\,\cos(2t).$$

◁

Example 4.2.6. Use the Laplace transform to find the solution y to the initial value problem

$$y^{(4)} - 4y = 0, \qquad \begin{aligned} y(0) &= 1, & y'(0) &= 0, \\ y''(0) &= -2, & y'''(0) &= 0. \end{aligned}$$

Solution: Compute the Laplace transform of the differential equation,

$$\mathcal{L}[y^{(4)} - 4y] = \mathcal{L}[0] = 0.$$

The Laplace transform is a linear operation,

$$\mathcal{L}[y^{(4)}] - 4\,\mathcal{L}[y] = 0,$$

and the Laplace transform relates derivatives with multiplications,

$$\Big[s^4\,\mathcal{L}[y] - s^3\,y(0) - s^2\,y'(0) - s\,y''(0) - y'''(0)\Big] - 4\,\mathcal{L}[y] = 0.$$

From the initial conditions we get

$$\Big[s^4\,\mathcal{L}[y] - s^3 - 0 + 2s - 0\Big] - 4\,\mathcal{L}[y] = 0 \quad \Rightarrow \quad (s^4 - 4)\mathcal{L}[y] = s^3 - 2s \quad \Rightarrow \quad \mathcal{L}[y] = \frac{(s^3 - 2s)}{(s^4 - 4)}.$$

In this case we are lucky, because

$$\mathcal{L}[y] = \frac{s(s^2 - 2)}{(s^2 - 2)(s^2 + 2)} = \frac{s}{(s^2 + 2)}.$$

Since

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2},$$

we get that

$$\mathcal{L}[y] = \mathcal{L}[\cos(\sqrt{2}t)] \quad \Rightarrow \quad y(t) = \cos(\sqrt{2}t).$$

◁

So why we are using Laplace transform instead of previous methods?

They are working well.

first derivative - velocity
second derivative - acceleration

Example 4.4.3. Use Newton's equation of motion and Dirac's delta to describe the change of momentum when a particle is hit by a hammer.

$ma = F$

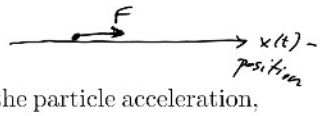
$mx''(t) = F(t, x(t))$

$a(t) = x''(t)$

what is happening if F is an impulse force.

Solution: A point particle with mass m, moving on one space direction, x, with a force F acting on it is described by

$$\boxed{ma = F \quad \Leftrightarrow \quad mx''(t) = F(t, x(t)),}$$



where $x(t)$ is the particle position as function of time, $a(t) = x''(t)$ is the particle acceleration, and we will denote $v(t) = x'(t)$ the particle velocity. ~~We saw in § 1.1 that Newton's second law of motion is a second order differential equation for the position function x.~~ Now it is more convenient to use the *particle momentum*, $p = mv$, to write the Newton's equation,

$$mx'' = mv' = (mv)' = F \quad \Rightarrow \quad p' = F.$$

So the force F changes the momentum, P. If we integrate on an interval $[t_1, t_2]$ we get

$$\cancel{\Delta p = p(t_2) - p(t_1) = \int_{t_1}^{t_2} F(t, x(t))\,dt.}$$

$p_0 \to \Delta p$?

Suppose that an impulsive force is acting on a particle at t_0 transmitting a finite momentum, say p_0. This is where the Dirac delta is uselful for, because we can write the force as

$$\boxed{F(t) = p_0\,\delta(t - t_0),}$$

then $F = 0$ on $\mathbb{R} - \{t_0\}$ and the momentum transferred to the particle by the force is

$$\Delta p = \int_{t_0 - \Delta t}^{t_0 + \Delta t} p_0\,\delta(t - t_0)\,dt = p_0.$$

00

Example 4.4.2. Consider now the following sequence, $\{u_n\}$, for $n \geqslant 1$,

$$u_n(t) = \begin{cases} 0, & t < 0 \\ nt, & 0 \leqslant t \leqslant \dfrac{1}{n} \\ 1, & t > \dfrac{1}{n}. \end{cases} \tag{4.4.1}$$

0
nt
1

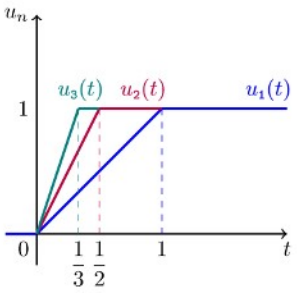


FIGURE 8. A few functions in the sequence $\{u_n\}$.

$\lim_{n\to\infty} u_n(t) = \tilde{u}(t)$, where

$$\tilde{u}(t) = \begin{cases} 0 & \text{for} \quad t \leqslant 0, \\ 1 & \text{for} \quad t > 0. \end{cases}$$

Definition 4.4.1. *The* ***Dirac delta*** *generalized function is the limit*

$$\delta(t) = \lim_{n\to\infty} \delta_n(t),$$

for every fixed $t \in \mathbb{R}$ of the sequence functions $\{\delta_n\}_{n=1}^{\infty}$,

$$\delta_n(t) = n\left[u(t) - u(t - \frac{1}{n})\right]. \tag{4.4.2}$$

The sequence of bump functions introduced above can be rewritten as follows,

There are other possibilities to define δ_n convergent to Dirac delta.

$$\delta_n(t) = \begin{cases} 0, & t < 0 \\ n, & 0 \leqslant t < \dfrac{1}{n} \\ 0, & t \geqslant \dfrac{1}{n}. \end{cases}$$

We then obtain the equivalent expression,

$$\delta(t) = \begin{cases} 0 & \text{for} \quad t \neq 0, \\ \infty & \text{for} \quad t = 0. \end{cases}$$

Remark: It can be shown that there exist infinitely many sequences $\{\tilde{\delta}_n\}$ such that their limit as $n \to \infty$ is Dirac's delta. For example, another sequence is

$$\tilde{\delta}_n(t) = n\left[u(t + \frac{1}{2n}) - u(t - \frac{1}{2n})\right]$$
$$= \begin{cases} 0, & t < -\dfrac{1}{2n} \\ n, & -\dfrac{1}{2n} \leqslant t \leqslant \dfrac{1}{2n} \\ 0, & t > \dfrac{1}{2n}. \end{cases}$$

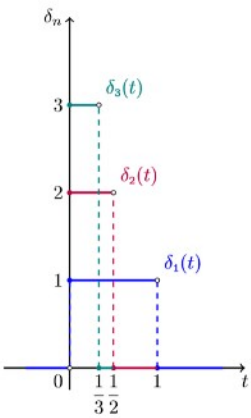


FIGURE 9. A few functions in the sequence $\{\delta_n\}$.

Some properties

Definition 4.4.2. *We introduce the following operations on the Dirac delta:*

$$f(t)\,\delta(t-c) + g(t)\,\delta(t-c) = \lim_{n\to\infty}\left[f(t)\,\delta_n(t-c) + g(t)\,\delta_n(t-c)\right],$$
$$\int_a^b \delta(t-c)\,dt = \lim_{n\to\infty}\int_a^b \delta_n(t-c)\,dt,$$
$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty}\mathcal{L}[\delta_n(t-c)].$$

since Dirac Delta is not a function we cannot use Riemann or Lebesgue definitions of integrals.

Theorem 4.4.3. *For every $c \in \mathbb{R}$ and $\epsilon > 0$ holds,* $\displaystyle\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\,dt = 1.$

Proof: $\displaystyle\int_{c-\varepsilon}^{c+\varepsilon} \delta(t-c)\,dt = \lim_{n\to\infty}\int_{c-\varepsilon}^{c+\varepsilon} \delta_n(t-c)\,dt$

For $n > \frac{1}{\varepsilon}$ $\displaystyle\int_{c-\varepsilon}^{c+\varepsilon} \delta(t-c)\,dt = \lim \int_{c}^{c+\frac{1}{n}} n\,dt = \lim n(c+\tfrac{1}{n}-c) = 1$ $\square$

For $n > \frac{1}{\varepsilon}$ $\quad \int_{c-\varepsilon}^{c+\varepsilon} \delta(t-c)\,dt = \lim_{n\to\infty} \int_{c}^{c+\frac{1}{n}} n\,dt = \lim_{n\to\infty} n\left(c+\frac{1}{n}-c\right) = 1.$ ■

Theorem 4.4.4. *If f is continuous on (a,b) and $c \in (a,b)$, then* $\int_a^b f(t)\,\delta(t-c)\,dt = f(c)$.

Proof: $\int_a^b f(t)\delta(t-c)\,dt = \lim_{n\to\infty} \int_a^b f(t)\delta_n(t-c)\,dt =$

$= \lim_{n\to\infty} \int_a^b \delta_n(t-c) f(t)\,dt = \lim_{n\to\infty} \int_a^b n\left[u(t-c) - u\left(t-c-\frac{1}{n}\right)\right] f(t)\,dt$

$= \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n f(t) = \text{For } F' = f \quad \lim_{n\to\infty} n\left[F\left(c+\frac{1}{n}\right) - F(c)\right] =$

$= \lim_{n\to\infty} \dfrac{F\left(c+\frac{1}{n}\right) - F(c)}{\frac{1}{n}} = F'(c) = f(c)$ ■

$b - c > \frac{1}{n}$ if $c \in [a,b]$

Laplace transform of Dirac delta.

Theorem 4.4.5. *For all $s \in \mathbb{R}$ holds* $\mathcal{L}[\delta(t-c)] = \begin{cases} e^{-cs} & \text{for} \quad c \geqslant 0, \\ 0 & \text{for} \quad c < 0. \end{cases}$

Second Proof of Theorem 4.4.5: The Laplace transform of a Dirac's delta is computed as a limit of Laplace transforms,

$$\begin{aligned} \mathcal{L}[\delta(t-c)] &= \lim_{n\to\infty} \mathcal{L}[\delta_n(t-c)] \\ &= \lim_{n\to\infty} \mathcal{L}\Big[n\big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\big]\Big] \\ &= \lim_{n\to\infty} \int_0^\infty n\big[u(t-c) - u\big(t-c-\frac{1}{n}\big)\big]\, e^{-st}\,dt. \end{aligned}$$

The case $c < 0$ is simple. For $\frac{1}{n} < |c|$ holds

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_0^\infty 0\,dt \quad \Rightarrow \quad \mathcal{L}[\delta(t-c)] = 0, \qquad \text{for } s \in \mathbb{R}, \quad c < 0.$$

Consider now the case $c \geqslant 0$. We then have,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\, e^{-st}\,dt.$$

For $s = 0$ we get

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\,dt = 1 \quad \Rightarrow \quad \mathcal{L}[\delta(t-c)] = 1 \quad \text{for } s = 0, \quad c \geqslant 0.$$

In the case that $s \neq 0$ we get,

$$\mathcal{L}[\delta(t-c)] = \lim_{n\to\infty} \int_c^{c+\frac{1}{n}} n\, e^{-st}\,dt = \lim_{n\to\infty} -\frac{n}{s}\big(e^{-cs} - e^{-(c+\frac{1}{n})s}\big) = e^{-cs} \lim_{n\to\infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)}.$$

The limit on the last line above is a singular limit of the form $\frac{0}{0}$, so we can use the l'Hôpital rule to compute it, that is,

$$\lim_{n\to\infty} \frac{(1 - e^{-\frac{s}{n}})}{\left(\frac{s}{n}\right)} = \lim_{n\to\infty} \frac{\left(-\frac{s}{n^2} e^{-\frac{s}{n}}\right)}{\left(-\frac{s}{n^2}\right)} = \lim_{n\to\infty} e^{-\frac{s}{n}} = 1.$$

We then obtain,

$$\mathcal{L}[\delta(t-c)] = e^{-cs} \quad \text{for } s \neq 0, \quad c \geqslant 0.$$

This establishes the Theorem. □

How to solve problems with Dirac delta as a right-hand-side function?

Problem:

$$y'' + a_1 y' + a_0 y = \delta(t-c)$$

Definition 4.4.6. *The* ***impulse response function*** *at the point $c \geqslant 0$ of the constant coefficients linear operator $L(y) = y'' + a_1\, y' + a_0\, y$, is the solution y_δ of*

$$L(y_\delta) = \delta(t-c), \qquad y_\delta(0) = 0, \qquad y_\delta'(0) = 0.$$

Remark: Impulse response functions are also called *fundamental solutions.*

Theorem 4.4.7. *The function y_δ is the impulse response function at $c \geqslant 0$ of the constant coefficients operator $L(y) = y'' + a_1\, y' + a_0\, y$ iff holds*

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{e^{-cs}}{p(s)}\Big].$$

where p is the characteristic polynomial of L.

Remark: The impulse response function y_δ at $c = 0$ satifies

$$y_\delta = \mathcal{L}^{-1}\Big[\frac{1}{p(s)}\Big].$$

Ex $\quad y'' + 2y' + 2y = \delta(t) \quad y_\delta(0) = 0 \quad y_\delta'(0) = 0$

$\mathcal{L}[\delta(t-c)] = e^{-cs}$ f

0 is not a solution :)

$\mathcal{L}(y'') + 2\mathcal{L}(y') + 2\mathcal{L}(y) = \mathcal{L}[\delta(t)] = e^0 = 1$

$(s^2 + 2s + 2)\,Y(s) = 1 \qquad Y(s) = \dfrac{1}{s^2+2s+2} = \dfrac{1}{(s+1)^2+1} = \mathcal{L}(e^{-t}\sin t)$

$e^{ct} f(t) \longrightarrow F(s-c)$

$\dfrac{1}{(s-(-1))^2+1}$

$F(s+c)$

$c = -1$