LECTURE 17. 03. 2020

Last time we have discussed the Bernoulli DE

$$y' + p(x)\,y = r(x)\,y^{\alpha}, \tag{B}$$

where functions p and q are continuous and $\alpha \in \mathbb{R}\setminus\{0,1\}$ is given real exponent. We have learnt a two-step method of solving (B):

1^{st} step: solve the corresponding (LH) $\;y' + p(x)\,y = 0$ getting the GS

$$y = C\varphi(x),$$

where C is arbitrary constant and $\varphi-$ a fundamental solution of (LH).

2^{nd} step (variation of parameter, variable constant, varied constant): predict the GS of (B) in the form

$$y = C(x)\,\varphi(x),$$

where $C(x)$ stands for a new unknown function.

Today I am going to present you a second method of solving the (B) DE. We use proper substitution. Namely, let us consider the function $z = z(x)$ given by relation

$$z = y^{1-\alpha}. \tag{1}$$

Differentiating (1) by dx we have

$$z'(x) = (1-\alpha)\cdot y^{-\alpha}(x)\cdot y'(x)$$

and therefore

$$y^{-\alpha}\cdot y' = \frac{1}{(1-\alpha.)}\cdot z' \tag{2}$$

Divide the (B) DE by $y^{\alpha} \neq 0$ and get

$$y'\cdot y^{-\alpha} + p(x)\,y^{1-\alpha} = r(x),$$

Using the relations (1) and (2) we obtain

$$\frac{1}{(1-\alpha.)}\cdot z' + p(x)\,z = r(x).$$

Notice that the above DE is linear and 1^{st} order, so we can apply the already known methods.

Example 1 *Find the GS of the following (B)* $y' + xy = \frac{x}{y^3}$.

Solution: In this case we have $\alpha = -3$. Take $z = y^{1-\alpha} = y^{1-(-3)} = y^4$.

Then $z' = 4y^3y'$ and so $4y^3y' = z'$. Multiplying both sides of the given DE by $4y^3$ we have

$$4y^3y' + 4xy^4 = 4x.$$

Plugging just evaluated relations to the above DE we obtain

$$z' + 4xz = 4x. \tag{3}$$

It is LDE of 1^{st} order. Let us solve it by the variation of parameter.

1^{st} step. The corresponding (H) is

$$z' + 4xz = 0.$$

We solve it in the following way

$$\begin{aligned}
\frac{dz}{dx} &= -4xz, \\
\int \frac{dz}{z} &= -\int 4x dx, \\
\ln|z| &= -2x^2 + \ln|C| = \ln\left|C\exp\left(-2x^2\right)\right|, \quad C \ \ is \ \ arbitrary \ \ \text{real}, \\
z &= C\exp\left(-2x^2\right), \ C \ \ is \ \ arbitrary \ \ \text{real}.
\end{aligned}$$

2^{nd} step. The GS of the (3) we predict in the form

$$z = C(x)\exp\left(-2x^2\right).$$

Differentiating we get

$$z' = C'\exp\left(-2x^2\right) + C(-4x)\exp\left(-2x^2\right).$$

Plugging to (3) we have

$$C'\exp\left(-2x^2\right) + C(-4x)\exp\left(-2x^2\right) + 4xC\exp\left(-2x^2\right) = 4x,$$

$$C'\exp\left(-2x^2\right) = 4x,$$

$$C' = 4x\exp\left(2x^2\right)$$

$$C(x) = \int 4x\exp\left(2x^2\right)dx = \exp\left(2x^2\right) + D, \quad D \ is \ \ arbitrary \ \text{real}.$$

Hence

$$y^4 = z = \left(\exp\left(2x^2\right) + D\right)\exp\left(-2x^2\right) = 1 + D\exp\left(-2x^2\right), \ D \ \ is \ \ arbitrary \ \text{real}.$$

Finally, the GS is

$$y = \pm\sqrt[4]{1 + D\exp\left(-2x^2\right)}, \quad D \ \ is \ \ arbitrary \ \text{real}.$$

Example 2 *Solve the IVP for the (B)* $y' + \frac{2y}{x} = 2\sqrt{y}, \ \ y(1) = 0.$

Solution: In this case $\alpha = \frac{1}{2}$. Take $z = y^{1-\alpha} = y^{1-\frac{1}{2}} = \sqrt{y}$. Therefore

$$y = z^2 \quad and \quad y' = 2z{\cdot}z'.$$

Plugging just evaluated relations we obtain the DE

$$2z{\cdot}z' + \frac{2z^2}{x} = 2z.$$

Divide both sides by $2z$ and get

$$z' + \frac{z}{x} = 2. \tag{4}$$

This time we solve the above DE by Integrating Factor Method. The integrating factor μ is a solution of the conjugated DE

$$\mu' - \frac{\mu}{x} = 0.$$

Solving we obtain

$$\begin{aligned} \frac{d\mu}{dx} &= \frac{\mu}{x}, \\ \int \frac{d\mu}{\mu} &= \int \frac{dx}{x}, \\ \ln|\mu| &= \ln|x|, \\ \mu &= x. \end{aligned}$$

Now we have to multiply both sides of (4) by x and end up with

$$(xz)' = xz' + z = 2x.$$

Thus

$$\begin{aligned} xz &= \int 2x dx = x^2 + C, \quad C \ \ is \ \ arbitrary \\ z &= x + \frac{C}{x}, \quad C \ \ is \ \ arbitrary. \end{aligned}$$

The GS of the given DE is

$$y = z^2 = \left(x + \frac{C}{x}\right)^2 \quad C \ \ is \ \ arbitrary.$$

Now for IVP we have $y(1) = 0 = \left(1 + \frac{C}{1}\right)^2$. Therefore $C = -1$.

Answer: The solution of the IVP is $y = z^2 = \left(x - \frac{1}{x}\right)^2$.

EXACT DE

The normal form of DE of the first order is $y' = f(x, y)$, where $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given function of two variables. Assume that $f(x, y) = -\frac{P(x,y)}{Q(x,y)}$, for certain functions $P, Q : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}$ of two variables. Then we have DE in the form $y' = \frac{dy}{dx} = -\frac{P(x,y)}{Q(x,y)}$. This can be rewritten in total differential form

$$P(x, y)\,dx + Q(x, y)\,dy = 0. \tag{5}$$

We shall assume that $P, Q \in C^1(D)$, i.e. they possess continuous partial derivatives on D.

Definition 3 *DE (5) is called EXACT if there is a function $\Phi(x, y) \in C^2(D)$ whose total differential is equal to*

$$d\Phi = P(x, y)\,dx + Q(x, y)\,dy. \tag{6}$$

Such function Φ is called the potential.

In the potential does not exists we call the DE (5) NONEXACT.

Since the total differential $d\Phi = \Phi_x dx + \Phi_y dy$ then the following equalities hold

$$\Phi_x = P \quad and \quad \Phi_y = Q. \tag{7}$$

Therefore

$$P_y = \Phi_{xy} = \Phi_{yx} = Q_x.$$

So we have the following:

Theorem 4 *The DE (5) is exact if and only if the partial derivatives $P_y(x, y) = Q_x(x, y)$.*

Example 5 *Consider DE $(x + y)\,dx + (2y + x)\,dy = 0$. We have $P = x + y$ and $Q = 2x + y$. Then*

$$P_y(x, y) = 1 = Q_x(x, y).$$

So the given DE is exact.

Example 6 *Consider DE $\left(x^2 + y^2 + x\right)dx + (xy)\,dy = 0$. We have $P = \left(x^2 + y^2 + x\right)$ and $Q = xy$. Therefore*

$$P_y(x, y) = 2y \quad and \quad Q_x(x, y) = y$$

and thus $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, so the equation is nonexact.

SOLVING the EXACT DE

How to solve the exact DE (5) ? If we know the potential $\Phi \in C^2(D)$ then the exactness of DE means that for the total variation is

$$d\Phi = 0.$$

Therefore the solution is in implicite form $\Phi(x,y) = C = constant$. Such form is called a prime integral. So the main problem is to find the potential. We illustrate such method by examples.

Example 7 *Consider the exact DE* $(x+y)\,dx + (2y+x)\,dy = 0$. *We have* $P = x+y$ *and* $Q = 2x+y$. *For the potential* Φ *we have to solve the system*

$$\Phi_x = P = x+y \quad and \quad \Phi_y = Q = 2y+x.$$

Integrating the first relation we get

$$\Phi(x,y) = \int (x+y)\,dx = \frac{1}{2}x^2 + xy + C(y). \tag{8}$$

The arbitrary constant depends on y, *because differentiation by* dx *is done fixing* y. *Differentiating (8) by* dy *we obtain the equation*

$$2y + x = \Phi_y = \left(\frac{1}{2}x^2 + xy + C(y)\right)_y = x + C'(y).$$

Thus $C'(y) = 2y$ *and therefore* $C(y) = y^2$. *There is no need to take an arbitrary constant, because we want to find just one potential. Finally*

$$\Phi(x,y) = \frac{1}{2}x^2 + xy + y^2$$

and the solution is in implicite form

$$\Phi(x,y) = \frac{1}{2}x^2 + xy + y^2 = C.$$

If needed, we can determine the function $y = y(x)$ *or* $x = x(y)$.

Remark 8 *In finding the potential we can start with the second equation* $\Phi_y = Q = 2y + x$. *Integrating we have*

$$\Phi(x,y) = \int (2y+x)\,dy = y^2 + xy + D(x). \tag{9}$$

This time the arbitrary constant is dependent on x. *Differentiating (9) by* dx *we obtain the equation*

$$x + y = \Phi_x = \left(y^2 + xy + D(x)\right)_x = y + D'(x).$$

Hence $D'(x) = x$ and therefore we get $D(x) = \frac{1}{2}x^2$. Thus the potential is

$$\Phi(x,y) = y^2 + xy + \frac{1}{2}x^2$$

and we obtain the solution $\Phi(x,y) = y^2 + xy + \frac{1}{2}x^2 = C$.

Example 9 *Solve the DE*

$$\left(3x^2 + 4xy^2\right)dx + \left(2y - 3y^2 + 4x^2y\right)dy = 0.$$

Solution: Here

$$P = 3x^2 + 4xy^2 \quad and \quad Q = 2y - 3y^2 + 4x^2y.$$

Therefore

$$P_y = 8xy \quad and \quad Q_x = 8xy.$$

Since

$$P_y = 8xy = Q_x,$$

then the given DE is exact. To find the potential Φ we have to solve the system

$$\Phi_x = P = 3x^2 + 4xy^2 \quad and \quad \Phi_y = Q = 2y - 3y^2 + 4x^2y.$$

Integrating the first relation by dx we obtain

$$\Phi(x,y) = \int \left(3x^2 + 4xy^2\right)dx = x^3 + 2x^2y^2 + C(y) \tag{10}$$

with "arbitrary constant" depending on y. Differentiating (10) by dy we obtain the equation

$$2y - 3y^2 + 4x^2y = \Phi_y = \left(x^3 + 2x^2y^2 + C(y)\right)_y = 4xy^2 + C'(y).$$

Thus simplifying we get

$$C'(y) = 2y - 3y^2$$

and hence

$$C(y) = \int \left(2y - 3y^2\right)dy = y^2 - y^3.$$

So we have obtained

$$\Phi(x,y) = x^3 + 2x^2y^2 + y^2 - y^3$$

and the solution is in the implicite form

$$\Phi(x,y) = x^3 + 2x^2y^2 + y^2 - y^3 = C.$$

Solving, we can evaluate $y = y(x)$ or $x = x(y)$, but the formulas are complicated.

NONEXACT DE

Recall that the DE (5)

$$P(x,y)\,dx + Q(x,y)\,dy = 0.$$

is nonexact if

$$P_y(x,y) \neq Q_x(x,y).$$

In certain situations it can be solved by, so called, integrating factor. By integrating factor we mean the function $\mu(x,y) \neq 0$ such that the equivalent DE

$$(\mu(x,y)\,P(x,y))\,dx + (\mu(x,y)\,Q(x,y))\,dy = 0 \tag{11}$$

is already exact. Note that the form (11) comes from (5) by multiplying both sides by $\mu(x,y)$. The integrating factor can be found in two important cases:

Theorem 10 *Assume that* $P, Q \in C^2(D)$.

a) if the function $\frac{P_y - Q_x}{Q}$ *is the function of* x *alone then as the integrating factor*

we can choose

$$\mu(x) = \exp\left(\int \frac{P_y - Q_x}{Q}dx\right).$$

b) if the expression $\frac{P_y - Q_x}{P}$ *is the function of* y *alone then as the integrating factor*

one can choose

$$\mu(y) = \exp\left(-\int \frac{P_y - Q_x}{P}dy\right).$$

Example 11 *Find the integrating factor and solve the DE*

1) $ydx - xdy = 0$;

2) $ydx + (2x - y^2)\,dy = 0$.

<u>Solution:</u> 1) We have $P = y$, $Q = -x$. Then $P_y = 1$ and $Q_x = -1$. So $P_y \neq Q_x$ and the given equation is nonexact. Calculating $\frac{P_y - Q_x}{Q}$ we obtain $\frac{P_y - Q_x}{Q} = \frac{1-(-1)}{-x} = -\frac{2}{x}$ $(x > 0)$ and this is the situation a). Therefore we have the integrating factor

$$\mu(x) = \exp\left(\int \frac{P_y - Q_x}{Q}dx\right) = \exp\left(-\int \frac{2}{x}dx\right) = \exp(-2\ln x) = \frac{1}{x^2}.$$

Multiplying both sides of the given DE by $\frac{1}{x^2}$ we get

$$\frac{y}{x^2}dx - \frac{1}{x}dy = 0.$$

This DE is already exact. Let us check this. We have

$$\widetilde{P} = \frac{y}{x^2} \quad \textit{and} \quad \widetilde{Q} = -\frac{1}{x}.$$

Hence

$$\widetilde{P}_y = \frac{1}{x^2} = \widetilde{Q}_x,$$

what gives the exactness. Now, in the new situation, we need to find the potential Φ such that

$$\Phi_x = \widetilde{P} = \frac{y}{x^2} \quad \textit{and} \quad \Phi_y = \widetilde{Q} = -\frac{1}{x}.$$

Starting with the second relation we have

$$\Phi = \int \left(-\frac{1}{x}\right) dy = -\frac{y}{x} + C(x),$$

where the "arbitrary" constant depends on x. Differentiating Φ by dx we obtain the equation

$$\frac{y}{x^2} = \widetilde{P} = \Phi_x = \frac{y}{x^2} + C'(x).$$

Hence $C'(x) = 0$ and we can take $C = 0$. Finally the potential $\Phi = \frac{y}{x^2}$ and the solutions are

$$\Phi = \frac{y}{x^2} = K.$$

Equivalently $y = Kx^2$, where K is arbitrary constant.

Solution: 2) $ydx + (2x - y^2)\,dy = 0$

Solution 12 *We have* $P = y$, $Q = 2x - y^2$. *Then* $P_y = 1$ *and* $Q_x = 2$. *So* $P_y \neq Q_x$ *and the given equation is nonexact. Calculating we have*

$$\frac{P_y - Q_x}{Q} = \frac{1-2}{x^2 - y} = \frac{1}{y - x^2}$$

and

$$\frac{P_y - Q_x}{P} = \frac{1-2}{y} = -\frac{1}{y} \quad (y > 0).$$

This time we have the situation (b) *and therefore the integrating factor is*

$$\mu(y) = \exp\left(\int \frac{1}{y}dy\right) = \exp(\ln y) = y.$$

Multiply the both sides of the given DE by $\mu(y) = y$ *we get*

$$y^2dx + (2xy - y^3)\,dy = 0.$$

This DE is already exact. Let us check this. We have

$$\widetilde{P} = y^2 \quad \text{and} \quad \widetilde{Q} = 2xy - y^3.$$

Hence

$$\widetilde{P}_y = 2y = \widetilde{Q}_x,$$

what gives the exactness. Now, in the new situation, we need to find the potential Φ *such that*

$$\Phi_x = \widetilde{P} = y^2 \quad \text{and} \quad \Phi_y = \widetilde{Q} = 2xy - y^3.$$

Starting with the first relation we have

$$\Phi = \int y^2 dx = y^2 x + C(y),$$

where the "arbitrary" constant depends on y*. Differentiating* Φ *by* dy *we obtain the equation*

$$2xy - y^3 = \widetilde{Q} = \Phi_y = 2yx + C'(y).$$

Hence $C'(y) = -y^3$ *and therefore* $C = \int \left(-y^3\right) dy = \frac{-y^4}{4}$*. Finally we pbtain the potential* $\Phi = y^2 x - \frac{y^4}{4}$ *and the solutions are*

$$\Phi = y^2 x - \frac{y^4}{4} = K,$$

where K *is an arbitrary constant.*