2.

Bernoulli equation $y' + p(x)y + q(x)y^n = 0$

$$xy' - y^2\ln x + y = 0 \qquad y(e) = 1$$

$$y' + y^2\left(-\frac{\ln x}{x}\right) + y\left(\frac{1}{x}\right) = 0 \;\Big|\cdot\left(\frac{1}{y^2}\right) \qquad p(x) = \frac{1}{x} \quad q(x) = \left(-\frac{\ln x}{x}\right)$$

let's just plug in the values to F1 formula derrived at the side

$$z' + (-1)\cdot\left(\frac{1}{x}\right)z = -\frac{\ln x}{x}$$

$$z' - \frac{z}{x} = -\frac{\ln x}{x}$$

$$z' - \frac{z}{x} = -\frac{\ln x}{x}$$

Nonhomogenous Linear equation

$$z' = \left(\frac{1}{x}\right)z + \left(-\frac{\ln x}{x}\right)$$

P.S. (NH) $y = \frac{1}{-\ln(x) + 1 + \frac{x}{e}}$

$$z' - \left(\frac{1}{x}\right)z = \underbrace{-\frac{\ln x}{x}}_{h(t)}$$

1. Corresponding (H)

$$z' - \frac{1}{x}z = 0$$

$$\frac{dz}{dx} = \frac{z}{x}$$

$$\int \frac{1}{z}\,dz = \int \frac{1}{x}\,dx$$

$$\ln|z| = \ln|x| + C$$

$$\ln|z| = \ln|x| + \ln|C|$$

$$\ln|z| = \ln|xC|$$

$$z = Cx \qquad \text{so } \varphi(x) = x$$

G.S. of (H) $y = \frac{1}{Cx}$

2. (NH)

$$z = C(x)\cdot x$$

$$z' = C'(x)\cdot x + C(x)$$

$\Rightarrow$ (H) ← we plug both (y & it's derrivetive) into homogenous

$$C'(x)\cdot x + \cancel{C(x)} - \cancel{\left(\frac{1}{x}\right)\cdot C(x)\cdot x} = -\frac{\ln x}{x}$$

$$x\cdot C'(x) = -\frac{\ln x}{x}$$

$$C'(x) = -\frac{\ln x}{x^2}$$

$$C(x) = -\int \frac{\ln x}{x^2}\,dx = -\frac{\ln(x)+1}{x} + D$$

← arbitrary constant of second integration

$$y = \frac{1}{z} = \frac{1}{C(x)\cdot x} = \frac{1}{-\frac{\ln(x)+1}{\cancel{x}}\cdot\cancel{x} + Dx} = \frac{1}{-\ln(x)+1+Dx} \quad \text{G.S(NH)}$$

$$y(e) = 1 \Rightarrow 1 = \frac{1}{\cancel{-1+1}+eD} \Rightarrow 1 = \frac{1}{eD} \Rightarrow eD = 1 \qquad D = \frac{1}{e}$$

since $n > 1$ substitute

$$z = y^{1-n} = \frac{1}{y} \Rightarrow y = \frac{1}{z}$$

$$z' = (1-n)y^{-n}y'$$

$$z' = -1y^{-2}y' = -\frac{1}{y^2}y'$$

$$y' = -z'\frac{1}{z^2} \Leftarrow -z'y^2 = y'$$

$$y' = -\frac{z'}{z^2}$$

$$z' + (1-n)p(x)z = (n-1)q(x)$$

F1 $\boxed{z' + (-1)\cdot p(x)z = q(x)}$

$$\int fg' = fg - \int f'g$$

$$f = \ln(x) \qquad g' = \frac{1}{x^2}$$

$$f' = \frac{1}{x} \qquad g = -\frac{1}{x}$$

$$= -\frac{\ln(x)}{x} - \int -\frac{1}{x^2}\,dx$$

$$= -\frac{\ln(x)}{x} - \frac{1}{x} + C$$

5. $xy' + y = x\sqrt{x} \;\Big|:x \;,\; y(1) = 2$ $\qquad u = \frac{y}{x} \qquad y' = u'x + u$

$$y' + \frac{y}{x} = \sqrt{x}$$

(NH) $y' + p(x)y = r(x)$ | (L) $\mu \cdot p(x) = 0$

(NH) $[\mu y] = r(x)\cdot\mu$

$$y' + \frac{y}{x} = \sqrt{x}$$

$$\mu' - \frac{\mu}{x} = 0$$

$$\frac{d\mu}{dx} = \frac{\mu}{x} \Rightarrow \int \frac{d\mu}{\mu} = \int \frac{dx}{x} \qquad \mu = x$$

$$\ln|\mu| = \ln|x|$$

$$\mu = x$$

$$[xy]' = \frac{1}{x}\cdot x = 1$$

$$xy = \int 1\,dx = x + C$$

$$y = \frac{C}{x} + 1$$

G.S. (NH) $y = \frac{C}{x} + 1$

$$2 = \frac{C}{1} + 1$$

$$1 = C$$

P.S. $y = \frac{1}{x} + 1$

3. $(x^2+xy)y' + (3xy+y^2)=0$

$$\underbrace{(3xy+y^2)}_{P(x,y)}dx + \underbrace{(x^2+xy)}_{Q(x,y)}dy = 0$$

$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$ it's not exact

$\frac{\partial P}{\partial y} = 3x+2y \qquad \frac{\partial Q}{\partial x} = 2x+y$

integration factor

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{P} = \frac{3x+2y-2x-y}{3xy+y^2} = \frac{x+y}{3xy+y^2} = \frac{x}{3xy+y^2} + \frac{1}{3x+y}$$

$$\frac{\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}}{Q} = \frac{x+y}{x^2+xy} = \frac{1}{x+y} + \frac{y}{x^2+xy}$$

$\int \frac{1}{x+y} + \frac{y}{x^2+xy}dx =$

$= \int \frac{1}{x+y}dx + \int \frac{1}{x}dx + \int \frac{-1}{x+y}dx =$

$= \cancel{\ln|x+y|} + \ln|x| - \cancel{\ln|x+y|}$

$\mu = e^{\ln|x|} = x$

$$x(3xy+y^2)dx + x(x^2+xy)dy = 0$$

$$\underbrace{(3x^2y+y^2x)}_{P}dx + \underbrace{(x^3+x^2y)}_{Q}dy = 0$$

$\frac{\partial P}{\partial y} = 3x^2+2yx \qquad \frac{\partial Q}{\partial x} = 3x^2+2yx$

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ now it's exact DE

$$\phi(x,y) = C = \int P\,dx + g(y) = \int 3x^2y + y^2x\,dx + g(y) =$$
$$= x^3y + \frac{1}{2}y^2x^2 + g(y)$$

$$\frac{\partial \phi(x,y)}{\partial y} = \frac{\partial}{\partial y}\left[x^3y + \frac{1}{2}y^2x^2 + g(y)\right] = x^3 + yx^2 + g'(y)$$

$$\frac{\partial \phi(x,y)}{\partial y} = Q$$

$$\cancel{x^3} + \cancel{yx^2} + g'(y) = \cancel{x^3} + \cancel{x^2y}$$

$g'(y) = 0 \Rightarrow g(y) = 0$

$$\phi(x,y) = x^3y + \frac{1}{2}y^2x^2 + 0 = x^3y + \frac{1}{2}y^2x^2 = C$$

$y - y \ln x + y = 0 \quad \dots (a)$

4. $y' = \frac{x^2+3y^2}{2xy}$

$$\frac{dy}{dx} = \frac{1}{2}\left(\left(\frac{x}{y}\right) + 3\left(\frac{y}{x}\right)\right) \qquad v = \frac{y}{x}$$

$$vx = y$$

$$v'\cancel{y}x + v = \frac{1}{2}\left(\frac{1}{v} + 3v\right)$$

$$v'x + v = \frac{1}{2v} + \frac{3}{2}v$$

$$v'x = \frac{1}{2v} + \frac{1}{2}v$$

$$v'x = \frac{1}{2}\left(\frac{1}{v} + v\right)$$

$$v'x = \frac{1}{2}\left(\frac{1+v^2}{v}\right)$$

$$v'x = \left(\frac{1+v^2}{2v}\right)$$

$$\cancel{y^2 = x^2(x-1)} \qquad \cancel{y = x\sqrt{x-1}}$$

$$\frac{dv}{dx}x = \left(\frac{1+v^2}{2v}\right)$$

$$\int \frac{2v}{1+v^2}\,dv = \int \frac{1}{x}\,dx$$

$$\ln(v^2+1) = \ln(x) + C$$

$$v^2+1 = xC \rightarrow$$

G.S.

$$\boxed{y = \sqrt{(Cx-1)x^2}}$$

1. $xy' = y\left(\ln\frac{y}{x} + 1\right)$ $\qquad u = \frac{y}{x}$ $\qquad x \neq 0 \notin D$ since $\frac{y}{x}$

$y = u \cdot x$

$y' = u'x + u \cdot 1$

$x(u'x + u) = y(\ln u + 1$

$\cancel{x}(u'x + u) = u\cancel{x}(\ln u + 1)$

$u'x = u\ln u + u - u$

$u'x = u\ln u$

$\frac{u'}{u\ln u} = \frac{1}{x}$

$\int \frac{1}{u\ln u}\,du = \int \frac{1}{x}\,dx$

$\ln(|\ln|u||) = \ln|x| + C$

$\ln(\ln|u|) = \ln|x| + C$

$\int \frac{1}{u\ln u}\,du = \left\{ \begin{array}{l} t = \ln u \\ \frac{dt}{du} = \frac{1}{u} \\ du = dt \cdot u \end{array} \right.$

$= \int \frac{1}{\cancel{u}t}\cancel{u}\,dt = \ln|t| + C =$

$= \ln(|\ln(u)|) + C$

$\ln|\ln|u|| = \ln|x| + \ln C$

$\ln|\ln|u|| = \ln(xC)$ ← here drop ln on both sides

$\ln|u| = xC$

$u = e^{xC}$

G.S. $y = xe^{xC}$

$e = 1e^{1C}$

$e = e^{C}$

$\underline{\underline{C = 1}}$ → so $\underline{\underline{y = xe^{x}}}$ particular solution

$y(1) = e$

$y = xe^{Cx}$ G.S.