nmi | spring 2024

lecture 18 : finite difference methods

7.2 finite difference methods

finite difference methods (FDM) replace derivatives with simpler, discrete approximations.

let $y(t)$ be a function with at least four continuous derivatives.

$$y'(t) = \frac{y(t+h) - y(t-h)}{2h} - \frac{h^2}{3!}y'''(c)$$
$$y''(t) = \frac{y(t+h) - 2y(t) + y(t-h)}{h^2} - \frac{h^2}{2 \cdot 3!}y^{(iv)}(c).$$

oc both first and second derivitives have error proportional to h^2.

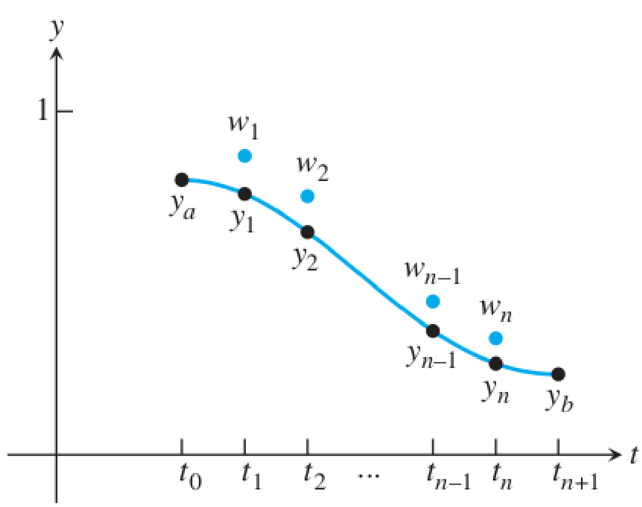

Figure 7.6 The Finite Difference Method for BVPs. Approximations $w_i, i = 1, \ldots, n$ for the correct values y_i at discrete points t_i are calculated by solving a linear system of equations.

if linear BVP, then solve with gauss elimination or iterative, else an algebraic system of nonlinear equations.

7.2.1 linear BVP

example 08

example 08 revisits example 07 using finite differences.

$$\text{example 08} \quad \begin{cases} y'' = 4y \\ y(0) = 1 \\ y(1) = 3. \end{cases}$$

consider the discrete form of differential equation $y'' = 4y$ using the centered-difference form for the second derivative. ie,

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - 4w_i = 0$$

$$\Downarrow$$

$$w_{i+1} + (-4h^2 - 2)w_i + w_{i-1} = 0.$$

for $n = 3$ points between interval end points, then step-size $h = \frac{1}{n+1} = \frac{1}{3+1} = \frac{1}{4}$. including boundary conditions gives three equations in three unknowns:

$$\begin{aligned} 1 + (-4h^2 - 2)w_1 + w_2 &= 0 \quad y(0) = 1 \\ w_1 + (-4h^2 - 2)w_2 + w_3 &= 0 \\ w_2 + (-4h^2 - 2)w_3 + 3 &= 0 \quad y(1) = 3. \end{aligned}$$

$\Downarrow \quad h = \frac{1}{4}$

$$\begin{bmatrix} -\frac{9}{4} & 1 & 0 \\ 1 & -\frac{9}{4} & 1 \\ 0 & 1 & -\frac{9}{4} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}.$$

$\Downarrow$ by gaussian elimination,

$$w = [1.0249, 1.3061, 1.9138]^T.$$

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> example 08, part 2: error

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usw

for any fixed t and step-size h, the FDM approximation $w_h(t)$ is order two in h and can be extrapolated with a simple formula.

7.2.2 nonlinear BVP

FDM applied to nonlinear DEQ results in a system of nonlinear algebra for multivariate newtons method (ch 2).

example 09

example 09 looks fresh. solve nonlinear BVP

$$\text{example 09} \quad \begin{cases} y'' = y - y^2 \\ y(0) = 1 \\ y(1) = 4. \end{cases}$$

the discretized form at t_i

$$\frac{w_{i+1} - 2w_i + w_{i-1}}{h^2} - w_i + w_i^2 = 0$$

$$\Downarrow$$

$$w_{i-1} - (2 + h^2)w_i + h^2 w_i + w_{i+1} = 0.$$

for $2 \leq i \leq n - 1$ and the boundary conditions:

$$\begin{aligned} y_a - (2 + h^2)w_1 + h^2 w_1^2 + w_2 &= 0 \\ w_{n-1} - (2 + h^2)w_n + h^2 w_n^2 + y_b &= 0. \end{aligned}$$

$\Downarrow$ solve $F(w) = 0$ with newtons multivariate,

$$w^{k+1} = w^k - DF(w^k)^{-1} F w^k$$

$$\Downarrow$$

$$DF(w^k)\Delta w = -F(w^k), \quad \Delta w = w^{k+1} - w^k$$
$$\Downarrow$$

$$F\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{n-1} \\ w_n \end{bmatrix} = \begin{bmatrix} y_a - (2+h^2)w_1 + h^2 w_1^2 + w_2 \\ w_1 - (2+h^2)w_2 + h^2 w_2^2 + w_3 \\ \vdots \\ w_{n-2} - (2+h^2)w_{n-1} + h^2 w_{n-1}^2 + w_n \\ w_{n-1} - (2+h^2)w_n + h^2 w_n^2 + y_b \end{bmatrix}$$

where $y_a = 1$, $y_b = 4$. the jacobian $DF(w)$ of F is

$$\begin{bmatrix} 2h^2 w_1 - (2+h^2) & 1 & 0 & \cdots & 0 \\ 1 & 2h^2 w_2 - (2+h^2) & \ddots & \ddots & \vdots \\ 0 & 1 & \ddots & 1 & 0 \\ \vdots & \ddots & \ddots & 2h^2 w_{n-1} - (2+h^2) & 1 \\ 0 & \cdots & 0 & 1 & 2h^2 w_n - (2+h^2) \end{bmatrix}$$

the ith row is the partial of the ith component of F with respect to each w_j.

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> code, python

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> example 10

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