LECTURE 19. 05. 2020

LINEAR SYSTEMS OF ODE'S (SLDE)

The system of LDE's is in the matrix form

$$Y' = AY + R(x),$$

where A is the matrix of coefficients, $R(x)$ is given vector function and Y is the column unknown vector (vector of unknowns). If $R(x) \equiv 0$ we deal with the homogeneous (H) system

$$Y' = AY.$$

Otherwise the system is nonhomogeneous (NH). The GS of the system is the set of all solutions of all LDE. We denote it GSS.

For the system of LDE we also consider IVP (Cauchy problem). It is in the form

$$Y(x_0) = Y_0,$$

where Y_0 is given vector column. So the IVP is

$$Y' = AY + R(x), \quad Y(x_0) = Y_0.$$

Let us recall the method of elimination for

$$\begin{cases} y_1' = ay_1 + by_2 + r_1(x) \\ y_2' = cy_1 + dy_2 + r_2(x) \end{cases},$$

where a, b, c, d are given coefficients. This we do by the following scheme:

a) from 1^{st} DE evaluate y_2 or from 2^{nd} - y_1;

b) differentiate and plug both to the remaining DE.

Example 1 *Solve the system*

$$\begin{cases} y_1' = y_2 + 2x, \\ y_2' = -2y_1 + 3y_2 + 6x. \end{cases}$$

In matrix form we have

$$Y' = AY + R(x),$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$ and $R(x) = \begin{bmatrix} 2x \\ 6x \end{bmatrix}$. From the first DE we find y_2 having

$$y_2 = y_1' - 2x. \tag{1}$$

This function we differentiate obtaining

$$y_2' = y_1'' - 2.$$

Plug both to the second equation and get

$$y_1'' - 2 = -2y_1 + 3\left(y_1' - 2x\right) + 6x.$$

Equivalently,

$$y_1'' - 3y_1' + 2y_1 = 2. \tag{2}$$

This is LDE of the 2^{nd} order with the unknown y_1. We solve it by (CH)

$$\lambda^2 - 3\lambda + 2 = 0.$$

Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ and FS

$$\varphi = e^x \quad and \quad \psi = e^{2x}.$$

One can also check that the PS is $y_{1p} = 1$. So, the GSNH is

$$y_1 = Ce^x + De^{2x} + 1,$$

where C, D are arbitrary constants.The evaluation of y_2 we do by (1)

$$y_2 = y_1' - 2x = Ce^x + 2De^{2x} - 2x.$$

Writing the solution as the column vector we obtain

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Ce^x + De^{2x} + 1 \\ Ce^x + 2De^{2x} - 2x \end{bmatrix} = C\begin{bmatrix} e^x \\ e^x \end{bmatrix} + D\begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix} + \begin{bmatrix} 1 \\ -2x \end{bmatrix},$$

where C, D are arbitrary constants. So the GSS is a linear combination of FSS

$$\Phi_1 = \begin{bmatrix} e^x \\ e^x \end{bmatrix}, \ \Phi_2 = \begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix}$$

plus a PSS

$$Y_p = \begin{bmatrix} 1 \\ -2x \end{bmatrix}.$$

PSS we obtain by taking $C = D = 0$. FSS are linearly indedendent vector functions. We check this evaluating the Wrońskian

$$W(x) = \det\left[\Phi_1(x), \Phi_2(x)\right] = \det\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} = e^{3x} \neq 0.$$

The GSS we can also write in the following form

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Ce^x + De^{2x} + 1 \\ Ce^x + 2De^{2x} - 2x \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} + \begin{bmatrix} 1 \\ -2x \end{bmatrix},$$

where

$$\Phi(x) = \left[\Phi_1(x), \Phi_2(x)\right] = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$$

is the Wroński matrix. It is also called the fundamental matrix (FM). The FM satisfies the relation

$$\Phi'(x) = A\Phi(x).$$

Let us check this. We have

$$L = \Phi'(x) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}' = \begin{bmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{bmatrix}$$

and

$$R = A\Phi(x) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} = \begin{bmatrix} e^x & 2e^{2x} \\ e^x & 4e^{2x} \end{bmatrix} = L.$$

How we can find in general the GSHS Y' = AY ?

We write down the characteristic equation

$$\det[A - \lambda I] = 0.$$

We do the list of the corresponding FS in the same way as for $n - th$ order LDE. Call them

$$\varphi_1, \varphi_2, ..., \varphi_n.$$

Each of the unknowns is a linear combination of functions

$$\varphi_1, \varphi_2, ..., \varphi_n$$

i.e. each y_i, $i = 1, 2, ..., n$ is in the form

$$y_i = C_{i1}\varphi_1 + C_{i2}\varphi_2 + ... + +C_{in}\varphi_n, \tag{3}$$

with certain constants $C_{i1}, C_{i2}, ..., C_{in}$. We need n arbitrary constants but the above representation take n^2 constants. So, most of them have to be eliminated. We plug the form (3) to the given system and compare the coefficients. This way we eliminate some of them and obtain the GSHS. After this procedure we obtain the GSHS in the form

$$Y(x) = \Phi(x) K,$$

where $\Phi(x)$ is the Wroński matrix (FM) and K arbitrary constant vector. The FM satisties the matrix relation

$$\Phi'(x) = A\Phi(x).$$

Example 2 *Solve the system*

$$\begin{cases} y_1' = y_2 + 2x, \\ y_2' = -2y_1 + 3y_2 + 6x. \end{cases}$$

Solution 3 *We have the matrix* $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$. *In this case*

$$\det \begin{bmatrix} 0-\lambda & 1 \\ -2 & 3-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 2 = 0. \tag{CH}$$

Let us observe that it is the same CH, which we obtain while looking for y_1. *The eigenvalues are*

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = 2.$$

The list of "FS" is,

$$\varphi = e^x \quad \text{and} \quad \psi = e^{2x}.$$

Both unknows are linear combinations of the above functions:

$$y_1 = Ce^x + De^{2x} \quad \text{and} \quad y_2 = Ke^x + Le^{2x}.$$

We know that the GSHS depends on two constants, but we have four. Two of them we have to eliminate. We plug that functions to the given HS

$$\begin{cases} y_1' = y_2, \\ y_2' = -2y_1 + 3y_2 \end{cases}$$

and have

$$\begin{cases} \left(Ce^x + De^{2x}\right)' = Ke^x + Le^{2x}, \\ \left(Ke^x + Le^{2x}\right)' = -2\left(Ce^x + De^{2x}\right) + 3\left(Ke^x + Le^{2x}\right) \end{cases} \iff$$

$$\begin{cases} Ce^x + 2De^{2x} = Ke^x + Le^{2x}, \\ Ke^x + 2Le^{2x} = -2Ce^x - 2De^{2x} + 3Ke^x + 3Le^{2x} \end{cases} \iff$$

$$\begin{cases} Ce^x + 2De^{2x} = Ke^x + Le^{2x}, \\ -2Ke^x - Le^{2x} = -2Ce^x - 2De^{2x} \end{cases}$$

Comparing coefficients we have the following relations

$$C = K, \quad 2D = L, \quad -2K = -2C, \quad -L = -2D.$$

Therefore

$$K = C, \quad L = 2D$$

and finally

$$y_1 = Ce^x + De^{2x} \quad \text{and} \quad y_2 = Ce^x + 2De^{2x}.$$

In vector form it means that

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} Ce^x + De^{2x} \\ Ce^x + 2De^{2x} \end{bmatrix} = C \begin{bmatrix} e^x \\ e^x \end{bmatrix} + D \begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}.$$

The Wroński matrix is $\Phi(x) = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$ *and arbitrary constant vector* $K = \begin{bmatrix} C \\ D \end{bmatrix}$.

How we can find in general the GSNHS
Y' = AY + R(x) ?

Finding the GSNHS we do in two steps:

1. We find the GSHS

$$Y = \Phi(x) K,$$

where K is arbitrary constant vector (column $(n \times 1)$).

2. The GSNHS we are looking for by variation of parameters (variable constants, varied constants). This time it is in the following form

$$Y(x) = \Phi(x) K(x),$$

where $K(x)$ is new unknown vector function. The derivative $Y'(x)$ we evaluate by "the product rule" (be careful, keep the order of factors):

$$Y'(x) = \Phi'(x) K(x) + \Phi(x) K'(x).$$

But

$$\Phi'(x) = A\Phi(x).$$

Thus

$$Y'(x) = A\Phi(x) K(x) + \Phi(x) K'(x) = A\Phi(x) K(x) + R(x)$$

and hence we have the system

$$\Phi(x) K'(x) = R(x). \tag{4}$$

The FM $\Phi(x)$ is invertible and therefore

$$\begin{aligned} K'(x) &= [\Phi(x)]^{-1} R(x), \\ K(x) &= \int [\Phi(x)]^{-1} R(x)\, dx. \end{aligned}$$

Remember that we integrate a vector function and we have to do that by coordinates.

Example 4 *Let us explain the method of variation of parameters for the system*

$$\begin{cases} y_1' = y_2 + 2x, \\ y_2' = -2y_1 + 3y_2 + 6x. \end{cases}$$

The GSHS is

$$Y = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}, \quad \textit{while} \quad R(x) = \begin{bmatrix} 2x \\ 6x \end{bmatrix}$$

The FM is $\Phi(x)=\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$. *The system (4) we can read as*

$$\begin{aligned} K'(x) &= \begin{bmatrix} C'(x) \\ D'(x) \end{bmatrix} = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}^{-1} \begin{bmatrix} 2x \\ 6x \end{bmatrix} \\ &= \frac{1}{e^{3x}} \begin{bmatrix} 2e^{2x} & -e^{2x} \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 2x \\ 6x \end{bmatrix} = \begin{bmatrix} -2xe^{-x} \\ 4xe^{-2x} \end{bmatrix}. \end{aligned}$$

Integrating by coordinates we have

$$\begin{aligned} C(x) &= \int \left(-2xe^{-x}\right) dx = 2e^{-x}(x+1)+E, \\ D(x) &= \int \left(4xe^{-2x}\right) dx = -e^{-2x}(2x+1)+F. \end{aligned}$$

Thus the GSNHS is

$$Y=\Phi(x)K(x)=\begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} \begin{bmatrix} 2e^{-x}(x+1)+E \\ -e^{-2x}(2x+1)+F \end{bmatrix} = \begin{bmatrix} Fe^{2x}+Ee^x+1 \\ 2Fe^{2x}+Ee^x-2x \end{bmatrix},$$

where E,F *are arbitrary constants.*

Example 5 *Solve the IVP*

$$\begin{cases} y_1' = y_2+2, & y_1(0)=1 \\ y_2' = -y_1+1, & y_2(0)=2. \end{cases}$$

Solution 6 *1) We have*

$$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad R(x)=\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

The HS is

$$\begin{cases} y_1' = y_2, \\ y_2' = -y_1. \end{cases}$$

The CH is

$$\det[A-\lambda I]=\det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2+1=0.$$

So the eigenvalues are $\lambda_1=i$ *and* $\lambda_2=-i$. *The list of corresponding FS is*

$$\cos x \quad \text{and} \quad \sin x.$$

So the solutions of the given system are in the form

$$\begin{aligned} y_1 &= C\cos x + D\sin x, \\ y_2 &= A\cos x + B\sin x. \end{aligned}$$

Plugging to HS we have

$$\begin{aligned} & \begin{cases} (C\cos x + D\sin x)' = A\cos x + B\sin x, \\ (A\cos x + B\sin x)' = -(C\cos x + D\sin x) \end{cases} \\ \Longleftrightarrow\ & \begin{cases} -C\sin x + D\cos x = A\cos x + B\sin x, \\ -A\sin x + B\cos x = -C\cos x - D\sin x \end{cases} \\ \Longleftrightarrow\ & A = D \quad \textit{and} \quad B = -C. \end{aligned}$$

Thus

$$\begin{aligned} y_1 &= C\cos x + D\sin x, \\ y_2 &= D\cos x - C\sin x, \end{aligned}$$

where C and D are arbitrary constants. Therefore the GSHS is

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix}.$$

2. *GSNHS we predict in the form*

$$Y = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} C(x) \\ D(x) \end{bmatrix},$$

where by (4)

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} C'(x) \\ D'(x) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} C'(x) \\ D'(x) \end{bmatrix} = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2\cos x - \sin x \\ \cos x + 2\sin x \end{bmatrix}.$$

Therefore

$$\begin{bmatrix} C(x) \\ D(x) \end{bmatrix} = \begin{bmatrix} \int (2\cos x - \sin x)\, dx \\ \int (\cos x + 2\sin x)\, dx \end{bmatrix} = \begin{bmatrix} \cos x + 2\sin x + K, \\ \sin x - 2\cos x + L, \end{bmatrix}$$

where K and L are arbitrary constants. Finally

$$Y = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} \cos x + 2\sin x + K \\ \sin x - 2\cos x + L \end{bmatrix} = \begin{bmatrix} K\cos x + L\sin x + 1 \\ L\cos x - K\sin x - 2 \end{bmatrix}.$$

3. *The IC is* $Y(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} K\cos 0 + L\sin 0 + 1 \\ L\cos 0 - K\sin 0 - 2 \end{bmatrix} = \begin{bmatrix} K+1 \\ L-2 \end{bmatrix}$. *Hence*

$$K = 1 \quad \textit{and} \quad L = 3.$$

Finally we have the answer:

$$Y = \begin{bmatrix} \cos x + 3\sin x + 1 \\ 3\cos x - \sin x - 2 \end{bmatrix}.$$

Example 7 *Find the GSS of*

$$\begin{cases} y_1' = -y_1 + y_2 + y_3 + 8, & y_1(0) = 0, \\ y_2' = y_1 - y_2 + y_3 + 4, & y_2(0) = 1, \\ y_3' = y_1 + y_2 + y_3 + 12, & y_3(0) = 0. \end{cases}$$

Solution 8 *1) We have*

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad R(x) = \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} \quad \text{and} \quad Y(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The HS is

$$\begin{cases} y_1' = -y_1 + y_2 + y_3, \\ y_2' = y_1 - y_2 + y_3, \\ y_3' = y_1 + y_2 + y_3. \end{cases}$$

The CH is

$$\det[A - \lambda I] = \det \begin{bmatrix} -1-\lambda & 1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{bmatrix} = -(\lambda - 2)(\lambda + 2)(\lambda + 1) = 0.$$

So the eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -2$ *and* $\lambda_3 = -1$. *The list of "the corresponding FS" is*

$$e^{2x}, \quad e^{-2x} \quad \text{and} \quad e^{-x}.$$

So the solutions of the given system are in the form

$$\begin{aligned} y_1 &= Ce^{2x} + De^{-2x} + Ee^{-x}, \\ y_2 &= Ke^{2x} + Le^{-2x} + Me^{-x} \\ y_3 &= Pe^{2x} + Qe^{-2x} + Re^{-x}. \end{aligned}$$

Plugging to HS we have

$$\begin{cases} 2Ce^{2x} - 2De^{-2x} - Ee^{-x} = -\left(Ce^{2x} + De^{-2x} + Ee^{-x}\right) + \left(Ke^{2x} + Le^{-2x} + Me^{-x}\right) + \left(Pe^{2x} + Qe^{-2x} + Re^{-x}\right. \\ 2Ke^{2x} - 2Le^{-2x} - Me^{-x} = \left(Ce^{2x} + De^{-2x} + Ee^{-x}\right) - \left(Ke^{2x} + Le^{-2x} + Me^{-x}\right) + \left(Pe^{2x} + Qe^{-2x} + Re^{-x}\right. \\ 2Pe^{2x} - 2Qe^{-2x} - Re^{-x} = \left(Ce^{2x} + De^{-2x} + Ee^{-x}\right) + \left(Ke^{2x} + Le^{-2x} + Me^{-x}\right) + \left(Pe^{2x} + Qe^{-2x} + Re^{-x}\right. \end{cases}$$

$$\begin{cases} 2Ce^{2x} - 2De^{-2x} - Ee^{-x} = (-C + K + P)e^{2x} + (-D + Q + L)e^{-2x} + (-E + M + R)e^{-x}, \\ 2Ke^{2x} - 2Le^{-2x} - Me^{-x} = (C - K + P)e^{2x} + (D + Q - L)e^{-2x} + (E - M + R)e^{-x}, \\ 2Pe^{2x} - 2Qe^{-2x} - Re^{-x} = (C + K + P) + (D + L + Q)e^{-2x} + (E + M + R)e^{-x}. \end{cases}$$

Comparing the coefficients we have

$$\begin{cases} 2C=-C+K+P, \quad -2D=-D+L+Q, \quad -E=-E+M+R, \\ 2K=C-K+P, \quad -2L=D+Q-L, \quad -M=E-M+R, \\ 2P=C+K+P, \quad -2Q=D+L+Q, \quad -R=E+M+R. \end{cases}$$

$$\Longleftrightarrow \begin{cases} 3C=K+P, \quad -D=L+Q, \quad 0=M+R, \\ 3K=C+P, \quad -L=D+Q, \quad 0=E+R, \\ P=C+K, \quad -3Q=D+L, \quad -2R=E+M. \end{cases}$$

$$\Longleftrightarrow \begin{cases} 3C=K+P, \quad 3K=C+P, \quad P=C+K, \\ -D=L+Q, \quad -L=D+Q, \quad -3Q=D+L \\ 0=M+R, \quad 0=E+R, \quad -2R=E+M. \end{cases}$$

$$\Longleftrightarrow \begin{cases} 4C=C+K+P, \quad 4K=C+K+P, \quad 2P=C+K+P, \\ 0=D+L+Q, \quad 0=D+L+Q, \quad -2Q=D+L+Q, \\ E=E+M+R, \quad M=E+M+R, \quad -R=E+M+R. \end{cases}$$

$$\Longleftrightarrow \quad 4C=4K=2P, \quad Q=0, \quad 0=D+L=0, \quad E=M=-R,$$

$$\Longleftrightarrow \quad 2C=2K=P, \quad Q=0, \quad L=-D, \quad E=M=-R.$$

$$\Longleftrightarrow \quad P=2C, \quad K=C, \quad Q=0, \quad L=-D, \quad M=E, \quad R=-E.$$

Thus

$$\begin{aligned} y_1 &= Ce^{2x}+De^{-2x}+Ee^{-x}, \\ y_2 &= Ce^{2x}-De^{-2x}+Ee^{-x} \\ y_3 &= Ce^{2x}-Ee^{-x}. \end{aligned}$$

where C, D *and* E *are arbitrary constants. Therefore the GSHS is*

$$Y=\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} e^{2x} & e^{-2x} & e^{-x} \\ e^{2x} & -e^{-2x} & e^{-x} \\ 2e^{2x} & 0 & -e^{-x} \end{bmatrix}\begin{bmatrix} C \\ D \\ E \end{bmatrix}.$$

2. GSNHS we predict in the form

$$Y=\begin{bmatrix} e^{2x} & e^{-2x} & e^{-x} \\ e^{2x} & -e^{-2x} & e^{-x} \\ 2e^{2x} & 0 & -e^{-x} \end{bmatrix}\begin{bmatrix} C(x) \\ D(x) \\ E(x) \end{bmatrix},$$

where by (4)

$$\begin{bmatrix} e^{2x} & e^{-2x} & e^{-x} \\ e^{2x} & -e^{-2x} & e^{-x} \\ 2e^{2x} & 0 & -e^{-x} \end{bmatrix}\begin{bmatrix} C'(x) \\ D'(x) \\ E'(x) \end{bmatrix}=\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix},$$

$$\begin{bmatrix} C'(x) \\ D'(x) \\ E'(x) \end{bmatrix}=\begin{bmatrix} e^{2x} & e^{-2x} & e^{-x} \\ e^{2x} & -e^{-2x} & e^{-x} \\ 2e^{2x} & 0 & -e^{-x} \end{bmatrix}^{-1}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix}=\begin{bmatrix} 9e^{-2x} \\ 2e^{2x} \\ -3e^{x} \end{bmatrix}.$$

But

$$\begin{bmatrix} e^{2x} & e^{-2x} & e^{-x} \\ e^{2x} & -e^{-2x} & e^{-x} \\ 2e^{2x} & 0 & -e^{-x} \end{bmatrix}^{-1} = \frac{1}{6e^{-x}} \begin{bmatrix} e^{-3x} & e^{-3x} & 2e^{-3x} \\ 3e^{x} & -3e^{x} & 0 \\ 2 & 2 & -2 \end{bmatrix}$$

Therefore

$$\begin{bmatrix} C'(x) \\ D'(x) \\ E'(x) \end{bmatrix} = \frac{1}{6e^{-x}} \begin{bmatrix} e^{-3x} & e^{-3x} & 2e^{-3x} \\ 3e^{x} & -3e^{x} & 0 \\ 2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 6e^{-2x} \\ 2e^{2x} \\ 0 \end{bmatrix}.$$

$$\begin{bmatrix} C(x) \\ D(x) \\ E(x) \end{bmatrix} = \begin{bmatrix} \int 6e^{-2x}dx \\ \int 2e^{2x}dx \\ \int 0dx \end{bmatrix} = \begin{bmatrix} -3e^{-2x} + K \\ e^{2x} + L \\ M \end{bmatrix}$$

where K, L and M are arbitrary constants. Finally

$$Y = \begin{bmatrix} e^{2x} & e^{-2x} & e^{-x} \\ e^{2x} & -e^{-2x} & e^{-x} \\ 2e^{2x} & 0 & -e^{-x} \end{bmatrix} \begin{bmatrix} -3e^{-2x} + K \\ e^{2x} + L \\ M \end{bmatrix} = \begin{bmatrix} Ke^{2x} + Le^{-2x} + Me^{-x} - 2 \\ Ke^{2x} - Le^{-2x} + Me^{-x} - 4 \\ 2Ke^{2x} - Me^{-x} - 6 \end{bmatrix},$$

3. The IC is

$$Y(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} K + L + M - 2 \\ K - L + M - 4 \\ 2K - M - 6 \end{bmatrix}.$$

Hence

$$K = \frac{9}{4}, \quad L = \frac{5}{4}, \quad M = -\frac{3}{2}$$

Finally we have the answer:

$$Y = \begin{bmatrix} \frac{9}{4}e^{2x} + \frac{5}{4}e^{-2x} - \frac{3}{2}e^{-x} - 2, \\ \frac{9}{4}e^{2x} - \frac{5}{4}e^{-2x} - \frac{3}{2}e^{-x} - 4, \\ \frac{9}{2}e^{2x} + \frac{3}{2}e^{-x} - 6. \end{bmatrix}$$