1 Central Limit Theorem

The central limit theorem basically says that the mean of independent random variables converges to normal distribution. More specifically, see the following theorem.

Theorem 1.1 (Central Limit Theorem)**.** *Let $X_1, \cdots X_n$ be a sequence of iid random variables. Let $E(X) = \mu < \infty$ and $\mathbb{V}(X) = \sigma^2 < \infty$, and mgfs exists in a neighborhood of 0. Define $\bar{X}_n = \frac{1}{n}\sum X_i$. Then,*

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N(0,1) \tag{1}$$

i.e. it converges in distribution to the standard normal distribution.

Let me sketch the proof.

Proof. Let $Y_i = \frac{X_i - \mu}{\sigma}$, so $\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} = \frac{1}{\sqrt{n}}\sum Y_i$. Let $M_Y(t)$ be the moment generating function of Y. Then,

$$M_{\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma}}(t) = \Big(M_Y(\frac{t}{\sqrt{n}})\Big)^n \tag{2}$$

We can expand this in Taylor series around 0.

$$M_Y(\frac{1}{\sqrt{n}}) = \sum_{k=0}^{\infty} M_Y^{(t)}(0)\frac{(t/\sqrt{n})^k}{k!} \tag{3}$$

Because $E[Y] = 0$ and $\mathbb{V}(Y) = 1$, we know $M_Y^{(0)}(0) = 1$, $M_Y^{(1)}(0) = 0$, $M_Y^{(2)}(0) = 1$. So,

$$M_Y(\frac{t}{\sqrt{n}}) = 1 + \frac{(t/\sqrt{2})^2}{2!} + R_Y(t/\sqrt{n}) \tag{4}$$

where R_Y is the remainder of the Taylor series. For fixed t, we have

$$\lim_{n\to\infty} \frac{R_Y(t/\sqrt{n})}{(t/\sqrt{n})} = 0 \tag{5}$$

Since t is fixed, we also have

$$\lim_{n\to\infty} \frac{R_Y(t/\sqrt{n})}{(1/\sqrt{n})^2} = \lim_{n\to\infty} nR_Y(t/\sqrt{n}) = 0 \tag{6}$$

Therefore,

$$\lim_{n\to\infty}\Big(M_Y(t/\sqrt{n})\Big)^n = \lim_{n\to\infty}\Big[1 + \frac{(t/\sqrt{n})^2}{2!} + R_Y(t/\sqrt{n})\Big]^n \tag{7}$$

$$= \lim_{n\to\infty}\Big[1 + \frac{1}{n}(\frac{t^2}{2} + nR_Y(t/\sqrt{n}))\Big]^n \tag{8}$$

$$= e^{t^2/2} \tag{9}$$

This is the mgf of the standard normal distribution.

□

References

[1] George Casella and Roger L Berger. *Statistical inference*, volume 2. Duxbury Pacific Grove, CA, 2002.