LECTURE 02. 06. 2020

THE LAPLACE TRANSFORM APPLIED TO ODE's

The application of the Laplace transform to IVP's for LDE or SLDE with constant coefficients is based on the following theorem:

Theorem 1 *If* $L[y(x)] = Y(s)$ *then*
a) $L[y'(x)] = sY(s) - y(0)$,
b) $L[y''(x)] = s^2 Y(s) - sy(0) - y'(0)$.
c) $L\left[y^{(n)}(x)\right] = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - ... - sy^{(n-2)}(0) - y^{(n-1)}(0)$.

We present the methods just on examples. Basically, because of the above formulas the IC's have to be given at 0, but at the end we shall show, how to transform arbitrary IC to IC at 0.

Example 2 *Solve the IVP*

$$y' + 2y = e^x, \quad y(0) = -3.$$

Solution 3 *Start with* $L[y(x)] = Y(s)$. *Then*

$$L[y'(x)] = sY(s) - y(0) = sY(s) + 3.$$

The Laplace transform from both sides of the given ODE having

$$\begin{aligned} L[y' + 2y] &= L[e^x], \\ L[y'] + 2L[y] &= L[e^x], \\ sY(s) + 3 + 2Y(s) &= \frac{1}{s-1}, \\ sY(s) + 2Y(s) &= \frac{1}{s-1} - 3 = \frac{4-3s}{s-1}, \\ (s+2)Y(s) &= \frac{4-3s}{s-1}. \end{aligned}$$

The ODE is therefore transformed to an algebraic equation. Hence

$$\begin{aligned} (s+2)Y(s) &= \frac{4-3s}{s-1}, \\ L[y] &= Y(s) = \frac{4-3s}{(s-1)(s+2)} \\ y(x) &= L^{-1}\left[\frac{4-3s}{(s-1)(s+2)}\right]. \end{aligned}$$

We need to evaluate the Laplace inverse transform we decompose $\frac{4-3s}{(s-1)(s+2)}$ *into the partial fractions:*

$$\begin{aligned}\frac{4-3s}{(s-1)(s+2)} &= \frac{A}{s-1}+\frac{B}{s+2},\\ 4-3s &= A(s+2)+B(s-1),\\ 4-3s &= (A+B)s+(2A-B).\end{aligned}$$

Comparing the coefficients we have the system

$$A+B=-3, \quad 2A-B=4.$$

Solve it and get

$$A=\frac{1}{3},\;\; B=-\frac{10}{3}.$$

Therefore

$$\begin{aligned} y(x) &= L^{-1}\left[\frac{4-3s}{(s-1)(s+2)}\right] = L^{-1}\left[\frac{\frac{1}{3}}{s-1}-\frac{\frac{10}{3}}{s+2}\right] = \\ &= \frac{1}{3}L^{-1}\left[\frac{1}{s-1}\right]-\frac{10}{3}L^{-1}\left[\frac{1}{s+2}\right] = \frac{1}{3}e^{x}-\frac{10}{3}e^{-2x}. \end{aligned}$$

Answer: The IVP possessess the solution $y(x)=\frac{1}{3}e^{x}-\frac{10}{3}e^{-2x}$.

Example 4 *Solve the IVP*

$$y'-3y=\sin 2x, \quad y(0)=1.$$

Solution 5 *Start with* $L[y(x)]=Y(s)$. *Then*

$$L[y'(x)]=sY(s)-y(0)=sY(s)-1.$$

The Laplace transform from both sides of the given ODE having

$$\begin{aligned} L[y'-3y] &= L[\sin x],\\ L[y']-3L[y] &= L[\sin x],\\ sY(s)-1-3Y(s) &= \frac{2}{s^2+4},\\ sY(s)-3Y(s) &= \frac{2}{s^2+4}+1=\frac{s^2+6}{s^2+4}. \end{aligned}$$

Solution 6 *The ODE is transformed to an algebraic equation. Hence*

$$\begin{aligned} (s-3)Y(s) &= \frac{s^2+6}{s^2+4}\\ L[y] &= Y(s)=\frac{s^2+6}{(s^2+4)(s-3)}\\ y(x) &= L^{-1}\left[\frac{s^2+6}{(s^2+4)(s-3)}\right]. \end{aligned}$$

To evaluate the Laplace inverse transform we decompose $\frac{s^2+6}{(s^2+4)(s-3)}$ *into the partial fractions:*

$$\begin{aligned}
\frac{s^2+6}{(s^2+4)(s-3)} &= \frac{As+B}{s^2+4}+\frac{C}{s-3}, \\
s^2+6 &= (As+B)(s-3)+C\left(s^2+4\right), \\
s^2+6 &= (A+C)s^2+(B-3A)s+(4C-3B).
\end{aligned}$$

Comparing the coefficients we have the system

$$A+C=1, \quad B-3A=0, \quad 4C-3B=6.$$

Solving we get

$$A=-\frac{2}{13}, \quad B=-\frac{6}{13}, \quad C=\frac{15}{13}.$$

Therefore

$$\begin{aligned}
y(x) &= L^{-1}\left[\frac{s^2+6}{(s^2+4)(s-3)}\right]=L^{-1}\left[\frac{-\frac{2}{13}s-\frac{6}{13}}{s^2+4}+\frac{\frac{15}{13}}{s-3}\right] \\
&= -\frac{2}{13}L^{-1}\left[\frac{s}{s^2+4}\right]-\frac{6}{13}L^{-1}\left[\frac{1}{s^2+4}\right]+\frac{15}{13}L^{-1}\left[\frac{1}{s-3}\right]= \\
&= -\frac{2}{13}\cos 2x-\frac{3}{13}\sin 2x+\frac{15}{13}e^{3x}
\end{aligned}$$

Answer: The IVP has the solution $y(x)=-\frac{2}{13}\cos 2x-\frac{3}{13}\sin 2x+\frac{15}{13}e^{3x}$.

Example 7 *Solve the IVP*

$$y'+y=2x, \quad y(1)=1.$$

Solution 8 *We have the IVP*

$$y'(x)+y(x)=2x, \quad y(1)=1.$$

First shift the unknown function $y(x)$ *by* -1 *in* x*. Namely, we take*

$$z(x)=y(x+1).$$

Hence $z'(x)=y'(x+1)$ *and* $z(0)=y(0+1)=y(1)=1$*. Therefore*

$$z'(x)+z(x)=y'(x+1)+y(x+1)=2(x+1), \quad z(0)=1.$$

Start with $L[z(x)]=Z(s)$*. Then*

$$L[z'(x)]=sZ(s)-z(0)=sZ(s)-1.$$

The Laplace transform from both sides of the given ODE having

$$\begin{aligned} L\left[z' + z\right] &= L\left[2x+2\right], \\ L\left[z'\right] + L\left[z\right] &= \frac{2}{s^2} + \frac{2}{s} = \frac{2+2s}{s^2}, \\ sZ(s) - 1 + Z(s) &= \frac{2+2s}{s^2}, \\ sZ(s) + Z(s) &= \frac{2+2s}{s^2} + 1 = \frac{s^2+2s+2}{s^2}. \end{aligned}$$

The ODE is therefore transformed to an algebraic equation. Hence

$$\begin{aligned} (s+1)\,Z(s) &= \frac{s^2+2s+2}{s^2}, \\ L\left[z\right] &= Z(s) = \frac{s^2+2s+2}{s^2(s+1)} \\ z(x) &= L^{-1}\left[\frac{s^2+2s+2}{s^2(s+1)}\right]. \end{aligned}$$

We need to evaluate the Laplace inverse transform. We decompose $\frac{s^2+2s+2}{s^2(s+1)}$ *into the partial fractions:*

$$\begin{aligned} \frac{s^2+2s+2}{s^2(s+1)} &= \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s^2}, \\ s^2+2s+2 &= As^2 + Bs(s+1) + C(s+1), \\ s^2+2s+2 &= (A+B)s^2 + (B+C)s + C. \end{aligned}$$

Comparing the coefficients we have the system

$$A+B=1, \quad B+C=2 \qquad C=2.$$

Solve it and get

$$A=1, \quad B=0 \qquad C=2.$$

Therefore

$$\begin{aligned} y(x+1) &= z(x) = L^{-1}\left[\frac{s^2+2s+2}{s^2(s+1)}\right] = L^{-1}\left[\frac{1}{s+1} + \frac{2}{s^2}\right] = \\ &= L^{-1}\left[\frac{1}{s+1}\right] + 2L^{-1}\left[\frac{1}{s^2}\right] = e^{-x} + 2x. \end{aligned}$$

Thus shifting back we have

$$\begin{aligned} y(x+1) &= e^{-x} + 2x, \\ y(x) &= e^{-(x-1)} + 2(x-1). \end{aligned}$$

Answer: The IVP possessess the solution $y(x) = e^{-(x-1)} + 2(x-1)$.

Example 9 *Solve the IVP*

$$y'' - 3y' + 2y = e^x, \quad y(0) = 1, \quad y'(0) = 1.$$

Solution 10 *Start with* $L[y(x)] = Y(s)$. *Then*

$$\begin{aligned} L[y'(x)] &= sY(s) - y(0) = sY(s) - 1, \\ L[y''(x)] &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 1 \end{aligned}$$

The Laplace transform from both sides of the given ODE having

$$\begin{aligned} L[y'' - 3y' + 2y] &= L[e^x], \\ L[y''] - 3L[y'] + 2L[y] &= L[e^x] \\ s^2Y(s) - s - 1 - 3(sY(s) - 1) + 2Y(s) &= \frac{1}{s-1}, \\ s^2Y(s) - 3sY(s) + 2Y(s) &= \frac{1}{s-1} + s - 5 = \frac{s^2 - 6s + 6}{s-1}. \end{aligned}$$

The ODE is therefore transformed to an algebraic equation. Hence

$$\begin{aligned} \left(s^2 - 3s + 2\right) Y(s) &= \frac{s^2 - 6s + 6}{s-1}, \\ L[y] &= Y(s) = \frac{s^2 - 6s + 6}{(s-1)(s^2 - 3s + 2)} = \frac{s^2 - 6s + 6}{(s-1)^2 (s-2)} \\ y(x) &= L^{-1}\left[\frac{s^2 - 6s + 6}{(s-1)^2 (s-2)}\right]. \end{aligned}$$

We need to evaluate the Laplace inverse transform we decompose $\frac{s^2-6s+6}{(s-1)^2(s-2)}$ *into the partial fractions:*

$$\begin{aligned} \frac{s^2 - 6s + 6}{(s-1)^2 (s-2)} &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2}, \\ s^2 - 6s + 6 &= A(s-1)(s-2) + B(s-2) + C(s-1)^2, \\ s^2 - 6s + 6 &= (A + C)s^2 + (B - 3A - 2C)s + (2A - 2B + C) \end{aligned}$$

Comparing the coefficients we have the system

$$A + C = 1, \quad B - 3A - 2C = -6, \quad 2A - 2B + C = 6.$$

Solve it and get

$$A = 3, \quad B = -1, \quad C = -2, \quad .$$

Therefore

$$\begin{aligned} y(x) &= L^{-1}\left[\frac{3}{s-1} - \frac{1}{(s-1)^2} - \frac{2}{s-2}\right] = 3L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{(s-1)^2}\right] - 2L^{-1}\left[\frac{1}{s-2}\right] = \\ &= 3e^x - xe^x - 2e^{2x} \end{aligned}$$

Answer: The IVP possessess the solution $y(x) = 3e^x - xe^x - 2e^{2x}$.

Example 11 *Solve the IVP*

$$y'' - 2y' + y = e^x, \quad y(0) = 1, \quad y'(0) = 2.$$

Solution 12 *Start with* $L[y(x)] = Y(s)$. *Then*

$$\begin{aligned} L[y'(x)] &= sY(s) - y(0) = sY(s) - 1, \\ L[y''(x)] &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s - 2 \end{aligned}$$

Take the Laplace transform from both sides of the given ODE having

$$\begin{aligned} L[y'' - 2y' + y] &= L[e^x], \\ L[y''] - 2L[y'] + L[y] &= \frac{1}{s-1} \\ s^2Y(s) - s - 2 - 2(sY(s) - 1) + Y(s) &= \frac{1}{s-1}, \\ s^2Y(s) - 2sY(s) + Y(s) &= \frac{1}{s-1} + s. \end{aligned}$$

The ODE is therefore transformed to an algebraic equation. Hence

$$\begin{aligned} \left(s^2 - 2s + 1\right) Y(s) &= \frac{s^2 - s + 1}{s-1} \\ L[y] &= Y(s) = \frac{s^2 - s + 1}{(s-1)^3} \\ y(x) &= L^{-1}\left[\frac{s^2 - s + 1}{(s-1)^3}\right]. \end{aligned}$$

To evaluate the Laplace inverse transform we apply the shifting is s :

$$\begin{aligned} y(x) &= L^{-1}\left[\frac{s^2 - s + 1}{(s-1)^3}\right] = \left\{ \begin{array}{c} \overline{s} = s - 1, \\ s = \overline{s} + 1, \quad a = 1 \end{array} \right\} \\ &= e^x L^{-1}\left[\frac{(\overline{s}+1)^2 - (\overline{s}+1) + 1}{(\overline{s})^3}\right] = \left\{ \begin{array}{c} \textit{skip} \\ \textit{the} \quad \textit{bar} \end{array} \right\} =, \\ &= e^x L^{-1}\left[\frac{s^2 + s + 1}{s^3}\right] = e^x \left\{1 + x + \frac{x^2}{2}\right\}. \end{aligned}$$

Answer: The IVP possessess the solution $y(x) = e^x \left\{1 + x + \frac{x^2}{2}\right\}$.

Example 13 *Solve the IVP*

$$\begin{cases} y_1' = -3y_1 + 4y_2, & y_1(0) = -1 \\ y_2' = -2y_1 + 3y_2, & y_2(0) = 3 \end{cases}$$

Solution 14 *We start with* $L[y_1(x)] = Y_1(s)$ *and* $L[y_2(x)] = Y_2(s)$. *Then*

$$\begin{aligned} L[y_1'(x)] &= sY_1(s) - y_1(0) = sY_1(s) + 1, \\ L[y_2'(x)] &= sY_2(s) - y_2(0) = sY_2(s) - 3. \end{aligned}$$

Take the Laplace transform from both sides of both given ODE's having

$$\begin{cases} L[y_1'] = -3L[y_1] + 4L[y_2], \\ L[y_2'] = -2L[y_1] + 3L[y_2]. \end{cases}$$

$$\begin{cases} sY_1(s) + 1 = -3Y_1(s) + 4Y_2(s), \\ sY_2(s) - 3 = -2Y_1(s) + 3Y_2(s). \end{cases}$$

$$\begin{cases} (s+3)Y_1(s) - 4Y_2(s) = -1, \\ 2Y_1(s) + (s-3)Y_2(s) = 3. \end{cases}$$

So we have a linear system of two equations with two unknowns $Y_1(s)$ *and* $Y_2(s)$. *The simplest way of solving this system is by Cramer's rule.*

$$\begin{aligned} W(s) &= \det \begin{bmatrix} s+3 & -4 \\ 2 & s-3 \end{bmatrix} = s^2 - 1, \\ W_1(s) &= \det \begin{bmatrix} -1 & -4 \\ 3 & s-3 \end{bmatrix} = 15 - s, \\ W_2(s) &= \det \begin{bmatrix} s+3 & -1 \\ 2 & 3 \end{bmatrix} = 3s + 11. \end{aligned}$$

Thus

$$\begin{aligned} Y_1(s) &= \frac{W_1(s)}{W(s)} = \frac{15-s}{s^2-1}, \quad Y_2(s) = \frac{W_2(s)}{W(s)} = \frac{3s+11}{s^2-1}. \\ y_1(x) &= L^{-1}\left[\frac{15-s}{s^2-1}\right], \quad y_2(x) = L^{-1}\left[\frac{3s+11}{s^2-1}\right]. \end{aligned}$$

Both unknowns we ewaluate by decomposition in partial fractions. We have

a)

$$\begin{aligned} \frac{15-s}{s^2-1} &= \frac{15-s}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{A(s+1) + B(s-1)}{(s-1)(s+1)}, \\ 15 - s &= (A+B)s + (A-B), \\ A + B &= -1, \quad A - B = 15, \\ A &= 7, \quad B = -8. \end{aligned}$$

Therefore

$$y_1(x) = L^{-1}\left[\frac{15-s}{s^2-1}\right] = L^{-1}\left[\frac{7}{s-1} - \frac{8}{s+1}\right] = 7e^x - 8e^{-x}.$$

Example 15 *b)*

$$\begin{aligned}
\frac{3s+11}{s^2-1} &= \frac{3s+11}{(s-1)(s+1)} = \frac{C}{s-1} + \frac{D}{s+1} = \frac{C(s+1)+D(s-1)}{(s-1)(s+1)}, \\
3s+11 &= (C+D)s + (C-D), \\
C+D &= 3, \qquad C-D = 11, \\
C &= 7, \qquad B = -4.
\end{aligned}$$

Therefore

$$y_2(x) = L^{-1}\left[\frac{7s-4}{s^2-1}\right] = L^{-1}\left[\frac{7}{s-1} - \frac{4}{s+1}\right] = 7e^{x} - 4e^{-x}.$$

Answer:

$$y_1(x) = 7e^{x} - 8e^{-x}, \quad y_2(x) = 7e^{x} - 4e^{-x}.$$

Example 16 *Solve the IVP*

$$\begin{cases} y_1' = y_2 + 2, & y_1(0) = 1 \\ y_2' = -y_1 + 1, & y_2(0) = 2. \end{cases}$$

Solution 17 *We start with* $L[y_1(x)] = Y_1(s)$ *and* $L[y_2(x)] = Y_2(s)$. *Then*

$$\begin{aligned}
L[y_1'(x)] &= sY_1(s) - y_1(0) = sY_1(s) - 1, \\
L[y_2'(x)] &= sY_2(s) - y_2(0) = sY_2(s) - 2.
\end{aligned}$$

Take the Laplace transform from both sides of both given ODE's having

$$\begin{cases} L[y_1'] = L[y_2] + L[2], \\ L[y_2'] = -L[y_1] + L[1] \;. \end{cases}$$

$$\begin{cases} sY_1(s) - 1 = Y_2(s) + \frac{2}{s}, \\ sY_2(s) - 2 = -Y_1(s) + \frac{1}{s}. \end{cases}$$

$$\begin{cases} sY_1(s) - Y_2(s) = \frac{2}{s} + 1 = \frac{s+2}{s}, \\ Y_1(s) + sY_2(s) = \frac{1}{s} + 3 = \frac{3s+1}{s}. \end{cases}$$

So we have abain a linear system of two equations with two unknowns $Y_1(s)$ *and* $Y_2(s)$. *We solve it by Cramer's rule.*

$$\begin{aligned}
W(s) &= \det\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = s^2 + 1, \\
W_1(s) &= \det\begin{bmatrix} \frac{s+2}{s} & -1 \\ \frac{3s+1}{s} & s \end{bmatrix} = \frac{s^2+5s+1}{s}, \\
W_2(s) &= \det\begin{bmatrix} s & \frac{s+2}{s} \\ 1 & \frac{3s+1}{s} \end{bmatrix} = \frac{3s^2-2}{s}.
\end{aligned}$$

Thus

$$\begin{aligned} Y_1(s) &= \frac{W_1(s)}{W(s)} = \frac{s^2+5s+1}{s(s^2+1)}, \quad Y_2(s) = \frac{W_2(s)}{W(s)} = \frac{3s^2+8s-1}{s(s^2+1)}. \\ y_1(x) &= L^{-1}\left[\frac{s^2+5s+1}{s(s^2+1)}\right], \quad y_2(x) = L^{-1}\left[\frac{3s^2-2}{s(s^2+1)}\right]. \end{aligned}$$

Both unknowns we ewaluate by decomposition in partial fractions. We have
a)

$$\begin{aligned} \frac{s^2+5s+1}{s(s^2+1)} &= \frac{K}{s} + \frac{Ls+M}{s^2+1}, \\ s^2+5s+1 &= (K+L)s^2 + Ms + K, \\ K+L &= 1, \quad M=5, \quad K=1, \\ K &= 1, \quad L=0, \quad M=5 \end{aligned}$$

Therefore

$$y_1(x) = L^{-1}\left[\frac{1}{s} + \frac{5}{s^2+1}\right] = 1 + 5\sin x.$$

b)

$$\begin{aligned} \frac{3s^2-2}{s(s^2+1)} &= \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{A\left(s^2+1\right) + (Bs+C)s}{s(s^2+1)}, \\ 3s^2-2 &= (A+B)s^2 + Cs + A, \\ A+B &= 3, \quad C=0, \quad A=-2, \\ A &= -2, \quad B=5, \quad C=0. \end{aligned}$$

Therefore

$$y_2(x) = L^{-1}\left[\frac{3s^2-2}{s(s^2+1)}\right] = L^{-1}\left[\frac{-2}{s} + \frac{5s}{s^2+1}\right] = -2 + 5\cos x.$$

Answer:

$$y_1(x) = 1 + 5\sin x, \quad y_2(x) = -1 + 5\cos x.$$

CONVOLUTION

Given integrable functions $f(x)$ and $g(x)$ for $x \geq 0$.

Definition 18 *By convolution of $f(x)$ and $g(x)$ it is meant the function*

$$(f * g)(x) = \int_0^x f(t)\, g(x-t)\, dt$$

defined for $x \geq 0$.

Example 19 *a)* $(e^x * 1)(x) = \int_0^x e^t \cdot 1 dt = e^x - 1$;

b) $(e^x * e^x) = \int_0^x e^t \cdot e^{x-t} dt = \int_0^x e^x dt = xe^x$;

c) $(x * x) = \int_0^x t \cdot (x-t)\, dt = \int_0^x \left(xt - t^2\right) dt = \frac{1}{6}x^3$.

PROPERTIES:

a) $f * g = g * f$;

b) $(f * g) * h = f * (g * h)$;

c) $(af + bg) * h = a(f * h) + b(g * h)$;

d) $f * 0 = 0 * f = 0$.

Theorem 20 *a)* $L[f * g] = L[f] \cdot L[g]$;

b) $L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$.

Example 21 *a) Find* $L^{-1}\left[\frac{1}{s(s-1)}\right] = L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s-1}\right] = 1 * e^x = e^x - 1$;

b) Find $L^{-1}\left[\frac{1}{s(s^2+1)}\right] = L^{-1}\left[\frac{1}{s}\right] * L^{-1}\left[\frac{1}{s^2+1}\right] = 1 * \sin x = \int_0^x \sin t \cdot 1 dt = 1 - \cos x$;

c) Find $L^{-1}\left[\frac{1}{s^4}\right] = L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s^2}\right] = x * x = \frac{x^3}{6}$.