nmi | spring 2024

lecture 20 : hyperbolic

8.2 hyperbolic equations

hyperbolic equations put less stringent constraints on explicit methods. consider the wave equation.

$$\text{the wave equation} \quad \begin{cases} u_{tt} = c^2 u_{xx} & a \le x \le b, t \ge 0 \\ u(x,0) = f(x) & a \le x \le b \\ u_t(x,0) = g(x) & a \le x \le b \\ u(a,t) = l(t) & t \ge 0 \\ u(b,t) = r(t) & t \ge 0 \end{cases}.$$

the wave equation describes the propagation of a wave along x with velocity c. if the wave in question is the oscillation of a violin string, u is displacement; if the wave is sound, u is the local air pressure.

for this higher-order derivative, initial velocity $g(x)$ is required in addition to initial shape $f(x)$.

to apply FDM to the hyperbolic wave equation, back to the grid $x_i = a + ih, t_j = jk$ for step sizes h, k where w_{ij} approximates solution $u(x_i, t_j)$.

to discretize the wave equation, the second partials are replaced by the centered-difference formula in both x, t directions.

$$\frac{w_{i,j+1} - 2w_{ij} + w_{i,j-1}}{k^2} - c^2 \frac{w_{i-1,j} - 2w_{ij} + w_{i+1,j}}{h^2} = 0.$$

set $\sigma = \frac{ck}{h}$, we can solve for the solution at the next time step and write the discretized equation as

$$w_{i,j+1} = (2-2\sigma^2)w_{ij} + \sigma^2 w_{i-1,j} + \sigma^2 w_{i+1,j} - w_{i,j-1}.$$

bc you need $j, j-1$ use three-point centered difference to approximate the first time derivative of solution u.

$$u_t(x_i, t_j) \approx \frac{w_{i,j+1} - w_{i,j-1}}{2k}$$

$$\Downarrow$$

$$g(x_i) = u_t(x_i, t_0) \approx \frac{w_{i1} - w_{i,-1}}{2k} \Rightarrow w_{i,-1} \approx w_{i1} - 2kg(x_i)$$

$$\Downarrow$$

$$w_{i1} = (2-2\sigma^2)w_{i0} + \sigma^2 w_{i-1,0} + \sigma^2 w_{i+1,0} - w_{i1} + 2kg(x_i)$$

$$\Downarrow \quad \text{algebra!}$$

$$w_{i1} = (1-\sigma^2)w_{i0} + kg(x_i) + \frac{\sigma^2}{2}(w_{i-1,0} + w_{i+1,0})$$

$$\Downarrow$$

$$A = \begin{bmatrix} 2-2\sigma^2 & \sigma^2 & 0 & \cdots & 0 \\ \sigma^2 & 2-2\sigma^2 & \sigma^2 & \ddots & \vdots \\ 0 & \sigma^2 & 2-2\sigma^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \sigma^2 \\ 0 & \cdots & 0 & \sigma^2 & 2-2\sigma^2 \end{bmatrix}$$

with initial and subsequent steps

$$\begin{bmatrix} w_{11} \\ \vdots \\ w_{m1} \end{bmatrix} = \frac{1}{2}A \begin{bmatrix} w_{10} \\ \vdots \\ w_{m0} \end{bmatrix} + k \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_m) \end{bmatrix} + \frac{1}{2}\sigma^2 \begin{bmatrix} w_{00} \\ 0 \\ \vdots \\ 0 \\ w_{m+1,0} \end{bmatrix}$$

$$\begin{bmatrix} w_{1,j+1} \\ \vdots \\ w_{m,j+1} \end{bmatrix} = \frac{1}{2}A \begin{bmatrix} w_{1j} \\ \vdots \\ w_{mj} \end{bmatrix} - \begin{bmatrix} w_{1,j-1} \\ \vdots \\ w_{m,j-1} \end{bmatrix} + \frac{1}{2}\sigma^2 \begin{bmatrix} w_{0j} \\ 0 \\ \vdots \\ 0 \\ w_{m+1,j} \end{bmatrix}$$

and their final form

$$\begin{bmatrix} w_{11} \\ \vdots \\ w_{m1} \end{bmatrix} = \frac{1}{2}A \begin{bmatrix} f(x_1) \\ \vdots \\ f(x_m) \end{bmatrix} + k \begin{bmatrix} g(x_1) \\ \vdots \\ g(x_m) \end{bmatrix} + \frac{1}{2}\sigma^2 \begin{bmatrix} l(t_0) \\ 0 \\ \vdots \\ 0 \\ r(t_0) \end{bmatrix}$$

$$\begin{bmatrix} w_{1,j+1} \\ \vdots \\ w_{m,j+1} \end{bmatrix} = \frac{1}{2}A \begin{bmatrix} w_{1j} \\ \vdots \\ w_{mj} \end{bmatrix} - \begin{bmatrix} w_{1,j-1} \\ \vdots \\ w_{m,j-1} \end{bmatrix} + \frac{1}{2}\sigma^2 \begin{bmatrix} l(t_j) \\ 0 \\ \vdots \\ 0 \\ r(t_j) \end{bmatrix}.$$

example 06

apply FDM to wave equation with $c = 2, f(x) = sin\pi x, g(x) = l(t) = r(t) = 0$.

code, matlab

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> code, python

[ ] ↳ *1 cell hidden*

usw

however if k is too large relative to h, that plot goes to hell.

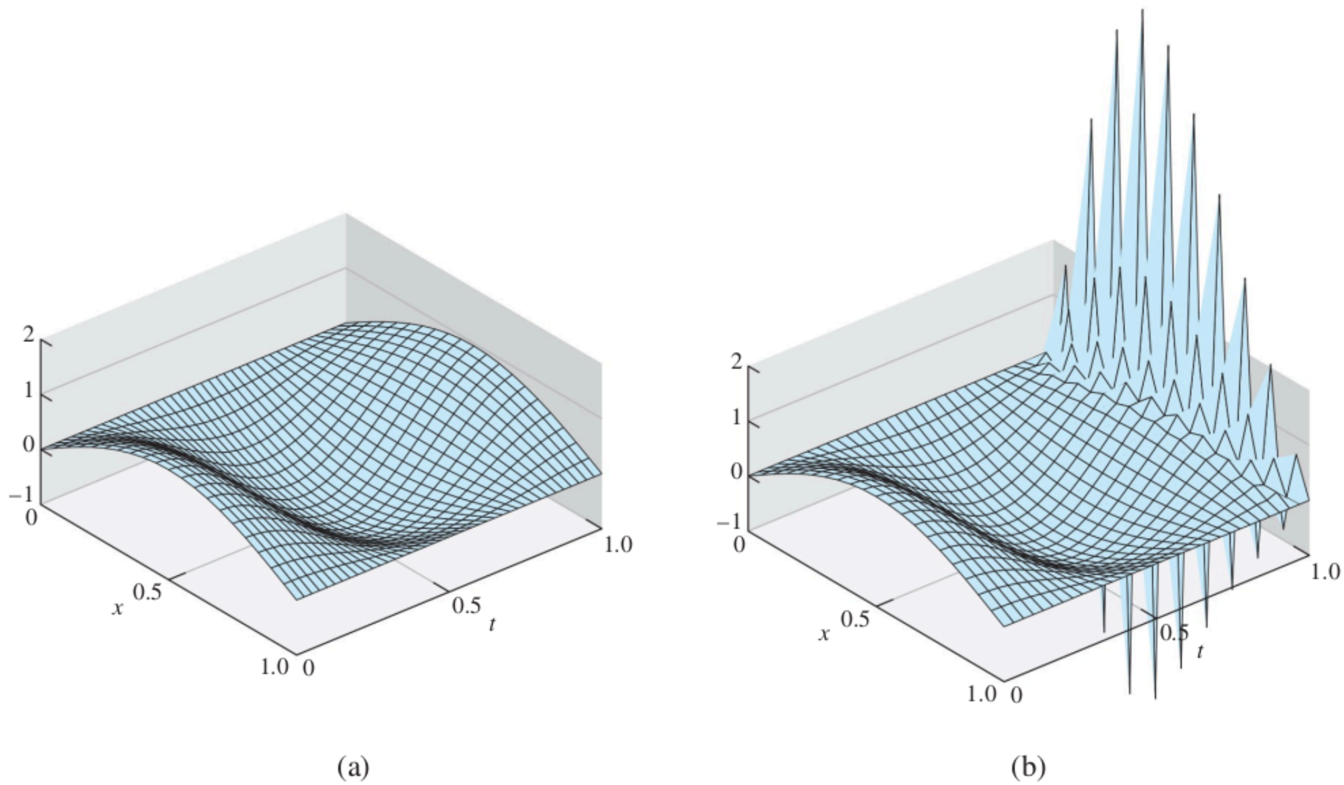

Figure 8.10 Wave Equation in Example 8.6 approximated by explicit Finite Difference Method. Space step size is $h = 0.05$. (a) Method is stable for time step $k = 0.025$, (b) unstable for $k = 0.032$.

8.1.2 CLF

theorem 05

FDM applied to the wave equation with speed $c > 0$ is stable if $\sigma = \frac{ck}{h} \leq 1$.

> proof

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usw

$\frac{ck}{h}$ is the **CFL number** of the method, named after courant, friedrichs, lewy (1928). in general CFL must be at most 1 for solver to be stable. that means if c is wave speed, the distance traveled ck should not exceed the space step h. otherwise bad graphs happen. the constraint $ck \leq h$ is the **CFL condition** for the wave equation.