Lineare Algebra

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1 Orthogonality, Projections and Least squares

1.1 Projection

Problem description

subspace $S \subset \mathbb{R}^m$ and $b \in \mathbb{R}^m$ and $b \notin S$

$$\textit{the closest point } p \in S \textit{ to } b \textit{ is orthogonally beneath } b$$
$$b - p \perp S$$

$S = span(a_1, ..., a_n)$

let A be:

$$A = \begin{bmatrix} | & & | \\ a_1 & ... & a_n \\ | & & | \end{bmatrix}$$

Note: columns of A are linearly independent

Normal equation

$$\begin{aligned} \text{since } b - p \perp S \quad &\Rightarrow \quad a_k \perp b - p \\ &\Rightarrow \quad A^T(b-p) = 0 \\ &\Leftrightarrow \quad A^T A\hat{x} = A^T b \quad | \quad p = A\hat{x} \end{aligned}$$

This is called the Normal equation.

Invertibility of $A^T A$

Note A has independent columns $\quad \Rightarrow \quad N(A) = \{0\} \quad | \quad (Ax = a_1x_1 + a_2x_2 + ... + a_nx_n)$

Hence if $N(A^T A) = N(A) \quad \Rightarrow \quad A^T A$ has full rank $\quad \Rightarrow \quad A^T A$ is invertible

Proof *that A^TA is invertible, i.e.* $N(A^TA) = N(A)$:

$N(A) \subset N(A^TA)$:

$$x \in N(A) \quad \Rightarrow \quad Ax = 0 \quad \Rightarrow \quad A^T(Ax) = A^T 0 = 0$$

$N(A^TA) \subset N(A)$:

$$x \in N(A^TA) \quad \Rightarrow \quad A^TAx = 0$$

$$\Rightarrow \quad x^TA^TAx = 0$$

$$\Leftrightarrow \quad (Ax)^TAx = 0$$

$$\Leftrightarrow \quad \|Ax\|^2 = 0$$

$$\Leftrightarrow \quad \|b\|^2 = 0 \quad | \quad Ax = b$$

$$\Leftrightarrow \quad {b_1}^2 + \ldots + {b_k}^2 = 0$$

$$\Rightarrow \quad 0 + \ldots + 0 = 0$$

$$\Rightarrow \quad b = 0$$

$$\Leftrightarrow \quad Ax = 0$$

$$\Rightarrow \quad x \in N(A)$$

Hence A^TA is invertible

Final Solution

$\hat{x} = (A^TA)^{-1}A^Tb$

$\Rightarrow \quad p = A\hat{x} = A(A^TA)^{-1}A^Tb$

Hence our projection matrix is $A(A^TA)^{-1}A^Tb$