Euler homogeneous eq.

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Idea:

Eul. eq.: $ax^2y'' + bxy' + cy = 0$

notation: $\frac{dy}{dt} = \dot{y}$

$x = e^t$

$y'(x) = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\cdot\dot{y} = \frac{1}{x}\dot{y}$

By chain rule:

Statement:

If $y = f(u)$ and $u = g(x)$, then this abbreviated form is written in Leibniz notation as:

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}.$$ [1]

For the second derivative we are getting:

$$\begin{aligned} y''(x) &= \frac{dy'}{dx} = \frac{dy'}{dt}\frac{dt}{dx} = \left[\frac{d}{dt}\left(e^{-t}\dot{y}\right)\right]e^{-t} = \left[\left(-e^{-t}\right)\dot{y} + e^{-t}\ddot{y}\right]e^{-t} \\ &= e^{-2t}\left(-\dot{y}+\ddot{y}\right) = \frac{1}{x^2}\left(\ddot{y}-\dot{y}\right), \end{aligned}$$

(From the 05.05 lecture)

Plugging both derivatives into our eq. we get:

$$a\cancel{x^2}\cdot\underbrace{\frac{1}{\cancel{x^2}}(\ddot{y}-\dot{y})}_{y''} + b\cancel{x}\underbrace{\frac{1}{\cancel{x}}\dot{y}}_{y'} + c\cdot y = 0$$

As you can see we have second order diff. eq. with constant coefficients.

Example: $2x^2y''+3xy'-15y=0$, $y(1)=0$, $y'(1)=1$.

Let us put: $x=e^t$. Then:

$$y'(x)=\frac{dy}{dx}=\frac{dy}{dt}\frac{dt}{dx}=e^{-t}\dot{y}=\frac{1}{x}\dot{y},$$

$$\begin{aligned} y''(x) &= \frac{dy'}{dx}=\frac{dy'}{dt}\frac{dt}{dx}=\left[\frac{d}{dt}\left(e^{-t}\dot{y}\right)\right]e^{-t}=\left[\left(-e^{-t}\right)\dot{y}+e^{-t}\ddot{y}\right]e^{-t} \\ &= e^{-2t}\left(-\dot{y}+\ddot{y}\right)=\frac{1}{x^2}\left(\ddot{y}-\dot{y}\right). \end{aligned}$$

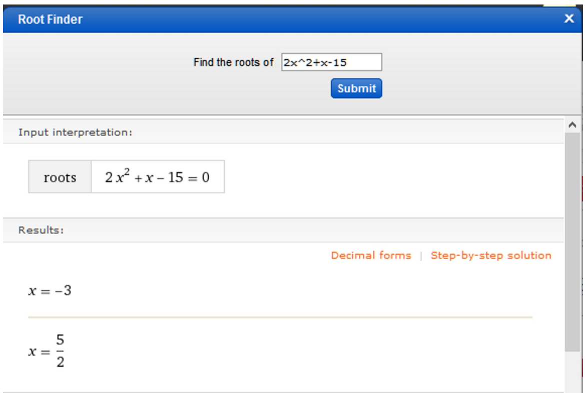
We can rewrite our eq:

$$2\cancel{x^2}\cdot\frac{1}{\cancel{x^2}}(\ddot{y}-\dot{y})+3\cancel{x}\frac{1}{\cancel{x}}\dot{y}-15y=0$$

$2\ddot{y}+\dot{y}-15y=0$ (We are looking for $y(t)$).

Char. eq:

$2\lambda^2+\lambda-15=0$



So $y_1(t)=e^{-3t}$, $y_2(t)=e^{\frac{5}{2}t}$

$\quad\quad\; = (e^t)^{-3}$, $\quad = (e^t)^{\frac{5}{2}}$

So: $y(x)=C_1\cdot x^{-3}+C_2x^{\frac{5}{2}}$ $\quad C_1,C_2\in\mathbb{R}$ $\quad x>0$