Euler homogeneous eq.

środa, 13 maja 2020 12:24

Idea:

Eul. eq.: $ax^2y''+bxy'+cy=0$

notation: $\frac{dy}{dt}=\dot{y}$

$x=e^t$

$$y'(x)=\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=e^{-t}\cdot\dot{y}=\frac{1}{x}\dot{y}$$

By chain rule:

Statement:

If $y = f(u)$ and $u = g(x)$, then this abbreviated form is written in Leibniz notation as:

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$ [1]

For the second derivative we are getting:

$$\begin{aligned} y''(x) &= \frac{dy'}{dx}=\frac{dy'}{dt}\frac{dt}{dx}=\left[\frac{d}{dt}\left(e^{-t}\dot{y}\right)\right]e^{-t}=\left[\left(-e^{-t}\right)\dot{y}+e^{-t}\ddot{y}\right]e^{-t} \\ &= e^{-2t}\left(-\dot{y}+\ddot{y}\right)=\frac{1}{x^2}\left(\ddot{y}-\dot{y}\right), \end{aligned}$$

(From the 05-05 lecture)

Plugging both derivatives into our eq. we get:

$$a\cancel{x^2}\cdot\underbrace{\frac{1}{\cancel{x^2}}(\ddot{y}-\dot{y})}_{y''}+bx\underbrace{\frac{1}{x}y}_{y'}+c\cdot y=0$$

As you can see we have second order diff. eq. with constant coefficients.

Example: $2x^2y'' + 3xy' - 15y = 0$, $y(1) = 0$, $y'(1) = 1$.

Let us put: $x = e^t$. Then:

$$y'(x) = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\dot{y} = \frac{1}{x}\dot{y},$$

$$\begin{aligned} y''(x) &= \frac{dy'}{dx} = \frac{dy'}{dt}\frac{dt}{dx} = \left[\frac{d}{dt}\left(e^{-t}\dot{y}\right)\right]e^{-t} = \left[\left(-e^{-t}\right)\dot{y} + e^{-t}\ddot{y}\right]e^{-t} \\ &= e^{-2t}\left(-\dot{y} + \ddot{y}\right) = \frac{1}{x^2}\left(\ddot{y} - \dot{y}\right). \end{aligned}$$

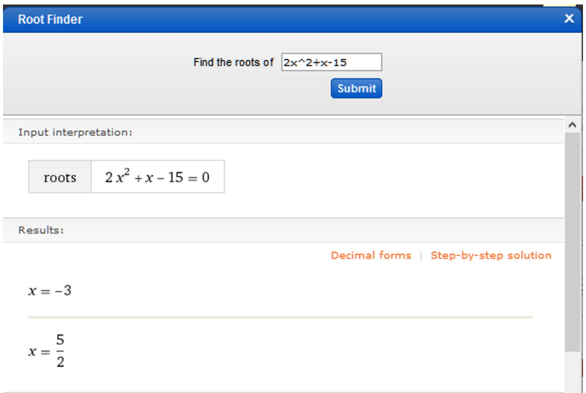
We can rewrite our eq:

$$2\cancel{x^2} \cdot \frac{1}{\cancel{x^2}}(\ddot{y} - \dot{y}) + 3\cancel{x}\frac{1}{\cancel{x}}\dot{y} - 15y = 0$$

$2\ddot{y} + \dot{y} - 15y = 0$ (We are looking for $y(t)$).

Char. eq:

$2\lambda^2 + \lambda - 15 = 0$



So $y_1(t) = e^{-3t}$, $y_2(t) = e^{\frac{5}{2}t}$

$\qquad\quad \| \qquad\qquad\qquad \|$

$\quad (e^t)^{-3}$, $\qquad (e^t)^{\frac{5}{2}}$

So: $y(x) = C_1 \cdot x^{-3} + C_2 x^{\frac{5}{2}}$ $\quad C_1, C_2 \in \mathbb{R}$ $\quad x > 0$

<u>Example</u> Solve $x^3y''' - x^2y'' - 2xy' + 6y = 0.$

Let us put $x = e^t$ and let us recall the third derivative:

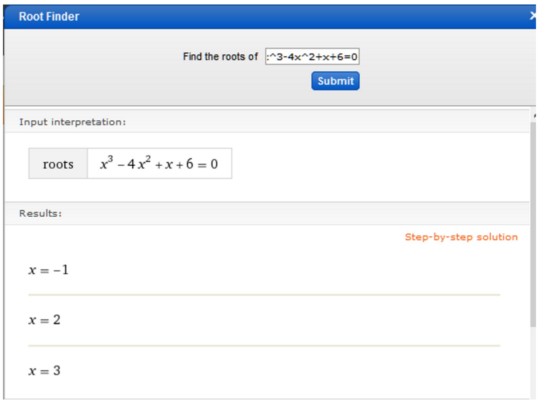
$$
\begin{aligned}
y'''(x) &= \frac{dy''}{dx} = \frac{dy''}{dt}\frac{dt}{dx} = \frac{d}{dt}\left[e^{-2t}\left(-\dot{y} + \ddot{y}\right)\right]e^{-t} \\
&= \left[-2e^{-2t}\left(-\dot{y} + \ddot{y}\right) + e^{-2t}\left(-\ddot{y} + \dddot{y}\right)\right]e^{-t}. \\
&= e^{-3t}\left(2\dot{y} - 2\ddot{y} - \ddot{y} + \dddot{y}\right) = \frac{1}{x^3}\left(\dddot{y} - 3\ddot{y} + 2\dot{y}\right).
\end{aligned}
$$

using our substitution we get:

$$\cancel{x^3}\cdot\frac{1}{\cancel{x^3}}(\dddot{y} - 3\ddot{y} + 2\dot{y}) - \cancel{x^2}\cdot\frac{1}{\cancel{x^2}}(\ddot{y} - \dot{y}) - 2\cancel{x}\cdot\frac{1}{\cancel{x}}\dot{y} + 6y = 0.$$

$$\dddot{y} - 4\ddot{y} + \dot{y} + 6y = 0$$

$$\lambda^3 - 4\lambda^2 + \lambda + 6 = 0$$



$y_1(t) = (e^t)^{-1}$

$y_2(t) = (e^t)^2$

$y_3(t) = (e^t)^3$

Answ: $y(x) = C_1\cdot x^{-1} + C_2\cdot x^2 + C_3x^3$, $x > 0$ $C_1, C_2, C_3 \in \mathbb{R}$.