# LECTURE 26. 05. 2020

# THE LAPLACE TRANSFORM

Today we start another method for solving the LDE's. This is called the Laplace Transform Method.

Given a function $y = y(x) : [0, \infty) \longrightarrow \mathbb{R}$. Recall that by the improper integral of the first kind we mean the limit

$$\int_0^{\infty} y(x)\, dx = \lim_{b \longrightarrow +\infty} \int_0^b y(x)\, dx.$$

We say that it is convergent, if the limit exists and is a number. Otherwise we say that the improper integral is divergent.

**Example 1** *a)* $\int_0^{\infty} e^{-3x} dx = \lim_{b \longrightarrow +\infty} \int_0^b e^{-3x} dx = \lim_{b \longrightarrow +\infty} \left[ \frac{e^{-3x}}{-3} \Big|_0^b \right. = \lim_{b \longrightarrow +\infty} \left[ \frac{e^{-3b}}{-3} - \frac{1}{-3} \Big|_0^b \right. = \frac{1}{3}$, *convergent;*

*b)* $\int_0^{\infty} e^{3x} dx = \lim_{b \longrightarrow +\infty} \int_0^b e^{3x} dx = \lim_{b \longrightarrow +\infty} \left[ \frac{e^{3x}}{3} \Big|_0^b \right. = \lim_{b \longrightarrow +\infty} \left[ \frac{e^{3b}-1}{3} \right] = \infty$, *divergent;*

*c) fix* $s \in \mathbb{R} \setminus \{0\}$. *Then*

$$\int_0^{\infty} e^{-sx} dx = \lim_{b \longrightarrow +\infty} \int_0^b e^{-sx} dx = \lim_{b \longrightarrow +\infty} \left[ \frac{e^{-sx}}{-s} \Big|_0^b \right. = \lim_{b \longrightarrow +\infty} \left[ \frac{e^{-sb}}{-s} - \frac{1}{-s} \Big|_0^b \right. = \begin{cases} \frac{1}{s} & if \quad s > 0, \\ \infty & if \quad s < 0. \end{cases}$$

**Definition 2** *Given a function* $y = y(x) : [0, \infty) \longrightarrow \mathbb{R}$. *The Laplace transform of* $y$ *we mean the function* $Y(s) = L[y(x)]$ *defined by*

$$Y(s) = L[y(x)] = \int_0^{\infty} e^{-sx} y(x)\, dx.$$

*By the domain of* $y$ *we mean the set* $D_Y = \{s : Y(s) \quad is \quad finite\}$.

**Example 3** *a) Let* $y(x) = 1,\ x \geq 0$. *Then* $Y(s) = L[1] = \int_0^{\infty} e^{-sx} \cdot (1)\, dx = \frac{1}{s}$ *for* $s > 0$. *Domain of* $Y(s)$ *is* $D_Y = \{s : s > 0\} = (0, \infty)$.

*b) Let* $y(x) = e^{2x},\ x \geq 0$. *Then* $Y(s) = L\left[e^{2x}\right] = \int_0^{\infty} e^{-sx} \cdot \left(e^{2x}\right) dx = \int_0^{\infty} e^{-(s-2)x} dx = \frac{1}{s-2}$ *for* $s > 2$. *Domain of* $Y(s)$ *is* $D_Y = \{s : s > 2\} = (2, \infty)$.

*c) Let* $y(x) = e^{ax},\ x \geq 0$. *Ten* $Y(s) = L[e^{ax}] = \int_0^{\infty} e^{-sx} \cdot (e^{ax})\, dx = \int_0^{\infty} e^{-(s-a)x} dx = \frac{1}{s-a}$ *for* $s > a$. *Domain of* $Y(s)$ *is* $D_Y = \{s : s > a\} = (a, \infty)$.

*d) Let* $y(x) = 0,\ x \geq 0$. *Then* $Y(s) = L[0] = \int_0^{\infty} e^{-sx} \cdot (0)\, dx = \int_0^{\infty} 0 dx = 0$ *for* $s \in \mathbb{R}$. *Domain of* $Y(s)$ *is* $D_Y = \mathbb{R}$.

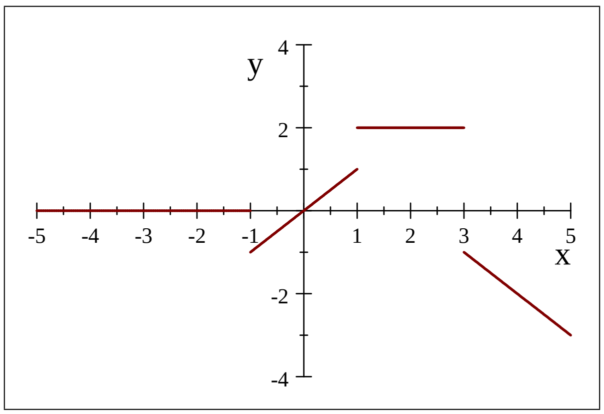
*e) Let $y(x) = x$, $x \geq 0$. Then*

$$\begin{aligned} Y(s) &= L[x] = \int_0^{\infty} e^{-sx} x dx = \lim_{b \longrightarrow +\infty} \int_0^b e^{-sx} x dx = \lim_{b \longrightarrow +\infty} \left[ \frac{x\left(e^{-sx}\right)}{-s} - \frac{\left(e^{-sx}\right)}{s^2} \right|_0^b \\ &= \lim_{b \longrightarrow +\infty} \left\{ \left[ \frac{b\left(e^{-sb}\right)}{-s} - \frac{\left(e^{-sb}\right)}{s^2} \right] - \left[ -\frac{1}{s^2} \right] \right\} = \frac{1}{s^2} \quad for \quad s > 0.. \end{aligned}$$

*Domain of $Y(s)$ is $D_Y = (0, \infty)$.*

*What functions possess the Laplace transform?*

**Definition 4** *A function $y = y(x)$ is said to be piecewise continuous on an interval $I$, if $I$ can be subdivided into finite number ob subintervals, in each of which $f$ is continuous and has left and right limits.*




**Definition 5** *A function $y = y(x)$ is piecewise continuous on interval $[0, \infty)$ and of exponential order $\alpha$ as $x$ tends to $\infty$, if there exist $M$, $A$ such that*

$$y(x) \leq M e^{\alpha x} \quad for \quad x > A.$$

**Theorem 6** *If $y = y(x)$ is of exponential order $\alpha$ as $x$ tends to $\infty$ then the Laplace transform $Y(s)$ is defined for $s > \alpha$, i.e. $D_Y \supset [\alpha, \infty)$. Furthermore, if $y(x)$ and $z(x)$ are piecewise continuous and*

$$L[y](s) = L[z](s)$$

*then $y(x) \equiv z(x)$.*

**Theorem 7 (Linearity of the Laplace transform)**

$$L[ay + bz](s) = aL[y](s) + bL[z](s).$$

**Example 8** *Evaluate the Laplace transforms of the following functions:*

*a)* $y = 2 - 3x,\ x \geq 0;$

*b)* $y = \cosh ax;$

*c)* $y = \sinh(ax).$

**Solution 9** *a)* $L[2-3x] = 2L[1] - 3L[x] = 2\left(\frac{1}{s}\right) - 3\left(\frac{1}{s^2}\right) = \frac{2s-3}{s^2};$

*b)* $\cosh(ax) = \frac{e^{ax}+e^{-ax}}{2}$. *Hence*

$$L[\cosh(ax)] = \frac{1}{2}L[e^{ax}] + \frac{1}{2}L\left[e^{-ax}\right] = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{s}{s^2-a^2};$$

*c)* $\sinh(ax) = \frac{e^{ax}-e^{-ax}}{2}$. *Hence*

$$L[\sinh(ax)] = \frac{1}{2}L[e^{ax}] - \frac{1}{2}L\left[e^{-ax}\right] = \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} = \frac{a}{s^2-a^2}.$$

**Theorem 10 (shifting in s)** *If* $L[y(x)] = Y(s)$ *then* $L[e^{ax}y(x)] = Y(s-a).$

**Proof.** $L[e^{ax}y(x)] = \int_0^\infty e^{-sx}(e^{ax}y(x))\,dx = \int_0^\infty e^{-(s-a)x}y(x)\,dx = Y(s-a).$

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**Example 11** *Evaluate* $L\left[e^{2x}x\right].$

**Solution 12** *Since* $L[x] = \frac{1}{s^2}$ *then* $L\left[e^{2x}x\right] = \frac{1}{(s-2)^2}.$

**Theorem 13** *If* $L[y(x)] = Y(s)$ *then* $L[y'(x)] = sY(s) - y(0).$

**Proof.** We have

$$\begin{aligned}
L[y'(x)] &= \int_0^\infty e^{-sx}(y'(x))\,dx = \lim_{b\longrightarrow+\infty}\int_0^b e^{-sx}(y'(x))\,dx \\
&= \lim_{b\longrightarrow+\infty}\left\{\left[e^{-sx}y(x)\right|_0^b - \int_0^b \left(e^{-sx}\right)' y(x)\,dx\right\} \\
&= \lim_{b\longrightarrow+\infty}\left\{\left[e^{-sb}y(b)\right] - \left[e^0 y(0)\right] - \int_0^b (-s)\,e^{-sx}y(x)\,dx\right\} \\
&= -y(0) + s\int_0^\infty e^{-sx}y(x)\,dx = sL[y](s) - y(0).
\end{aligned}$$

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**Example 14** *Check the above formula for* $y(x) = x.$

**Solution 15** *Denote* $L[y](s) = Y(s)$. *Then we have* $y'(x) = 1$ *and therefore*

$$L[y'(x)] = L[1] = \frac{1}{s}.$$

*From the other hand on the left we have* $L[y'(x)] = sY(s) - y(0) = s \cdot \frac{1}{s^2} - 0 = \frac{1}{s}.$

**Example 16** *Find* $L\left[x^2\right].$

**Solution 17** *Denote* $y(x) = x^2$. *Then* $y'(x) = 2x$ *and therefore* $L[y'(x)] = L[2x] = 2L[x] = \frac{2}{s^2}$ *for* $s > 0$. *Therefore*

$$\begin{aligned} sL[y(x)] - y(0) &= L[y'(x)] = \frac{2}{s^2}. \\ sL[y(x)] &= \frac{2}{s^2} \\ L\left[x^2\right] &= L[y(x)] = \frac{2}{s^3} \quad for \quad s > 0. \end{aligned}$$

**Example 18** *Using the above formula evaluate* $L[e^{ax}].$

**Solution 19** *Denote* $y(x) = e^{ax}$. *Then* $y'(x) = ae^{ax} = ay(x)$. *Taking the Laplace transform from both sides we obtain*

$$L[y'(x)] = aL[y(x)].$$

*Hence*

$$\begin{aligned} sL[y(x)] - y(0) &= aL[y(x)], \\ sL[y(x)] - aL[y(x)] &= y(0) = 1, \\ (s-a)L[y(x)] &= 1, \\ L[e^{ax}] &= L[y(x)] = \frac{1}{s-a}. \end{aligned}$$

**Theorem 20** $L[x^n] = \frac{n!}{s^{n+1}}$ *for* $s > 0.$

**Example 21** *Find* $L\left[2x^3 - 4x^2 + 3x - 7\right]$

**Solution 22** *We proceed as follows*

$$\begin{aligned} L\left[2x^3 - 4x^2 + 3x - 7\right] &= 2L\left[x^3\right] - 4L\left[x^2\right] + 3L[x] - 7L[1] \\ &= 2\frac{3!}{s^4} - 4\frac{2!}{s^3} + \frac{3}{s^2} - \frac{7}{s} = \frac{-7s^3 + 3s^2 - 8s + 12}{s^4}. \end{aligned}$$

**Example 23** *Derive the formula for* $L[e^{ax}x^n].$

**Solution 24** *Since* $L[x^n] = \frac{n!}{s^{n+1}}$ *then* $L[e^{ax}x^n] = \frac{n!}{(s-a)^{n+1}}$.

APPLICATION

**Example 25** *Solve the IVP* $y' - 2y = e^{2x}, \;\; y(0) = 1$

**Solution 26** *Denote* $L[y](s) = Y(s)$. *Then* $L[y'] = sL[y] - y(0) = sY(s) - 1$. *Taking the Laplace transforms from both sides of the given ODE we proceed as follows:*

$$\begin{aligned}
L[y' - 2y] &= L\left[e^{2x}\right], \\
L[y'] - 2L[y] &= \frac{1}{s-2}, \\
sY(s) - 1 - 2Y(s) &= \frac{1}{s-2}, \\
sY(s) - 2Y(s) &= \frac{1}{s-2} + 1, \\
(s-2)Y(s) &= \frac{1}{s-2} + 1, \\
Y(s) &= \frac{1}{(s-2)^2} + \frac{1}{s-2}, \\
L[y(x)] &= L\left[e^{2x}x\right] + L[e^x] = L\left[e^{2x}x + e^{2x}\right], \\
y(x) &= e^{2x}x + e^{2x}.
\end{aligned}$$

**Theorem 27** *If* $L[y(x)] = Y(s)$ *then*

$$L[y''(x)] = s^2Y(s) - sy(0) - y'(0).$$

*In general*

$$L\left[y^{(n)}(x)\right] = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - sy^{(n-2)}(0) - y^{(n-1)}(0).$$

**Proof.** We have already derived

$$L[y'(x)] = sY(s) - y(0) = sL[y(x)] - y(0).$$

Thus

$$\begin{aligned}
L[y''(x)] &= L\left[(y')'(x)\right] = sL[y'(x)] - y'(0) = s(sY(s) - y(0)) - y'(0) \\
&= s^2Y(s) - sy(0) - y'(0).
\end{aligned}$$

The proof of the general situation goes by mathematical induction.

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**Example 28** *Derive the Laplace transforms of the following functions:*
*a)* $y = \sin bx$;
*b)* $y = \cos bx$.

**Solution 29** *a) We have* $y' = b\cos bx$ *and* $y'' = -b^2 \sin bx = -b^2 y$. *Taking the Laplace transforms from both sides we proceed as follows:*

$$\begin{aligned} L[y''] &= L\left[-b^2 y\right] = -b^2 L[y], \\ s^2 L[y] - sy(0) - y'(0) &= -b^2 L[y], \\ s^2 L[y] - b &= -b^2 L[y], \\ s^2 L[y] + b^2 L[y] &= b, \\ \left(s^2 + b^2\right) L[y] &= b, \\ L[\sin bx] &= L[y] = \frac{b}{s^2 + b^2} \end{aligned}$$

*b) For* $y = \cos bx$ *we have:* $y' = -b\sin bx$ *and* $y'' = -b^2 \cos bx = -b^2 y$. *Hence*

$$\begin{aligned} L[y''] &= L\left[-b^2 y\right] = -b^2 L[y], \\ s^2 L[y] - sy(0) - y'(0) &= -b^2 L[y], \\ s^2 L[y] - s &= -b^2 L[y], \\ s^2 L[y] + b^2 L[y] &= s, \\ \left(s^2 + b^2\right) L[y] &= s, \\ L[\cos bx] &= L[y] = \frac{s}{s^2 + b^2}. \end{aligned}$$

**Conclusion 30**

$$\begin{aligned} L[\sin bx] &= \frac{b}{s^2 + b^2}; \\ L[\cos bx] &= \frac{s}{s^2 + b^2}. \end{aligned}$$

**Example 31** *By Shifting we have*

$$\begin{aligned} L\left[e^{ax} \sin bx\right] &= \frac{b}{(s-a)^2 + b^2}; \\ L\left[e^{ax} \cos bx\right] &= \frac{s-a}{(s-a)^2 + b^2}. \end{aligned}$$

**Example 32** $L\left[2e^{3x} \sin 4x - 5e^{3x} \cos 4x\right] = 2L\left[e^{3x} \sin 4x\right] - 5L\left[e^{3x} \cos 4x\right]$

$$= 2\frac{4}{(s-3)^2 + 4^2} - 5\frac{s-3}{(s-3)^2 + 4^2} = \frac{8 - 5(s-3)}{(s-3)^2 + 4^2} = \frac{23 - 5s}{s^2 - 6s + 25}.$$

INVERSE LAPLACE TRANSFORM

By Theorem 6 we know that the correspondence $y(x) \longrightarrow Y(s) = L[y](s)$ is surjective, i. e. for given $Y(s)$ the function $y(x)$ such that $Y(s) = L[y](s)$ is uniquely defined.

**Definition 33** *For given $Y(s)$ such function $y(x)$ that $L[y(x)] = Y(s)$ we denote*

$$y(x) = L^{-1}[Y(s)]$$

*and call the Laplace inverse transform of $Y(s)$.*

**Example 34** *a)* $L^{-1}\left[\frac{1}{s}\right] = 1,$ *because* $L[1] = \frac{1}{s}$;
*b)* $L^{-1}\left[\frac{1}{s^2}\right] = x,$ *because* $L[x] = \frac{1}{s^2}$;
*c)* $L^{-1}\left[\frac{2}{s^3}\right] = x^2,$ *because* $L\left[x^2\right] = \frac{2}{s^3}$;
*d)* $L^{-1}\left[\frac{n!}{s^{n+1}}\right] = x^n,$ *because* $L[x^n] = \frac{n!}{s^{n+1}}$;
*e)* $L^{-1}\left[\frac{1}{s-a}\right] = e^{ax},$ *because* $L[e^{ax}] = \frac{1}{s-a}$;
*f)* $L^{-1}\left[\frac{1}{s^2+b^2}\right] = \frac{\sin bx}{b},$ *because* $L[\sin bx] = \frac{b}{s^2+b^2}$, $L\left[\frac{\sin bx}{b}\right] = \frac{1}{s^2+b^2}$;
*g)* $L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bx,$ *because* $L[\cos bx] = \frac{s}{s^2+b^2}$.

EVALUATION OF THE INVERSE LAPLACE TRANSFORMS

**Theorem 35 (Linearity of the inverse Laplace transform)**

$$L^{-1}[aY(s) + bZ(s)] = aL^{-1}[Y(s)] + bL^{-1}[Z(s)].$$

**Example 36** *a)* $L^{-1}\left[\frac{5+4s}{s^3}\right] = 5L^{-1}\left[\frac{1}{s^3}\right] + 4L^{-1}\left[\frac{1}{s^2}\right] = 5\frac{x^2}{2!} + 4\frac{x^1}{1!} = \frac{5}{2}x^2 + 4x$;
*b)* $L^{-1}\left[\frac{5+4s}{s^2+9}\right] = 5L^{-1}\left[\frac{1}{s^2+3^2}\right] + 4L^{-1}\left[\frac{s}{s^2+3^2}\right] = 5\frac{\sin 3x}{3} + 4\cos 3x.$

**Example 37** *Evaluate* $L^{-1}\left[\frac{5+4s}{s^2+2s}\right]$

**Solution 38** *First we decompose* $\frac{5+4s}{s^2+2s}$ *into the partial fractions. We have*

$$\frac{5+4s}{s^2+2s} = \frac{5+4s}{(s+2)s} = \frac{A}{s+2} + \frac{B}{s}.$$

*Hence*

$$\begin{aligned} 5+4s &= As + B(s+2), \\ 5+4s &= (A+B)s + 2B, \\ A+B &= 4, \quad 2B = 5, \\ A &= \frac{3}{2}, \quad B = \frac{5}{2}. \end{aligned}$$

*So*

$$\frac{5+4s}{s^2+2s} = \frac{\frac{3}{2}}{s+2} + \frac{\frac{5}{2}}{s}.$$

*Therefore*

$$L^{-1}\left[\frac{5+4s}{s^2+2s}\right] = \frac{3}{2}L^{-1}\left[\frac{1}{s+2}\right] + \frac{5}{2}L^{-1}\left[\frac{1}{s}\right] = \frac{3}{2}e^{-2x} + \frac{5}{2}.$$

**Theorem 39 (shifting in s)** *If* $L^{-1}[Y(s)] = y(x)$ *then* $L^{-1}[Y(s-a)] = e^{ax}y(x)$.

**Example 40** *a)* $L^{-1}\left[\frac{5+4s}{s^2+2s+1}\right] = L^{-1}\left[\frac{5+4s}{(s+1)^2}\right] = \left\{\begin{array}{c} \overline{s} = s+1, \\ s = \overline{s}-1, \quad a=-1 \end{array}\right\}$

$$\begin{aligned} &= e^{-x}L^{-1}\left[\frac{5+4(\overline{s}-1)}{(\overline{s})^2}\right] = \left\{\begin{array}{c} forget \\ the \quad bar \end{array}\right\} = e^{-x}L^{-1}\left[\frac{4s+1}{s^2}\right] \\ &= e^{-x}\left\{4L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s^2}\right]\right\} = e^{-x}(4+x). \end{aligned}$$

*b)* $L^{-1}\left[\frac{5+4s}{s^2-2s+10}\right] = L^{-1}\left[\frac{5+4s}{(s-1)^2+9}\right] = \left\{\begin{array}{c} \overline{s} = s-1, \\ s = \overline{s}+1, \quad a=1 \end{array}\right\}$

$$\begin{aligned} &= e^{x}L^{-1}\left[\frac{5+4(\overline{s}+1)}{(\overline{s})^2+9}\right] = \left\{\begin{array}{c} forget \\ the \quad bar \end{array}\right\} = e^{x}L^{-1}\left[\frac{4s+9}{s^2+9}\right] \\ &= e^{x}\left\{4L^{-1}\left[\frac{s}{s^2+9}\right] + 9L^{-1}\left[\frac{1}{s^2+9}\right]\right\} = e^{x}\{4\cos 3x + 3\sin 3x\}. \end{aligned}$$