Study on water rockets
Comparison of theory and experiment

To what extent do theoretical values of the flight of the water rocket correspond to experimental ones? Comparison of two theoretical approaches to the experiment.

Word count: 3983

Subject: Physics

Candidate personal code: jxm461

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1. Introduction

How can we know what is true and what is not? We tried to answer this question on Theory of Knowledge (TOK) classes, using concepts, which include for example personal and shared knowledge and coherence test. This led me to the question whether equations we learn in physics lessons are really true. As a result I decided to conduct an experiment, whose aim was to determine, to what extent do theoretical model correspond to practical results. In order to investigate this problem, I used a water rocket as a model. Water rocket is a type of rocket that uses water and compressed air as a "fuel" to drive the rocket upwards [1]. In most cases it is built out of a plastic bottle and a launchpad. There are also examples of 2-stage water rockets, which use two bottles filled with water.

Watching videos on Tom Stanton's YouTube [2] channel made me interested in pressure driven models. He gave me the inspiration to build a model similar to his, which use pressurized air as a main source of energy. Perhaps one of the simplest objects of this kind are water rockets, which are also enjoyable design and construct.

The aim of this essay is to plot the graph of altitude reached by the rocket versus time and initial volume of the water (later referred to as IVW) in a rocket and compare theoretical results to experimental ones. In the derivation of the theoretical formulas I based my derivation on the works of Ivan Meshchersky [3] and Konstantin Tsiolkovsky [4]. Both scientists contributed to the description of bodies with variable mass, which is especially helpful in astrophysics when a rocket decreases its mass by expelling a propellant.

Most of the existing research papers rely mainly on analysis of either height versus time, or height versus IVW. Therefore my aim was to put together two parameters that define the height of the rocket – time and IVW – and compare them to experimental results. This is crucial, as the theory cannot explain any phenomenon, if it has no resemblance in reality.

2. Theory

The motion of the water rocket can be divided into 2 stages: acceleration stage and flight after burnout. The former one occurs, when there is water inside the rocket and the latter one can be described using projectile motion with initial speed at the moment of burnout. The acceleration stage is more complex, as it required analysis of multiple factors, including consideration of the type of air expansion – isothermal or adiabatic.

In order to investigate a water rocket, an analysis of the forces acting on it is crucial. There are 3 forces acting on a rocket (see Figure 1.). These are thrust, weight (F_g) and drag force (F_d) caused by air resistance. The drag on the rocket is rather small compared to weight. Reason for this fact is the small horizontal cross-sectional area of the rocket, small air density and relatively low speed of the rocket. Therefore the drag force will be neglected in further calculations, but evaluated in the "Evaluation" section.

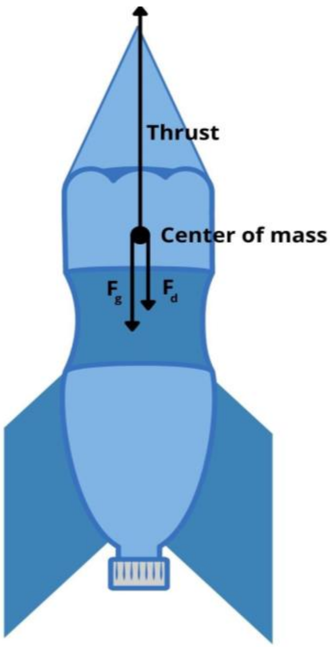

Figure 3. Model of forces acting on the rocket.

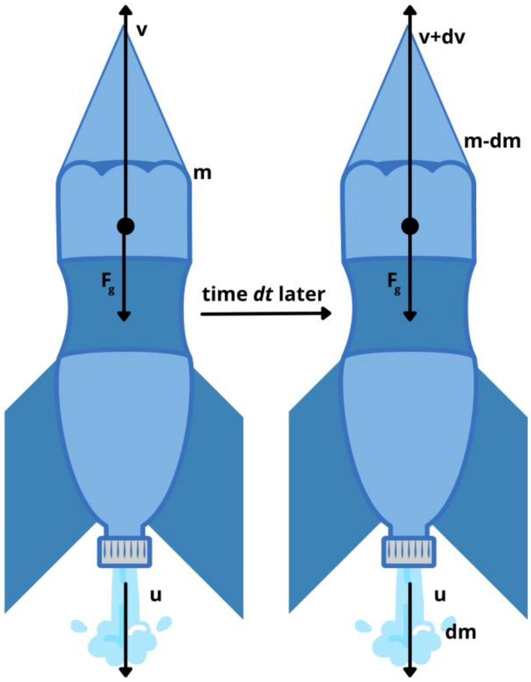

Figure 2. Water rocket in given time (on the left) and some small time *dt* later (on the right).

The theory behind the flight of water rocket can be derived from Newton's third law of motion, which states:

Whenever one object exerts a force on a second object, the second object exerts an equal and opposite force on the first [5].

So, a water exiting the nozzle of the rocket exerts an upward force on the rocket, resulting in propelling the rocket upwards. These forces form a relation, which is described by a first order linear differential equation, known as Meschersky equation [6]:

$$m\frac{d\vec{v}}{dt} = \vec{F} + \frac{dm}{dt}\vec{u} \tag{1}$$

Where:

m - the mass of the rocket.

$\vec{v}$ - the velocity of the rocket.

$\vec{F}$ - external net force acting on the rocket.

$\vec{u}$ - velocity, with which the propellant is exiting the nozzle with respect to the rocket.

t – the time

In order to find a maximum value of altitude of the rocket a function of velocity would be helpful, which is a derivative of displacement with respect to time. This requires solving differential equation (1). In the specific case of the rocket, the external force F is the weight of the rocket described as:

$$\vec{F} = -mg \qquad (2)$$

Equation one is uses vector quantities, so in order to get scalar ones one has to put minus sign in front of $\vec{u}$. This is because the velocity vector of exiting water is opposite to the velocity vector of the rocket. The transformation together with substitution of equation (2) into equation (1) gives:

$$m\frac{dv}{dt} = -mg - \frac{dm}{dt}u \qquad (3)$$

Note that now u and v became a scalars instead of vectors. Firstly let me define crucial parameters:

v_i- instantaneous velocity of the rocket.

v_0- initial velocity of the rocket.

u - velocity, with which the propellant is exiting the nozzle with respect to the rocket.

g- acceleration of free fall $g = 9.81$ [m s^{-2}] [7]

m_0 - initial mass of the rocket, which consists of mass of the rocket itself and mass of the propellant.

m_i - the instantaneous mass of the rocket, which depends on time.

t_i – instantaneous time

t_0 – initial time

Then, solving differential equation and working towards obtaining a function $v(t)$.

$$\frac{dv}{dt} = -g - \frac{1}{m}\frac{dm}{dt}u \tag{4}$$

$$dv = -g\ dt - \frac{1}{m}\ dm\, u \tag{5}$$

$$\int_{v_0}^{v_i} dv = -\int_{t_0}^{t_i} g\ \ dt - u\int_{m_0}^{m_i}\frac{dm}{m} \tag{6}$$

$$v_i - v_0 = -g\ (t_i - t_0) - u\ (\ln|m_i| - \ln|m_o|) \tag{7}$$

Now, if there were no forces acting on the rocket, e.g. it travelled in space, then equation (7) could be written in shortened form as Tsiolkovsky rocket equation [6]:

$$\Delta v = u \ln\left|\frac{m_0}{m_i}\right| \tag{8}$$

Where Δv is change of velocity.

However, near the Earth's surface, gravitation plays an important role. Further rearrangement of the equation (7) leads to:

$$v_i = v_0\ - g(t_i - t_0) - u \ln\left|\frac{m_i}{m_0}\right| \tag{9}$$

$$v_i = v_0\ - g(t_i - t_0) + u \ln\left|\frac{m_0}{m_i}\right| \tag{10}$$

Now the term *u* must be defined, which can be done by using Bernoulli's equation:

$$P = \rho\frac{u^2}{2} \tag{11}$$

Where *P* is the instantaneous pressure inside the rocket and ρ is the density of water ($\rho = 997$ [kg m^{-3}]). Rearranging (11) gives expression for *u*:

$$u = \sqrt{\frac{2\,P}{\rho}} \tag{12}$$

The pressure however depends on the volume of the air inside the rocket. The relation then depends on, whether one takes into consideration isothermal or adiabatic process of air expansion. The equations of both assumptions will be derived here, but the real life situation will be discussed in “Evaluation” section.

2.1. Isothermal expansion

Isothermal expansion of air will be considered first, as it is easier to derive. During the isothermal processes, the temperature remains constant, hence pressure and volume of a gas obey the following relation [8]:

$$P\,V = P_0\,V_0 \tag{13}$$

$$P = \frac{P_0\,V_0}{V} \tag{14}$$

Where V is the volume of air inside the rocket, P_0 is the initial pressure of air and V_0 is the initial volume of air. Now the term V must be determined. Let me consider another relation, which is the volume of water exiting the nozzle in small time interval dt:

$$\frac{dV}{dt} = S\,u \tag{15}$$

Where S is the area of the nozzle given by:

$$S = \frac{\pi}{4} {D_n}^2 \tag{16}$$

Where D_n is the diameter of the nozzle. Substituting expression for u (12) into (15) and working towards expression for V:

$$\frac{dV}{dt} = S\sqrt{\frac{2\,P}{\rho}} \tag{17}$$

$$\frac{dV}{dt} = S\sqrt{\frac{2\,P}{\rho\,V}} \tag{18}$$

$$dV = S\sqrt{\frac{2\,P_0\,V_0}{\rho\,V}}\;dt \tag{19}$$

$$\int \sqrt{V}\,dV = \int S\sqrt{\frac{2\,P_0\,V_0}{\rho}}\;dt \tag{20}$$

$$\frac{2\,V^{\frac{3}{2}}}{3} = S\sqrt{\frac{2\,P_0\,V_0}{\rho}}\;t + C \tag{21}$$

$$V = \left(\frac{3}{2}\,S\sqrt{\frac{2\,P_0\,V_0}{\rho}}\;t + \frac{3}{2}C\right)^{\frac{2}{3}} \tag{22}$$

The constant C is the initial volume of the air, hence:

$$V_0 = \frac{3\,C^{\frac{2}{3}}}{2} \tag{23}$$

$$C = \left(\frac{2}{3}V_0\right)^{\frac{3}{2}} \tag{24}$$

Substituting C into (22):

$$V = \left(\frac{3}{2}\,S\sqrt{\frac{2\,P_0\,V_0}{\rho}}\;t + \frac{3}{2}\left(\frac{2}{3}V_0\right)^{\frac{3}{2}}\right)^{\frac{2}{3}} \tag{25}$$

Substituting (25) into (14) and (14) into (12)gives:

$$u = \sqrt{\frac{2\,P_0\,V_0}{\rho\left(\frac{3}{2}\,S\sqrt{\frac{2\,P_0\,V_0}{\rho}}\;t + \frac{3}{2}\left(\frac{2}{3}V_0\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}}} \tag{26}$$

Now let me consider terms m_0 and m_i. The initial mass of rocket consists of the mass of the rocket itself (m_r) and initial mass of the water (m_w) or initial volume of water (V_w):

$$m_0 = m_r + m_w = m_r + \rho V_w \tag{27}$$

While initial mass is constant, the instantaneous mass depends on the volume of the air inside the rocket, hence time:

$$m_i = m_r + \rho\,(V_r - V) \tag{28}$$

$$m_i = m_r + \rho\left(V_r - \left(\frac{3}{2}\,S\sqrt{\frac{2\,P_0\,V_0}{\rho}}\;t + \frac{3}{2}\left(\frac{2}{3}V_0\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}\right) \tag{29}$$

It is important to note, that the initial volume of the water and initial volume of the air sum up to volume of the rocket:

$$V_r = V_0 + V_w \tag{30}$$

Using this relation and substituting the terms *u,* m_0 and m_f into equation (10) leads to expression for the velocity of the rocket during the acceleration stage assuming isothermal expansion of the air (note that v_0 is zero and $t_0=0$):

$$v = -gt + \sqrt{\frac{2\,P_0\,(V_r - V_w)}{\rho\left(\frac{3}{2}\,S\sqrt{\frac{2\,P_0\,\,(V_r - V_w)}{\rho}}\,\,t + \frac{3}{2}\left(\frac{2}{3}(V_r - V_w)\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}}} \times$$

$$\times \ln\left|\frac{m_r + \rho V_w}{m_r + \rho\left(V_r - \left(\frac{3}{2}\,S\sqrt{\frac{2\,P_0\,\,(V_r - V_w)}{\rho}}\,\,t + \frac{3}{2}\left(\frac{2}{3}(V_r - V_w)\right)^{\frac{3}{2}}\right)^{\frac{2}{3}}\right)}\right| \tag{31}$$

However, this expression works only if there is water inside the rocket. Hence, the expression for velocity when $V=V_r$ takes the following form:

$$v = -gt + \sqrt{\frac{2\,P_0\,(V_r - V_w)}{\rho\,V_r}}\ln\left|\frac{m_r + \rho V_w}{m_r}\right| \tag{32}$$

Note that now the last term in the expression is constant and does not depend on time, hence the motion can be described using projectile equations. The expression given by the equation (31) describes the velocity in time, however in my experiment I am looking for height *h* versus time *t* and initial volume of water V_w , so the equation must be integrated in order to find expression of *h*. The integration would be very difficult to calculate analytically, so easier an solution might be to use numerical Euler's method. The calculations would still be tedious to do by hand, so the technology comes with help. I wrote a code in "Python" to solve and plot the graph in 3 dimensions, which is shown in Figure 3. (See Appendix A for code).

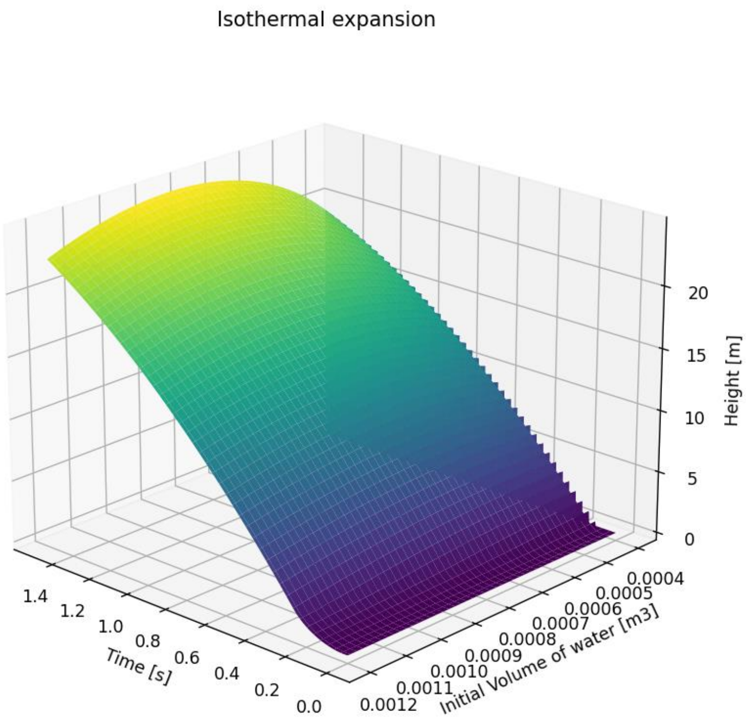

Figure 4.

2.2. Adiabatic expansion

The derivation of the expression for adiabatic expansion remains the same until equation (13). In adiabatic process, the pressure and volume obey following relation [9]:

$$P\,V^{\gamma} = P_0\,{V_0}^{\gamma} \tag{33}$$

$$P = P_0\,\frac{{V_0}^{\gamma}}{V^{\gamma}} \tag{34}$$

Where γ is a constant, which for air takes the value of 1.4 [10]. Now, expression for u becomes:

$$u = \sqrt{\frac{2\,P_0\,{V_0}^{\gamma}}{\rho\,V^{\gamma}}} \tag{35}$$

Solving differential equation (15) with new expression for *u* gives the expression for *V* as follows:

$$V = \left(\frac{17}{10}\, S \sqrt{\frac{2\, P_0\, {V_0}^{\gamma}}{\rho}}\; t + \frac{17}{10} \left(\frac{10}{17} V_0 \right)^{\frac{17}{10}} \right)^{\frac{10}{17}} \tag{36}$$

Substituting (36) into (35) and into (28), then all other necessary equations into (10) leads to:

$$v = -gt + \sqrt{\frac{2\, P_0\, (V_r - V_w)^{\gamma}}{\rho \left(\frac{17}{10}\, S \sqrt{\frac{2\, P_0\, {V_0}^{\gamma}}{\rho}}\; t + \frac{17}{10} \left(\frac{10}{17} (V_r - V_w) \right)^{\frac{17}{10}} \right)^{\frac{10\,\gamma}{17}}}} \times$$

$$\times \ln \left| \frac{m_r + \rho V_w}{m_r + \rho \left(\frac{17}{10}\, S \sqrt{\frac{2\, P_0\, {V_0}^{\gamma}}{\rho}}\; t + \frac{17}{10} \left(\frac{10}{17} (V_r - V_w) \right)^{\frac{17}{10}} \right)^{\frac{10}{17}}} \right| \tag{37}$$

Given that $V_r > V_w$. Otherwise:

$$v = -gt + \sqrt{\frac{2\, P_0\, (V_r - V_w)^{\gamma}}{\rho\, {V_r}^{\gamma}}} \ln \left| \frac{m_r + \rho V_w}{m_r} \right| \tag{38}$$

The graph of these equations is shown in Figure 4.

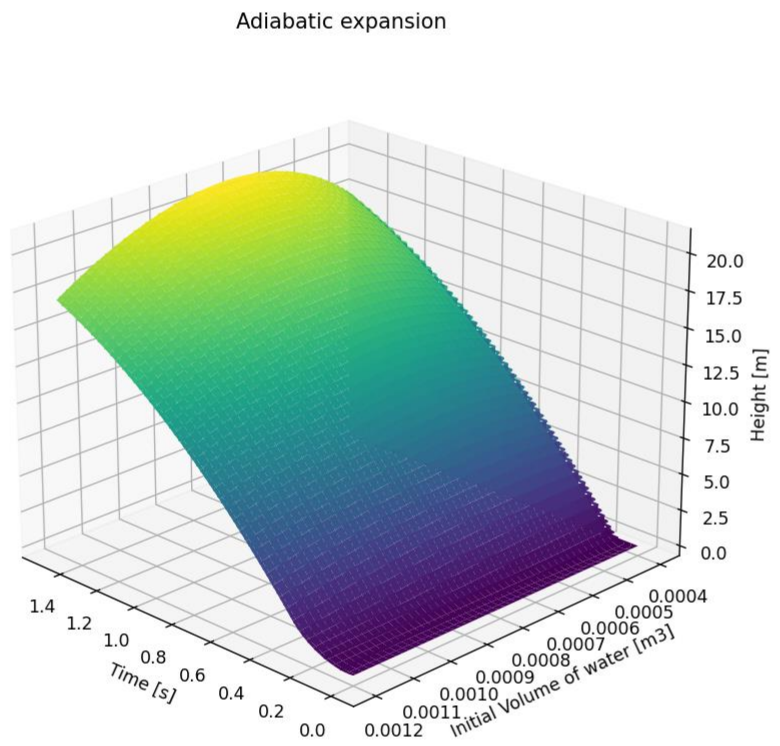

Figure 5.

3. Methods

3.1. Design of the rocket

For a rocket to be able to fly, a simple condition must be fulfilled: the centre of mass (CM) must be above the centre of the vertical cross-sectional area (CA) [11]. This prevents small turbulences that occur during flight to change the course of the rocket and improves stability. The mechanism of this phenomena is shown on Figure 5.

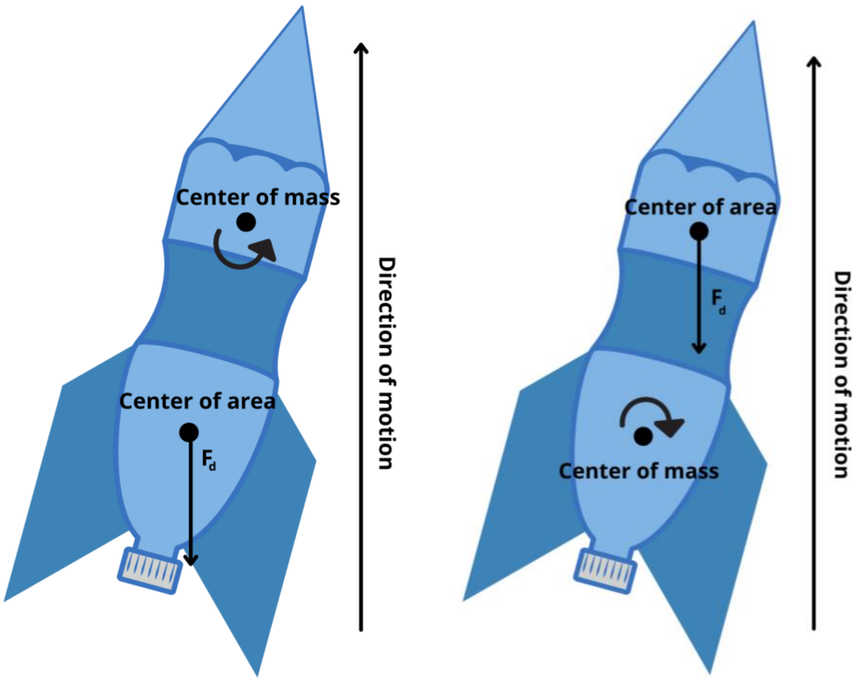

Figure 6.

Let me consider the following example: CM is above the CA. In this case then when a small change of the direction (in this example rotation to the right) is applied to the rocket by external force (for example wind), then the drag force increases. The vector of that force has its origin in the CA. It is important to note that this force will change the direction of a rocket and the rotation axis is centred at the CM of the rocket. Therefore, if the CM is above the CA, the rocket will turn left and the original course will be maintained. However, if the CM is below the CA, then the drag force acting on a rocket will cause the rocket to turn left resulting in complete change of direction of motion.

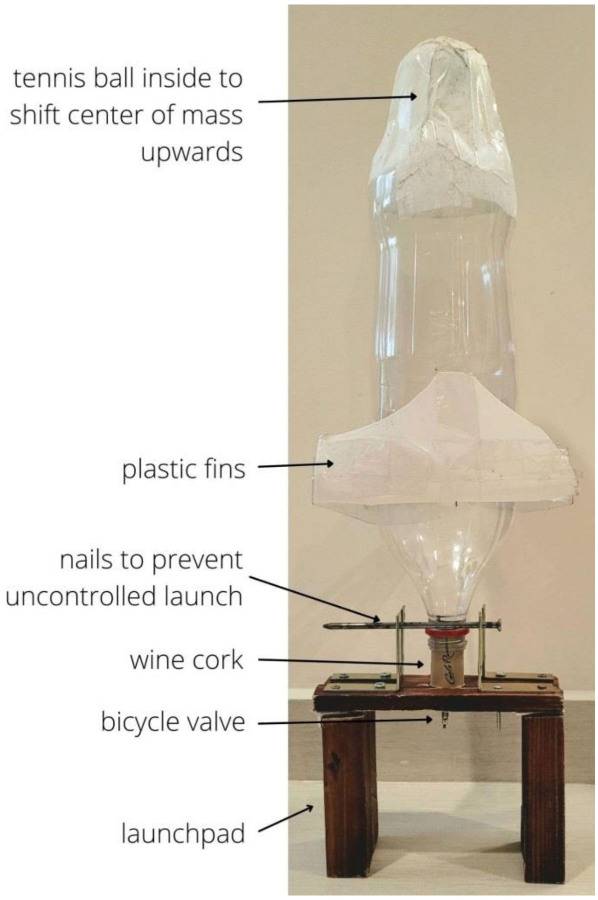

Figure 7. Components of the rocket and launchpad.

3.2. Apparatus

Materials needed to construct the launchpad:

1. Wooden blocks
2. Metal builder’s square
3. Nails
4. String
5. Bicycle valve
6. Wine cork
7. Tape

This was actually the second design of the launchpad. The first one, (developed by myself independently) involved using garden hose connections and a bicycle valve to pump air. However, this design was inefficient due to issues with controlled launch from distance. The second one was inspired by YouTube video by "The Q" [12]. It uses iron nails connected to the string which prevents the rocket from uncontrolled launch.

The wine cork was covered with tape in order to prevent any leakage of water when air was pumped and the rocket was ready to be launched.

Materials used to build a rocket:

1. Plastic bottle
2. Plastic fins
3. Tennis ball
4. Tape

Numeral characteristics of the rocket:

1. $m_r = 116.827$ [g]
2. $D_n = 22$ [mm]
3. $V_r = 2$ [l]

Plastic fins were made of used plastic packaging. Tennis ball was mounted on top of the rocket in order to satisfy the condition that the centre of mass should be above the centre of area. The plastic fins, which increase the centre of area at the bottom of the rocket, were mounted for the same purpose.

Figure 8. The rocket ready to be launched.

Additional apparatus employed:

1. Camera to record the height
2. Bicycle pump [uncertainty: $\pm$ 2 psi]
3. Cup [uncertainty: $\pm$ 25 ml]
4. Ladder and plate to measure height

The use of the camera has both advantages and disadvantages. The advantage of the camera is its accessibility. On the other hand, the disadvantage of the camera may be time uncertainty of 30 shots per second, which may not be enough for exact measurements. This issue will be discussed more deeply in the "Evaluation" section.

3.3. Experimental procedure

1. Set the launchpad and the measuring plate.
2. Pour the desired amount of water into the rocket and plug the wine cork with a bicycle valve into the rocket.
3. Set the rocket on the launchpad and place the nails to prevent the rocket from uncontrolled launch.
4. Pump the air inside the rocket to achieve gauge pressure P_0= 20 psi = 1380 hPa = 138000 Pa.
5. Screw the bicycle valve to prevent water leakage.
6. Turn on the camera and start recording.
7. Launch the rocket.
8. Stop recording after the rocket has landed.
9. Repeat steps 2.-8. five times for each value of volume of water added.

It is worth mentioning that the experimental procedure was developed by myself.

3.4. Gathering data

After the footage of the rocket was recorded it had to be analyzed in order to provide data of the position of the rocket versus time. For each value of initial amount of water inside the rocket, 5 trials were conducted. In each trials the values of height versus time were recorded from the camera footage. These values were then averaged and plotted on a graph to obtain Figure 8. It is also important to justify the restriction of *Time* axis to the value of *1.5* seconds. In each trial, due to the varying wind, the total time of flight of the rocket was different, hence the maximum height was obtained in different time. As a result, the maximum height would be less than in reality, so the best option was to restrict the time domain to *1.5* seconds.

3.5. Safety consideration

Table 1. Safety consideration

Possible risks	Measures to provide safety
Uncontrolled launch	Nails together with builder’s square
Explosion of the bottle due to extreme pressure	Pressure was kept below the critical point (≈ 150 psi)

4. Experimental data

4.1. Qualitative data

During the phase of pumping the air inside the rocket, a tiny leakage of water between the wine cork and the nozzle of the rocket could be observed. In addition, due to the wind present during the experiment the rocket landed up to few meters away from the launchpad, instead of ideally few centimetres away. The design of the launchpad also may have cause this issue, as the course of the rocket may have been deflected up to few degrees. Moreover, after the landing of the rocket, a fog could be seen inside the bottle.

4.2. Quantitative data

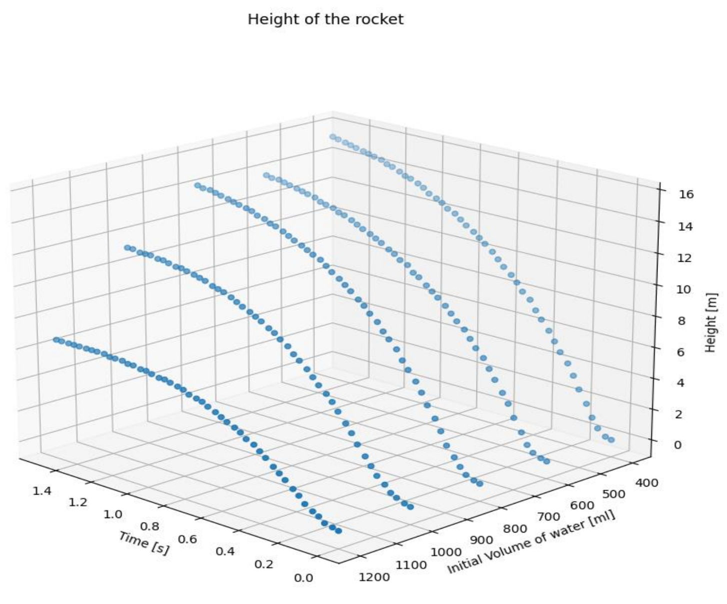

Figure 9.

The Quantitative data for each trial and IVW are included in the Appendix B, because they occupy too much space.

5. Data analysis

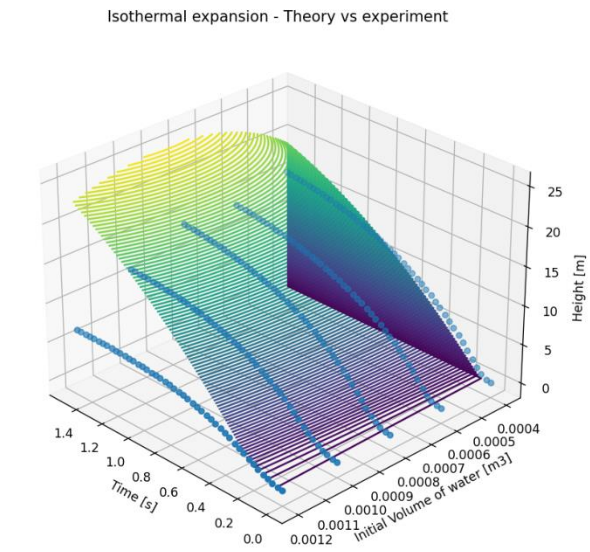

Figure 10.

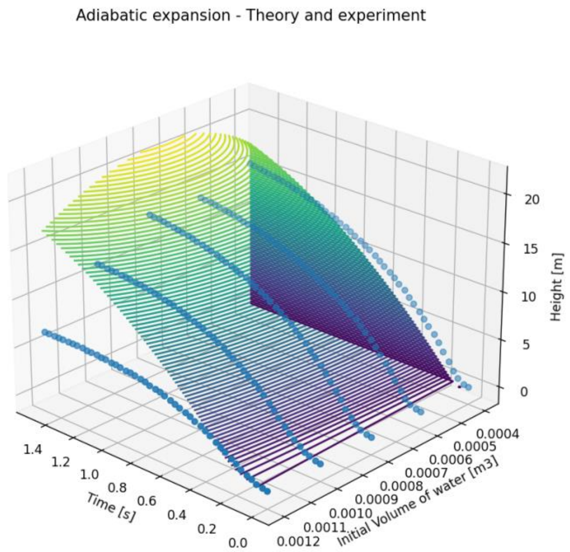

Figure 11.

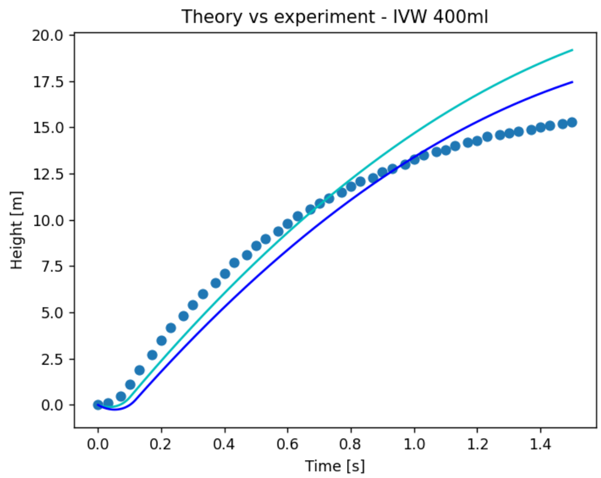

Figure 13. Experimental graph (scattered blue curve), Adiabatic expansion (blue continuous), Isothermal (cyan)

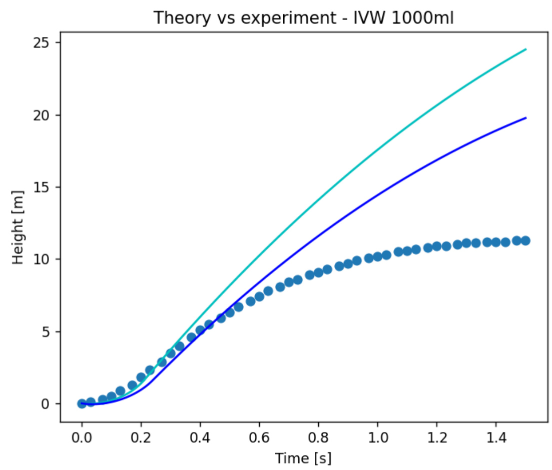

Figure 12. Experimental graph (scattered blue curve), Adiabatic expansion (blue continuous), Isothermal (cyan)

In my point of view, looking at the theoretical and experimental graph, the theoretical model predicts reality quite well. Moreover, it can be seen that the adiabatic model gives better agreement with the experimental values for Initial Volume of Water (IVW) of 1200 ml than the isothermal model. However, the height for IVW of 600 ml remains similar. In addition, both theoretical graphs predict the height of the rocket at time $t=1.5\ s$ to be above 20 m, while experimental values show the height of around 14 m. This difference will be discussed in "Evaluation" section, however one of possible reasons for this may be the negligence of drag force in theoretical models or other limitations such as wind or design of the launchpad.

From Figures 12. and 13. it can be gathered that the theoretical curves underestimate the altitude during the acceleration stage, but overestimate the displacement roughly after *1s* of flight. The problem of underestimation was also faced by the David Kagan and other researchers [9] and may be due to difficulties in assessing the "ignition" stage of the rocket. A possible solution to deal with this problem may be shifting experimental data in time to the right, however it would make the problem of overestimation of the flight after burnout even worse.

Let me now discuss random errors that occur in experimental data. Looking sorely at the experimental graph, it seems that the IVW of 600 ml is a random error, which should take slightly higher value in the decreasing pattern of final height. However, looking at theoretical graph it can be clearly seen that the IVW of 400 ml is a random error, not IVW of 600 ml. The value of height at IVW of 400 ml should be around 12 m instead of 15m. This random error may have been caused by the wind or possibly the uncertainty of pumping apparatus.

However the most important feature of all graphs is that the general trend of theory and experiment with regards to IVW is conserved. If there is too much water in the rocket, the rocket is too heavy and does not fly upwards. On the other hand, if the water level is too low, the rocket does not have enough fuel to be propelled upwards.

6. Conclusion

The theoretical and experimental values correspond with each other to the great extent, which may prove the validation of theory. However, there are some factors, that if accounted for, would give even better prediction.

The investigation of water rockets led me to deep thoughts about natural sciences and knowledge as a whole. Up to this point in time, everything I learned I took for granted, because I trusted teachers, who have the authority. However, now I have a chance to verify by myself, that this knowledge is in fact true. If only one of the equations used to derive formula for velocity was not true, the theoretical and experimental graphs would not match each other. The remarkable thing about it is that when the correct values are substituted, the output is the very similar to the experimental one. This also led me to the question how the knowledge can be produced. Firstly, scientists come up with theory in the area of metaphysics, and then the theory is verified by the real life experiment, which either proves the theory right or wrong. Another question would be why scientists come up with theory. One possible answer may be: to be able to predict behaviour of a particular system in unknown conditions or that it is simpler to solve a problem theoretically first and then experimentally. The discussion of these questions is still open and still many more can be explored in the subject of water rockets.

7. Evaluation

There are a few factors which cause the difference between experimental and theoretical values. These are for example consideration of external forces, such as drag force and force of the wind. Consideration of adiabatic and isothermal process may also influence theoretical values.

During the derivation of the theoretical equations drag force was neglected, which caused overestimation of theoretical models. Taking this force into consideration in equation (1) would decrease the height of the rocket in time or shift the graph downwards. However, the frag force depends on the speed with which the rocket is flying, so the final expression for velocity would be even more complicated than it already is.

Another difference between theory and experiment is that theory does not predict any random errors that may occur during the experiment. This however may be caused by the weather conditions. Because a water rocket can fly up to the height of 16 meters, the experiment had to be conducted outside. However, this increased the probability of wind influencing the trajectory of the rocket. Theoretically, wind would affect only horizontal component of the rocket, while in the analysis only vertical component is considered. On the other hand, in reality the force of wind could also potentially increase or decrease vertical component of velocity of the rocket at any random point. It is also important to note that the wind had the biggest influence on the rocket in the stage after burnout, when the mass is the smallest. The support of this claim may be that the standard deviation of the altitude of the rocket increases with time (see Appendix B). Therefore, the mean value of displacement was decreased by the trials, in which the wind was particularly strong, thus causing underestimation in experimental results.

There is one more factor, that could disturb the data, which may be a faulty design of launchpad, due to which the rocket landed few meters away from its initial position. This may indicate that the trajectory of the rocket was deviated few degrees from perfectly vertical one, therefore decreasing vertical component of velocity, leading to lower height value compared to the theoretical model. Moreover, the theoretical model accurately predicts the acceleration stage and then parabolic increase of height similar to the projectile motion, which were determined experimentally.

As previously stated, when the rocket was ready to launch, a small leakage of water could be observed, which would decrease both initial volume of water and initial pressure of water. The amount was so small that I did not consider it important, nevertheless it does influence the flight and could potentially decrease the experimental values compared to theoretical.

Another factor influencing the differences and random error in experimental values may be the initial pressure uncertainty. The uncertainty of height at $t=1.5\ s$ due to uncertainty of bicycle pump was calculated to be about $\pm$ 1.5 m using *Python* code.

Let me now discuss the expansion process of air. The isothermal expansion occurs when the system is allowed to exchange heat with the surroundings. In the adiabatic expansion the system does not exchange any heat with its surroundings. In the case of the experiment, the rocket theoretically may exchange heat with the surrounding air, however there is some evidence to suggest the different scenario [9]. First of all, after the landing, a fog could be seen inside the rocket, which was created by condensation of water vapour due to fall in temperature. The explanation for this fact may be that the expansion happens so fast, the rocket does not have enough time to exchange heat with the surroundings. Therefore in reality the process is most likely to be between isothermal and adiabatic.

7.1. Limitations and Improvements

I believe that the biggest limitation to the investigation were the weather conditions, that deflected the trajectory of the rocket. The experiment can be repeated in a closed space, however it would require special conditions. Further improvements can be done by improving measuring equipment, for example swap bicycle pump for apparatus with lower uncertainty. The improvements can also be made by swapping the camera with higher frame rate. Another possibility is to install an altimeter inside the rocket to measure height. When used with the high frame rate camera, the height uncertainty would be reduced to minimum. Improvements can also be done to the rocket itself and the launchpad. Both were home-made with the use of available resources, so there can be much improvement made for example by using different materials.

7.2. Extensions

There are a few possible extensions to the experiment, which may include dependence between height of the rocket in time and initial pressure inside the rocket and dependence between diameter of the nozzle and height in time. These were kept constant throughout the experiment in order to simplify the investigation. Let me explain the impact of rocket nozzle on the height. If the nozzle diameter is too small, the volume, hence mass of water exiting the nozzle is small, therefore the change of momentum of water is not sufficient to propel the rocket. On the other hand, if the nozzle is too big, then the rocket experiences large impulse in short time, decreasing the instantaneous velocity at the time that all water is ejected. On the other hand, increasing initial pressure increases the height up to the point, when the pressure inside the rocket exceeds the pressure a plastic bottle can maintain. Furthermore, taking all factors influencing the height of the rocket into considerations would result in 5-dimensional graph. Further extension of the experiment may include comparison of experimental and theoretical values with water rocket simulations available on the internet. The problem with these simulations is that they act as some kind of “magic black box”. One gives the input and the “box” gives an output, but the mechanism behind the simulation remains unknown.

8. References

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Water rocket simulators:

1. <https://www.npl.co.uk/water-rockets>
2. <https://www.grc.nasa.gov/www/k-12/rocket/BottleRocket/sim.htm>
3. <http://www.sciencebits.com/RocketCalculator?fromForm=yes>

9. List of Figures

10. Appendix A

10.1. Python 3 code for isothermal expansion

```
import numpy as np
import math
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d
```

```
ax = plt.axes(projection="3d")

#axis

x0 = 0
y0 = 0.0004
z0 = 0

xf = 1.5
n = 501
deltax = (xf-x0)/(n-1)

yf = 0.0012
m = 501
deltay = (yf-y0)/(m-1)

x = np.linspace(x0 , xf, n)
y = np.linspace(y0, yf, m)

z = np.zeros([n,m])

z[0,0] = z0

Y, X = np.meshgrid(y, x)

#define parameters:

g = 9.81
p0 = 137895.15
r = 997
dh = 0.022
dr = 0.1
mr = 0.1268727
T = 293
R = 8.31
vr = 0.00208

for i in  range (1, n):

    for j in range(1, m):

#define equations

        v0 = vr-y[j]
        a = v0*p0
        u_a = math.sqrt((2*a)/r)
        s = ((1/4)*math.pi*((dh)**2))
        v = (3/2*s*u_a*x[i-1]+3/2*(2/3*v0)**(3/2))**(2/3)
        u = math.sqrt((2*a)/(r*v))
        m0 = mr+r*y[j]
        mf = mr+r*(vr-v)
```

```
        e = np.log(m0/mf)
        uf = math.sqrt((2*a)/(r*vr))
        ef = np.log(m0/mr)

        if vr>v:

#final equation for displacement:

            z[i,j] = deltax*(-g*x[i-1]+u*e) + z[i-1,j]

        else:

            z[i,j] = deltax*(-g*x[i-1]+uf*ef) + z[i-1,j]

print(x[i], y[j], z[i,j])

Z = z


#set axis titles

ax.set_title("Isothermal expansion")
ax.set_xlabel("Time [s]")
ax.set_ylabel("Initial Volume of water [m3]")
ax.set_zlabel("Height [m]")

surf = ax.plot_surface(X, Y, Z, cmap='viridis')
plt.show()
```

10.2. Python 3 code for adiabatic expansion

```
import numpy as np
import math
import matplotlib.pyplot as plt
from mpl_toolkits import mplot3d

ax = plt.axes(projection="3d")

#axis

x0 = 0
y0 = 0.0004
z0 = 0

xf = 1.5
n = 501
deltax = (xf-x0)/(n-1)

yf = 0.0012
m = 501
```

```
deltay = (yf-y0)/(m-1)

x = np.linspace(x0 , xf, n)
y = np.linspace(y0, yf, m)

z = np.zeros([n,m])

z[0,0] = z0

Y, X = np.meshgrid(y, x)

#define parameters:

g = 9.81
p0 = 137895.15
r = 997
dh = 0.022
dr = 0.1
mr = 0.1268727
T = 293
R = 8.31
vr = 0.00208
k = 1.4

for i in  range (1, n):

    for j in range(1, m):

#define equations

        v0 = (vr-y[j])
        a = p0*(v0)**(k)
        u_a = np.sqrt((2*a)/r)
        s = ((1/4)*math.pi*((dh)**2))
        v = (17/10*s*u_a*x[i-1]+17/10*(10/17*v0)**(17/10))**(10/17)
        u = np.sqrt((2*a)/(r*(v)**(k)))
        m0 = mr+r*y[j]
        mf = mr+r*(vr-v)
        e = np.log(m0/mf)
        uf = np.sqrt((2*a)/(r*(vr)**(k)))
        ef = np.log(m0/mr)

        if vr>v:

#final equation for displacement during acceleration:

            z[i,j] = deltax*(-g*x[i-1]+u*e) + z[i-1,j]

        else:

#final equation for displacement after burnout:
```

```
            z[i,j] = deltax*(-g*x[i-1]+uf*ef) + z[i-1,j]

print(x[i], y[j], z[i,j])

Z = z

#set axis titles

ax.set_title("Adiabatic expansion")
ax.set_xlabel("Time [s]")
ax.set_ylabel("Initial Volume of water [m3]")
ax.set_zlabel("Height [m]")

surf = ax.plot_surface(X, Y, Z, cmap='viridis')
plt.show()
```

11. Appendix B

Table 2. IVW 400 ml

Time	Displacement						
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Mean	σ
0,00	0,0	0,0	0,0	0,0	0,0	0,0	0,00
0,03	0,1	0,0	0,0	0,2	0,2	0,1	0,08
0,07	0,5	0,2	0,3	0,7	0,6	0,5	0,19
0,10	0,9	0,7	1,0	1,5	1,4	1,1	0,34
0,13	1,8	1,5	1,8	2,3	2,3	1,9	0,34
0,17	2,6	2,3	2,7	3,1	3,1	2,7	0,34
0,20	3,3	3,1	3,4	3,8	3,8	3,5	0,33
0,23	4,0	3,7	4,1	4,4	4,5	4,2	0,31
0,27	4,7	4,4	4,8	5,0	5,1	4,8	0,28
0,30	5,3	5,1	5,4	5,7	5,8	5,4	0,29
0,33	5,9	5,7	6,0	6,2	6,4	6,0	0,31
0,37	6,4	6,2	6,6	6,8	7,0	6,6	0,31
0,40	7,0	6,8	7,1	7,3	7,6	7,1	0,31
0,43	7,5	7,3	7,6	7,8	8,1	7,7	0,32
0,47	8,0	7,7	8,0	8,2	8,6	8,1	0,34
0,50	8,5	8,2	8,4	8,6	9,1	8,6	0,33
0,53	9,0	8,6	8,8	9,1	9,6	9,0	0,34
0,57	9,4	9,1	9,2	9,5	10,0	9,4	0,35
0,60	9,9	9,5	9,5	9,9	10,4	9,8	0,37
0,63	10,3	9,9	9,9	10,2	10,8	10,2	0,37

0,67	10,7	10,2	10,2	10,6	11,1	10,6	0,39
0,70	11,1	10,5	10,5	10,9	11,5	10,9	0,41
0,73	11,4	10,9	10,8	11,2	11,8	11,2	0,43
0,77	11,8	11,2	11,0	11,5	12,1	11,5	0,46
0,80	12,1	11,5	11,2	11,8	12,4	11,8	0,48
0,83	12,5	11,8	11,5	12,0	12,7	12,1	0,51
0,87	12,8	12,0	11,7	12,2	13,0	12,3	0,54
0,90	13,1	12,3	11,9	12,5	13,3	12,6	0,58
0,93	13,3	12,6	12,0	12,7	13,5	12,8	0,60
0,97	13,6	12,8	12,2	12,8	13,8	13,0	0,63
1,00	13,8	13,1	12,4	13,0	14,0	13,3	0,66
1,03	14,1	13,3	12,5	13,2	14,2	13,5	0,70
1,07	14,3	13,5	12,7	13,4	14,5	13,7	0,73
1,10	14,5	13,7	12,8	13,5	14,6	13,8	0,76
1,13	14,8	13,9	12,9	13,7	14,8	14,0	0,79
1,17	14,9	14,1	13,1	13,8	15,0	14,2	0,81
1,20	15,1	14,2	13,2	13,9	15,1	14,3	0,83
1,23	15,3	14,4	13,3	14,1	15,3	14,5	0,86
1,27	15,4	14,6	13,4	14,2	15,4	14,6	0,89
1,30	15,6	14,8	13,5	14,3	15,6	14,7	0,91
1,33	15,7	14,9	13,5	14,4	15,7	14,8	0,92
1,37	15,8	15,0	13,6	14,5	15,8	14,9	0,94
1,40	16,0	15,1	13,7	14,5	15,9	15,0	0,96
1,43	16,1	15,3	13,7	14,6	16,0	15,1	0,98
1,47	16,2	15,4	13,8	14,7	16,1	15,2	1,00
1,50	16,3	15,5	13,8	14,8	16,1	15,3	1,02

Table 3. IVW 600 ml

Time	Displacement						
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Mean	σ
0,00	0,0	0,0	0,0	0,0	0,0	0,0	0,00
0,03	0,1	0,0	0,0	0,1	0,2	0,1	0,05
0,07	0,4	0,3	0,2	0,3	0,5	0,3	0,09
0,10	0,8	0,6	0,6	0,7	1,0	0,7	0,16
0,13	1,6	1,3	1,2	1,4	1,7	1,4	0,22
0,17	2,4	2,1	2,0	2,2	2,4	2,2	0,20
0,20	3,2	2,8	2,8	2,9	3,1	3,0	0,17
0,23	3,8	3,5	3,5	3,6	3,7	3,6	0,14
0,27	4,5	4,2	4,2	4,2	4,3	4,3	0,13
0,30	5,1	4,8	4,8	4,9	4,9	4,9	0,13

0,33	5,7	5,4	5,4	5,4	5,4	5,5	0,14
0,37	6,3	5,9	6,0	6,0	5,9	6,0	0,15
0,40	6,8	6,3	6,5	6,5	6,3	6,5	0,17
0,43	7,2	6,8	7,0	7,0	6,8	7,0	0,18
0,47	7,7	7,2	7,5	7,5	7,2	7,4	0,21
0,50	8,0	7,6	7,9	7,9	7,6	7,8	0,21
0,53	8,4	8,0	8,3	8,3	7,9	8,2	0,23
0,57	8,8	8,3	8,7	8,8	8,3	8,6	0,27
0,60	9,1	8,6	9,1	9,2	8,6	8,9	0,30
0,63	9,4	8,9	9,5	9,5	8,9	9,2	0,32
0,67	9,7	9,2	9,9	9,9	9,1	9,6	0,36
0,70	10,0	9,5	10,2	10,2	9,4	9,8	0,39
0,73	10,2	9,9	10,5	10,6	9,6	10,2	0,39
0,77	10,5	10,2	10,8	10,9	9,9	10,4	0,43
0,80	10,7	10,3	11,1	11,2	10,1	10,7	0,48
0,83	10,9	10,5	11,4	11,5	10,3	10,9	0,53
0,87	11,2	10,7	11,7	11,8	10,5	11,2	0,59
0,90	11,4	10,8	12,0	12,1	10,7	11,4	0,65
0,93	11,5	11,0	12,3	12,3	10,9	11,6	0,70
0,97	11,7	11,1	12,5	12,6	11,0	11,8	0,75
1,00	11,9	11,2	12,8	12,8	11,2	12,0	0,80
1,03	12,1	11,4	13,0	13,1	11,3	12,2	0,86
1,07	12,2	11,5	13,3	13,3	11,5	12,4	0,92
1,10	12,4	11,6	13,5	13,6	11,6	12,5	0,98
1,13	12,5	11,7	13,7	13,8	11,7	12,7	1,03
1,17	12,6	11,8	13,9	14,0	11,8	12,8	1,08
1,20	12,7	11,9	14,1	14,2	11,9	12,9	1,13
1,23	12,8	11,9	14,2	14,4	12,0	13,1	1,19
1,27	12,9	12,0	14,4	14,6	12,1	13,2	1,24
1,30	13,0	12,0	14,5	14,7	12,2	13,3	1,28
1,33	13,1	12,1	14,7	14,9	12,3	13,4	1,33
1,37	13,1	12,1	14,8	15,1	12,4	13,5	1,38
1,40	13,2	12,1	14,9	15,2	12,4	13,6	1,43
1,43	13,2	12,2	15,1	15,4	12,5	13,7	1,48
1,47	13,3	12,2	15,2	15,5	12,5	13,7	1,53
1,50	13,3	12,2	15,3	15,7	12,6	13,8	1,58

Table 4. IVW 800 ml

Time	Displacement						
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Mean	σ

0,00	0,0	0,0	0,0	0,0	0,0	0,0	0,00
0,03	0,1	0,1	0,1	0,1	0,1	0,1	0,03
0,07	0,4	0,4	0,3	0,2	0,3	0,3	0,05
0,10	0,7	0,7	0,7	0,6	0,7	0,7	0,04
0,13	1,2	1,1	1,2	1,1	1,2	1,2	0,04
0,17	1,8	1,6	2,0	1,8	1,8	1,8	0,12
0,20	2,4	2,4	2,8	2,6	2,6	2,6	0,18
0,23	3,1	3,0	3,6	3,5	3,4	3,3	0,24
0,27	3,7	3,6	4,4	4,3	4,1	4,0	0,34
0,30	4,3	4,2	5,1	5,1	4,7	4,7	0,44
0,33	4,8	4,7	5,8	5,8	5,4	5,3	0,53
0,37	5,4	5,2	6,5	6,5	5,9	5,9	0,62
0,40	5,8	5,7	7,1	7,2	6,4	6,4	0,72
0,43	6,3	6,1	7,7	7,8	7,0	7,0	0,81
0,47	6,7	6,5	8,3	8,4	7,4	7,5	0,90
0,50	7,0	6,8	8,9	8,9	7,9	7,9	1,00
0,53	7,4	7,2	9,5	9,5	8,4	8,4	1,09
0,57	7,7	7,5	10,0	9,9	8,8	8,8	1,18
0,60	8,0	7,8	10,5	10,4	9,2	9,2	1,28
0,63	8,3	8,0	11,0	10,8	9,6	9,5	1,37
0,67	8,6	8,3	11,4	11,2	10,0	9,9	1,46
0,70	8,8	8,5	11,9	11,6	10,3	10,2	1,56
0,73	9,0	8,7	12,3	12,0	10,7	10,5	1,66
0,77	9,2	8,9	12,7	12,3	11,0	10,8	1,75
0,80	9,4	9,0	13,1	12,6	11,3	11,1	1,83
0,83	9,6	9,2	13,5	12,9	11,6	11,4	1,92
0,87	9,7	9,3	13,8	13,2	11,9	11,6	2,00
0,90	9,9	9,5	14,1	13,5	12,2	11,8	2,09
0,93	10,0	9,6	14,4	13,7	12,5	12,0	2,18
0,97	10,1	9,6	14,7	13,9	12,8	12,2	2,26
1,00	10,2	9,7	15,0	14,2	13,0	12,4	2,34
1,03	10,4	9,8	15,2	14,4	13,2	12,6	2,41
1,07	10,5	9,9	15,5	14,6	13,5	12,8	2,48
1,10	10,6	10,0	15,7	14,8	13,7	12,9	2,56
1,13	10,6	10,0	15,9	15,0	13,9	13,1	2,63
1,17	10,7	10,1	16,1	15,1	14,1	13,2	2,70
1,20	10,7	10,1	16,4	15,3	14,3	13,4	2,78
1,23	10,8	10,2	16,5	15,4	14,5	13,5	2,85
1,27	10,8	10,2	16,7	15,6	14,6	13,6	2,92
1,30	10,9	10,2	16,9	15,7	14,8	13,7	2,99
1,33	10,9	10,2	17,1	15,8	14,9	13,8	3,06

1,37	10,9	10,2	17,2	16,0	15,1	13,9	3,12
1,40	10,9	10,2	17,4	16,1	15,2	14,0	3,19
1,43	10,9	10,2	17,5	16,2	15,4	14,1	3,26
1,47	10,9	10,2	17,7	16,3	15,5	14,1	3,32
1,50	10,9	10,2	17,8	16,4	15,6	14,2	3,38

Table 5. IVW 1000 ml

Time	Displacement						
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Mean	σ
0,00	0,0	0,0	0,0	0,0	0,0	0,0	0,00
0,03	0,1	0,1	0,1	0,1	0,1	0,1	0,02
0,07	0,3	0,2	0,2	0,3	0,2	0,3	0,06
0,10	0,6	0,5	0,5	0,6	0,4	0,5	0,07
0,13	1,0	0,8	0,9	1,0	0,7	0,9	0,10
0,17	1,4	1,3	1,3	1,4	1,1	1,3	0,15
0,20	1,9	1,8	1,8	1,9	1,5	1,8	0,18
0,23	2,5	2,4	2,4	2,5	1,9	2,3	0,24
0,27	3,1	3,0	3,0	3,1	2,4	2,9	0,30
0,30	3,7	3,6	3,5	3,8	2,9	3,5	0,36
0,33	4,2	4,2	4,1	4,4	3,4	4,0	0,40
0,37	4,7	4,7	4,6	5,0	3,8	4,6	0,43
0,40	5,2	5,2	5,1	5,5	4,3	5,1	0,47
0,43	5,6	5,7	5,5	6,0	4,7	5,5	0,50
0,47	6,0	6,1	5,9	6,6	5,1	5,9	0,53
0,50	6,3	6,5	6,3	7,0	5,5	6,3	0,56
0,53	6,7	6,9	6,7	7,5	5,8	6,7	0,61
0,57	7,0	7,2	7,1	8,0	6,2	7,1	0,64
0,60	7,4	7,6	7,4	8,4	6,5	7,4	0,68
0,63	7,7	7,9	7,7	8,8	6,8	7,8	0,71
0,67	8,0	8,2	8,0	9,1	7,0	8,1	0,74
0,70	8,2	8,5	8,3	9,5	7,3	8,4	0,78
0,73	8,5	8,8	8,6	9,8	7,5	8,6	0,82
0,77	8,7	9,0	8,8	10,1	7,7	8,9	0,85
0,80	8,9	9,2	9,1	10,4	8,0	9,1	0,89
0,83	9,0	9,4	9,3	10,7	8,1	9,3	0,92
0,87	9,2	9,6	9,5	11,0	8,3	9,5	0,96
0,90	9,4	9,8	9,7	11,2	8,5	9,7	0,98
0,93	9,5	10,0	9,9	11,4	8,6	9,9	1,01
0,97	9,6	10,2	10,1	11,6	8,8	10,1	1,03

1,00	9,8	10,3	10,2	11,9	8,9	10,2	1,07
1,03	9,9	10,4	10,4	12,0	9,0	10,3	1,08
1,07	9,9	10,6	10,5	12,2	9,2	10,5	1,11
1,10	10,0	10,7	10,6	12,3	9,3	10,6	1,13
1,13	10,1	10,8	10,8	12,5	9,3	10,7	1,16
1,17	10,2	10,9	10,9	12,6	9,4	10,8	1,19
1,20	10,2	10,9	10,9	12,8	9,5	10,9	1,21
1,23	10,2	11,0	11,0	12,9	9,6	10,9	1,24
1,27	10,3	11,1	11,1	13,0	9,6	11,0	1,26
1,30	10,3	11,1	11,2	13,1	9,7	11,1	1,28
1,33	10,3	11,2	11,2	13,2	9,7	11,1	1,30
1,37	10,4	11,2	11,3	13,2	9,7	11,2	1,32
1,40	10,4	11,3	11,4	13,3	9,8	11,2	1,34
1,43	10,4	11,3	11,4	13,4	9,8	11,2	1,36
1,47	10,4	11,3	11,4	13,4	9,8	11,3	1,37
1,50	10,4	11,3	11,5	13,4	9,8	11,3	1,39

Table 6. IVW 1200 ml

Time			Displacement				
	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5	Mean	σ
0,00	0,0	0,0	0,0	0,0	0,0	0,0	0,00
0,03	0,1	0,1	0,1	0,1	0,0	0,1	0,03
0,07	0,2	0,2	0,3	0,2	0,2	0,2	0,04
0,10	0,5	0,4	0,5	0,5	0,4	0,3	0,07
0,13	0,8	0,6	0,8	0,8	0,6	0,5	0,09
0,17	1,0	0,9	1,2	1,1	0,9	0,7	0,13
0,20	1,3	1,2	1,6	1,5	1,2	0,9	0,16
0,23	1,6	1,5	2,0	1,9	1,6	1,1	0,20
0,27	1,9	1,8	2,4	2,3	1,9	1,3	0,26
0,30	2,1	2,1	2,8	2,7	2,3	1,5	0,32
0,33	2,4	2,4	3,2	3,1	2,6	1,7	0,39
0,37	2,6	2,7	3,6	3,6	2,9	1,9	0,47
0,40	2,8	2,9	4,0	4,0	3,2	2,0	0,55
0,43	3,0	3,1	4,3	4,4	3,5	2,2	0,64
0,47	3,2	3,4	4,7	4,7	3,7	2,3	0,72
0,50	3,4	3,5	5,0	5,0	4,0	2,5	0,80
0,53	3,5	3,7	5,3	5,4	4,2	2,6	0,88
0,57	3,6	3,9	5,6	5,7	4,4	2,7	0,97
0,60	3,7	4,0	5,9	6,0	4,6	2,8	1,04

0,63	3,8	4,1	6,1	6,3	4,8	2,9	1,12
0,67	3,9	4,2	6,3	6,5	5,0	2,9	1,20
0,70	4,0	4,3	6,6	6,8	5,1	3,0	1,29
0,73	4,0	4,4	6,8	7,0	5,3	3,0	1,38
0,77	4,0	4,5	7,0	7,3	5,4	3,1	1,46
0,80	4,1	4,5	7,2	7,5	5,6	3,1	1,54
0,83	4,1	4,6	7,4	7,7	5,7	3,2	1,62
0,87	4,1	4,6	7,6	7,9	5,8	3,2	1,70
0,90	4,1	4,6	7,7	8,0	5,9	3,2	1,78
0,93	4,1	4,6	7,9	8,2	5,9	3,2	1,86
0,97	4,1	4,6	8,0	8,4	6,0	3,2	1,94
1,00	4,1	4,6	8,1	8,5	6,0	3,2	2,01
1,03	4,1	4,6	8,3	8,7	6,1	3,2	2,07
1,07	4,1	4,6	8,3	8,8	6,1	3,3	2,13
1,10	4,1	4,6	8,4	8,9	6,2	3,3	2,18
1,13	4,1	4,6	8,5	9,0	6,2	3,3	2,23
1,17	4,1	4,6	8,6	9,1	6,2	3,3	2,28
1,20	4,1	4,6	8,7	9,2	6,2	3,3	2,32
1,23	4,1	4,6	8,7	9,3	6,2	3,3	2,35
1,27	4,1	4,6	8,8	9,3	6,2	3,3	2,38
1,30	4,1	4,6	8,8	9,4	6,1	3,3	2,41
1,33	4,1	4,6	8,8	9,4	6,1	3,3	2,42
1,37	4,1	4,6	8,9	9,5	6,1	3,4	2,44
1,40	4,1	4,6	8,9	9,5	6,1	3,4	2,45
1,43	4,1	4,6	8,9	9,5	6,1	3,4	2,46
1,47	4,1	4,6	8,9	9,5	6,1	3,4	2,46
1,50	4,1	4,6	8,9	9,5	6,1	3,4	2,46