LECTURE 09. 06. 2020

THE DIFFERENCE (RECCURENCE) EQUATIONS

Let $\mathbb{N} = \{1, 2, 3, ...\}$ be a set of natural numbers, while $\mathbb{R}$ - set of real numbers. By a sequence we mean a function $f : \mathbb{N} \to \mathbb{R}$. This function we identify with the set

$$\{y_1, y_2, y_3, ..., y_n, ...\}$$

and denote $(y_n)_{n\in N}$, $(y_n)_{n=1}^{\infty}$ or, shortly, (y_n). The number y_n is called the $n-th$ term of the sequence. The sequences are usually given by formulas.

Example 1 *Let $y_n = \frac{n}{n+1}$, $n = 1, 2, 3, ...$. Then the 9-th term of the sequence is $y_9 = \frac{9}{10}$. And opposite, the number $\frac{13}{14}$ is 13-th term of the sequence, while $\frac{18}{23}-$ the 18-th term of the sequence.*

Example 2 *Given is sequence $\left\{\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, ...\right\}$. Find the general formula of that sequence.*

Solution 3 *we have $y_1 = \frac{2^0}{3^1} = \frac{2^{1-1}}{3^1}$, $y_2 = \frac{2^1}{3^2} = \frac{2^{2-1}}{3^2}$, $y_3 = \frac{2^2}{3^3} = \frac{2^{3-1}}{3^3}$, $y_4 = \frac{2^3}{3^4} = \frac{2^{4-1}}{3^4}$. So we conclude that $y_n = \frac{2^{n-1}}{3^n}$. Obviously that formula has to be checked by the mathematical induction.*

The summation $\sum$.

The sum of consequtive terms of the given sequence starting from $k-th$, and finishing on $m-th$ is equal to

$$y_k + y_{k+1} + y_{k+2} + ... + y_{m-2} + y_{m-1} + y_m.$$

This sum we shortly denote by $\sum_{n=k}^{m} y_n$, i. e.

$$\sum_{n=k}^{m} y_n = y_k + y_{k+1} + y_{k+2} + ... + y_{m-2} + y_{m-1} + y_m.$$

Example 4 *Let* $y_n = \frac{1}{n^2+n}$. *Find* $\sum_{n=1}^{9} y_n = \sum_{n=1}^{9} \frac{1}{n^2+n}$.

Solution*:* We have $y_n = \frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Hence

$$\begin{aligned} y_1 &= 1 - \frac{1}{2}, \quad y_2 = \frac{1}{2} - \frac{1}{3}, \\ y_3 &= \frac{1}{3} - \frac{1}{4}, \qquad , y_8 = \frac{1}{8} - \frac{1}{9}, \quad y_9 = \frac{1}{9} - \frac{1}{10}. \end{aligned}$$

Thus

$$\sum_{n=1}^{9} y_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{8} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) = 1 - \frac{1}{10} = \frac{9}{10}.$$

IMPORTANT FORMULAS FOR SUMS

1. $\sum_{n=k}^{m} 0 = 0, \quad \sum_{n=k}^{m} 1 = m - k + 1;$
2. $\sum_{n=k}^{m} (x_n + y_n) = \sum_{n=k}^{m} x_n + \sum_{n=k}^{m} y_n;$
3. $\sum_{n=k}^{m} c y_n = c \left(\sum_{n=k}^{m} y_n\right);$
4. $\sum_{n=k}^{m} n = \frac{(m+k)(m-k+1)}{2};$
5. $\sum_{n=1}^{m} n^2 = \frac{m(m+1)(2m+1)}{6};$
6. If (y_n) is the arithmetic sequence with the first term y_1 and the difference r then the general term is given by

$$y_n = y_1 + (n-1)\,r$$

and

$$\sum_{n=1}^{m} y_n = m\frac{y_1 + y_m}{2} = m\frac{2y_1 + (m-1)\,r}{2};$$

7. If (y_n) is the geometric sequence with the first term y_1 (or y_0) and the quotient $q \neq 1,$, then the general term is given by

$$y_n = y_1 q^{n-1} = y_0 q^n.$$

and

$$\sum_{n=1}^{m} y_n = y_1 \frac{1-q^m}{1-q};$$

8. $\sum\limits_{n=1}^{m} q^{n-1} = \frac{1-q^m}{1-q}$, $q \neq 1$; see the formula from (7);

9. $\sum\limits_{n=1}^{m} nq^{n-1} = \frac{1-q^{m+1}}{(1-q)^2} - \frac{(m+1)q^m}{1-q}$, $q \neq 1$.

DIFFERENCE (RECCURENCE) EQUATIONS

By the difference (reccurence) equation of the $k-th$ order for a sequence (y_n) we mean the equation describing the relation between an arbitrary set of $(k+1)$ consequtive terms of (y_n), i. e. betwen y_n, y_{n+1}, ..., y_{n+k}. In particular, the first order reccurency is describing the relation between any pair y_n and y_{n+1}. In general it is described by

$$F(n, y_n, y_{n+1}) = 0 \quad or \quad y_{n+1} = f(y_n) \quad for \;\; each \;\; n,$$

where F or f are given functions.

The second order reccurency is describing the relation between any triple elements y_n, y_{n+1} and y_{n+2}. In general it is described by

$$F(n, y_n, y_{n+1}, y_{n+2}) = 0 \quad or \quad y_{n+2} = f(y_n, y_{n+1}) \quad for \;\; each \;\; n,$$

where F or f are given functions.

Example 5 *a)* $y_{n+1} - y_n = r$, $n = 1, 2, ...$ *(arithmetic sequence);*
b) $\frac{y_{n+1}}{y_n} = q$, $n = 1, 2, ...$ *(geometric sequence);*
c) $y_{n+1} = y_n^2$, $n = 1, 2, ...$ *;*
d) $y_{n+1} - ny_n = 2^n$, $n = 1, 2, ...$ *;*
e) $y_{n+2} = y_{n+1} + y_n$, $n = 1, 2, ...$ *(Fibonacci sequence) - 2^{nd} order reccurency;*
f) $y_{n+3} - 3y_{n+2} + 3y_{n+1} - 3y_n = n$, $n = 1, 2, ...$ *3^{rd} order reccurency.*

Usually we do not write that $n = 1, 2, ...$, but only in the case that it is not misleading. Having a difference equation we want to find the general fomula for the terms of the unknown sequence and then conclude what are their properties.

Each reccurence equation can be rewritten with the use of the difference operator.

Definition 6 *By the difference operator we mean an operator Δ assigning to each sequence (y_n) a sequence of their increases (differences) $\Delta y_n = y_{n+1} - y_n$.*

Example 7 *a) for the sequence $y_n = 3n + 7$ we have $\Delta y_n = [3(n+1) + 7] - (3n+7) = 3$;*

b) for the sequence $y_n = n^2$ we have $\Delta y_n = (n+1)^2 - n^2 = 2n + 1$;

c) for the arithmetic sequence we have the equation $\Delta y_n = r$;

d) for the geometric sequence we have the equation $\Delta y_n = (q-1) y_n$;

e) the sequence $y_{n+1} = y_n^2$ satisfies the equation $\Delta y_n = y_n (y_n - 1)$;

f) the sequence $y_{n+1} - n y_n = 2^n$ satisfies the equation $\Delta y_n = (n-1) y_n + 2^n$.

The difference operator Δy_n describes the increase of the sequence $y_n = f(n)$. It is a subsitute of the derivative of the discrete function: $\Delta y_n = \frac{y_{n+1} - y_n}{(n+1)-(n)} = \frac{f(n+1) - f(n)}{(n+1)-(n)}$ possess many interesting properties, but we omit them, since we are not going to use them.

Linear first order difference equations

Searching for the general formula of the difference equations in many situations can be desribed by certain procedures. This is the case cf. for first order linear difference equations. By this we mean difference equations of the form

$$y_{n+1} - a_n y_n = r_n, \ n \in \mathbb{N} \tag{1}$$

or, equivalently, for

$$\Delta y_n + (1 - a_n) y_n = r_n, \ n \in \mathbb{N}$$

The sequences (a_n) are (r_n) are known, while the sequence (y_n) is unknown. The equation is called linear since the terms y_n, y_{n+1} and Δy_n are in the first power. Knowing the first term y_1 and having the relation (1) one can evaluate all the terms or find the general formula for y_n. Below we describe this procedure:

Step I. First we are looking for the sequences (y_n) satisfying, so called, homogeneous linear difference equation

$$y_{n+1} - a_n y_n = 0, \ n = 1, 2, ... \tag{2}$$

i.e. the equation with the sequence r_n replaced by constant sequence equal to 0. Equivalently (if all terms $y_n \neq 0$), it means that

$$\frac{y_{n+1}}{y_n} = a_n, \ n = 1, 2, ...$$

Taking $n = 1, 2, ..., (n-1)$ we have a table of relations

$$\begin{aligned} \frac{y_2}{y_1} &= a_1, \quad \frac{y_3}{y_2} = a_2, \quad \frac{y_4}{y_3} = a_3, \\ &\dots \\ \frac{y_{n-1}}{y_{n-2}} &= a_{n-2}, \quad \frac{y_n}{y_{n-1}} = a_{n-1}. \end{aligned}$$

Multiplying "side by side" we obtain

$$\frac{y_n}{y_1} = \frac{y_n}{y_{n-1}} \cdot \frac{y_{n-1}}{y_{n-2}} \cdot \ldots \cdot \frac{y_4}{y_3} \cdot \frac{y_3}{y_2} \cdot \frac{y_2}{y_1} = a_{n-1} a_{n-2} \cdot \ldots \cdot a_3 a_2 a_1.$$

Denote by (φ_n) the sequence of products of the RHS, i.e.

$$\varphi_n = a_{n-1} a_{n-2} \cdot \ldots \cdot a_3 a_2 a_1.$$

Then we have the solution

$$y_n = y_1 \varphi_n.$$

Step II. <u>(variation of parameter)</u>. We are searching for the solutions of (NH) $y_{n+1} - a_n y_n = r_n$ in the form

$$y_n = c_n \varphi_n,$$

where (φ_n) is previously obtained sequence and (c_n) is a new unknown sequence. Substituting this form to the given (NH) we get

$$c_{n+1} \varphi_{n+1} - a_n c_n \varphi_n = r_n.$$

But

$$\varphi_{n+1} = a_n \cdot a_{n-1} a_{n-2} \cdot \ldots \cdot a_3 a_2 a_1 = a_n \varphi_n$$

and therefore

$$c_{n+1} a_n \varphi_n - a_n c_n \varphi_n = r_n.$$

Equivalently,

$$\Delta c_n = c_{n+1} - c_n = \frac{r_n}{a_n \varphi_n}.$$

Taking this relations for consequtive $n \in \mathbb{N}$ we have a table

$$\begin{aligned} c_2 - c_1 &= \frac{r_1}{a_1 \varphi_1}, \\ c_3 - c_2 &= \frac{r_2}{a_2 \varphi_2}, \\ &\dots \\ c_n - c_{n-1} &= \frac{r_{n-1}}{a_{n-1} \varphi_{n-1}}, \\ c_{n+1} - c_n &= \frac{r_n}{a_n \varphi_n}. \end{aligned}$$

Now adding all equalities "side by side" we obtain

$$c_{n+1} - c_1 = \sum_{i=1}^{n} (c_{i+1} - c_i) = \sum_{i=1}^{n} \frac{r_i}{a_i \varphi_i}.$$

Thus

$$c_{n+1} = c_1 + \sum_{i=1}^{n} \frac{r_i}{a_i \varphi_i}$$

and hence, after renumerating,

$$c_n = c_1 + \sum_{i=1}^{n-1} \frac{r_i}{a_i \varphi_i}.$$

Thus the equation possess the solutions in the form

$$y_n = \left(c_1 + \sum_{i=1}^{n-1} \frac{r_i}{a_i \varphi_i} \right) \varphi_n,$$

where the term c_1 can be considered as an arbitrary real number. Note that there is no need to remember the above formulas but only the scheme.

Example 8 *Find all sequences y_n satisfying the equation*

$$y_{n+1} - \frac{y_n}{2} = \frac{1}{3}.$$

Solution 9 *Step I. We have the homogeneous equation $y_{n+1} - \frac{y_n}{2} = 0$. Equivalently $\frac{y_{n+1}}{y_n} = \frac{1}{2}$, and this means that (y_n) is a geometric sequence with the quotient $q = \frac{1}{2}$. Thus $y_n = \frac{y_1}{2^{n-1}} = \frac{c}{2^n}$.*

Step II. We are looking for the solutions of $y_{n+1} - \frac{y_n}{2} = \frac{1}{3}$ in the form

$$y_n = \frac{c_n}{2^n},$$

where (c_n) is new unknown. Plugging to the original (NH) we have

$$\frac{c_{n+1}}{2^{n+1}} - \frac{c_n}{2^{n+1}} = \frac{1}{3}.$$

Equivalently,

$$c_{n+1} - c_n = \frac{2^{n+1}}{3}.$$

Writing these relations for $n = 1, 2, ...$ we have

$$\begin{aligned} c_2 - c_1 &= \frac{2^2}{3}, \\ c_3 - c_2 &= \frac{2^3}{3}, \\ &\ldots\ldots. \\ c_n - c_{n-1} &= \frac{2^n}{3} \\ c_{n+1} - c_n &= \frac{2^{n+1}}{3}. \end{aligned}$$

Adding "side by side" we get

$$c_{n+1} - c_1 = \frac{2^{n+1}}{3} + \frac{2^n}{3} + \frac{2^{n-1}}{3} + ... + \frac{2^2}{3} = \frac{2^2}{3}\left(\sum_{i=0}^{n-1} 2^i\right) = \frac{4\,(2^n - 1)}{3}. \quad (3)$$

Hence

$$c_{n+1} = c_1 + \frac{2^{n+2} - 4}{3}$$

and, after renumerating,

$$c_n = c_1 + \frac{2^{n+1} - 4}{3}.$$

Thus

$$y_n = \frac{c_1 + \frac{2^{n+1}-4}{3}}{2^n} = \frac{3c_1 + 2^{n+1} - 4}{3 \cdot 2^n},$$

where c_1 can be an arbitrary real number.

Example 10 *Find the sequence y_n satisfying the difference equation $y_{n+1} - \frac{y_n}{2} = \frac{1}{3}$, whose the first term $y_1 = -\frac{1}{3}$.*

Solution 11 *From the previous example we know that **GS** is in the form $y_n = \frac{3c_1+2^{n+1}-4}{3 \cdot 2^n}$. Then $y_1 = -\frac{1}{3} = \frac{c_1}{2}$ and therefore $c_1 = -\frac{2}{3}$. So the solution is $y_n = \frac{2^{n+1}-6}{3 \cdot 2^n}$.*

Answer: Given equation possess the only solution $y_n = \frac{2^{n+1}-6}{3 \cdot 2^n}$.

Example 12 *Find the sequence y_n satisfying the difference equation $y_{n+1} - \frac{y_n}{2n} = \frac{2}{6^n n!}$, whose the first term $y_1 = -\frac{1}{3}$.*

Solution 13 *Step I. We have the homogeneous equation* $y_{n+1} - \frac{y_n}{2n} = 0$. *Equivalently* $\frac{y_{n+1}}{y_n} = \frac{1}{2n}$. *Taking* $n = 1, 2, ..., (n-1)$ *we have a table of relations*

$$\begin{aligned} \frac{y_2}{y_1} &= \frac{1}{2\cdot 1}, \quad \frac{y_3}{y_2} = \frac{1}{2\cdot 2}, \quad \frac{y_4}{y_3} = \frac{1}{2\cdot 3}, \\ &\cdots \\ \frac{y_{n-1}}{y_{n-2}} &= \frac{1}{2\cdot (n-2)}, \quad \frac{y_n}{y_{n-1}} = \frac{1}{2\cdot (n-1)}. \end{aligned}$$

Multiplying side by side we get

$$\frac{y_2}{y_1} \cdot \frac{y_3}{y_2} \cdot \frac{y_4}{y_3} \cdot \ldots \cdot \frac{y_{n-1}}{y_{n-2}} \cdot \frac{y_n}{y_{n-1}} = \frac{1}{2\cdot 1} \cdot \frac{1}{2\cdot 2} \cdots \frac{1}{2\cdot (n-2)} \cdot \frac{1}{2\cdot (n-1)}.$$

Hence

$$\frac{y_n}{y_1} = \frac{1}{2^{n-1}\cdot (n-1)!}.$$

Thus GS of (H)

$$y_n = \frac{y_1}{2^{n-1}\cdot (n-1)!} = \frac{c}{2^{n-1}\cdot (n-1)!}.$$

Step II. We are looking for the solutions of $y_{n+1} - \frac{y_n}{2n} = \frac{2}{6^n n!}$, *in the form*

$$y_n = \frac{c_n}{2^{n-1}\cdot (n-1)!}.$$

Thus

$$y_{n+1} = \frac{c_{n+1}}{2^n \cdot (n!)}.$$

Plugging both to the given (NH) *we obtain*

$$\begin{aligned} y_{n+1} - \frac{y_n}{2n} &= \frac{2}{6^n n!}, \\ \frac{c_{n+1}}{2^n \cdot (n!)} - \frac{\frac{c_n}{2^{n-1}\cdot (n-1)!}}{2n} &= \frac{2}{6^n n!}, \\ \frac{c_{n+1}}{2^n \cdot (n!)} - \frac{c_n}{2^n \cdot (n!)} &= \frac{2}{6^n n!} \Big| \times [2^n \cdot (n!)], \\ c_{n+1} - c_n &= \frac{2}{3^n} = 2\left(\frac{1}{3}\right)^n. \end{aligned}$$

Writing these relations for $n = 1, 2, ...$ *we have*

$$\begin{aligned} c_2 - c_1 &= 2\left(\frac{1}{3}\right)^1, \\ c_3 - c_2 &= 2\left(\frac{1}{3}\right)^2, \\ &\cdots\cdots \\ c_n - c_{n-1} &= 2\left(\frac{1}{3}\right)^{n-1} \\ c_{n+1} - c_n &= 2\left(\frac{1}{3}\right)^n. \end{aligned}$$

$$c_{n+1} - c_1 = 2\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^2 + ... + 2\left(\frac{1}{3}\right)^n.$$

This time we need to add side by side getting

$$c_{n+1} - c_1 = 2\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^2 + ... + 2\left(\frac{1}{3}\right)^n = \frac{2}{3}\left(\sum_{i=0}^{n-1}\left(\frac{1}{3}\right)^i\right) = \frac{2}{3}\frac{1-\left(\frac{1}{3}\right)^n}{1-\left(\frac{1}{3}\right)} = 1 - \left(\frac{1}{3}\right)^n.$$

Therefore

$$\begin{aligned} c_{n+1} &= c_1 + 1 - \left(\frac{1}{3}\right)^n = d - \left(\frac{1}{3}\right)^n \\ &\textit{and} \\ c_n &= d - \left(\frac{1}{3}\right)^{n-1}. \end{aligned}$$

Hence

$$y_n = \frac{c_n}{2^{n-1} \cdot (n-1)!} = \frac{d - \left(\frac{1}{3}\right)^{n-1}}{2^{n-1} \cdot (n-1)!}.$$

Apply now the IC $y_1 = -\frac{1}{3}$. *We then have*

$$d - 1 = -\frac{1}{3} \Longrightarrow d = \frac{2}{3}$$

and thus

$$y_n = \frac{\frac{2}{3} - \left(\frac{1}{3}\right)^{n-1}}{2^{n-1} \cdot (n-1)!}.$$

MOTIVATION

Why we consider the difference equations? For ODE's we have discussed the situations, when the method of finding the GS is known. But there are some types of ODE's that the method is unknown. But from the physical context we know that the solution should be represented as a MacLaurin or Taylor series

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \quad for \quad |x - x_0| < R,$$

where $R > 0$ is the radius of convergence. Then we know that

$$y'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n \quad for \quad |x - x_0| < R.$$

Thus we can compare both sides of the given ODE and have the relation between a_n and a_{n+1}. This difference eq-n of the 1^{st} order. If we can solve it then we have the solution in series form. But is enough.

Example 14 *Solve the IVP*

$$y' = 2y, \quad y(0) = 3.$$

Solution 15 *Assume that*

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \quad for \quad |x| < R.$$

Then

$$y'(x) = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n \qquad for \quad |x| < R$$

and we have the relation

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = 2 \sum_{n=0}^{\infty} a_n x^n, \qquad a_0 = 3.$$

Comparing coefficients we have for all $n = 0, 1, 2, ...$ *that*

$$(n+1) a_{n+1} = 2a_n, \qquad a_0 = 3.$$

Equivalently we get the 1st *order differece eq-n*

$$\frac{a_{n+1}}{a_n} = \frac{2}{n+1}, \qquad a_0 = 3.$$

Taking $n = 0, 1, 2, ..., (n-1)$ *we have a table of relations*

$$\begin{aligned} \frac{a_1}{a_0} &= \frac{2}{1}, \quad \frac{a_2}{a_1} = \frac{2}{2}, \quad \frac{a_3}{a_2} = \frac{2}{3}, \\ &\cdots \\ \frac{a_{n-1}}{a_{n-2}} &= \frac{2}{n-1}, \quad \frac{a_n}{a_{n-1}} = \frac{2}{n}. \end{aligned}$$

Multiplying side by side we get

$$\frac{a_1}{a_0} \cdot \frac{a_2}{a_1} \cdot \frac{a_3}{a_2} \cdot \ldots \cdot \frac{a_{n-1}}{a_{n-2}} \cdot \frac{a_n}{a_{n-1}} = \frac{2}{1} \cdot \frac{2}{2} \cdot \ldots \cdot \frac{2}{n-1} \cdot \frac{2}{n}.$$

Hence

$$\frac{a_n}{a_0} = \frac{2^n}{n!}.$$

Thus

$$a_n = a_0 \cdot \frac{2^n}{n!} = 3 \frac{2^n}{n!}.$$

Therefore

$$y(x) = 3 \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n = 3 \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 3e^{2x}.$$

LINEAR 2^{ND} ORDER DIFFERENCE EQ-S WITH CC

By this we mean the difference equations in the form

$$y_{n+2} + py_{n+1} + qy_n = r_n, \tag{4}$$

where coefficients p, q and sequence (r_n) are given, while the sequence (y_n) is unknown. If the terms $r_n = 0$ then the equation (3) has the form

$$y_{n+2} + py_{n+1} + qy_n = 0 \tag{5}$$

and is called homogeneous (H). Otherwise we have nonhomogeneous difference equation (NH).

Solving (H) linear second order difference equations with CC

By this we mean the difference equation

$$y_{n+2} + py_{n+1} + qy_n = 0. \tag{6}$$

We start with the following

Example 16 *For the equation*

$$y_{n+2} - 5y_{n+1} + 6y_n = 0$$

Check that given below sequences are solutions

a) $\varphi_n = 2^n$, *b)* $\psi_n = 3^n$ *c)* $y_n = a{\cdot}2^n + b{\cdot}3^n$, *where* a, b *are arbitrry real numbers.*

Solution 17 *a) we have* $L[\varphi_n] = 2^{n+2} - 5{\cdot}2^{n+1} + 6{\cdot}2^n = 4{\cdot}2^n - 10{\cdot}2^n + 6{\cdot}2^n = 0 = P$ *(for each* n*);*

b) we have $L[\psi_n] = 3^{n+2} - 5{\cdot}3^{n+1} + 6{\cdot}3^n = 9{\cdot}3^n - 15{\cdot}3^n + 6{\cdot}3^n = 0 = P$ *(for each* n*);*

c) this time

$$\begin{aligned} L[y_n] &= \left(a{\cdot}2^{n+2} + b{\cdot}3^{n+2}\right) - 5\left(a{\cdot}2^{n+1} + b{\cdot}3^{n+1}\right) + 6\left(a{\cdot}2^n + b{\cdot}3^n\right) \\ &= \left(4a{\cdot}2^n + 9b{\cdot}3^n\right) - 5\left(2a{\cdot}2^n + 3b{\cdot}3^n\right) + 6\left(a{\cdot}2^n + b{\cdot}3^n\right) \\ &= a\left(4{\cdot}2^n - 10{\cdot}2^n + 6{\cdot}2^n\right) + b\left(9{\cdot}3^n - 15{\cdot}3^n + 6{\cdot}3^n\right) = 0 = P \end{aligned}$$

(for each n*).*

The situation escribed in the previous Example 11. is typical for the solution sets of the equation (6) (also with variable coefficients). Namely, we have the following:

Theorem 18 *If sequences* (φ_n) *and* (ψ_n) *are solutions of (6) then the sequence* (y_n) *given by* $y_n = a\varphi_n + b\psi_n$*, where* a *and* b *are arbitrary reals, is a solution of (6) as well.*

Proof. Since (φ_n) and (ψ_n) are *solutions,* then

$$\begin{aligned} L[(\varphi_n)] &= \varphi_{n+2} + p\varphi_{n+1} + q\varphi_n = 0, \\ L[(\psi_n)] &= \psi_{n+2} + p\psi_{n+1} + q\psi_n = 0. \end{aligned}$$

Hence for $y_n = a\varphi_n + b\psi_n$ we have

$$\begin{aligned} L[(y_n)] &= L[(a\varphi_n + b\psi_n)] = \left(a\varphi_{n+2} + b\psi_{n+2}\right) + p\left(a\varphi_{n+1} + b\psi_{n+1}\right) + q\left(a\varphi_n + b\psi_n\right) \\ &= a\left(\varphi_{n+2} + p\varphi_{n+1} + q\varphi_n\right) + b\left(\psi_{n+2} + p\psi_{n+1} + q\psi_n\right) = 0, \end{aligned}$$

and this means that (y_n) is a solution. ■

In order to find the all possible solutions of (6) we have to examine the set $\mathcal{Y}$ of all possible possible solutions. This set is called the **GS**. To describe the set $\mathcal{Y}$ we need to recall the notion of linearly independence in the context of sequences.

Definition 19 *Sequences* $\left(\varphi_n^1\right), ..., \left(\varphi_n^k\right)$ *are linearly independent if for certain reals* $\lambda_1, ..., \lambda_k$ *the relation*

$$\lambda_1\varphi_n^1 + ... + \lambda_k\varphi_n^k = 0$$

holds for $n = 1, 2, 3, ...$ *iff* $\lambda_1 = \lambda_2 = ... = \lambda_k = 0$.

Example 20 *Sequences* $\left(\varphi_n^1\right) = (1), \left(\varphi_n^2\right) = (n), ..., \left(\varphi_n^{k+1}\right) = \left(n^k\right)$ *are linearly independent.*

Proof. Let for certain reals $\lambda_0, \lambda_1, ..., \lambda_k$ we have

$$\lambda_0 + \lambda_0 n + ... + \lambda_k n^k = 0$$

for all $n = 1, 2, ...$. Therefore each number $x = 1, 2, ...$ is the zero of the polynomial $p(x) = \lambda_0 + \lambda_0 x + ... + \lambda_k x^k$. Hence $\lambda_0 = \lambda_1 = ... = \lambda_k = 0$. ■

There is a simple criterion of the linear independence of the sequences $\left(\varphi_n^1\right), ..., \left(\varphi_n^k\right)$. Namely, the following result holds:

Theorem 21 *If for each* $n = 1, 2, ...$ *the determinants*

$$C_n = \det\begin{bmatrix} \varphi_n^1 & \varphi_{n+1}^1 & \cdots & \varphi_{n+k}^1 \\ \varphi_n^2 & \varphi_{n+1}^2 & \cdots & \varphi_{n+k}^2 \\ . & . & . & . \\ \varphi_n^k & \varphi_{n+1}^k & \cdots & \varphi_{n+k}^k \end{bmatrix} \neq 0,$$

then the sequences $\left(\varphi_n^1\right), ..., \left(\varphi_n^k\right)$ *are linearly independent.*

Example 22 *For sequences* (1), (n) *and* $\left(n^2\right)$ *we have*

$$C_n = \det \begin{bmatrix} 1 & 1 & 1 \\ n & n+1 & n+2 \\ n^2 & (n+1)^2 & (n+2)^2 \end{bmatrix} = 2 \neq 0.$$

Therefore they are linearly independent.

Definition 23 *The determinant* $C_n = \det \begin{bmatrix} \varphi_n^1 & \varphi_{n+1}^1 & \cdots & \varphi_{n+k}^1 \\ \varphi_n^2 & \varphi_{n+1}^2 & \cdots & \varphi_{n+k}^2 \\ \cdot & \cdot & \cdot & \cdot \\ \varphi_n^k & \varphi_{n+1}^k & \cdots & \varphi_{n+k}^k \end{bmatrix}$ *is called the Casorati'an. It plays the similar role as the Wrońskian for the functions of real variable.*

In case of difference equation (4) we have the following result:

Theorem 24 *Let* $\mathcal{Y}$ *be the* ***GS*** *of (6) . Then* $\mathcal{Y}$ *is linear space and dim*$\mathcal{Y} = 2$*. In particular, it means that there are two linearly independent solutions* (φ_n) *and* (ψ_n) *of* (4) *and the* ***GS*** *is in the form* $y_n = a\varphi_n + b\psi_n$*, where* a *and* b *are arbitrary reals.*

Definition 25 *Each couple of linearly independent solutions* (φ_n) *and* (ψ_n) *of* (4) *is called the* ***fundamental solutions (FS).***

The question arises, how to find the set of **FS.** The Example 11 suggests that they should be looked for in the form

$$y_n = \lambda^n,$$

where $\lambda \neq 0$. Assuming such form we have

$$L\left[(\lambda^n)\right] = \lambda^{n+2} + p\lambda^{n+1} + q\lambda^n = \lambda^n \left(\lambda^2 + p\lambda + q\right).$$

One can notice the following

Theorem 26 *The sequence* $y_n = \lambda^n$*, where* $\lambda \neq 0$*, is solution of*

$$y_{n+2} + py_{n+1} + qy_n = 0$$

if and only if

$$\lambda^2 + p\lambda + q = 0. \tag{7}$$

The equation (7) is called the characteristic equation **(CH)** for (6) and it easy to recognize how it is formed. Solving **CH** we can meet the following three cases:

Case A. $\Delta = p^2 - 4q > 0$. Then we have two distinct real roots λ_1 and λ_2. In this case the **FS** are $\varphi_n = (\lambda_1)^n$ and $\psi_n = (\lambda_2)^n$. The **GS** is $y_n = a(\lambda_1)^n + b(\lambda_2)^n$, *where* a and b are arbitrary constants.

Example 27 *For the equation*

$$y_{n+2}-5y_{n+1}+6y_n=0$$

***the CH** is* $\lambda^2-5\lambda+6=0$. *The eigenvalues are* $\lambda_1=2$, $\lambda_2=3$ *and hence* $\varphi_n=2^n$ *and* $\psi_n=3^n$ *are solutions. The functions* $\{\varphi_n,\psi_n\}=\{2^n,3^n\}$ *are **FS** because*

$$C_n=\begin{vmatrix} 2^n & 3^n \\ 2^{n+1} & 3^{n+1}\end{vmatrix}=6^n\neq 0.$$

*The **GS** is the sequence* $y_n=a\cdot 2^n+b\cdot 3^n$, *where* a *and* b *are arbitrary constants.*

Case B. $\Delta=p^2-4q<0$. Then the **CH** possess have two distinct complex roots (eigenvalues) $\lambda_1=r(\cos\alpha+i\sin\alpha)$ and $\lambda_2=r(\cos\alpha+i\sin\alpha)$. In this case the **FS** in real form are sequences $\varphi_n=r^n\cos(n\alpha)$ and $\psi_n=r^n\sin(n\alpha)$. The **GS** is the sequence $y_n=r^n(a\cos(n\alpha)+b\sin(n\alpha))$, where a and b are arbitrary constants.

Example 28 *For the equation*

$$y_{n+2}-2y_{n+1}+2y_n=0$$

*the **CH** is*

$$\lambda^2-2\lambda+2=0.$$

The eigenvalues are $\lambda_1=1+i=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)$ *and* $\lambda_2=1-i=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)-i\sin\left(\frac{\pi}{4}\right)\right)$. *We shall verify that the sequences* $\varphi_n=\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right)$ *and* $\psi_n=\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)$ *are solutions.*

a) for $\varphi_n=\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right)$ *we have*

$$\begin{aligned}
L(\varphi_n) &= \left(\sqrt{2}\right)^{n+2}\cos\left(\frac{(n+2)\pi}{4}\right)-2\left(\sqrt{2}\right)^{n+1}\cos\left(\frac{(n+1)\pi}{4}\right)+2\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right)\\
&= 2\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}+\frac{\pi}{2}\right)-2\sqrt{2}\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}+\frac{\pi}{4}\right)+2\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right)\\
&= -2\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)-2\sqrt{2}\left(\sqrt{2}\right)^n\frac{\sqrt{2}}{2}\left(\cos\frac{n\pi}{4}-\sin\left(\frac{n\pi}{4}\right)\right)+2\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right)\\
&= 2\left(\sqrt{2}\right)^n\left[-\sin\left(\frac{n\pi}{4}\right)-\left(\cos\frac{n\pi}{4}-\sin\left(\frac{n\pi}{4}\right)\right)+\cos\left(\frac{n\pi}{4}\right)\right]=0=R.
\end{aligned}$$

b) for $\psi_n=\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)$ *we have*

$$\begin{aligned}
L(\psi_n) &= \left(\sqrt{2}\right)^{n+2}\sin\left(\frac{(n+2)\pi}{4}\right)-2\left(\sqrt{2}\right)^{n+1}\sin\left(\frac{(n+1)\pi}{4}\right)+2\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)\\
&= 2\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}+\frac{\pi}{2}\right)-2\sqrt{2}\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}+\frac{\pi}{4}\right)+2\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)\\
&= 2\left(\sqrt{2}\right)^n\cos\left(\frac{n\pi}{4}\right)-2\sqrt{2}\left(\sqrt{2}\right)^n\frac{\sqrt{2}}{2}\left(\sin\left(\frac{n\pi}{4}\right)+\cos\left(\frac{n\pi}{4}\right)\right)+2\left(\sqrt{2}\right)^n\sin\left(\frac{n\pi}{4}\right)\\
&= 2\left(\sqrt{2}\right)^n\left[\cos\left(\frac{n\pi}{4}\right)-\left(\sin\left(\frac{n\pi}{4}\right)+\cos\left(\frac{n\pi}{4}\right)\right)+\sin\left(\frac{n\pi}{4}\right)\right]=0=R.
\end{aligned}$$

The functions $\{\varphi_n, \psi_n\} = \left\{ \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right), \left(\sqrt{2}\right)^n \sin\left(\frac{n\pi}{4}\right) \right\}$ *are* ***FS*** *because the Casoratian*

$$C_n = \begin{vmatrix} \left(\sqrt{2}\right)^n \cos\left(\frac{n\pi}{4}\right) & \left(\sqrt{2}\right)^n \sin\left(\frac{n\pi}{4}\right) \\ \left(\sqrt{2}\right)^{n+1} \cos\left(\frac{(n+1)\pi}{4}\right) & \left(\sqrt{2}\right)^{n+1} \sin\left(\frac{(n+1)\pi}{4}\right) \end{vmatrix} = 2^n \neq 0.$$

So the ***GS*** *is the sequence* $y_n = \left(\sqrt{2}\right)^n \left(a\cos\left(\frac{n\pi}{4}\right) + b\sin\left(\frac{n\pi}{4}\right)\right)$, *where* a, b *are arbitrary constants.*

Case C. $\Delta = p^2 - 4q = 0$. Then the **CH** possess double root $\lambda = \lambda_0$. Obviously, the sequence $\varphi_n = (\lambda_0)^n$ is a solution. Second **FS** is $\psi_n = n(\lambda_0)^n$. The **GS** is $y_n = (\lambda_0)^n (a + bn)$, where a and b are arbitrary constants.

Example 29 *Find* ***FS*** *and* ***GS*** *of the equation*

$$y_{n+2} - 4y_{n+1} + 4y_n = 0.$$

Solution 30 *The* ***CH*** *is*

$$\lambda^2 - 4\lambda + 4 = 0$$

and it possess double root $\lambda = 2$. *One can easily verify that the sequence* $\varphi_n = 2^n$ *is a solution. We shall check that the sequence* $\psi_n = n2^n$ *is a solution as well. We proceed as follows:*

$$\begin{aligned} L[(\psi_n)] &= (n+2)\,2^{n+2} - 4\cdot(n+1)\,2^{n+1} + 4\cdot n2^n \\ &= 2^{n+2}\left[(n+2) - 2\cdot(n+1) + n\right] = 0 = R. \end{aligned}$$

The functions $\{\varphi_n, \psi_n\} = \{2^n, n2^n\}$ *are* ***FS*** *because the Casoratian*

$$C_n = \begin{vmatrix} 2^n & n2^n \\ 2^{n+1} & (n+1)\,2^{n+1} \end{vmatrix} = 2^{2n+1} \neq 0.$$

Therefore the ***GS*** *is* $y_n = (a + bn)\,2^n$, *where* a *and* b *are arbitrary constants.*

As we have seen every **GS** of (4) involves two arbitrary constants. The exact solution (y_n) we obtain whenever we know the first and the second term y_1 and y_2. So we have the IVP

$$\begin{gathered} y_{n+2} + py_{n+1} + qy_n = 0 \\ y_1 = A, \quad y_2 = B. \end{gathered} \tag{8}$$

Example 31 *By the Fibonacci sequence it is usuall meant the sequence* (f_n) *given by the relations*

$$\begin{gathered} f_1 = 1, \quad f_2 = 1 \\ f_{n+2} = f_n + f_{n+1}. \end{gathered}$$

Find the formula for that sequence.

Solution 32 *The given relation can be rewritten as*

$$f_{n+2} - f_{n+1} - f_n = 0.$$

*Notice tah it is 2^{nd} order difference equation. The **CH** is*

$$\lambda^2 - \lambda - 1 = 0$$

and the eigenvalues are

$$\lambda_1 = \frac{1+\sqrt{5}}{2}, \quad \lambda_2 = \frac{1-\sqrt{5}}{2}$$

*Therefore the **FS** are*

$$\varphi_n = \left(\frac{1+\sqrt{5}}{2}\right)^n, \quad \psi_n = \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The solutions $\{\varphi_n, \psi_n\}$ form FS because Casoratian

$$C_n = \det \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & \left(\frac{1-\sqrt{5}}{2}\right)^n \\ \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} & \left(\frac{1-\sqrt{5}}{2}\right)^{n+1} \end{bmatrix} = \sqrt{5} \neq 0$$

*The **GS** is in the form*

$$f_n = a\left(\frac{1+\sqrt{5}}{2}\right)^n + b\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

where a and b are arbitrary constants. Conditions $f_1 = 1, \quad f_2 = 1$ lead to linear system

$$\begin{cases} a\left(\frac{1+\sqrt{5}}{2}\right)^1 + b\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1, \\ a\left(\frac{1+\sqrt{5}}{2}\right)^2 + b\left(\frac{1-\sqrt{5}}{2}\right)^2 = 1. \end{cases}$$

Solving we get $a = \frac{\sqrt{5}}{5}, \quad b = -\frac{\sqrt{5}}{5}$. Therefore we have obtained

$$f_n = \frac{\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The above formula was derived in XVIII century by Binet.

Solving (NH) linear second order difference equations with CC

$$y_{n+2} + py_{n+1} + qy_n = r_n. \tag{9}$$

The difference equation of the type (9) we solve, similarly as equations of the 1^{st} order in two steps:

Step I. We are first finding the **GS** of the corresponding (H)

$$y_{n+2} + py_{n+1} + py_n = 0.$$

We then have

$$y_n = a\varphi_n + b\psi_n,$$

where φ_n and ψ_n are **FS**, while a and b are arbitrary constants.

Step II. The **GS** of (9) we are searching in the form

$$y_n = a_n\varphi_n + b_n\psi_n,$$

where (a_n) i (b_n) are new unknown sequences. We use the following result

Theorem 33 *The sequence* $y_n = a_n\varphi_n + b\psi_n$ *is a solution of (9) if and only if, when*

$$\begin{bmatrix} \varphi_{n+1} & \psi_{n+1} \\ \varphi_{n+2} & \psi_{n+2} \end{bmatrix} \begin{bmatrix} \Delta a_n \\ \Delta b_n \end{bmatrix} = \begin{bmatrix} 0 \\ r_n \end{bmatrix}.$$

Example 34 *Find the* ***GS*** *of*

$$y_{n+2} - 5y_{n+1} + 6y_n = 12^n.$$

Solution 35 *Step I. For the corresponding (H)*

$$y_{n+2} - 5y_{n+1} + 6y_n = 0$$

we have as the ***FS*** *the sequences* $\varphi_n = 2^n$ *and* $\psi_n = 3^n$ *(see Ex. 10).*

Step II. The **FS** of (NH) we are looking for in the form

$$y_n = a_n\varphi_n + b_n\psi_n = a_n 2^n + b_n 3^n,$$

where sequences (a_n) are (b_n) solutions of the following system

$$\begin{bmatrix} 2^{n+1} & 3^{n+1} \\ 2^{n+2} & 3^{n+2} \end{bmatrix} \begin{bmatrix} \Delta a_n \\ \Delta b_n \end{bmatrix} = \begin{bmatrix} 0 \\ 12^n \end{bmatrix}.$$

Hence

$$\Delta a_n = -\frac{1}{2}6^n, \qquad \Delta b_n = \frac{1}{3}4^n.$$

In other words

$$a_{n+1} - a_n = -\frac{1}{2}6^n, \quad b_{n+1} - b_n = \frac{1}{3}4^n.$$

The sequence (a_n) we may find as follows:

$$\begin{array}{rcl} a_{n+1} - a_n &=& -\frac{1}{2}6^n, \\ a_n - a_{n-1} &=& -\frac{1}{2}6^{n-1}, \\ && \cdots\cdots \\ a_2 - a_1 &=& -\frac{1}{2}6^1. \end{array}$$

Adding "side by side" we get

$$a_{n+1} - a_1 = -\frac{1}{2}\left(6^n + 6^{n-1} + ... + 6\right) = -\frac{3}{5}\left(6^n - 1\right)$$

and therefore

$$a_n = a_1 - \frac{3}{5}\left(6^{n-1} - 1\right) = a - \frac{1}{10}6^n,$$

where a is arbitrary constant.

Similarly for the sequence (b_n) we have

$$\begin{array}{rcl} b_{n+1} - b_n &=& \frac{1}{3}4^n, \\ b_n - b_{n-1} &=& \frac{1}{3}4^{n-1}, \\ && \cdots\cdots \\ b_2 - b_1 &=& \frac{1}{3}4^1 \end{array}$$

and thus

$$b_n = b_1 + \frac{4}{9}\left(4^{n-1} - 1\right) = b + \frac{1}{9}4^n,$$

where b is is arbitrary constant. Finally we obtain

$$y_n = \left(a - \frac{1}{10}6^n\right)2^n + \left(b + \frac{1}{9}4^n\right)3^n = a{\cdot}2^n + b{\cdot}3^n + \frac{1}{90}12^n,$$

where a and b are arbitrary constants.

Let us notice that the **GS** of (NH) involves two arbitrary constants. Therefore also in an IVP for (NH) there are needed the first and the second terms of the unknown sequence. So the IVP looks like

$$\begin{array}{c} y_{n+2} + py_{n+1} + qy_n = r_n, \\ y_1 = A, \quad y_2 = B. \end{array}$$

Example 36 *Solve the IVP*

$$y_{n+2} - 2y_{n+1} + y_n = -1,$$
$$y_1 = 0, \quad y_2 = -2.$$

Solution 37 *Step I. We begin with the* ***GS*** *of the corresponding (H)*

$$y_{n+2} - 2y_{n+1} + y_n = 0.$$

The ***CH*** *is*

$$\lambda^2 - 2\lambda + 1 = 0.$$

It possess the double root $\lambda = 1$. So the ***FS*** *form the sequences*

$$\varphi_n = 1 \quad \text{and} \quad \psi_n = n.$$

The Casoratian is

$$C_n = \det \begin{bmatrix} 1 & n \\ 1 & n+1 \end{bmatrix} = 1 \neq 0$$

Stage I. We are looking for the ***GS*** *of (NH) in the form*

$$y_n = a_n + b_n n,$$

where

$$\begin{bmatrix} 1 & n+1 \\ 1 & n+2 \end{bmatrix} \begin{bmatrix} \Delta a_n \\ \Delta b_n \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

solving we obtain

$$\Delta a_n = n+1, \qquad \Delta b_n = -1.$$

Hence

$$a_n = a + \frac{n\,(n+1)}{2}, \qquad \Delta b_n = b - n,$$

where a and b are arbitrary constants. Therefore

$$y_n = a + bn + \frac{n\,(1-n)}{2},$$

where a and b are arbitrary constants. Using the initial conditions $y_1 = 0, \; y_2 = -2$ we evaluate that $a = 1$ and $b = -1$. Finally

Answer: $y_n = \frac{(1-n)(2+n)}{2}$.