nmi | spring 2024

lecture 20 : elliptic

8.3 elliptic

the heat and wave equations were functions of time; elliptic equations model steady states. eg, a steady-state distribution of heat on a plane whose boundary is held at a specific temperature.

definition 06 laplacian

let $u(x, y)$ be 2x differentiable function and define the **laplacian** of u as

$$\Delta u = u_{xx} + u_{yy}.$$

for continuous function $f(x, y)$, the partial differential equation

$$\Delta u(x, y) = f(x, y)$$

is called the **poisson equation**. the poisson equation with $f(x, y) = 0$ is called the **laplace equation** and its solution is called a **harmonic** function.

usw

the extra conditions to for a single solution are typically boundary conditions. dirichlet boundary conditions specify values of the solution $u(x, y)$ on boundary ∂R of region R. neumann boundary conditions specify values of the direction derivative $\frac{\partial u}{\partial n}$ on boundary where n denotes the outward unit normal vector.

example 07

show $u(x,y) = x^2 - y^2$ is a solution of the laplace equation on $[0,1] \times [0,1]$ with dirichlet boundary conditions

$$\begin{cases} u(x,0) = x^2 \\ u(x,1) = x^2 - 1 \\ u(0,y) = -y^2 \\ u(1,y) = 1 - y^2. \end{cases}$$

laplacian $\Delta u = u_{xx} + u_{yy} = 2 - 2 = 0$ can be verified by its boundary conditions. ✓

- usw

poisson and laplace equations are ubiquitous in classical physics bc their solutions represent potential energy. eg, an electric field E is the gradient of an electrostatic potential u

$$E = -\nabla u.$$

the gradient of an electric field is related to charge density ρ by maxwells equation

$$\nabla E = \frac{\rho}{\epsilon},$$

where ϵ is the electrical permittivity.

$$\Rightarrow \quad \Delta u = \nabla(\nabla u) = -\frac{\rho}{\epsilon},$$

the poisson equation for potential u. in the special case of zero charge, the potential satisfies the laplace equation $\Delta u = 0$.

other instances of potential energy are modeled by the poisson equation: aerodynamics at low speeds are the solution of the laplace equation. and gravitational potential u generated by a distribution of mass density ρ satisfies the poisson equation

$$\Delta u = 4\pi G\rho,$$

where G is the gravitational constant.

8.3.1 FDM

consider the poisson equation $\Delta u = f$ on rectangle $[x_l, x_r] \times [y_b, y_t]$ in the plane with dirichlet boundary conditions.

$$\begin{cases} u(x, y_b) = g_1(x) \\ u(x, t_t) = g_2(x) \\ u(x_l, y) = g_3(x) \\ u(x_r, y) = g_4(x). \end{cases}$$

with $M = m - 1$ steps along x and $N = n - 1$ steps along t with mesh sizes $h = \frac{x_r - x_l}{M}, k = \frac{y_t - y_b}{N}$.

a centered-difference formula can approxiate both second derivatives in the laplacian operator. the poisson equation has the FDM form

$$\frac{u(x-h, y) - 2u(x, y) + u(x+h, y)}{h^2} + \mathcal{O}(h^2) + \frac{u(x, y-k) - 2u(x, y) + u(x, y+k)}{k^2} +$$

and $w_{ij} \approx u(x_i, y_j)$

$$\frac{w_{i-1,j} - 2w_{ij} + w_{i+1,j}}{h^2} + \frac{w_{i,j-1} - 2w_{ij} + w_{i,j+1}}{k^2} = f(x_i, y_j)$$

where $x_i = x_l + (i-1)h, y_j = y_b + (j-1)k$ for $1 \leq i \leq m, 1 \leq j \leq n$.

bc the unknowns are $m \cdot n$, use alternative system for solution values

$$v_{i+(j-1)m} = w_{ij} \Rightarrow A_{mn\times mn} v = b.$$

ie, A_{pq} is qth linear coefficient of pth equation. ie, at point (i,j), equation $p = i + (j-1)m$ with coefficients $w_{i-1,j}, w_{ij}, \ldots$.

x	y	Equation number p
i	j	$i+(j-1)m$

x	y	Coefficient number q
i	j	$i+(j-1)m$
$i+1$	j	$i+1+(j-1)m$
$i-1$	j	$i-1+(j-1)m$
i	$j+1$	$i+jm$
i	$j-1$	$i+(j-2)m$

Table 8.1 Translation table for two-dimensional domains. The equation at grid point (i,j) is numbered p, and its coefficients are A_{pq} for various q, with p and q given in the right column of the table. The table is simply an illustration of (8.39).

by p, q, matrix entries A_{pq}

$$\begin{aligned}
A_{i+(j-1)m,i+(j-1)m} &= \frac{2}{h^2} - \frac{2}{k^2} \\
A_{i+(j-1)m,i+1+(j-1)m} &= \frac{1}{h^2} \\
A_{i+(j-1)m,i-1+(j-1)m} &= \frac{1}{h^2} \\
A_{i+(j-1)m,i+jm} &= \frac{1}{k^2} \\
A_{i+(j-1)m,i+(j-2)m} &= \frac{1}{k^2}
\end{aligned}$$

and RHS for (i,j),

$$b_{i+(j-1)m} = f(x_i, y_i)$$

for interior points $1 < i < m, 1 < j < n$.

for boundary points - in this case, dirichlet boundary conditions,

$$\begin{aligned}
\text{bottom} \quad & w_{ij} = g_1(x_i), \quad j = 1, 1 \le i \le m \\
\text{top} \quad & w_{ij} = g_2(x_i), \quad j = n, 1 \le i \le m \\
\text{left} \quad & w_{ij} = g_3(y_j), \quad i = 1, 1 < j < n \\
\text{right} \quad & w_{ij} = g_4(y_j), \quad i = m, 1 < j < n
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\text{bottom} \quad & A_{i+(j-1)m, i+(j-1)m} = 1, b_{i+(j-1)m} = g_1(x_i), \quad j = 1, 1 \le i \le m \\
\text{top} \quad & A_{i+(j-1)m, i+(j-1)m} = 1, b_{i+(j-1)m} = g_2(x_i), \quad j = n, 1 \le i \le m \\
\text{left} \quad & A_{i+(j-1)m, i+(j-1)m} = 1, b_{i+(j-1)m} = g_3(y_j), \quad i = 1, 1 < j < n \\
\text{right} \quad & A_{i+(j-1)m, i+(j-1)m} = 1, b_{i+(j-1)m} = g_4(y_j), \quad i = m, 1 < j < n
\end{aligned}$$

all other entries of A, b are zero.

example 08

apply FDM with $m = n = 5$ to approximate laplace equation $\Delta u = 0$ on $[0, 1] \times [1, 2]$ with dirichlet boundary conditions

$$\begin{cases}
u(x, 1) = ln(x^2 + 1) \\
u(x, 2) = ln(x^2 + 4) \\
u(0, y) = 2lny \\
u(1, y) = ln(y^2 + 1).
\end{cases}$$

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example 09

electrostatic potential on $[0,1] \times [0,1]$ with no interior charge and the following boundary conditions

$$\begin{cases} u(x,0) = sin\pi x \\ u(x,1) = sin\pi x \\ u(0,y) = 0 \\ u(1,y) = 0. \end{cases}$$

use mesh size $h = k = 0.1$ or $M = N = 10$.

only the boundary functions change! and the step-size but thats just a gimme.

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8.3.2 FEM

FEM converts the differential equation into a variational equivalent called the weak form of the equation.

consider the dirichlet problem for elliptic equation

$$\begin{aligned} \Delta u + r(x,y)u = f(x,y) \quad & \text{in } R \\ u = g(x,y) \quad & \text{on } S \end{aligned}$$

where solution $u(x,y)$ is defined on a region R in the plane bounded by a piecewise-smooth closed curve S. use L^2 function space over region R. let

$$L^2(R) = \left\{ \text{functions } \phi(x,y) \text{ on } R \quad \middle| \quad \iint_R \phi(x,y)^2 dxdy \text{ exists and is finite} \right\}.$$

consider subspace $L_0^2(R)$ consisting of functions zero on boundary S of region R. minimize the squared error of the elliptic equation by forcing residual $\Delta u(x,y) + r(x,y)u(x,y) - f(x,y)$ to be orthogonal to a large subspace of $L^2(R)$. let $\phi_1(x,y), \ldots, \phi_p(x,y)$ be elements of $L^2(R)$. ie,

$$\iint_R (\Delta u + ru - f)\phi_p dxdy = 0$$

$$\Rightarrow \quad \iint_R (\Delta u + ru)\phi_p dxdy = \iint_R f\phi_p dxdy$$

for each $1 \leq p \leq P$. this is the weak form.

theorem 07 greens first identity

let R be bounded region with piecewise smooth boundary S. let u, v be smooth functions and let n denote outward unit normal along boundary.

$$\iint_R v\Delta u = \int_S v\frac{\partial u}{\partial n} dS - \iint_R \nabla u \cdot \nabla v.$$

with directional derivative, $\frac{\partial u}{\partial n} = \nabla u \cdot (n_x, n_y)$, apply greens theorem to the weak form.

$$\Rightarrow \quad \int_S \phi_P \frac{\partial u}{\partial n} dS - \iint_R (\nabla u \cdot \nabla \phi_P) dxdy + \iint_R ru\phi_P dxdy = \iint_R f\phi_P dxdy.$$

replace u with FEM,

$$w(x,y)=\sum_{q=1}^{P} v_q\phi_q(x,y)$$

and determine unknown v_q. assume $\phi_p \in L_0^2(R)$ - ie, $\phi_p(S)=0$. for each $\phi_p \in L_0^2(R)$

$$\iint_R \left(\sum_{q=1}^{P} v_q \nabla\phi_q\right)\cdot\nabla\phi_p dxdy - \iint_R r\left(\sum_{q=1}^{P} v_q\phi_q\right)\phi_p dxdy = \iint_R f\phi_p dxdy$$

$$\Downarrow$$

$$\sum_{q=1}^{P} v_q \underbrace{\left[\iint_R \nabla\phi_q\cdot\nabla\phi_p dxdy - \iint_R r\phi_q\phi_p dxdy\right]}_{\text{matrix } A} = \underbrace{-\int_R f\phi_p dxdy.}_{\text{matrix } b}$$

ie, for each $\phi_p \in L_0^2(R)$, a system of linear equations in the unknowns $v_1,\ldots,v_p$.

choose linear b-splines for ϕ. let R be a rectangular $M\times N$ mesh of $m=M+1, n=N+1$ points and form a triangulation with nodes (x_i,y_j).

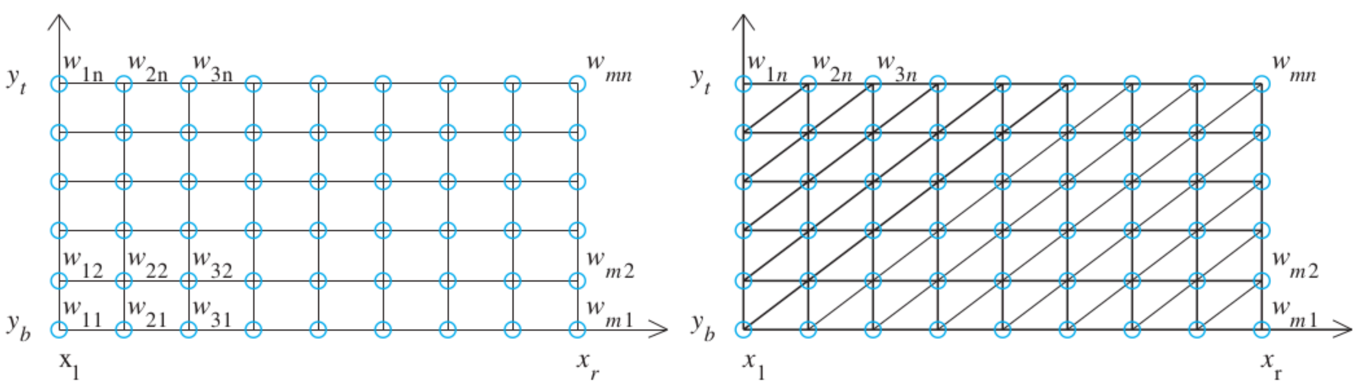

Figure 8.16 Finite element solver of elliptic equation with Dirichlet boundary conditions. (a) Mesh is same as used for finite difference solver. (b) A possible triangulation of the region. Each interior point is a vertex of six different triangles.

for $P=mn$ piecewise linear functions, $\phi_1,\ldots,\phi_{mn}$,

$$\begin{aligned} \phi_{i+(j-1)m}(x_i, y_j) &= 1 \qquad \text{for point}(x_i, y_j) \\ \phi_{i+(j-1)m}(x_i', y_j') &= 0 \qquad \text{for all other points}(x_i', y_j'). \end{aligned}$$

each $\phi_p(x, y)$ is differentiable except for triangle edges and is riemann-integrable function belonging to $L^2(R)$. for every non-boundary point (x_i, y_j) of rectangle R, $\phi_{i+(j-1)m}$ belongs to $L_0^2(R)$.

$$w(x_i, y_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} v_{i+(j-1)m} \phi_{i+(j-1)m}(x_i, y_j) = v_{i+(j-1)m} \qquad i = 1, \ldots, m, \quad j = 1, \ldots,$$

to solve that collection of boundary value problems, approximate the integrals of A, b by 2d midpoint. define the barycenter of a region as the point $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{\iint_R x dx dy}{\iint_R 1 dx dy}, \quad \bar{y} = \frac{\iint_R y dx dy}{\iint_R 1 dx dy}.$$

oc if region is a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}, \quad \bar{y} = \frac{y_1 + y_2 + y_3}{3}.$$

lemma 08

the average value of a linear function $L(x, y)$ on plane region R is $L(\bar{x}, \bar{y})$, the value at the barycenter. ie, $\iint_R L(x, y) dx dy = L(\bar{x}, \bar{y})$.

proof

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also, taylors for functions of two variables,

$$f(x,y) = f(\bar{x},\bar{y}) + \frac{\partial f}{\partial x}(\bar{x},\bar{y})(x-\bar{x}) + \frac{\partial f}{\partial y}(\bar{x},\bar{y})(y-\bar{y}) + \mathcal{O}((x-\bar{x})^2, (x-\bar{x})$$

$$\Downarrow$$

$$\begin{aligned}\iint_R f(x,y)dxdy &= \iint_R L(x,y)dxdy + \iint_R \mathcal{O}((x-\bar{x})^2, (x-\bar{x})(y-\bar{y}), (y-\bar{y})^2)dxdy \\ &= \text{area}(R)\cdot L(\bar{x},\bar{y}) + \mathcal{O}(h^4) \\ &= \text{area}(R)\cdot f(\bar{x},\bar{y}) + \mathcal{O}(h^4)\end{aligned}$$

where h is the **diameter** of R, the largest distance between two points of R.

midpoint rule in two dimensions

$$\iint_R f(x,y)dxdy = \text{area}(R)\cdot f(\bar{x},\bar{y}) + \mathcal{O}(h^4)$$

where $(\bar{x},\bar{y})$ is barycenter of bounded region R and $h = \text{diam}(R)$.

ie, for midpoint rule applied to FEM with $\mathcal{O}(h^2)$, approximate integrals by evaluating integrands at triangle barycenters. which is why b-splines were chosen.

lemma 09

let $\phi(x,y)$ be linear function on triangle T with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ satisfying $\phi(x_1,y_1)=1, \phi(x_2,y_2)=0, \phi(x_3,y_3)=0$. then $\phi(\bar{x},\bar{y}) = \frac{1}{3}$.

lemma 10

let $\phi_1(x,y), \phi_2(x,y)$ be linear functions on triangle T with vertices $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ satisfying
$\phi_1(x_1,y_1)=1, \phi_1(x_2,y_2)=0, \phi_1(x_3,y_3)=0, \phi_2(x_1,y_0)=1, \phi_2(x_2,y_2)=1, \phi_2(x_3,y_3)$
. let $f(x,y)$ be twice differentiable. set

$$d = det \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}.$$

then

(a) triangle T has area $\frac{|d|}{2}$

(b) $\nabla\phi_1(x,y) = \left(\frac{y_2-y_3}{d}, \frac{x_3-x_2}{d}\right)$

(c) $\iint_T \nabla\phi_1 \cdot \nabla\phi_1 dxdy = \frac{(x_2-x_3)^2+(y_2-y_3)^2}{2|d|}$

(d) $\iint_T \nabla\phi_1 \cdot \nabla\phi_2 dxdy = \frac{-(x_1-x_3)(x_2-x_3)-(y_1-y_3)(y_2-y_3)}{2|d|}$

(e) $\iint_T f\phi_1\phi_2 dxdy = f(\bar{x},\bar{y})\frac{|d|}{18} + \mathcal{O}(h^4) = \iint_T f\phi_1^2 dxdy$

(f) $\iint_T f\phi_1 dxdy = f(\bar{x},\bar{y})\frac{|d|}{6} + \mathcal{O}(h^4)$

where $(\bar{x},\bar{y})$ is barycenter of T and $h = \text{diam}(T)$.

usw

now calculate A.

consider vertex (x_i, y_j) not on boundary S of rectangle R. then $\phi_{i+(j-1)m}$ belongs to $L_0^2(R)$ and $p = i + (j-1)m$ and A_{pp} is composed of two integrals. the integrands are zero outside of the six triangles shown.

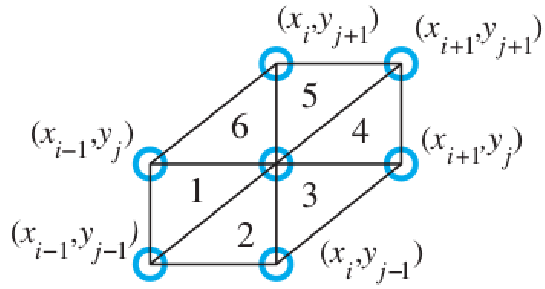

Figure 8.17 Detail of the (i, j) interior point from Figure 8.16(b). Each interior point (x_i, y_j) is surrounded by six triangles, numbered as shown. The B-spline function $\phi_{i+(j-1)m}$ is linear, takes the value 1 at the center, and is zero outside of these six triangles.

the triangles have horizontal and vertical sides h, k respectively. for first integral, summing from triangle 1 to triangle 6, use lemma 8.10.c

$$\frac{k^2}{2hk} + \frac{h^2}{2hk} + \frac{h^2+k^2}{2hk} + \frac{k^2}{2hk} + \frac{h^2}{2hk} + \frac{h^2+k^2}{2hk} = \frac{2(h^2+k^2)}{hk}.$$

for second integral use lemma 10 (e). the barycenters of the six triangles are

$$\begin{aligned}
B_1 &= (x_i - \frac{2}{3}h, y_j - \frac{1}{3}k) \\
B_2 &= (x_i - \frac{1}{3}h, y_j - \frac{2}{3}k) \\
B_3 &= (x_i + \frac{1}{3}h, y_j - \frac{1}{3}k) \\
B_4 &= (x_i + \frac{2}{3}h, y_j + \frac{1}{3}k) \\
B_5 &= (x_i + \frac{1}{3}h, y_j + \frac{2}{3}k) \\
B_6 &= (x_i - \frac{1}{3}h, y_j + \frac{1}{3}k)
\end{aligned}$$

$$\Downarrow$$

$$\sum B = -\frac{hk}{18}\left[r(B_1) + r(B_2) + r(B_3) + r(B_4) + r(B_5) + r(B_6)\right].$$

$$A_{i+(j-1)m,i+(j-1)m} = \frac{2(h^2+k^2)}{hk} - \frac{hk}{18}[r(B_1)+r(B_2)+r(B_3)+r(B_4)+r(B_5)+r($$

$$\Downarrow \text{similarly}$$

$$\begin{aligned} A_{i+(j-1)m,i-1+(j-1)m} &= -\frac{k}{h} - \frac{hk}{18}[r(B_6)+r(B_1)] \\ A_{i+(j-1)m,i-1+(j-2)m} &= -\frac{hk}{18}[r(B_1)+r(B_2)] \\ A_{i+(j-1)m,i+(j-2)m} &= -\frac{h}{k} - \frac{hk}{18}[r(B_2)+r(B_3)] \\ A_{i+(j-1)m,i+1+(j-1)m} &= -\frac{h}{k} - \frac{hk}{18}[r(B_3)+r(B_4)] \\ A_{i+(j-1)m,i+1+jm} &= -\frac{hk}{18}[r(B_4)+r(B_5)] \\ A_{i+(j-1)m,i+jm} &= -\frac{hk}{18}[r(B_5)+r(B_6)] \end{aligned}$$

for b, use lemma 10 (f).

$$b_{i+(j-1)m} = -\frac{hk}{6}[f(B_1)+f(B_2)+f(B_3)+f(B_4)+f(B_5)+f(B_6)].$$

for FEM functions on boundary, $\phi_{i+(j-1)m}$ does not belong to $L_0^2(R)$ equations

$$\begin{aligned} A_{i+(j-1)m,i+(j-1)m} &= 1 \\ b_{i+(j-1)m} &= g(x_i, y_j) \end{aligned}$$

will guarantee dirichlet boundary condition $v_{i+(j-1)m} = g(x_i, y_j)$ where (x_i, y_j) is a boundary point.

- example 10

apply FEM with $M = N = 4$ to laplace quation $\Delta u = 0$ on $[0,1] \times [1,2]$ with dirichley boundary conditions

$$\begin{cases} u(x,1) = ln(x^2+1) \\ u(x,2) = ln(x^2+4) \\ u(0,y) = 2lny \\ u(1,y) = ln(y^2+1). \end{cases}$$

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example 11

apply FEM $M = N = 16$ to approximate elliptic dirichlet problem

$$\begin{cases} \Delta u + 4\pi^2 u = 2sin2\pi y \\ u(x,0) = 0, \quad 9 \le x \le 1 \\ u(x,1) = 0, \quad 9 \le x \le 1 \\ u(0,y) = 0, \quad 9 \le y \le 1 \\ u(1,y) = sin2\pi y, \quad 9 \le y \le 1. \end{cases}$$

define $r(x,y) = 4\pi^2$, $f(x,y) = 2sin2\pi y$. actual solution $u(x,y) = x^2 sin2\pi y$.

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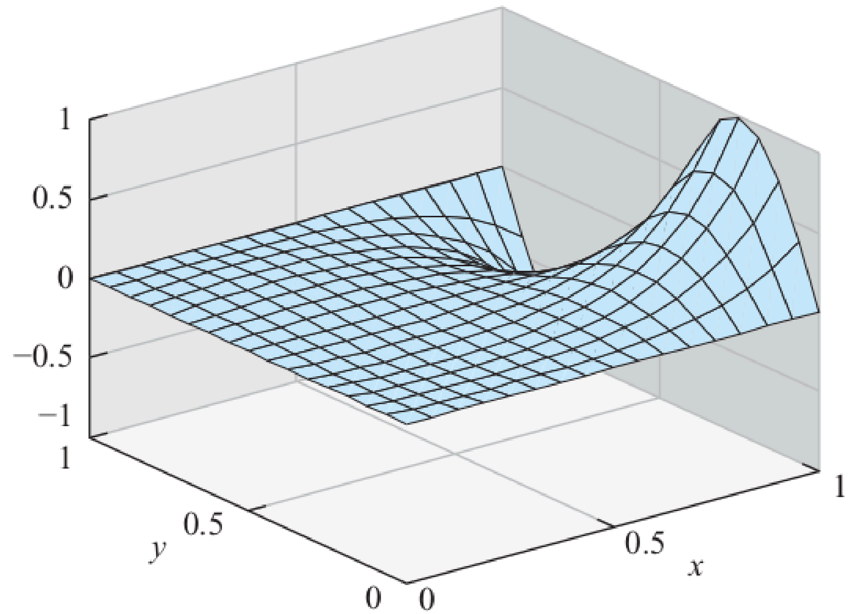

Figure 8.18 Finite element solution of Example 8.11. Maximum error on $[0, 1] \times [0, 1]$ is