7 tutorials 2nd order variation o parameters

wtorek, 21 kwietnia 2020 08:48

wtorek, 26 listopada 2019
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Example $y'' - 2y' + y = \frac{e^x}{x^2+1}$

1 Step: We are looking for hom. solution i.e.

$$y'' - 2y' + y = 0$$

Ch. eq: $\lambda^2 - 2\lambda + 1 = 0 \quad (\lambda - 1)^2 = 0$

$\lambda = 1$ is a double root.

Hence $y_0(x) = C_1 e^x + C_2 x e^x, \quad C_1, C_2 \in \mathbb{R}$

2 Step: For particular solution, for non-homogeneous problem, we will use VARIATION OF PARAMETERS method: i.e:

$y_p = C_1(x) \underbrace{e^x}_{y_1} + C_2(x) \underbrace{x e^x}_{y_2}$, where

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^x}{x^2+1} \end{bmatrix}$$

$\Downarrow$

we have formula for derivatives of C_1, C_2

In our case:

$$\begin{bmatrix} e^x & x e^x \\ e^x & e^x + x e^x \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{e^x}{x^2+1} \end{bmatrix}$$

$$\begin{bmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{bmatrix} \cdot \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{e^x}{x^2+1} \end{bmatrix}$$

General case [edit]

Consider a system of n linear equations for n unknowns, represented in matrix multiplication form as follows:

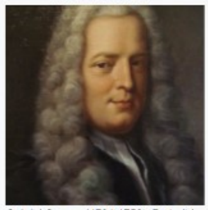
$$Ax = b$$

where the $n \times n$ matrix A has a nonzero determinant, and the vector $x = (x_1, \ldots, x_n)^{\mathrm{T}}$ is the column vector of the variables. Then the theorem states that in this case the system has a unique solution, whose individual values for the unknowns are given by:

$$x_i = \frac{\det(A_i)}{\det(A)} \qquad i = 1, \ldots, n$$

where A_i is the matrix formed by replacing the i-th column of A by the column vector b.

Gabriel Cramer



Gabriel Cramer (1704-1752). Portrait by Robert Gardelle, year unknown.

Using CRAMER'S RULE we get:

$$C_1'(x) = \frac{\begin{vmatrix} 0 & xe^x \\ \frac{e^x}{x^2+1} & e^x+xe^x \end{vmatrix}}{\begin{vmatrix} e^x & xe^x \\ e^x & e^x+xe^x \end{vmatrix}} = -\frac{x\cancel{e^{2x}}}{x^2+1} \cdot \frac{1}{\cancel{e^{2x}}} = \frac{x}{x^2+1}$$

$$e^{2x} + xe^{2x} - xe^{2x} = e^{2x}$$

$$C_2'(x) = \frac{\begin{vmatrix} e^x & 0 \\ e^x & \frac{e^x}{x^2+1} \end{vmatrix}}{e^{2x}} = \frac{\frac{e^{2x}}{x^2+1}}{e^{2x}} = \frac{1}{x^2+1}$$

$$C_1(x) = \int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1)$$

$$C_2(x) = \operatorname{arctg} x = \tan^{-1}(x)$$

Answ: $y(x) = y_0(x) + y_p(x) = C_1 \cdot e^x + C_2 xe^x + \frac{1}{2}\ln(x^2+1)e^x + \operatorname{arctg} x \cdot xe^x$