nmi

2024 spring : lecture 02 : revisit rounding error wrt stability

theorem 01 wrt bound, aggregate error

suppose $i = 1,\ldots,n$ and $0 < \delta_i \le \mu_M$ and $e_i \in \{-1,+1\}$. additionally suppose $n\mu_M < 1$. then

$$\prod^n (1 + \delta_i)^{e\_i} = 1 + \Theta_n,$$

where $|\Theta_n| \le \gamma_n \coloneqq n\mu_M / (1 - n\mu_M)$. **ie, Θ_n aggregates error and γ_n is its bound.**

note: μ_M is rounding error. in FPS, $\mu_M = ½\, \varepsilon_M$, machine error.

proof-lite. (its just a sketch.)

$\prod^n (1 + \delta_i)^{e\_i} \le \prod^n (1 + \delta_i) \le \prod^n (1 + n\mu_M) = (1 + n\mu_M)^n$.

then by binomial theorem ⇒

$(1 + n\mu_M)^n \le n\mu_M / (1 - n\mu_M)$.

more intensely:

[Formal Proofs of Rounding Error Bounds]

eg 01

forward stability

forward stability. ie, $\eta > 0$ such that $||\Delta y|| \leq \eta ||y||$.

$$\Delta y = \hat{y} - y$$

$$\phi : \mathbb{R} \to \mathbb{R}, x \overset{\phi}{\mapsto} x^3$$

$$\hat{\phi} : \mathbb{F} \to \mathbb{F}, \hat{x} \overset{\hat{\phi}}{\mapsto} \hat{x}^3 \Rightarrow \hat{y} = \hat{\phi}(\hat{x}) = \hat{x}^3$$

$$\begin{aligned}
\hat{y} = \hat{x}^3 &= (\hat{x} \otimes \hat{x}) \otimes \hat{x} \\
&= (x(1+\delta_x) \otimes x(1+\delta_x)) \otimes (x(1+\delta_x)) \\
&= x^2(1+\delta_x)^2(1+\delta_\otimes) \otimes (x(1+\delta_x)) \\
&= x^3(1+\delta_x)^3(1+\delta_\otimes)^1(1+\delta_\otimes)^1 \\
&= x^3(1+\theta_5) \text{ [theorem 01]}
\end{aligned}$$

$$\hat{y} - y = x^3(1+\theta_5) - x^3 = x^3\theta_5$$

$$|\hat{y} - y| = |x^3||\theta_5| \leq |y|\gamma_5 \text{ [theorem 01]}$$

$$\gamma_5 = \frac{n\mu_M}{1-n\mu_M} = \frac{5\mu_M}{1-5\mu_M}.$$

let $\eta = \gamma_5$, then $|\hat{y} - y| \leq \eta |y|$. ✓

eg 02 numerical stability

numerical stability. ie, $\eta > 0$ such that $|\Delta y| \leq \eta |y|$ and $\epsilon > 0$ such that $|\Delta x| \leq \epsilon |x|$.

$y = \phi(x) = x^3$. need Δx, Δy.

note: $\hat{y} = x^3(1 + \theta_5)$, γ_5 from eg 01.

$$\begin{aligned}
\hat{y} + \Delta y &= \phi(x + \Delta x) \\
x^3(1 + \theta_5) + \Delta y &= (x + \Delta x)^3 = x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 \\
\Rightarrow \Delta y &= 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3\theta_5
\end{aligned}$$

what η for $|\Delta y| \leq \eta |y|$?

$\Rightarrow$ what ϵ for $|\Delta x| \leq \epsilon |x|$?

$$\begin{aligned}
|\Delta y| &\leq 3|x|^2|\Delta x| + 3|x||\Delta x|^2 + |\Delta x|^3 + |x|^3\theta_5 \\
&\leq 3|x|^2\epsilon|x| + 3|x|\epsilon^2|x|^2 + \epsilon^3|x|^3 + |x|^3\gamma_5 \\
&\Rightarrow 3\epsilon|x|^3 + 3\epsilon^2|x|^3 + \epsilon^3|x|^3 + |x|^3\gamma_5 \Rightarrow (3\epsilon + 3\epsilon^2 + \epsilon^3 + \gamma_5)|y|
\end{aligned}$$

$$\Rightarrow |\Delta y| \leq \eta |y| \Rightarrow \eta = 3\epsilon + 3\epsilon^2 + \epsilon^3 + \gamma_5$$

ie, take any $\epsilon > 0$, any Δx such that $|\Delta x| \leq \epsilon |x|$. then let $\eta = 3\epsilon + 3\epsilon^2 + \epsilon^3 + \gamma_5$, $\Delta y = 3x^2(\Delta x) + 3x(\Delta x)^2 + (\Delta x)^3 - x^3\theta_5$.

by construction (ie, by choice), we know $|\Delta y| \leq \eta |y|$, $\hat{y} + \Delta y = \phi(x + \Delta x)$. ✓

ie, $\hat{\phi}(x) = \hat{x}^3$ is numerically stable.

also, "any" ϵ vs "the best, smallest" ϵ, η. note that the only part of the derived η that you cannot control is γ_5, so choose $\epsilon \ll 1$ and η will barely above γ_5. ie, $\eta = \gamma_5 \cdot c_0$ where $c_0 \geq 1$.

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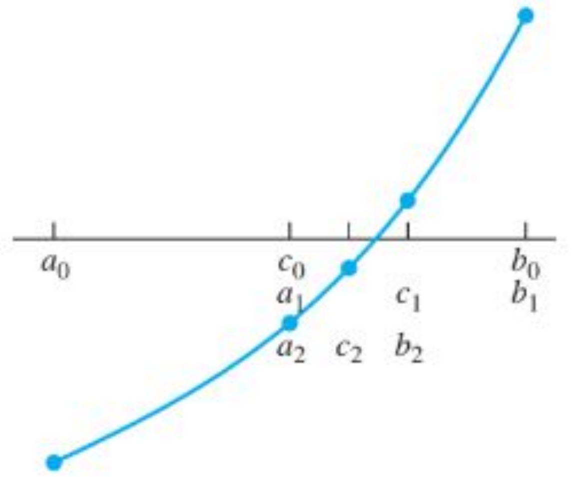
2024 spring : lecture 03 : bisection, fixed-point iteration

root finding, bisection

given $f(x) = x^3 + x - 1$, find its root on the interval $[0, 1]$. let a, b represent the endpoints of the interval considered and c as its bisector, $\frac{a+b}{2}$.

a	f(a)	b	f(b)	c	f(c)
0	-1	1	1	$\frac{1}{2}$	$-\frac{3}{8}$
$\frac{1}{2}$	$-\frac{3}{8}$	1	1	$\frac{3}{4}$	+
$\frac{1}{2}$	$-\frac{3}{8}$	$\frac{3}{4}$	+	$\cdots$	$\cdots$

how will you stop this function? when is x good enough? or is when $f(x)$ is good enough?
thats on you! an error less than 0.5×10^{-p} for precision of p decimal places.



root finding, fixed-point iteration, usw

$x_{n+1} = f(x_n)$ with n=0,1,2,...

$\Rightarrow x_0, x_1, x_2, \ldots = x_0, f(x_0), f(f(x_0)), \ldots$

$\Rightarrow f(x_{fix}) = x_{fix}$.

eg, rearrange the previous $f(x) = x^3 + x - 1$

$\Rightarrow x = 1 - x3 = g(x)$

$\Rightarrow x_1 = g(x_0), x_2 = g(x_1), x_3 = g(x_2), \ldots$

image right shows cobweb pattern of convergence.

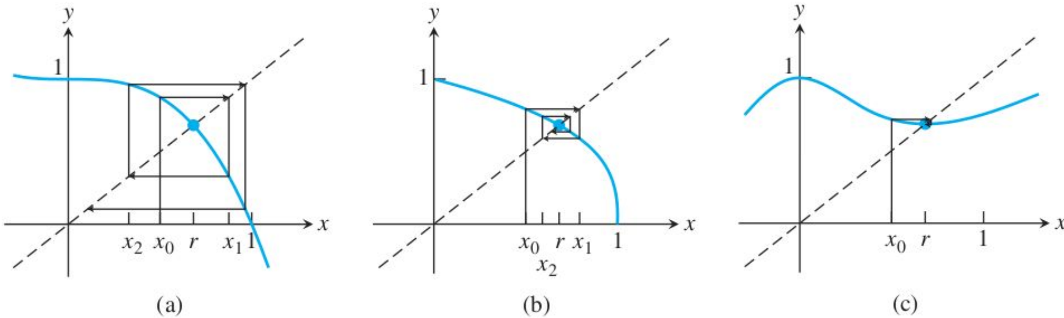


Figure 1.3 Geometric view of FPI. The fixed point is the intersection of $g(x)$ and the diagonal line. Three examples of $g(x)$ are shown together with the first few steps of FPI. (a) $g(x) = 1 - x^3$ (b) $g(x) = (1-x)^{1/3}$ (c) $g(x) = (1+2x^3)/(1+3x^2)$

and heres another fpi, newton-raphson.

$x_{n+1} = x_n - f(x_n)/f'(x_n) = g(x_n)$.

fpi, extended

for the first part of this revisitation, lets re-rig that failed example.

start by considering that $f(x) = x^3 + x - 1$ has three roots. (thats three roots - not necessarily three distinct roots.) so

$f(x) = 0 \Rightarrow x = 1 - x^3 = g(x) \Rightarrow x_{n+1} = 1 - x_n^3$.

thats the one that barfs, so lets look at what happens when x^3 is on the left side of that equation - in colab… *refer to blackboard for python notebook, demo_02_rootfinding.*

what makes those last two methods increasingly better? #2 addresses that there are three roots and #3 reduces the order of g(x).

next time

newtons method

secant method