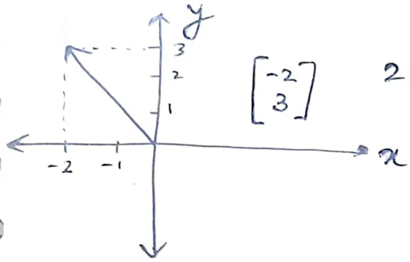
LINEAR ALGEBRA

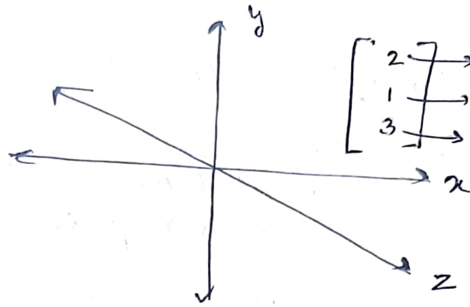
Reffered to 3Blue1Brown "Essence of Linear Algebra" playlist.

CHAPTER 1 : ~~Essence~~ Vectors

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \Leftrightarrow$ 

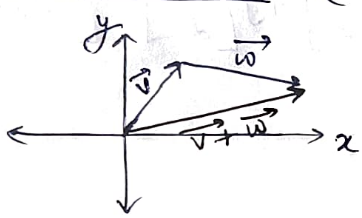


$\begin{bmatrix} -2 \\ 3 \end{bmatrix}$ 2D dimension

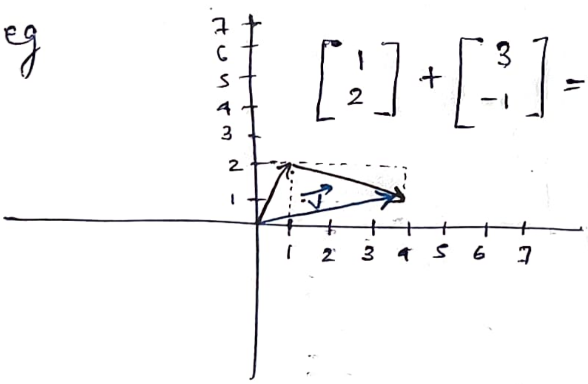


$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$
2 → x distance
1 → y distance
3 → z distance

- VECTOR ADDITION $(\vec{v} + \vec{w})$

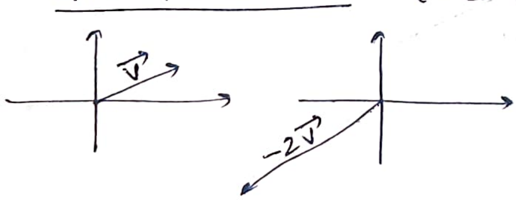


eg



$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

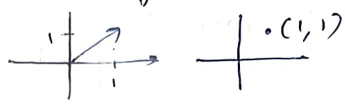
- MULTIPLICATION (scaling)



$\hat{i}$ → i cap → unit vector in x direction
$\hat{j}$ → j cap → unit vector in y direction.
$\hat{i}$ and $\hat{j}$ are the "basis vectors" of the xy coordinate system.

- Linear combination of $\vec{v}$ and $\vec{w}$ is $a\vec{v} + b\vec{w}$ where a, b are scalars.

- The "span" of $\vec{v}$ and $\vec{w}$ is the set of all their linear combinations $a\vec{v} + b\vec{w}$
  ↓
  all possible vectors possible.
- Representation of vectors as point → imagine tip as a point



- Two vectors are said to be "linearly dependent" if one of the vectors can be expressed as a linear combination of the other vector.

CHAPTER 3 - Linear Transformations & matrices

possible when
- all lines must remain lines and not curve
- origin must remain fixed

2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

→ place where 2nd basis ($\hat{j}$) vector lands

↳ where first basis ($\hat{i}$) vector lands

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} a \\ c \end{bmatrix} + y\begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

- "Shear":

Chapter 4
Matrix Multiplication as composition

Shear & Rotation

← Read Right to left $f(g(x))$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}}_{\text{shear}}\left(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{composition}}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\overset{M_2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}\overset{M_1}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} = \begin{bmatrix} ae+bg & af+bh \\ ce+gd & cf+dh \end{bmatrix}$$

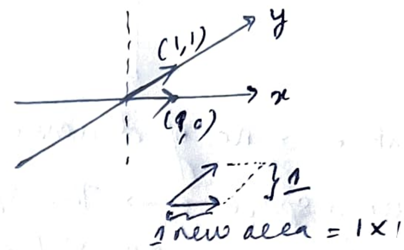
$$M_2 M_1 \neq M_1 M_2$$

Associativity $(AB)C = A(BC)$

Chapter 5 - Three dimensional transformation

eg $\underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}}_{\text{second transformation}} \underbrace{\begin{bmatrix} l & m & n \\ o & p & q \\ r & s & t \end{bmatrix}}_{\text{first transformation}}$

$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

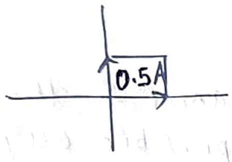


Chp 6 Determinant

determinant of a transformation

$\det\left(\begin{bmatrix} 0.5 & 0.5 \\ -0.5 & 0.5 \end{bmatrix}\right) =$

$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = ad - bc$



Negative determinant → inverting (change) orientation
↳ orientation opposite $\hat{j}$ $\hat{i}$

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a[ei - fh] - b[di - fg] + c[dh - eg]$

- $\det(M_1 M_2) = \det(M_1)\det(M_2)$

Chp 7 Inverse matrices, column space & null space

$\underbrace{2x + 5y + 3z}_{} = -3$

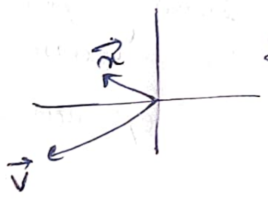
$4x + 0y + 8z = 0$

$\underbrace{1x + 3y + 0z}_{\text{all variables on left}} = \underbrace{2}_{\text{constants}}$

$\Rightarrow \underbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\vec{v}}$

Linear system of eq's

$A\vec{x} = \vec{v}$] means that



So basically after transformation, $\vec{x}$ lands on $\vec{v}$

$A^{-1}A = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ The transformation that does nothing i.e after transformation comes back to original.

$$\boxed{A^{-1}A = I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}$$

$A\vec{x} = \vec{v}$

$A^{-1}A\vec{x} = A^{-1}\vec{v}$

$\vec{x} = A^{-1}\vec{v}$

$A^{-1} = \frac{\vec{x}}{\vec{v}}$

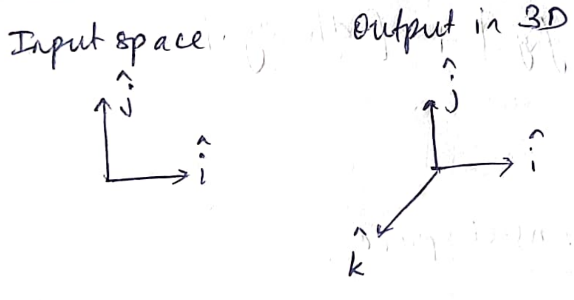
"Rank" ⟷ no. of dimensions in the output of a transformation.

"Column space" of A ⟷ The span of all columns of matrix X OR Set of all possible outputs $A\vec{v}$

eg. $\begin{array}{cc} \begin{bmatrix}3 & 1\\ 4 & 1\end{bmatrix} \\ \uparrow\ \text{where } \hat{i} \text{ lands} & \leftarrow \text{where } \hat{j} \text{ lands}\end{array}$

The columns of matrix tell where the basis vector lands & the span of all those transformed basis vectors gives all possible outputs.

Chapter 8 - Nonsquare matrices as transformations between dimensions



$$\begin{bmatrix}2 & 0\\ -1 & 1\\ -2 & 1\end{bmatrix}$$

↑ where $\hat{i}$ lands ↖ where $\hat{j}$ lands

→ has geometric interpretation of mapping 2 dimensions into 3 dimensions.

Since 2 columns indicate that the input space has two basis vectors, and the three rows indicate that the landing spot for each of these basis vectors is described with 3 separate coordinates

$$\underbrace{\begin{bmatrix} 3 & 1 & 4 \\ 1 & 5 & 9 \end{bmatrix}}_{\text{3 basis vectors}} \Big\} \text{ 2 coordinates for each landing spots}$$

→ 2 rows indicate the landing spot for each of those basis vectors.

3 columns indicate 3 basis vectors. i.e we are starting with 3 dimensions.

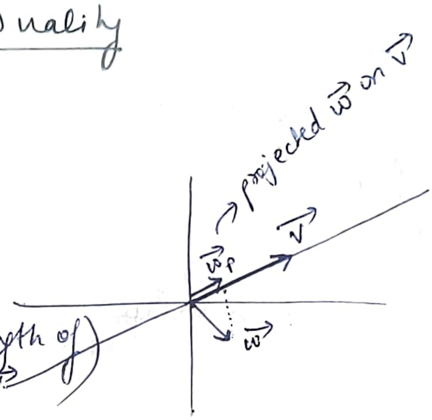
(3D → 2D transformation)

Chapter 9 – Dot Products & Duality

Two vectors of same dimension

$$\underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\vec{v}} \cdot \underbrace{\begin{bmatrix} d \\ e \\ f \end{bmatrix}}_{\vec{w}} = a \cdot d + b \cdot e + c \cdot f$$

$= \left(\text{length of projected } \vec{w}\right)\left(\text{length of } \vec{v}\right)$

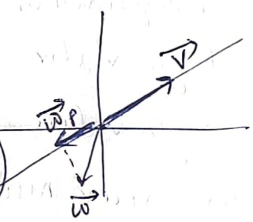


Same dir^n

$\vec{v} \cdot \vec{w} > 0$

(+ve)

$= -\left(\text{length of projected } \vec{w}\right)\left(\text{length of } \vec{v}\right)$



Opp dir^n

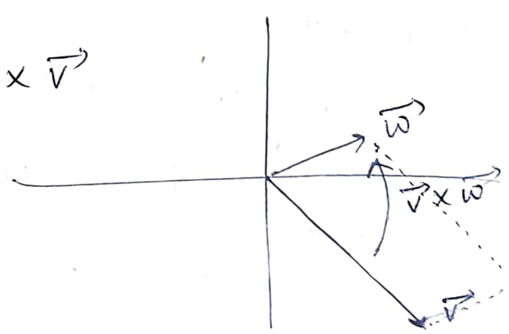
$\vec{v} \cdot \vec{w} < 0$

(−ve)

Note: You can project $\vec{v}$ on $\vec{w}$ as well or vice versa.
Not necessarily project $\vec{w}$ on $\vec{v}$

Chapter 10 – Cross Products

$\underbrace{\vec{v} \times \vec{w}}_{\vec{v} \text{ on right of } \vec{w}}$ = Area of parallelogram $= -\vec{w} \times \vec{v}$



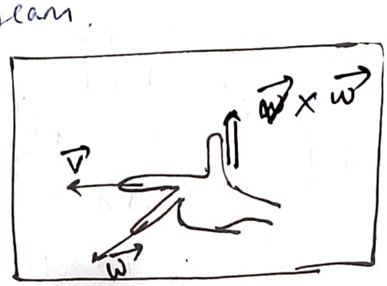
eg $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$$\vec{v} \times \vec{w} = \underbrace{\det\left(\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}\right)}_{\text{area of parallelogram.}}$$

$\hat{i}$ should be on right of $\hat{j}$

$\hat{i} \times \hat{j} = +1$

• $3\vec{v} \times \vec{w} = 3(\vec{v} \times \vec{w})$



$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \det\left(\begin{bmatrix} \overset{(+)}{\hat{i}} & \overset{(-)}{v_1} & \overset{(+)}{w_1} \\ \overset{(-)}{\hat{j}} & \overset{(+)}{v_2} & \overset{(-)}{w_2} \\ \hat{k} & v_3 & w_3 \end{bmatrix}\right)$$

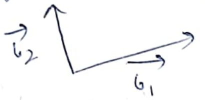
$$= \hat{i}\,(v_2 w_3 - v_3 w_2) - \hat{j}\,(v_1 w_3 - v_3 w_1) + \hat{k}\,(v_1 w_2 - v_2 w_1)$$

Chapter 12 → skip

Chapter 13 → Change of basis

$\hat{i}, \hat{j}$ → basis vectors of standard coordinate system

eg $\vec{b_1}, \vec{b_2}$ are basis vectors



Chp 14 → Eigen vectors & Eigen values

$\boxed{A\vec{v} = \lambda\vec{v}}$

A → Transformation matrix

$\vec{v}$ = Eigen vector

λ → Eigen value

This says that the matrix vector multiplication gives the same result as just the scaling the eigen vector $\vec{v}$ by some value lambda.

LHS : matrix vector multiplication

RHS : scalar vector multiplication

$A\vec{v} = (\lambda I)\vec{v}$

$A\vec{v} - (\lambda I)\vec{v} = \vec{0}$

$(A - \lambda I)\vec{v} = \vec{0}$ $\qquad\qquad$ $\det(A - \lambda I) = 0$

This matrix looks like this

$$\begin{bmatrix} 3-\lambda & 1 & 4 \\ 1 & 5-\lambda & 9 \\ 2 & 6 & 5-\lambda \end{bmatrix}$$

Q Find the eigenvalues of $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

Ans $\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = (3-\lambda)(1-\lambda) - 1(4)$

$= (3 - 4\lambda + \lambda^2) - 4$

$= \lambda^2 - 4\lambda - 1$ → characteristic polynomial

$\lambda^2 - 4\lambda - 1 = 0$

$\lambda = 2 \pm \sqrt{2}$

Learn 3 Points

(1) $\frac{1}{2}\operatorname{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{a+d}{2} = \frac{\lambda_1+\lambda_2}{2} = \overset{(\text{mean})}{m}$

tr → trace of matrix
↳ sum of diagonal values $= a+d$

Mean of the eigen values (λ_1, λ_2) is equal to the mean of the diagonal entries.

(2) $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad-bc = \lambda_1\lambda_2 = \overset{(\text{product})}{p}$

The determinant of a matrix $(ad-bc)$ is equal to the product of the eigen values.

(3) $\boxed{\lambda_1, \lambda_2 = m \pm \sqrt{m^2-p}}$

$m = \frac{a+d}{2}$

$p = ad-bc$