1.) $y'' + y' = e^{-x} + 5x + 2$

$y'' + y' = 0$

$t^2 + t = 0 \Rightarrow t(t+1) = 0$

$t = 0 \quad m = 1$

$t = -1 \quad m = 1 \rightarrow$ AD1

G.S. (H)

$y = C_1 + C_2 x + C_3 e^{-x}$

(NH) Und. coeff.

a) for $r(x) = 5x + 2$

(PS) $y_s = (ax + b)\, x^{1} = ax^2 + bx$

$y' = 2ax + b$

$y'' = 2a$

$2a + 2ax + b = 5x + 2$

$2ax + 2a + b = 5x + 2$

$\begin{cases} 2a = 5 \\ 2a + b = 2 \end{cases} \Rightarrow \begin{matrix} a = \frac{5}{2} \\ b = -3 \end{matrix}$

$\underline{y_s = \frac{5}{2}x^2 - 3x}$

b) for $r(x) = e^{-x}$

$y_s = A \cdot e^{-x} \cdot x^{1}$ AD1

$y' = Ae^{-x} - Axe^{-x}$

$y'' = -2Ae^{-x} + Axe^{-x}$

$Ae^{-x} \cancel{- Axe^{-x}} - 2Ae^{-x} \cancel{+ Axe^{-x}} = e^{-x}$

$-Ae^{-x} = e^{-x}$

$-A = 1$

$A = -1$

$\underline{y_s = -1 \cdot e^{-x} \cdot x = -xe^{-x}}$

c) summing up

$y_s = \underline{-xe^{-x} + \frac{5}{2}x^2 - 3x}$

GS (NH)

$y = y_H + y_s$

$y = C_1 + C_2 x + C_3 e^{-x} + \frac{5}{2}x^2 - 3x - xe^{-x}$

2.) $y'' + y = \frac{1}{\sqrt{\cos(2x)}}$

I Part

(H) $y'' + y = 0$ $\quad \lambda^2 + 1 = 0 \Rightarrow \lambda = i$ or $\lambda = -i$

if $\lambda = i$ then

$e^{ix} = \cos x + i \sin x$

czy forma standardowa to tylko $\underline{1} \cdot y''$???

(HS) $y_c(x) = C_1 \cos 2x + C_2 \sin 2x$

II part $\quad r(x) = \frac{1}{\sqrt{\cos(2x)}} = \frac{\cancel{\sqrt{\cos(2x)}}}{\cos(2x)} = \cancel{\sqrt{\sec}} \sqrt{\frac{1}{\cos(2x)}} = \sqrt{\sec(2x)}$

$\alpha = 0 \quad \beta = 2$

$W = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x)$

$= 2(\cos^2(x) + \sin^2(x)) = \underline{2}$

$y_1 = \cos(2x) \qquad y_2 = \sin(2x)$

$y_1' = -2\sin(2x) \qquad y_2' = 2\cos(2x)$

$f(x) = \sqrt{\sec(2x)}$

$W_1 = \begin{vmatrix} 0 & \sin(2x) \\ \sqrt{\sec(2x)} & 2\cos(2x) \end{vmatrix} = -\sqrt{\sec(2x)} \cdot \sin(2x) = -\sqrt{\frac{\sin^2(2x)}{\cos(2x)}}$

$W_2 = \begin{vmatrix} \cos(2x) & 0 \\ -2\sin(2x) & \sqrt{\sec(2x)} \end{vmatrix} = \cos(2x) \cdot \sqrt{\sec(2x)} = \cancel{\sec(2x)} \sqrt{\cos(2x)}$

$U_1 = \int \frac{W_1}{W} dx = \frac{1}{2} \int \frac{-\sqrt{\sin^2(2x)}}{\cos 2x} dx = -\frac{1}{2} \int \sqrt{\frac{\sin^2(2x)}{\cos(2x)}} dx = -\frac{1}{2} \int \sqrt{\frac{1}{\cos(2x)} - \cos(2x)}\, dx = $

$-\frac{1}{2} \int \frac{\sin(2x)}{\sqrt{\cos(2x)}} dx = \left\{ \begin{array}{l} u = \cos(2x) \\ du = -2\sin(2x)dx \\ dx = \frac{du}{-2\sin(2x)} \end{array} \right\} = -\frac{1}{2} \int \frac{\sin(2x)}{\sqrt{u}} \cdot \frac{du}{-2\sin(2x)} = \frac{1}{4}\sqrt{u} + C = \frac{1}{4}\sqrt{\cos(2x)}$

$U_2 = \int \frac{W_2}{W} dx = \frac{1}{2} \int \cancel{\frac{\cos(2x)}{\sqrt{\cos 2x}}} \sqrt{\cos(2x)}\, dx$ ← tu uderzam głową w ścianę, bezpośrednio do tego że to niekompletne eliptyczne całka drugiego rodzaju

3.) $y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 5$

~~first~~

(H)

$x^2 y'' - 3xy' + 4y = 0$

it's clearly Euler-Cauchy eq so we look for solutions of the form $y = x^m$

$y = x^m$

$y' = mx^{m-1}$

$y'' = m(m-1)x^{m-2}$

$m(m-1) - 3m + 4 = 0$

$m^2 - 4m + 4 = 0$

$(m-2)^2 = 0$

the double root is $m = 2$

the basic solution is

$y_1 = x^2 \ln(x) \quad y_2 = x^2$

So the solutions have the shape of

$y = C_1 x^m \ln(x) + C_2 x^m$

that is

$y_p = \underbrace{C_1 x^2 (\ln(x))}_{y_1} + \underbrace{C_2 x^2}_{y_2}$

y_1, y_2 are clearly linearly dependent

$y_1' = (x^2 \ln(x))' = 2x\ln(x) + \frac{1}{x}\cdot x^2 = x(2\ln(x)+1) \qquad y_2' = 2x$

$$W = \begin{vmatrix} x^2\ln(x) & x^2 \\ \ln(e^x x^{2x}) & 2x \end{vmatrix} = 2x^3\ln(x) - x^2\ln(e^x x^{2x}) = x^3(\ln(x^2) - \ln(e\cdot x^2)) = x^3\ln\left(\frac{x^2}{ex^2}\right) =$$

$$= x^3 \ln\left(\frac{1}{e}\right) = x^3\cdot(-\ln e) = -x^3$$

$C_1' = W_1 = \dfrac{\begin{vmatrix} 0 & x^2 \\ 5 & 2x \end{vmatrix}}{-x^3} = \dfrac{-5x^2}{-x^3} = \dfrac{5}{x} \qquad C_1(x) = \int \frac{5}{x}dx = 5\ln(x)$

$C_2'(x) = W_2 = \dfrac{\begin{vmatrix} x^2\ln(x) & 0 \\ \ln(e^x x^{2x}) & 5 \end{vmatrix}}{-x^3} = \dfrac{5x^2\ln(x)}{-x^3} = -\dfrac{5\ln(x)}{x}$ $\qquad C_2(x) = -5\int\frac{1}{x}\cdot\ln(x)dx$

$\left\{ \begin{array}{l} t = \ln(x) \\ dt' = \frac{1}{x} \\ \frac{dt}{dx} = \frac{1}{x} \Rightarrow dx = x\,dt \end{array} \right.$

$= -5\cdot\frac{1}{2}\ln^2 x$

Answ:

$y(x) = y_0(x) + y_p(x) = C_1 x^2 \ln(x) + C_2 x^2 + 5\ln(x) - \frac{5}{2}\ln^2(x)$

5.) $x(3xy-2)dx + (x^3+2y)\,dy = 0$

$P = 3x^2y - 2x \qquad Q = x^3+2y$

$\frac{\partial P}{\partial y} = 3x^2 = \frac{\partial Q}{\partial x} = 3x^2$

eq. is exact

$\int 3x^2y - 2x\,dx = x^3y - x^2 \qquad \int x^3+2y\,dy = x^3y+y^2$

$\cancel{x^3y - x^2 + x^3y + y^2 = D}$

$\cancel{2x^3y - x^2 = D - y^2 \quad | \cdot(-1)}$

$\cancel{-2x^3y + x^2 = -D + y^2}$

$\cancel{x^3\left(-2y + \frac{1}{x}\right) = -D + y^2}$

$\int P\,dx + \int Q(y)\,dy = C$

$x^3y - x^2 + \int 2y\,dy = C$

$x^3y - x^2 + y^2 = C \quad |:x^2$ Solution

$\cancel{xy - 1 + \frac{y^2}{x^2} = \frac{C}{x^2}}$