Useful theorems in probability theory

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Sum of exponential random variables is a gamma random variable

Theorem: If $X_i \overset{i.i.d}{\sim} Expo(\lambda)$, $i = 1...n$, then $Y = \sum_{i=1}^n X_i \sim Gamma(n, \lambda)$.

Proof:

First, recall that if $X \sim Expo(\lambda)$,

$$M_X(t) = \frac{\lambda}{\lambda - t}, \quad (t < \lambda)$$

if $Z \sim Gamma(\alpha, \beta)$,

$$M_Z(t) = (\frac{1}{1 - \beta t})^{-\alpha}$$

Observe that

$$\begin{aligned}
M_Y(t) &= \mathbb{E}[e^{tY}] \\
&= \mathbb{E}[e^{t\sum_{i=1}^n X_i}] \\
&= \mathbb{E}[\prod_{i=1}^n e^{tX_i}] \\
&= \prod_{i=1}^n \mathbb{E}[e^{tX_i}] \\
&= \prod_{i=1}^n \frac{\lambda}{\lambda - t} \\
&= (\frac{\lambda}{\lambda - t})^n \\
&= (\frac{\lambda - t}{\lambda})^{-n} \\
&= (1 - \frac{t}{\lambda})^{-n}
\end{aligned}$$

This is an mgf of Gamma distribution. So, $Y \sim Gamma(n, \lambda)$.

Linear combination of Gaussian random variables is also Gaussian

Theorem: if $X \sim N(\mu, \sigma^2)$, and $Y = aX + b$, $(a \neq 0)$, then $Y \sim N(a\mu + b, a^2\sigma^2)$.

Proof: Recall that if $X \sim N(\mu, \sigma^2)$

$$M_X(t) = e^{\mu t + \frac{\sigma^2}{2} t^2}$$

Then,

$$\begin{aligned}
M_Y(t) &= \mathbb{E}[e^{t(aX+b)}] \\
&= \mathbb{E}[e^{taX+tb}] \\
&= e^{tb}\mathbb{E}[e^{taX}] \\
&= e^{tb}M_X(ta) \\
&= e^{tb}e^{\mu t+\frac{\sigma^2}{2}t^2} \\
&= exp(tb + t\mu a + \frac{\sigma^2}{2}t^2) \\
&= exp((a\mu + b)t + \frac{a^2\sigma^2}{2}t^2)
\end{aligned}$$

This is also an mgf of Gaussian. $Y \sim N(a\mu + b, a^2\sigma^2)$.

Distribution of maximum values

Theorem: If $X_1...X_n \overset{i.i.d}{\sim} f_X(x)$, and $Y \equiv max\{X_1...X_n\}$, then $f_Y(y) = nF_X^{n-1}(y)f_X(y)$.

Proof:

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) \\
&= P(max\{X_1...X_n\} \leq y) \\
&= P(X_1 \leq y)...P(X_n \leq y) \\
&= F_X(y)...F_X(y) \\
&= F_X^n(y)
\end{aligned}$$

Then, because $f_Y(y) = \frac{\partial}{\partial y}F_Y(y)$,

$$f_Y(y) = nF_X^{n-1}(y)f_X(y)$$