Example 1. $\begin{cases} y_1'(x) = y_2(x) + x \\ y_2'(x) = -y_1(x) - x \end{cases}$

Matrix form: $R(x) = \begin{bmatrix} x \\ -x \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = A$$

Characteristic eq: $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0 \equiv \lambda = \pm i$

For $\lambda = -i$ we get:

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad i \cdot v_1 + v_2 = 0 \quad \text{taking } v_1 = 1 \quad v_2 = -i$$

$$v = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\tilde{y}(x) = e^{-ix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = (\cos x - i \sin x) \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} \cos x \\ \sin x \end{bmatrix} + i \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix}$$

using polar form of complex numb.

Fundamental matrix: $\begin{bmatrix} \underbrace{\begin{matrix}\cos x \\ \sin x\end{matrix}}_{Re\,\tilde{y}} & \underbrace{\begin{matrix}-\sin x \\ \cos x\end{matrix}}_{Im\,\tilde{y}(x)} \end{bmatrix}$

Nonhomogeneous case:

$$\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \cdot \begin{bmatrix} c_1'(x) \\ c_2'(x) \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix} = R(x)$$

Using Cramer's method:

$$c_1'(x) = \frac{\begin{vmatrix} x & -\sin x \\ -x & \cos x \end{vmatrix}}{\begin{vmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{vmatrix}} = x\cos x - x \sin x$$

$$c_2'(x) = \frac{\begin{vmatrix} \cos x & x \\ \sin x & -x \end{vmatrix}}{1} = -x\cos x - x \sin x$$