Solving Triangular Systems

A Lower Triangular System of Equations

Consider

$$Lx = b$$

where

$$L = [l_{ij}] \longleftarrow n \times n \text{ lower triangular matrix}$$

$$b \longleftarrow n \times 1 \text{ vector.}$$

In expanded form,

$$\begin{array}{lcl} l_{11}x_1 & = & b_1 \\ l_{21}x_1 + l_{22}x_2 & = & b_2 \\ & \vdots & \\ l_{k1}x_1 + l_{k2}x_2 + \cdots + l_{kk}x_k & = & b_k \\ & \vdots & \\ l_{n1}x_1 + l_{n2}x_2 + \cdots + l_{nk}x_k + \cdots + l_{nn}x_n & = & b_n \end{array}$$

Row Oriented Forward Substitution

$$x_1 = b_1 / l_{11}; \quad x_k = \left(b_k - \sum_{j=1}^{k-1} l_{kj} b_j \right) / l_{kk}, \; k = 2, \ldots, n;$$

for $k = 1 : n$
    for $j = 1 : k - 1$
        $b_k = b_k - l_{kj} b_j;$
    end
            if $l_{kk} \neq 0$
                $b_k = b_k / l_{kk};$
            else
                exit {'matrix is singular'}
            end
end

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for $k = 1 : n$
    for $j = 1 : k-1$ $\quad\leftarrow$ entries of L are being accessed row wise
        $b_k = b_k - l_{kj} b_j;$ (2 flops)
    end
        if $l_{kk} \neq 0$
            $b_k = b_k / l_{kk};$ (1 flop)
        else
            exit {'matrix is singular'}
        end
end

Total flops:

$$1 + \sum_{k=2}^{n} \left\{ \left(\sum_{j=1}^{k-1} 2 \right) + 1 \right\} = \sum_{k=1}^{n} 2(k-1) + n = n^2.$$

Column Oriented Forward Substitution

for $j = 1 : n$
  if $l_{jj} \neq 0$
    $b_j = b_j / l_{jj};$ (1 flop)
  end
 for $i = j + 1 : n$ $\leftarrow$ entries of L are being accessed column wise
  $b_i = b_i - l_{ij} b_i;$ (2 flops)
 end
end

Total flops: n^2.

Likewise, $n \times n$ upper triangular systems may be solved in n^2 flops each by performing row and column oriented backward substitutions.

Blocked forward subtitution:

The $n \times n$ lower triangular system system $Lx = b$ may be rewritten as

$$\begin{bmatrix} L_{11} & \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},$$

where the blocks of L are

$$L_{11} \to k \times k \text{ lower triangular; } L_{21} \to (n-k) \times k;$$
$$L_{22} \to (n-k) \times (n-k) \text{ lower triangular}$$

and the partitions x_1, x_2 of x and b_1, b_2 of b are such that

$$x_1, b_1 \to k \times 1 \text{ and } x_2, b_2 \to (n-k) \times 1.$$

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Pseudocode for blocked forward substitution:

1. Solve $L_{11}x_1 = b_1$ for x_1.
2. Compute $b_2 - L_{21}x_1$.
3. Solve $L_{22}x_2 = b_2 - L_{21}x_1$ for x_2.