LECTURE 21. 04. 2020

SOLVING (H) LDE OF THE n-th ORDER

$$D\left[y\right] = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_2y'' + a_1y' + a_0y = 0, \tag{1}$$

where $a_0,\ a_1, ..., a_{n-1}$ are given reals. The GS of (H) is the set

$$\mathcal{Y} = \left\{y : D\left[y\right] = 0\right\}.$$

Theorem 1 *The set* $\mathcal{Y}$ *is a linear space and* $\dim \mathcal{Y} = n$.

Definition 2 *Equation*

$$p\left(\lambda\right) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_2\lambda^2 + a_1\lambda + a_0 = 0$$

is called the characteristic equation (CH) of DE

$$D\left[y\right] = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_2y'' + a_1y' + a_0y = 0, \tag{2}$$

while $p\left(\lambda\right) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_2\lambda^2 + a_1\lambda + a_0$ *is the characteristic polynomial. The zeros of (6) are called the eigenvalues. We say that the eigenvalue* $\lambda = r$ *has the multiplicity* m *iff*

$$\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_2\lambda^2 + a_1\lambda + a_0 = \left(\lambda - r\right)^m w\left(\lambda\right) \quad \text{and} \quad w\left(r\right) \neq 0$$

Theorem 3 *If* $\lambda = r$ *is real eigenvalue of multiplicity* m*, then it gives the following* m *FS:*

$$e^{rx}, xe^{rx}, x^2e^{rx}, ..., x^{m-1}e^{rx}$$

If $\lambda = \alpha + i\beta$ *is complex eigenvalue of multiplicity* m*, then it gives the following* $2m$ *FS:*

$$\begin{aligned} &e^{\alpha x}\cos\left(\beta x\right), xe^{\alpha x}\cos\left(\beta x\right), x^2e^{\alpha x}\cos\left(\beta x\right) ..., x^{m-1}e^{\alpha x}\cos\left(\beta x\right), \\ &e^{\alpha x}\sin\left(\beta x\right), xe^{\alpha x}\sin\left(\beta x\right), x^2e^{\alpha x}\sin\left(\beta x\right) ..., x^{m-1}e^{\alpha x}\sin\left(\beta x\right). \end{aligned}$$

This way we obtain all FS $\varphi_1, \varphi_2, ..., \varphi_n \in \mathcal{Y}$ *and the GS is a linear combination of them*

$$y = C_1\varphi_1 + C_2\varphi_2 + ... + C_n\varphi_n \in \mathcal{Y}.$$

Example 4 *Determine the GS of*

$$D\left[y\right] = y''' - 3y'' + 3y' - y = 0.$$

The CH is

$$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = \left(\lambda - 1\right)^3 = 0.$$

So the FS are

$$e^x, xe^x, x^2e^x.$$

The GS is

$$y = Ce^x + Dxe^x + Ex^2e^x,$$

where C, D, E are abitrary constants.

Example 5 *Determine the GS of*

$$D[y] = y^{(4)} + 2y'' + y = 0.$$

Solution 6 *The CH is*

$$\lambda^4 + 2\lambda^2 + 1 = \left(\lambda^2 + 1\right)^2 = 0.$$

The eigenvalues are $\lambda = i$ and $\lambda = -i$, both are double roots. The real part is $\alpha = 0$ and imaginary part - $\beta = 1$. So we obtain FS

$$\begin{aligned} \varphi_1 &= \cos x, \quad \varphi_2 = x\cos x, \\ \varphi_3 &= \sin x, \quad \varphi_4 = x\sin x. \end{aligned}$$

Check for $\varphi_2 = x\cos x$. Then

$$\begin{aligned} \varphi_2' &= \frac{d}{dx}(x\cos x) = \cos x - x\sin x, \\ \varphi_2'' &= \frac{d}{dx}(\cos x - x\sin x) = -2\sin x - x\cos x, \\ \varphi_2''' &= \frac{d}{dx}(-2\sin x - x\cos x) = x\sin x - 3\cos x \\ \varphi_2'''' &= \frac{d}{dx}(x\sin x - 3\cos x) = 4\sin x + x\cos x. \end{aligned}$$

Hence

$$D[\varphi_2] = \varphi_2^{(4)} + 2\varphi_2'' + \varphi_2 =$$

$$= (4\sin x + x\cos x) + 2(-2\sin x - x\cos x) + (x\cos x) = 0.$$

The FS are linearly independent. We need to check the Wrońskian

$$
\begin{aligned}
W(x) &= \det \begin{bmatrix} \varphi_1(x) & \varphi_2(x) & \varphi_3(x) & \varphi_4(x) \\ \varphi_1'(x) & \varphi_2'(x) & \varphi_3'(x) & \varphi_4'(x) \\ \varphi_1''(x) & \varphi_2''(x) & \varphi_2''(x) & \varphi_4''(x) \\ \varphi_1'''(x) & \varphi_2'''(x) & \varphi_2'''(x) & \varphi_4'''(x) \end{bmatrix} = \\
&= \det \begin{array}{c} \begin{bmatrix} \cos x & x\cos x & \sin x & x\sin x \\ -\sin x & \cos x - x\sin x & \cos x & \sin x + x\cos x \\ -\cos x & -2\sin x - x\cos x & -\sin x & 2\cos x - x\sin x \\ \sin x & x\sin x - 3\cos x & -\cos x & -3\sin x - x\cos x \end{bmatrix} \end{array} \begin{array}{c} r_1 \\ r_2 \\ r_3 + r_1 \\ r_4 + r_2 \end{array} = \\
&= \det \begin{bmatrix} \cos x & x\cos x & \sin x & x\sin x \\ -\sin x & \cos x - x\sin x & \cos x & \sin x + x\cos x \\ 0 & -2\sin x & 0 & 2\cos x \\ 0 & -2\cos x & 0 & -2\sin x \end{bmatrix} \\
&= (-1)^{1+1}\cos x \det \begin{bmatrix} \cos x - x\sin x & \cos x & \sin x + x\cos x \\ -2\sin x & 0 & 2\cos x \\ -2\cos x & 0 & -2\sin x \end{bmatrix} \\
&\quad + (-1)^{2+1}(-\sin x)\det \begin{bmatrix} x\cos x & \sin x & x\sin x \\ -2\sin x & 0 & 2\cos x \\ -2\cos x & 0 & -2\sin x \end{bmatrix} \\
&= \cos^2 x \cdot (-1)^{1+2} \det \begin{bmatrix} -2\sin x & 2\cos x \\ -2\cos x & -2\sin x \end{bmatrix} + \sin^2 x \cdot (-1)^{1+2} \det \begin{bmatrix} -2\sin x & 2\cos x \\ -2\cos x & -2\sin x \end{bmatrix} \\
&= -\cos^2 x \cdot (4) - \sin^2 x \cdot (4) = -4 \neq 0.
\end{aligned}
$$

Therefore the GS is

$$y = C_1 \cos x + C_2 x \cos x + C_3 \sin x + C_4 x \sin x,$$

where C_1, C_2, C_3, C_4 are arbitrary constants.

Example 7 *Knowing that the CH of certain LDE $D[y] = 0$ is equal to*

$$p(\lambda) = \lambda^4 (\lambda - 1)^3 \left(\lambda^2 - 4\lambda + 13\right)^3 = 0$$

determine the FS and the GS.

Solution 8 *The $\deg p = 13$. So the order of LDE is 13 and we need 13 FS. The eigenvalues are:*

$$
\begin{aligned}
\lambda &= 0, \quad m = 4, \\
\lambda &= 1, \quad m = 3, \\
\lambda &= 2 + 3i, \quad m = 3, \quad \alpha = 2, \quad \beta = 3 \\
\lambda &= 2 - 3i, \quad m = 3.
\end{aligned}
$$

The FS are

$$\begin{aligned}&1, x, x^2, x^3,\\ &e^x, xe^x, x^2e^x,\\ &e^{2x}\cos 3x, xe^{2x}\cos 3x, x^2e^{2x}\cos 3x,\\ &e^{2x}\sin 3x, xe^{2x}\sin 3x, x^2e^{2x}\sin 3x.\end{aligned}$$

Therefore the GS is

$$\begin{aligned} y &= C_1 + C_2x + C_3x^2 + C_4x^3 + C_5e^x + C_6xe^x + C_7x^2e^x + C_8e^{2x}\cos 3x + C_9xe^{2x}\cos 3x + \\ &\quad + C_{10}x^2e^{2x}\cos 3x + C_{11}e^{2x}\sin 3x + C_{12}xe^{2x}\sin 3x + C_{13}x^2e^{2x}\sin 3x. \end{aligned}$$

SOLVING IVP's FOR (H) LDE OF $n-th$ ORDER

The IVP problem for the (H) n-th order LDE is the following:

$$D[y] = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_2y'' + a_1y' + a_0y = 0, \tag{3}$$

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \quad y''(x_0) = y_2, ..., \quad y^{(n-1)}(x_0) = y_{n-1} \tag{4}$$

where $x_0, y_0, y_1, ..., y_{n-1}$ are given reals. The GS of (H) is involving n arbitrary constants and they have to be evaluated with the use IC's.

Example 9 *Determine the solutions of the following IVP's:*
a) $D[y] = y''' - 3y'' + 3y' - y = 0$; $y(0) = 1$, $y'(0) = -2$, $y''(0) = 3$;
b) $D[y] = y^{(4)} + 2y'' + y = 0$; $y(0) = 1$, $y'(0) = -2$, $y''(0) = 3$, $y'''(0) = -4$.

Solution 10 *a) The GS is*

$$y = y(x) = Ce^x + Dxe^x + Ex^2e^x = \left(C + Dx + Ex^2\right)e^x,$$

with

$$\begin{aligned} y'(x) &= \left[(C + D) + (2E + D)x + Ex^2\right]e^x, \\ y''(x) &= \left[(C + 2E + 2D) + (4E + D)x + Ex^2\right]e^x. \end{aligned}$$

Plugging the IC's we get the system

$$\begin{aligned} C &= 1, \\ C + D &= -2, \\ C + 2E + 2D &= 3. \end{aligned}$$

Thus

$$C = 1, D = -3, E = 4$$

and the answer:

$$y = \left(1 - 3x + 4x^2\right) e^x.$$

b) the GS is

$$y = C_1 \cos x + C_2 x \cos x + C_3 \sin x + C_4 x \sin x,$$

with

$$\begin{aligned}
y' &= -C_1 \sin x + C_2 \left(\cos x - x \sin x\right) + C_3 \cos x + C_4 \left(\sin x + x \cos x\right), \\
y'' &= -C_1 \cos x + C_2 \left(-2 \sin x - x \cos x\right) - C_3 \sin x - C_4 \left(2 \cos x - x \sin x\right), \\
y''' &= -C_1 \sin x + C_2 \left(x \sin x - 3 \cos x\right) - C_3 \cos x + C_4 \left(-3 \sin x - x \cos x\right).
\end{aligned}$$

Plugging the IC's we get the system

$$\begin{aligned}
C_1 &= 1, \\
C_2 + C_3 &= -2, \\
-C_1 - 2C_4 &= 3, \\
-3C_2 - C_3 &= -4.
\end{aligned}$$

Thus

$$C_1 = 1, C_2 = 3, C_3 = -5, C_4 = -2$$

and finally the answer:

$$y = \cos x + 3x \cos x - 5 \sin x - 2x \sin x$$

SOLVING (NH) LDE OF THE n-th ORDER

$$D\left[y\right] = y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_2 y'' + a_1 y' + a_0 y = r\left(x\right), \tag{5}$$

and $r(x)$ is given continuous function. The way of solving them is, similarly to the 1^{st} and 2^{nd} order, a two step method.

Step 1: We solve the corresponding (H)

$$D\left[y\right] = y^{(n)} + a_{n-1} y^{(n-1)} + ... + a_2 y'' + a_1 y' + a_0 y = 0$$

getting the FS $\varphi_1, \varphi_2, ..., \varphi_n$ and the GS(H)

$$y = C_1 \varphi_1 + C_2 \varphi_2 + ... + C_n \varphi_n.$$

where $C_1, C_2, ..., C_n$ are arbitrary real constants.

Step 2: The GS of (NH) we predict in the form

$$y = y\left(x\right) = C_1\left(x\right) \varphi_1\left(x\right) + C_2\left(x\right) \varphi_2\left(x\right) + ... + C_n\left(x\right) \varphi_n\left(x\right).$$

where $C_1(x), C_2(x), ..., \varphi_n(x)$ are new variable unknowns. This method is again called "variation of parameters", "variable constants" or "varied constants".

Theorem 11 *Let $C_1(x), C_2(x), ..., C_n(x)$ are solutions of the system*

$$\begin{bmatrix} \varphi_1(x) & \varphi_2(x) & ... & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & ... & \varphi_n'(x) \\ ... & ... & ... & ... \\ \varphi_1^{(n-2)}(x) & \varphi_2^{(n-2)}(x) & ... & \varphi_n^{(n-2)}(x) \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & ... & \varphi_x^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \\ ... \\ C_{n-1}'(x) \\ C_n'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ ... \\ 0 \\ r(x) \end{bmatrix}. \quad (6)$$

Then the GS of (NH) is

$$y = y(x) = C_1(x)\varphi_1(x) + C_2(x)\varphi_2(x) + ... + C_n(x)\varphi_n(x).$$

Remark 12 *This system possesses always the solution, because, for each x, the Wrońskian is*

$$W(x) = \det \begin{bmatrix} \varphi_1(x) & \varphi_2(x) & ... & \varphi_i(x) & ... & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & ... & \varphi_i'(x) & ... & \varphi_n'(x) \\ ... & ... & ... & ... & ... & ... \\ \varphi_1^{(n-2)}(x) & \varphi_2^{(n-2)}(x) & ... & \varphi_i^{(n-2)}(x) & ... & \varphi_n^{(n-2)}(x) \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & ... & \varphi_i^{(n-1)}(x) & ... & \varphi_x^{(n-1)}(x) \end{bmatrix} \neq 0.$$

The best way of solving (6) is by the Cramer's rules. Take for $i = 1, 2, ..., n$

$$W_i(x) = \det \begin{bmatrix} & & & i-th\ \ column & & \\ \varphi_1(x) & \varphi_2(x) & ... & 0 & ... & \varphi_n(x) \\ \varphi_1'(x) & \varphi_2'(x) & ... & 0 & ... & \varphi_n'(x) \\ ... & ... & ... & ... & ... & ... \\ \varphi_1^{(n-2)}(x) & \varphi_2^{(n-2)}(x) & ... & 0 & ... & \varphi_n^{(n-2)}(x) \\ \varphi_1^{(n-1)}(x) & \varphi_2^{(n-1)}(x) & ... & r(x) & ... & \varphi_x^{(n-1)}(x) \end{bmatrix}.$$

Then for $i = 1, 2, ..., n$ we have

$$C_i'(x) = \frac{W_i(x)}{W(x)}.$$

Having known all $C_i'(x)$ we obtain $C_i(x)$ by integration.

Example 13 *Determine the GS of the following LDE's*
a) $D[y] = y''' - 3y'' + 3y' - y = e^x$;
b) $D[y] = y^{(4)} + 2y'' + y = x$.

Solution 14 *a) The GS of (H) is*

$$y = y(x) = Ce^x + Dxe^x + Ex^2e^x.$$

We predict GS of (NH) in the form

$$y = y(x) = C(x)e^x + D(x)xe^x + E(x)x^2e^x.$$

The unknown functions $C(x), D(x), E(x)$ *are solutions of the following system*

$$\begin{bmatrix} e^x & xe^x & x^2e^x \\ e^x & (x+1)e^x & \left(x^2+2x\right)e^x \\ e^x & (x+2)e^x & \left(x^2+4x+2\right)e^x \end{bmatrix} \begin{bmatrix} C'(x) \\ D'(x) \\ E'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ e^x \end{bmatrix}. \qquad (7)$$

The Wrońskian is

$$W(x) = 2e^{3x},$$

$$\begin{aligned} W_1(x) &= \det \begin{bmatrix} 0 & xe^x & x^2e^x \\ 0 & (x+1)e^x & \left(x^2+2x\right)e^x \\ e^x & (x+2)e^x & \left(x^2+4x+2\right)e^x \end{bmatrix} = x^2e^{3x}, \\ W_2(x) &= \det \begin{bmatrix} e^x & 0 & x^2e^x \\ e^x & 0 & \left(x^2+2x\right)e^x \\ e^x & e^x & \left(x^2+4x+2\right)e^x \end{bmatrix} = -2xe^{3x}, \\ W_3(x) &= \det \begin{bmatrix} e^x & xe^x & 0 \\ e^x & (x+1)e^x & 0 \\ e^x & (x+2)e^x & e^x \end{bmatrix} = e^{3x}, \end{aligned}$$

The Cramer's rule gives

$$\begin{aligned} C'(x) &= \frac{W_1(x)}{W(x)} = \frac{x^2e^{3x}}{2e^{3x}} = \frac{1}{2}x^2, \\ D'(x) &= \frac{W_1(x)}{W(x)} = \frac{-2xe^{3x}}{2e^{3x}} = -x, \\ E'(x) &= \frac{W_1(x)}{W(x)} = \frac{e^{3x}}{2e^{3x}} = \frac{1}{2}. \end{aligned}$$

Integrating we obtain

$$\begin{aligned} C(x) &= \int \frac{1}{2}x^2dx = \frac{1}{6}x^3 + K, \\ D(x) &= -\int xdx = -\frac{1}{2}x^2 + L, \\ E(x) &= \int \frac{1}{2}dx = \frac{1}{2}x + M, \end{aligned}$$

where K, L, M *are arbitrary constants. Substituting we get*

$$\begin{aligned} y &= y(x) = C(x)e^x + D(x)xe^x + E(x)x^2e^x \\ &= \left(\frac{1}{6}x^3 + K\right)e^x + \left(-\frac{1}{2}x^2 + L\right)xe^x + \left(\frac{1}{2}x + M\right)x^2e^x = \\ &= \frac{1}{6}x^3e^x + Ke^x + Lxe^x + Mx^2e^x. \end{aligned}$$

The function $y = \frac{1}{6}x^3e^x$ *is a PS of* (NH), *obtained for* $K = L = M = 0$, *while* $Ke^x + Lxe^x + Mx^2e^x$ *represents the GS of* (H). *So the rule is*

$$GS(NH) = PS(NH) + GS(H).$$

b) The GS of (H) is

$$y = y(x) = C_1 \cos x + C_2 x \cos x + C_3 \sin x + C_4 x \sin x.$$

We predict GS of (NH) in the form

$$y(x) = C_1(x) \cos x + C_2(x) x \cos x + C_3(x) \sin x + C_4(x) x \sin x.$$

The unknown functions $C_1(x), ..., C_4(x)$ are solutions of the following system

$$\begin{bmatrix} \cos x & x\cos x & \sin x & x\sin x \\ -\sin x & \cos x - x\sin x & \cos x & \sin x + x\cos x \\ -\cos x & -2\sin x - x\cos x & -\sin x & 2\cos x - x\sin x \\ \sin x & x\sin x - 3\cos x & -\cos x & -3\sin x - x\cos x \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \\ C_3'(x) \\ C_4'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x \end{bmatrix}$$

We apply the Cramer's rule:

$$W(x) = \det \begin{bmatrix} \cos x & x\cos x & \sin x & x\sin x \\ -\sin x & \cos x - x\sin x & \cos x & \sin x + x\cos x \\ -\cos x & -2\sin x - x\cos x & -\sin x & 2\cos x - x\sin x \\ \sin x & x\sin x - 3\cos x & -\cos x & -3\sin x - x\cos x \end{bmatrix} = -4$$

$$W_1(x) = \det \begin{bmatrix} 0 & x\cos x & \sin x & x\sin x \\ 0 & \cos x - x\sin x & \cos x & \sin x + x\cos x \\ 0 & -2\sin x - x\cos x & -\sin x & 2\cos x - x\sin x \\ x & x\sin x - 3\cos x & -\cos x & -3\sin x - x\cos x \end{bmatrix} = 2x\sin x - 2x^2\cos x$$

$$W_2(x) = \det \begin{bmatrix} \cos x & 0 & \sin x & x\sin x \\ -\sin x & 0 & \cos x & \sin x + x\cos x \\ -\cos x & 0 & -\sin x & 2\cos x - x\sin x \\ \sin x & x & -\cos x & -3\sin x - x\cos x \end{bmatrix} = 2x\cos x$$

$$W_3(x) = \det \begin{bmatrix} \cos x & x\cos x & 0 & x\sin x \\ -\sin x & \cos x - x\sin x & 0 & \sin x + x\cos x \\ -\cos x & -2\sin x - x\cos x & 0 & 2\cos x - x\sin x \\ \sin x & x\sin x - 3\cos x & x & -3\sin x - x\cos x \end{bmatrix} = -2x^2\sin x - 2x\cos x$$

$$W_4(x) = \det \begin{bmatrix} \cos x & x\cos x & \sin x & 0 \\ -\sin x & \cos x - x\sin x & \cos x & 0 \\ -\cos x & -2\sin x - x\cos x & -\sin x & 0 \\ \sin x & x\sin x - 3\cos x & -\cos x & x \end{bmatrix} = 2x\sin x.$$

Therefore

$$C_1'(x) = \frac{W_1(x)}{W(x)} = \frac{2x\sin x - 2x^2\cos x}{-4} = \frac{-x\sin x + x^2\cos x}{2}$$

$$C_2'(x) = \frac{W_2(x)}{W(x)} = \frac{2x\cos x}{-4} = -\frac{x\cos x}{2}$$

$$C_3'(x) = \frac{W_3(x)}{W(x)} = \frac{-2x^2\sin x - 2x\cos x}{-4} = \frac{x^2\sin x + x\cos x}{2}$$

$$C_4'(x) = \frac{W_4(x)}{W(x)} = -\frac{x\sin x}{2}$$

Integrating we obtain

$$C_1(x) = \int \frac{-x\sin x + x^2\cos x}{2}dx = \frac{-3\sin x + x^2\sin x + 3x\cos x}{2} + D_1,$$

$$C_2(x) = -\int \frac{x\cos x}{2}dx = -\frac{\cos x + x\sin x}{2} + D_2,$$

$$C_3(x) = \int \frac{x^2\sin x + x\cos x}{2}dx = \frac{3\cos x - x^2\cos x + 3x\sin x}{2} + D_3,$$

$$C_4(x) = -\int \frac{x\sin x}{2}dx = \frac{-\sin x + x\cos x}{2} + D_4.$$

So

$$y(x) = C_1(x)\cos x + C_2(x)x\cos x + C_3(x)\sin x + C_4(x)x\sin x =$$

$$= \left(\frac{-3\sin x + x^2\sin x + 3x\cos x}{2} + D_1\right)\cos x + \left(-\frac{\cos x + x\sin x}{2} + D_2\right)x\cos x$$
$$+\left(\frac{3\cos x - x^2\cos x + 3x\sin x}{2} + D_3\right)\sin x + \left(\frac{-\sin x + x\cos x}{2} + D_4\right)x\sin x =$$

$$= x + D_1\cos x + D_2 x\cos x + D_3\sin x + D_4 x\sin x.$$

If all $D_i = 0$, *then* $y = x$ *is a PS of* (NH), *while* $D_1\cos x + D_2x\cos x + D_3\sin x + D_4x\sin x$ - *GS of* (H). *Again we conclude the rule*

$$GS(NH) = PS(NH) + GS(H).$$

SOLVING IVP's FOR (NH) LDE OF $n-th$ ORDER

The IVP problem for the (H) n-th order LDE is the following:

$$D[y] = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_2y'' + a_1y' + a_0y = r(x), \quad (8)$$
$$y(x_0) = y_0,\ \ y'(x_0) = y_1,\ \ y''(x_0) = y_2, ...,\ \ y^{(n-1)}(x_0) = y_{n-1} \quad (9)$$

where $x_0, y_0,\ y_1, ..., y_{n-1}$ are given reals. The GS of (NH) is involving n arbitrary constants and they have to be evaluated with the use IC's.

Example 15 *Determine the solutions of the following IVP's:*

a) $D[y] = y''' - 3y'' + 3y' - y = e^x$; $y(0) = 1$, $y'(0) = -2$, $y''(0) = 3$;

b) $D[y] = y^{(4)} + 2y'' + y = x$; $y(0) = 1$, $y'(0) = -2$, $\quad y''(0) = 3$, $\quad y''(0) = -4$.

Solution 16 *a) The GS is*

$$y = y(x) = \frac{1}{6}x^3e^x + Ke^x + Lxe^x + Mx^2e^x,$$

with

$$\begin{aligned} y'(x) &= \frac{1}{6}x^2e^x(x+3) + \left[(K+L) + (2M+L)x + Mx^2\right]e^x, \\ y''(x) &= \frac{1}{6}xe^x\left(x^2+6x+6\right) + \left[(K+2M+2L) + (4M+L)x + Mx^2\right]e^x. \end{aligned}$$

Plugging the IC's we get the system

$$\begin{aligned} K &= 1, \\ K+L &= -2, \\ K+2M+2L &= 3 \end{aligned}$$

Thus

$$K = 1, L = -3, M = 4.$$

and the answer is:

$$y = y(x) = \frac{1}{6}e^x\left(-18x + 24x^2 + x^3 + 6\right),$$

b) the GS is

$$y = x + D_1\cos x + D_2x\cos x + D_3\sin x + D_4x\sin x.$$

with

$$\begin{aligned} y' &= 1 - D_1\sin x + D_2(\cos x - x\sin x) + D_3\cos x + D_4(\sin x + x\cos x), \\ y'' &= -D_1\cos x + D_2(-2\sin x - x\cos x) - D_3\sin x - D_4(2\cos x - x\sin x), \\ y''' &= -D_1\sin x + D_2(x\sin x - 3\cos x) - D_3\cos x + D_4(-3\sin x - x\cos x). \end{aligned}$$

Plugging the IC's we get the system

$$\begin{aligned} D_1 &= 1, \\ 1 + D_2 + D_3 &= -2, \\ -D_1 - 2D_4 &= 3 \\ -3D_2 - D_3 &= -4 \end{aligned}$$

Thus

$$D_1 = 1, D_2 = \frac{7}{2}, D_3 = -\frac{13}{2}, D_4 = -2$$

and the final the answer is:

$$y = x + \cos x + \frac{7}{2}x\cos x - \frac{13}{2}\sin x - 2x\sin x.$$