Assignment 1

MA-423 : Matrix Computations September-November 2020 S. Bora

Total Marks: 20

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1. Let A be any $n \times n$ matrix. For $k = 1, \ldots, n-1$, let P_k and M_k be respectively the transposition matrix and multiplier matrix used in the k$^{\text{th}}$ step of Gaussian Elimination with Partial Pivoting (GEPP) on A such that

$$U := M_{n-1}P_{n-1}M_{n-2}P_{n-1} \cdots M_k P_k \cdots M_2 P_2 M_1 P_1 A.$$

is upper triangular. Prove the following.

(a) Let $\mathcal{P}_k = P_{k+1} \cdots P_{n-1}$, $k = 1, \ldots, n-2$. Then,

$$\mathcal{P}_k = \left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P}_{k+1} \cdots \widetilde{P}_{n-1} \end{array}\right]$$

where for all $j = k+1, \ldots, n-1$, $\widetilde{P}_j$ are transpositions of size $n-k \times n-k$. **(2 marks)**

(b) Let $\widetilde{M}_k = \mathcal{P}_k^T M_k \mathcal{P}_k$, $k = 1, \ldots, n-2$. Then,

$$\widetilde{M}_k = I_n - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \widetilde{m}_{k+1,k} \\ \vdots \\ \widetilde{m}_{nk} \end{bmatrix} e_k^T,$$

where

$$\begin{bmatrix} \widetilde{m}_{k+1,k} \\ \vdots \\ \widetilde{m}_{nk} \end{bmatrix} = \widetilde{P}_{n-1} \cdots \widetilde{P}_{k+1} \begin{bmatrix} m_{k+1,k} \\ \vdots \\ m_{nk} \end{bmatrix},$$

where $m_{k+1,k}, \ldots, m_{nk}$ are the multipliers used in step k, $1 \leq k \leq n-2$. **(2 marks)**

(c) $PA = LU$ where $P = P_{n-1}P_{n-2}\cdots P_1$ and

$$L = \widetilde{M}_1^{-1}\cdots\widetilde{M}_{n-2}^{-1}M_{n-1}^{-1}$$
$$= \begin{bmatrix} 1 & & & & & & & \\ \widetilde{m}_{21} & 1 & & & & & & \\ \widetilde{m}_{31} & \widetilde{m}_{32} & \ddots & & & & & \\ \vdots & \vdots & & \ddots & & & & \\ \widetilde{m}_{k1} & \widetilde{m}_{k2} & \cdots & \cdots & 1 & & & \\ \widetilde{m}_{k+1,1} & \widetilde{m}_{k+1,2} & \cdots & \cdots & \widetilde{m}_{k+1,k} & \ddots & & \\ \vdots & \vdots & & & \vdots & & 1 & \\ \widetilde{m}_{n1} & \widetilde{m}_{n2} & \cdots & \cdots & \widetilde{m}_{nk} & \cdots & m_{n,n-1} & 1 \end{bmatrix}$$

(6 marks)

2. Let $A = [a_{ij}]$ be an $n \times n$ positive definite matrix and $u \in \mathbb{R}^n \setminus \{0\}$ such that $u^T A^{-1} u \neq 1$.

   (a) Prove that
   $$(A - uu^T)^{-1} = A^{-1} + \frac{(A^{-1}u)u^T A^{-1}}{1 - u^T A^{-1} u}.$$
   (3 marks)

   (b) Use the relation in part (a) to write a pseudocode (outline only and no *for loops*!) for solving the $n \times n$ system of equations $(A - uu^T)x = b$ in $n^3/3 + O(n^2)$ flops. **(4 marks)**

   (c) Prove that the entries of A satisfy $a_{ii}a_{jj} - a_{ij}^2 > 0$ for all $i \neq j$. **(3 marks)**