LECTURE 24. 03. 2020

We are going to discuss the DE of the $n-th$ order. By this type we mean ODE of the form

$$F\left(x, y, y', ..., y^{(n)}\right) = 0,$$

where the unknown is a function $y = y(x)$. By solution we mean any function $y = y(x)$ such that

$$F\left(x, y(x), y'(x), ..., y^{(n)}(x)\right) \equiv 0, \quad for \quad all \quad x \in I.$$

Example 1 *Consider DE* $y'' = y$. *Solutions are:*

1) $y = e^x$, *because* $L = y'' = (e^x)'' = e^x$ *and* $R = y = e^x$; $L = R$;

2) $y = e^{-x}$, *because* $L = y'' = (e^{-x})'' = e^{-x}$ *and* $R = y = e^{-x}$; $L = R$;

3) $y = 0$, *because* $L = y'' = (0)'' = 0$ *and* $R = y = 0$; $L = R$;

4) $y = Ce^x + De^{-x}$, *where* C, D *are arbitrary constants. It is that because* $L = y'' = (Ce^x + De^{-x})'' = Ce^x + De^{-x}$ *and* $R = y = Ce^x + De^{-x}$; $L = R$;

Example 2 *Consider DE* $y''' = -y'$. *Solutions are:*

1) $y = \sin x$, *because* $L = y''' = (\sin x)''' = -\cos x$ *and* $R = -y' = -\cos x$; $L = R$;

2) $y = \cos x$, *because* $L = y'' = (\cos x)''' = \sin x$ *and* $R = -y' = \sin x$; $L = R$;

3) $y = C$, *because* $L = y''' = (C)''' = 0$ *and* $R = -y' = 0$; $L = R$;

4) $y = C + D\sin x + E\cos x$, *where* C, D, E *are arbitrary constants. We check*

$$\begin{aligned} L &= y''' = (C + D\sin x + E\cos x)''' = -D\cos x + E\sin x, \\ R &= -y' = -(C + D\sin x + E\cos x)' = -D\cos x + E\sin x. \end{aligned}$$

Observation:

$$the\ order\ of\ ODE = numbers\ of\ arbitrary\ constants.$$

The IVP for ODE of the $n - th$ order. $\quad F\left(x, y, y', ..., y^{(n)}\right) = 0,$

$$\begin{aligned} y(x_0) &= y_0, \ y'(x_0) = y_1, \ y''(x_0) = y_2, ..., \ y^{(n-1)}(x_0) = y_{n-1}, \\ & where \ \ x_0, \ y_0, \ y_1, ..., y_{n-1} \ \ are\ given\ numbers. \end{aligned}$$

Example 3 $y'' = y$, $y(0) = 2$, $y'(0) = -4$.

The GS is $y = Ce^x + De^{-x}$, *where* C, D *are arbitrary constants. Then* $y' = Ce^x - De^{-x}$. *In the last two reltions we plug:* $x = 0$, $y = 2$, $y' = -4$. *We then get the system*

$$C + D = 2, C - D = -4.$$

Solving we obtain $C = -1, D = 3$ *and the answer:* $y = -e^x + 3e^{-x}$.

ODE REDUCIBLE TO THE FIRST ORDER

This subject includes the following types of ODE:

- $y^{(n)} = f(x)$, where $f(x)$ is given function;
- $F\left(x, y^{(n)}, y^{(n+1)}\right) = 0$;
- $F(y, y', y'') = 0$.

ODE OF THE FORM $y^{(n)} = f(x)$

We solve then integrating $n - times$.

Example 4 *Find the DE:*
a) $y'' = 12x^2$; *b)* $y''' = \sin x$;
c) solve the IVP $y'' = 6x^2$, $y(1) = 2$; $y'(1) = -1$.

Solution 5 *a) Rewrite the ODE as* $(y')' = 6x^2$. *consequtively we obtain*

$$\begin{aligned} y' &= \int 12x^2 dx = 4x^3 + C, \\ y &= \int \left(4x^3 + C\right) dx = x^4 + Cx + D, \quad C, D \ \ arbitrary \ \ constants. \end{aligned}$$

b) Integrating three times we obtain

$$\begin{aligned} y'' &= \int y'''(x)\, dx = \int \sin x dx = -\cos x + C, \\ y' &= \int y''(x)\, dx = \int (-\cos x + C)\, dx = -\sin x + Cx + D, \\ y &= \int y'(x)\, dx = \int (-\sin x + Cx + D)\, dx = \cos x + \frac{C}{2}x^2 + Dx + E, \\ y &= \cos x + Cx^2 + Dx + E, \quad C, D, E \ \ arbitrary \ \ constants. \end{aligned}$$

c) From Example a) we have the GS

$$\begin{aligned} y &= x^4 + Cx + D, \\ y' &= 4x^3 + C, \quad C, D \ \ arbitrary \ \ constants \end{aligned}$$

Plugging $x = 1, y = 2, y' = -1$ *we have the system*

$$2 = 1 + C + D, \quad -1 = 4 + C.$$

Solving we get $C = -5 \quad D = 6$ *and hence*

$$y = x^4 - 5x + 6.$$

ODE OF THE FORM $F\left(x, y^{(n)}, y^{(n+1)}\right) = 0.$

We solve it in two steps:

1st step: Sustibute $u = y^{(n)}$. Then $y^{(n+1)} = u'$. Plugging to the given DE we have

$$F\left(x, u, u'\right) = 0.$$

Just obtained DE is of the 1st order. We can find the GS $u = u(x)$.

2nd step: Having known the function $u = u(x)$ we have the DE

$$y^{(n)} = u.$$

This is the first discussed case and we need to integrate it $n - times$.

Example 6 *Solve the following ODE's:*

a) $(1+x)\, y'' = y'$; *b)* $y''' - \frac{y''}{x} = xe^{x}$;

c) $y'' = \frac{y'}{x} + \frac{x^2}{y'}$, $y(1) = 0$, $y'(1) = 4$.

Solution 7 *a) Substitute $y' = u$. Then $y'' = u'$ and hence*

$$(1+x)\, u' = u.$$

It is separable ODE with variable x, u to be separated. We the proceed:

$$\begin{aligned} (1+x)\frac{du}{dx} &= u; \\ \frac{du}{u} &= \frac{dx}{1+x}; \\ \int \frac{du}{u} &= \int \frac{dx}{1+x}; \end{aligned}$$

$$\begin{aligned} \ln|u| &= \ln|1+x| + \ln|C| = \ln\left(C(1+x)\right); \\ y' &= u = C(1+x); \\ y &= \int C(1+x)\, dx = C\left(x + \frac{x^2}{2}\right) + D, \quad C, D \ \ arbitrary \ \ constants \end{aligned}$$

Solution 8 *b) $y''' - \frac{y''}{x} = xe^{x}$. Substitute $y'' = u$. Then $y''' = u'$ and hence*

$$u' - \frac{u}{x} = xe^{x}.$$

It is linear of the 1st order with the unknown $u = u(x)$. We solve it cf. by integrating factor. The CONJ. EQ. is

$$\mu' + \frac{\mu}{x} = 0.$$

Thus

$$\frac{d\mu}{dx} = -\frac{\mu}{x} \quad and \quad \frac{d\mu}{\mu} = -\frac{dx}{x}.$$

Integrating we have $\int \frac{d\mu}{\mu} = -\int \frac{dx}{x}$,

$$\ln|\mu| = -\ln|x| = \ln\left|\frac{1}{x}\right| \quad and \quad \mu = \frac{1}{x}.$$

So

$$\begin{aligned} \left[u \cdot \frac{1}{x}\right]' &= xe^x \cdot \frac{1}{x} = e^x, \\ u \cdot \frac{1}{x} &= \int e^x dx = e^x + C, \\ y'' &= u = xe^x + Cx. \end{aligned}$$

If we want to find y, *we need to integrate twice by parts:*

$$\begin{aligned} y'(x) &= \int y''(x)\,dx = \int (xe^x + C)\,dx = (x-1)e^x + Cx + D, \\ y(x) &= \int y'(x)\,dx = \int ((x-1)e^x + Cx + D)\,dx = (x-2)e^x + C\frac{x^2}{2} + Dx + E. \end{aligned}$$

Therefore the GS is $y(x) = (x-2)e^x + Cx^2 + Dx + E$, C, D, E *arbitrary constants.*

c) $y'' = \frac{y'}{x} + \frac{x^2}{y'}$, $y(1) = 0$, $y'(1) = 4$.
Substitute $y' = u$. *Then* $y'' = u'$ *and hence*

$$\begin{aligned} u' &= \frac{u}{x} + \frac{x^2}{u}, \\ u' - \frac{u}{x} &= \frac{x^2}{u} = x^2u^{-1}, \quad u(1) = y'(1) = 4. \end{aligned}$$

This is the Bernoulli DE, $\alpha = -1$. *We solve it by substitution*

$$z = u^{1-\alpha} = u^2, \quad u = \sqrt{z}, \quad u' = \frac{z'}{2\sqrt{z}}.$$

Thus

$$\frac{z'}{2\sqrt{z}} - \frac{\sqrt{z}}{x} = \frac{x^2}{\sqrt{z}}.$$

Multiply both sides by $2\sqrt{z}$ *and get*

$$z' - \frac{2z}{x} = 2x^2.$$

This time we solve it by the variation of parameters.

Step 1: We have (H) $\qquad z' - \frac{2z}{x} = 0$,

$$\begin{aligned} \frac{dz}{dx} &= \frac{2z}{x}, \\ \int \frac{dz}{z} &= \int \frac{2dx}{x}, \\ \ln|z| &= 2\ln|x| + \ln|C| = \ln\left|Cx^2\right|, \\ GS\,(H) \quad z &= Cx^2. \end{aligned}$$

Step 2: The GS of (NH) *we predict in the form*

$$z = C(x)\,x^2 \qquad \text{and} \qquad z' = C'x^2 + 2xC.$$

Plugging to $z' - \frac{2z}{x} = 2x^2$ *we obtain*

$$C'x^2 + 2xC - \frac{2Cx^2}{x} = 2x^2.$$

Hence

$$\begin{aligned} C' &= 2, \\ C &= 2x + D, \\ z &= (2x + D)\,x^2 \end{aligned}$$

Thus

$$y' = u = \sqrt{z} = \sqrt{(2x + D)\,x^2} = x\sqrt{2x + D}.$$

At this moment we can use the IC $u(1) = y'(1) = 4$. *Hence*

$$4 = \sqrt{2 + D} \qquad \text{and} \qquad D = 14.$$

So

$$\begin{aligned} y' &= x\sqrt{2x + 14}, \\ y &= \int x\sqrt{2x + 14}dx = \frac{2}{15}\sqrt{2}\,(3x - 14)\,(x + 7)^{\frac{3}{2}} + K. \end{aligned}$$

Now we use the IC $y(1) = 0$. *So*

$$0 = \frac{2}{15}\sqrt{2}\,(3 - 14)\,(1 + 7)^{\frac{3}{2}} + K \quad \text{and} \quad K = -\frac{704}{15}.$$

Finally the answer is:

$$y = \int x\sqrt{2x + 14}dx = \frac{2\sqrt{2}}{15}\,(3x - 14)\,(x + 7)^{\frac{3}{2}} - \frac{704}{15}.$$

ODE OF THE FORM $F(y, y', y'') = 0.$

We solve them by substitution $y' = z(y)$. Therefore

$$y'' = \frac{d}{dx}[z(y)] = \frac{d}{dy}[z(y)]\frac{dy}{dx} = \frac{dz}{dy}y' = \frac{dz}{dy}z.$$

Plugging we obtain

$$F\left(y, z, \frac{dz}{dy} \cdot z\right) = 0.$$

This is ODE of the 1^{st} order with the uknown $z = z(y)$. We solve it and having known the function $z(y)$ we next solve the ODE

$$y' = z(y).$$

Example 9 *Solve the ODE:*
a) $yy'' + (y')^2 = 0$; *b)* $y'' - e^{2y} = 0,\ y(0) = 0,\ y'(0) = 1.$

Solution 10 *a) take* $y' = z(y) \neq 0$ *and then* $y'' = \frac{dz}{dy} \cdot z$. *Plugging the get*

$$\begin{aligned} y\frac{dz}{dy} \cdot z + z^2 &= 0, \\ y\frac{dz}{dy} + z &= 0. \end{aligned}$$

It is separable ODE with variable y *and* z *to be separated.*

$$\begin{aligned} y\frac{dz}{dy} &= -z, \\ \frac{dz}{z} &= -\frac{dy}{y}, \\ \int \frac{dz}{z} &= -\int \frac{dy}{y}, \\ \ln|z| &= -\ln|y| + \ln|C| = \ln\left|\frac{C}{y}\right| \\ z &= \frac{C}{y}. \end{aligned}$$

Therefore

$$y' = z(y) = \frac{C}{y}.$$

It is again separable ODE with variable y *and* x *to be separated*

$$\begin{aligned}
\frac{dy}{dx} &= \frac{C}{y}, \\
ydy &= Cdx, \\
\int 2ydy &= \int 2Cdx, \\
y^2 &= Cx + D, \quad C, D \ \ \textit{arbitrary} \ \ \textit{constants.}
\end{aligned}$$

Solution 11 *b)* $\qquad y'' - e^{2y} = 0, \ y(0) = 0, \ y'(0) = 1.$

Take $y' = z(y) \neq 0$ *and then* $y'' = \frac{dz}{dy} \cdot z$ *and moreover, because of IC's*

$$y(0) = 0, \ y'(0) = 1$$

we have

$$1 = y'(0) = z(y(0)) = z(0).$$

Plugging the get

$$\frac{dz}{dy} \cdot z - e^{2y} = 0 \quad \textit{and} \quad z(0) = 1$$

It is separable DE with variable y *and* z *to be separated. We then have*

$$\begin{aligned}
zdz &= e^{2y} dy, \\
\int 2zdz &= \int 2e^{2y} dy, \\
z^2 &= e^{2y} + C \quad \textit{and} \quad z(0) = 1.
\end{aligned}$$

Hence

$$1^2 = e^0 + C = 1 + C, \quad C = 0.$$

Therefore

$$\begin{aligned}
z^2 &= e^{2y} \quad \textit{and} \quad z > 0, \\
y' &= z = e^y.
\end{aligned}$$

This is again the separable ODE

$$y' = e^y, \quad y(0) = 0.$$

$$\begin{aligned}
\frac{dy}{dx} &= e^y, \\
\left(-e^{-y}\right) dy &= -dx, \\
\int \left(-e^{-y}\right) dy &= -\int dx, \\
e^{-y} &= D - x, \\
y &= -\ln(D - x) \quad \textit{and} \quad y(0) = 0.
\end{aligned}$$

Finally $D = 1$ *and* $y = -\ln(1 - x) = \ln \frac{1}{1-x}$.