Matrix Computations

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September-November 2020

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- Direct methods
- Iterative methods

General Techniques and Issues

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- Speed of algorithms

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- The Singular Value decomposition (SVD) ($A = USV^*$).

Rounding: The Silent Killer

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IEEE standard allows to track small errors made when two numbers are added, subtracted, multiplied or divided on a computer.

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On my computer MATLAB produces:

$$\begin{aligned}(\frac{4}{3} - 1) * 3 - 1 &= -2.2204 \times 10^{-16}\\ 5 \times \frac{(1 + \exp(-50)) - 1}{(1 + \exp(-50)) - 1} &= \mathbf{NaN}.\\ \frac{\log(\exp(750))}{100} &= \mathbf{Inf}.\end{aligned}$$

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Prevention is better than cure!

Stability of algorithms

Analysing the errors caused by the algorithm itself requires knowing the effect of rounding errors during the execution of the algorithm. A desirable property of algorithms is *backward stability:*

> *If an algorithm alg(x) is used to compute $f(x)$, then including the effect of rounding error, alg(x) is said to be backward stable if alg$(x) = f(x + \delta x)$ for small δx. Here δx is called the backward error.*

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Thus a backward stable algorithm provides the exact answer to a slightly perturbed problem.

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Thus if the algorithm is backward stable, then the *forward error* which is the difference between its exact and computed solutions is small if the solution is not too sensitive to perturbation.

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If an algorithm is *iterative*, then it is necessary to know the number of iterations necessary to accept any approximate solution as an answer. This is decided by the quality of the convergence, whether *linear, quadratic or cubic....*

Class timings

- The class timings are slot A (9a.m.-9:55 a.m. of Monday-Wednesday) for theory and slot A1 (2-4 p.m. on Tuesday) for lab.
- Additionally (9 a.m.-9:55 a.m) of Saturdays with Monday/Tuesday/Wednesday timetables are also reserved for the course.
- There will be no live classes. Instead lecture videos and slides will be posted to the class group on MS Teams on a regular basis.
- There will be interaction sessions on MS teams every week on Wednesdays (9-9:55 a.m. for theory) and Tuesdays (2p.m.-4 p.m. for lab).
- The other class timings will be utilized for assessments/tests.
- In the first week, the interaction will be on Monday (9a.m.-9:55 a.m.) for theory.

Texts & References and Evaluation Policy

Textbook:

- Fundamentals of Matrix Computations by D. S. Watkins (2nd or 3rd edition)

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A detailed evaluation policy has already been posted.