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MULTIPLE PERIOD OPTIMIZATION OF BUS TRANSIT SYSTEMS

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Abstract—Analytic models are developed for optimizing bus services with time dependence and elasticity in their demand characteristics. Some supply parameters, i.e. vehicle operating costs and speeds are also allowed to vary over time. The multiple period models presented here allow some of the optimized system characteristics (e.g. route structure) to be fixed at values representing the best compromise over different time periods, while other characteristics (e.g. service headways) may be optimized within each period. In a numerical example the demand is assumed to fluctuate over a daily cycle (e.g. peak, offpeak and night), although the same models can also be used for other cyclical or noncyclical demand variations over any number of periods. Models are formulated and compared for four types of conditions, which include steady fixed demand, cyclical fixed demand, steady equilibrium demand and cyclical equilibrium demand. When fixed demand is assumed, the optimization objective is minimum total system cost, including operator cost and user cost, while operator profit and social welfare are the objective functions maximized for equilibrium demand. The major results consist of closed form solutions for the route spacings, headways, fares and costs for optimized feeder bus services under various demand conditions. A comparison of the optimization results for the four cases is also presented. When demand and bus operating characteristics are allowed to vary over time, the optimal functions are quite similar to those for steady demand and supply conditions. The optimality of a constant ratio between the headway and route spacing, which is found at all demand densities if demand is steady, is also maintained with a multi-period adjustment factor in cyclical demand cases, either exactly or with a relatively negligible approximation. These models may be used to analyze and optimize fairly complex feeder or radial bus systems whose demand and supply characteristics may vary arbitrarily over time.

1. INTRODUCTION

A considerable amount of literature exists on analytic approaches for optimizing public transportation systems. Most studies have assumed a fixed demand (i.e. insensitive to service quality or price) and minimized a total cost function consisting of the sum of operator cost and user cost (see, for examples, Vuchic and Newell, 1968; Byrne, 1976; Wirasinghe, 1980; Tsao and Schonfeld, 1984; Kuah and Perl, 1988). An assumption of zero demand elasticity may be reasonable for systems on which ridership is mostly captive or fixed. With this assumption, a model might be sufficiently simplified to be solved analytically, but its applicability may be limited. For instance, a model without demand elasticity cannot properly address fare policy or optimize systems for profitability or objectives that include consumer surplus.

Kocur and Hendrickson (1982) have analyzed bus services with ridership sensitive to bus service levels and fares, and developed closed-form solutions for the optimal route spacing, headway and fare for different objective functions. Using a linearized approximation of a logit mode share model, that work has demonstrated the tractability of an optimization approach to transit system design with demand elasticity. An analytic model considering demand elasticity, financial constraints and the effect of fleet size on congestion, has been developed by Oldfield and Bly (1988) to determine the optimal vehicle size for an urban bus service. The studies mentioned above did not consider time-dependent demand.

Treating demand as a smooth function of time with one or more peaks, Newell (1971) optimized the schedule for a transit route with a given fixed fleet size by minimizing user wait time. To extend that work, Salzborn (1972) used a two-stage process, including

a primary objective of minimizing fleet size and then a secondary objective of minimizing passenger wait time, to determine the optimal headway as a function of time. Hurdle (1973) treated demand as a smooth function of time and space to optimize the route spacing and headway for parallel feeder bus routes by minimizing total system cost. The optimized headways varied over time while the optimized route spacing could either be fixed or variable over time. Clarens and Hurdle (1975) have considered time-dependent demand in analyzing a commuter bus system without demand elasticity; however, analytic solutions were obtained only when the system parameters (e.g. operating cost) were assumed to be independent of time.

The multiple period models presented here rely on discrete distributions for demand over time rather than the smooth distributions used in some previous analytic optimization models such as Newell (1971) and Hurdle (1973). Such discrete distributions are easier to obtain from empirical data than fitted smooth distributions, and can yield twice differentiable objective functions that can be optimized analytically. Discrete distributions are also used in this paper to account for the time dependence of certain supply variables, i.e. bus speeds and operating costs. Recent extensions of these models with location-dependent variables, many-to-many demand and financial constraints have also used discrete distributions (Chang and Schonfeld, 1989c; Chang, 1990). The global optimality of the closed-form solutions for such models is examined in Chang (1990).

It should be noted that many studies have used numerical rather than analytic methods to optimize public transportation systems (for example, Boyd, Asher and Wetzler, 1973; Keeler et al., 1975; Nihan and Morlok, 1976; Schonfeld, 1977; Furth and Wilson, 1981; Viton, 1980; Viton, 1982; Morlok and Viton, 1984). However, closed-form analytic solutions were sought in this study in order to identify fundamental optimality relations.

In this paper, analytic models with various assumptions about demand elasticity and time variability in demand and supply characteristics are developed for a feeder bus system, and closed-form analytic solutions for the optimal system, including headway, fare, fleet size, route spacing, operator cost and user cost, are presented and compared for different objectives. A general optimality relation indicating a constant ratio between headway and route spacing, which has been discussed by Hurdle (1973), Schonfeld (1981), Kocur and Hendrickson (1982), and Tsao and Schonfeld (1984), is also assessed for the bus systems considered here.

2. ASSUMPTIONS

The models developed in this paper are based on the same geometric configuration assumptions, but differ in their demand and supply assumptions, as discussed below. They may be used to analyze a wide variety of bus service types, including feeder services to and from transfer stations, zone structure services and radial services to activity centers.

2.1. Bus system characteristics

In this analysis, a branched zone bus system is assumed to provide service for a rectangular area with dimensions $L \times W$, from which trip ends are assumed to access a single point, such as a mass transit station or activity center. The same model formulation may be used for one-directional demand (many-to-one or one-to-many) or for a two-directional demand combination. Figure 1 illustrates this bus system. The variables and their typical values are defined in Table 1.

The service area has N zones, each of length L and width $r = W/N$. A vehicle round trip to Zone j during period t consists of: (1) a line haul distance J traveled at express speed yV_t from the starting point to a corner of rectangular service area;(2) a distance of W_j miles traveled at local nonstop speed bV_t from the corner to the assigned zone; (3) a collection route L miles traveled at local speed V_t along the middle of the zone stopping for passengers every d miles; and (4) a retracing in reverse order of the first three stages. It is also assumed that bus layover time and externalities are negligible, although certain layover and externality effects may be added fairly simply to these models.

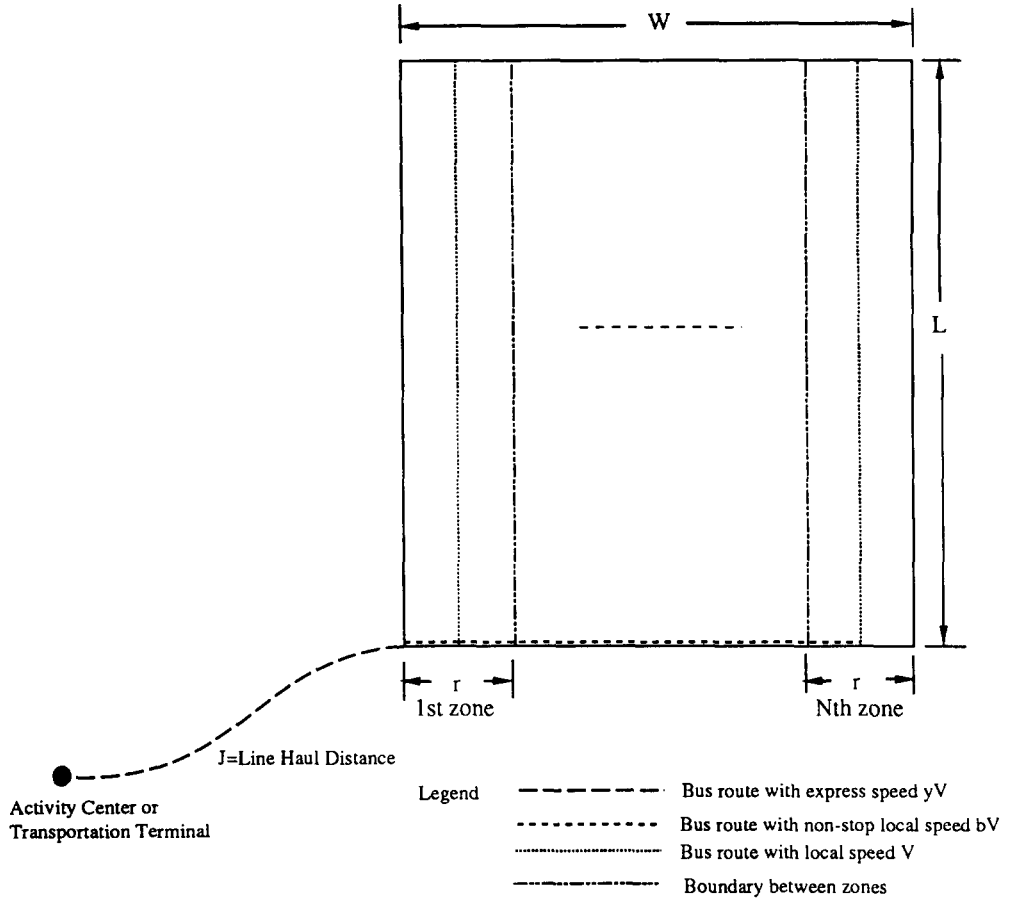

Fig. 1. Bus system configuration.

Passengers are assumed to walk at speed g between their trip ends and the nearest bus stop along a rectangular street network (parallel and perpendicular to the feeder route) with negligible street spacing. This assumption implies an access distance of $(r + d)/4$ and an access time of $(r + d)/4g$.

The geometric structure assumed in this model and illustrated in Fig. 1 may be used to analyze feeder bus services in urban areas with nonuniform geographic characteristics. Such areas may be subdivided into approximately rectangular areas that are fairly homogeneous internally and may differ considerably from each other in size, shape and other characteristics. The optimal route length and zone shape may be derived as closed-form functions of the demand density and line haul distance from the activity center to each specific zone (Chang, 1990). Figure 2 illustrates qualitatively how an urban area might be partitioned into 15 service areas, based on demand density, line haul distance and other local characteristics. In Fig. 2 several zones (i.e. *F, G, I* and *J*) have zero line haul distance from the activity center. Narrow zones, such as *I, K* and *L,* might be served by only one bus route.

These models may also be used in some cases to analyze bus systems with many-to-many demand patterns if the system can be separated into subsystems in which many-through-one analysis is applicable (as in the bus network discussed by Newell, 1979). It can also be shown by inspection of the derived functions for optimized system characteristics, or by numerical sensitivity analysis, that relatively small deviations from the optimal values of decision variables do not significantly affect the objective function values, because the objective functions are relatively flat in the vicinity of the optima. Thus, relatively small changes in the optimal route spacing to provide an integer number of routes or to satisfy constraints imposed by an existing street network would produce far less than proportional (and practically negligible) deviations from optimal total cost, profits or net benefits.

Table 1. Variable definitions

Variable	Definition	Typical Value
A	avg. travel cost insensitive to design variables = xd/4g + vM	-
B	bus operating cost (\$/veh hr)	32.5
B_t	bus operating cost in period t B_1, B_2, and B_3 = 50, 25, and 25 dollars/veh hr, respectively.	-
b	non-stop ratio = non-stop speed/local speed	2
C	system total cost (\$/period)	-
C_o	total operator cost (\$/period)	-
C_u	total user cost (\$/period)	-
c	system average cost (\$/trip)	-
D_t	bus avg. round trip time during period t (hrs) $=2L/V_t+W/bV_t+2J/yV_t$	
d	bus-stop spacing (miles)	0.25
E_t	bus load in period t (passengers/veh)	-
e_p	demand elasticity parameter for fare	0.07
e_v	demand elasticity parameter for in-vehicle time	0.35
e_w	demand elasticity parameter for wait time	0.7
e_x	demand elasticity parameter for access time	0.7
F	fleet size (vehicles)	-
F_t	fleet size in period t (vehicles)	-
f	fare (\$)	-
G	consumer surplus (\$/period)	-
g	average walk speed (miles/hr)	2.5
h	avg. headway over all periods (hrs/veh)	-
h_t	headway in period t (hrs/veh)	-
j	zone index	-
J	line haul distance (miles)	4.0
k_t	invariant components of the demand function $=1-e_xd/4g-e_vM_t$	-
L	length of service area (miles)	3.0
M	passenger avg. trip time over all periods (hrs)	-
M_t	passenger avg. trip time during period t (hrs) $=L/2V_t+W/2bV_t+J/yV_t$	
m	number of time periods	3
N	number of zones	-
P	profit (\$/period)	-
Q	demand function for steady equilibrium case (Eq. 1a)	-
Q_t	demand function for period t (Eq. 1b)	-
Q_T	total daily ridership (passengers/day) $=TLWq$ or $LW\Sigma_tq_tT_t$	-
q	fixed or potential demand density (trips/sq. mile/hr)	67.8
q_t	fixed or potential demand density in period t q_1, q_2, and q_3 = 150, 60, and 12 trips/sq. mile/hr, respectively.	-
R	total revenue (\$/period)	-
r	route spacing (miles)	-
S	vehicle capacity (seats/veh)	50
T	total service hours (hrs/day or hrs/analysis period)	10
T_i	service time in period i (hrs) T_1, T_2, and T_3 = 3, 3, 4 hours, respectively.	-
u	load factor at the peak load point (passengers/seat)	1.0
V_t	local speed during period t (miles/hr)	15.0
v	value of in-vehicle time (\$/passenger hr)	5.0
W	width of service area (miles)	2.0
w	value of wait time (\$/passenger hr)	10.0
X	composite variable $= \Sigma_tT_t(D_tB_tq_t)^{1/2}/\Sigma_tT_tq_t$	-
x	value of access time (\$/passenger hr)	10.0
Y	social welfare (\$/period) $=G+R-C_o$	-
y	express ratio = express speed/local speed	2
Z_t	approximation term defined in Eq. 61	-
z	ratio of wait time/headway	0.5

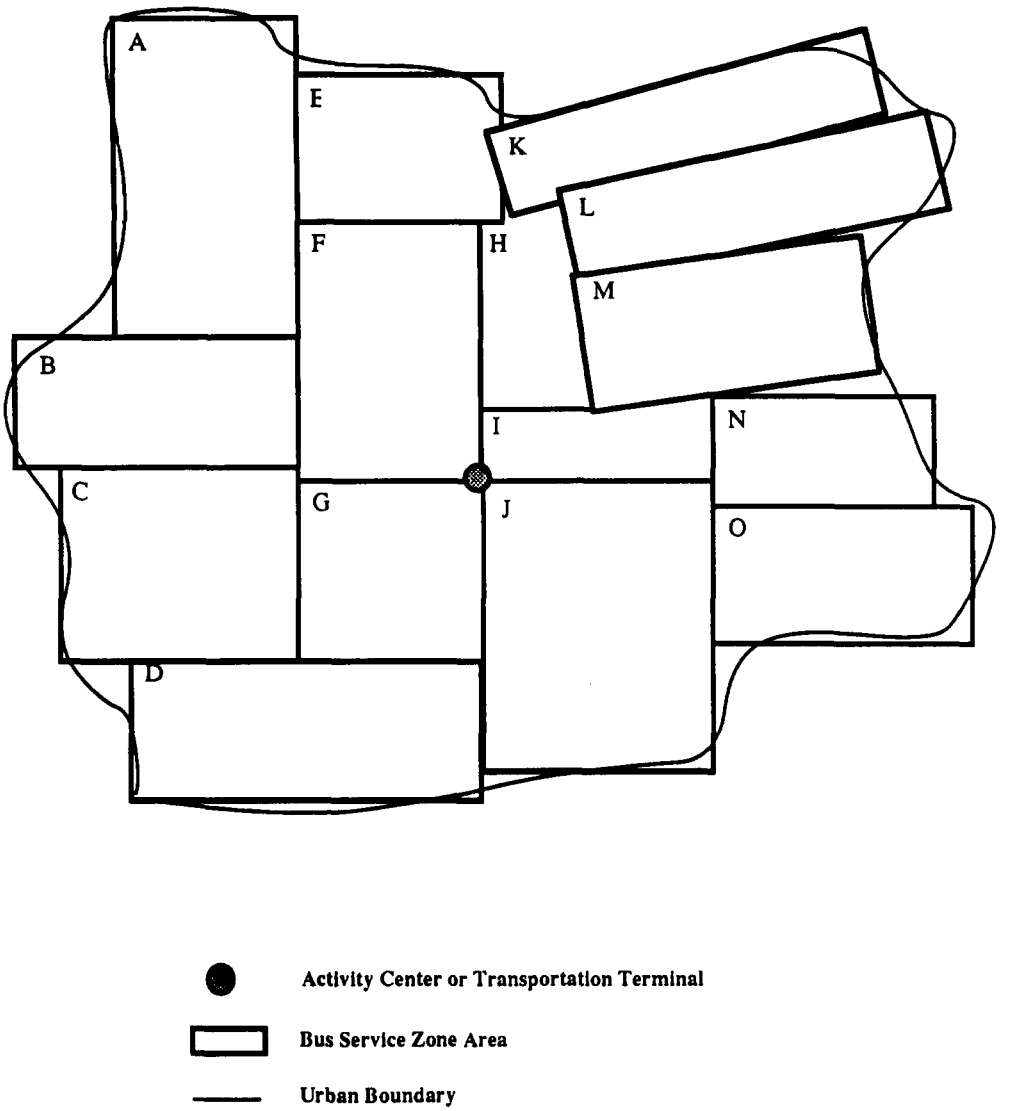

Fig. 2. Partition of urban area into bus service areas.

2.2. *Demand functions*

For comparison, four trip demand conditions are considered in this analysis:

Case 1 – steady fixed demand q,
Case 2 – cyclical fixed demand q_t,
Case 3 – steady equilibrium demand Q,
Case 4 – cyclical equilibrium demand Q_t.

In Case 1, q is a fixed demand density, comparable to that used in most past studies. Here q_t is the fixed demand density during period t in Case 2, and could be determined from the time distribution of demand shown in Fig. 3. It is shown that a step demand distribution (Fig. 3b) relating monotonically volume levels and their durations can be directly obtained from the empirical demand distribution (Fig. 3a). Although only three periods are used as an example of the demand distribution in Fig. 3e, the number and duration of periods is unlimited in the following models derived from Cases 2 and 4, and may be selected to represent variations over time with whatever precision is desired (and supported by empirical data). Other models (Newell, 1971; Hurdle, 1973) have used smoothed functions (e.g. Fig. 3c) to represent demand variation over time. However,

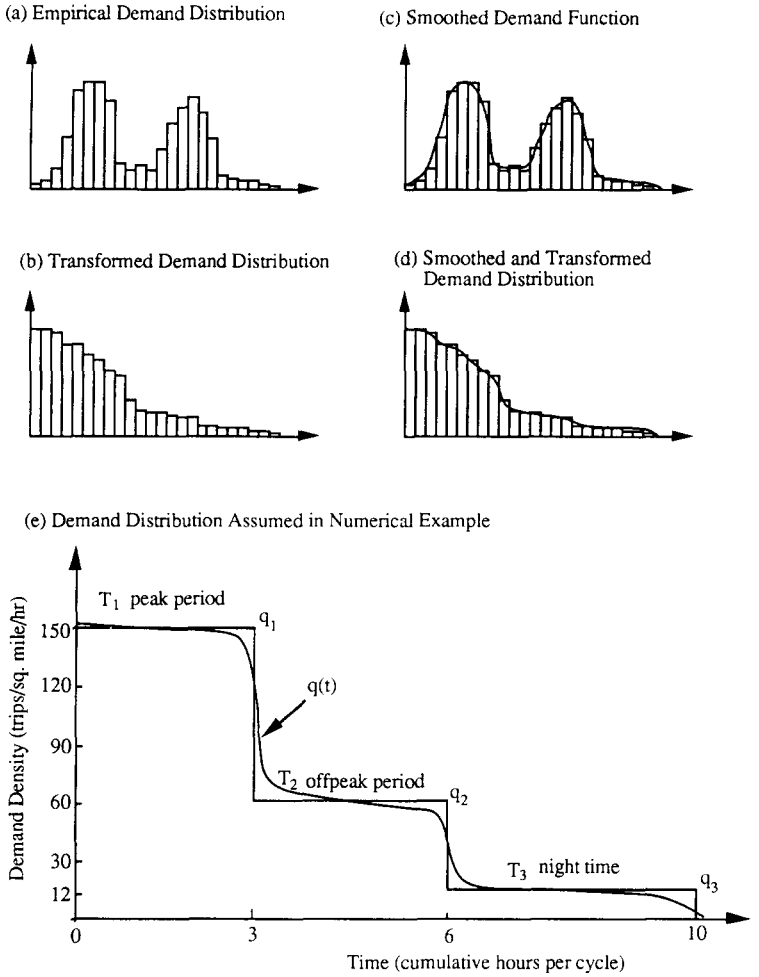

Fig. 3. Distribution of demand assumed in the analysis.

such smoothing is not necessary to formulate objective functions that are twice differentiable, and hence appears to be an unnecessary complication. The approach used in this work relies on step functions for demand, costs, speeds and, potentially, other variables, that are obtained directly from the empirical data.

In the equilibrium demand cases, a linear demand function which is sensitive to various travel time components and fare is formulated as follows:
Case 3:

$$Q = q(1 - e_w zh - e_x(r + d)/4g - e_v M - e_p f). \tag{1a}$$

Case 4:

$$Q_t = q_t(1 - e_w zh_t - e_x(r + d)/4g - e_v M_t - e_p f). \tag{1b}$$

The demand models include the following variables and parameters: potential demand density of the transit system $= q_t$ (or q); wait time $= zh_t$ (or zh), which is assumed to be a constant factor (usually $z = 0.5$) multiplied by the headway; access time $= (r + d)/4g$; in-vehicle travel time $M_t = (L + W/b + 2J/y)/2V_t$, which is assumed to change over time for cyclical demand cases due to different traffic conditions represented by different speed V_t in each period; and fare $= f$. The optimizable decision variables in Case 4 are the headway for each time period h_t, the route spacing r and the fare f. This implies that in Case 4 with equilibrium time-dependent demand, the optimized bus route

structure and fare are assumed to be the same in all periods, while the headways are separately optimized for each period. It should be noted that the elasticity factors e_w, e_x, e_v and e_p for wait time, access time, in-vehicle time, and fare, respectively, are not direct measures of elasticities in such a linear model. However, the ratio e_w/e_p determines the implied value of wait time (w), the ratio e_x/e_p determines the implied value of access time (x) and the ratio e_v/e_p determines the value of in-vehicle time (v). From the parameter values assumed in Table 1, the implied values of time are \$10 per passenger hour for access time and wait time, and \$5 per passenger hour for in-vehicle time.

It should be noted that the linear demand function has been extensively used in demand models (Oum, 1989) because it is relatively simple to manipulate (as in Kocur and Hendrickson, 1982). Alternative demand functions such as exponential types yield models which are far more difficult to solve in closed form. Alternative demand functions may be approximated in the range of interest by the linear demand functions used here.

The same models may be used for many-to-one, one-to-many or two-directional demand combinations by defining q, q_t, Q or Q_t as the sum of trips in both directions. Thus, in the multiple period models, the directional mix can vary in each period. In all cases the demand is assumed to be deterministic and uniformly distributed over time during each specified period. (However, the periods may be arbitrarily short.) The demand is also assumed to be uniformly distributed over space within each specified service area. The latter assumption has been relaxed in subsequent work (Chang, 1990).

2.3. Operator costs

The operator costs per analysis period (e.g. per day) consist of fleet size F, which is the bus round trip time D divided by the headway h, multiplied by the hourly operating cost and total daily service time. The bus round trip time D is assumed to be constant in the single period cases (1 and 3), while different round trip times D_t are assumed for different periods in Cases 2 and 4 due to different traffic conditions represented by different speeds:

Single period cases 1 and 3.

$$D = \frac{2L}{V} + \frac{W}{bV} + \frac{2J}{yV}. \tag{2a}$$

Multiple period cases 2 and 4.

$$D_t = \frac{2L}{V_t} + \frac{W}{bV_t} + \frac{2J}{yV_t}. \tag{2b}$$

The hourly operating cost is assumed to be a constant B in Cases 1 and 3, while different bus operating costs B_t are assumed for different periods t in the Cases 2 and 4. Therefore the operator costs per day (or for any other analysis period) are as follows:

Single period cases 1 and 3.

$$C_o = FBT = (WD/rh)BT \tag{3a}$$

Multiple Period Cases 2 and 4.

$$C_o = \sum_{t=1}^{m} F_t B_t T_t = \sum_{t=1}^{m} \frac{WD_t B_t T_t}{rh_t}. \tag{3b}$$

It should be noted that for time-dependent conditions, i.e. Cases 2 and 4, we assume that the bus operating costs B_t for each period are given, which presupposes that an acceptable cost allocation method is available to determine the operating cost for each period and that these costs B_t are independent of the variables optimized in this analysis.

3. OBJECTIVE FUNCTIONS

The alternative objective functions used in this analysis are (1) cost minimization, (2) profit maximization, and (3) social welfare maximization. Objectives (2) and (3) are generally, though not universally, accepted as the most appropriate ones for private operators and public agencies, respectively (Viton, 1982; Wohl and Hendrickson, 1984). Objective (1) is acceptable when demand elasticity is negligible. Hence, in this paper total cost minimization is used in fixed demand cases, while the other two objectives are used for equilibrium demand.

3.1. Total system cost

When demand is fixed, the objective is to minimize system total cost C, including operator cost C_o and user cost C_u.

$$C = C_o + C_u. \tag{4}$$

This objective function is very common in analytic studies of system optimization with fixed demand. The operator cost is shown in eqns (3a) and (3b) for the four cases. The user cost consists of access cost C_x, wait cost C_w and in-vehicle cost C_v, which can be formulated as follows:

Case 1:

$$C_x = TLWqx(r + d)/4g, \tag{5a}$$

$$C_w = TLWqwzh, \tag{6a}$$

$$C_v = TLWqvM. \tag{7a}$$

Case 2:

$$C_x = \sum_{t=1}^{m} LWq_tT_tx\left(\frac{r + d}{4g}\right), \tag{5b}$$

$$C_w = \sum_{t=1}^{m} LWq_tT_twzh_t, \tag{6b}$$

$$C_v = \sum_{t=1}^{m} LWq_tT_tvM_t. \tag{7b}$$

In eqns. (5–7) above the access cost is average access distance $(r + d)/4$ divided by access speed g and multiplied by the total demand TLW_q and the value of access time x, the wait cost is the average wait time zh(or zh_t) multiplied by the total demand and the value of wait time w, and the in-vehicle cost is the average in-vehicle travel time $M = L/2V + W/2bV + J/yV$ multiplied by the total demand and the value of in-vehicle time v. Therefore, the system total costs [eqn (4)] can be formulated as summations of eqns (5a), (6a) and (7a) for Case 1 and eqns (5b), (6b) and (7b) for Case 2:

Case 1:

$$C = \frac{WDBT}{rh} + TLWqx\left(\frac{r + d}{4g}\right) + TLWqwzh + TLWqvM. \tag{4a}$$

Case 2:

$$C = \frac{W}{r}\sum_{t=1}^{m}\frac{D_tB_tT_t}{h_t} + xLW\left(\frac{r + d}{4g}\right)\sum_{t=1}^{m} q_tT_t$$

$$+ wzLW\sum_{t=1}^{m} h_tq_tT_t + vLW\sum_{t=1}^{m} M_tq_tT_t. \tag{4b}$$

3.2. *Profit and social welfare*

Two objectives, namely operator profit maximization and social welfare maximization, are analyzed here for equilibrium demand conditions. Profit is total revenue R minus operator cost C_o:

$$P = R - C_o. \tag{8}$$

The total revenue is the fare times the total demand Q [from eqn 1a] or Q_t [from eqn (1b)]:
Case 3:

$$R = fTLWQ. \tag{9a}$$

Case 4:

$$R = \sum_{t=1}^{m} fLWQ_tT_t. \tag{9b}$$

Therefore, the profit of eqn (8) for the equilibrium cases can be formulated as follows:
Case 3:

$$P = fTLWq\left(k - e_wzh - e_x\left(\frac{r}{4g}\right) - e_pf\right) - \frac{WDBT}{rh}. \tag{8a}$$

Case 4:

$$P = fLW\sum_{t=1}^{m} T_tq_t\left(k_t - e_wzh_t - e_x\left(\frac{r}{4g}\right) - e_pf\right) - \frac{W}{r}\sum_{t=1}^{m}\frac{D_tB_tT_t}{h_t}. \tag{8b}$$

In eqn (8b), k_t is a constant representing a potential demand component insensitive to optimized variables:

$$k_t = 1 - e_xd/4g - e_vM_t. \tag{10}$$

The social welfare Y (also known as net social benefit) is the consumer surplus G plus the total profit P:

$$Y = G + P. \tag{11}$$

By inverting the demand functions shown in eqns (1a) and (1b) to find fare as a function of demand and by integrating the inverted functions over demand, the total social benefit (also known as the users' willingness to pay) can be obtained. Then, the consumer surplus can be derived as the total social benefit minus the total cost that the users actually pay [as in the theoretical analysis by Wohl and Hendrickson (1984) and application by Kocur and Hendrickson (1982)]:
Case 3:

$$G = \frac{TLWq}{2e_p}\left(k - e_wzh - e_x\left(\frac{r}{4g}\right) - e_pf\right)^2. \tag{12a}$$

Case 4:

$$G = \frac{LW}{2e_p}\sum_{t=1}^{m} T_tq_t\left(k_t - e_wzh_t - e_x\left(\frac{r}{4g}\right) - e_pf\right)^2. \tag{12b}$$

Therefore the social welfare [eqn (11)] for the two cases can be formulated as follows:
Case 3:

$$Y = \frac{TLWq}{2e_p}\left(k - e_w zh - e_x\left(\frac{r}{4g}\right) - e_p f\right)^2$$

$$+ fTLWq\left(k - e_w zh - e_x\left(\frac{r}{4g}\right) - e_p f\right) - \frac{WDBT}{rh}. \tag{11a}$$

Case 4:

$$Y = \frac{LW}{2e_p}\sum_{t=1}^{m} T_t q_t\left(k_t - e_w zh_t - e_x\left(\frac{r}{4g}\right) - e_p f\right)^2 + fLW$$

$$\sum_{t=1}^{m} T_q q_t\left(k_t - e_w zh_t - e_x\left(\frac{r}{4g}\right) - e_p f\right) - \frac{W}{r}\sum_{t=1}^{m}\frac{D_t B_t T_t}{h_t}. \tag{11b}$$

4. OPTIMIZATION RESULTS FOR FIXED DEMAND CASES

For fixed demand conditions (i.e. Cases 1 and 2 where the demand elasticity is zero) the objective functions are eqns (4a) and (4b), respectively. The optimized decision variables are the headway $H(h_t)$ and the route spacing r, which can be determined by setting the partial derivatives of the total cost functions equal to zero and solving them.

4.1. Case 1 – Steady fixed demand

The first-order conditions for an optimum are

$$\frac{\partial C}{\partial r} = -\frac{DWBT}{r^2 h} + \frac{xTLWq}{4g} = 0, \tag{13}$$

$$\frac{\partial C}{\partial h} = -\frac{DWBT}{rh^2} + wzTLWq = 0. \tag{14}$$

Solving eqns (13) and (14) simultaneously, the following optimized route spacing r^* and headway h^* can be obtained:

$$r^* = \left(\frac{16DBzwg^2}{x^2 Lq}\right)^{1/3}, \tag{15}$$

$$h^* = \left(\frac{DBx}{4gLqz^2w^2}\right)^{1/3}. \tag{16}$$

The objective function in this case is strictly convex and the second-order conditions are generally satisfied (Chang, 1990), as they are in other cases analyzed in this paper.

A relation between h^* and r^* can also be obtained from these results:

$$h^* = (xr^*/4gzw). \tag{17}$$

This relation indicates that the headway h and the route spacing r should be kept in constant proportion regardless of demand density in such an optimized system. Equation (17) can also be rewritten as

$$wzh^* = (xr^*/4g), \tag{17a}$$

which indicates that the average wait cost equals the average lateral access cost. Similar relations have been obtained by Hurdle (1973), Schonfeld (1981), Kocur and Hendrickson (1982) and Chang and Schonfeld (1989a) for different types of transit systems. This relation is assessed in this paper for other conditions and under different objective functions, as shown later.

When the optimal headway and route spacing are substituted into eqns (2a), (5a), (6a) and (4a), the optimal access cost C_o^*, wait cost C_w^*, operator cost C_x^* and total system cost C^* can be obtained:

$$C_o^* = Q_T(DBzwx/4gLq)^{1/3}, \tag{18}$$

$$C_w^* = C_o^*, \tag{19}$$

$$C_x^* = C_w^* + Q_T(xd/4g), \tag{20}$$

$$C^* = Q_T\left[3\left(\frac{DBzwx}{4gLq}\right)^{1/3} + \frac{xd}{4g} + vM\right], \tag{21}$$

where Q_T is the total daily demand $TLWq$. It should be noted that the optimized operator cost C_o^*, the user wait cost C_w^* and the lateral component of user access cost are all equal, each of these being $Q_T(BDxwz/4gLq)^{1/3}$. This optimality condition is also assessed later in this paper for other conditions.

Then the optimal average cost c^*, in dollars per trip, is the total system cost divided by the total daily demand:

$$c^* = \frac{C^*}{Q_T} = 3\left(\frac{DBzwx}{4gLq}\right)^{1/3} + \frac{xd}{4g} + vM. \tag{22}$$

The optimal fleet size F^*, i.e. number of vehicles required, can also be derived from eqn (2a):

$$F^* = \frac{C_o^*}{BT} = LWq\left(\frac{Dzwx}{4gLqB^2}\right)^{1/3}. \tag{23}$$

These optimized results can identify the relations among design variables, costs and system parameters. For instance, eqn (23) indicates that fleet size should vary with the ⅓ power of the round trip time D, wait time value(w) and access time value (x), with the $-$⅓ power of access speed g and with the $-$⅔ power of the bus operating cost B.

4.2. Case 2—Cyclical Fixed Demand

Using eqn (4b) as the objective function, the first-order conditions for an optimum in this case are

$$\frac{\partial C}{\partial r} = -\frac{W}{r^2}\sum_{t=1}^{m}\frac{D_tB_tT_t}{h_t} + \frac{xLW}{4g}\sum_{t=1}^{m} q_tT_t = 0, \tag{24}$$

$$\frac{\partial C}{\partial h_t} = -\frac{WD_tB_tT_t}{rh_t^2} + wzLWq_tT_t = 0, \qquad t = 1,2,\ldots,m. \tag{25}$$

By solving eqns (24) and (25), the relation between headway in each period h_t and route spacing r is found to be

$$h_t^* = \left(\frac{D_tB_t}{q_t}\right)^{1/2}\left(\frac{xr^*}{4gzwX}\right), \tag{26}$$

where X is defined as

$$X = \sum_{t=1}^{m} T_t(D_tB_tq_t)^{1/2} \Big/ \sum_{t=1}^{m} T_tq_t. \tag{27}$$

This relation is very similar to eqn (17) obtained for Case 1, except for the demand-dependent factor X and the specific cost and demand density $(D_tB_t/q_t)^{1/2}$ of each period.

Then the optimal route spacing r^* can be derived by substituting eqn (26) into eqn (25), and solving them:

$$r^* = (16X^2zwg^2/x^2L)^{1/3}. \tag{28}$$

The optimized headway for each period can be obtained by substituting eqn (28) into eqn (26):

$$h_t^* = \left(\frac{D_tB_t}{q_t}\right)^{1/2}\left(\frac{x}{4gLz^2w^2X}\right)^{1/3}. \tag{29}$$

By substituting eqns (28) and (29) into eqns (3b), (5b), (6b) and (4b), the optimal wait time cost, operator cost, access cost and total system cost with cyclical fixed demand can be obtained:

$$C_o^* = Q_T(X^2zwx/4gL)^{1/3}, \tag{30}$$

$$C_w^* = C_o^*, \tag{31}$$

$$C_x^* = C_w^* + Q_T(xd/4g), \tag{32}$$

$$C^* = Q_T\left[3\left(\frac{X^2xwz}{4gL}\right)^{1/3} + \frac{xd}{4g} + vM\right], \tag{33}$$

where Q_T is again the total daily demand, defined as $LW\Sigma_tT_tq_t$ for the cyclical demand case, and M is the average in-vehicle time over all periods, defined as $\Sigma_tM_tT_tq_t/\Sigma\, T_tq_t$ for the cyclical case. Equations (30)–(32) indicate that the optimality condition that the operator cost, user wait cost and user lateral access cost should be equal is still true in this case.

The optimized average cost in dollars per trip is the optimized total cost divided by the total demand:

$$c^* = \frac{C^*}{Q_T} = 3\left(\frac{X^2zwx}{4gL}\right)^{1/3} + \frac{xd}{4g} + vM. \tag{34}$$

The optimal fleet size for each period can also be obtained by substituting eqns (28) and (29) into eqn (3b):

$$F_t^* = LW\left(\frac{D_tq_t}{B_t}\right)^{1/2}\left(\frac{zwx}{4gLX}\right)^{1/3}. \tag{35}$$

Hence, the required fleet size for this optimized system is F_1^*, which by our definition is the peak period fleet size.

5. OPTIMIZATION RESULTS FOR EQUILIBRIUM CASES

In cases 3 and 4, profit and social welfare maximization are analyzed while taking into account the demand elasticity. The optimization results are presented below for each case and for each objective.

5.1. Case 3 – Steady Equilibrium Demand

Profit Maximization. The objective function for profit maximization with steady equilibrium demand is shown in eqn (8b). As in the previous analysis, the partial derivatives of the objective function are

$$\frac{\partial P}{\partial r} = -\frac{e_x fTLWq}{4g} + \frac{WDBT}{r^2 h} = 0, \tag{36}$$

$$\frac{\partial P}{\partial h} = -e_w zfTLWq + \frac{WDBT}{rh^2} = 0, \tag{37}$$

$$\frac{\partial P}{\partial f} = TLWq\left(k - e_w zh - \frac{e_x r}{4g} - e_p f\right) - e_p fTLWq = 0. \tag{38}$$

From eqns (36) and (37), the relation between headway and number of zones can be obtained:

$$h^* = (e_x r^*/4gze_w). \tag{39}$$

This relation is identical to eqn (17) except that the values of access time (x) and wait time (w) in eqn (17) are replaced by the elasticity parameters of access time (e_x) and wait time (e_w), respectively. Substituting this relation into eqn (37), we can obtain:

$$-\frac{e_x r^4}{2gk} + r^3 - \frac{32DBze_w e_p g^2}{e_x^2 Lkq} = 0. \tag{40}$$

The first term in eqn (40) is relatively small and, given the parameter values in Table 1, can be ignored for route spacings less than two miles. Therefore, the remaining equation yields the following result for the optimized route spacing:

$$r^* \cong (32DBze_w e_p g^2/e_x^2 Lkq)^{1/3}. \tag{41}$$

The optimized headway can then be obtained by substituting eqn (41) into eqn (39):

$$h^* \cong (DBe_x e_p/2gLkqz^2 e_w^2)^{1/3}. \tag{42}$$

The optimal fare can also be obtained by substituting eqns (41) and (42) into (38):

$$f^* = \frac{k}{2e_p} - \frac{e_x r^*}{4ge_p} \cong \frac{k}{2e_p} - \left(\frac{DBze_w e_x}{2gLkqe_p^2}\right)^{1/3}. \tag{43}$$

The optimal fleet size becomes:

$$\mathrm{F}^* \cong LWq(Dze_w e_x k^2/16gLqB^2 e_p^2)^{1/3}. \tag{44}$$

Substituting the results shown above into eqns (2a) and (9a), we can obtain the optimal total revenue R^* and operator cost C_o^* for the steady elastic demand case, as summarized in Table 2a later.

Social welfare optimization. For this objective, eqn (11a) is the objective function, and the first-order conditions for an optimum are

$$\frac{\partial Y}{\partial r} = -\frac{e_x TLWq}{4ge_p}\left(k - e_w zh - \frac{e_x r}{4g} - e_p f\right) - \frac{e_x fTLWq}{4g} + \frac{WDBT}{r^2 h} = 0, \tag{45}$$

Table 2a. Summary of analytic models – Optimized objective functions

Elasticity \ Time		Single Period	Multiple Periods
Fixed Demand	Total Cost Minimization	$C = \frac{WDBT}{rh} + TLWqx(\frac{r+d}{4g}) + TLWqwzh + TLWqvM$ $C_o^* = C_w^* = Q_T\left(\frac{DBzwx}{4gLq}\right)^{1/3}$ $c^* = 3\left(\frac{DBzwx}{4gLq}\right)^{1/3} + A$	$C = \frac{W}{r}\sum_{i=1}^{m}\frac{D_iB_iT_i}{h_i} + xLW(\frac{r+d}{4g})\sum_{i=1}^{m}q_iT_i + wzLW\sum_{i=1}^{m}h_iq_iT_i + vLW\sum_{i=1}^{m}M_iq_iT_i$ $C_o^* = C_w^* = Q_T\left(\frac{X^2zwx}{4gL}\right)^{1/3}$ $c^* = 3\left(\frac{X^2zwx}{4gL}\right)^{1/3} + A$
Equilibrium Demand	Profit Maximization	$P = fTLWq(k - e_wzh - e_x(\frac{r}{4g}) - e_pf) - \frac{WDBT}{rh}$ $R^* = \frac{Q_T}{4e_p}\left(k - \left(\frac{DBze_we_xe_p}{2gLkq}\right)^{1/3}\right)^2$ $C_o^* > C_w^* = Q_T\left(\frac{DBzwx}{2gLkq}\right)^{1/3}$ $c^* = \left(2 + \frac{1}{1 - e_xr^*/2gk}\right)\left(\frac{DBzwx}{2gLkq}\right)^{1/3} + A$	$P = fLW\sum_{i=1}^{m}T_iq_i(k_i - e_wzh_i - e_x(\frac{r}{4g}) - e_pf) - \frac{W}{r}\sum_{i=1}^{m}\frac{D_iB_iT_i}{h_i}$ $R^* = \frac{Q_T}{4e_p}\left(k - \left(\frac{X^2ze_we_xe_p}{2gLk}\right)^{1/3}\right)^2$ $C_o^* > C_w^* = Q_T\left(\frac{X^2zwx}{2gLk}\right)^{1/3}$ $c^* = \left(2 + \frac{1}{1 - e_xr^*/2gk}\right)\left(\frac{X^2zwx}{2gLk}\right)^{1/3} + A$
	Social Welfare Maximization	$Y = \frac{TLWq}{2e_p}(k - e_wzh - e_x(\frac{r}{4g}) - e_pf)^2 + P$ $G^* = \frac{Q_T}{2e_p}\left(k - \left(\frac{2DBze_we_xe_p}{gLkq}\right)^{1/3}\right)^2$ $C_o^* > C_w^* = Q_T\left(\frac{DBzwx}{4gLkq}\right)^{1/3}$ $c^* = \left(2 + \frac{1}{1 - e_xr^*/2gk}\right)\left(\frac{DBzwx}{4gLkq}\right)^{1/3} + A$	$Y = \frac{LW}{2e_p}\sum_{i=1}^{m}T_iq_i(k - e_wzh_i - e_x(\frac{r}{4g}) - e_pf)^2 + P$ $G^* = \frac{Q_T}{2e_p}\left(k - \left(\frac{2X^2ze_we_xe_p}{gLk}\right)^{1/3}\right)^2$ $C_o^* > C_w^* = Q_T\left(\frac{X^2zwx}{4gLk}\right)^{1/3}$ $c^* = \left(2 + \frac{1}{1 - e_xr^*/2gk}\right)\left(\frac{X^2zwx}{4gLk}\right)^{1/3} + A$

Table 2b. Summary of analytic models – Optimized decision variables

Elasticity \ Time		Single Period	Multiple Periods
Fixed Demand	Total Cost Minimization	$r^* = \left(\frac{16DBzwg^2}{x^2Lq}\right)^{1/3}$ (15) $h^* = \left(\frac{DBx}{4gLqz^2w^2}\right)^{1/3}$ (16) $h^* = \frac{x\,r^*}{4gzw}$ (17) $F^* = LWq\left(\frac{Dzwx}{4gLqB^2}\right)^{1/3}$ (23)	$r^* = \left(\frac{16X^2zwg^2}{x^2L}\right)^{1/3}$ (28) $h_t^* = \left(\frac{DB_t}{q_t}\right)^{1/2}\left(\frac{x}{4gLz^2w^2X}\right)^{1/3}$ (29) $h_t^* = \left(\frac{DB_t}{q_t}\right)^{1/2}\left(\frac{x\,r^*}{4gzwX}\right)$ (26) $F_t^* = LW\left(\frac{Dq_t}{B_t}\right)^{1/2}\left(\frac{zwx}{4gLX}\right)^{1/3}$ (35)
Equilibrium Demand	Profit Maximization	$r^* = \left(\frac{32DB\varkappa_w e_p g^2}{e_x^2Lkq}\right)^{1/3}$ (41) $h^* = \left(\frac{DBe_xe_p}{2gLkqz^2e_w^2}\right)^{1/3}$ (42) $f^* = \frac{k}{2e_p}\cdot\left(\frac{DB\varkappa_w e_x}{2gLqe_p^2}\right)^{1/3}$ (43) $h^* = \frac{e_x r^*}{4g\varkappa_w}$ (39) $F^* = LWq\left(\frac{Dz\varkappa_w e_x k^2}{16gLqB^2e_p^2}\right)^{1/3}$ (44)	$r^* = \left(\frac{32X^2\varkappa_w e_p g^2}{e_x^2Lk}\right)^{1/3}$ (55) $h_t^* = \left(\frac{DB_t}{q_t}\right)^{1/2}\left(\frac{e_xe_p}{2gLkz^2e_w^2X}\right)^{1/3}$ (56) $f^* = \frac{k}{2e_p}\cdot\left(\frac{\varkappa_w e_x X}{2gLke_p^2}\right)^{1/3}$ (57) $h_t^* = \left(\frac{DB_t}{q_t}\right)^{1/2}\left(\frac{e_x r^*}{4g\varkappa_w X}\right)$ (54) $F_t^* = LW\left(\frac{Dq_t}{B_t}\right)^{1/2}\left(\frac{\varkappa_w e_x k^2}{16gLe_p^2X}\right)^{1/3}$ (58)
	Social Welfare Maximization	$r^* = \left(\frac{16\,DB\varkappa_w e_p g^2}{e_x^2Lkq}\right)^{1/3}$ (51) $h^* = \left(\frac{DBe_xe_p}{4gLkqz^2e_w^2}\right)^{1/3}$ (52) $f^* = 0$ (48) $h^* = \frac{e_x r^*}{4g\varkappa_w}$ (49) $F^* = LWq\left(\frac{Dz\varkappa_w e_x k^2}{4gLqB^2e_p^2}\right)^{1/3}$ (53)	$r^* = \left(\frac{16X^2\varkappa_w e_p g^2}{e_x^2Lk}\right)^{1/3}$ (63) $h_t^* = \left(\frac{DB_t}{q_t}\right)^{1/2}\left(\frac{e_xe_p}{4gLkz^2e_w^2X}\right)^{1/3}$ (64) $f^* = 0$ (59) $h_t^* = \left(\frac{DB_t}{q_t}\right)^{1/2}\left(\frac{e_x r^*}{4g\varkappa_w X}\right)Z_t$ (60) $F_t^* = LW\left(\frac{Dq_t}{B_t}\right)^{1/2}\left(\frac{\varkappa_w e_x k^2}{4gLe_p^2X}\right)^{1/3}$ (65)

$$\frac{\partial Y}{\partial h} = -\frac{e_w zTLWq}{e_p}\left(k - e_w zh - \frac{e_x r}{4g} - e_p f\right) - e_w zfTLWq + \frac{WDBT}{rh^2} = 0, \quad (46)$$

$$\frac{\partial Y}{\partial f} = -e_p fTLWq = 0. \quad (47)$$

Equation (47) indicates that the optimal fare is zero, i.e.

$$f^* = 0. \quad (48)$$

The zero fare result is not surprising since the marginal operator cost is zero in bus systems considered here at least until a vehicle capacity constraint becomes binding. Similar zero fare results have been obtained by Nash (1982) and by Kocur and Hendrickson (1982), and discussed by Viton (1980).

Substituting eqn (48) into eqns (45) and (46) and solving them, we obtain the relation between headway and route spacing:

$$h^* = (e_x r^*/4gze_w). \quad (49)$$

Equation (49) is identical to that in eqn (39). This indicates that the headway and route spacing should be kept in the same constant ratio for both profit maximization and social welfare optimization. Substituting eqn (49) into eqn (46), we can obtain the following results:

$$-\frac{3e_x r^4}{4gk} + r^3 - \frac{16DB\,ze_w e_p g^2}{e_x^2 Lkq} = 0. \quad (50)$$

The same analysis and approximation used in eqn (40) yield the optimal route spacing, and then the optimal headway for maximizing social welfare for a steady equilibrium demand:

$$r^* \cong (16DBze_w e_p g^2/e_x^2 Lkq)^{1/3}, \quad (51)$$

$$h^* \cong (DBe_x e_p/4gLkqz^2 e_w^2)^{1/3}. \quad (52)$$

The optimal fleet size becomes:

$$F^* = LWq(Dze_w e_x k^2/4gLqB^2 e_p^2)^{1/3}. \quad (53)$$

The optimal operator cost C_o^* can be derived from eqns (2a), (51) and (52), and the optimal consumer surplus G^* can be derived from eqns (12a), (51) and (52), as summarized in Table 2a.

5.2. *Case 4 – Cyclical Equilibrium Demand*

Profit maximization. As in Case 3, two objectives are analyzed in this case with time varying equilibrium demand. The profit maximization objective function is shown in eqn (8b). Again, from the first-order conditions, the relation between headway h_t and the route spacing r can still be found:

$$h_t^* = \left(\frac{D_t B_t}{q_t}\right)^{1/2}\left(\frac{e_x r^*}{4gze_w X}\right), \quad (54)$$

where X is defined as eqn (27).

With an approximation similar to that used in eqn (40), the following results for the optimal route spacing, headway and fare can be obtained:

$$r^* \cong \left(\frac{32X^2ze_we_pg^2}{e_x^2Lk}\right)^{1/3}, \tag{55}$$

$$h_t^* \cong \left(\frac{D_tB_t}{q_t}\right)^{1/2}\left(\frac{e_xe_p}{2gLkz^2e_w^2X}\right)^{1/3}, \tag{56}$$

$$f^* = \frac{k}{2e_p} - \frac{e_xr^*}{4ge_p} \cong \frac{k}{2e_p} - \left(\frac{ze_we_xX}{2gLke_p^2}\right)^{1/3}. \tag{57}$$

In eqns (55)–(57) k is an average quantity of k_t over all periods and is defined as $\Sigma_t k_t T_t q_t / \Sigma T_t q_t$. The optimized fleet size for each period becomes:

$$F_t^* \cong LW\left(\frac{D_tq_t}{B_t}\right)^{1/2}\left(\frac{ze_we_xk^2}{16gLe_p^2X}\right)^{1/3}. \tag{58}$$

Substituting the results shown above into eqns (2b) and (9b), the optimal total revenue R^* and operator cost C_o^* can be obtained, as shown in Table 2a.

Social welfare maximization. Here eqn (11b) is the objective function, and from the first-order conditions for f, we obtain:

$$f^* = 0. \tag{59}$$

Substituting eqn (59) into the first-order conditions for r and h_t and solving them, the following relation between headway h_t and the route spacing r can be found:

$$h_t^* = \left(\frac{D_tB_t}{q_t}\right)^{1/2}\left(\frac{e_xr^*}{4gze_wX}\right)Z_t, \tag{60}$$

where

$$Z_t = \frac{k - e_wzh - e_xr/4g}{k_t - e_wzh_t - e_xr/4g}, \qquad t = 1,2, \ldots, m. \tag{61}$$

In eqn (61) k is the same as in eqns (55), (56) and (57), while h is the average headway over all periods, defined as $\Sigma_t h_t T_t q_t / \Sigma_t T_t q_t$.

Again, eqn (60) is similar to eqn (26) except for the factors Z_t. Since the Z_t are also approximately 1.0 for reasonable parameter values (with the typical parameter values in Table 1 we obtain $Z_1 = 0.9785$, $Z_2 = 1.0071$ and $Z_3 = 1.0974$), we can simplify eqn (60) as follows:

$$h_t^* \cong \left(\frac{D_tB_t}{q_t}\right)^{1/2}\left(\frac{e_xr^*}{4gze_wX}\right). \tag{62}$$

Again, with the above relation and an approximation process similar to that for eqn (50), we obtain the optimal route spacing and headway:

$$r^* \cong \left(\frac{16X^2ze_we_pg^2}{e_x^2Lk}\right)^{1/3}, \tag{63}$$

$$h_t^* \cong \left(\frac{D_t B_t}{q_t}\right)^{1/2} \left(\frac{e_x e_p}{4gLkz^2 e_w{}^2 X}\right)^{1/3}. \tag{64}$$

The fleet size can then be obtained as follows:

$$F_t^* \cong LW\left(\frac{D_t q_t}{B_t}\right)^{1/2} \left(\frac{z e_w e_x k^2}{4gLe_p{}^2 X}\right)^{1/3}. \tag{65}$$

Substituting eqns (59, (63) and (64) into eqns (2b) and (12b) the optimal operator cost C_o^* and consumer surplus G^* can be obtained, as summarized in Table 2a.

6. DISCUSSION OF ANALYTIC RESULTS

The optimized analytic functions for all cases are summarized in Table 2. The functions in Table 2a include the objective functions (C = total cost, c = average cost, P = profit and Y = social welfare) and their optimized results. The functions in Table 2b include the optimized decision variables (h = headway, r = route spacing, f = fare), the relation between the optimal headway and route spacing and the optimized fleet size F.

The functions summarized in Table 2 show more clearly than any numerical results, what the relations among variables should be for optimized feeder bus services with the general configuration analyzed in this paper. To a large extent, the sensitivity of optimized decision variables to the various parameters can be determined by visually inspecting the functions rather than by numerical analysis. For example, Case 4 (multiple period equilibrium), which accounts for demand elasticity and in which demand, operating costs and travel times are time dependent, is the most realistic and complex case analyzed here. For this case we can observe that profit is maximized with a headway in each time period [eqn (60)] that varies with the ½ power of the bus round trip time and operating cost during that period, the ⅓ power of a fare elasticity factor, the $-$ ½ power of the potential demand density and the $-$ ⅓ power of a headway elasticity factor. The optimal route spacing varies with the ⅓ power of the wait time and fare elasticity factors, the ⅔ power of access speed, the $-$ ⅓ power of the route length and the $-$ ⅔ power of access time elasticity factor. The relation obtained between the optimized headway and the demand density in multi-period cases is similar to that found by Newell (1971) for continuously varying demand. With a constant multiplier that is smaller by a factor of $2^{-1/3}$, i.e. $16^{1/3}$ instead of $32^{1/3}$, the above results are also true for the social welfare maximization objective.

The functions in Table 2b indicate that the effects of system parameters on optimized variables are very similar, except that: (1) the ratio DB/q (operating cost/demand density) in single period cases is replaced in multiple period cases by either $(D_t B_t/q_t)^{1/2}/X$ (for h_t^*) of by X^2 (for all other functions); and (2) the time value parameters w, x and v in fixed demand cases are replaced by the elasticity factors e_w, e_x, e_v and e_p in equilibrium demand cases. It is also notable in Table 2b that as the more general multiple period solutions for cyclical demand cases are reduced to single period cases, the powers of the time-dependent variables change from ½ to ⅓. These time-dependent variables are the vehicle round travel times D_t (which are based on time-dependent speeds), the vehicle operating costs B_t and the demand densities q_t. The difference in powers are accounted by the composite variable X, which acts as a constant in the multiple period solutions.

It can also be shown that the optimized headway [Eqns (52) and (64)] and route spacing [Eqns (51) and (63)] for the maximum social welfare objective are identical to those [Eqns (16), (29), (15) and (28), respectively] for the minimum total cost objective when the parameter k in the equilibrium demand cases becomes 1.0, which implies a change from equilibrium to fixed demand.

The optimal fare is found to be very sensitive to the fare elasticity parameter for profit maximization and it is zero for social welfare maximization in equilibrium cases.

These findings are similar to those obtained by Kocur and Hendrickson (1982) for steady conditions. The results also indicate that the constant ratio between route spacing and headway, which has been found for optimized transit system in several studies under various assumptions (Hurdle, 1973; Schonfeld, 1981; Kocur and Hendrickson, 1982; Tsao and Schonfeld, 1984; Chang and Schonfeld, 1989a), is maintained exactly for steady demand conditions, as shown in eqns (17), (39) and (49). This optimality condition is also found to be maintained with an adjustment factor for multiple period cases, as shown in eqns (26), (54) and (60).

For cost minimization at fixed demand the average operator cost, the user wait cost and the lateral component of user access cost (perpendicular to routes) are all exactly equal in an optimized system, each of these being $(BDzwx/4gLg)^{1/3}$ for single period cases [see eqns (18)–(22)] and $(X^2zwx/4gL)^{1/3}$ for multiple period cases [see eqns (30–34)]. This optimality condition, which was found in some previous studies (Holroyd, 1965; Schonfeld, 1981; Tsao and Schonfeld, 1984), does not extend to elastic demand cases, in which average user wait cost is equal to average lateral access cost, but both are less than the average operator cost.

We can expect that any two cost components will be equalized by an optimal solution whenever the objective function reaches its optimum value where the functions of those two components intersect, as shown qualitatively in Fig. 4. It can be shown that such behavior holds for all objective functions of the form $\alpha + \beta/x + \gamma x$ and, more generally, $\alpha + \beta x^{-\epsilon} + \gamma x^{\epsilon}$.

The optimized average cost functions for Cases 3 and 4 [for which detailed derivations are shown in Chang and Schonfeld (1989b)] are also worth comparing:

Case 3 — steady conditions.

(1) Profit maximization

$$c^* = \left(2 + \frac{1}{1 - e_x r^*/2gk}\right)\left(\frac{DBzwx}{2gLkq}\right)^{1/3} + A \tag{66}$$

$$\cong 3.23\left(\frac{DBzwx}{4gL(kq/2)}\right)^{1/3} + A. \tag{66a}$$

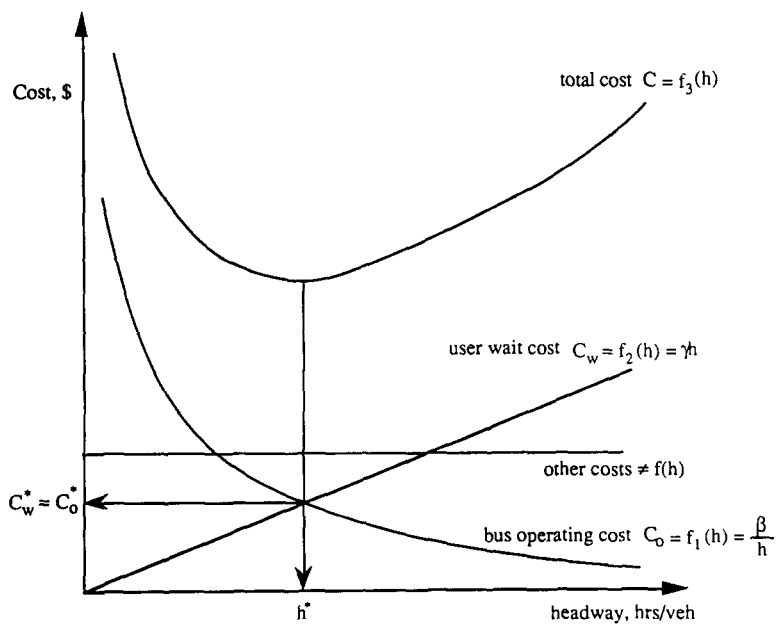

Fig. 4. Equal optimized cost components.

(2) Social welfare maximization

$$c^* = \left(2 + \frac{1}{1 - e_x r^*/2gk}\right)\left(\frac{DBzwx}{4gLkq}\right)^{1/3} + A \tag{67}$$

$$\cong 3.18\left(\frac{DBzwx}{2gLkq}\right)^{1/3} + A, \tag{67a}$$

where A is the component of average cost that is insensitive to the design variables and defined as $(xd/4g + vM)$.

Case 4 – Time-dependent conditions.

(1) Profit maximization

$$c^* = \left(2 + \frac{1}{1 - e_x r^*/2gk}\right)\left(\frac{X^2zwx}{4gLk}\right)^{1/3} + A. \tag{68}$$

(2) Social welfare maximization

$$c^* = \left(2 + \frac{1}{1 - e_x r^*/2gk}\right)\left(\frac{X^2zwx}{4gLk}\right)^{1/3} + A. \tag{69}$$

It should be noted from the components of eqns (66) and (67) that the average user cost is equal to the user lateral access cost, each of which is $(BDxwz/2gLkq)^{1/3}$, and both are smaller that the average operator cost, which can be derived as $(BDzwx/2gLkq)^{1/3}/(1 - e_x r^*/2gk)$for profit maximization and $(BDzwx/4gLkq)^{1/3}/(1 - e_x r^*/2gk)$ for social welfare maximization.

It is also notable that eqns (66a) and (67a) are very similar to eqn (22) for total cost minimization except for a factor "$kq/2$" instead of q in the first term of eqn (66a), and for a factor "kq" instead of q in the first term of eqns (67a). It should be noted that the terms "$kq/2$" and "kq" in the equations are approximations of equilibrium demand density for profit and social welfare maximization, respectively. They correspond to "q" in eqn (22) when these two cases are compared at equal demand levels. The actual demand is always overestimated slightly by this approximation.

Cases 1 and 3 indicate that average cost is always higher for equilibrium demand compared to fixed demand and for maximum profit compared to maximum social welfare. The same holds in comparing Case 2 with Case 4. It can also be shown that at equal actual demand densities the relative magnitudes of the optimal headway h^*, and route spacing r^*, are ranked similarly to the optimal average costs c^* for the various objectives.

7. VEHICLE LOAD CONSTRAINT

It should be noted that no vehicle capacity constraint has been considered in the previous analysis. When overloaded buses are not allowable, the optimized headway obtained should be adjusted. Newell (1971) has proposed an approach to optimize variable headways subject to a vehicle capacity constraint on a transit route with time-varying demand. In this paper, passenger demand is time dependent but is assumed to be constant within each period. The following approaches may be used to satisfy a vehicle capacity constraint for multiple period cases with fixed demand. The procedure for incorporating a vehicle capacity or allowable load factor constraint is as follows:

(1) Check if the bus load E_t exceeds the allowable bus load uS, which is given by the maximum allowable load factor u multiplied by vehicle seat capacity S.

The bus load may be derived as

$$E_t = h_t^* r^* L q_t, \tag{70}$$

where r^*, h^*_t are the optimized route spacing and headway, as determined in eqns (28) and (29), respectively. It should be noted that the specifiable load factor allows for (a) a safety factor to accommodate overload passengers due to some stochastic fluctuations in demand (whereby $u < 1$), and (b) some standees as an economizing policy during peaks ($u > 1$).

(2) If $E_t \leq uS$, then load constraints are satisfied.

(3) If $E_t > uS$, then vehicles are overloaded and the load constrained headway and route spacing should be derived by substituting the following equation into the total cost function [eqn (4b)]:

$$h_t = uS/r^*Lq_t. \tag{71}$$

By definition, if the vehicle load is satisfied for the overloaded periods, it can be satisfied at all other times. Therefore, the total cost function [Eqn (4b)] becomes

$$C = \frac{LW}{uS}\sum_{t=1}^{j-1} D_tB_tq_tT_t + \frac{W}{r}\sum_{t=j}^{m}\frac{D_tB_tT_t}{h_t} + \frac{wzuSW}{r}\sum_{t=1}^{j-1} T_t + wzLW\sum_{t=j}^{m} h_tq_tT_t \tag{72}$$

$$+ \frac{xrLW}{4g}\sum_{t=1}^{m} q_tT_t + vMLW\sum_{t=1}^{m} q_tT_t,$$

in which Periods 1 to $j - 1$ are assumed to be overloaded, while Periods j to m are not overloaded.

Solving the first-order conditions of eqn (72), the following relations between route spacing r^* and the headways h_t can be obtained:

$$h^*_t = \frac{uS}{r^*Lq_t}, \qquad t = 1 \sim j - 1, \tag{73}$$

$$h^*_t = \left(\frac{D_tB_t}{r^*wzLq_t}\right)^{1/2}, \qquad t = j \sim m. \tag{74}$$

Substituting eqns (73) and (74) into the first-order condition with respect to r^*, the following result can be obtained:

$$\left(\frac{xL}{4g}\right)r^2 - \left(\frac{(wzL)^{1/2}\sum_{t=j}^{m}(D_tB_tq_t)^{1/2}T_t}{\sum_{t=1}^{m} q_tT_t}\right)r^{1/2} - \frac{\sum_{t=1}^{j-1} wzuST_t}{\sum_{t=1}^{m} q_tT_t} = 0. \tag{75}$$

The optimal route spacing r^* can be obtained numerically from eqn (75). Then the load constrained headways for Periods 1 to $j - 1$ can be obtained by substituting r^* into eqn (73), while the headways for other periods can be obtained by substituting r^* into eqn (74). Numerical results are shown later in Table 4.

An analogous approach for incorporating vehicle load constraints in multiple period cases with equilibrium demand is presented in Chang (1990).

8. NUMERICAL RESULTS

The baseline parameter values shown in Table 1 were selected for our numerical example since they seemed reasonably typical. The numerical results obtained are shown in Table 3. These results are for a 3 × 2-mile rectangular service area. For cyclical demand cases a three period demand distribution was assumed in which demand densities of 150, 60 and 12 trips/sq. mile/hour applied during service periods of 3, e and 4 hours,

Table 3. Summary of optimized bus systems analyzed

Cases	Part I - Baseline Results						Part II - Results for Equal Demand Comparison[3]			
	1	2	3		4		1		2	
Objectives	Min. Cost (1)	Min. Cost (2)	Max. Profit (3)	Max. S. W.[2] (4)	Max. Profit (5)	Max. S. W.[2] (6)	Min. Cost (7)	(8)	Min. Cost (9)	(10)
Headway (hours)	0.177	0.150[1] 0.168 0.375	0.234	0.168	0.199[1] 0.222 0.497	0.158[1] 0.176 0.394	0.251	0.196	0.209[1] 0.236 0.617	0.166[1] 0.186 0.440
Route Spacing (miles)	0.88	0.85	1.17	0.93	1.13	0.90	1.26	0.98	1.19	0.94
Fare (dollars)	—	—	4.98	0	4.86	0	—	—	—	—
Fleet Size (vehicles)	11	14[1] 12 6	7	10	8[1] 7 3	13[1] 11 5	6	9	8[1] 7 3	12[1] 10 5
Actual Demand (trips/sq. mile/hr)	67.8	150[1] 60 12	23.7	49.6	56.0[1] 21.9 3.2	111.6[1] 44.3 7.9	23.7	49.6	56.0[1] 21.9 3.2	111.6[1] 44.3 7.9
Total Cost ($/day)	18797	18423	8175	14655	8198	14393	8162	14602	8157	14361
Average Cost ($/trip)	4.62	4.53	5.76	4.92	5.73	4.80	5.74	4.91	5.52	4.79
Avg. User Cost ($/trip)	3.74	3.68	4.31	3.83	4.34	3.75	4.48	3.93	4.33	3.85
Avg. Wait Cost ($/trip)	0.88	0.85	1.17	0.93	1.14	0.89	1.26	0.98	1.19	0.94
Avg. Oper. Cost ($/trip)	0.88	0.85	1.45	1.09	1.39	1.05	1.26	0.98	1.19	0.94
Total Revenue ($/day)			7088	0	6959	0				
Total Profit ($/day)			5034	-3244	5218	-3146		—		
Cons. Surplus ($/day)				15577		15769				
Soc. Welfare ($/day)				12333		12623				
Pass. Load per Bus	32	58[1] 26 12	20	26	38[1] 17 6	48[1] 21 9	23	29	42[1] 19 7	50[1] 24 10

Note: 1. results for peak, offpeak, and night time, respectively.
2. S. W. is Social Welfare.
3. Columns 7, 8, 9, and 10 minimize total cost at ridership levels corresponding to equilibrium levels in Columns 3, 4, 5, and 6, respectively.

respectively. These conditions are shown graphically in Fig. 2. The bus operating costs during those three periods (peak, offpeak and night) are assumed to be 50, 25 and 25 dollars/vehicle hour, respectively.

The first part of Table 3 presents results for the four demand cases optimized for various objective functions. The second part of Table 3 (Columns 7, 8, 9 and 10) presents the results obtained with the inelastic demand models and minimum total cost objective while using the actual demand densities obtained in the equilibrium cases (Columns 3, 4, 5 and 6) in order to compare the equilibrium and fixed models at equal ridership levels.

Several observations about these numerical results should be noted:

(a) Given the assumption of equal values for wait time and access time, the constant ratio between the optimal headway and route spacing in Cases 1 and 3 indicates that the optimized wait time and lateral access time should be equal. For example, in Case 1 the average wait time of 0.088 hours (=0.88 dollars/trip divided by 10 dollars/passenger hour), which is the half of the headway, is equal to the average lateral access time to a bus route, which is the ¼ of route spacing divided by the access speed.

(b) In Cases 1 and 2, the operator cost is equal to the passenger wait cost, as required by eqns (19) and (31). For example, in Case 1 average operator cost and wait cost are each 0.88 dollar/trip. This relation is not maintained in the elastic demand conditions in which the operator cost is higher than the user wait cost. For example, in Case 3 with the maximum profit objective the average operator cost of 1.45 dollar/trip is larger than the user wait cost of 1.17 dollar/trip.

(c) The actual demand density is 23.7 trip/sq. mile/hour, or 35% of the potential bus demand (i.e. the weighted average demand density of 67.8 trips/sq. mile/hour) for the maximum profit objective. It is 49.6 trips/sq. mile/hour, or 73% of potential demand for the optimum social welfare objective in Case 3. In the cyclical demand conditions the peak ridership constitutes a larger fraction of potential demand.

(d) The results for profit maximization indicate that the fare is very sensitive to the elasticity factors especially when demand is relatively inelastic with respect to fares. The fares are zero for the optimum social welfare objective. For the profit maximization in Cases 3 and 4 the fares are \$4.98 and \$4.86, respectively, which are much higher than the typical fares charged by bus operators.It should be noted that most U.S. bus systems operate with considerable deficits, and that the optimal fare for maximizing profit (without price regulation) approaches infinity as the price elasticity of demand approaches zero.

(e) In Part II of the numerical results shown in Table 3, the average operator costs are all equal to the wait costs. This has been shown to be an optimality condition for fixed demand situations. In comparing the fixed demand results (Columns 7, 8, 9, 10) with the corresponding equilibrium demand results (Columns 3, 4, 5 and 6) we see that better service (e.g. lower headways), and hence lower user costs and higher operator costs are provided with equilibrium demand.

(f) As expected from the discussion of analytic results, the average costs per trip for equilibrium cases are higher than for fixed demand cases. Since in the cyclical demand case we have assumed different bus operating costs for different periods, i.e. \$50 per vehicle hour for peak periods and \$25 per vehicle hour for other periods, while the bus operating costs for steady demand cases are \$32.5 per vehicle hour at all times, the average costs difference between the cyclical demand Cases 1 and 3 is more significant than that between the steady demand Cases 2 and 4.

In Fig. 5, the effects of the route spacing on the total cost minimization and profit maximization objectives are evaluated. It is shown that small deviations from the optimal route spacing have relatively slight effects on the objective functions.The effects of the approximations made in deriving analytic solutions have been evaluated by numerically comparing the optimized analytic functions with and without those approximations (Chang and Schonfeld, 1989b). It has been shown that the effects of the approximations on objective functions are negligible. For example, the optimized route spacing r^* is 1.13 and the corresponding value of the objective function (i.e. maximum profit P^*) is \$5218 per day in Case 4 with profit maximization, as shown in Fig. 5. When the problem is

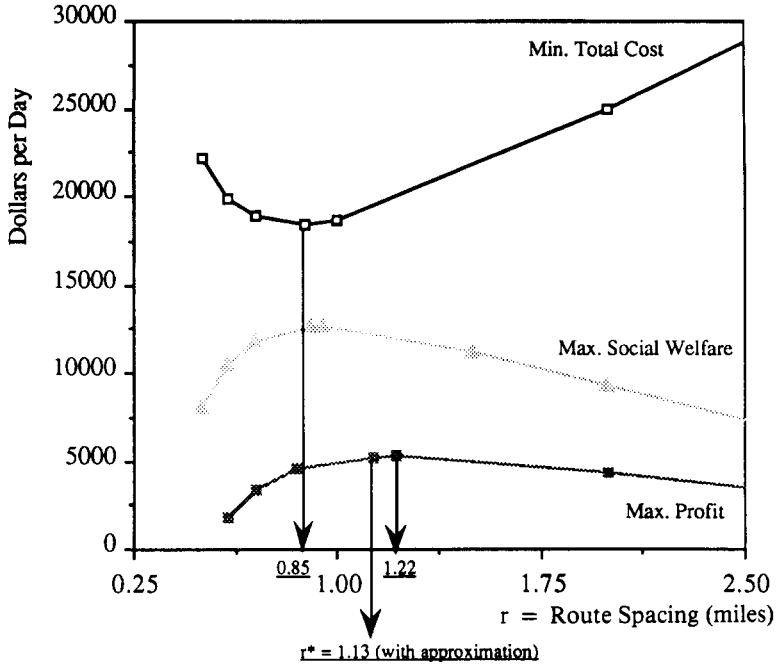

Fig. 5. Effects of route spacing on objective functions.

solved numerically without the approximation shown in eqn (59), we obtain $r^* = 1.22$ and $P^* = \$5245$ per day. Thus, the approximation error is less than 0.1% of the value of the objective function.

It should be noted that the bus load in Case 2 of the numerical example exceeds the seat capacity of standard buses ($S = 50$ seats and $u = 1.0$ for the numerical example). Considering the load-constrained conditions and applying the analytic models developed in eqns (73)–(75), the load-constrained design results for the cyclical fixed demand case can be obtained, as presented in Table 4. It is shown that a load constraint reduces the optimal route spacing, while the headways decrease in the peak period and increase in the other two periods. The headway increase in the other two periods partly offsets the cost of the reduced route spacing, which has to be an optimal compromise over all periods. The fleet size needed for peak period increases significantly from 14 to 20 vehicles, while

Table 4. Load constrained results for a multiple period fixed demand case

	Unconstrained Results	Load Constrained Results
Route Spacing (miles)	0.85	0.80
Headway (hrs)*	0.150 0.168 0.375	0.139 0.160 0.357
Fleet Size (vehs)*	14 12 6	20 16 7
Avg. Cost ($/trip)	4.53	4.78
Avg. User Cost ($/trip)	3.68	3.58
Avg. Wait Cost ($/trip)	0.85	0.80
Avg. Operator Cost ($/trip)	0.85	1.20
Bus Load (Passengers/veh)*	58 26 12	50 23 11

Note: * Results for the three periods.

it also increases for the other two periods.It is also shown that the average operator cost increases from \$0.85 to \$1.20 per trip, while the average user cost decreases from \$3.68 to \$3.58 per trip and the average wait cost decreases from \$0.85 to \$0.80 per trip. The bus loads for all periods decrease and, as expected, the peak hour bus load is 50 passengers. The optimality of equal average wait cost, operator cost and lateral access cost has not been maintained. However, the average wait cost and access cost are still identical.

9. CONCLUSIONS

Analytic models have been developed for optimizing feeder bus services whose demand may be sensitive to service quality and price and whose supply and demand characteristics may differ in various time periods. Variations in demand and supply characteristics over space have been treated in some related analytic optimization models (Chang and Schonfeld, 1989c; Chang, 1990). The resulting models may be used to analyze and optimize relatively complex feeder or radial bus systems.

The multi-period optimization models developed here may be used to determine the best route structure that is invariant over time while optimizing service headways within each period. That is more difficult than a separate unconstrained optimization of all decision variables within each period.

Different objective functions have been used in the analysis for comparison purposes: Total system cost, including operator cost and user cost, is the objective function used when fixed demand is assumed; operator profit and social welfare are the objective functions under equilibrium demand conditions. Although the presentation in this paper treats the various time periods as components of a daily cycle, the analytic models are also applicable to other cyclical or noncyclical demand variations.

The equations for all the models analyzed are summarized in Tables 2a and 2b. The relations among variables in these optimized systems can be determined by inspection of the closed-form solutions provided. The comparison of cyclical demand models (Cases 2 and 4) with steady demand models (Cases 1 and 3) shows that the effects of system parameters on optimized variables are very similar, except that:(1) the demand dependent factor X^2 defined in eqn (27) replaces the ratio DB/q; and (2) the elasticity factors e_w, e_x, e_v and e_p replace the time value parameters w, x and v.

A cost comparison indicates that the average costs for equilibrium cases (Cases 3 and 4) are always higher than for fixed demand cases (Cases 1 and 2) when these are compared at the equal demand levels. Average costs are also invariably higher for profit maximization than for social welfare maximization. Similar rankings hold for the optimal headway h^* and route spacing r^*.

In these models, the optimality of a constant ratio between headway and route spacing is strictly maintained in single period (steady) cases and maintained in time-varying cases with a multi-period adjustment factor. For multi-period social welfare maximization, that optimality relation is maintained with a relatively negligible approximation. Three cost components, namely operator cost, user cost and user access cost, are equalized when costs are minimized at a fixed demand, but this result does not extend to equilibrium cases, for which the operator costs are higher than other cost components. Such results might serve, even without the models presented here, as evaluation and optimization guidelines, for example in heuristic search algorithms or in relatively informal assessments of transit systems.

The interrelations among the optimization results for various objectives are also presented. For example, when setting the elasticity coefficient k of equilibrium cases equal to 1.0, the optimized results for social welfare maximization are reduced exactly to those for total cost minimization. The route spacings and headways that maximize profit maintain a constant ratio of $2^{1/3}$, with respect to those that maximize social welfare regardless of the values of other parameters such as demand density, elasticity factors, value of time or access speeds.

Bus systems with a vehicle load constraint are also optimized for cyclical fixed demand. Numerical results indicate how the optimal route spacing may change with a

load constraint: the optimal headways for overloaded periods decrease, while headways for other periods increase. The optimality of equal average wait cost, lateral access cost and operator cost is not maintained; however average wait cost is still equal to average lateral access cost.

The analytic models presented here might still be extended by incorporating more complex system configurations, costs and demand functions. These might include competing modes, operating costs that are functions of vehicle size and many-to-many demand patterns.

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