# MA423 Theory Assignment 1

Group 4

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**A1.(a)**

$P_k = \left[\begin{array}{c|c} I_{k-1} & \\ \hline & \hat{P_k} \end{array}\right]$, $\hat{P_k}$ is a $(n-k+1)\times(n-k+1)$ transposition.
$\mathcal{P}_k = P_{k+1}\cdots P_{n-1}$, $k = 1, 2\cdots n-2$

here $P_k$ is also a transposition in which no row exchanges happen in the first $k-1$ rows. So for any transposition $P$ the matrix $\left[\begin{array}{c|c} I & \\ \hline & P \end{array}\right]$ is also a transposition matrix.

So we can write each $P_j = \left[\begin{array}{c|c} I_k & \\ \hline & \tilde{P_j} \end{array}\right]$ for $j = k+1, \cdots, n-1$

$\mathcal{P}_k = P_{k+1}\cdots P_{n-1}$, $k = 1, 2\cdots n-2$

$\implies \mathcal{P}_k = \left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{k+1}} \end{array}\right]\left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{k+2}} \end{array}\right]\cdots\left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{n-1}} \end{array}\right]$

$\therefore \mathcal{P}_k = \left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{k+1}}\widetilde{P_{k+2}}\cdots\widetilde{P_{n-1}} \end{array}\right]$

**A1.(b)**

To prove if $\widetilde{M_k} = {\mathcal{P}_k}^T M_k \mathcal{P}_k$, then

$$\widetilde{M_k} = I_n - \begin{bmatrix} 0 \\ \vdots \\ \widetilde{m_{k+1,k}} \\ \vdots \\ \widetilde{m_{n,k}} \end{bmatrix} {e_k}^T$$

where,

$$\begin{bmatrix} \widetilde{m_{k+1,k}} \\ \vdots \\ \widetilde{m_{n,k}} \end{bmatrix} = \widetilde{\mathcal{P}_{n-1}}\cdots\widetilde{\mathcal{P}_{k+1}} \begin{bmatrix} \widetilde{m_{k+1,k}} \\ \vdots \\ \widetilde{m_{n,k}} \end{bmatrix}$$

we know that , $M_k = I_n - \begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} {e_k}^T$ so

$${\mathcal{P}_k}^T M_k \mathcal{P}_k = {\mathcal{P}_k}^T I_n \mathcal{P}_k - {\mathcal{P}_k}^T \begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} \mathcal{P}_k$$

$$\mathcal{P}_k{}^T M_k \mathcal{P}_k = I_n - \ \mathcal{P}_k{}^T \begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} e_k^T \mathcal{P}_k$$

so we know that $\begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} e_k^T = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \cdots & \vdots \\ 0 & \cdots & m_{k+1,k} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & m_{n,k} & \cdots & 0 \end{bmatrix}$

since the last $n-k$ columns are 0 and $A\mathcal{P}_k$ permutes only the columns $k+1 \cdots n$ of a matrix $A$

$$\begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} e_k{}^T \mathcal{P}_k = \begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} e_k^T$$

$$\mathcal{P}_k{}^T = \left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{k+1}}\widetilde{P_{k+2}} \cdots \widetilde{P_{n-1}} \end{array}\right]^T = \left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{n-1}}^T \widetilde{P_{n-2}}^T \cdots \widetilde{P_{k+1}}^T \end{array}\right] = \left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{n-1}}\widetilde{P_{n-2}} \cdots \widetilde{P_{k+1}} \end{array}\right]$$

as $P^T = P$ for permutation matrix

$$\therefore \mathcal{P}_k \begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} e_k^T = \left[\begin{array}{c|c} I_k & \\ \hline & \widetilde{P_{n-1}}\widetilde{P_{n-2}} \cdots \widetilde{P_{k+1}} \end{array}\right] \begin{bmatrix} 0 \\ \vdots \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix} e_k^T = \widetilde{P_{n-1}}\widetilde{P_{n-2}} \cdots \widetilde{P_{k+1}} \begin{bmatrix} m_{k+1,k} \\ \vdots \\ m_{n,k} \end{bmatrix}$$

Hence proved

**A1.(c)** We know that

$$U = M_{n-1}P_{n-1}M_{n-2}P_{n-2} \cdots M_k P_k \cdots M_1 P_1 A$$

and

$$\widetilde{M}_k^{-1} = P_{n-1}P_{n-2} \cdots P_{k+1} M_k^{-1} P_{k+1} \cdots P_{n-1}$$

$\implies \widetilde{M}_{n-1}^{-1} U = M_{n-1}^{-1} M_{n-1}P_{n-1}M_{n-2}P_{n-2} \cdots M_k P_k \cdots M_1 P_1 A$

$\implies \widetilde{M}_{n-1}^{-1} U = P_{n-1}M_{n-2}P_{n-2} \cdots M_k P_k \cdots M_1 P_1 A$

$\implies \widetilde{M}_{n-2}^{-1}\widetilde{M}_{n-1}^{-1} U = (P_{n-1}M_{n-2}^{-1}P_{n-1})P_{n-1}M_{n-2}P_{n-2} \cdots M_k P_k \cdots M_1 P_1 A$

using $P^{-1} = P$ for a permutation matrix

$\implies \widetilde{M}_{n-2}^{-1}\widetilde{M}_{n-1}^{-1} U = P_{n-1}P_{n-2}M_{n-3} \cdots M_k P_k \cdots M_1 P_1 A$

$\vdots$

At the $(n-k)^{\text{th}}$ step, we have

$$\widetilde{M}_{k+1}^{-1}\cdots\widetilde{M}_{n-1}^{-1}U = P_{n-1}\cdots P_{k+1}M_k\cdots M_1P_1A = \mathcal{P}_k^T M_k\cdots M_1P_1A$$

$$\widetilde{M}_{k}^{-1}\cdots\widetilde{M}_{n-1}^{-1}U = \mathcal{P}_k^T\,(M_k^{-1}\mathcal{P}_k\mathcal{P}_k^T M_k)\cdots\ M_1P_1A = \mathcal{P}_k^T\,M_{k-1}\cdots\ M_1P_1A$$

so we finally get

$$\widetilde{M}_{1}^{-1}\cdots\widetilde{M}_{k}^{-1}\cdots\widetilde{M}_{n-1}^{-1}U = P_{n-1}\cdots P_1A$$

And, since $LU = P_{n-1}\cdots P_1A$

$(L-\widetilde{M}_{1}^{-1}\cdots\widetilde{M}_{k}^{-1}\cdots\widetilde{M}_{n-1}^{-1})U = 0$

$\therefore\ L = \widetilde{M}_{1}^{-1}\cdots\widetilde{M}_{k}^{-1}\cdots\widetilde{M}_{n-1}^{-1}$ (Since $U$ is invertible)

now we prove

$$\widetilde{M}_{k}^{-1} = I_n + \begin{bmatrix} 0 \\ \vdots \\ \widetilde{m}_{k+1,k} \\ \vdots \\ \widetilde{m}_{n,k} \end{bmatrix} e_k^T$$

let $x = \begin{bmatrix} 0 \\ \vdots \\ \widetilde{m}_{k+1,k} \\ \vdots \\ \widetilde{m}_{n,k} \end{bmatrix}$ which gives

$\widetilde{M}_{k}^{-1}\widetilde{M}_k = (I_n + xe_k^T)(I_n - xe_k^T)$

$\implies\ \widetilde{M}_{k}^{-1}\widetilde{M}_k = I_n + I_nxe_k^T - I_nxe_k^T - x(e_k^Tx)e_k^T$

here $e_k^Tx = x_k = 0$ where $x_k$ is the $k^{\text{th}}$ element in column vector $x$

so we get our result $\widetilde{M}_{k}^{-1}\widetilde{M}_k = I_n$

$$\therefore \widetilde{M}_{1}^{-1}\cdots\widetilde{M}_{n-1}^{-1} = \left(I_n + \begin{bmatrix} 0 \\ \widetilde{m}_{2,1} \\ \vdots \\ \widetilde{m}_{n,1} \end{bmatrix} e_1^T\right)\cdots\left(I_n + \begin{bmatrix} 0 \\ \vdots \\ \widetilde{m}_{k+1,k} \\ \vdots \\ \widetilde{m}_{n,k} \end{bmatrix} e_k^T\right)\cdots\left(I_n + \begin{bmatrix} 0 \\ \vdots \\ \widetilde{m}_{n-1,n} \end{bmatrix} e_{n-1}^T\right)$$

$$\widetilde{M}_{1}^{-1}\cdots\widetilde{M}_{n-1}^{-1} = \begin{bmatrix} 1 & & & & & \\ \widetilde{m}_{21} & 1 & & & & \\ \widetilde{m}_{31} & \widetilde{m}_{32} & \ddots & & & \\ \vdots & \vdots & & \ddots & & \\ \widetilde{m}_{k1} & \widetilde{m}_{k2} & \cdots & \cdots & 1 & \\ \vdots & \vdots & & \vdots & & \ddots \\ \widetilde{m}_{n1} & \widetilde{m}_{n2}\cdots & \cdots & \widetilde{m}_{n,k-1} & \cdots\widetilde{m}_{n,n-1} & 1 \end{bmatrix}$$

**Q2.(a)** Prove $(A^{-1} - uu^T)^{-1} = A^{-1} + \frac{(A^{-1}u)u^T A^{-1}}{1-u^T A^{-1} u}$

Since $A$ is positive definite, So is $A^{-1}$.

Let $x = A^{-1}u$. Then $x^T = u^T (A^{-1})^T = u^T A^{-1}$

$\therefore A^{-1} + \frac{(A^{-1}u)u^T A^{-1}}{1-u^T A^{-1} u} = A^{-1} + \frac{xx^T}{1-u^T x}$

Then, $(A - uu^T)(A^{-1} + \frac{xx^T}{1-u^T x}) = I + \frac{Axx^T}{1-u^T x} - uu^T A^{-1} - \frac{uu^T xx^T}{1-u^T x}$

$= I + \frac{Axx^T}{1-u^T x} - (Ax)(x^T A)A^{-1} - \frac{(Ax)u^T xx^T}{1-u^T x}$ $(x = A^{-1}u \implies u = Ax$ and $u^T = x^T A^T = x^T A)$

$= I + \frac{Axx^T}{1-u^T x} - (1 - u^T x)\frac{Axx^T}{1-u^T x} - (u^T x)\frac{Axx^T}{1-u^T x}$ ($u^T x$ is scalar)

$= I + 0 = I$

$\implies A^{-1} + \frac{(A^{-1}u)u^T A^{-1}}{1-u^T A^{-1} u}$ is the inverse of $A - uu^T$

**Q2.(b)** Write pseudo code to solve $(A - uu^T)x = b$, using the relation established in part (a) in $\frac{n^3}{3} + O(n^2)$ flops.

Here, $x = (A - uu^T)^{-1}b$, substituting the value of $(A - uu^T)^{-1}$ from part (a)

$x = A^{-1}b + \frac{(A^{-1}u)(A^{-1}u)^T b}{1-u^T A^{-1} u}$

Let $x_1 = A^{-1}b$ and $x_2 = A^{-1}u$

Then $x = x_1 + \frac{(x_2)(x_2)^T b}{1-u^T x_2}$

We can compute the value of $x$ by solving these two systems.We find the cholesky factor of $A$ and use it to solve for $x_1$ and $x_2$.

Algorithm

1. Find the Cholesky factor G of A. $\frac{n^3}{3}$+O($n^2$) flops
2. Solve $G^T y_1 = b$,where $y_1 = Gx_1$ O($n^2$) flops
3. Solve $Gx_1 = y_1$ O($n^2$) flops
4. Solve $G^T y_2 = u$,where $y_2 = Gx_2$ O($n^2$) flops
5. Solve $Gx_2 = y_2$ O($n^2$) flops
6. Using the values of $x_1$ and $x_2$, compute the value of $x$. O($n^2$) flops

Total flop count =$\frac{n^3}{3}$+O($n^2$)

**Q2.(c)**

A = $[a_{i,j}]$ is a positive definite matrix.
Consider x = $\alpha e_i + \beta e_j$ , $1 \leq i < j \leq n$, where $e_i$ is the $i^{th}$ column vector of $I_n$.

Since A is positive definite $x^T Ax > 0 \ \forall \, x \, \epsilon \, R^n \setminus 0$.

$$\implies (\alpha e_i + \beta e_j)^T A(\alpha e_i + \beta e_j) > 0$$

$$\implies \begin{bmatrix} 0 & \cdots & \alpha & \cdots & \beta & \cdots & 0 \end{bmatrix} \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ \alpha \\ \vdots \\ \beta \\ \vdots \\ 0 \end{bmatrix} > 0.$$

$\implies \alpha^2 a_{i,i} + \beta^2 a_{j,j} + \alpha\beta(a_{i,j} + a_{j,i}) > 0$

Since A is symmetric positive definite $a_{i,j} = a_{j,i}$.

$\implies \alpha^2 a_{i,i} + \beta^2 a_{j,j} + 2\alpha\beta a_{i,j} > 0$

For $\alpha = a_{i,j}$ & $\beta = -a_{i,i}$

$a_{i,j}^2 a_{i,i} + a_{i,i}^2 a_{j,j} - 2a_{i,i}a_{i,j}^2 > 0$

$\implies a_{i,i}(a_{i,i}a_{j,j} - a_{i,j}^2) > 0$

Here $a_{i,i} > 0 \ \forall \ 1 \leq i \leq n$ as A is positive definite.

$\therefore a_{i,i}a_{j,j} - a_{i,j}^2 > 0 \ \forall \ 1 \leq i < j \leq n.$