Differentiation

$y = f(x)$	$\frac{\mathrm{d}y}{\mathrm{d}x} = f'(x)$
k, constant	0
x^n, any constant n	nx^{n-1}
e^x	e^x
$\ln x = \log_\mathrm{e} x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x = \frac{\sin x}{\cos x}$	$\sec^2 x$
$\mathrm{cosec}\, x = \frac{1}{\sin x}$	$-\mathrm{cosec}\, x \cot x$
$\sec x = \frac{1}{\cos x}$	$\sec x \tan x$
$\cot x = \frac{\cos x}{\sin x}$	$-\mathrm{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cosh x$	$\sinh x$
$\sinh x$	$\cosh x$
$\tanh x$	$\mathrm{sech}^2 x$
$\mathrm{sech}\, x$	$-\mathrm{sech}\, x \tanh x$
$\mathrm{cosech}\, x$	$-\mathrm{cosech}\, x \coth x$
$\coth x$	$-\mathrm{cosech}^2 x$
$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$
$\tanh^{-1} x$	$\frac{1}{1-x^2}$

The linearity rule for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(au + bv) = a\frac{\mathrm{d}u}{\mathrm{d}x} + b\frac{\mathrm{d}v}{\mathrm{d}x} \quad a, b \text{ constant}$$

The product and quotient rules for differentiation

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$

The chain rule for differentiation

If $y = y(u)$ where $u = u(x)$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x}$

For example,
if $y = (\cos x)^{-1}$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -1(\cos x)^{-2}(-\sin x)$

Integration

$f(x)$	$\int f(x)\,\mathrm{d}x = F(x) + c$	
k, constant	$kx + c$	
x^n, $(n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$	
$x^{-1} = \frac{1}{x}$	$\begin{cases} \ln x + c & x > 0 \\ \ln(-x) + c & x < 0 \end{cases}$	
e^x	$\mathrm{e}^x + c$	
$\cos x$	$\sin x + c$	
$\sin x$	$-\cos x + c$	
$\tan x$	$\ln(\sec x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\sec x$	$\ln(\sec x + \tan x) + c$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$
$\mathrm{cosec}\, x$	$\ln(\mathrm{cosec}\, x - \cot x) + c$	$0 < x < \pi$
$\cot x$	$\ln(\sin x) + c$	$0 < x < \pi$
$\cosh x$	$\sinh x + c$	
$\sinh x$	$\cosh x + c$	
$\tanh x$	$\ln\cosh x + c$	
$\coth x$	$\ln\sinh x + c$	$x > 0$
$\frac{1}{x^2+a^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\frac{x-a}{x+a} + c$	$\lvert x\rvert > a > 0$
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\frac{a+x}{a-x} + c$	$\lvert x\rvert < a$
$\frac{1}{\sqrt{x^2+a^2}}$	$\sinh^{-1}\frac{x}{a} + c$	$a > 0$
$\frac{1}{\sqrt{x^2-a^2}}$	$\cosh^{-1}\frac{x}{a} + c$	$x \geqslant a > 0$
$\frac{1}{\sqrt{x^2+k}}$	$\ln(x + \sqrt{x^2+k}) + c$	
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a} + c$	$-a \leqslant x \leqslant a$
$f(ax+b)$	$\frac{1}{a}F(ax+b) + c$	$a \neq 0$
e.g. $\cos(2x-3)$	$\frac{1}{2}\sin(2x-3) + c$	

The linearity rule for integration

$$\int (af(x) + bg(x))\,\mathrm{d}x = a\int f(x)\,\mathrm{d}x + b\int g(x)\,\mathrm{d}x, \quad (a, b \text{ constant})$$

Integration by substitution

$$\int f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(u)\mathrm{d}u \quad \text{and} \quad \int_a^b f(u)\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int_{u(a)}^{u(b)} f(u)\mathrm{d}u$$

Integration by parts

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = [uv]_a^b - \int_a^b \frac{\mathrm{d}u}{\mathrm{d}x}v\,\mathrm{d}x$$

Alternative form:

$$\int_a^b f(x)g(x)\,\mathrm{d}x = \left[f(x)\int g(x)\mathrm{d}x\right]_a^b - \int_a^b \frac{\mathrm{d}f}{\mathrm{d}x}\left\{\int g(x)\mathrm{d}x\right\}\mathrm{d}x$$

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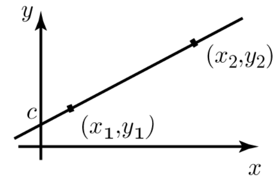
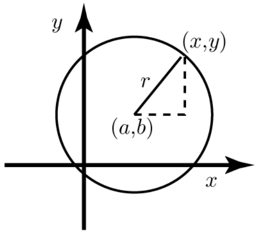
Graphs of common functions

Linear $y = mx + c$, m=gradient, c = vertical intercept



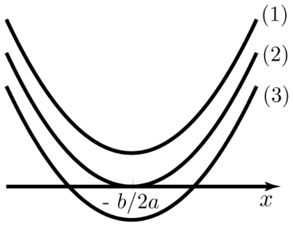
$$m = \frac{y_2-y_1}{x_2-x_1}$$

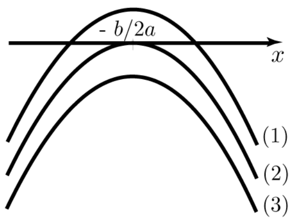
The equation of a circle centre (a, b), radius r



$$(x-a)^2 + (y-b)^2 = r^2$$

Quadratic functions $y = ax^2 + bx + c$





$a > 0$

(1) $b^2 - 4ac < 0$
(2) $b^2 - 4ac = 0$
(3) $b^2 - 4ac > 0$

$a < 0$

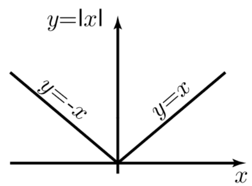
(1) $b^2 - 4ac > 0$
(2) $b^2 - 4ac = 0$
(3) $b^2 - 4ac < 0$

Completing the square

$$\text{If } a \neq 0, \quad ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

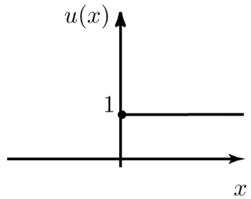
The modulus function

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

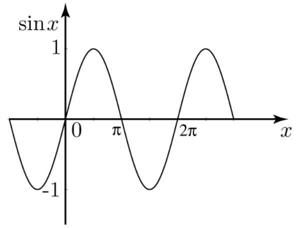


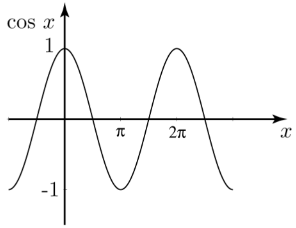
The unit step function, $u(x)$

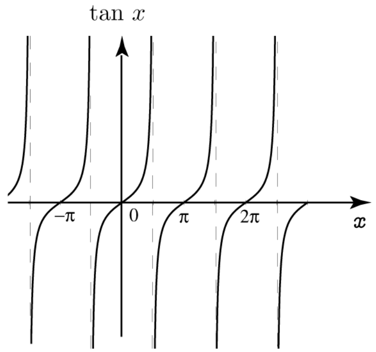
$$u(x) = \begin{cases} 1 & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$



Trigonometric functions

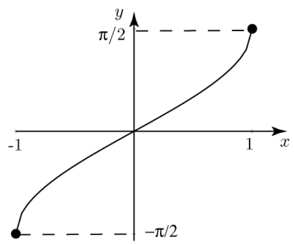




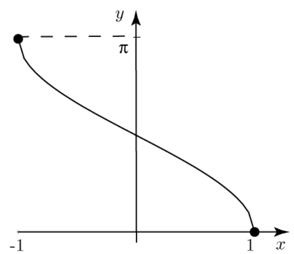


The sine and cosine functions are periodic with period 2π.
The tangent function is periodic with period π.

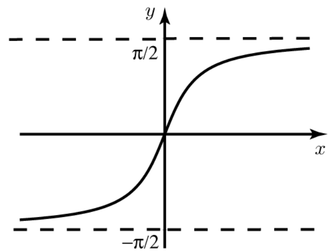
Inverse trigonometric functions



$y = \sin^{-1} x$



$y = \cos^{-1} x$



$y = \tan^{-1} x$

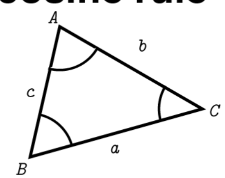
The sine rule and cosine rule

The sine rule

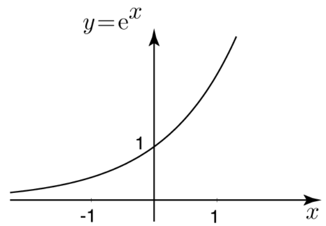
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

The cosine rule

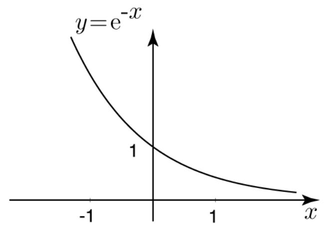
$$a^2 = b^2 + c^2 - 2bc\cos A$$



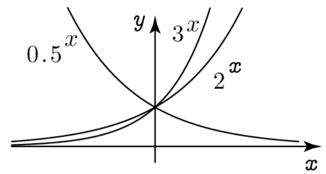
Exponential functions



Graph of $y = e^x$ showing exponential growth

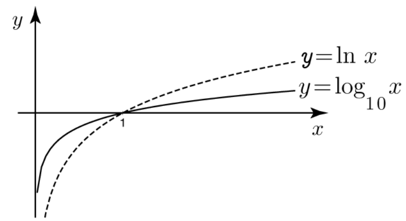


Graph of $y = e^{-x}$ showing exponential decay



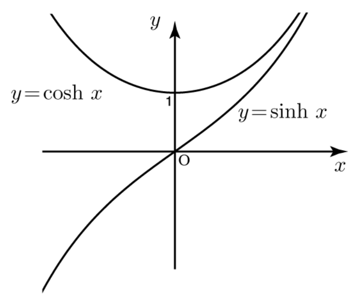
Graphs of $y = 0.5^x$, $y = 3^x$, and $y = 2^x$

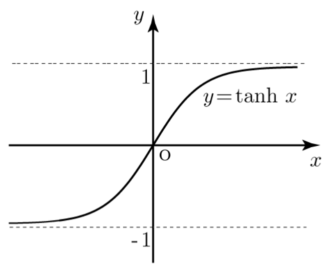
Logarithmic functions



Graphs of $y = \ln x$ and $y = \log_{10} x$

Hyperbolic functions

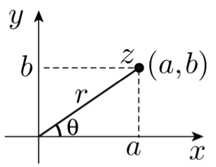




Graphs of $y = \sinh x$, $y = \cosh x$ and $y = \tanh x$

Complex Numbers

Cartesian form: $z = a + bj$
where $j = \sqrt{-1}$
Polar form:
$z = r(\cos\theta + j\sin\theta) = r\angle\theta$
$a = r\cos\theta, \; b = r\sin\theta,$
$\tan\theta = \frac{b}{a}$



Exponential form: $z = re^{j\theta}$

Euler's relations
$e^{j\theta} = \cos\theta + j\sin\theta, \quad e^{-j\theta} = \cos\theta - j\sin\theta$

Multiplication and division in polar form

$$z_1 z_2 = r_1 r_2 \angle(\theta_1 + \theta_2), \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2)$$

If $z = r\angle\theta$, then $z^n = r^n\angle(n\theta)$

De Moivre's theorem

$$(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$$

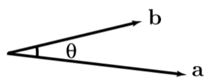
Relationship between hyperbolic and trig functions
$\cos jx = \cosh x, \quad \sin jx = j\sinh x$
$\cosh jx = \cos x, \quad \sinh jx = j\sin x$
i rather than j may be used to denote $\sqrt{-1}$.

Vectors

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$

Scalar product

$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta$

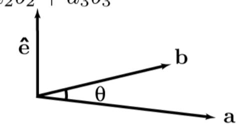


If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Vector product

$\mathbf{a}\times\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{e}}$



$\hat{\mathbf{e}}$ is a unit vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$ in a sense defined by the right hand screw rule.

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then

$$\mathbf{a}\times\mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Sequences and Series

Arithmetic progression: $a, a+d, a+2d, \ldots$
a = first term, d = common difference,
kth term $= a + (k-1)d$
Sum of n terms, $S_n = \frac{n}{2}(2a + (n-1)d)$

Sum of the first n integers,
$1 + 2 + 3 + \ldots + n =$

$$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$$

Sum of the squares of the first n integers,
$1^2 + 2^2 + 3^2 + \ldots + n^2 =$

$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$$

Geometric progression: $a, ar, ar^2, \ldots$
a = first term, r = common ratio,
kth term $= ar^{k-1}$
Sum of n terms, $S_n = \frac{a(1-r^n)}{1-r}$, provided $r \neq 1$

Sum of an infinite geometric series:
$S_\infty = \frac{a}{1-r}, \quad -1 < r < 1$

The binomial theorem
If n is a positive integer

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots + x^n$$

When n is negative or fractional, the series is infinite and converges when $-1 < x < 1$

Standard power series expansions

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \text{ for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \text{ for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \text{ for all } x$$

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ for } -1 < x \leqslant 1 \text{ only}$$

The exponential function as the limit of a sequence

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x$$

Matrices and Determinants

The 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

The 3×3 matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ has determinant

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

(expanded along the first row).

The inverse of a 2×2 matrix
If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
provided that $ad - bc \neq 0$.

Matrix multiplication: for 2×2 matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix} = \begin{pmatrix} a\alpha + b\beta & a\gamma + b\delta \\ c\alpha + d\beta & c\gamma + d\delta \end{pmatrix}$$

Remember that $AB \neq BA$ except in special cases.

The Binomial Coefficients

The coefficient of x^k in the binomial expansion of $(1+x)^n$ when n is a positive integer is denoted by $\binom{n}{k}$ or ${}^nC_k$.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

$0! = 1, \quad n! = n(n-1)!$
so, for example, $4! = 1.2.3.4$

The pattern of the coefficients is seen in

Pascal's triangle:

$$\begin{array}{ccccccccccc} & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \end{array}$$

${}^nC_k$ is the number of subsets with k elements that can be chosen from a set with n elements.

Algebra

$$(x+k)(x-k) = x^2 - k^2$$
$$(x+k)^2 = x^2 + 2kx + k^2, \quad (x-k)^2 = x^2 - 2kx + k^2$$
$$x^3 \pm k^3 = (x \pm k)(x^2 \mp kx + k^2)$$

Formula for solving a quadratic equation:

$$\text{if } ax^2 + bx + c = 0 \text{ then } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Laws of Indices

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$
$$a^0 = 1 \qquad a^{-m} = \frac{1}{a^m} \qquad a^{1/n} = \sqrt[n]{a} \qquad a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

Laws of Logarithms

For any positive base b (with $b \neq 1$)

$$\log_b A = c \quad \text{means} \quad A = b^c$$
$$\log_b A + \log_b B = \log_b AB, \qquad \log_b A - \log_b B = \log_b \frac{A}{B},$$
$$n \log_b A = \log_b A^n, \qquad \log_b 1 = 0, \qquad \log_b b = 1$$

Formula for change of base: $\log_a x = \frac{\log_b x}{\log_b a}$

Logarithms to base e, denoted $\log_e$ or alternatively ln are called *natural logarithms*. The letter e stands for the exponential constant which is approximately 2.718.

Partial fractions

For *proper fractions* $\frac{P(x)}{Q(x)}$ where P and Q are polynomials with the degree of P less than the degree of Q:

a *linear factor* $ax+b$ in the denominator produces a partial fraction of the form $\frac{A}{ax+b}$

repeated linear factors $(ax+b)^2$ in the denominator produce partial fractions of the form $\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$

a *quadratic factor* ax^2+bx+c in the denominator produces a partial fraction of the form $\frac{Ax+B}{ax^2+bx+c}$

Improper fractions require an additional term which is a polynomial of degree $n-d$ where n is the degree of the numerator and d is the degree of the denominator.

Inequalities:

$a > b$ means a is greater than b

$a < b$ means a is less than b

$a \geqslant b$ means a is greater than or equal to b

$a \leqslant b$ means a is less than or equal to b

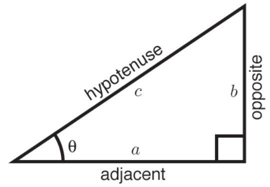
Trigonometry

Degrees and radians

$$360° = 2\pi \text{ radians}, \quad 1° = \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians}$$
$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees} \approx 57.3°$$

Trig ratios for an acute angle θ:

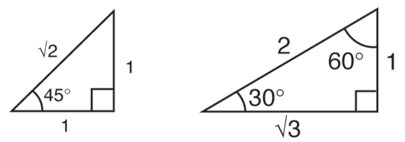
$$\sin\theta = \frac{\text{side opposite to } \theta}{\text{hypotenuse}} = \frac{b}{c}$$
$$\cos\theta = \frac{\text{side adjacent to } \theta}{\text{hypotenuse}} = \frac{a}{c}$$
$$\tan\theta = \frac{\text{side opposite to } \theta}{\text{side adjacent to } \theta} = \frac{b}{a}$$



Pythagoras' theorem

$$a^2 + b^2 = c^2$$

Standard triangles:



$$\sin 45° = \frac{1}{\sqrt{2}}, \qquad \cos 45° = \frac{1}{\sqrt{2}}, \qquad \tan 45° = 1$$
$$\sin 30° = \frac{1}{2}, \qquad \cos 30° = \frac{\sqrt{3}}{2}, \qquad \tan 30° = \frac{1}{\sqrt{3}}$$
$$\sin 60° = \frac{\sqrt{3}}{2}, \qquad \cos 60° = \frac{1}{2}, \qquad \tan 60° = \sqrt{3}$$

Common trigonometric identities

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

$2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

$2 \cos A \cos B = \cos(A-B) + \cos(A+B)$

$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

$\sin^2 A + \cos^2 A = 1$

$1 + \cot^2 A = \text{cosec}^2 A, \quad \tan^2 A + 1 = \sec^2 A$

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$\sin 2A = 2 \sin A \cos A$

$\sin^2 A = \frac{1 - \cos 2A}{2}, \qquad \cos^2 A = \frac{1 + \cos 2A}{2}$

$\sin^2 A$ is the notation used for $(\sin A)^2$. Similarly $\cos^2 A$ means $(\cos A)^2$ etc. This notation is used with trigonometric and hyperbolic functions but with positive integer powers only.

Hyperbolic functions

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\text{sech}\, x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$
$$\text{cosech}\, x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Hyperbolic identities

$e^x = \cosh x + \sinh x, \quad e^{-x} = \cosh x - \sinh x$

$\cosh^2 x - \sinh^2 x = 1$

$1 - \tanh^2 x = \text{sech}^2 x$

$\coth^2 x - 1 = \text{cosech}^2 x$

$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

$\sinh 2x = 2 \sinh x \cosh x$

$\cosh 2x = \cosh^2 x + \sinh^2 x$

$\cosh^2 x = \frac{\cosh 2x + 1}{2}$

$\sinh^2 x = \frac{\cosh 2x - 1}{2}$

Inverse hyperbolic functions

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}) \quad$ for $x \geqslant 1$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$

$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \quad$ for $-1 < x < 1$

The Greek alphabet

A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

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Vector Calculus

grad $\equiv \nabla$ div $\equiv \nabla \cdot$ curl $\equiv \nabla\times$

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Laplacian $\equiv \nabla^2 \equiv \mathrm{div}(\mathrm{grad}) \equiv \nabla\cdot\nabla \equiv \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

If $\Phi(x,y,z)$ is a scalar field and $\mathbf{v}(x,y,z) = v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}$ is a vector field

$$\mathrm{grad}\,\Phi = \nabla\Phi = \frac{\partial\Phi}{\partial x}\mathbf{i} + \frac{\partial\Phi}{\partial y}\mathbf{j} + \frac{\partial\Phi}{\partial z}\mathbf{k} \qquad \text{a vector.}$$

$$\mathrm{div}\,\mathbf{v} = \nabla\cdot\mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \qquad \text{a scalar.}$$

$$\mathrm{curl}\,\mathbf{v} = \nabla\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix} \qquad \text{a vector.}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}.$$

$$\nabla^2\mathbf{v} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k}).$$

Vector calculus identities:

$\mathrm{grad}(\Phi\psi) = \Phi\,\mathrm{grad}\,\psi + \psi\,\mathrm{grad}\,\Phi$
$\mathrm{div}\,(\Phi\mathbf{a}) = \Phi\,\mathrm{div}\,\mathbf{a} + \mathbf{a}\cdot\mathrm{grad}\,\Phi$
$\mathrm{curl}\,(\Phi\mathbf{a}) = \Phi\,\mathrm{curl}\,\mathbf{a} + \mathrm{grad}\,\Phi\times\mathbf{a}$
$\mathrm{curl}\,\mathrm{grad}\,\Phi = \mathbf{0}, \qquad \mathrm{div}\,\mathrm{curl}\,\mathbf{a} = 0$
$\mathrm{curl}\,\mathrm{curl}\,\mathbf{a} = \mathrm{grad}\,\mathrm{div}\,\mathbf{a} - \nabla^2\mathbf{a}$
$\mathrm{grad}(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{b}\cdot\mathrm{grad})\mathbf{a} + (\mathbf{a}\cdot\mathrm{grad})\,\mathbf{b} + \mathbf{b}\times\mathrm{curl}\,\mathbf{a} + \mathbf{a}\times\mathrm{curl}\,\mathbf{b}$
$\mathrm{div}(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot\mathrm{curl}\,\mathbf{a} - \mathbf{a}\cdot\mathrm{curl}\,\mathbf{b}$
$\mathrm{curl}(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\mathrm{grad})\mathbf{a} - (\mathbf{a}\cdot\mathrm{grad})\mathbf{b} + \mathbf{a}\,\mathrm{div}\,\mathbf{b} - \mathbf{b}\,\mathrm{div}\,\mathbf{a}$

Green's theorem in the plane:

$$\oint_C (P\mathrm{d}x + Q\mathrm{d}y) = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y.$$

Stokes' theorem:

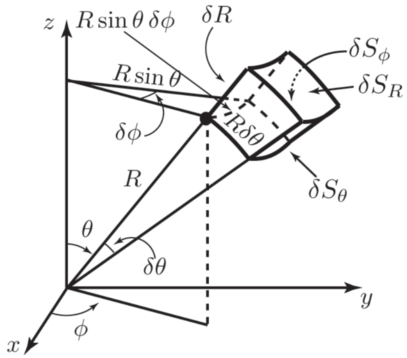
$$\oint_C \mathbf{v}\cdot\mathrm{d}\mathbf{r} = \int_S \mathrm{curl}\,\mathbf{v}\cdot\mathrm{d}\mathbf{S}.$$

The divergence theorem:

$$\oint_S \mathbf{v}\cdot\mathrm{d}\mathbf{S} = \int_V \mathrm{div}\,\mathbf{v}\,\mathrm{d}V.$$

Spherical polar coordinates

The diagram shows spherical polar coordinates (R,θ,ϕ).



$$\left.\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned}\right\} \qquad \begin{aligned} R &\geq 0 \\ 0 \leq &\theta \leq \pi \\ 0 \leq &\phi < 2\pi \end{aligned}$$

If $\mathbf{v} = v_R\hat{\mathbf{e}}_R + v_\theta\hat{\mathbf{e}}_\theta + v_\phi\hat{\mathbf{e}}_\phi$:

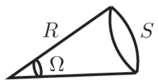
$$\nabla\Phi = \frac{\partial\Phi}{\partial R}\hat{\mathbf{e}}_R + \frac{1}{R}\frac{\partial\Phi}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{1}{R\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi.$$

$$\nabla\cdot\mathbf{v} = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 v_R\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right) + \frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}\left(v_\phi\right).$$

$$\nabla\times\mathbf{v} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_R & R\hat{\mathbf{e}}_\theta & R\sin\theta\,\hat{\mathbf{e}}_\phi \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ v_R & Rv_\theta & R\sin\theta v_\phi \end{vmatrix}.$$

$$\nabla^2\Phi = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial\Phi}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}.$$

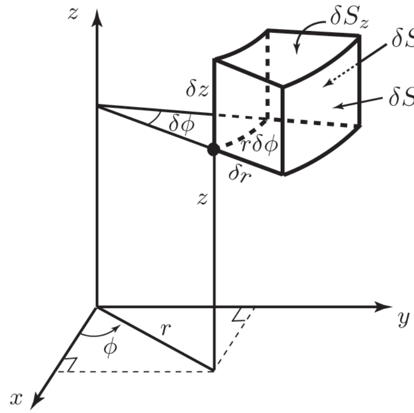
Volume element: $\delta V = R^2\sin\theta\,\delta R\,\delta\theta\,\delta\phi$.
Surface elements:
$\delta S_R = R^2\sin\theta\,\delta\theta\,\delta\phi$,
$\delta S_\theta = R\sin\theta\,\delta R\,\delta\phi$,
$\delta S_\phi = R\,\delta R\,\delta\theta$.



Solid angles: Consider part of a sphere of radius R. If the area cut off on the surface is S, the **solid angle** at the centre is $\Omega = \frac{S}{R^2}$ steradians. The solid angle at the apex of a cone of semi-vertical angle θ is $\Omega = 2\pi(1-\cos\theta)$.

Cylindrical polar coordinates

The diagram shows cylindrical polar coordinates (r,ϕ,z).



$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \\ z &= z \end{aligned}\right\} \qquad \begin{aligned} r &\geq 0 \\ 0 \leq &\phi < 2\pi \\ -\infty < &z < \infty \end{aligned}$$

If $\mathbf{v} = v_r\hat{\mathbf{e}}_r + v_\phi\hat{\mathbf{e}}_\phi + v_z\hat{\mathbf{e}}_z$:

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{\mathbf{e}}_\phi + \frac{\partial\Phi}{\partial z}\hat{\mathbf{e}}_z.$$

$$\nabla\cdot\mathbf{v} = \frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(v_\phi\right) + \frac{\partial v_z}{\partial z}.$$

$$\nabla\times\mathbf{v} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\phi & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ v_r & rv_\phi & v_z \end{vmatrix}.$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}.$$

Volume element: $\delta V = r\,\delta r\,\delta\phi\,\delta z$.
Surface elements:
$\delta S_r = r\,\delta\phi\,\delta z$,
$\delta S_\phi = \delta r\,\delta z$,
$\delta S_z = r\,\delta r\,\delta\phi$.

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FOURIER SERIES: DEFINITION

If $f(t)$ is a periodic function, with period T, and $\omega = \dfrac{2\pi}{T}$, then the *Fourier series* of f is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + b_n \sin n\omega t$$

where

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega t \, \mathrm{d}t$$
$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega t \, \mathrm{d}t$$

for $n = 1, 2, 3, \ldots$

FOURIER SERIES: HALF-RANGE SERIES

If $f(t)$ is a function defined for $t \in [0, L]$, then the *half-range cosine series* of f is

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{L}$$

where

$$a_n = \frac{2}{L} \int_0^L f(t) \cos \frac{n\pi t}{L} \, \mathrm{d}t$$

for $n = 1, 2, 3, \ldots$, and the *half-range sine series* of f is

$$f(t) \sim \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$$

where

$$b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi t}{L} \, \mathrm{d}t$$

for $n = 1, 2, 3, \ldots$

FOURIER TRANSFORMS

Definition

The Fourier transform of a function $f(t)$ is denoted by $\mathcal{F}[f(t)]$ and defined by

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, \mathrm{e}^{-j\omega t}\ \mathrm{d}t$$

where $j = \sqrt{-1}$.

The Fourier transform $\mathcal{F}[f(t)]$ of $f(t)$ is, therefore, a complex function of the variable ω and is commonly denoted by $F(\omega)$.

The inverse Fourier transform is given by

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)\, \mathrm{e}^{j\omega t}\ \mathrm{d}\omega.$$

Magnitude and phase spectra

Given a Fourier transform $F(\omega) = R(\omega)\, \mathrm{e}^{j\phi(\omega)} = X(\omega) + jY(\omega)$ (written in polar and Cartesian form, respectively), the magnitude and phase spectra are defined to be

$$\begin{aligned} |F(\omega)| &= R(\omega) = \sqrt{((X(\omega))^2 + (Y(\omega))^2}, \\ \angle F(\omega) &= \phi(\omega) = \tan^{-1}\left(\frac{Y(\omega)}{X(\omega)}\right), \end{aligned}$$

respectively.

Some Fourier transform theorems

1. Linearity of the Fourier transform:

$$\mathcal{F}[af(t) + bg(t)] = a\mathcal{F}[f(t)] + b\mathcal{F}[g(t)] = aF(\omega) + bG(\omega).$$

2. Symmetry of the Fourier transform: if $\mathcal{F}[f(t)] = F(\omega)$ then

$$\mathcal{F}[F(t)] = 2\pi f(-\omega)$$

3. The Fourier transform of the derivatives of a function:

$$\mathcal{F}\left[\frac{\mathrm{d}^n f}{\mathrm{d}t^n}\right] = (j\omega)^n F(\omega).$$

4. The derivatives of a Fourier transform:

$$\mathcal{F}[t^n f(t)] = j^n \frac{\mathrm{d}^n F}{\mathrm{d}\omega^n}.$$

5. Scaling in the time domain: for real $a \neq 0$

$$\mathcal{F}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

6. Shift in the time domain:

$$\mathcal{F}[f(t - t_0)] = \mathrm{e}^{-j\omega t_0}\, F(\omega).$$

7. Shift in the frequency domain:

$$\mathcal{F}[\mathrm{e}^{j\omega_0 t}\, f(t)] = F(\omega - \omega_0),$$

8. The convolution theorem:

$$\mathcal{F}[(f \star g)(t)] = F(\omega)G(\omega),$$

where $(f \star g)(t) = \displaystyle\int_{-\infty}^{\infty} f(\tau)g(t - \tau)\,\mathrm{d}\tau$ is the convolution of $f(t)$ and $g(t)$.

LAPLACE TRANSFORMS

Definition

The Laplace transform of a function $x(t)$ is denoted by $L[x(t)]$ and defined by

$$L[x(t)] = \int_0^\infty x(t)\,\mathrm{e}^{-st}\ \mathrm{d}t.$$

The Laplace transform $L[x(t)]$ of $x(t)$ is, therefore, a function of the variable s and is commonly denoted by $X(s)$. The inverse Laplace transform is written as $L^{-1}[X(s)] = x(t)$.

Basic table of Laplace transform pairs

$X(s)$	$x(t)$
1	$\delta(t)$
$\dfrac{1}{s}$	$H(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$
$\dfrac{1}{s^{n+1}}$	$\dfrac{t^n}{n!}$ (for $n = 1, 2, 3, \ldots$)
$\dfrac{1}{s+a}$	e^{-at}
$\dfrac{1}{(s+a)^{n+1}}$	$\dfrac{t^n\,\mathrm{e}^{-at}}{n!}$ (for $n = 1, 2, 3, \ldots$)
$\dfrac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\dfrac{\omega}{s^2+\omega^2}$	$\sin(\omega t)$
$\dfrac{s}{s^2-a^2}$	$\cosh(at)$
$\dfrac{a}{s^2-a^2}$	$\sinh(at)$
$\dfrac{s}{(s^2+\omega^2)^2}$	$\dfrac{t}{2\omega}\sin(\omega t)$
$\dfrac{\omega^2}{(s^2+\omega^2)^2}$	$\dfrac{1}{2\omega}\left(\sin(\omega t) - \omega t\cos(\omega t)\right)$

Some Laplace transform theorems

1. Linearity of the Laplace transform:

$$L[ax(t) + by(t)] = aL[x(t)] + bL[y(t)] = aX(s) + bY(s).$$

2. The Laplace transform of the derivatives of a function:

$$\begin{aligned}
L\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] &= sX(s) - x(0),\\
L\left[\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right] &= s^2X(s) - sx(0) - \frac{\mathrm{d}x}{\mathrm{d}t}(0),\\
L\left[\frac{\mathrm{d}^nx}{\mathrm{d}t^n}\right] &= s^nX(s) - s^{n-1}x(0) - s^{n-2}\frac{\mathrm{d}x}{\mathrm{d}t}(0) - s^{n-3}\frac{\mathrm{d}^2x}{\mathrm{d}t^2}(0) - \ldots - \frac{\mathrm{d}^{n-1}x}{\mathrm{d}t^{n-1}}(0).
\end{aligned}$$

3. The first shifting theorem:

$$L[\mathrm{e}^{-at}\,x(t)] = X(s+a),$$

which is commonly used in the form

$$L^{-1}[X(s+a)] = \mathrm{e}^{-at}\,x(t).$$

4. The second shifting (or translation) theorem: for $a > 0$,

$$L[H(t-a)x(t-a)] = \mathrm{e}^{-as}\,X(s).$$

5. The derivative of the Laplace transform:

$$L[tx(t)] = -\frac{\mathrm{d}X}{\mathrm{d}s}.$$

6. The final value theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s).$$

7. The convolution theorem:

$$L[(x \star y)(t)] = X(s)Y(s),$$

where $(x \star y)(t) = \displaystyle\int_0^t x(t-\tau)y(\tau)\,\mathrm{d}\tau$ is the convolution of $x(t)$ and $y(t)$.

Applied Statistics Formulae

Engineering Mathematics 2

Common probability distributions

Binomial distribution: $X \sim \mathrm{Bin}(n, p)$

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad \mathrm{E}[X] = np, \qquad \mathrm{Var}[X] = np(1-p)$$

Geometric distribution: $X \sim \mathrm{Geom}(p)$

$$P(X = x) = (1-p)^{x-1} p, \qquad \mathrm{E}[X] = \frac{1}{p}, \qquad \mathrm{Var}[X] = \frac{1-p}{p^2}$$

Poisson distribution: $X \sim \mathrm{Po}(\lambda)$

$$P(X = x) = \mathrm{e}^{-\lambda} \frac{\lambda^x}{x!}, \qquad \mathrm{E}[X] = \lambda, \qquad \mathrm{Var}[X] = \lambda$$

Exponential distribution: $X \sim \mathrm{Expon}(\theta)$

$$p_X(x) = \theta\, \mathrm{e}^{-\theta x}, \qquad \mathrm{E}[X] = \frac{1}{\theta}, \qquad \mathrm{Var}[X] = \frac{1}{\theta^2}$$

Normal distribution: $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$

$$p_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \qquad \mathrm{E}[X] = \mu, \qquad \mathrm{Var}[X] = \sigma^2$$

Sample statistics

Sample mean

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

Sample variance (unbiased estimate of population variance)

$$s^2 = \frac{1}{n-1}\sum_{i=1}^{n} (X_i - \bar{x})^2$$

Pearson's correlation coefficient

$$r = \frac{\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2}\sqrt{\sum_{i=1}^{n} (y_i - \bar{y})^2}}$$

Test statistics

Sample mean $\bar{x}$ of n samples where $X_i \sim N(\mu, \sigma^2)$

$$\frac{\bar{x}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

When the population variance is unknown, the (Bessel corrected) sample variance s^2 may be used instead

$$\frac{\bar{x}-\mu}{s/\sqrt{n}} \sim T_{n-1}$$

Difference of two sample means ($n = \min(n_1, n_2)$)

$$\frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} \sim T_{n-1}$$

Sample variance (Bessel corrected)

$$(n-1)\frac{s^2}{\sigma^2} \sim \chi^2_{n-1}$$

Pearson's χ^2 test where O_i are observed outcomes, E_i are expected outcomes, and p is the number of parameters estimated (e.g., sample mean, sample variance) from the data

$$\sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i} \sim \chi^2_{n-1-p}$$

The Fisher transformation for Pearson's correlation coefficient r, where ρ is the true or hypothesised correlation coefficient and n is the number of samples

$$\frac{1}{2}\ln\left(\frac{1+r}{1-r}\right) \sim N\left(\frac{1}{2}\ln\left(\frac{1+\rho}{1-\rho}\right), \frac{1}{n-3}\right)$$

Linear regression

For the linear model

$$y = X\beta$$

where y is an $n \times 1$ matrix of outputs, X is an $n \times p$ matrix of inputs, and β is a $p \times 1$ matrix of parameters, the least-squares best fit with n samples is given by

$$\hat{\beta} = (X^T X)^{-1} X^T y$$

The covariance of the estimate for β is given by

$$\text{Covar}\left(\hat{\beta}\right) = (X^T X)^{-1}\sigma^2$$

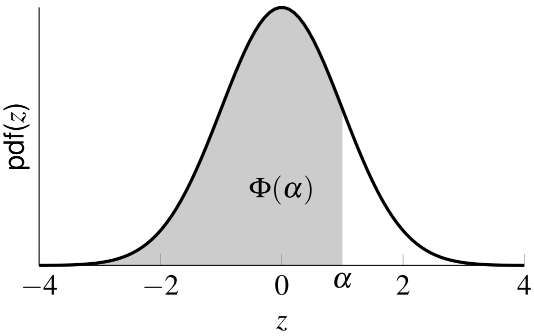
where σ^2 is the variance of the noise. An estimate for σ^2 is provided by the residuals $e = y - X\beta$

$$\hat{\sigma}^2 = \frac{1}{n-p}\|y - X\beta\|^2 = \frac{1}{n-p}\|e\|^2$$

Cumulative standard normal distribution

$\Phi(\alpha)$ is the integral of the standard normal distribution between $-\infty$ and α.

$$P(z \leq \alpha) = \Phi(\alpha) = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-z^2/2} \, \mathrm{d}z$$



Critical values

α	$\Phi(\alpha)$	One-tailed	Two-tailed
1.645	0.9500	5%	10%
1.960	0.9750	2.5%	5%
2.326	0.9900	1%	2%
2.576	0.9950	0.5%	1%

Values of $\Phi(\alpha)$

α	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998
3.5	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998	0.9998
3.6	0.9998	0.9998	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999

Critical values of Student's t distribution

Degrees of freedom		Significance level					
	Two-tailed:	10%	5%	2%	1%	0.2%	0.1%
	One-tailed:	5%	2.5%	1%	0.5%	0.1%	0.05%
1		6.314	12.706	31.821	63.657	318.309	636.619
2		2.920	4.303	6.965	9.925	22.327	31.599
3		2.353	3.182	4.541	5.841	10.215	12.924
4		2.132	2.776	3.747	4.604	7.173	8.610
5		2.015	2.571	3.365	4.032	5.893	6.869
6		1.943	2.447	3.143	3.707	5.208	5.959
7		1.895	2.365	2.998	3.499	4.785	5.408
8		1.860	2.306	2.896	3.355	4.501	5.041
9		1.833	2.262	2.821	3.250	4.297	4.781
10		1.812	2.228	2.764	3.169	4.144	4.587
11		1.796	2.201	2.718	3.106	4.025	4.437
12		1.782	2.179	2.681	3.055	3.930	4.318
13		1.771	2.160	2.650	3.012	3.852	4.221
14		1.761	2.145	2.624	2.977	3.787	4.140
15		1.753	2.131	2.602	2.947	3.733	4.073
16		1.746	2.120	2.583	2.921	3.686	4.015
17		1.740	2.110	2.567	2.898	3.646	3.965
18		1.734	2.101	2.552	2.878	3.610	3.922
19		1.729	2.093	2.539	2.861	3.579	3.883
20		1.725	2.086	2.528	2.845	3.552	3.850
21		1.721	2.080	2.518	2.831	3.527	3.819
22		1.717	2.074	2.508	2.819	3.505	3.792
23		1.714	2.069	2.500	2.807	3.485	3.768
24		1.711	2.064	2.492	2.797	3.467	3.745
25		1.708	2.060	2.485	2.787	3.450	3.725
26		1.706	2.056	2.479	2.779	3.435	3.707
27		1.703	2.052	2.473	2.771	3.421	3.690
28		1.701	2.048	2.467	2.763	3.408	3.674
29		1.699	2.045	2.462	2.756	3.396	3.659
30		1.697	2.042	2.457	2.750	3.385	3.646
40		1.684	2.021	2.423	2.704	3.307	3.551
50		1.676	2.009	2.403	2.678	3.261	3.496
60		1.671	2.000	2.390	2.660	3.232	3.460
70		1.667	1.994	2.381	2.648	3.211	3.435
80		1.664	1.990	2.374	2.639	3.195	3.416
90		1.662	1.987	2.368	2.632	3.183	3.402
100		1.660	1.984	2.364	2.626	3.174	3.390
150		1.655	1.976	2.351	2.609	3.145	3.357
200		1.653	1.972	2.345	2.601	3.131	3.340
300		1.650	1.968	2.339	2.592	3.118	3.323
400		1.649	1.966	2.336	2.588	3.111	3.315
500		1.648	1.965	2.334	2.586	3.107	3.310
600		1.647	1.964	2.333	2.584	3.104	3.307
∞		1.645	1.960	2.326	2.576	3.090	3.291

Critical values of the chi-squared distribution

Degrees of freedom	Significance level (one-tailed) 5%	2.5%	1%	0.5%	0.1%	0.05%
1	3.841	5.024	6.635	7.879	10.828	12.116
2	5.991	7.378	9.210	10.597	13.816	15.202
3	7.815	9.348	11.345	12.838	16.266	17.730
4	9.488	11.143	13.277	14.860	18.467	19.997
5	11.070	12.833	15.086	16.750	20.515	22.105
6	12.592	14.449	16.812	18.548	22.458	24.103
7	14.067	16.013	18.475	20.278	24.322	26.018
8	15.507	17.535	20.090	21.955	26.124	27.868
9	16.919	19.023	21.666	23.589	27.877	29.666
10	18.307	20.483	23.209	25.188	29.588	31.420
11	19.675	21.920	24.725	26.757	31.264	33.137
12	21.026	23.337	26.217	28.300	32.909	34.821
13	22.362	24.736	27.688	29.819	34.528	36.478
14	23.685	26.119	29.141	31.319	36.123	38.109
15	24.996	27.488	30.578	32.801	37.697	39.719
16	26.296	28.845	32.000	34.267	39.252	41.308
17	27.587	30.191	33.409	35.718	40.790	42.879
18	28.869	31.526	34.805	37.156	42.312	44.434
19	30.144	32.852	36.191	38.582	43.820	45.973
20	31.410	34.170	37.566	39.997	45.315	47.498
21	32.671	35.479	38.932	41.401	46.797	49.011
22	33.924	36.781	40.289	42.796	48.268	50.511
23	35.172	38.076	41.638	44.181	49.728	52.000
24	36.415	39.364	42.980	45.559	51.179	53.479
25	37.652	40.646	44.314	46.928	52.620	54.947
26	38.885	41.923	45.642	48.290	54.052	56.407
27	40.113	43.195	46.963	49.645	55.476	57.858
28	41.337	44.461	48.278	50.993	56.892	59.300
29	42.557	45.722	49.588	52.336	58.301	60.735
30	43.773	46.979	50.892	53.672	59.703	62.162
40	55.758	59.342	63.691	66.766	73.402	76.095
50	67.505	71.420	76.154	79.490	86.661	89.561
60	79.082	83.298	88.379	91.952	99.607	102.695
70	90.531	95.023	100.425	104.215	112.317	115.578
80	101.879	106.629	112.329	116.321	124.839	128.261
90	113.145	118.136	124.116	128.299	137.208	140.782
100	124.342	129.561	135.807	140.169	149.449	153.167