Francis's (Implicit) QR Algorithm

Single shift or degree one: Let $A_0 = A$.

for $j = 1, 2, \ldots$

(i) Find reflector $Q_{j-1}^{(1)}$ *such that*

$$Q_{j-1}^{(1)}(A_{j-1} - \rho_{j-1} I)e_1 = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and compute $Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)}$.

(i) Find reflectors $\hat{Q}_{j-1}^{(2)}, \ldots, \hat{Q}_{j-1}^{(n-2)}$ *such that*

$$A_j = \hat{Q}_{j-1}^{(n-1)} \cdots \hat{Q}_{j-1}^{(2)} Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)} \hat{Q}_{j-1}^{(2)} \cdots \hat{Q}_{j-1}^{(n-1)}$$

is upper Hessenberg.

Francis's (Implicit) QR Algorithm

Double shift or degree two: Let $A_0 = A$.

for $j = 1, 2, \ldots$

(i) Find reflector $Q_{j-1}^{(1)}$ *such that*

$$Q_{j-1}^{(1)}(A_{j-1} - \rho_{j-1} I)(A_{j-1} - \tau_{j-1} I)e_1 = \begin{bmatrix} \alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and compute $Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)}$.

(i) Find reflectors $\hat{Q}_{j-1}^{(2)}, \ldots, \hat{Q}_{j-1}^{(p)}$ *such that*

$$A_j = \hat{Q}_{j-1}^{(p)} \cdots \hat{Q}_{j-1}^{(2)} Q_{j-1}^{(1)} A_{j-1} Q_{j-1}^{(1)} \hat{Q}_{j-1}^{(2)} \cdots \hat{Q}_{j-1}^{(p)}$$

is upper Hessenberg.

Francis's QR Algorithm of degree 2

Let

$$A_0 = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}.$$

Then $(A_0 - \rho I)(A_0 - \tau I)e_1 = \begin{bmatrix} (a_{11} - \rho_1)(a_{11} - \tau) + a_{12}a_{21} \\ a_{21}((a_{11} + a_{22}) - (\rho + \tau)) \\ a_{32}a_{21} \\ 0 \end{bmatrix}.$

Let $Q_0^{(1)} = \begin{bmatrix} \tilde{Q}_0^{(1)} & 0 \\ 0 & 1 \end{bmatrix}$ where

$$\tilde{Q}_0^{(1)} \begin{bmatrix} (a_{11} - \rho)(a_{11} - \tau) + a_{12}a_{21} \\ a_{21}((a_{11} + a_{22}) - (\rho + \tau)) \\ a_{32}a_{21} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \\ 0 \end{bmatrix}.$$

Francis's QR Algorithm of degree 2

Then $Q_0^{(1)} A_0 = \begin{bmatrix} \tilde{Q}_0^{(1)} & 0 \\ 0 & 1 \end{bmatrix} A_0 = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ b & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}$

and $Q_0^{(1)} A_0 Q_0^{(1)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ b_1 & \times & \times & \times \\ b_2 & b_3 & \times & \times \end{bmatrix}.$

Bulge chasing:

Let $\hat{Q}_0^{(2)} = \begin{bmatrix} 1 & 0 \\ 0 & \tilde{Q}_0^{(2)} \end{bmatrix}$ where $\tilde{Q}_0^{(2)} \begin{bmatrix} \times \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \times \\ 0 \\ 0 \end{bmatrix}.$

Francis's QR Algorithm of degree 2

Then $\hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & b_4 & \times & \times \end{bmatrix}$, and

$$\hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & b_5 & \times & \times \end{bmatrix}.$$

Let $\hat{Q}_0^{(3)} = \begin{bmatrix} I_2 & 0 \\ 0 & \tilde{Q}_0^{(3)} \end{bmatrix}$ where $\tilde{Q}_0^{(3)} \begin{bmatrix} \times \\ b_5 \end{bmatrix} = \begin{bmatrix} \times \\ 0 \end{bmatrix}$.

Francis's QR Algorithm of degree 2

Then $\hat{Q}_0^{(3)} \hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix}$, and

$$\hat{Q}_0^{(3)} \hat{Q}_0^{(2)} Q_0^{(1)} A_0 Q_0^{(1)} \hat{Q}_0^{(2)} \hat{Q}_0^{(3)} = \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} = A_1 .$$

Francis's QR Algorithm of degree 2

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The bulge is gone and A_1 is in upper Hessenberg form!

Why is Francis's QR algorithm the same as the theoretical QR algorithm?

Francis's QR $\equiv$ QR

Let $A \in \mathbb{C}^{n\times n}$ and $x \in \mathbb{C}^n$. Then,

$$K(A, x) = [x \quad Ax \; \cdots \; A^{n-1}x] \in \mathbb{C}^{n\times n}$$

is called the Krylov matrix associated with A and x.

Key properties of $K(A, x)$:

(1) For any $\alpha \in \mathbb{C}$, $\alpha K(A, x) = K(A, \alpha x)$.

(2) If A is upper Hessenberg, then $K(A, e_1)$ is upper triangular.

(3) If A is properly or irreducible upper Hessenberg, then $K(A, e_1)$ is upper triangular and non singular.

(4) For any polynomial $p(z)$, $p(A)K(A, x) = K(A, p(A)x)$.

(5) For any nonsingular matrix $S \in \mathbb{C}^{n\times n}$,
$K(S^{-1}AS, x) = S^{-1}K(A, Sx)$.

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Exercise: Prove (1) - (5)

Francis's QR $\equiv$ QR

Theorem Let A be a properly upper Hessenberg matrix and $p(x)$ be a polynomial over $\mathbb{R}$ or $\mathbb{C}$. Let Q be a unitary matrix such that $\hat{A} := Q^* AQ$ is upper Hessenberg and the first column of Q is proportional to the first column of $p(A)$. Then there exists an upper triangular matrix R such that $p(A) = QR$.

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Theorem Let A be a properly upper Hessenberg matrix and $\hat{A} = \hat{Q}^* A\hat{Q}$ be the matrix obtained after a single iteration of Francis's implicit QR algorithm of degree 1 or 2. Let $p(A) = A - \rho I$ for degree 1 and $p(A) = (A - \rho I)(A - \tau I)$ for degree 2. Then $\hat{Q}e_1 = \alpha p(A)e_1$ for some $\alpha \in \mathbb{C} \setminus \{0\}$.

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Corollary Let A be a properly upper Hessenberg matrix and $\hat{A} = \hat{Q}^* A \hat{Q}$ be the matrix obtained after a single iteration of Francis's implicit QR algorithm of degree 1 or 2. Let $p(A) = A - \rho I$ for degree 1 and $p(A) = (A - \rho I)(A - \tau I)$ for degree 2. Then

$$p(A) = \hat{Q} R$$

where $R = \frac{1}{\alpha} K(\hat{A}, e_1)[K(A, e_1)]^{-1}$ is upper triangular.