nmi | 2024 spring

lecture 06 : systems of equations : iterative

why stop at one? part two.

2.5 iterative methods

gauss elimination is a finite sequence of $\mathcal{O}(n^3)$ operations that result in a solution. ie, it is a direct method. iterative methods solve systems of linear equations by refining an initial guess.

2.5.1 jacobi method

jacobi is fixed-point iteration for a system of equations and FPI rewrites equations then solves for the unknown.

ex 19

apply jacobi to system $3u + v = 5, u + 2v = 5$ starting with $(u_0, v_0) = (0, 0)$. solve first equation for u first.

$$
\begin{aligned}
u &= \frac{5 - v}{3} \\
v &= \frac{5 - u}{2}
\end{aligned}
$$

$$
\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$
$$
\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-0}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{2} \end{bmatrix}
$$
$$
\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-\frac{5}{2}}{3} \\ \frac{5-\frac{5}{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{6} \\ \frac{5}{3} \end{bmatrix}
$$
$$
\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-\frac{5}{3}}{3} \\ \frac{5-\frac{5}{6}}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{25}{12} \end{bmatrix}.
$$

further steps show convergence to solution, $[1,2]^T$.

ex 20

apply jacobi to system $u+2v=5, 3u+v=5$ starting with $(u_0, v_0) = (0,0)$. same equations, flip the order before solving for u.

$$
\begin{aligned} u &= 5-2v \\ v &= 5-3u \end{aligned}
$$

$$
\begin{bmatrix} u_0 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$
$$
\begin{bmatrix} u_1 \\ v_1 \end{bmatrix} = \begin{bmatrix} 5-2v_0 \\ 5-3u_0 \end{bmatrix} = \begin{bmatrix} 5-2(0) \\ 5-3(0) \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}
$$
$$
\begin{bmatrix} u_2 \\ v_2 \end{bmatrix} = \begin{bmatrix} 5-2v_1 \\ 5-3u_1 \end{bmatrix} = \begin{bmatrix} 5-2(5) \\ 5-3(5) \end{bmatrix} = \begin{bmatrix} -5 \\ -10 \end{bmatrix}
$$
$$
\begin{bmatrix} u_3 \\ v_3 \end{bmatrix} = \begin{bmatrix} 5-2v_2 \\ 5-3u_2 \end{bmatrix} = \begin{bmatrix} 5-2(-10) \\ 5-3(-5) \end{bmatrix} = \begin{bmatrix} 25 \\ 20 \end{bmatrix}.
$$

obviously, this one is not your bff. you need some rules.

def 09 $n \times n$ matrix $A = (a_{ii})$ is **strictly diagonally dominant** if for each $1 \le i \le n, |a_{ii}| > \sum_{j \ne i} |a_{ij}|$.

th 10 if $n \times n$ matrix A is strictly diagonally dominant, then (1) A is nonsingular and (2) for every vector b and every starting guess, the jacobi method applied to $Ax = b$ converges to the (unique) solution.

ex 21

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & -5 & 2 \\ 1 & 6 & 8 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 8 & 1 \\ 9 & 2 & -2 \end{bmatrix}$$

where A is and B isnt.

usw

let D denote main diagonal of A, L denote lower triangle of A (below the diagonal) and U denote upper triangle of A (above the triangle) such that $A = L + D + U$. then

$$\begin{aligned} Ax &= b \\ (D + L + U)x &= b \\ Dx = b - (L + U)x \\ x = D^{-1}(b - (L + U)x). \end{aligned}$$

jacobi method

$$\begin{aligned} x_0 &= \text{initial vector} \\ x_{k+1} &= D^{-1}(b - (L + U)x_k) \quad k = 0, 1, 2, \ldots \end{aligned}$$

ex 19, continued

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$
$$\Downarrow$$

$$\begin{aligned} x_k &= \begin{bmatrix} u_k \\ v_k \end{bmatrix} \\ x_{k+1} &= D^{-1}(b - (L+U)x_k) \\ &= \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} 5 \\ 5 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_k \\ v_k \end{bmatrix} \right) \\ &= \begin{bmatrix} \frac{5-v_k}{3} \\ \frac{5-u_k}{2} \end{bmatrix}. \end{aligned}$$

gauss-seidel and SOR

with gauss-seidel, values are used as they are available.

ex 19, continued

$$\begin{aligned} \begin{bmatrix} u_0 \\ v_0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} u_1 \\ v_1 \end{bmatrix} &= \begin{bmatrix} \frac{5-v_0}{3} \\ \frac{5-u_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-0}{3} \\ \frac{5-\frac{5}{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix} \\ \begin{bmatrix} u_2 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \frac{5-v_1}{3} \\ \frac{5-u_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-\frac{5}{3}}{3} \\ \frac{5-\frac{10}{9}}{2} \end{bmatrix} = \begin{bmatrix} \frac{10}{9} \\ \frac{35}{18} \end{bmatrix} \\ \begin{bmatrix} u_3 \\ v_3 \end{bmatrix} &= \begin{bmatrix} \frac{5-v_2}{3} \\ \frac{5-u_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{5-\frac{35}{18}}{3} \\ \frac{5-\frac{55}{54}}{2} \end{bmatrix} = \begin{bmatrix} \frac{55}{54} \\ \frac{215}{108} \end{bmatrix}. \end{aligned}$$

usw

gauss-seidel

$$
\begin{aligned}
x_0 &= \text{initial vector} \\
x_{k+1} &= D^{-1}(b - Ux_k - Lx_{k+1}) \quad k = 0, 1, 2, \ldots
\end{aligned}
$$

ex 22

apply gauss-seidel to system

$$
\begin{bmatrix} 3 & 1 & -1 \\ 2 & 4 & 1 \\ -1 & 2 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}.
$$

iteration:

$$
\begin{aligned}
u_{k+1} &= \frac{4 - v_k + w_k}{3} \\
v_{k+1} &= \frac{1 - 2u_{k+1} - w_k}{4} \\
w_{k+1} &= \frac{1 + u_{k+1} - 2v_{k+1}}{5}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} &= \begin{bmatrix} \frac{4-0-0}{3} = \frac{4}{3} \\ \frac{1-\frac{8}{3}-0}{4} = -\frac{5}{12} \\ \frac{1+\frac{4}{3}+\frac{5}{6}}{5} = \frac{19}{30} \end{bmatrix} \approx \begin{bmatrix} 1.3333 \\ -0.4167 \\ 0.6333 \end{bmatrix} \\
\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} &= \begin{bmatrix} \frac{101}{60} \\ -\frac{3}{4} \\ \frac{251}{300} \end{bmatrix} \approx \begin{bmatrix} 1.6833 \\ -0.7500 \\ 0.8367 \end{bmatrix}
\end{aligned}
$$

this system is strictly diagonally dominant and will converge to $[2, -1, 1]^T$.

further, if weights are applied to gauss-seidel to speed convergence, thats **successive over-relaxation (SOR)**.

let $\omega \in \mathbb{R}$ and define each component of new guess x_{k+1} as a weighted average of ω times gauss-seidel formula and $1 - \omega$ times the current guess x_k. ω is the **relaxation parameter** and $\omega \in [0, 2]$. $\omega > 1$ is referred to as **over-relaxation**; and $\omega < 1$ is under-relaxation; and $\omega = 1$ is gauss-seidel, :).

ex 23

apply SOR with $\omega = 1.25$ to example 22.

$$
\begin{aligned}
u_{k+1} &= (1-\omega)u_k + \omega\frac{4 - v_k + w_k}{3} \\
v_{k+1} &= (1-\omega)v_k + \omega\frac{1 - 2u_{k+1} - w_k}{4} \\
w_{k+1} &= (1-\omega)w_k + \omega\frac{1 + u_{k+1} - 2v_{k+1}}{5}
\end{aligned}
$$

$$
\begin{aligned}
\begin{bmatrix} u_0 \\ v_0 \\ w_0 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} u_1 \\ v_1 \\ w_1 \end{bmatrix} &\approx \begin{bmatrix} 1.6667 \\ -0.7292 \\ 1.0312 \end{bmatrix} \\
\begin{bmatrix} u_2 \\ v_2 \\ w_2 \end{bmatrix} &\approx \begin{bmatrix} 1.9835 \\ -1.0672 \\ 1.0216 \end{bmatrix}
\end{aligned}
$$

usw

and

$$
\begin{aligned}
(\omega L+\omega D+\omega U)x &= \omega b \\
(\omega L+D)x &= \omega b-\omega Ux+(1-\omega)Dx \\
x &= (\omega L+D)^{-1}[(1-\omega)Dx-\omega Ux]+\omega(D+\omega L)^{-1}b.
\end{aligned}
$$

successive over-relaxation (SOR)

$$
\begin{aligned}
x_0 &= \text{initial vector} \\
x_{k+1} &= (\omega L+D)^{-1}[(1-\omega)Dx_k-\omega Ux_k]+\omega(D+\omega L)^{-1}b
\end{aligned}
$$

also

- relaxation parameter with gauss-seidel @themathguy.

ex 24

compare jacobi, gauss-seidel, SOR with system

$$
\begin{bmatrix} 3 & -1 & 0 & 0 & 0 & \frac{1}{2} \\ -1 & 3 & -1 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 3 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & 0 \\ 0 & \frac{1}{2} & 0 & -1 & 3 & -1 \\ \frac{1}{2} & 0 & 0 & 0 & -1 & 3 \end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}
=
\begin{bmatrix} \frac{5}{2} \\ \frac{3}{2} \\ 1 \\ 1 \\ \frac{3}{2} \\ \frac{5}{2} \end{bmatrix}.
$$

solution is $x=[1,1,1,1,1,1]^T$.

Jacobi	Gauss–Seidel	SOR
0.9879	0.9950	0.9989
0.9846	0.9946	0.9993
0.9674	0.9969	1.0004
0.9674	0.9996	1.0009
0.9846	1.0016	1.0009
0.9879	1.0013	1.0004

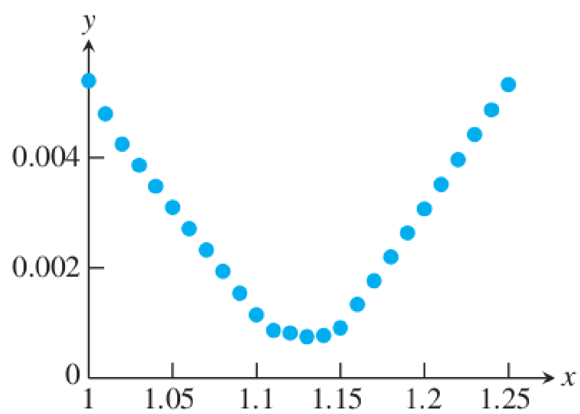

Figure 2.3 Infinity norm error after six steps of SOR in Example 2.24, as a function of over-relaxation parameter ω. Gauss–Seidel corresponds to $\omega = 1$. Minimum error occurs for $\omega \approx 1.13$

2.5.3 convergence

why approximate with iterative methods vs solve directly with gaussian elimination methods? bc its operationally cheaper. its even more so if starting with a good guess.

iterative methods also avoid the bleed over that happens with sparse matrices that implement gauss elimination. (there are also structures for sparse matrices.)