I. THE ORIGINAL HEOM

A. The correlation function

$$\alpha_R(t) = \frac{\hbar}{\pi}\int_0^\infty d\omega J(\omega)\coth\left(\frac{\hbar\omega\beta}{2}\right)\cos\omega t$$
$$= \sum_{k=0}^{\infty} d_k e^{-\nu_k t} \tag{1}$$

Under the Debye spectral density:

$$J(\omega) = \frac{\eta\gamma\omega}{\gamma^2+\omega^2} \quad , \tag{2}$$

the expansion coefficients are

$$d_0 = \frac{\eta\hbar\gamma}{2}\cot\left(\frac{\beta\hbar\gamma}{2}\right) \tag{3}$$

and

$$d_k = \frac{4k\pi\eta\hbar\gamma}{(2k\pi)^2-(\beta\hbar\gamma)^2}$$
$$= \frac{2\nu_k\eta\hbar\gamma\hbar\beta}{(\nu_k\beta\hbar)^2-(\beta\hbar\omega)^2} \quad ; \tag{4}$$

with

$$\nu_0 = \gamma \quad , \tag{5}$$

and

$$\nu_k = \frac{2k\pi}{\hbar\beta} \quad . \tag{6}$$

B. The influence functional

$$\mathcal{F} = \exp\left(-\frac{1}{\hbar^2}\int_0^t ds\int_0^s du V^\times(s)[\alpha_R(s-u)V^\times(u) + i\alpha_I(s-u)V^\circ(u)]\right)$$
$$= \exp\left(-\frac{1}{\hbar^2}\int_0^\infty d\omega J(\omega)\int_0^t ds\int_0^s du V^\times\Big[\big(\coth(\frac{\beta\hbar\omega}{2})\cos\omega(s-u)\big)V^\times - i\sin(s-u)V^\circ(u)\Big]\right) \tag{7}$$

Here, $V^{\times}(t) = \phi^{+}(t) - \phi^{-}(t)$ and $V_{\circ}(t) = \phi^{+}(t) + \phi^{-}(t)$.

$$
\begin{aligned}
\mathcal{F} = \exp\Bigg(&-\int_0^t ds \int_0^s du V^{\times}(s)\Big[d_0 e^{-\gamma(s-u)} V^{\times}(u) - i\frac{\hbar\eta\gamma}{2} e^{-\gamma(s-u)} V^{\circ}(u)\Big]\Bigg) \\
&\times \prod_{k=1}^{K} \exp\Bigg(-\int_0^t ds \int_0^s du V^{\times}(s) d_k e^{-\nu_k(s-u)} V^{\times}(u)\Bigg) \\
&\times \prod_{k=K+1}^{\infty} \exp\Bigg(-\int_0^t ds V^{\times}(s) \frac{2\eta\hbar\gamma\hbar\beta}{(\beta\hbar\nu_k)^2 - (\beta\hbar\gamma)^2} V^{\times}(s)\Bigg)
\end{aligned}
\tag{8}
$$

Build the auxiliary density operators as follows:

$$
\begin{aligned}
\rho^{n}_{j_1,\ldots,j_K}(q^+, q^-, t) &= \int \mathcal{D}q^+(t) \int \mathcal{D}q^-(t) e^{iS[q^+(t)] - iS[q^-(t)]} \mathcal{F}[q^+, q^-] \\
&\quad \times \Bigg(-\int_0^t du \Big[d_0 e^{-\gamma(t-u)} V^{\times}(u) - i\frac{\hbar\eta\gamma}{2} e^{-\gamma(t-u)} V^{\circ}(u)\Big]\Bigg)^{n} \\
&\quad \times \prod_{k=1}^{K} \Bigg(-\int_0^t du \Big[d_k\Big] e^{-\nu_k(t-u)} V^{\times}(u)\Bigg)^{j_k} \rho(q^+, q^-, t) \\
&= \int \mathcal{D}q^+(t) \int \mathcal{D}q^-(t) e^{iS[q^+(t)] - iS[q^-(t)]} \mathcal{F}[q^+, q^-] \\
&\quad \times \Bigg(-\int_0^t du \Big[d_0 e^{-\gamma(t-u)} V^{\times}(u) - i\frac{\hbar\eta\gamma}{2} e^{-\gamma(t-u)} V^{\circ}(u)\Big]\Bigg)^{n} \\
&\quad \times \prod_{k=1}^{K} \Bigg(-\int_0^t du \Big[\frac{2\eta\hbar\gamma\hbar\beta\nu_k}{(\beta\hbar\nu_k)^2 - (\beta\hbar\gamma)^2}\Big] e^{-\nu_k(t-u)} V^{\times}(u)\Bigg)^{j_k} \rho(q^+, q^-, t)
\end{aligned}
\tag{9}
$$

Differentiating it with respect to t, we obtain the following HEOM:

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho^{n}_{j_1,\ldots,j_K}(q^+, q^-, t) = &-\Bigg[i\mathcal{L} + n\gamma + \sum_{k=1}^{K} j_k\nu_k + \sum_{k=K+1}^{\infty} V^{\times}(t)\Big[\frac{2\eta\hbar\gamma\hbar\beta\nu_k}{(\beta\hbar\nu_k)^2 - (\beta\hbar\gamma)^2}\Big] V^{\times}(t)\Bigg] \rho^{n}_{j_1,\ldots,j_K} \\
&+ V^{\times}(t)\rho^{n+1}_{j_1,\ldots,j_K} + \sum_{k=1}^{K} V^{\times}(t)\rho^{n}_{j_1,\ldots j_k+1,\ldots,j_K} \\
&+ \sum_{k=1}^{K} \Big[-\frac{2\eta\hbar\gamma\hbar\beta\nu_k}{(\beta\hbar\nu_k)^2 - (\beta\hbar\gamma)^2}\Big] V^{\times}(t)\rho^{n}_{j_1,\ldots j_k-1,\ldots,j_K} \\
&- n\Big[d_0 V^{\times}(t) - i\frac{\hbar\gamma\eta}{2} V^{\circ}(t)\Big]\rho^{n-1}_{j_1,\ldots,j_K}
\end{aligned}
\tag{10}
$$

II. SHI'S FORM

A. Before the scaling

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho^{n}_{j_1,\dots,j_K} &= -\Big(i\mathcal{L}+\sum_{k=0}^{K} n_k\nu_k\Big)\rho_{n} - i\sum_{k=0}^{K}[Q,\rho_{n_k^+}] - i\sum_{k=0}^{K} n_k(c_kQ\rho_{n_k^-} - c_k^*\rho_{n_k^-}Q)\\
&= -\Big(i\mathcal{L}+n\gamma+\sum_{k=1}^{K} n_k\nu_k\Big)\rho^{n}_{j_1,\dots,j_K} - i[Q,\rho^{n+1}_{j_1,\dots,j_K}] - i\sum_{k=1}^{K}[Q,\rho^{n}_{j_1,\dots j_k+1\dots,j_K}]\\
&- in_0(c_0Q\rho^{n-1}_{j_1,\dots,j_K} - c_0^*\rho^{n-1}_{j_1,\dots,j_K}Q) - i\sum_{k=1}^{K} n_kc_k(Q\rho^{n}_{j_1,\dots j_k-1,\dots,j_K} - \rho^{n}_{j_1,\dots j_k-1,\dots,j_K}Q)\\
&= -\Big(i\mathcal{L}+n\gamma+\sum_{k=1}^{K} n_k\nu_k\Big)\rho^{n}_{j_1,\dots,j_K} - i[Q,\rho^{n+1}_{j_1,\dots,j_K}] - i\sum_{k=1}^{K}[Q,\rho^{n}_{j_1,\dots j_k+1\dots,j_K}]\\
&- in_0d_0(Q\rho^{n-1}_{j_1,\dots,j_K} - \rho^{n-1}_{j_1,\dots,j_K}Q) - n_0(\frac{\hbar\eta\gamma}{2}Q\rho^{n-1}_{j_1,\dots,j_K} + \frac{\hbar\eta\gamma}{2}\rho^{n-1}_{j_1,\dots,j_K}Q)\\
&- i\sum_{k=1}^{K} n_kc_k(Q\rho^{n}_{j_1,\dots j_k-1,\dots,j_K} - \rho^{n}_{j_1,\dots j_k-1,\dots,j_K}Q)
\end{aligned}
\tag{11}
$$

Please keep in mind the corresponding notations, c_ks are the expansion coefficients of

$$C(t>0) = \sum_{k=0}^{K} c_k e^{-\gamma_k t} \tag{12}$$

B. Add the scaling

$$\tilde{\rho}_n(t) = \big(\prod_k n_k!|c_k|^{n_k}\big)^{-1/2}\rho_n(t) \tag{13}$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho^{n}_{j_1,\ldots,j_K} &= -\Big(i\mathcal{L}+\sum_{k=0}^{K} n_k\nu_k\Big)\rho_n - i\sum_{k=0}^{K}\sqrt{(n_k+1)|c_k|)}[Q,\rho_{n_k^+}] - i\sum_{k=0}^{K}\sqrt{n_k/|c_k|}(c_kQ\rho_{n_k^-}-c_k^*\rho_{n_k^-}Q) \\
&= -\Big(i\mathcal{L}+n\gamma+\sum_{k=1}^{K} n_k\nu_k\Big)\rho^{n}_{j_1,\ldots,j_K} - i\sqrt{(n_0+1)|c_0|}[Q,\rho^{n+1}_{j_1,\ldots,j_K}] \\
&- i\sum_{k=1}^{K}\sqrt{(n_k+1)|c_k|)}[Q,\rho^{n}_{j_1,\ldots j_k+1\ldots,j_K}] - i\sqrt{n_0/|c_0|}(c_0Q\rho^{n-1}_{j_1,\ldots,j_K}-c_0^*\rho^{n-1}_{j_1,\ldots,j_K}Q) \\
&- i\sum_{k=1}^{K}\sqrt{n_k/|c_k|}c_k(Q\rho^{n}_{j_1,\ldots j_k-1\ldots,j_K}-\rho^{n}_{j_1,\ldots j_k-1\ldots,j_K}Q) \\
&= -\Big(i\mathcal{L}+n\gamma+\sum_{k=1}^{K} n_k\nu_k\Big)\rho^{n}_{j_1,\ldots,j_K} - i\sqrt{(n_0+1)|c_0|}[Q,\rho^{n+1}_{j_1,\ldots,j_K}] \\
&- i\sum_{k=1}^{K}\sqrt{(n_k+1)|c_k|)}[Q,\rho^{n}_{j_1,\ldots j_k+1\ldots,j_K}] \\
&- i\sqrt{n_0/|c_0|}d_0(Q\rho^{n-1}_{j_1,\ldots,j_K}-\rho^{n-1}_{j_1,\ldots,j_K}Q) + i\sqrt{n_0/|c_0|}(\frac{i\hbar\eta\gamma}{2}Q\rho^{n-1}_{j_1,\ldots,j_K}+\frac{i\hbar\eta\gamma}{2}\rho^{n-1}_{j_1,\ldots,j_K}Q) \\
&- i\sum_{k=1}^{K}\sqrt{n_k/|c_k|}c_k(Q\rho^{n}_{j_1,\ldots j_k-1,\ldots,j_K}-\rho^{n}_{j_1,\ldots j_k-1,\ldots,j_K}Q) \\
&= -\Big(i\mathcal{L}+n\gamma+\sum_{k=1}^{K} n_k\nu_k\Big)\rho^{n}_{j_1,\ldots,j_K} - i\sqrt{(n_0+1)|c_0|}[Q,\rho^{n+1}_{j_1,\ldots,j_K}] \\
&- i\sqrt{n_0/|c_0|}d_0(Q\rho^{n-1}_{j_1,\ldots,j_K}-\rho^{n-1}_{j_1,\ldots,j_K}Q) - \sqrt{n_0/|c_0|}(\frac{\hbar\eta\gamma}{2}Q\rho^{n-1}_{j_1,\ldots,j_K}+\frac{\hbar\eta\gamma}{2}\rho^{n-1}_{j_1,\ldots,j_K}Q) \\
&- i\sum_{k=1}^{K}\sqrt{(n_k+1)|c_k|}[Q,\rho^{n}_{j_1,\ldots j_k+1,\ldots,j_K}] \\
&- i\sum_{k=1}^{K}\sqrt{n_k|c_k|}(Q\rho^{n}_{j_1,\ldots j_k-1,\ldots,j_K}-\rho^{n}_{j_1,\ldots j_k-1,\ldots,j_K}Q)
\end{aligned}
\tag{14}
$$

where in the last step, we've inserted

$$
c_0 = d_0 - \frac{i\eta\gamma}{2} \quad . \tag{15}
$$

Further, we complete it with the **dephasing term**:

$$
\begin{aligned}
\frac{\partial}{\partial t}\rho^{n}_{j_1,\ldots,j_K} = &-\Big(i\mathcal{L} + n\gamma + \sum_{k=1}^{K} n_k\nu_k\Big)\rho^{n}_{j_1,\ldots,j_K} - i\sqrt{(n_0+1)|c_0|}[Q,\rho^{n+1}_{j_1,\ldots,j_K}] \\
&- i\sqrt{n_0/|c_0|}d_0(Q\rho^{n-1}_{j_1,\ldots,j_K} - \rho^{n-1}_{j_1,\ldots,j_K}Q) - \sqrt{n_0/|c_0|}(\frac{\hbar\eta\gamma}{2}Q\rho^{n-1}_{j_1,\ldots,j_K} + \frac{\hbar\eta\gamma}{2}\rho^{n-1}_{j_1,\ldots,j_K}Q) \\
&- i\sum_{k=1}^{K}\sqrt{(n_k+1)|c_k|}[Q,\rho^{n}_{j_1,\ldots j_k+1\ldots,j_K}] \\
&- i\sum_{k=1}^{K}\sqrt{n_k|c_k|}(Q\rho^{n}_{j_1,\ldots j_k-1\ldots,j_K} - \rho^{n}_{j_1,\ldots j_k-1\ldots,j_K}Q)
\end{aligned}
$$

$>>>>>>$

$$
\begin{aligned}
&= -\Big(i\mathcal{L} + n\gamma + \sum_{k=1}^{K} n_k\nu_k\Big)\rho^{n}_{j_1,\ldots,j_K} \underbrace{-\Big(\frac{\eta k_B T - c_0}{\gamma} - \sum_{k=1}^{K}\frac{c_k}{\nu_k}\Big)[Q,[Q,\rho^{n}_{j_1,\ldots,j_K}]]}_{\text{dephasing term}} \\
&- i\sqrt{(n_0+1)|c_0|}[Q,\rho^{n+1}_{j_1,\ldots,j_K}] \\
&- i\sqrt{n_0/|c_0|}d_0(Q\rho^{n-1}_{j_1,\ldots,j_K} - \rho^{n-1}_{j_1,\ldots,j_K}Q) - \sqrt{n_0/|c_0|}(\frac{\hbar\eta\gamma}{2}Q\rho^{n-1}_{j_1,\ldots,j_K} + \frac{\hbar\eta\gamma}{2}\rho^{n-1}_{j_1,\ldots,j_K}Q) \\
&- i\sum_{k=1}^{K}\sqrt{(n_k+1)|c_k|}[Q,\rho^{n}_{j_1,\ldots j_k+1\ldots,j_K}] \\
&- i\sum_{k=1}^{K}\sqrt{n_k|c_k|}(Q\rho^{n}_{j_1,\ldots j_k-1\ldots,j_K} - \rho^{n}_{j_1,\ldots j_k-1\ldots,j_K}Q)
\end{aligned}
\tag{16}
$$

(16)

The final form is in accordance with the Fortran code.

III. FIRST DERIVATION: A LITTLE TEDIOUS

$$\tilde{\alpha}_R(t-s) = \alpha_R(t-s) - \alpha_1(t-s)$$

$$\begin{aligned}
&= \frac{\hbar}{\pi}\int_0^\infty d\omega J(\omega)\tanh\left(\frac{\beta\hbar\omega}{4}\right)\cos\omega(t-s) \\
&= \frac{\hbar}{\pi}\int_0^\infty d\omega \frac{\eta\gamma\omega}{\omega^2+\gamma^2}\tanh\left(\frac{\beta\hbar\omega}{4}\right)\cos\omega t \\
&= \frac{\hbar}{\pi}\frac{1}{2}\mathrm{Re}\left[\int_{-\infty}^\infty d\omega \frac{\eta\gamma\omega}{\omega^2+\gamma^2}\frac{e^{\beta\hbar\omega/2}-1}{e^{\beta\hbar\omega/2}+1}e^{i\omega t}\right] \qquad (17)
\end{aligned}$$

The singular points are $\pm i\gamma$ and $i2(2k-1)\pi/\beta\hbar$ $(k\geq 1)$, so the value is:

$$\begin{aligned}
\tilde{\alpha}_R(t-s) &= \alpha_R(t-s) - \alpha_1(t-s) \\
&= \frac{\hbar}{\pi}\frac{1}{2}\mathrm{Re}\left[2\pi i\sum_j \mathrm{Res(j)}\right] \\
&= \frac{\hbar}{\pi}\mathrm{Re}\left[\pi i\left[\frac{\eta\gamma}{2}\frac{e^{i\beta\hbar\gamma/2}-1}{e^{i\beta\hbar\gamma/2}+1}e^{-\gamma t} + \sum_{k=1}^\infty \frac{\eta\gamma\frac{i2(2k-1)\pi}{\beta\hbar}}{\gamma^2+\left(\frac{i2(2k-1)\pi}{\beta\hbar}\right)^2}\frac{2}{\beta\hbar/2}e^{-\frac{2(2k-1)\pi}{\beta\hbar}t}\right]\right] \\
&= \frac{\hbar}{\pi}\mathrm{Re}\left[\pi i\left[\frac{\eta\gamma}{2}i\tan\left(\frac{\beta\hbar\gamma}{4}\right)e^{-\gamma t} + \sum_{k=1}^\infty \frac{\eta\gamma\beta\hbar\nu_k}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}4e^{-\nu_k t}\right]\right] \\
&= \frac{\hbar}{\pi}\mathrm{Re}\left[\pi i\left[\frac{\eta\gamma}{2}i\tan\left(\frac{\beta\hbar\gamma}{4}\right)e^{-\gamma t} + \sum_{k=1}^\infty \frac{\eta\gamma i2(2k-1)\pi}{(\beta\hbar\gamma)^2+(i2(2k-1)\pi)^2}4e^{-\nu_k t}\right]\right] \\
&= \frac{\hbar}{\pi}\left[-\frac{\pi\eta\gamma}{2}\tan\left(\frac{\beta\hbar\gamma}{4}\right)e^{-\gamma t} - \sum_{k=1}^\infty \frac{4\pi\eta\gamma\beta\hbar\nu_k}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}e^{-\nu_k t}\right] \\
&= \sum_{k=0}^\infty d_k e^{-\nu_k(t-s)} \quad , \qquad (18)
\end{aligned}$$

with

$$\nu_0 = \gamma \quad , \qquad (19)$$

$$\nu_k = 2(2k-1)\pi/\beta\hbar \quad , \qquad (20)$$

and

$$d_0 = -\hbar\frac{\eta\gamma}{2}\tan\left(\frac{\beta\hbar\gamma}{4}\right) \quad , \qquad (21)$$

$$
\begin{aligned}
d_k &= -\hbar\frac{4\eta\gamma\beta\hbar\nu_k}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2} && (22)\\
&= -\hbar\frac{8\eta\gamma(2k-1)\pi}{(\beta\hbar\gamma)^2+(i2(2k-1)\pi)^2} \quad . && (23)
\end{aligned}
$$

$$
\begin{aligned}
i\alpha_I(t) &= i\frac{\hbar}{\pi}\int_0^\infty d\omega J(\omega)\sin\omega t\\
&= i\frac{\hbar\eta\gamma}{2}e^{-\gamma t} && (24)
\end{aligned}
$$

The reduced density matrix element in the path integral with the factorized initial condition:

$$
\rho(t)=\int\mathcal{D}\phi^+(t)\int\mathcal{D}\phi^-(t)e^{\frac{i}{\hbar}S[\phi^+(\tau)]}\mathcal{F}e^{-\frac{i}{\hbar}S[\phi^-(\tau)]}\rho(0) \tag{25}
$$

where $\mathcal{F}$ is the Feynman-Vernon influence functional:

$$
\begin{aligned}
\mathcal{F} &= \exp\left(-\frac{1}{\hbar^2}\int_0^t ds\int_0^s duV^\times[\alpha_R(s-u)V^\times - i\alpha_I(s-u)V^\circ(u)]\right)\\
&= \exp\left(-\frac{1}{\hbar^2}\int_0^\infty d\omega\frac{1}{\pi}J(\omega)\int_0^t ds\int_0^s duV^\times\Big[\big(\coth(\frac{\beta\hbar\omega}{2})\cos\omega(s-u)\big)V^\times - i\sin(s-u)V^\circ(u)\Big]\right)
\end{aligned} \tag{26}
$$

After the unraveling, the equation of motion now reads as

$$
\frac{\partial\Psi(t)}{\partial t}=\frac{\partial}{\partial t}\Big[\mathcal{G}(t,0)\Psi(0)\Big] \tag{27}
$$

where the propagator $\mathcal{G}$ is:

$$
\begin{aligned}
\mathcal{G} &= \int\mathcal{D}z\int\mathcal{D}q^+(t)\exp\left(iS[q^+(t)]+i\int_0^t dsq(s)z(s)\right)\tilde{\mathcal{F}}^+_{res}[q^+]\\
&= \int\mathcal{D}z\int\mathcal{D}q^+(t)\exp\left(iS[q^+(t)]+i\int_0^t dsq(s)z(s)-\int_0^t ds\int_0^s duq^+(s)\tilde{\alpha}(s-u)q^+(u)\right)
\end{aligned} \tag{28}
$$

Note:$\tilde{\mathcal{F}}^+_{res}[q^+]=\exp(-\int_0^t ds\int_0^s duq^+(s)\tilde{\alpha}(s-u)q^+(u))$

Now, insert $\tilde{\alpha}$

$$
\tilde{\alpha}=\tilde{\alpha}_R+i\alpha_I \tag{29}
$$

Under Debye spectral density:

$$
\begin{aligned}
i\alpha_I(t) &= i\frac{\hbar}{\pi}\int_0^\infty d\omega J(\omega)\sin\omega t\\
&= i\frac{\hbar\eta\gamma}{2}e^{-\gamma t} && (30)
\end{aligned}
$$

Put together

$$
\begin{aligned}
\tilde{\alpha}(t) &= \sum_{k=0}^{\infty} d_k e^{-\nu_k t} - i\frac{\hbar\eta\gamma}{2} e^{-\gamma t} \\
&= d_0 e^{-\gamma t} - i\frac{\hbar\eta\gamma}{2} e^{-\gamma t} + \sum_{k=1}^{\infty} d_k e^{-\nu_k t} \\
&= (d_0 - i\frac{\hbar\eta\gamma}{2}) e^{-\gamma t} + \sum_{k=1}^{\infty} d_k e^{-\nu_k t} \qquad (31)
\end{aligned}
$$

Now

$$
\begin{aligned}
\mathcal{G} &= \int \mathcal{D}z \int \mathcal{D}q^+(t) \exp\left(iS[q^+(t)] + i\int_0^t dsq^+(s)z(s) \right) \tilde{\mathcal{F}}^+_{res}[q^+] \\
&= \int \mathcal{D}z \int \mathcal{D}q^+(t) \exp\left(iS[q^+(t)] + i\int_0^t dsq(s)z(s) - \int_0^t ds \int_0^s duq^+(s)\tilde{\alpha}(s-u)q^+(u) \right) \\
&= \int \mathcal{D}z \int \mathcal{D}q^+ \exp\Bigg(iS[q^+(t)] + i\int_0^t dsq(s)z(s) \\
&\quad - \int_0^t ds \int_0^s duq^+(s)\Big[(d_0 - i\frac{\hbar\eta\gamma}{2}) e^{-\gamma t} + \sum_{k=1}^{\infty} d_k e^{-\nu_k t} \Big] q^+(u)
\end{aligned}
\qquad (32)
$$

where we've inserted $\tilde{\mathcal{F}}^+_{res}[q^+]$

$$
\tilde{\mathcal{F}}^+_{res}[q^+] = \exp\left(-\int_0^t ds \int_0^s duq^+(s)\Big[(d_0 - i\frac{\hbar\eta\gamma}{2}) e^{-\gamma t} + \sum_{k=1}^{\infty} d_k e^{-\nu_k t} \Big] q^+(u) \right) \qquad (33)
$$

Now introduce the K-space truncation

$$
\nu_k e^{-\nu_k (s-u)} \simeq \delta(s-u) \qquad (34)
$$

$$
\begin{aligned}
\mathcal{F} = \exp\Big(&-\int_0^t ds \int_0^s du q^+(s)(d_0 - i\frac{\hbar\eta\gamma}{2})e^{-\gamma(s-u)}q^+(u)\Big) \\
&\times \prod_{k=1}^{K} \exp\Big(-\int_0^t ds \int_0^s du q^+(s) d_k e^{-\nu_k(s-u)} q^+(u)\Big) \\
&\times \prod_{k=K+1}^{\infty} \exp\Big(\int_0^t ds q^+(s) \frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2 - (\beta\hbar\nu_k)^2} q^+(s)\Big) \\
= \exp\Big(&-\int_0^t ds \int_0^s du q^+(s)(d_0 - i\frac{\hbar\eta\gamma}{2})e^{-\gamma(s-u)}q^+(u)\Big) \\
&\times \prod_{k=1}^{K} \exp\Big(-\int_0^t ds \int_0^s du q^+(s)[-\frac{4\eta\gamma\beta\hbar\nu_k}{(\beta\hbar\gamma)^2 - (\beta\hbar\nu_k)^2}]e^{-\nu_k(s-u)} q^+(u)\Big) \\
&\times \prod_{k=K+1}^{\infty} \exp\Big(\int_0^t ds q^+(s) \frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2 - (\beta\hbar\nu_k)^2} q^+(s)\Big)
\end{aligned}
\tag{35}
$$

Now introduce the auxiliary wavefunction $\Psi^n_{j_1,\ldots,j_K}$

$$
\begin{aligned}
\Psi^n_{j_1,\ldots,j_K}(q^+,t) = \int \mathcal{D}z \int &\mathcal{D}q^+(t) e^{iS[q^+(t)] + i\int_0^t ds q^+(s)z(s)} \tilde{\mathcal{F}}^+_{res}[q^+] \\
&\times \Big(-\int_0^t du (d_0 - i\frac{\hbar\eta\gamma}{2})e^{-\gamma(t-u)}q^+(u)\Big)^n \\
&\times \prod_{k=1}^{K} \Big(-\int_0^t du[-\frac{4\eta\gamma\beta\hbar\nu_k}{(\beta\hbar\gamma)^2 - (\beta\hbar\nu_k)^2}]e^{-\nu_k(t-u)} q^+(u)\Big)^{j_k} \Psi(q^+,t)
\end{aligned}
\tag{36}
$$

Then we obtain the following hierarchy of equations

$$
\begin{aligned}
\frac{\partial}{\partial t}\Psi^n_{j_1,\ldots,j_K} = &-\Bigg[iH - iq^+(t)z(t) + n\gamma + \sum_{k=1}^{K} j_k\nu_k - \sum_{k=K+1}^{\infty}\Big[\frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2 - (\beta\hbar\nu_k)^2}\Big] q^+(t)q^+(t)\Bigg]\Psi^n_{j_1,\ldots,j_K} \\
&+ q^+(t)\Psi^{n+1}_{j_1,\ldots,j_K} + \sum_{k=1}^{K} q^+(t)\Psi^n_{j_1,\ldots j_k+1,\ldots,j_K} \\
&+ \sum_{k=1}^{K} j_k \Big[\frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2 - (\beta\hbar\nu_k)^2}\Big] q^+(t)\Psi^n_{j_1,\ldots j_k-1,\ldots,j_K} \\
&- n\Big[d_0 - i\frac{\hbar\gamma\eta}{2}\Big] q^+(t)\Psi^{n-1}_{j_1,\ldots,j_K}
\end{aligned}
\tag{37}
$$

Furthermore,insert

$$\sum_{k=K+1}^{\infty}\Big[\frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}\Big]=\Big[\frac{1}{2}\eta\tan\frac{\beta\hbar\gamma}{4}-\sum_{k=1}^{K}\frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}\Big] \tag{38}$$

and denote

$$\Gamma=-\frac{1}{2}\eta\tan\frac{\beta\hbar\gamma}{4}q^+(t)q^+(t) \tag{39}$$

we arrive at a relatively good-looking form:

$$\begin{aligned}\frac{\partial}{\partial t}\Psi^n_{j_1,\ldots,j_K}=&-\left[iH-iq^+(t)z(t)+n\gamma+\sum_{k=1}^{K}\Big(j_k\nu_k+\frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}q^+(t)q^+(t)\Big)+\Gamma\right]\Psi^n_{j_1,\ldots,j_K}\\&+q^+(t)\Psi^{n+1}_{j_1,\ldots,j_K}+\sum_{k=1}^{K}q^+(t)\Psi^n_{j_1,\ldots j_k+1,\ldots,j_K}\\&+\sum_{k=1}^{K}j_k\Big[\frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}\Big]q^+(t)\Psi^n_{j_1,\ldots j_k-1,\ldots,j_K}\\&-n\Big[d_0-i\frac{\hbar\gamma\eta}{2}\Big]q^+(t)\Psi^{n-1}_{j_1,\ldots,j_K}\end{aligned} \tag{40}$$

Please note the differences (let's call it the *0-order term* and the *minus*) exist between

$$\tan x=8x\sum_{k=1}^{\infty}\frac{1}{((2k-1)\pi)^2-(2x)^2} \tag{41}$$

and

$$\cot x=\frac{1}{x}+2x\sum_{k=1}^{\infty}\frac{1}{x^2-(\pi k)^2}\quad . \tag{42}$$

A. Add the scaling

Let's now add the scaling to our HOPS.

$$\begin{aligned}\frac{\partial}{\partial t}\Psi^n_{j_1,\ldots,j_K}=&-\left[iH-iq^+(t)z(t)+n\gamma+\sum_{k=1}^{K}\Big(j_k\nu_k+\frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}q^+(t)q^+(t)\Big)+\Gamma\right]\Psi^n_{j_1,\ldots,j_K}\\&-\sqrt{n_0/|c_0|}d_0q^+(t)\Psi^{n-1}_{j_1,\ldots,j_K}+i\sqrt{n_0/|c_0|}\frac{\hbar\gamma\eta}{2}q^+(t)\Psi^{n-1}_{j_1,\ldots,j_K}\\&+\sqrt{(n_0+1)|c_0|}q^+(t)\Psi^{n+1}_{j_1,\ldots,j_K}\\&+\sum_{k=1}^{K}\sqrt{(n_k+1)|c_k|}q^+(t)\Psi^n_{j_1,\ldots j_k+1,\ldots,j_K}\\&+\sum_{k=1}^{K}\sqrt{n_k|c_k|}q^+(t)\Psi^n_{j_1,\ldots j_k-1,\ldots,j_K}\quad .\end{aligned} \tag{43}$$

The final form is in accordance with the Fortran code. Please do remember:

$$j_1 + j_2 + ... + j_K = n \ . \tag{44}$$

We did not use this $sum_n level$: n, but used the filtering algorithm instead.

IV. THE NONLINEAR FORM

$$\begin{aligned}
\frac{\partial}{\partial t}\Psi^n_{j_1,\ldots,j_K} = &-\left[iH - iq^+(t)\tilde{z}(t) + n\gamma + \sum_{k=1}^{K}\Big(j_k\nu_k + \frac{4\hbar\eta\gamma\beta}{(\beta\hbar\gamma)^2-(\beta\hbar\nu_k)^2}q^+(t)q^+(t)\Big) + \Gamma\right]\Psi^n_{j_1,\ldots,j_K} \\
&- \sqrt{n_0/|c_0|}d_0 q^+(t)\Psi^{n-1}_{j_1,\ldots,j_K} + i\sqrt{n_0/|c_0|}\frac{\hbar\gamma\eta}{2}q^+(t)\Psi^{n-1}_{j_1,\ldots,j_K} \\
&+ \sqrt{(n_0+1)|c_0|}(q^+(t)-\langle q^+\rangle_t)\Psi^{n+1}_{j_1,\ldots,j_K} \\
&+ \sum_{k=1}^{K}\sqrt{(n_k+1)|c_k|}(q^+(t)-\langle q^+\rangle_t)\Psi^n_{j_1,\ldots j_k+1,\ldots,j_K} \\
&+ \sum_{k=1}^{K}\sqrt{n_k|c_k|}q^+(t)\Psi^n_{j_1,\ldots j_k-1,\ldots,j_K} \quad .
\end{aligned} \tag{45}$$

Here,

$$\tilde{z}(t) = z(t) + \int_0^t ds\tilde{\alpha}(t-s)\langle q^+\rangle_s \quad , \tag{46}$$

with

$$\begin{aligned}
\tilde{\alpha}(t) &= \tilde{\alpha}_R + i\alpha_I \\
&= \sum_{k=0}^{\infty} d_k e^{-\nu_k t} - i\frac{\hbar\eta\gamma}{2}e^{-\gamma t} \\
&= d_0 e^{-\gamma t} - i\frac{\hbar\eta\gamma}{2}e^{-\gamma t} + \sum_{k=1}^{\infty} d_k e^{-\nu_k t} \\
&= (d_0 - i\frac{\hbar\eta\gamma}{2})e^{-\gamma t} + \sum_{k=1}^{\infty} d_k e^{-\nu_k t} \quad .
\end{aligned} \tag{47}$$

V. A SECOND DERIVATION:FORMALLY SIMPLE

introduce the auxiliary wavefunction Ψ^n as:

$$\begin{aligned}
\Psi^n = \int \mathcal{D}z \int \mathcal{D}q^+(t) \exp\left(iS[q^+(t)] + i\int_0^t ds q^+(s)z(s)\right) \\
\tilde{\mathcal{F}}^+_{res}[q^+]\Big[\int_0^t du\tilde{\alpha}(t-u)q^+(u)\Big]^n
\end{aligned} \tag{48}$$

The hierarchy equations are:

$$\begin{aligned}\frac{\partial}{\partial t}\Psi^n(t) &= -\Big(iH - iq^+ z(t)\Big)\Psi^n - q^+(t)\Psi^{n+1}\\ &\quad + \Big[\tilde{\alpha}(0)q^+(t) + \int_0^t du[\frac{\partial}{\partial t}\tilde{\alpha}(t-u)]q^+(u)\Big]\Psi^{n-1} \end{aligned} \tag{49}$$