LECTURE 05. 05. 2020

THE EULER DE

By the Euler DE we mean a LDE of the form

$$D[y] = x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + ... + a_2 x^2 y'' + a_1 x y' + a_0 y = 0, \qquad (1)$$

where $a_0, a_1, ..., a_{n-1}$ are given reals. It is LDE with variable coefficients, but in special form. It is defined either only for $x > 0$ or only for $x < 0$.

Example 1 *a)* $D[y] = x^2 y'' - 3xy' + 2y = 0$; *order 2;*
b) $D[y] = x^3 y''' - x^2 y'' + xy' - y = 0$, *order 3;*
c) $D[y] = x^4 y^{(iv)} + x^2 y'' + y = 0$, *order 4.*

Differential operator $D[y]$ is linear, the GS $\mathcal{Y} = \{y : D[y] = 0\}$ is again $n-$ dimensional linear space and we will discuss, how to find the FS.

The Euler DE can be solved by substitution

$$x = \begin{cases} e^t & for \;\; x > 0 \\ -e^t & for \;\; x < 0 \end{cases}.$$

Assume first that $x > 0$. Using the transformation $x = e^t$ we have $\frac{dx}{dt} = e^t$. Therefore $\frac{dt}{dx} = e^{-t} = \frac{1}{x}$. Let us consider the new unknown $y = y(e^t)$. Then we have, by the chain rule,

$$y'(x) = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = e^{-t}\dot{y} = \frac{1}{x}\dot{y},$$

where dots denote derivatives wits respect to t. Further we have

$$\begin{aligned} y''(x) &= \frac{dy'}{dx} = \frac{dy'}{dt}\frac{dt}{dx} = \left[\frac{d}{dt}\left(e^{-t}\dot{y}\right)\right]e^{-t} = \left[\left(-e^{-t}\right)\dot{y} + e^{-t}\ddot{y}\right]e^{-t} \\ &= e^{-2t}\left(-\dot{y} + \ddot{y}\right) = \frac{1}{x^2}\left(\ddot{y} - \dot{y}\right), \\ y'''(x) &= \frac{dy''}{dx} = \frac{dy''}{dt}\frac{dt}{dx} = \frac{d}{dt}\left[e^{-2t}\left(-\dot{y} + \ddot{y}\right)\right]e^{-t} \\ &= \left[-2e^{-2t}\left(-\dot{y} + \ddot{y}\right) + e^{-2t}\left(-\ddot{y} + \dddot{y}\right)\right]e^{-t}. \\ &= e^{-3t}\left(2\dot{y} - 2\ddot{y} - \ddot{y} + \dddot{y}\right) = \frac{1}{x^3}\left(\dddot{y} - 3\ddot{y} + 2\dot{y}\right). \end{aligned}$$

For higher order derivatives we can also derive the formulas.

The same formulas we obtain for $x < 0$, substituting $x = -e^t$.

With the above transformations the Euler LD of the 2^{nd} order

$$x^2y'' + axy' + by = 0$$

transforms in the following way

$$\begin{aligned} x^2\frac{1}{x^2}\left(\ddot{y} - \dot{y}\right) + ax\frac{1}{x}\dot{y} + by &= 0, \\ \left(\ddot{y} - \dot{y}\right) + a\dot{y} + by &= 0, \\ \ddot{y} + (a-1)\,\dot{y} + by &= 0. \end{aligned}$$

It occurs to be LDE of 2^{nd} order with constant coefficients.

Similarly, the Euler LD of the 3^{rd} order

$$x^3y''' + ax^2y'' + bxy' + cy = 0$$

gives

$$\begin{aligned} x^3\frac{1}{x^3}\left(\dddot{y} - 3\ddot{y} + 2\dot{y}\right) + ax^2\frac{1}{x^2}\left(\ddot{y} - \dot{y}\right) + bx\frac{1}{x}\dot{y} + cy &= 0, \\ \left(\dddot{y} - 3\ddot{y} + 2\dot{y}\right) + a\left(\ddot{y} - \dot{y}\right) + b\dot{y} + cy &= 0, \\ \dddot{y} + (a-3)\,\ddot{y} + (2-a+b)\,\dot{y} + cy &= 0 \end{aligned}$$

and this is the LDE of 3^{rd} order with constant coefficients.

In general, the above sibstitution leads to $n-th$ order LDE with constant coefficients, but such one we know how to solve.

Example 2 *Solve the Euler equation*

$$x^2y'' - 3xy' + 3y = 0, \quad x > 0.$$

Solution 3 *Taking* $x = e^t$ *we have* $y'(x) = \frac{1}{x}\dot{y}$, $y''(x) = \frac{1}{x^2}\left(\ddot{y} - \dot{y}\right)$. *Therefore*

$$\begin{aligned} x^2\frac{1}{x^2}\left(\ddot{y} - \dot{y}\right) - 3x\frac{1}{x}\dot{y} + 3y &= 0, \\ \left(\ddot{y} - \dot{y}\right) - 3\dot{y} + 3y &= 0, \\ \ddot{y} - 4\dot{y} + 3y &= 0. \end{aligned}$$

(CH)

$$\lambda^2 - 4\lambda + 3 = 0.$$

The eigenvalues are $\lambda_1 = 1$ *and* $\lambda_2 = 3$. *Therefore we have (FS)*

$$\varphi = e^t = x \ \ \textit{and} \ \ \psi = e^{3t} = x^3.$$

(GS) is $y = Cx + Dx^3$, *where* C, D *are arbitrary constants.*

Example 4 *Solve the Euler equation*

$$x^3 y''' + 4x^2 y'' - 8xy' + 8y = 0, \quad x > 0.$$

Solution 5 *Taking* $x = e^t$ *we have* $y'(x) = \frac{1}{x}\dot{y}$, $y''(x) = \frac{1}{x^2}\left(\ddot{y} - \dot{y}\right)$ *and* $\frac{1}{x^3}\left(\dddot{y} - 3\ddot{y} + 2\dot{y}\right)$. *Therefore*

$$\begin{aligned} \left(\dddot{y} - 3\ddot{y} + 2\dot{y}\right) + 4\left(\ddot{y} - \dot{y}\right) - 8\dot{y} + 8y &= 0, \\ \dddot{y} + \ddot{y} - 10\dot{y} + 8y &= 0. \end{aligned}$$

(CH)

$$\lambda^3 + \lambda^2 - 10\lambda + 8 = 0.$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$ *and* $\lambda_2 = -4$. *Therefore we have (FS)*

$$\varphi = e^t = x, \quad \psi = e^{2t} = x^2 \quad and \quad \eta = e^{-4t} = x^{-4}.$$

(GS) is $y = Cx + Dx^2 + Ex^{-4}$, *where* C, D, E *are arbitrary constants.*

Example 6 *Solve the Euler equation*

$$x^2 y'' - 3xy' + 4y = 0, \quad x > 0.$$

Solution 7 *Taking* $x = e^t$ *we have* $y'(x) = \frac{1}{x}\dot{y}$, $y''(x) = \frac{1}{x^2}\left(\ddot{y} - \dot{y}\right)$. *Therefore*

$$\begin{aligned} x^2 \frac{1}{x^2}\left(\ddot{y} - \dot{y}\right) - 3x\frac{1}{x}\dot{y} + 4y &= 0, \\ \left(\ddot{y} - \dot{y}\right) - 3\dot{y} + 4y &= 0, \\ \ddot{y} - 4\dot{y} + 4y &= 0. \end{aligned}$$

(CH)

$$\lambda^2 - 4\lambda + 4 = 0.$$

We have double eigenvalue $\lambda = 2$. *Therefore we have (FS)*

$$\varphi = e^{2t} = x^2 \quad and \quad \psi = te^{2t} = x^2 \ln x.$$

(GS) is $y = Cx^2 + Dx^2 \ln x$, *where* C, D *are arbitrary constants.*

LINEAR SYSTEMS OF ODE'S (SLDE)

The system of LDE's is in the form

$$\begin{aligned} y_1'(x) &= a_{11}y_1(x) + a_{12}y_2(x) + ... + a_{1n}y_n(x) + r_1(x), \\ y_2'(x) &= a_{21}y_1(x) + a_{22}y_2(x) + ... + a_{2n}y_n(x) + r_2(x), \\ &\ldots\ldots \\ y_{n-1}'(x) &= a_{n-1,1}y_1(x) + a_{n-1,2}y_2(x) + ... + a_{n-1,n}y_n(x) + r_{n-1}(x), \\ y_n'(x) &= a_{n1}y_1(x) + a_{n2}y_2(x) + ... + a_{nn}y_n(x) + r_n(x), \end{aligned}$$

where $a_0, a_1, ..., a_{n-1}$ are given reals and all $r_i(x)$ are given continuous functions. The unknowns are functions $y_1,\ y_2, ...,\ y_n$. Shortly we say

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + ... + a_{1n}y_n + r_1(x), \\ y_2' &= a_{21}y_1 + a_{22}y_2 + ... + a_{2n}y_n + r_2(x), \\ &...... \\ y_{n-1}' &= a_{n-1,1}y_1 + a_{n-1,2}y_2 + ... + a_{n-1,n}y_n + r_{n-1}(x), \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + ... + a_{nn}y_n + r_n(x), \end{aligned}$$

In matrix form it can written down as

$$\begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_{n-1} \\ y_n \end{bmatrix}' = \begin{bmatrix} y_1' \\ y_2' \\ ... \\ y_{n-1}' \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n-1,1} & a_{n-1,2} & ... & a_{n-1,n} \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} r_1(x) \\ r_2(x) \\ ... \\ r_{n-1}(x) \\ r_n(x) \end{bmatrix}.$$

Denote

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ ... \\ y_{n-1} \\ y_n \end{bmatrix},\ A = \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n-1,1} & a_{n-1,2} & ... & a_{n-1,n} \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix},\ R(x) = \begin{bmatrix} r_1(x) \\ r_2(x) \\ ... \\ r_{n-1}(x) \\ r_n(x) \end{bmatrix}.$$

The vector column Y is called the column of unknowns or unknown column vector, A is the matrix of coefficients and $R(x)$ - the column of right-hand sides or right-hand side (column vector). We should also be aware that

$$Y'(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ ... \\ y_{n-1}(x) \\ y_n(x) \end{bmatrix}' = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ ... \\ y_{n-1}'(x) \\ y_n'(x) \end{bmatrix}.$$

Using that notation we have a vector form of the system

$$Y' = AY + R(x).$$

If $R(x) \equiv 0$ we deal with the homogeneous (H) system

$$Y' = AY.$$

Otherwise the system is nonhomogeneous (NH). The GS of the system is the set of all solutions of all LDE. We denote it GSS.

For the system of LDE we also consider IVP (Cauchy problem). It is in the form

$$y_1(x_0) = y_{01},\ y_2(x_0) = y_{02}, ...,\ y_n(x_0) = y_{0n}.$$

In vector form we have

$$Y(x_0) = Y_0,$$

where $Y_0 = \begin{bmatrix} y_{10} \\ y_{20} \\ \dots \\ y_{n-1,0} \\ y_{n0} \end{bmatrix}$ is given vector column. So the IVP is

$$\begin{aligned} Y' &= AY + R(x), \\ Y(x_0) &= Y_0. \end{aligned}$$

The discussion about the methods of solving the (H) from system of two LDE with two unknows. We shall describe the, so called, method of elimination for

$$\begin{cases} y_1' = ay_1 + by_2 \\ y_2' = cy_1 + dy_2 \end{cases},$$

where a, b, c, d are given coefficients. This we do by the following scheme:
a) from 1^{st} DE evaluate y_2 or from 2^{nd} - y_1;
b) differentiate and plug both to the remaining DE.

Example 8 *Solve the system*

$$\begin{cases} y_1' = y_2, \\ y_2' = -2y_1 + 3y_2. \end{cases}$$

Solution 9 *In matrix form we have*

$$Y' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = AY,$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$. *From the first DE we find* y_2 *having*

$$y_2 = y_1'. \tag{2}$$

This function we differentiate obtaining

$$y_2' = y_1''.$$

Plug both to the second equation and get

$$y_1'' = -2y_1 + 3y_1'.$$

Equivalently,

$$y_1'' - 3y_1' + 2y_1 = 0. \tag{3}$$

This is LDE of the 2^{nd} *order with the unknown* y_1. *We solve it by (CH)*

$$\lambda^2 - 3\lambda + 2 = 0.$$

Eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 2$ *and FS*

$$\varphi = e^x \quad and \quad \psi = e^{2x}.$$

The GS is

$$y_1 = Ce^x + De^{2x},$$

where C, D *are arbitrary constants. The evaluation of* y_2 *we do by (2)*

$$y_2 = y_1' = Ce^x + 2De^{2x}.$$

Writing the solution as the column vector we obtain

$$Y = \left[\begin{array}{c} y_1 \\ y_2 \end{array}\right] = \left[\begin{array}{c} Ce^x + De^{2x} \\ Ce^x + 2De^{2x} \end{array}\right] = C\left[\begin{array}{c} e^x \\ e^x \end{array}\right] + D\left[\begin{array}{c} e^{2x} \\ 2e^{2x} \end{array}\right],$$

where C, D *are arbitrary constants. So the GS of the system (GSS) is linear combination of vectors*

$$\Phi = \left[\begin{array}{c} e^x \\ e^x \end{array}\right] \quad and \quad \Psi = \left[\begin{array}{c} e^{2x} \\ 2e^{2x} \end{array}\right].$$

Both are PS of the given system (write) PSS. Φ *we obtain taking* $C = 1, D = 0$, *while* Ψ *for* $C = 0, D = 1$. *We check* $\Psi = \left[\begin{array}{c} e^{2x} \\ 2e^{2x} \end{array}\right]$ *is a vector solution. Namely, we have the solution* $\psi_1 = e^{2x}$ *and* $\psi_2 = 2e^{2x}$.

1st *equation* $L_1 = \psi_1' = 2e^{2x} = \psi_2 = R_1$;

2nd *equation* $L_2 = \psi_2' = 4e^{2x},\ R_2 = -2\psi_1 + 3\psi_2 = -2\left(e^{2x}\right) + 3\left(2e^{2x}\right) = 4e^{2x}$,

$$L_2 = R_2.$$

Observe that used eigenvalues are $\lambda_1 = 1,\ \lambda_2 = 3$. *If we evaluate the characteristic equation of the matrix* $\left[\begin{array}{cc} 0 & 1 \\ -2 & 3 \end{array}\right]$, *we have*

$$\det\left[\begin{array}{cc} 0-\lambda & 1 \\ -2 & 3-\lambda \end{array}\right] = \lambda^2 - 3\lambda + 2 = 0$$

and this the (CH) of the DE (3).

The above considerations lead to the general scheme for arbitrary HS

$$Y' = AY.$$

Definition 10 *The vector functions* $Y_1(x), Y_2(x), ..., Y_n(x)$ *are linearly independent iff*

$$a_1 Y_1(x) + a_2 Y_2(x) + ... + a_n Y_n(x) \equiv 0 \Longrightarrow a_1 = a_2 = ... = a_n = 0.$$

Definition 11 *The matrix* $\mathcal{W}(x) = [Y_1(x), Y_2(x), ..., Y_n(x)]$ *is called the Wroński matrix, while the wrońskian is*

$$W(x) = \det[Y_1(x), Y_2(x), ..., Y_n(x)].$$

Theorem 12 *The GSS(H)* $\mathcal{Y} = \{Y : Y' = AY\}$ *is linear space of the dimension* n*. It means that there exist* $n-$ *linearly independent vector solutions* Φ_1, $\Phi_2, ..., \Phi_n$*. They are called FS of the system - FSS. The GSS(H) is*

$$Y = C_1\Phi_1 + C_2\Phi_2 + ... + C_n\Phi_n,$$

where C_1, $C_2, ..., C_n$ *are arbitrary constants.*

Theorem 13 *If* $W(x) = det[\Phi_1(x), \Phi_2(x), ..., \Phi_n(x)] \neq 0$, *the vector solutions* Φ_1, $\Phi_2, ..., \Phi_n$ *are linearly independent.*

Example 14 *In the previous example we have*

$$FSS: \ \Phi = \begin{bmatrix} e^x \\ e^x \end{bmatrix} \ \ and \ \ \Psi = \begin{bmatrix} e^{2x} \\ 2e^{2x} \end{bmatrix}.$$

The Wroński matrix equals $\mathcal{W}(x) = [\Phi, \Psi] = \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix}$, *while the wrońskian is*

$$W(x) = \det[\Phi, \Psi] = \det \begin{bmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{bmatrix} = e^{3x} \neq 0.$$

It means that FSS Φ *and* Ψ *are linearly independent.*

Example 15 *Write down the solution of the IVP*

$$\begin{cases} y_1' = 2y_1 + y_2, & y_1(0) = 3, \\ y_2' = y_1 + 2y_2, & y_2(0) = -1. \end{cases}$$

Solution 16 *The matrix of coefficients is* $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, *the IC* $Y(0) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$.

From the second DE we evaluate y_1 *having*

$$y_1 = y_2' - 2y_2. \tag{4}$$

Differentiate and obtain

$$y_1' = y_2'' - 2y_2'.$$

Plug both realtions to the first DE and get

$$y_2'' - 2y_2' = 2(y_2' - 2y_2) + y_2.$$

Simplifying we have

$$y_2'' - 4y_2' + 3y_2 = 0. \tag{5}$$

It is again the LDE of 2^{nd} order with the unknown y_2. The (CH) is

$$\lambda^2 - 4\lambda + 3 = 0.$$

The eigenvalue are $\lambda_1 = 1$, $\lambda_2 = 3$. So the FS are e^x and e^{3x}, while the GS is

$$y_2 = Ce^x + De^{3x},$$

where C, D are arbitrary constants. The function y_1 we find from (4).

$$y_1 = y_2' - 2y_2 = \left(Ce^x + De^{3x}\right)' - 2\left(Ce^x + De^{3x}\right).$$

$$y_1 = -Ce^x + De^{3x}.$$

Therefore the GSS in the vector form is

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -Ce^x + De^{3x} \\ Ce^x + De^{3x} \end{bmatrix} = C\begin{bmatrix} -e^x \\ e^x \end{bmatrix} + D\begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}$$

The FSS are

$$\Phi = \begin{bmatrix} -e^x \\ e^x \end{bmatrix} \quad \textit{and} \quad \Psi = \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}.$$

Check for $\Phi = \begin{bmatrix} -e^x \\ e^x \end{bmatrix}$. We have $\psi_1 = -e^x$ and $\psi_2 = e^x$.

$$\begin{array}{rcl} 1^{st} \ \textit{equation} \ L_1 & = & \psi_1' = -e^x, \\ R_1 & = & 2\psi_1 + \psi_2 = -2e^x + e^x = -e^x \\ R_1 & = & L_1; \\ 2^{nd} \ \textit{equation} \ L_2 & = & \psi_2' = e^x, \\ R_2 & = & \psi_1 + 2\psi_2 = -e^x + 2e^x = e^x, \\ R_2 & = & L_2. \end{array}$$

The wronskian is

$$W(x) = \det\begin{bmatrix} -e^x & e^{3x} \\ e^x & e^{3x} \end{bmatrix} = -2e^{4x} \neq 0,$$

so $\Phi = \begin{bmatrix} -e^x \\ e^x \end{bmatrix}$ and $\Psi = \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}$ are linearly independent. Observe also that the used eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$. If we evaluate the characteristic equation of the matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, we have

$$\det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = \lambda^2 - 4\lambda + 3 = 0$$

and this the (CH) of the DE (5).

Now let pass to the IVP. The GSS is

$$Y = Y(x) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C \begin{bmatrix} -e^x \\ e^x \end{bmatrix} + D \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix}.$$

Thus

$$Y(0) = C \begin{bmatrix} -1 \\ 1 \end{bmatrix} + D \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} D - C \\ C + D \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.$$

Solving we get $C = -2$, $D = 1$. *So the answer is*

$$Y = Y(x) = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -2 \begin{bmatrix} -e^x \\ e^x \end{bmatrix} + 1 \begin{bmatrix} e^{3x} \\ e^{3x} \end{bmatrix} = \begin{bmatrix} 2e^x + e^{3x} \\ e^{3x} - 2e^x \end{bmatrix}$$

or $y_1 = 2e^x + e^{3x}$, $y_2 = e^{3x} - 2e^x$.

METHOD OF ELIMINATION IN SOLVING SLDE(NH)

The SLDE(NH) can also be solved by the method of elimination and the scheme is the same.

Example 17 *Solve the system*

$$\begin{cases} y_1' = y_1 - 2y_2 + e^x, \\ y_2' = 2y_1 + y_2 + e^x. \end{cases}$$

Solution 18 *From the second DE we evaluate* y_1 *having*

$$y_1 = \frac{1}{2}\left(y_2' - y_2 - e^x\right). \tag{6}$$

Differentiate and obtain

$$y_1' = \frac{1}{2}\left(y_2'' - y_2' - e^x\right).$$

Plug both realtions to the first DE and get

$$\frac{1}{2}\left(y_2'' - y_2' - e^x\right) = \frac{1}{2}\left(y_2' - y_2 - e^x\right) - 2y_2 + e^x.$$

Simplifying we have

$$y_2'' - 2y_2' + 5y_2 = 2e^x.$$

This time it is again the LDE(NH) of 2^{nd} *order with the unknown* y_2. *The (CH) is*

$$\lambda^2 - 2\lambda + 5 = 0.$$

The eigenvalue are $\lambda_1 = 1 + 2i$, $\lambda_2 = 1 - 2i$. *So the FS are* $e^x \cos(2x)$ *and* $e^x \sin(2x)$, *while the GS(H) is*

$$y_2 = Ce^x \cos(2x) + De^x \sin(2x),$$

where C, D *are arbitrary constants. The PS of (NH) is in the form* $y_p = Ke^x$. *Plugging we have*

$$Ke^x - 2Ke^x + 5Ke^x = 2e^x \Longrightarrow K = \frac{1}{2} \quad and \quad y_p = \frac{1}{2}e^x.$$

So we have the GS of (NH)

$$y_2 = Ce^x \cos(2x) + De^x \sin(2x) + \frac{1}{2}e^x,$$

The function y_1 *we find from (6).*

$$\begin{aligned} y_1 &= \frac{1}{2}\left(y_2' - y_2 - e^x\right) \\ &= \frac{1}{2}\left(Ce^x \cos(2x) + De^x \sin(2x) + \frac{1}{2}e^x\right)' - \left(Ce^x \cos(2x) + De^x \sin(2x) + \frac{1}{2}e^x\right) - \frac{1}{2}e^x \\ &= C\left(-\frac{1}{2}\cos 2x - \sin 2x\right)e^x + D\left(\cos 2x - \frac{1}{2}\sin 2x\right)e^x - \frac{3}{4}e^x \end{aligned}$$

Therefore the GSS in the vector form is

$$\begin{aligned} Y &= \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C\left(-\frac{1}{2}\cos 2x - \sin 2x\right)e^x + D\left(\cos 2x - \frac{1}{2}\sin 2x\right)e^x - \frac{3}{4}e^x \\ Ce^x \cos(2x) + De^x \sin(2x) + \frac{1}{2}e^x \end{bmatrix} \\ &= C\begin{bmatrix} -\frac{1}{2}e^x \cos 2x - e^x \sin 2x \\ e^x \cos 2x \end{bmatrix} + D\begin{bmatrix} e^x \cos 2x - \frac{1}{2}e^x \sin 2x \\ e^x \sin 2x \end{bmatrix} + \begin{bmatrix} -\frac{3}{4}e^x \\ \frac{1}{2}e^x \end{bmatrix} \end{aligned}$$

The FSS are

$$\Phi = \begin{bmatrix} -\left(\frac{1}{2}\cos 2x + \sin 2x\right)e^x \\ e^x \cos 2x \end{bmatrix} \quad and \quad \Psi = \begin{bmatrix} \left(\cos 2x - \frac{1}{2}\sin 2x\right)e^x \\ e^x \sin(2x) \end{bmatrix}.$$

PSS is

$$Y_p = \begin{bmatrix} -\frac{3}{4}e^x \\ \frac{1}{2}e^x \end{bmatrix}.$$

The wrońskian is

$$W(x) = \det\begin{bmatrix} -\left(\frac{1}{2}\cos 2x + \sin 2x\right)e^x & \left(\cos 2x - \frac{1}{2}\sin 2x\right)e^x \\ e^x \cos 2x & e^x \sin(2x) \end{bmatrix} = -2e^{2x} \neq 0.$$