LECTURE 28. 04. 2020

METHOD OF UNDETERMINED COEFFICIENTS

The method is used when we want to find a PS of (NH) LDE

$$D[y] = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_2y'' + a_1y' + a_0y = r(x) \tag{1}$$

where a_0, $a_1, ..., a_{n-1}$ are given reals and $r(x)$ is given continuous function. A theoretical background of the method is based on the following results:

Theorem 1 *Let φ and ψ are solutions of NH (1). Then $y = \varphi - \psi$ is a solution of corresponding (H)*

$$D[y] = y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_2y'' + a_1y' + a_0y = 0. \tag{2}$$

Theorem 2 *Let φ is a solution of NH (1), while ψ of (H) (2). Then $z = \varphi + \psi$ is a solution of (1).*

Theorem 3 *Let the LDE $D[y] = r_1(x)$ possess a PS $\varphi_1(x)$, and $D[y] = r_2(x)$ possess a PS $\varphi_2(x)$. Then the LDE $D[y] = ar_1(x) + br_2(x)$ possess a PS $y_p(x) = a\varphi_1(x) + b\varphi_2(x)$.*

Proof. (THM 1) The functions φ and ψ satisfy the relations

$$D[\varphi] = D[\phi] = r(x).$$

Therefore

$$D[y] = D[\varphi - \psi] = D[\varphi] - D[\psi] \equiv r(x) - r(x) \equiv 0.$$

■

Proof. (THM 2) For φ we have $D[\varphi] = r(x)$, while for ψ is $D[\varphi] = 0$. Hence for $z = \varphi + \psi$ we conclude that

$$D[z] = D[\varphi + \psi] = D[\varphi] + D[\psi] \equiv r(x) + 0 \equiv r(x).$$

■

Proof. (THM 3) $D[y_p] = D[a\varphi_1(x) + b\varphi_2(x)] = aD[\varphi_1] + bD[\varphi_2] = ar_1(x) + br_2(x)$.

■

Fix ψ a PS(NH). Then for any solution φ of (NH) we have $\varphi - \psi \in GS\,(H)$ (from THM 1)

$$\varphi \in \psi + GS\,(H)\,.$$

Hence

$$GS(NH) \subset PS(NH) + GS(H). \tag{3}$$

From THM 2 we obtain

$$PS(NH) + GS(H) \subset GS(NH). \tag{4}$$

Both Conclusions (1) and (2) yield

$$GS(NH) = PS(NH) + GS(H). \tag{5}$$

So, finding GS of (NH) can be split into two steps: finding PS(NH) and finding GS(H). Finding the GS(H) we do by the characteristic equation (CH) of LDE

$$p\,(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + ... + a_2\lambda^2 + a_1\lambda + a_0 = 0. \tag{6}$$

The method of Undetermined Coefficients allows to find PS(NH). We can do that only for special types of right-hand sides $r\,(x)$. Namely, it can be applicable only for

$$r\,(x) = (p\,(x)\cos(\beta x) + q\,(x)\sin(\beta x))\cdot e^{\alpha x},$$

where $p\,(x)$ and $q\,(x)$ are polynomials.

If it is the case, then a PS(NH) y_p is of the form

$$y_p\,(x) = [(w\,(x)\cos(\beta x) + v\,(x)\sin(\beta x))]\cdot e^{\alpha x}\cdot x^{\boldsymbol{m}},$$

where $w\,(x)$ and $v\,(x)$ are, in general, polynomials of degree equal to $\max(\deg p, \deg q)$. It also a factor $x^{\boldsymbol{m}}$. To determine the exponent $\boldsymbol{m}$, we need to take into account (TiA) the number

$$\lambda = \alpha + i\beta$$

and comparing it with the possible eigenvalues.

$$If \;\; \lambda \;\; is \;\; an \;\; eigenvalue \;\; then \;\; \boldsymbol{m} \;\; is \;\; the \;\; multiplicity \;\; of \;\; it.$$

$$Otherwise, \;\; we \; take \;\; \boldsymbol{m} = 0.$$

Example 4 *The LDE* $D\,[y] = r\,(x)$ *has the (CH)*

$$p\,(\lambda) = \lambda^3\,(\lambda - 1)^4\left(\lambda^2 - 4\lambda + 13\right)^5 = 0.$$

Determine the form of PS of knowing that

a) $r\,(x) = x^2$;
b) $r\,(x) = \left(x^2 + x\right)e^{3x}$;
c) $r\,(x) = (x + 1)\,e^x$;
d) $r\,(x) = x^3 e^x \sin 3x$;
e) $r\,(x) = ((2x + 1)\cos 3x + (3x - 2)\sin 3x)\cdot e^{2x}$

Solution 5 *Below there are listed the eigenvalues of (CH):*

$$\begin{aligned}
\lambda &= 0 \ \ with \ \ the \ \ multiplicity \ \ 3; \\
\lambda &= 1 \ \ with \ \ the \ \ multiplicity \ \ 4; \\
\lambda &= 2+3i \ \ with \ \ the \ \ multiplicity \ \ 5; \\
\lambda &= 2-3i \ \ with \ \ the \ \ multiplicity \ \ 5.
\end{aligned}$$

a) $r(x) = x^2 = \left[x^2 \cos(0x) + 0 \sin(0x)\right] e^{0x}$. *In this case there is no* $e^{\alpha x}$, *no* $\cos(\beta x)$ *and no* $\sin(\beta x)$. *This means that*

$$\alpha = \beta = 0.$$

So TiA

$$\lambda = 0 + 0 \cdot i = 0 \ \ and \ \ \boldsymbol{m} = 3.$$

Moreover $p(x) = x^2$ *and* $q(x) = 0$. *Thus*

$$\max(\deg p, \deg q) = \max(2, 0) = 2.$$

Therefore both polynomials $w(x)$ *and* $v(x)$ *have to be of degree 2. Hence the PS should be in the form*

$$\begin{aligned}
y_p &= \left[\left(Ax^2 + Bx + C\right)\cos(0x) + \left(Kx^2 + Lx + M\right)\sin(0x)\right] \cdot \left(e^{0x}\right) \cdot \boldsymbol{x^3} \\
&= Ax^5 + Bx^4 + Cx^3.
\end{aligned}$$

b) $r(x) = \left(x^2 + x\right) e^{3x} = \left[\left(x^2 + x\right)\cos(0x) + 0 \sin(0x)\right] e^{3x}$. *In this case there is* e^{3x} *but no* $\cos(\beta x)$ *and no* $\sin(\beta x)$. *This means that*

$$\alpha = 3, \quad \beta = 0 \ \ and \ \ we \ \ TiA \ \ \lambda = 3 + 0 \cdot i = 3, \quad \boldsymbol{m} = 0.$$

Moreover $p(x) = x^2 + x$ *and* $q(x) = 0$. *Thus*

$$\max(\deg p, \deg q) = \max(2, 0) = 2.$$

Therefore both polynomials $w(x)$ *and* $v(x)$ *have to be of degree 2. Hence the PS should be in the form*

$$\begin{aligned}
y_p &= \left[\left(Ax^2 + Bx + C\right)\cos(0x) + \left(Kx^2 + Lx + M\right)\sin(0x)\right] \cdot \left(e^{3x}\right) \cdot x^0 \\
&= \left(Ax^2 + Bx + C\right) e^{3x}.
\end{aligned}$$

c) $r(x) = (x+1)e^x = [(x+1)\cos(0x) + 0 \sin(0x)] e^{1x}$. *In this case there is* e^x *but no* $\cos(\beta x)$ *and no* $\sin(\beta x)$. *This means that*

$$\alpha = 1, \quad \beta = 0 \ \ and \ \ we \ \ TiA \ \ \lambda = 1 + 0 \cdot i = 1, \quad \boldsymbol{m} = 4.$$

Moreover $p(x) = x + 1$ *and* $q(x) = 0$. *Thus*

$$\max(\deg p, \deg q) = \max(1, 0) = 1.$$

Therefore both polynomials $w(x)$ and $v(x)$ have to be of degree 1. Hence the PS should be in the form

$$\begin{aligned} y_p &= [(Ax+B)\cos(0x)+(Kx+L)\sin(0x)]\cdot(e^x)\cdot\boldsymbol{x^4} \\ &= \left(Ax^5+Bx^4\right)e^x. \end{aligned}$$

d) $r(x) = x^3e^x\sin 3x = \left[0\cos(3x)+x^3\sin(3x)\right]e^{1x}$. *In this case there is* e^x *and* $\sin(3x)$ *but no* $\cos(3x)$. *This means that*

$$\alpha = 1, \quad \beta = 3 \quad and \quad we \quad TiA \quad \lambda = 1+3i = 0, \quad \boldsymbol{m} = 0.$$

Moreover $p(x) = 0$ *and* $q(x) = x^3$. *Thus*

$$\max(\deg p, \deg q) = \max(0,3) = 3.$$

Therefore both polynomials $w(x)$ and $v(x)$ have to be of degree 3. Hence the PS should be in the form

$$\begin{aligned} y_p &= \left[\left(Ax^3+Bx^2+Cx+D\right)\cos(3x)+\left(Kx^3+Lx^2+Mx+N\right)\sin(3x)\right]\cdot\left(e^{1x}\right)\cdot x^0 \\ &= \left[\left(Ax^3+Bx^2+Cx+D\right)\cos(3x)+\left(Kx^3+Lx^2+Mx+N\right)\sin(3x)\right]\cdot\left(e^x\right). \end{aligned}$$

e) $r(x) = ((2x+1)\cos 3x+(3x-2)\sin 3x)\,e^{2x}$. *In this case there is* e^{2x}, $\sin(3x)$ *and* $\cos(3x)$. *This means that*

$$\alpha = 2, \quad \beta = 3 \quad and \quad we \quad TiA \quad \lambda = 2+3i, \quad \boldsymbol{m} = 5.$$

Moreover $p(x) = (2x+1)$ *and* $q(x) = (3x-2)$. *Thus*

$$\max(\deg p, \deg q) = \max(1,1) = 1.$$

Therefore both polynomials $w(x)$ and $v(x)$ have to be of degree 1. Hence the PS should be in the form

$$\begin{aligned} y_p &= [(Ax+Bx)\cos(3x)+(Kx+L)\sin(3x)]\cdot\left(e^{2x}\right)\cdot\boldsymbol{x^5} \\ &= \left[\left(Ax^6+Bx^5\right)\cos(3x)+\left(Kx^6+Lx^5\right)\sin(3x)\right]\cdot e^{2x}. \end{aligned}$$

Having properly established the form of the PS, we plug it into LDE $D[y] = r(x)$ and evaluate the coefficients by comparing them.

Example 6 *Find the GS of* $D[y] = y''' = 24x^3+12x = \left[\left(24x^3+12x\right)\cos(0x)+0\sin(0x)\right]e^{0x}$. *The corresponding (H) is*

$$y''' = 0.$$

The (CH) is

$$\lambda^3 = 0.$$

The eigenvalue is $\lambda = 0$ with the multiplicity $m = 3$. Therefore FS are $1, x, x^2$ and GS(H) is

$$y = C+Dx+Ex^2,$$

where C, D, E are arbitrary constants. Now let us pass to (NH). In this case there we have $\alpha = \beta = 0$ and we TiA

$$\lambda = 0 + 0 \cdot i = 0 \quad \textit{and} \quad \boldsymbol{m} = 3.$$

Further we have $p(x) = 24x^3 + 12x$ and $q(x) = 0$. Thus

$$\max(\deg p, \deg q) = \max(3, 0) = 3.$$

Therefore both polynomials $w(x)$ and $v(x)$ have to be of degree 3. Hence the PS should be in the form

$$\begin{aligned} y_p &= \left[\left(Ax^3 + Bx^2 + Kx + L\right)\cos(0x) + \left(Ex^3 + Fx^2 + Gx + H\right)\sin(0x)\right] \cdot \left(e^{0x}\right) \cdot x^3 \\ &= Ax^6 + Bx^5 + Kx^4 + Lx^3 \end{aligned}$$

We need to evaluate such constants A, B, K, L in such way that

$$D[y_p] = 24x^3 + 12x.$$

We have

$$\begin{aligned} y_p' &= \frac{d}{dx}\left(Ax^6 + Bx^5 + Kx^4 + Lx^3\right) = 6Ax^5 + 5Bx^4 + 4Kx^3 + 3Lx^2, \\ y_p'' &= \frac{d}{dx}\left(6Ax^5 + 5Bx^4 + 4Kx^3 + 3Lx^2\right) = 30Ax^4 + 20Bx^3 + 12Kx^2 + 6Lx, \\ y_p''' &= \frac{d}{dx}\left(30Ax^4 + 20Bx^3 + 12Kx^2 + 6Lx\right) = 120Ax^3 + 60Bx^2 + 24Kx + 6L. \end{aligned}$$

Plugging we have

$$D[y_p] = 24x^3 + 12x.$$

$$120Ax^3 + 60Bx^2 + 24Kx + 6L = 24x^3 + 12x.$$

Comparing we obtain

$$120A = 24, \quad 60B = 0, \quad 24K = 12, \quad 6L = 0.$$

Hence

$$A = \frac{1}{5}, \quad B = 0, \; K = \frac{1}{2}, \; L = 0.$$

So the PS is

$$y_p = Ax^6 + Bx^5 + Kx^4 + Lx^3 = \frac{1}{5}x^6 + \frac{1}{2}x^4$$

and finally the GS(NH)

$$y = C + Dx + Ex^2 + \frac{1}{5}x^6 + \frac{1}{2}x^4,$$

where C, D, E are arbitrary constants.

Remark 7 *Another way to finding the GS(NH) is to integrate it three times. Then*

$$\begin{aligned}
y''' &= 24x^3 + 12x, \\
y'' &= \int \left(24x^3 + 12x\right) dx = 6x^4 + 6x^2 + C, \\
y' &= \int \left(6x^4 + 6x^2 + C\right) dx = \frac{6}{5}x^5 + 2x^3 + Cx + D, \\
y &= \int \left(\frac{6}{5}x^5 + 2x^3 + Cx + D\right) dx = \frac{1}{5}x^6 + \frac{1}{2}x^4 + \frac{1}{2}Cx^2 + Dx + E,
\end{aligned}$$

where C, D, E are arbitrary constants. Is it the same solution?

Example 8 *Find the GS of*
a) $D[y] = y'' - 2y' + y = x^2$;
b) $D[y] = y'' - 2y' + y = xe^{2x}$;
c) $D[y] = y'' - 2y' + y = 12xe^{x}$;
d) $D[y] = y'' - 2y' + y = 3\sin(2x)$.

Solution 9 *The corresponding (H) is*

$$y'' - 2y' + y = 0.$$

The (CH) is

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0.$$

The eigenvalue is $\lambda = 1$ and the multiplicity $m = 2$. Therefore FS are e^x, xe^x and GS(H) is

$$y = Ce^x + Dxe^x,$$

where C, D are arbitrary constants.

Now let us pass to (NH).

a) $D[y] = y'' - 2y' + y = x^2 = \left[x^2\cos(0x) + 0\sin(0x)\right]e^{0x}$. *In this case we have $\alpha = \beta = 0$ and we TiA*

$$\lambda = 0 + 0 \cdot i = 0 \quad \text{and} \quad \boldsymbol{m} = 0.$$

Further we have $p(x) = x^2$ and $q(x) = 0$. Thus

$$\max(\deg p, \deg q) = \max(2, 0) = 2.$$

Therefore both polynomials $w(x)$ and $v(x)$ have to be of degree 2. Hence the PS should be in the form

$$\begin{aligned}
y_p &= \left[\left(Ax^2 + Bx + C\right)\cos(0x) + \left(Kx^2 + Lx + M\right)\sin(0x)\right] \cdot \left(e^{0x}\right) \cdot x^0 \\
&= Ax^2 + Bx + C.
\end{aligned}$$

We need to evaluate such constants A, B, C in such way that $D[y_p] = x^2$. We have

$$y_p' = 2Ax + B \quad \text{and} \quad y_p'' = 2A.$$

So $\qquad D[y_p] = y_p'' - 2y_p' + y_p = x^2.$ $\qquad$ *Plugging we have*

$$\begin{aligned} 2A - 2(2Ax + B) + \left(Ax^2 + Bx + C\right) &= x^2, \\ Ax^2 + x(B - 4A) + (2A - 2B + C) &= x^2. \end{aligned}$$

Comparing we obtain

$$A = 1, \quad B - 4A = 0, \quad 2A - 2B + C = 0.$$

Hence

$$A = 1, \quad B = 4, \; C = 6.$$

So the PS is

$$y_p = x^2 + 4x + 6$$

and finally the GS(NH)

$$y = Ce^x + Dxe^x + x^2 + 4x + 6,$$

where C, D *are arbitrary constants.*

b) $D[y] = y'' - 2y' + y = xe^{2x} = [x\cos(0x) + 0\sin(0x)]\, e^{2x}$. *In this case there is* e^{2x}, *no* $\cos(\beta x)$ *and no* $\sin(\beta x)$. *This means that* $\alpha = 2, \quad \beta = 0$ *and we TiA*

$$\lambda = 2 + 0{\cdot}i = 2 \quad \textit{and} \quad \boldsymbol{m} = 0.$$

Further we have $p(x) = x$ *and* $q(x) = 0$. *Thus*

$$\max(\deg p, \deg q) = \max(1, 0) = 1.$$

Therefore both polynomials $w(x)$ *and* $v(x)$ *have to be of degree 1. Hence the PS should be in the form*

$$y_p = [(Ax + B)\cos(0x) + (Kx + L)\sin(0x)] \cdot \left(e^{2x}\right) \cdot x^0 = (Ax + B)\, e^{2x}.$$

We need to evaluate such constants A, B *in such way that* $D[y_p] = xe^{2x}$. *We have*

$$y_p' = (A + 2B + 2Ax)\, e^{2x} \quad \textit{and} \quad y_p'' = 4(A + B + Ax)\, e^{2x}.$$

So plugging the the LDE we have

$$\begin{aligned} D[y_p] &= y_p'' - 2y_p' + y_p = xe^{2x} \\ 4(A + B + Ax)\, e^{2x} - 2(A + 2B + 2Ax)\, e^{2x} + (Ax + B)\, e^{2x} &= xe^{2x}, \\ 4(A + B + Ax) - 2(A + 2B + 2Ax) + (Ax + B) &= x, \\ Ax + (2A + B) &= x \end{aligned}$$

Comparing we obtain

$$A = 1, \quad 2A + B = 0$$

Hence

$$A = 1, \quad B = -2.$$

So the PS is

$$y_p = (x-2)\, e^{2x}$$

and finally the GS(NH)

$$y = Ce^x + Dxe^x + (x-2)\, e^{2x},$$

where C, D are arbitrary constants.

c) $D[y] = y'' - 2y' + y = 12xe^x = [(12x)\cos(0x) + 0\sin(0x)]\, e^{1x}$. *In this case there is e^x, no $\cos(\beta x)$ and no $\sin(\beta x)$. This means that $\alpha = 1, \quad \beta = 0$ and we TiA*

$$\lambda = 1 + 0{\cdot}i = 1 \quad \textit{and} \quad \boldsymbol{m} = 2.$$

Further we have $p(x) = 12x$ and $q(x) = 0$. Thus

$$\max(\deg p, \deg q) = \max(1, 0) = 1.$$

Therefore both polynomials $w(x)$ and $v(x)$ have to be of degree 1. Hence the PS should be in the form

$$y_p = [(Ax+B)\cos(0x) + (Kx+L)\sin(0x)] \cdot (e^x) \cdot \boldsymbol{x^2} = \left(Ax^3 + Bx^2\right) e^x.$$

We need to evaluate such constants A, B in such way that $D[y_p] = 12xe^{2x}$. We have

$$y_p' = e^x \left(Ax^3 + (3A+B)\, x^2 + 2Bx\right) \quad \textit{and} \quad y_p'' = e^x \left(Ax^3 + (B+6A)\, x^2 + (6A+4B)\, x + 2B\right).$$

So plugging the the LDE we get

$$D[y_p] = y_p'' - 2y_p' + y_p = 12xe^x.$$

$$\begin{aligned} e^x \left(Ax^3 + (B+6A)\, x^2 + (6A+4B)\, x + 2B\right) - 2e^x \left(Ax^3 + (3A+B)\, x^2 + 2Bx\right) + \left(Ax^3 + Bx^2\right) e^x &= 12xe^x, \\ Ax^3 + (B+6A)\, x^2 + (6A+4B)\, x + 2B - 2Ax^3 - 2(3A+B)\, x^2 - 4Bx + \left(Ax^3 + Bx^2\right) &= 12x, \end{aligned}$$

$$6Ax + 2B = 12x$$

Comparing we obtain

$$A = 2, \quad B = 0$$

Hence the PS is

$$y_p = 2x^3 e^x.$$

and finally the GS(NH)

$$y = Ce^x + Dxe^x + 2x^3 e^x,$$

where C, D are arbitrary constants.

d) $D[y] = y'' - 2y' + y = 3\sin(2x) = [0{\cdot}\cos(2x) + 3\sin(2x)]\, e^{0x}$. *In this case there is no $e^{\alpha x}$, no $\cos(\beta x)$ and is $\sin(2x)$. This means that $\alpha = 0, \quad \beta = 2$ and we TiA*

$$\lambda = 0 + 2{\cdot}i = 2i \quad \textit{and} \quad \boldsymbol{m} = 0.$$

Further we have $p(x) = 0$ and $q(x) = 3$. Thus

$$\max(\deg p, \deg q) = \max(0, 0) = 0.$$

Therefore both polynomials $w(x)$ and $v(x)$ have to be constants. Hence the PS should be in the form

$$y_p = [A\cos(2x) + B\sin(2x)] \cdot \left(e^{0x}\right) \cdot x^{\mathbf{0}} = A\cos(2x) + B\sin(2x).$$

We need to evaluate such constants A, B in such way that $D[y_p] = 3\sin(2x)$. We have

$$y_p' = 2B\cos 2x - 2A\sin 2x \quad \textit{and} \quad y_p'' = -4A\cos 2x - 4B\sin 2x.$$

So plugging the the LDE we get

$$D[y_p] = y_p'' - 2y_p' + y_p = 3\sin(2x),$$

$$\begin{aligned} -4A\cos 2x - 4B\sin 2x - 2(2B\cos 2x - 2A\sin 2x) + A\cos(2x) + B\sin(2x) &= 3\sin(2x), \\ (4A - 3B)\sin 2x - (4B + 3A)\cos 2x &= 3\sin(2x), \end{aligned}$$

Comparing we obtain

$$4A - 3B = 3, \quad 4B + 3A = 0$$

$$A = \frac{3}{22}, \quad B = -\frac{9}{88}.$$

Hence the PS is

$$y_p = \frac{3}{22}\cos(2x) - \frac{9}{88}\sin(2x).$$

and finally the GS(NH)

$$y = Ce^x + Dxe^x + \frac{3}{22}\cos(2x) - \frac{9}{88}\sin(2x),$$

where C, D are arbitrary constants.

Example 10 *Find the GS of*
a) $D[y] = y''' + y'' - 2y = 6x^2$;
b) $D[y] = y''' + y'' - 2y = 12xe^x$;
c) $D[y] = y''' + y'' - 2y = 6x^2 + 12xe^x$.

Solution 11 *The corresponding (H) is*

$$y''' + y'' - 2y' = 0.$$

The (CH) is

$$\lambda^3 + \lambda^2 - 2\lambda = \lambda(\lambda - 1)(\lambda + 2) = 0.$$

The eigenvalues is $\lambda = 0, 1, -2,$ *all with the multiplicity* $m = 1$. *Therefore FS are* $1, e^{x}, e^{-2x}$ *and GS(H) is*

$$y = C + De^{x} + Ee^{-2x},$$

where C, D, E *are arbitrary constants.*

Now let us pass to (NH).

a) $D[y] = y''' + y'' - 2y' = 6x^2 = \left[6x^2 \cos(0x) + 0 \sin(0x)\right] e^{0x}$. *In this case there is no* $e^{\alpha x}$, *no* $\cos(\beta x)$ *and no* $\sin(\beta x)$. *This means that* $\alpha = \beta = 0$ *and we TiA*

$$\lambda = 0 + 0{\cdot}i = 0 \quad \textit{and} \quad \boldsymbol{m} = 1.$$

Further we have $p(x) = 6x^2$ *and* $q(x) = 0$. *Thus*

$$\max(\deg p, \deg q) = \max(2, 0) = 2.$$

Therefore both polynomials $w(x)$ *and* $v(x)$ *have to be of degree 2. Hence the PS should be in the form*

$$\begin{aligned} y_p &= \left[\left(Ax^2 + Bx + C\right)\cos(0x) + \left(Kx^2 + Lx + M\right)\sin(0x)\right] \cdot \left(e^{0x}\right) \cdot \boldsymbol{x^1} \\ &= Ax^3 + Bx^2 + Cx. \end{aligned}$$

We need to evaluate such constants A, B, C *in such way that* $D[y_p] = 6x^2$. *We have*

$$y_p' = 3Ax^2 + 2Bx + C, \quad y_p'' = 6Ax + 2B \quad \textit{and} \quad y_p''' = 6A\ .$$

So $\qquad D[y_p] = y_p''' + y_p'' - 2y_p' = x^2. \qquad$ *Plugging we have*

$$\begin{aligned} 6A + (6Ax + 2B) - 2\left(3Ax^2 + 2Bx + C\right) &= 6x^2, \\ -6Ax^2 + (6A - 4B)x + (6A + 2B - 2C) &= 6x^2. \end{aligned}$$

Comparing we obtain

$$A = -1, \quad 6A - 4B = 0, \quad 6A + 2B - 2C = 0.$$

Hence

$$A = -1, \quad yB = -\frac{3}{2}, \quad C = -\frac{9}{2}.$$

So the PS is

$$y_p = -x^3 - \frac{3}{2}x^2 - \frac{9}{2}x$$

and we get the GS(NH)

$$y = C + De^{x} + Ee^{-2x} - x^3 - \frac{3}{2}x^2 - \frac{9}{2}x,$$

where C, D, E are arbitrary constants.

b) $D[y] = y''' + y'' - 2y = 12xe^x = [12x\cos(0x) + 0\sin(0x)]\, e^x$. *In this case there is* e^x, *no* $\cos(\beta x)$ *and no* $\sin(\beta x)$. *This means that* $\alpha = 1, \beta = 0$ *and we TiA*

$$\lambda = 1 + 0 \cdot i = 1 \quad and \quad \boldsymbol{m} = 1.$$

Further we have $p(x) = 12x$ *and* $q(x) = 0$. *Thus*

$$\max(\deg p, \deg q) = \max(1, 0) = 1.$$

Therefore both polynomials $w(x)$ *and* $v(x)$ *have to be of degree 1. Hence the PS should be in the form*

$$\begin{aligned} y_p &= [(Ax + B)\cos(0x) + (Kx + L)\sin(0x)] \cdot \left(e^{1x}\right) \cdot \boldsymbol{x^1} \\ &= \left(Ax^2 + Bx\right) e^x. \end{aligned}$$

We need to evaluate such constants A, B *in such way that* $D[y_p] = 12xe^x$. *We have*

$$\begin{aligned} y_p' &= \frac{d}{dx}\left(\left(Ax^2 + Bx\right)e^x\right) = e^x\left(B + Ax^2 + 2Ax + Bx\right), \\ y_p'' &= \frac{d}{dx}\left(e^x\left(B + Ax^2 + 2Ax + Bx\right)\right) = e^x\left(2A + 2B + Ax^2 + 4Ax + Bx\right), \\ y_p''' &= \frac{d}{dx}\left(e^x\left(2A + 2B + Ax^2 + 4Ax + Bx\right)\right) = e^x\left(6A + 3B + Ax^2 + 6Ax + Bx\right) \end{aligned}$$

So plugging we have

$$e^x\left(6A + 3B + Ax^2 + 6Ax + Bx\right) + e^x\left(2A + 2B + Ax^2 + 4Ax + Bx\right) - 2e^x\left(B + Ax^2 + 2Ax + Bx\right) = 12xe^x,$$

$$\begin{aligned} e^x(8A + 3B + 6Ax) &= 12xe^x, \\ (8A + 3B) + 6Ax &= 12x. \end{aligned}$$

Comparing we obtain

$$6A = 12 \quad and \quad 8A + 3B = 0$$

$$A = 2 \quad and \quad B = -\frac{16}{3}.$$

Hence the PS is

$$y_p = \left(2x^2 - \frac{16}{3}x\right)e^x.$$

and we get the GS(NH)

$$y = C + De^x + Ee^{-2x} + \left(2x^2 - \frac{16}{3}x\right)e^x,$$

where C, D, E *are arbitrary constants.*

c) $D[y] = y''' + y'' - 2y = 6x^2 + 12xe^x$. *The LDE* $D[y] = y''' + y'' - 2y = 6x^2$ *possesses the PS* $\varphi_1(x) = -x^3 - \frac{3}{2}x^2 - \frac{9}{2}x$, *while* $D[y] = y''' + y'' - 2y = 12xe^x$ *has* $\varphi_2(x) = \left(2x^2 - \frac{16}{3}x\right)e^x$. *THM 3 yields the conclusion that the PS for LDE*

$$D[y] = y''' + y'' - 2y = 6x^2 + 12xe^x$$

is

$$y_p = -x^3 - \frac{3}{2}x^2 - \frac{9}{2}x + \left(2x^2 - \frac{16}{3}x\right)e^x.$$

Thus we obtain the GS(NH)

$$y = C + De^x + Ee^{-2x} - x^3 - \frac{3}{2}x^2 - \frac{9}{2}x + \left(2x^2 - \frac{16}{3}x\right)e^x,$$

where C, D, E *are arbitrary constants.*