9-10 tutorials systems

niedziela, 8 grudnia 2019 17:05

Układy równań różniczkowych

Double eigenvalues $\vec{y}' = A\vec{y}$

$X_1(t) = e^{\lambda t} \cdot v^{(1)}$

$X_2(t) = te^{\lambda t} \cdot v^{(1)} + e^{\lambda t} \cdot w$, where w solves $(A - \lambda I)w = v^{(1)}$.

Uwaga. W notacji wektorowo-macierzowej układ równań (UL) przyjmuje post

$$\underbrace{\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix}}_{\vec{y}'} = \underbrace{\begin{bmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \dots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \dots & a_{nn}(t) \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{\vec{y}} + \underbrace{\begin{bmatrix} h_1(t) \\ h_2(t) \\ \vdots \\ h_n(t) \end{bmatrix}}_{\vec{h}(t)}$$

lub krótko

$$\vec{y}' = A(t)\vec{y} + \vec{h}(t).$$

Macierz $A(t)$ nazywamy macierzą tego układu.

○ **Ćwiczenie 3.1.11**

Rozwiązać podane zagadnienia początkowe:

a) $\begin{cases} x' = x \\ y' = x + y \end{cases}$, $\begin{cases} x(1) = e \\ y(1) = -e \end{cases}$; **b)** $\begin{cases} x' = 2xy \\ y' = y^2 \end{cases}$, $\begin{cases} x(0) = 1 \\ y(0) = 1 \end{cases}$;

c) $\begin{cases} x' = \frac{3}{2}x + \frac{1}{2}y \\ y' = \frac{1}{2}x + \frac{3}{2}y \end{cases}$, $\begin{cases} x(0) = 2 \\ y(0) = 1 \end{cases}$; **d)** $\begin{cases} x' = x^3 \\ y' = \frac{1}{yx^2} \end{cases}$, $\begin{cases} x(0) = \frac{1}{\sqrt{2}} \\ y(0) = -\sqrt{2} \end{cases}$.

○ **Ćwiczenie 3.4.11**

Rozwiązać układ równań różniczkowych $\vec{y}' = A\vec{y}$, jeżeli macierz A, o rzeczywistych wartościach własnych ma postać:

a) $\begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$; **b)** $\begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix}$; **c)** $\begin{bmatrix} -3 & 2 \\ -3 & 4 \end{bmatrix}$;

d) $\begin{bmatrix} 3 & -2 & 4 \\ -2 & 0 & -2 \\ 4 & 2 & 3 \end{bmatrix}$; **e)** $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$; **f)** $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & -7 \\ 0 & 2 & -4 \end{bmatrix}$.

$\Delta > 0$

○ **Ćwiczenie 3.4.14**

Rozwiązać układ równań różniczkowych $\vec{y}' = A\vec{y}$, jeżeli macierz A, o zespolonych wartościach własnych ma postać:

(*) **a)** $\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}$; **b)** $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$;

c) $\begin{bmatrix} 3 & 2 \\ -5 & 1 \end{bmatrix}$; **d)** $\begin{bmatrix} 4 & -2 \\ 5 & 2 \end{bmatrix}$;

e) $\begin{bmatrix} -2 & -8 \\ 1 & 2 \end{bmatrix}$; **f)** $\begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}$;

g) $\begin{bmatrix} 1 & -2 & -1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 3 & 2 & 0 \\ -1 & -1 & 2 & 1 \end{bmatrix}$; **h)** $\begin{bmatrix} 0 & 3 & -2 & 0 \\ 1 & -2 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 1 & -4 & 1 & 2 \end{bmatrix}$.

$\Delta < 0$

$$\begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix} \quad (*) \quad \begin{vmatrix} 2-\lambda & -1 \\ 5 & -2-\lambda \end{vmatrix} = -4 - 2\lambda + 2\lambda + \lambda^2 + 5 = \lambda^2 + 1 = 0$$

$\lambda = i$

$$\begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$v_1 = t \quad v_2 = i(2-i)t \qquad \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$

$$e^{i \cdot t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = (\cos t + i \sin t) \begin{bmatrix} 1 \\ 2-i \end{bmatrix} =$$

$$= \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} + i \begin{bmatrix} \sin t \\ -\cos t + 2\sin t \end{bmatrix}$$

$y_1 = C_1 \cos t + i \sin t$

$y_2 = C_1(\cos t + 2\sin t) + C_2(-\cos t + 2\sin t)$

$$\vec{y}'(t) = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{matrix} 2y_1 - y_2 = y_1' \\ 5y_1 - 2y_2 = y_2' \end{matrix}$$

$y_1' = -C_1 \cos t + C_2 \cos t$

$y_2' = -2C_1 \sin t + C_2 \cos t + C_2 \sin t + 2C_2 \cos t$

$2C_1 \cancel{\cos t} + 2C_2 \cancel{\sin t} - \cancel{2C_1 \cos t} - C_1 \sin t + C_2 \cos t - \cancel{2C_2 \sin t}$

Δ

$$\begin{pmatrix} 3 & 9 \\ -4 & -3 \end{pmatrix} \qquad \begin{pmatrix} 3-\lambda & 9 \\ -4 & -3-\lambda \end{pmatrix}$$

http://tutorial.math.lamar.edu/Classes/DE/SystemsIntro.aspx

● **Definicja* 3.4.18** *(uogólnione wektory własne macierzy)*

Niech A będzie rzeczywistą macierzą kwadratową i niech λ będzie jej wartością własną krotności k, tj. k krotnym pierwiastkiem wielomianu charakterystycznego tej macierzy. Niezerowy wektor $\vec{v}$ nazywamy uogólnionym wektorem własnym macierzy A odpowiadającym wartości własnej λ, jeżeli spełnia warunek

$$(A - \lambda I)^k \vec{v} = \vec{0}.$$

Uwaga*. Dla każdej k-krotnej wartości własnej macierzy kwadratowej istnieje dokładnie k liniowo niezależnych uogólnionych wektorów własnych. Zbiór tych wektorów nazywamy *serią* wektorów własnych odpowiadających wartości własnej. Jeżeli λ jest jednokrotną wartością własną macierzy, to odpowiadajacy jej wektor własny jest również jej wektorem uogólnionym.

○ **Ćwiczenie* 3.4.19**

Wyznaczyć wielokrotne wartości własne i odpowiadające im uogólnione wektory własne podanych macierzy:

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; **b)** $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$;

c) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$; **d)** $\begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix}$.

● **Fakt* 3.4.20** *(wielokrotne wartości własne; metoda Eulera)*

Jeżeli λ jest k-krotną rzeczywistą wartością własną macierzy A układu jednorodnego równań różniczkowych o stałych współczynnikach (US), to każda z k funkcji wektorowych

$$e^{\lambda t} B\vec{v}_1,\ e^{\lambda t} B\vec{v}_2,\ \dots,\ e^{\lambda t} B\vec{v}_k,$$

gdzie $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k$ oznacza serię uogólnionych wektorów własnych odpowiadających wartości własnej λ, a B macierz określoną wzorem

$$B = I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \dots + \frac{t^{k-1}}{(k-1)!}(A - \lambda I)^{k-1},$$

jest rozwiązaniem tego układu.

B. The defective case.

If the eigenvalue λ is a double root of the characteristic equation, but the system (2) has only one non-zero solution $\mathbf{v}_1$ (up to constant multiples), then the eigenvalue is said to be **incomplete** or **defective** and $\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1$ is the unique normal mode. However, a second order system needs two independent solutions. Our experience with repeated roots in second order ODE's suggests we try multiplying our normal solution by t. It turns out this doesn't quite work, but it can be fixed as follows: a second independent solution is given by

$$\mathbf{x}_2 = e^{\lambda_1 t}(t\mathbf{v}_1 + \mathbf{v}_2). \tag{3}$$

where $\mathbf{v}_2$ is any vector satisfying

$$(A - \lambda_1 I)\,\mathbf{v}_2 = \mathbf{v}_1\,.$$

(You can easily, if tediously, check by substitution that this does give a solution. You need to remember that $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$.)

Fact. The equation for $\mathbf{v}_2$ is guaranteed to have a solution, provided that the eigenvalue λ_1 really is defective. When solving for $\mathbf{v}_2 = (b_1, b_2)^T$, try setting $b_1 = 0$, and solving for b_2. If that does not work, try setting $b_2 = 0$ and solving for b_1.

Remarks 1. Some people do not bother with (3). When they encounter the defective case (at least when $n = 2$), they give up on eigenvalues, and simply solve the original system (1) by elimination.

2. Although we will not go into it in this course, there is a well developed theory of defective matrices which gives insight into where this formula comes from. You will learn about all this when you study linear algebra.

We will now do a worked example.

$\Delta = 0$

0 method: Transforming into second order d. eq.

$x' = x$

$y' = x + 1$

Nonhomogeneous Linear Systems of Differential Equations with Constant Coefficients

Objective: Solve

$$\boxed{\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t),} \longrightarrow \text{main problem}$$

where A is an $n \times n$ constant coefficient matrix A and $\vec{\mathbf{f}}(t) = \begin{bmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{bmatrix}$ is a given vector function.

$\neq 0$ nonhomogeneous system

The unknown is $\vec{\mathbf{x}}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$.

I method:

Solution Formula Using Fundamental Matrix: Suppose that $M(t)$ is a fundamental matrix solution of the corresponding homogeneous system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$; in other words,

- $M(t)$ satisfies $M'(t) = AM(t)$; that is, every column of $M(t)$ solves the homogeneous system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$;
- $M(t)$ is an invertible matrix for every t; that is, the n columns of $M(t)$ are linearly independent.

The general solutions of the nonhomogeneous system $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$ are

General sol. ←

$$\boxed{\vec{\mathbf{x}}(t) = M(t)\begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} + \int^t M(t)M(s)^{-1}\vec{\mathbf{f}}(s)ds,}$$

direct formula

where $C_1, \cdots, C_n$ are arbitrary constants.

The solution of the initial value problem

initial value problem

$$\boxed{\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0}$$

is given by

initial v. sol. ←

$$\boxed{\vec{\mathbf{x}}(t) = M(t)M(t_0)^{-1}\vec{\mathbf{x}}_0 + \int_{t_0}^t M(t)M(s)^{-1}\vec{\mathbf{f}}(s)ds.}$$

A-matrix.

II method:

Solution Formula Using Matrix Exponential: The general solutions of the nonhomogeneous system $d\vec{\mathbf{x}}/dt = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t)$ are

review of algebra

$$\vec{\mathbf{x}}(t) = e^{tA}\begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} + \int^t e^{(t-s)A}\vec{\mathbf{f}}(s)ds,$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

where $C_1, \cdots, C_n$ are arbitrary constants.

The solution of the initial value problem

$$\vec{\mathbf{x}}(t) = e^{tA}\begin{bmatrix} C_1 \\ \vdots \\ C_n \end{bmatrix} + \int^t e^{(t-s)A}\vec{\mathbf{f}}(s)ds,$$

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

where $C_1, \cdots, C_n$ are arbitrary constants.

The solution of the initial value problem

$$\frac{d\vec{\mathbf{x}}}{dt} = A\vec{\mathbf{x}} + \vec{\mathbf{f}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$$

decoupling - oddzielanie

$\vec{x}(t) = Pu(t)$

$A = PDP^{-1}$

$x'(t) = A(t)\cdot x(t) + \vec{f}(t)$

$x'(t) = PDP^{-1}\cdot P\cdot u(t)) + \vec{f}(t)$

I

$x'(t) = PD\cdot u(t) + \vec{f}(t)$

$Pu'(t) = PDu(t) + \vec{f}(t) \quad / \cdot P^{-1}$

$u'(t) = Du(t) + P^{-1}\vec{f}(t)$

is given by

$$\vec{\mathbf{x}}(t) = e^{(t-t_0)A}\vec{\mathbf{x}}_0 + \int_{t_0}^t e^{(t-s)A}\vec{\mathbf{f}}(s)ds.$$

II method:

Solution Method by Decoupling: If A is diagonalizable (i.e., $A = PDP^{-1}$ with an invertible P and a diagonal D), then the system can be decoupled by setting $\vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t)$. The system for $\vec{\mathbf{u}}(t)$ becomes

we know P

$u = ?$

$$\frac{d\vec{\mathbf{u}}}{dt} = D\vec{\mathbf{u}} + P^{-1}\vec{\mathbf{f}}(t).$$

EXAMPLE. Solve the initial value probelm

$P, \infty ?$

$$\frac{d\vec{\mathbf{x}}}{dt} = \begin{bmatrix} 6/7 & -15/14 \\ -5/7 & 37/14 \end{bmatrix}\vec{\mathbf{x}} + \begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix}, \quad \vec{\mathbf{x}}(0) = \begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

$$\frac{dx}{dt} = \frac{1}{14}\begin{bmatrix} 12 & -15 \\ -10 & 37 \end{bmatrix}\cdot\vec{x} + \begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix}$$

$$\begin{vmatrix} 12-\lambda & -15 \\ -10 & 37-\lambda \end{vmatrix} = 0$$

$$(12-\lambda)(37-\lambda) + 150 = 0$$

Solution 1 (Use a fundamental matrix): First find eigenvalues and eigenvectors of A.

The eigenvalues of A are $\lambda_1 = 1/2, \lambda_2 = 3$.

Vector $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is an eigenvector associated with $\lambda_1 = 1/2$.

Vector $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is an eigenvector associated with $\lambda_2 = 3$.

Thus,

$$M(t) = [e^{\lambda_1 t}\mathbf{v}_1 \quad e^{\lambda_2 t}\mathbf{v}_2] = \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}$$

is a fundamental matrix for the homogeneous system $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$.

The solution to the initial value problem is given by

$$\begin{aligned}
\vec{\mathbf{x}}(t) &= M(t)M(0)^{-1}\vec{\mathbf{x}}(0) + \int_0^t M(t)M(s)^{-1}\vec{\mathbf{f}}(s)ds \\
&= \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -1 \end{bmatrix} + \int_0^t \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\begin{bmatrix} 3e^{s/2} & -e^{3s} \\ e^{s/2} & 2e^{3s} \end{bmatrix}^{-1}\begin{bmatrix} e^{2s} \\ e^{-s} \end{bmatrix} ds \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \int_0^t \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\frac{1}{7}\begin{bmatrix} 2e^{3s/2} + e^{-3s/2} \\ -e^{-s} + 3e^{-4s} \end{bmatrix} ds \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \begin{bmatrix} 3e^{t/2} & -e^{3t} \\ e^{t/2} & 2e^{3t} \end{bmatrix}\frac{1}{7}\begin{bmatrix} \frac{4}{3}(e^{3t/2} - 1) - \frac{2}{3}(e^{-3t/2} - 1) \\ (e^{-t} - 1) - \frac{3}{4}(e^{-4t} - 1) \end{bmatrix} \\
&= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} - \frac{2}{7}e^{t/2} + \frac{1}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} - \frac{2}{21}e^{t/2} - \frac{1}{14}e^{3t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} + \frac{19}{7}e^{t/2} + \frac{29}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} + \frac{19}{21}e^{t/2} - \frac{29}{14}e^{3t} \end{bmatrix}.
\end{aligned}$$

$$\begin{cases} y_1'(x) = y_2(x) + x \\ y_2'(x) = -y_1(x) - x \end{cases} \qquad A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Leftrightarrow \lambda^2 + 1 = 0 \Leftrightarrow \lambda = \pm i$$

For $\lambda = -i$ we get $\begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $v_1 \cdot i + v_2 = 0$

$v_2 = t \quad v_1 \cdot i + t = 0 \quad v_1 \cdot i = -t \;/ i \quad v_1(-1) = -it \quad v_1 = i\cdot t \quad v = \begin{bmatrix} i \\ 1 \end{bmatrix}$

which is alway conjugate of the first one

$\lambda = i \quad \begin{bmatrix} -i & 1 \\ 1 & -i \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow -w_1 i + w_2 = 0$

$w_2 = t \quad w_1 i = t \quad w_1 = -ti$

$w = \begin{bmatrix} -i \\ 1 \end{bmatrix} \qquad y_1(x) = e^{-ix}\cdot\begin{bmatrix} i \\ 1 \end{bmatrix} = (\cos x - i\sin x)\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix} + i\begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$

$y_1^1(x) = C_1 \sin x + C_2 \cos x \quad \leftarrow$ From $\vec{v}$

$y_1^2(x) = C_1 \cos x - C_2 \sin x$

For $\vec{w} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ we get $e^{ix}\cdot\begin{bmatrix} -i \\ 1 \end{bmatrix} = (\cos x + i\sin x)\begin{bmatrix} -i \\ 1 \end{bmatrix} =$

$= \begin{bmatrix} -\sin x \\ \cos x \end{bmatrix} + i\begin{bmatrix} -\cos x \\ \sin x \end{bmatrix} \qquad M(t) = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix}$

$|M(x)| = -\sin^2 x - \cos^2 x = -1$

$$\begin{bmatrix} \cos x \end{bmatrix} \quad \begin{bmatrix} \sin x \end{bmatrix} \qquad M(t) \quad \begin{bmatrix} \cos x & \sin x \end{bmatrix}$$

$$|M(x)| = -\sin^2 x - \cos^2 x = -1$$

$$D_{M(t)} = \begin{bmatrix} -\sin x & -\cos x \\ -\cos x & \sin x \end{bmatrix} \qquad D^T = \begin{bmatrix} -\sin x & -\cos x \\ -\cos x & \sin x \end{bmatrix}$$

$$M^{-1}(x) = \begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \qquad M^{-1}(s) \cdot \begin{bmatrix} s \\ -s \end{bmatrix} = \begin{bmatrix} s\sin s - s\cos s \\ s\cos s + s\sin s \end{bmatrix}$$

$$\int_0^x M^{-1}(s) \begin{bmatrix} s \\ -s \end{bmatrix} ds = \begin{bmatrix} -x\cos x + \sin x - x\sin x - \cos x \\ x\sin x + \cos x - x\cos x + \sin x \end{bmatrix} \quad \text{de}$$

$$\begin{bmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{bmatrix} \cdot \begin{bmatrix} -x\cos x + \sin x - x\sin x - \cos x \\ x\sin x + \cos x - x\cos x + \sin x \end{bmatrix} = 1 - x$$

$$= -x\cancel{\cos x \sin x} + \sin^2 x - x\sin^2 x - \cancel{\sin x \cos x} + \cancel{x\sin x \cos x} + \cos^2 x - x\cos^2 x + \cancel{\sin x \cos x}$$

$$- x\cos^2 x + \sin x \cos x - \cancel{x\sin x \cos x} - \cos^2 x - x\sin^2 x - \cancel{\sin x \cos x} + \cancel{x\sin x \cos x} - \sin^2 x$$

$$-1 - x \quad \text{de}.$$

Przykład 1.17. Wyznaczymy rozwiązanie układu $\begin{cases} y_1' = 3y_1 - y_2 + 1, \\ y_2' = 2y_1 + x, \end{cases}$ spełniającego warunek początkowy $y_1(0) = 1$, $y_2(0) = -1$.

Rozwiązanie. Rozwiążemy równanie jednorodne nie wyznaczają explicite baz podprzestrzeni niezmienniczych. Wartościami własnymi macierzy $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ są liczby $\lambda_1 = 1$, $\lambda_2 = 2$. Rozwiązanie równania jednorodnego możemy zatem zapisać w postaci $y_1(x) = c_{11}e^x + c_{12}e^{2x}$, $y_2(x) = c_{21}e^x + c_{22}e^{2x}$. Obliczając pochodne tych funkcji i wstawiając do równania jednorodnego, otrzymujemy zwiazki między stałymi $c_{21} = 2c_{11}$ oraz $c_{12} = c_{22}$. Przyjmując $C_1 = c_{11}$ oraz $C_2 = c_{12}$ rozwiązanie układu jednorodnego zapisujemy w postaci $y_1(x) = C_1e^x + C_2e^{2x}$, $y_2(x) = 2C_1e^x + C_2e^{2x}$. Uzmienniamy teraz stałe przyjmując $y_1(x) = C_1(x)e^x + C_2(x)e^{2x}$, $y_2(x) = 2C_1(x)e^x + C_2(x)e^{2x}$. Po zróżniczkowaniu, wstawieniu do równania niejednorodnego i redukcji, otrzymujemy układ równań względem niewiadomych $C_1'(x)$, $C_2'(x)$ postaci

$$\begin{cases} C_1'(x)e^x + C_2'(x)e^{2x} = 1, \\ 2C_1'(x)e^x + C_2'(x)e^{2x} = x. \end{cases}$$

x'(t)=y+t, y'(t)=-x-t

Extended Keyboard Upload

Input:

$\{x'(t) = y(t) + t,\ y'(t) = -x(t) - t\}$

ODE classification:

First-order system of linear differential equations

Alternate form:

$\{y(t) + t = x'(t),\ x(t) + y'(t) + t = 0\}$

Differential equation solutions:

$x(t) = c_2 \sin(t) + c_1 \cos(t) - t + 1$

$y(t) = -c_1 \sin(t) + c_2 \cos(t) - t - 1$

Solution 2 (Use decoupling): The geiven coefficient matrix A is diagonalizable: $A = PDP^{-1}$ with

$$\cancel{P = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}, \qquad D = \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}.}$$

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \quad P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Set $\vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t)$. The system for $\vec{\mathbf{u}}(t)$ becomes

Solution 2 (Use decoupling): The geiven coefficient matrix A is diagonalizable: $A = PDP^{-1}$ with

$$\cancel{P = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}}, \qquad \cancel{D = \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}}.$$

$A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

Set $\vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t)$. The system for $\vec{\mathbf{u}}(t)$ becomes

$$\frac{d\vec{\mathbf{u}}}{dt} = D\vec{\mathbf{u}} + P^{-1}\vec{\mathbf{f}}(t) = \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}\vec{\mathbf{u}} + \begin{bmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{bmatrix}\begin{bmatrix} e^{2t} \\ e^{-t} \end{bmatrix}, \quad \vec{\mathbf{u}}(0) = P^{-1}\begin{bmatrix} 4 \\ -1 \end{bmatrix}.$$

or, equivalently,

$$\begin{aligned} u_1' &= \frac{1}{2}u_1 + \frac{2}{7}e^{2t} + \frac{1}{7}e^{-t}, \quad u_1(0) = 1, \\ u_2' &= 3u_2 - \frac{1}{7}e^{2t} + \frac{3}{7}e^{-t}, \quad u_2(0) = -1. \end{aligned}$$

These two equations can be solved separately (the method of integrating factor and the method of undetermined coefficients both work in this case). The solutions are

$$\begin{aligned} u_1(t) &= \frac{4}{21}e^{2t} - \frac{2}{21}e^{-t} + \frac{19}{21}e^{t/2}, \\ u_2(t) &= \frac{1}{7}e^{2t} - \frac{3}{28}e^{-t} - \frac{29}{28}e^{3t}. \end{aligned}$$

Finally, the solution to the original problem is given by

$$\begin{aligned} \vec{\mathbf{x}}(t) = P\vec{\mathbf{u}}(t) &= P\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} \\ &= \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \frac{4}{21}e^{2t} - \frac{2}{21}e^{-t} + \frac{19}{21}e^{t/2} \\ \frac{1}{7}e^{2t} - \frac{3}{28}e^{-t} - \frac{29}{28}e^{3t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} + \frac{19}{7}e^{t/2} + \frac{29}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} + \frac{19}{21}e^{t/2} - \frac{29}{14}e^{3t} \end{bmatrix}. \end{aligned}$$

Solution 3 (Use matrix exponential): First find e^{tA} using a fundamental matrix: $e^{tA} = M(t)M(0)^{-1}$. Or, one can also find e^{tA} by diagonalization:

$$e^{tA} = Pe^{tD}P^{-1} = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 2/7 & 1/7 \\ -1/7 & 3/7 \end{bmatrix} = \begin{bmatrix} \frac{6}{7}e^{t/2} + \frac{1}{7}e^{3t} & \frac{3}{7}e^{t/2} - \frac{3}{7}e^{3t} \\ \frac{2}{7}e^{t/2} - \frac{2}{7}e^{3t} & \frac{1}{7}e^{t/2} + \frac{6}{7}e^{3t} \end{bmatrix}.$$

The solution to the initial value problem is given by

$$\vec{\mathbf{x}}(t) = e^{tA}\vec{\mathbf{x}}(0) + \int_0^t e^{(t-s)A}\vec{\mathbf{f}}(s)ds$$

$$\begin{aligned} &= \begin{bmatrix} \frac{6}{7}e^{t/2} + \frac{1}{7}e^{3t} & \frac{3}{7}e^{t/2} - \frac{3}{7}e^{3t} \\ \frac{2}{7}e^{t/2} - \frac{2}{7}e^{3t} & \frac{1}{7}e^{t/2} + \frac{6}{7}e^{3t} \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} \\ &\quad + \int_0^t \begin{bmatrix} \frac{6}{7}e^{(t-s)/2} + \frac{1}{7}e^{3(t-s)} & \frac{3}{7}e^{(t-s)/2} - \frac{3}{7}e^{3(t-s)} \\ \frac{2}{7}e^{(t-s)/2} - \frac{2}{7}e^{3(t-s)} & \frac{1}{7}e^{(t-s)/2} + \frac{6}{7}e^{3(t-s)} \end{bmatrix}\begin{bmatrix} e^{2s} \\ e^{-s} \end{bmatrix} ds \\ &= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \int_0^t \begin{bmatrix} \frac{6}{7}e^{t/2+3s/2} + \frac{1}{7}e^{3t-s} + \frac{3}{7}e^{t/2-3s/2} - \frac{3}{7}e^{3t-4s} \\ \frac{2}{7}e^{t/2+3s/2} - \frac{2}{7}e^{3t-s} + \frac{1}{7}e^{t/2-3s/2} + \frac{6}{7}e^{3t-4s} \end{bmatrix} ds \\ &= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} \\ &\quad + \begin{bmatrix} \frac{6}{7}e^{t/2} \cdot \frac{2}{3}(e^{3t/2} - 1) + \frac{1}{7}e^{3t}(1 - e^{-t}) + \frac{3}{7}e^{t/2} \cdot \frac{2}{3}(1 - e^{-3t/2}) - \frac{3}{7}e^{3t} \cdot \frac{1}{4}(1 - e^{-4t}) \\ \frac{2}{7}e^{t/2} \cdot \frac{2}{3}(e^{3t/2} - 1) - \frac{2}{7}e^{3t}(1 - e^{-t}) + \frac{1}{7}e^{t/2} \cdot \frac{2}{3}(1 - e^{-3t/2}) + \frac{6}{7}e^{3t} \cdot \frac{1}{4}(1 - e^{-4t}) \end{bmatrix} \\ &= \begin{bmatrix} 3e^{t/2} + e^{3t} \\ e^{t/2} - 2e^{3t} \end{bmatrix} + \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} - \frac{2}{7}e^{t/2} + \frac{1}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} - \frac{2}{21}e^{t/2} - \frac{1}{14}e^{3t} \end{bmatrix} \\ &= \begin{bmatrix} \frac{3}{7}e^{2t} - \frac{5}{28}e^{-t} + \frac{19}{7}e^{t/2} + \frac{29}{28}e^{3t} \\ \frac{10}{21}e^{2t} - \frac{13}{42}e^{-t} + \frac{19}{21}e^{t/2} - \frac{29}{14}e^{3t} \end{bmatrix}. \end{aligned}$$

EXERCISES

[1] Solve

$$\begin{aligned} x_1' &= 5x_1 - 3x_2 + 8, \\ x_2' &= x_1 + x_2 + 32t, \\ x_1(0) &= 2, x_2(0) = 0. \end{aligned}$$

[2] Solve

$$\begin{aligned} x_1' &= -7x_1 - 9x_2 + 9x_3 + e^{-t}, \\ x_2' &= 3x_1 + 5x_2 - 3x_3 + 2e^{-t} + e^{t}, \\ x_3' &= -3x_1 - 3x_2 + 5x_3 + 3e^{t}, \\ x_1(0) &= 1, x_2(0) = 0, x_3(0) = 0. \end{aligned}$$

[3] Solve

$$\begin{aligned} x_1' &= -5x_1 - 8x_2 + 4x_3, \\ x_2' &= 2x_1 + 3x_2 - 2x_3 + e^{-t}, \\ x_3' &= 6x_1 + 14x_2 - 5x_3 + 9t, \\ x_1(0) &= 2, x_2(0) = 1, x_3(0) = 1. \end{aligned}$$

Answers:

[1] $x_1(t) = -10 - 12t + 3e^{4t} + 9e^{2t}$,
$x_2(t) = -10 - 20t + e^{4t} + 9e^{2t}$

[2] $x_1(t) = -9e^{t} + \frac{4}{3}e^{2t} + \frac{26}{3}e^{-t} + 9te^{-t}$,
$x_2(t) = 2e^{t} + \frac{5}{3}e^{2t} - \frac{11}{3}e^{-t} - 3te^{-t}$,
$x_3(t) = -6e^{t} + 3e^{2t} + 3e^{-t} + 3te^{-t}$

[3] $x_1(t) = -\frac{8}{3} + 4t + (4 + 6t)e^{-t} + (\frac{2}{3} - 18t)e^{-3t}$,
$x_2(t) = \frac{4}{3} - 2t - 2te^{-t} + (-\frac{1}{3} + 9t)e^{-3t}$,
$x_3(t) = \frac{1}{3} + t + (\frac{11}{2} + 2t)e^{-t} + (-\frac{29}{6} + 9t)e^{-3t}$

~~$x+5t$~~ ~~$x+t\times 2$~~

EDDE. PROBLEM SET 7

Solve the following systems of differential equations:

a)
$$\begin{cases} y_1'(x) = y_2(x) + x \\ y_2'(x) = -y_1(x) - x \end{cases}$$

b)
$$\begin{cases} y_1'(x) = y_2(x) + x \\ y_2'(x) = y_1(x) - x \end{cases}$$

c)
$$\begin{cases} y_1'(x) = y_2(x) + e^{x} \\ y_2'(x) = y_1(x) - e^{x} \end{cases}$$

d)
$$\begin{cases} y_1'(x) = y_1(x) + 2y_2(x) \\ y_2'(x) = y_1(x) \end{cases}$$

e)
$$\begin{cases} y_1'(x) = y_1(x) + 2y_2(x) + e^{x} \\ y_2'(x) = y_1(x) - e^{x} \end{cases}$$

f)
$$\begin{cases} y_1'(x) = y_1(x) + 2y_2(x) + e^{-x} \\ y_2'(x) = y_1(x) \end{cases}$$

g)
$$\begin{cases} y_1'(x) = 2y_1(x) + y_2(x) \\ y_2'(x) = -8y_1(x) - 2y_2(x) \end{cases}$$

h)
$$\begin{cases} y_1'(x) = 2y_1(x) + y_2(x) + \sin 3x \\ y_2'(x) = -8y_1(x) - 2y_2(x) \end{cases}$$

a)

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} =$$

$$= \lambda^2 + 1 = 0$$

$$\lambda = i$$

$$y(x) = e^{ix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = e^{ix} \begin{bmatrix} -i \\ 1 \end{bmatrix} =$$

$$-i \quad 1 \quad v_1 \quad v_2$$

$$v_2 = t \quad v_1(-i) = -t \,/\, i$$

$$v_1 = -it$$

$$= (\cos x + i\sin x) \begin{bmatrix} -i \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} +\sin x \\ \cos x \end{bmatrix} + i \begin{bmatrix} -\cos x \\ \sin x \end{bmatrix}$$

$$y_1 = C_1 \cdot \sin x + C_2(-\cos x) \;\wedge\; y_2 = C_1 \cos x + C_2 \sin x$$

$$C_1 \cos x + C_2 \sin x \quad \text{ok}$$

$$\begin{bmatrix} \sin x & -\cos x \\ \cos x & \sin x \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} x \\ -x \end{bmatrix}$$

$$C_1'(x) = \frac{1}{1} \cdot \begin{vmatrix} x & -\cos x \\ -x & \sin x \end{vmatrix} = x\sin x - x\cos x$$

$$1 \quad -\cos x \qquad 1 \quad \sin x$$

$$W = \sin^2 x + \cos^2 x = 1 \neq 0$$

$$C_2'(x) = \frac{1}{1} \begin{vmatrix} \sin x & x \\ \cos x & -x \end{vmatrix} = -x\sin x - x\cos x$$

$$C_1(x) = \underbrace{x(-\cos x) + \sin x}_{x\sin x} - \underbrace{[x\sin x + \cos x]}_{x\cos x}$$

$$C_2(x) = x \ldots\ldots$$

Zastosowania np. zanieczyszczenia

Pond Pollution

Consider three ponds connected by streams, as in Figure 4. The first pond has a pollution source, which spreads via the connecting streams to the other ponds. The plan is to determine the amount of pollutant in each pond.

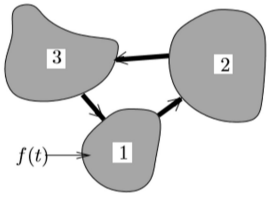

Figure 4. Three ponds 1, 2, 3 of volumes V_1, V_2, V_3 connected by streams. The pollution source $f(t)$ is in pond 1.

Assume the following.

- Symbol $f(t)$ is the pollutant flow rate into pond 1 (lb/min).
- Symbols f_1, f_2, f_3 denote the pollutant flow rates out of ponds 1, 2, 3, respectively (gal/min). It is assumed that the pollutant is well-mixed in each pond.
- The three ponds have volumes V_1, V_2, V_3 (gal), which remain constant.
- Symbols $x_1(t)$, $x_2(t)$, $x_3(t)$ denote the amount (lbs) of pollutant in ponds 1, 2, 3, respectively.

The pollutant flux is the flow rate times the pollutant concentration, e.g., pond 1 is emptied with flux f_1 times $x_1(t)/V_1$. A compartment analysis is summarized in the following diagram.

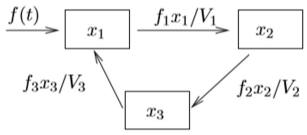

Figure 5. Pond diagram. The compartment diagram represents the three-pond pollution problem of Figure 4.

The diagram plus compartment analysis gives the following differential equations.

$$\begin{aligned} x_1'(t) &= \frac{f_3}{V_3}\,x_3(t) - \frac{f_1}{V_1}\,x_1(t) + f(t), \\ x_2'(t) &= \frac{f_1}{V_1}\,x_1(t) - \frac{f_2}{V_2}\,x_2(t), \\ x_3'(t) &= \frac{f_2}{V_2}\,x_2(t) - \frac{f_3}{V_3}\,x_3(t). \end{aligned}$$

For a specific numerical example, take $f_i/V_i = 0.001$, $1 \le i \le 3$, and let $f(t) = 0.125$ lb/min for the first 48 hours (2880 minutes), thereafter $f(t) = 0$. We expect due to uniform mixing that after a long time there will be $(0.125)(2880) = 360$ pounds of pollutant uniformly deposited, which is 120 pounds per pond.

Initially, $x_1(0) = x_2(0) = x_3(0) = 0$, if the ponds were pristine. The specialized problem for the first 48 hours is

$$\begin{aligned} x_1'(t) &= 0.001\,x_3(t) - 0.001\,x_1(t) + 0.125, \\ x_2'(t) &= 0.001\,x_1(t) - 0.001\,x_2(t), \\ x_3'(t) &= 0.001\,x_2(t) - 0.001\,x_3(t), \\ x_1(0) &= x_2(0) = x_3(0) = 0. \end{aligned}$$

The solution to this system is

$$\begin{aligned} x_1(t) &= e^{-\frac{3t}{2000}}\left(\frac{125\sqrt{3}}{9}\sin\left(\frac{\sqrt{3}t}{2000}\right) - \frac{125}{3}\cos\left(\frac{\sqrt{3}t}{2000}\right)\right) + \frac{125}{3} + \frac{t}{24}, \\ x_2(t) &= -\frac{250\sqrt{3}}{9}e^{-\frac{3t}{2000}}\sin\left(\frac{\sqrt{3}t}{2000}\right) + \frac{t}{24}, \\ x_3(t) &= e^{-\frac{3t}{2000}}\left(\frac{125}{3}\cos\left(\frac{\sqrt{3}t}{2000}\right) + \frac{125\sqrt{3}}{9}\sin\left(\frac{\sqrt{3}t}{2000}\right)\right) + \frac{t}{24} - \frac{125}{3}. \end{aligned}$$

After 48 hours elapse, the approximate pollutant amounts in pounds are

$$x_1(2880) = 162.30, \quad x_2(2880) = 119.61, \quad x_3(2880) = 78.08.$$

It should be remarked that the system above is altered by replacing 0.125 by zero, in order to predict the state of the ponds after 48 hours. The

Home Heating

Consider a typical home with attic, basement and insulated main floor.

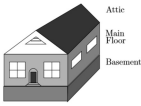

Figure 6. Typical home with attic and basement. The below-grade basement and the attic are un-insulated. Only the main living area is insulated.

It is usual to surround the main living area with insulation, but the attic area has walls and ceiling without insulation. The walls and floor in the basement are insulated by earth. The basement ceiling is insulated by air space in the joists, a layer of flooring on the main floor and a layer of drywall in the basement. We will analyze the changing temperatures in the three levels using Newton's cooling law and the variables

$z(t)$ = Temperature in the attic,
$y(t)$ = Temperature in the main living area,
$x(t)$ = Temperature in the basement,
t = Time in hours.

Initial data. Assume it is winter time and the outside temperature in constantly 35°F during the day. Also assumed is a basement earth temperature of 45°F. Initially, the heat is off for several days. The initial values at noon ($t = 0$) are then $x(0) = 45$, $y(0) = z(0) = 35$.

Portable heater. A small electric heater is turned on at noon, with thermostat set for 100°F. When the heater is running, it provides a 20°F rise per hour, therefore it takes some time to reach 100°F (probably never!). Newton's cooling law

$$\text{Temperature rate} = \text{k(Temperature difference)}$$

will be applied to five boundary surfaces: (0) the basement walls and floor, (1) the basement ceiling, (2) the main floor walls, (3) the main floor ceiling, and (4) the attic walls and ceiling. Newton's cooling law gives positive cooling constants k_0, k_1, k_2, k_3, k_4 and the equations

$$\begin{aligned} x' &= k_0(45 - x) + k_1(y - x), \\ y' &= k_1(x - y) + k_2(35 - y) + k_3(z - y) + 20, \\ z' &= k_3(y - z) + k_4(35 - z). \end{aligned}$$

The insulation constants will be defined as $k_0 = 1/2$, $k_1 = 1/2$, $k_2 = 1/4$, $k_3 = 1/4$, $k_4 = 1/2$ to reflect insulation quality. The reciprocal $1/k$ is approximately the amount of time in hours required for 63% of the temperature difference to be exchanged. For instance, 4 hours elapse for the main floor. The model:

$$\begin{aligned} x' &= \frac{1}{2}(45 - x) + \frac{1}{2}(y - x), \\ y' &= \frac{1}{2}(x - y) + \frac{1}{4}(35 - y) + \frac{1}{4}(z - y) + 20, \\ z' &= \frac{1}{4}(y - z) + \frac{1}{2}(35 - z). \end{aligned}$$

The homogeneous solution in vector form is given in terms of constants $a = (7 - \sqrt{21})/8$, $b = (7 + \sqrt{21})/8$, $c = -1 + \sqrt{21}$, $d = -1 - \sqrt{21}$ and arbitrary constants c_1, c_2, c_3 by the formula

$$\begin{pmatrix} x_h(t) \\ y_h(t) \\ z_h(t) \end{pmatrix} = c_1 e^{-t/2} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + c_2 e^{-at} \begin{pmatrix} 4 \\ c \\ 2 \end{pmatrix} + c_3 e^{-bt} \begin{pmatrix} 4 \\ d \\ 2 \end{pmatrix}.$$

A particular solution is an equilibrium solution

$$\begin{pmatrix} x_p(t) \\ y_p(t) \\ z_p(t) \end{pmatrix} = \begin{pmatrix} \frac{49}{2} \\ \frac{25}{2} \\ \frac{15}{4} \end{pmatrix}.$$

The homogeneous solution has limit zero at infinity, hence the temperatures of the three spaces hover around $x = 57$, $y = 69$, $z = 46$ degrees Fahrenheit. Specific information can be gathered by solving for c_1, c_2, c_3 according to the initial data $x(0) = 45$, $y(0) = z(0) = 35$. The answers are

$$c_1 = -\frac{85}{24}, \quad c_2 = -\frac{25}{24} - \frac{115\sqrt{21}}{168}, \quad c_3 = -\frac{25}{24} + \frac{115\sqrt{21}}{168}.$$

Underpowered heater. To the main floor each hour is added 20°F, but the heat escapes at a substantial rate, so that after one hour $y \approx 51$°F. After five hours, $y \approx 65$°F. The heater in this example is so inadequate that even after many hours, the main living area is still under 69°F.

Forced air furnace. Replacing the space heater by a normal furnace adds the difficulty of **switches** in the input, namely, the thermostat turns off the furnace when the main floor temperature reaches 70°F, and it turns it on again after a 4°F temperature drop. We will suppose that the furnace has four times the BTU rating of the space heater, which translates to an 80°F temperature rise per hour. The study of the forced air furnace requires two differential equations, one with 20 replaced by 80 (DE 1, furnace on) and the other with 20 replaced by 0 (DE 2, furnace off). The plan is to use the first differential equation on time interval $0 \le t \le t_1$, then switch to the second differential equation for time interval $t_1 \le t \le t_2$. The time intervals are selected so that $y(t_1) = 70$ (the thermostat setting) and $y(t_2) = 66$ (thermostat setting less 4 degrees). Numerical work gives the following results.

Time in minutes	Main floor temperature	Model	Furnace
31.6	70	DE 1	on
40.9	66	DE 2	off
45.3	70	DE 1	on
55.0	66	DE 2	off

The reason for the non-uniform times between furnace cycles can be seen from the model. Each time the furnace cycles, heat enters the main floor, then escapes through the other two levels. Consequently, the initial conditions applied to models 1 and 2 are changing, resulting in different solutions to the models on each switch.

Coupled Spring-Mass Systems

Three masses are attached to each other by four springs as in Figure 11.

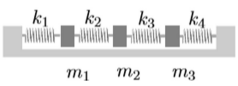

Figure 11. Three masses connected by springs. The masses slide along a frictionless horizontal surface.

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