MTL101 Lecture 11 and 12

(Sum & direct sum of subspaces, their dimensions, linear transformations, rank & nullity)

(39) Suppose W_1, W_2 are subspaces of a vector space V over $\mathbb{F}$. Then define

$$W_1 + W_2 := \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

This is a subspace of V and it is call the sum of W_1 and W_2. Students must verify that W_1+W_2 is a subspace of V (use the criterion for a subspace).

Examples:

(a) Let $V = \mathbb{R}^2$, $W_1 = \{(x,x) : x \in \mathbb{R}\}$ and $W_2 = \{(x,-x) : x \in \mathbb{R}\}$. Then $W_1 + W_2 = \mathbb{R}^2$. Indeed, $(x,y) = (\frac{x+y}{2}, \frac{x+y}{2}) + (\frac{x-y}{2}, -\frac{x-y}{2})$.

(b) Next, let $V = \mathbb{R}^4$, $W_1 = \{(x,y,z,w) : x+y+z = 0, x+2y-z = 0\}$, $W_2 = \{(s, 2s, 3s, t) : s,t \in \mathbb{R}\}$. How to describe $W_1 + W_2$ (e.g., find a basis)?

The following theorem tells us the dimension of W_1+W_2 and the proof of the theorem suggest how to write its bases.

Theorem: If W_1, W_2 are subspaces of a vector space V, then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2).$$

Proof: Let S be a basis of $W_1 \cap W_2$ (if $W_1 \cap W_2$ is the zero space then $S = \Phi$.). For each $i = 1, 2$, extend S to a basis B_i of W_i. Let $S = \{u_1, u_2, \ldots, u_r\}$, $B_1 = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s\}$ and $B_2 = \{u_1, u_2, \ldots, u_r, w_1, w_2, \ldots, w_t\}$. Then $\dim(W_1 \cap W_2) = r$, $\dim W_1 = r+s$, $\dim W_2 = r+t$. Let $B = \{u_1, u_2, \ldots, u_r, v_1, v_2, \ldots, v_s, w_1, w_2, \ldots, w_t\}$. It is enough to show that B is a basis of $W_1 + W_2$ because then we have $\dim(W_1 + W_2) = r+s+t = (r+s)+(r+t)-r = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

To show that B is linearly independent let

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j + \sum_{k=1}^{t} c_k w_k = 0.$$

Then

$$\sum_{i=1}^{r} a_i u_i + \sum_{j=1}^{s} b_j v_j = -\sum_{k=1}^{t} c_k w_k.$$

Now the LHS is in W_1 and the RHS is in W_2. So this element is in $W_1 \cap W_2$. Thus $-\sum\limits_{k=1}^{t} c_k w_k = \sum\limits_{k=1}^{t} d_i u_i$ so that $\sum\limits_{i=1}^{r} d_i u_i + \sum\limits_{k=1}^{t} c_k w_k = 0$ which implies $d_i = 0$ and $c_k = 0$ for each i and k (since B_2 is linearly independent). Therefore, $\sum\limits_{i=1}^{r} a_i u_i + \sum\limits_{j=1}^{s} b_j v_j = 0$ which implies $a_i = 0$, $b_j = 0$ for each i and each j (since B_1 is linearly independent). Thus, B is linearly independent. Let $w \in W_1 + W_2$. Then $w = w_1 + w_2$ for some $w_i \in W_i$ for $i = 1, 2$. Then $w_1 = \sum\limits_{i=1}^{r} p_i u_i + \sum\limits_{j=1}^{s} q_j v_j$ and $w_2 = \sum\limits_{i=1}^{r} g_i u_i + \sum\limits_{j=1}^{t} h_j w_j$ for $p_i, q_i, g_i, h_i \in \mathbb{F}$. Now $w = \sum\limits_{i=1}^{r} (p_i + g_i) u_i + \sum\limits_{j=1}^{s} q_j v_j + \sum\limits_{k=1}^{t} h_k w_k$ which is in span(B). $\square$

(40) The sum $W_1 + W_2$ is called direct if $W_1 \cap W_2 = \{0\}$. In particular, a vector space V is said to be the direct sum of two subspaces W_1 and W_2 if $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. When V is a direct sum of W_1 and W_2 we write $V = W_1 \oplus W_2$.

Theorem: Suppose W_1 and W_2 are subspaces of a vector space V so that $V = W_1 + W_2$. Then $V = W_1 \oplus W_2$ if and only if every vector in V can be written in a unique way as $w_1 + w_2$ where $w_i \in W_i$.

Proof: Since $V = W_1 + W_2$, every vector in V is a sum of a vector in W_1 and a vector in W_2. Suppose that for every $v \in V$, there is only one pair (w_1, w_2) with $w_i \in W_i$ such that $v = w_1 + w_2$. If $W_1 \cap W_2$ is nonzero, pick a nonzero vector $u \in W_1 \cap W_2$. Then $u = u + 0$

with $u \in W_1$, $0 \in W_2$ and $u = 0 + u$ with $0 \in W_1$, $u \in W_2$. This contradicts our uniqueness assumption.

Conversely, suppose $V = W_1 \oplus W_2$. Then $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$. If for $v \in V$ we have $v = w_1 + w_2 = w_1' + w_2'$ for $w_1, w_1' \in W_1$ and $w_2, w_2' \in W_2$ then $w_1 - w_1' = w_2' - w_2$. The LHS is in W_1 and the RHS in W_2; therefore, this vector is in $W_1 \cap W_2$. Since by assumption $W_1 \cap W_2 = \{0\}$, we have $w_1 - w_1' = 0$ and $w_2' - w_2 = 0$ so that $w_1 = w_1'$ and $w_2 = w_2'$. □

Examples:

(a) $V = \mathbb{R}^2$, $W_1 = \{(x, 2x) : x \in \mathbb{R}\}$, $W_2 = \{(x, 3x) : x \in \mathbb{R}\}$. Then $V = W_1 \oplus W_2$. Indeed, $(x, y) = (3x - y, 6x - 2y) + (y - 2x, 3y - 6x)$ (Hint: Find $a, b \in \mathbb{F}$ such that $(x, y) = (a, 2a) + (b, 3b)$) and $W_1 \cap W_2 = 0$ (Hint: Let $(x, y) \in W_1 \cap W_2$ then $(x, y) = (a, 2a) = (b, 3b)$ for some $a, b \in \mathbb{F}$. By comparing the first component we get $a = b$ and comparing the second we get $a = 0$).

(b) Suppose $n \geq 2$, $V = \mathbb{R}^n$, $W_1 = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n : \ a_1 + a_2 + \cdots a_n = 0\}$ and $W_2 = \{(a_1, a_2, \ldots, a_n) \in \mathbb{R}^n : \ -a_1 + a_2 - \cdots + (-1)^n a_n = 0\}$. Show that $V = W_1 + W_2$. Further show that when $n = 2$, $V = W_1 \oplus W_2$ and when $n > 2$ the sum is not direct.

(c) $V = M_n(\mathbb{R})$, W_1 is the subspace of all the upper trangular matrices and W_2 is the subspace of all the lower trangular matrices over $\mathbb{R}$ (this sum is not direct).

(d) $V = M_n(\mathbb{R})$, W_1 is the subspace of all the symmetric $n \times n$ matrices over $\mathbb{R}$ and W_2 is the subspace of all the skew-symmetric $n \times n$ matrices over $\mathbb{R}$ (in this, the sum is direct).

Exercise: Write a basis of $W_1 + W_2$ of example 39b.

(41) Suppose V and W are vector spaces over the same field $\mathbb{F}$. A map $T : V \to W$ is called a **linear transformation** if $T(au + bv) = aT(u) + bT(v)$ for any $a, b \in \mathbb{F}$ and any $u, v \in V$.

Examples:

(a) Let $V = W = \mathbb{R}^2$, $\mathbb{F} = \mathbb{R}$. Then $T_1(x, y) = (ax + by, cx + dy)$ for any $a, b, c, d \in \mathbb{R}$ is a linear transformation (verify). But $T_2(x, y) = (x + y + 1, x - y)$ is not a linear transformation (why?). If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation then $T = T_1$ for some choices of $a, b, c, d \in \mathbb{R}$. Indeed, $T(x, y) = T(xe_1 + ye_2) = xT(e_1) + yT(e_2)$. Since $T(e_1), T(e_2) \in \mathbb{R}^2$, we have $T(e_1) = (p, q)$ and $T(e_2) = (r, s)$ for some $p, q, r, s \in \mathbb{R}$. Thus, $T(x, y) = x(p, q) + y(r, s) = (px + ry, qx + sy)$.

(b) For any vector spaces V, W over $\mathbb{F}$, map from V to W defined by $v \mapsto 0$ for all $v \in V$ is a linear transformation and it is called the **zero map** (or zero transformation).

(c) $\mathbf{Id}_V : V \to V$ defined by $v \mapsto v$ for all $v \in V$ is a linear tranformation; it is called the **identity operator**.

(d) For each $1 \leq i \leq n$, $p_i : \mathbb{R}^n \to \mathbb{R}$ defined by $p_i(a_1, a_2, \ldots, a_n) = a_i$ is a linear transformation and it is called the **i-th projection**.

Remarks: If $T : V \to W$ is a linear transformation then $T(0) = 0$, i.e., the image of the zero vector of V is the zero vector of W. Indeed, $T(0) = T(00) = 0T(0) = 0$ (where we have written 00, the first zero is the scalar zero and the second zero is the zero vector in the domain space V of T. $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x, y) = (x + y + 1, x - y)$ is not linear because $T(0, 0) = (1, 0)$ which is not the zero vector of $\mathbb{R}^2$.

(42) Suppose $T : V \to W$ is a linear transformation. Then define

$$\ker(T) := \{v \in V : T(v) = 0\} \text{ and } T(V) := \{T(v) : v \in V\}.$$

Show that $\ker(T) < V$ and $T(V) < W$ for any linear transformation T. The spaces $\ker(T)$ and $T(V)$ are called respectively **null space** (or **kernel**) and **image space** (or **range space**) of T. The dimension of $\ker(T)$ is call the **nullity** of T and the dimension of $T(V)$ is call the **rank** of T.

Theorem: A linear tranformation is injective if and only if its null space is the zero space.

Proof: Suppose $T : V \to W$ is a linear tranformation. Suppose that the null space is the zero space. If $T(u) = T(v)$ for $u, v \in V$ then $T(u - v) = 0$ so that $u - v \in \ker(T) = \{0\}$ which implies $u - v = 0 \Rightarrow u = v$. Conversely, assume that T is injective. If $u \in \ker(T)$ then $T(u) = 0 = T(0)$ which implies $u = 0$ (since T is invertible). □

Theorem: (Rank-nullity Theorem) Suppose $T : V \to W$ is a linear tranformation. Then
$$\text{rank}(T) + \text{Nullity}(T) = \dim(V).$$

Proof: Suppose $\{u_1, u_2, \ldots, u_m\}$ is a basis of the null space $\ker(T)$. Extend this basis to a basis of V. Let $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_r\}$ be a basis of V so that $\dim(V) = s + r$. We will show that $B = \{T(v_1), T(v_2), \ldots, T(v_r)\}$ is a basis of the range space $T(V)$ (to prove that $\dim(T(V)) = r$). By definition $B \subset T(V)$. Suppose $\sum_{i=1}^{r} a_i T(v_i) = 0$. Then $T(\sum_{i=1}^{r} a_i v_i) = 0$ (since T is linear). Hence $\sum_{i=1}^{r} a_i v_i \in \ker(T)$ so that $\sum_{i=1}^{r} a_i v_i = \sum_{i=1}^{m} b_j u_j$ or $\sum_{j=1}^{m}(-b_j)u_j + \sum_{i=1}^{r} a_i v_i = 0$. Since $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_r\}$ is linearly independent, the coefficients are all zero. In particular, $a_i = 0$ for each $1 \leq i \leq r$. Thus, B is linearly independent. If $w \in T(V)$, there exists $v \in V$ such that $T(v) = w$. Since $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_r\}$ is a basis of V, $v = \sum_{i=1}^{m} p_i u_i + \sum_{j=1}^{r} q_j v_j$ where $p_i, q_j \in \mathbb{F}$. Since $T(u_i) = 0$ for each i, we get, by applying T both sides, $T(v) = \sum_{j=1}^{r} q_j T(v_j)$. $\square$

Examples: Let $T : \mathbb{R}^3 \to R^3$ defined by $T(x, y, z) = (x + y - z, x - y + z, y - z)$. The null of T is $\{(x, y, z) : x + y - z = 0, x - y + z = 0, y - z = 0\}$ which is the solution space of certain homogeneous system of linear equations. We know how to solve a system of homogenous linear equations. Show that $\ker(T) = \{(x, y, z) : x = 0, y = z\} = \{(0, t, t) : t \in \mathbb{R}\}$. $\{(0, 1, 1)\}$ is a basis of $\ker(T)$ so that $\text{Nullity}(T) = 1$. Using the rank-nullity theorem $\text{rank}(T) = \dim(\mathbb{R}^3) - 1 = 3 - 1 = 2$.

How to compute $\text{rank}(T)$ directly by displaying a basis of $T(V)$: First pick a basis of V, for instance, $\{e_1, e_2, e_3\}$. Then $T(V)$ is generated by $Y = \{T(e_1), T(e_2), T(e_3)\}$ which is $\{(1, 1, 0), (1, -1, 1), (-1, 1, -1)\}$. We have to pick a basis of $T(V)$ using Y. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix}$ which is obtained from Y by converting elements of Y into rows. Then a basis of the row space of A is a basis of the range space. To find a basis of the row space find the row reduced echelon form of A which is $\begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$. Hence $\{(1, 0, \frac{1}{2}), (0, 1, -\frac{1}{2})\}$ is a basis of the range space. Thus $\text{rank}(T) = 2$. Thus we have verified the rank-nullity theorem for the given linear tranformation.

Applications of rank-nullity theorem: Since for any linear transformation T, $\text{rank}(T) \geq 0$,$\text{Nullity}(T) \geq 0$, we have, $\text{rank}(T) \leq \dim(V)$ and $\text{Nullity}(T) \leq \dim(V)$. A linear tranformation $T : V \to W$ is injective if and only if $\text{Nullity} = \{0\}$; and T is surjective if and only if $\text{rank}(T) = \dim(W)$. Using rank-nullity theorem we have the following statements.

(a) There is no injective linear tranformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ if $m > n$.
(b) There is no surjective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$ if $n > m$.
(c) There is an isomorphism (a bijective linear tranformation) from $\mathbb{R}^m$ to $\mathbb{R}^n$ if and only if $m = n$.

Proof: (a) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is injective, then $\text{Nullity}(T) = 0$ so $\text{rank}(T) + 0 = \dim(\mathbb{R}^m) = m$ (by rank-nullity theorem). Since $T(V) < \mathbb{R}^n$, $\text{rank}(T) \leq n$. Therefore, $m \leq n$. Equivalentely, if $m > n$, there is no injective linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$.
(b) If $T : \mathbb{R}^m \to \mathbb{R}^n$ is surjective (i.e.,$T(V) = W$), then $\text{rank}(T) = n$. By rank-nullity theorem, $n + \text{Nullity}(T) = m$. So $m - n \geq 0$ or $m \geq n$ (since $\text{Nullity}(T) \geq 0$).
(c) Suppose $T : \mathbb{R}^m \to \mathbb{R}^n$ is an isomorphism (a bijective linear transformation). Since T is injective, by part (a), $m \leq n$ and since T is surjective $m \geq n$. Therefore, $m = n$.
Next, suppose $m = n$. Then the identity map is a linear transformation from $\mathbb{R}^m$ to itself.

Remark: This completes the syllabus of **Minor Test I**. Go through the questions with answers appeared in the course page. Soon more of QWA are to appear.