In homework 2, we show that

$$x \sim \mathcal{N}(0_D, \mathbb{I}_D) \implies \|x\|_2^2 \sim \chi_D^2$$

Therefore let $x \sim \mathcal{N}(0_D, \mathbb{I}_D)$. Let A be a matrix. Prove that

$$\mathbb{E}\|Ax\|_2^2 = \|A\|_F^2$$

And check that the generalization holds for the case $A = \mathbb{I}_D$.

Solution: We have

$$\mathbb{E}\|Ax\|_2^2 = \mathbb{E}(tr(x^T A^T Ax))$$

By invariance of trace under cyclic permutations, we obtain

$$\mathbb{E}\|Ax\|_2^2 = \mathbb{E}(tr(xx^T A^T A))$$

By linearity of trace as an operator, it commutes with expectation, so we get

$$tr(\mathbb{E}(xx^T A^T A))$$

By linearity of expectation, we obtain

$$tr(\mathbb{E}(xx^T)A^T A)$$

From the covariance matrix of x being $\mathbb{I}_D$, we get

$$tr(A^T A)$$

Which is another definition for the squared Frobenius norm, i.e.

$$\|A\|_F^2 = \sum_i \sum_j |A_{ij}|^2$$

In the case $A = \mathbb{I}_D$. we can apply homework 2 and get

$$\mathbb{E}(\|\mathbb{I}_D x\|_2^2) = \mathbb{E}(\|x\|_2^2) = \mathbb{E}(\chi_D^2) = D = \|\mathbb{I}_D\|_F^2$$