Lab Session 5

MA-423 : Matrix Computations September-November 2020 S. Bora

Instructions: The report question 4 is to be written clearly in a single pdf file.

1. Write a MATLAB function program $[\mathtt{Q}, \mathtt{R}] = \mathtt{cgs}(\mathtt{V})$ to orthonormalize the columns of an $n \times m$ matrix V, $(n \geq m)$ by the Classical Gram Schmidt procedure so that Q is an isometry satisfying

$$\begin{aligned} \operatorname{span}\{Q(:,1)\} &= \operatorname{span}\{V(:,1)\} \\ \operatorname{span}\{Q(:,1), Q(:,2)\} &= \operatorname{span}\{V(:,1), V(:,2)\}, \\ &\vdots \\ \operatorname{span}\{Q(:,1), Q(:,2), \ldots, Q(:,m)\} &= \operatorname{span}\{V(:,1), V(:,2), \ldots, V(:,m)\} \end{aligned}$$

   and R is an upper triangular matrix such that $R(i,j) = \langle V(:,j), Q(:,i) \rangle$.

2. A slight modification of the above program leads to the Modified Gram Schmidt procedure for orthonormalizing the columns of V. Perform this modification to obtain another function program $[\mathtt{Q}, \mathtt{R}] = \mathtt{mgs}(\mathtt{V})$.

3. Write a function program $[\mathtt{Q}, \mathtt{R}] = \mathtt{cgsrep}(\mathtt{V})$ that performs Classified Gram Schmidt with reorthogonalization by making appropriate changes to your function program `cgs`.

   Take care to replace *for loops* by matrix-vector multiplications as far as possible in each of the above programs.

4. Use the program $\mathtt{A} = \mathtt{condmat}(\mathtt{n}, \mathtt{kappa})$ written during the assessment for Module 2 to generate a random 50×50 positive definite matrix A with given condition number 10^5. Store the matrix in a workspace file `Q4-1.mat` and run the `cgs`, `mgs` and `cgsrep` codes written above on the matrix. Finally also find a condensed QR decomposition of the matrix using the appropriate MATLAB command. (Find out what this is!)

   Generate two more 50×50 positive definite matrices with condition numbers 10^7 and 10^{11} and store them in workspace files `Q4-2.mat` and `Q4-3.mat` respectively. Repeat the above process on those matrices also.

   Record the departure from orthonormality for the isometries obtained from each of the processes and record them in a properly labeled table that also mentions the condition numbers.

5. Given a vector $x \in \mathbb{R}^n$, write a MATLAB function program $[\mathtt{u}, \gamma, \tau] = \mathtt{reflect}(\mathtt{x})$ to compute $u \in \mathbb{R}^n$, and $\gamma \in \mathbb{R}$ so that $Qx = [-\tau, 0, \ldots, 0]^T$ where $\tau = \pm\|x\|_2$ and the sign is chosen so that it is the same as that of the first entry of x.

6. Write another function program $\mathtt{B} = \mathtt{applreflect}(\mathtt{u}, \mathtt{gamma}, \mathtt{A})$ to *efficiently* perform the multiplication QA where $Q = I - \gamma uu^T$.

7. Use the programs written above to write another function program $\mathtt{R} = \mathtt{reflectqr}(\mathtt{A})$ that computes the R of a QR decomposition of $A \in \mathbb{R}^{n \times n}$ via reflectors. The program should be such that the lower triangular part of A contains the vectors u (apart from the leading 1 entry) required to construct the reflectors used at each stage and the values of γ corresponding to each reflector are stored in a separate vector.