LECTURE 7

LECTURE OUTLINE

- Preservation of closure under partial minimization
- Hyperplanes
- Hyperplane separation
- Nonvertical hyperplanes
- Min common and max crossing problems

- Question: If $F(x, z)$ is closed, is $f(x) = \inf_z F(x, z)$ closed? Can be addressed by using the relation

$$P\big(\mathrm{epi}(F)\big) \subset \mathrm{epi}(f) \subset \mathrm{cl}\Big(P\big(\mathrm{epi}(F)\big)\Big),$$

where $P(\cdot)$ denotes projection on the space of (x, w), i.e., $P(x, z, w) = (x, w)$.

- Closedness of f is guaranteed if the closure of $\mathrm{epi}(F)$ is preserved under the projection operation $P(\cdot)$.

PARTIAL MINIMIZATION THEOREM

- Let $F : \Re^{n+m} \mapsto (-\infty, \infty]$ be a closed proper convex function, and consider $f(x) = \inf_{z \in \Re^m} F(x, z)$.
- If there exist $\overline{x} \in \Re^n$, $\overline{\gamma} \in \Re$ such that the set

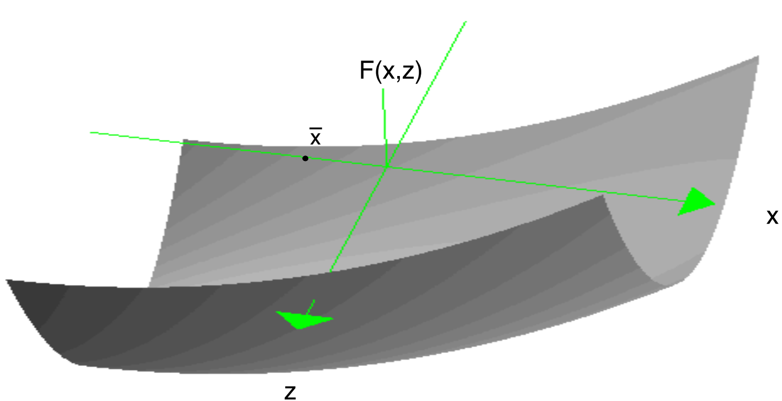
$$\big\{z \mid F(\overline{x}, z) \leq \overline{\gamma}\big\}$$

is nonempty and compact, then f is convex, closed, and proper. Also, for each $x \in \mathrm{dom}(f)$, the set of minima of $F(x, \cdot)$ is nonempty and compact.

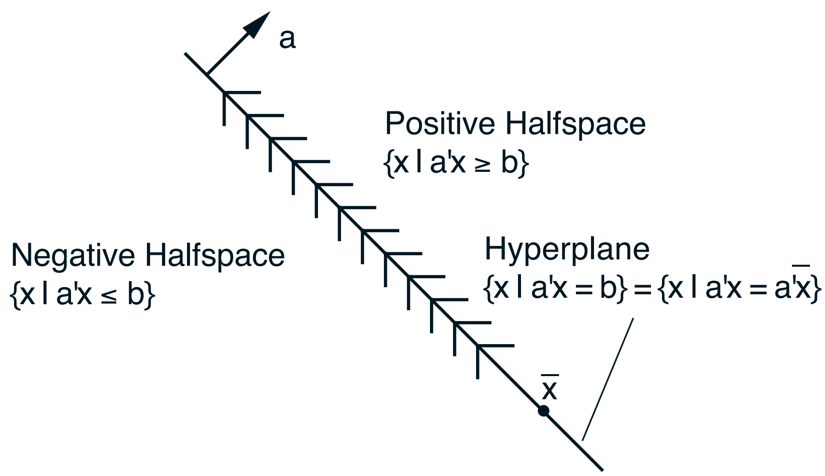
Proof: (Outline) By the hypothesis, there is no nonzero y such that $(0, y, 0) \in R_{\mathrm{epi}(F)}$. Also, all the nonempty level sets

$$\{z \mid F(x, z) \leq \gamma\}, \qquad x \in \Re^n, \;\; \gamma \in \Re,$$

have the same recession cone, which by the hypothesis, is equal to $\{0\}$.



HYPERPLANES



- A *hyperplane* is a set of the form $\{x \mid a'x = b\}$, where a is nonzero vector in $\Re^n$ and b is a scalar.

- We say that two sets C_1 and C_2 are *separated by a hyperplane* $H = \{x \mid a'x = b\}$ if each lies in a different closed halfspace associated with H, i.e.,

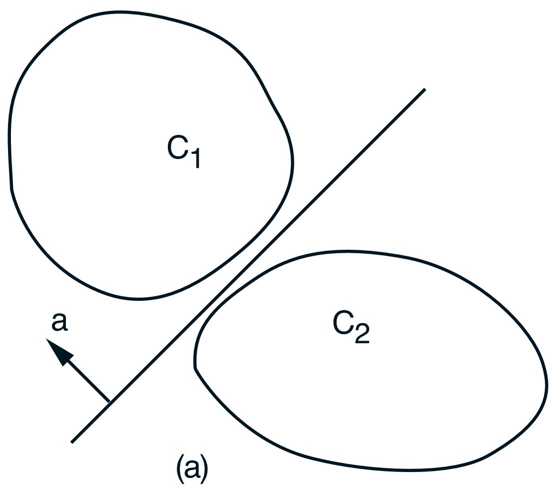
either $\quad a'x_1 \le b \le a'x_2, \qquad \forall\, x_1 \in C_1, \forall\, x_2 \in C_2,$

or $\quad a'x_2 \le b \le a'x_1, \qquad \forall\, x_1 \in C_1, \ \forall\, x_2 \in C_2.$

- If $\overline{x}$ belongs to the closure of a set C, a hyperplane that separates C and the singleton set $\{\overline{x}\}$ is said be *supporting* C *at* $\overline{x}$.

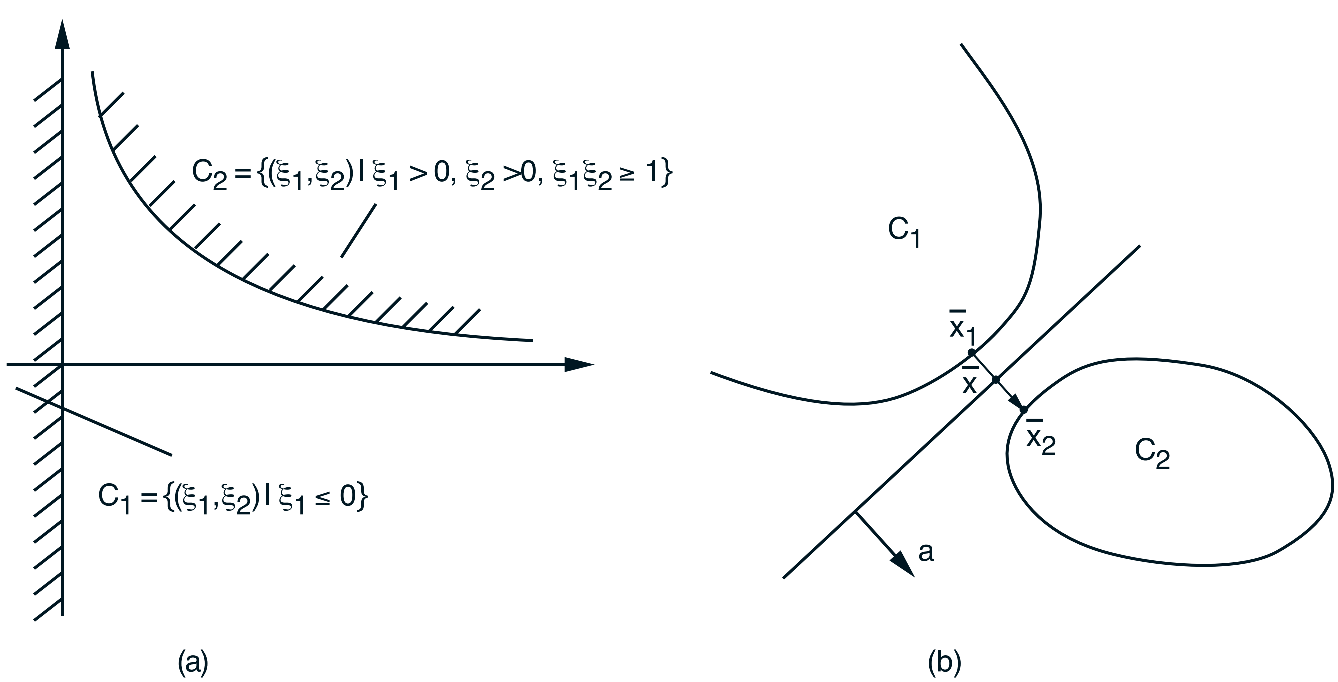
VISUALIZATION

- Separating and supporting hyperplanes:



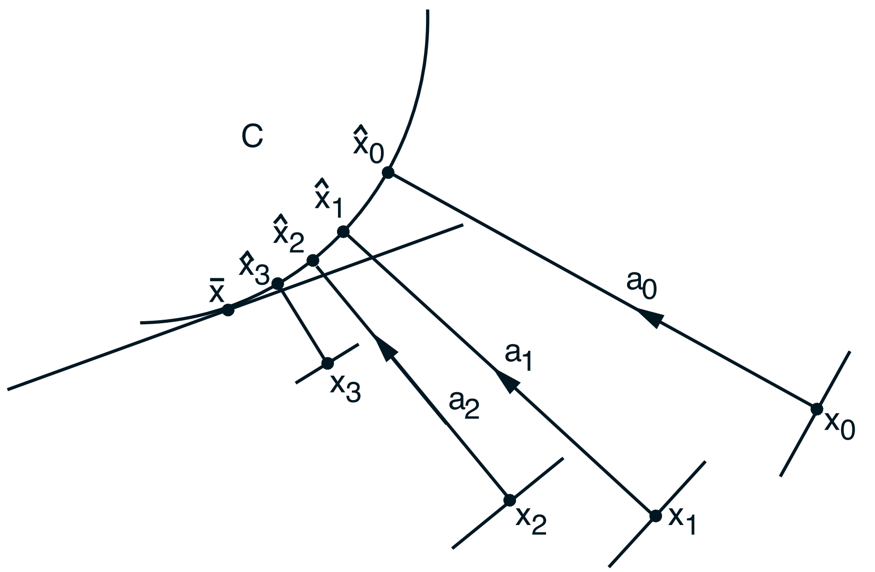
- A separating $\{x \mid a'x = b\}$ that is disjoint from C_1 and C_2 is called *strictly* separating:

$$a'x_1 < b < a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2.$$



SUPPORTING HYPERPLANE THEOREM

- Let C be convex and let $\overline{x}$ be a vector that is not an interior point of C. Then, there exists a hyperplane that passes through $\overline{x}$ and contains C in one of its closed halfspaces.



Proof: Take a sequence $\{x_k\}$ that does not belong to $\mathrm{cl}(C)$ and converges to $\overline{x}$. Let $\hat{x}_k$ be the projection of x_k on $\mathrm{cl}(C)$. We have for all $x \in \mathrm{cl}(C)$

$$a_k' x \geq a_k' x_k, \qquad \forall\, x \in \mathrm{cl}(C),\ \forall\, k = 0, 1, \ldots,$$

where $a_k = (\hat{x}_k - x_k)/\|\hat{x}_k - x_k\|$. Le a be a limit point of $\{a_k\}$, and take limit as $k \to \infty$. **Q.E.D.**

SEPARATING HYPERPLANE THEOREM

• Let C_1 and C_2 be two nonempty convex subsets of $\Re^n$. If C_1 and C_2 are disjoint, there exists a hyperplane that separates them, i.e., there exists a vector $a \neq 0$ such that

$$a'x_1 \leq a'x_2, \qquad \forall\, x_1 \in C_1,\ \forall\, x_2 \in C_2.$$

Proof: Consider the convex set

$$C_1 - C_2 = \{x_2 - x_1 \mid x_1 \in C_1,\, x_2 \in C_2\}.$$

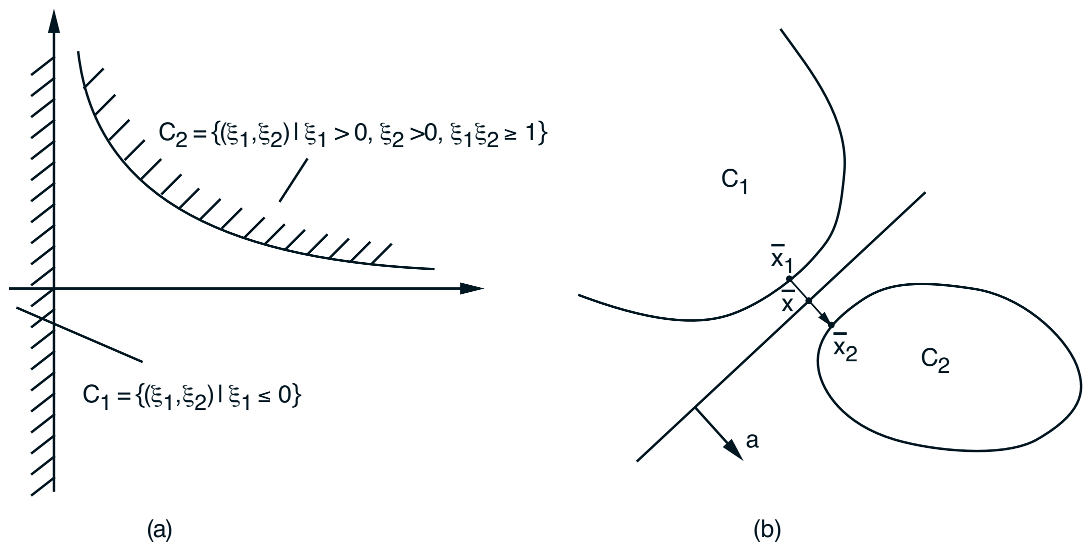
Since C_1 and C_2 are disjoint, the origin does not belong to $C_1 - C_2$, so by the Supporting Hyperplane Theorem, there exists a vector $a \neq 0$ such that

$$0 \leq a'x, \qquad \forall\, x \in C_1 - C_2,$$

which is equivalent to the desired relation. **Q.E.D.**

STRICT SEPARATION THEOREM

- *Strict Separation Theorem*: Let C_1 and C_2 be two disjoint nonempty convex sets. If C_1 is closed, and C_2 is compact, there exists a hyperplane that strictly separates them.



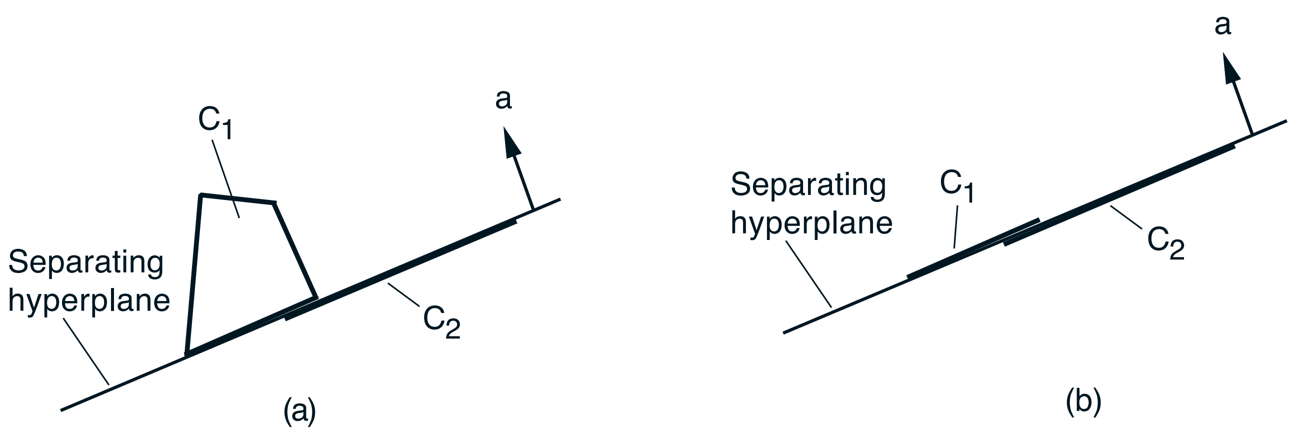
Proof: (Outline) Consider the set $C_1 - C_2$. Since C_1 is closed and C_2 is compact, $C_1 - C_2$ is closed. Since $C_1 \cap C_2 = \emptyset$, $0 \notin C_1 - C_2$. Let $\overline{x}_1 - \overline{x}_2$ be the projection of 0 onto $C_1 - C_2$. The strictly separating hyperplane is constructed as in (b).

- *Note*: Any conditions that guarantee closedness of $C_1 - C_2$ guarantee existence of a strictly separating hyperplane. However, there may exist a strictly separating hyperplane without $C_1 - C_2$ being closed.

ADDITIONAL THEOREMS

- *Fundamental Characterization*: The closure of the convex hull of a set $C \subset \Re^n$ is the intersection of the closed halfspaces that contain C.

- We say that a hyperplane *properly separates* C_1 *and* C_2 if it separates C_1 and C_2 and does not fully contain both C_1 and C_2.

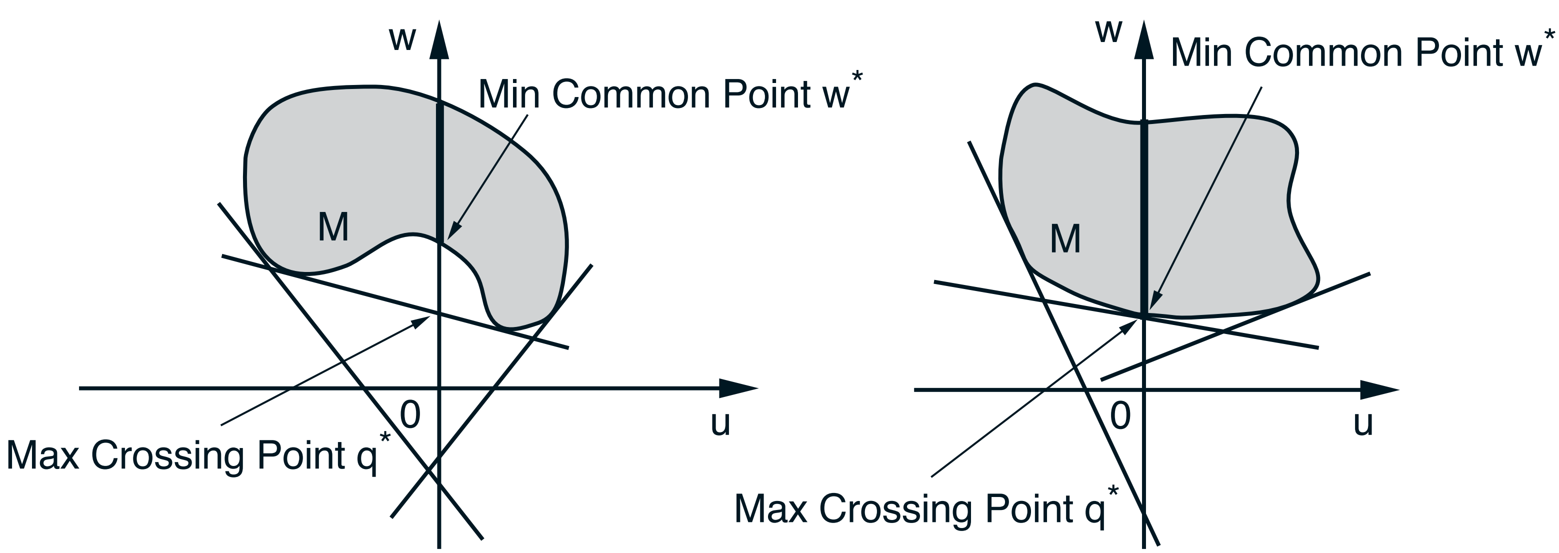


- *Proper Separation Theorem*: Let C_1 and C_2 be two nonempty convex subsets of $\Re^n$. There exists a hyperplane that properly separates C_1 and C_2 if and only if

$$\mathrm{ri}(C_1) \cap \mathrm{ri}(C_2) = \emptyset.$$

MIN COMMON / MAX CROSSING PROBLEMS

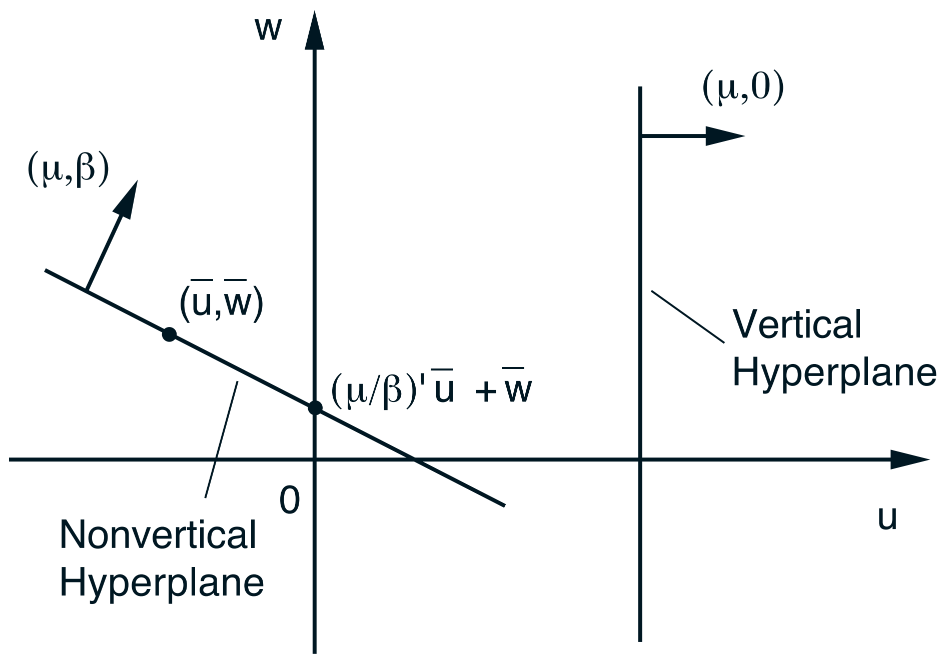
- We introduce a pair of fundamental problems:
- Let M be a nonempty subset of $\Re^{n+1}$
  - (a) *Min Common Point Problem*: Consider all vectors that are common to M and the $(n+1)$st axis. Find one whose $(n+1)$st component is minimum.
  - (b) *Max Crossing Point Problem*: Consider "nonvertical" hyperplanes that contain M in their "upper" closed halfspace. Find one whose crossing point of the $(n+1)$st axis is maximum.



- Need to study "nonvertical" hyperplanes.

NONVERTICAL HYPERPLANES

- A hyperplane in $\Re^{n+1}$ with normal (μ, β) is nonvertical if $\beta \neq 0$.
- It intersects the $(n+1)$st axis at $\xi = (\mu/\beta)'\overline{u} + \overline{w}$, where $(\overline{u}, \overline{w})$ is any vector on the hyperplane.



- A nonvertical hyperplane that contains the epigraph of a function in its "upper" halfspace, provides lower bounds to the function values.
- The epigraph of a proper convex function does not contain a vertical line, so it appears plausible that it is contained in the "upper" halfspace of some nonvertical hyperplane.

NONVERTICAL HYPERPLANE THEOREM

- Let C be a nonempty convex subset of $\Re^{n+1}$ that contains no vertical lines. Then:

  (a) C is contained in a closed halfspace of a nonvertical hyperplane, i.e., there exist $\mu \in \Re^n$, $\beta \in \Re$ with $\beta \neq 0$, and $\gamma \in \Re$ such that $\mu' u + \beta w \geq \gamma$ for all $(u, w) \in C$.

  (b) If $(\overline{u}, \overline{w}) \notin \mathrm{cl}(C)$, there exists a nonvertical hyperplane strictly separating $(\overline{u}, \overline{w})$ and C.

Proof: Note that $\mathrm{cl}(C)$ contains no vert. line [since C contains no vert. line, $\mathrm{ri}(C)$ contains no vert. line, and $\mathrm{ri}(C)$ and $\mathrm{cl}(C)$ have the same recession cone]. So we just consider the case: C closed.

(a) C is the intersection of the closed halfspaces containing C. If all these corresponded to vertical hyperplanes, C would contain a vertical line.

(b) There is a hyperplane strictly separating $(\overline{u}, \overline{w})$ and C. If it is nonvertical, we are done, so assume it is vertical. "Add" to this vertical hyperplane a small ϵ-multiple of a nonvertical hyperplane containing C in one of its halfspaces as per (a).