Some Essential Linear Algebra

The Cauchy-Schwarz inequality and the angle between vectors in $\mathbb{R}^n$

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with equality holding if and only if $\{x, y\}$ is a linearly dependent set.

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If $x, y \in R^n$, then by the Cauchy-Schwarz inequality

$$-1 \leq \frac{\langle x, y\rangle}{\|x\|_2 \|y\|_2} \leq 1.$$

Hence $\theta = \arccos \frac{\langle x, y\rangle}{\|x\|_2 \|y\|_2}$ is called the angle between x and y.

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Two vectors $x, y \in \mathbb{C}^n$ are said to be mutually orthogonal if $\langle x, y\rangle = 0$.

Orthogonal complements of sets

Let S be any nonempty subset of $\mathbb{F}^n$ where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$. Then the orthogonal complement of S is defined by

$$S^\perp := \{x \in \mathbb{F}^n : \langle x, y \rangle = 0 \text{ for all } y \in S\}.$$

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Examples:

1. In $\mathbb{C}^3$, $\{e_3\}^\perp = \operatorname{span}\{e_1, e_2\}$.
2. In $\mathbb{R}^4$, $\{e_2 + e_4\}^\perp = \operatorname{span}\{e_1, e_3, e_2 - e_4\}$.

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Exercise: Let S be any nonempty subset of $\mathbb{F}^n$. Prove that

1. $S^\perp$ is always a *subspace* of $\mathbb{F}^n$.
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Exercise: Given any $n \times n$ matrix A prove that $N(A)^\perp = R(A^T)$ where

$$N(A) = \{x \in \mathbb{F}^n : Ax = 0\}$$
$$R(A^T) = \{A^T x : x \in \mathbb{F}^n\}$$

with $\mathbb{F} = \mathbb{R}$ if A is real and $\mathbb{F} = \mathbb{C}$ if A is complex.

Sums and direct sums of subspaces

Sum of two subspaces: Given any two subspaces U, W of $\mathbb{F}^n$,

$$\mathbb{F}^n = U + W$$

if for every $x \in \mathbb{F}^n$, there exist $x_1 \in U$ and $x_2 \in W$ such that $x = x_1 + x_2$.

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Theorem Let U and W be two subspaces of $\mathbb{F}^n$. such that $\mathbb{F}^n = U + W$. Then

$$\dim U + \dim W - \dim(U \cap W) = n,$$

and $\mathrm{span}(U \cup W) = U + W$.

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$$\mathbb{F}^n = U \oplus W \tag{1}$$

if for every $x \in \mathbb{F}^n$, there exist *unique* $x_1 \in U$ and $x_2 \in W$ such that $x = x_1 + x_2$. The equation (1) is called a diect sum decomposition of $\mathbb{F}^n$.

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Theorem Suppose U, W are subspaces of $\mathbb{F}^n$ such that $\mathbb{F}^n = U + W$. Then $\mathbb{F}^n = U \oplus W$ if and only if $u + w = 0$ for $u \in U$, $w \in W$, implies that $u = w = 0$.

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Find more examples like example 2!

Projections

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In fact, any $n \times n$ idempotent matrix, ie., $A^2 = A$ defines a projection on $\mathbb{F}^n$.

Exercises:

1. Given any projection P on $\mathbb{F}^n$, prove the following.
   (a) $\mathbb{F}^n = N(P) \oplus R(P)$.
   (b) $I - P$ is also a projection.
   (c) $N(P) = R(I - P)$ and $R(P) = N(I - P)$.
2. If U and V are subspaces of $\mathbb{F}^n$ such that $\mathbb{F}^n = U \oplus V$ then $P : \mathbb{F}^n \mapsto \mathbb{F}^n$ defined by $Px = x_1$ where $x = x_1 + x_2, x_1 \in U, x_2 \in V$, is a projection onto U, that is, $R(P) = U$.

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1. Given any orthogonal projection P on $\mathbb{F}^n$, prove the following.
   (a) $I - P$ is also an orthogonal projection.
   (b) $N(P) = R(P)^\perp$.
   (c) $\mathbb{F}^n = R(P) \oplus R(P)^\perp$.
2. If U is a subspace of $\mathbb{F}^n$ such that $\mathbb{F}^n = U \oplus U^\perp$ then $P : \mathbb{F}^n \mapsto \mathbb{F}^n$ defined by $Px = x_1$ where $x = x_1 + x_2, x_1 \in U, x_2 \in U^\perp$, is an orthogonal projection onto U, that is $R(P) = U$.

Orthonormal sets and orthonormal bases

A nonempty subset $\{v_1, \ldots, v_m\}$ of $\mathbb{R}^n$ or $\mathbb{C}^n$ is said to be an *orthonormal set* if

$$\langle v_i, v_j \rangle = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

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Examples:

- The canonical basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$ or $\mathbb{C}^n$ where e_i is the i^{th} column of I_n.
- $\left\{ \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ or $\mathbb{C}^3$.

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Any linearly independent subset of $\mathbb{R}^n$ or $\mathbb{C}^n$ may be converted to an orthonormal set via the process of Gram-Schmidt orthonormalisation.

Classical Gram Schmidt Orthonormalisation

Let $\{v_1, \ldots, v_m\}$ be an ordered set of linearly independent vectors in $\mathbb{R}^n$. The Classical Gram Schmidt (CGS) process finds an ordered orthonormal set of vectors $\{q_1, \ldots, q_m\}$ in $\mathbb{R}^n$ such that

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Classical Gram Schmidt (CGS):

Step 1: $q_1 := v_1 / \|v_1\|_2$.

Step 2: $q_2 := \underbrace{(v_2 - (v_2^T q_1) q_1)}_{=:\hat{q}_2} / \|v_2 - (v_2^T q_1) q_1\|_2$.

Step k: Assuming that $q_1, \dots, q_{k-1}$ are calculated as above,

$$q_k = \underbrace{\left(v_k - \sum_{i=1}^{k-1} (v_k^T q_i) q_i\right)}_{=:\hat{q}_k} / \|v_k - \Sigma_{i=1}^{k-1} (v_k^T q_i) q_i\|_2.$$

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Exercise: Show that CGS applied to the basis $\{e_1 + e_2, e_2, e_2 + e_3\}$ in $\mathbb{R}^3$ produces the ordered orthonormal basis

$$\{(e_1 + e_2)/\sqrt{2}, (e_2 - e_1)/\sqrt{2}, e_3\}.$$

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- $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all $x, y \in \mathbb{C}^n$.
- $\|Qx\|_2 = \|x\|_2$.
- $\|QB\|_2 = \|B\|_2$ for any $B \in \mathbb{C}^{n \times m}$.
- $\|Q\|_2 = 1$ and $\|Q\|_F = \sqrt{n}$.
- $\kappa_2(Q) = 1$.
- $Q^* AQ$ is Hermitian if A is Hermitian.
- If A is real symmetric and Q is orthogonal, then $Q^T AQ$ is also real symmetric.
- In the presence of rounding errors, $fl(QA) = Q(A + E)$ where $\|E\|_2 / \|A\|_2$ is $O(u)$.

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- $\|Q\|_2 = 1$ and $\|Q\|_F = \sqrt{n}$.
- $\kappa_2(Q) = 1$. Prove these properties!
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Clearly a matrix Q with more rows than columns is an isometry if and only if its columns form an orthonormal set in $\mathbb{R}^n$ or $\mathbb{C}^n$.

Isometries have properties very similar to that of unitary matrices. Given an $n \times m$ isometry $Q = [q_1 \cdots q_m]$,

- $\langle Qx, Qy\rangle = \langle x, y\rangle$ for all $x, y \in \mathbb{C}^n$.
- $\|Qx\|_2 = \|x\|_2$.
- $\|QB\|_2 = \|B\|_2$ for any $B \in \mathbb{C}^{n\times m}$.
- $\|Q\|_2 = 1$ and $\|Q\|_F = \sqrt{m}$.
- $\kappa_2(Q) = 1$.
- In the presence of rounding errors, $fl(QA) = Q(A + E)$ where $\|E\|_2/\|A\|_2$ is $O(u)$.
- QQ^* is the orthogonal projection onto $\operatorname{span}\{q_1, \ldots, q_m\}$, that is, $QQ^*v = v$ for all $v \in \operatorname{span}\{q_1, \ldots, q_m\}$ and $QQ^*w = 0$ for all $w \in \{q_1, \ldots, q_m\}^\perp$. Prove this!

Suggested resources for further study

- G. Strang, Linear Algebra and Its Applications, Cengage Learning, 4th Edition, 2006.
- J. Gilbert and L. Gilbert, Linear Algebra and Matrix Theory, Academic Press, 1995.
- MIT OCW on Linear Algebra.